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Geometric generators for braid-like groups

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We study the problem of finding generators for the fundamental group G of a space of the following sort: one removes a family of complex hyperplanes from $\mathbb{C}^n$, or complex hyperbolic space $\mathbb{CH}^n$, or the Hermitian symmetric space for $\mathrm{O}(2,n)$, and then takes the quotient by a discrete group $P\Gamma$. The classical example is the braid group, but there are many similar "braid-like" groups that arise in topology and algebraic geometry. Our main result is that if $P\Gamma$ contains reflections in the hyperplanes nearest the basepoint, and these reflections satisfy a certain property, then G is generated by the analogues of the generators of the classical braid group. We apply this to obtain generators for G in a particular intricate example in $\mathbb{CH}^{13}$. The interest in this example comes from a conjectured relationship between this braid-like group and the monster simple group M, that gives geometric meaning to the generators and relations in the Conway–Simons presentation of $(M\times M):2$. We also suggest some other applications of our machinery.

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1 Introduction

The usual n–strand braid group of the plane was described by Fox and Neuwirth [19] as the fundamental group of $\mathbb{C}^n$, minus the hyperplanes $x_i = x_j$, modulo the action of the group generated by the reflections across them (the symmetric group S_n). The idea is that a path $(x_1(t),\dots,x_n(t))$ in this hyperplane complement corresponds to the braid whose strands are the graphs of the maps $t\mapsto x_i(t)$, regarded as curves in $[0,1]\times\mathbb{C}$. The removal of the hyperplanes corresponds to the condition that the strands do not meet each other. The fundamental group of the hyperplane complement is the pure braid group. Impure braids correspond to paths, not loops, in the hyperplane complement. But these paths become loops once we quotient by S_n, and one obtains the usual braid group.

The term "braid-like" in the title is meant to suggest groups that arise by this construction, generalizing the choices of $\mathbb{C}^n$ and this particular hyperplane arrangement. Artin groups (see Brieskorn [13]) and the braid groups of finite complex reflection groups

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(see Coxeter [17], Shephard and Todd [37] and Bessis [11]) are examples. The problem we address is how to find generators for groups of this sort. We are mainly interested in the case that there are infinitely many hyperplanes, for example coming from hyperplane arrangements in complex hyperbolic space $\mathbb{CH}^n$. Our specific motivation is a conjecture relating the monster finite simple group to the braid-like group associated to a certain hyperplane arrangement in $\mathbb{CH}^{13}$. By our results and those of Heckman [23] and Heckman and Rieken [25], this conjecture now seems approachable.

The general setting is the following: Let X be complex Euclidean space, or complex hyperbolic space, or the Hermitian symmetric space for an orthogonal group $\mathrm{O}(2,n)$. Let $\mathcal{M}$ be a locally finite set of complex hyperplanes in X, $\mathcal{H}$ their union, and $\mathrm{P}\Gamma \subseteq \mathrm{Isom}\, X$ a discrete group preserving $\mathcal{H}$. At this point we have no group Γ in mind; the notation $\mathrm{P}\Gamma$ is just for compatibility with Sections 4–5. Let $a \in X - \mathcal{H}$. Then the associated "braid-like group" means the orbifold fundamental group

$$G_a := \pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma, a\big).$$

See Section 3 for the precise definition of this. In many cases, $\mathrm{P}\Gamma$ acts freely on $X-\mathcal{H}$, so that the orbifold fundamental group is just the ordinary fundamental group.

Our first result describes generators for $\pi_1(X-\mathcal{H}, a)$. This is a subgroup of G_a, since $X-\mathcal{H}$ is an orbifold covering space of $(X-\mathcal{H})/\mathrm{P}\Gamma$. For $H \in \mathcal{M}$ we define in Section 2 a loop $\overrightarrow{aH}^{\circ}$ that travels from a to a point $c \in X-\mathcal{H}$ very near H, encircles H once, and then returns from c to a. We pronounce the notation "a loop H" or "a lasso H". See Section 2 for details and a slight generalization (Theorem 2.3) of the following result:

Theorem 1.1 *The loops $\overrightarrow{aH}^{\circ}$, with H varying over $\mathcal{M}$, generate $\pi_1(X-\mathcal{H}, a)$.* □

If a is generic enough then this follows easily from stratified Morse theory; see Goresky and MacPherson [22]. But in our applications it is very important to take a non-generic, because choosing it to have large $\mathrm{P}\Gamma$–stabilizer can greatly simplify the analysis of $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma, a\big)$. So we prove Theorem 1.1 with no genericity conditions on a. This lack of genericity complicates even the definition of $\overrightarrow{aH}^{\circ}$. For example, $\overrightarrow{aH}^{\circ}$ may encircle some hyperplanes other than H, and this difficulty cannot be avoided in any natural way (see Remark 2.1). One might view Theorem 1.1 as a first step toward a version of stratified Morse theory for non-generic basepoints.

Next we consider generators for the orbifold fundamental group G_a of $(X-\mathcal{H})/\mathrm{P}\Gamma$. In our motivating examples, $\mathrm{P}\Gamma$ is generated by complex reflections in the hyperplanes $H \in \mathcal{M}$. (A complex reflection means an isometry of finite order >1 that pointwise fixes a hyperplane, called its mirror.) So suppose $H \in \mathcal{M}$ is the mirror of some complex

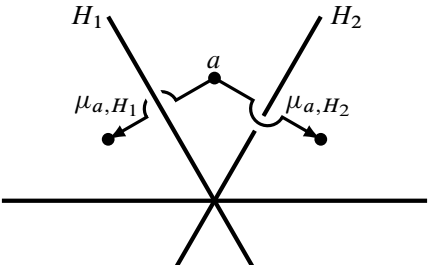

Figure 1: The standard generators for the 3–strand braid group, in terms of the A_2 hyperplane arrangement. The endpoints of the paths are the images of a under the reflections across H_1 and H_2.

reflection in $\mathrm{P}\Gamma$. Then there is a "best" such reflection R_H, characterized by the following properties: every complex reflection in $\mathrm{P}\Gamma$ with mirror H is a power of R_H, and R_H acts on the normal bundle of H by $\exp(2\pi i/n)$, where n is the order of R_H.

For each hyperplane $H \in \mathcal{M}$, we will define in Section 3 an element $\mu_{a,H}$ of G_a; these are the analogues of the standard generators for the classical braid group. Figure 1 illustrates the construction for the 3–strand braid group. (We have drawn the subset of $\mathbb{R}^3 \subseteq \mathbb{C}^3$ with coordinate sum 0, and our paths lie in this $\mathbb{R}^2$ except where they dodge the hyperplanes.) Recall that the definition of $\overrightarrow{aH}$ referred to a point $c \in X - \mathcal{H}$ very near H, and a circle around H based at c. We define $\mu_{a,H}$ to go from a to c as before, then along the portion of this circle from c to $R_H(c)$, then along the R_H–image of the inverse of the path from a to c. This is a path in $X - \mathcal{H}$, not a loop. But R_H sends its beginning point to its end point, so we have specified a loop in $(X - \mathcal{H})/\mathrm{P}\Gamma$. So we may regard $\mu_{a,H}$ as an element of G_a. (Because a may have nontrivial stabilizer, properly speaking we must record the ordered pair $(\mu_{a,H}, R_H)$ rather than just $\mu_{a,H}$; see Section 3 for background on the orbifold fundamental group.)

Referring again to Figure 1, generation of the 3–strand braid group $\pi_1^{\mathrm{orb}}\big((\mathbb{C}^2 - \mathcal{H})/S_3, a\big)$ requires only the loops $\mu_{a,H}$ for the hyperplanes H closest to the basepoint a. This is proven in Fox and Neuwirth [19] for the n–strand braid group. For this to hold, one should choose a as we did here, having the same distance to every facet of the Weyl chamber. Our next theorem shows that the same holds in our more general situation, provided that a is chosen so that there are "enough" mirrors closest to it. This analogy with the braid group is the source of the term "geometric generators" in our title.

Theorem 1.2 *Suppose $\mathcal{C} \subseteq \mathcal{M}$ is the set of hyperplanes closest to a, and that the complex reflections R_C generate $\mathrm{P}\Gamma$, where C varies over $\mathcal{C}$. Suppose that for each $H \in \mathcal{M} - \mathcal{C}$, some power of some R_C moves a closer to p, where p is the point of H closest to a. Then the loops $(\mu_{a,C}, R_C)$ generate $G_a = \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/\mathrm{P}\Gamma, a\big)$.*

The hypothesis of being able to move a closer to the various points p appears very hard to check in practice. But it can be done in some nontrivial cases. The verification of this hypothesis for our motivating example, in a slightly weakened form, occupies most of Section 5.

This motivating example is the setting for a conjectural relation between the monster simple group M and the following braid-like group. We take X to be complex hyperbolic space $\mathbb{CH}^{13}$. We take $\mathrm{P}\Gamma$ to be the group of projective automorphisms of the unique Hermitian lattice L over $\mathbb{Z}[e^{\pi i/3}]$ that has signature $(13,1)$ and whose dual lattice coincides with $(1/\sqrt{-3})L$. (See Definitions 4.1 and 4.2 for two explicit descriptions of L. These are more useful than the quick-but-nonconstructive definition just given, and are what we will actually use in the paper.) Until Section 4, it will be enough to know that $\mathrm{P}\Gamma$ is a finite-covolume discrete subgroup of $\mathrm{Aut}\, X$, generated by its complex reflections of order 3. We take $\mathcal{M}$ to be the set of mirrors of these complex reflections, and $\mathcal{H} := \bigcup_{H\in\mathcal{M}} H$ as usual. Conjecture 1.3 below, from Allcock [4], suggests a relationship between the braid-like group $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma\big)$ and the monster simple group.

It turns out that any two of the mirrors are $\mathrm{P}\Gamma$–equivalent, so the image of $\mathcal{H}$ in $X/\mathrm{P}\Gamma$ is irreducible. The positively oriented boundary, of a small disk in $X/\mathrm{P}\Gamma$ transverse to a generic point of this image, determines a conjugacy class in $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma\big)$. Following knot theory terminology, we call the elements of this conjugacy class *meridians*.

Conjecture 1.3 [4] *The quotient of $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma\big)$ by the normal subgroup generated by the squares of the meridians is the semidirect product of $M\times M$ by $\mathbb{Z}/2$. Here M is the monster simple group and $\mathbb{Z}/2$ exchanges the factors in the obvious way.*

Presumably, any proof of this will require generators and relations for $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma\big)$, which is the motivation for the current paper. In [9] the second author found a point $\tau\in X-\mathcal{H}$ (called $\overline{\rho}$ there) such that the set $\mathcal{C}$ of mirrors closest to τ has size 26, and showed that their complex reflections generate $\mathrm{P}\Gamma$. Describing τ precisely requires some preparation, so we refer to Definition 4.1 for details and for now just mention that τ has nontrivial $\mathrm{P}\Gamma$–stabilizer. Therefore the corresponding meridians are ordered pairs $(\mu_{\tau,C}, R_C)$ rather than just bare paths $\mu_{\tau,C}$. Taking τ as our basepoint, we announce the following result, which we regard as a significant step toward Conjecture 1.3.

Theorem 1.4 *The 26 meridians $(\mu_{\tau,C}, R_C)$, with C varying over the 26 mirrors of $\mathcal{M}$ closest to τ, generate $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma,\tau\big)$.* □

We wish this were a corollary of Theorem 1.2. Unfortunately the hypothesis there about being able to move τ closer to the various $p \in H \in \mathcal{M}$ fails badly. Instead, the proof goes as follows. First, in Section 5 we prove Theorem 1.5 below, which is the analogue of Theorem 1.4 with a different basepoint ρ in place of τ. For this basepoint, one can (almost) verify the hypothesis of Theorem 1.2 about being able to move ρ closer to the various points p. Considerable calculation is required, plus extra work dealing with the fact that this hypothesis almost holds but not quite.

Then, given Theorem 1.5, one joins τ and ρ by a path and uses it to identify the fundamental groups based at these points. One can then prove Theorem 1.4 by studying how these groups' generators are related. The argument is delicate, of more specialized interest, and has little in common with the ideas in this paper. Therefore it will appear separately [6].

After stating Theorem 1.5 we will explain our real reason for preferring τ over ρ as a basepoint. An additional reason is that ρ is not a point of $\mathbb{CH}^{13}$. Rather, it is a cusp of $\mathrm{P}\Gamma$, and in particular lies in the sphere at infinity $\partial\mathbb{CH}^{13}$. This complicates things in two ways. First, there are infinitely many mirrors "closest" to ρ, indexed by the elements of a certain 25–dimensional integral Heisenberg group. And second, the definition of the meridians "based at ρ" requires more care. This leads to Theorem 1.5 giving an infinite generating set, consisting of paths that are more complicated than those of Theorem 1.4.

One can work through these complications as follows. As we did for τ, we refer to Section 4 for the precise definition of ρ. All we need for now is that it is a cusp of $\mathrm{P}\Gamma$ and that there is a closed horoball A centered at ρ that misses $\mathcal{H}$ (Lemma 4.3). We choose any basepoint a inside A. We call the mirrors that come closest to A the "Leech mirrors". The name comes from the fact that they are indexed by the elements of (a central extension of) the complex Leech lattice Λ; in particular there are infinitely many of them. If C is a Leech mirror, let $b_C \in A$ be the point of A nearest it. Then $\mu_{a,A,C}$ is defined to be the geodesic $\overline{ab_C} \subseteq A$ followed by $\mu_{b_C,C}$ followed by $R_C(\overline{b_C a})$. See Figure 3 for a picture. These are meridians in the sense of Conjecture 1.3, and we call them the Leech meridians. As before, because a may have nontrivial $\mathrm{P}\Gamma$–stabilizer, the meridian associated to C is really the ordered pair $(\mu_{a,A,C}, R_C)$ rather than just the bare path $\mu_{a,A,C}$.

Theorem 1.5 *The orbifold fundamental group $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/\mathrm{P}\Gamma, a\big)$ is generated by the Leech meridians, that is, by the loops $(\mu_{a,A,C}, R_C)$ with C varying over the (infinitely many) mirrors closest to ρ.*

We promised to explain the real reason we prefer Theorem 1.4 to Theorem 1.5, that is, why we prefer the basepoint to be τ rather than ρ. It is because the 26 meridians of

Theorem 1.4 are closely related to the coincidences that motivated Conjecture 1.3. In particular, by results of Basak [10] they satisfy the braid and commutation relations specified by the incidence graph of the 13 points and 13 lines of $\mathbb{P}^2(\mathbb{F}_3)$. That is, two generators x, y commute ($xy = yx$) or braid ($xyx = yxy$) according to whether the corresponding nodes of this graph are unjoined or joined. We call the abstract group with 26 generators, subject to these relations, the Artin group of $\mathbb{P}^2(\mathbb{F}_3)$.

There are a family of presentations of the "bimonster" $(M \times M) \rtimes \mathbb{Z}/2$ as a quotient of this Artin group, due to Conway, Ivanov, Norton, Simons and Soicher in various combinations. All of them impose the relations that the generators have order 2, which yields the (infinite) Coxeter group whose Coxeter diagram is the same incidence graph. Modding out by the squares of meridians in Conjecture 1.3 corresponds to taking this quotient. There are several different ways to specify additional relations that collapse this Coxeter group to the bimonster. The most natural one for our purposes seems to be the "$\widetilde{A}_{11}$ deflation" relations of Conway and Simons [16], because these have a good geometric interpretation in terms of the $\mu_{\tau,C}$ and a certain copy of $\mathbb{CH}^9$ in $\mathbb{CH}^{13}$. See Heckman [23] and Heckman and Rieken [25] for more details. This complex-hyperbolic reinterpretation of the bimonster's deflation relations convinces us that τ is the "right" basepoint for further work on Conjecture 1.3.

We also hope that our techniques will be useful more generally. For example, they might be used to give generators for the fundamental group of the moduli space of Enriques surfaces. Briefly, this is the quotient of the Hermitian symmetric space for $\mathrm{O}(2,10)$, minus a hyperplane arrangement, by a certain discrete group. See Namikawa [34] for the original result and Allcock [3] for a simpler description of the arrangement. The symmetric space has two orbits of 1–dimensional cusps, one of which misses all the hyperplanes. Taking this as the base "point", the hyperplanes nearest it are analogues of the Leech mirrors. It seems reasonable to hope that the meridians associated to these mirrors generate the orbifold fundamental group.

There are many spaces in algebraic geometry with a description $(X - \mathcal{H})/\mathrm{P}\Gamma$ of the sort we have studied. For example, the discriminant complements of many hypersurface singularities (see Looijenga [29; 30]), the moduli spaces of del Pezzo surfaces (see Allcock, Carlson and Toledo [7], Kondō [27] and Heckman and Looijenga [24]), the moduli space of curves of genus four (see Kondō [28]), the moduli spaces of smooth cubic threefolds (see Allcock, Carlson and Toledo [8] and Looijenga and Swierstra [33]) and cubic fourfolds (see Looijenga [32]), and the moduli spaces of lattice-polarized K3 surfaces (see Nikulin [35] and Dolgachev [18]). The orbifold fundamental groups of these spaces are "braid-like" in the sense of this paper, and we hope that our methods will be useful in understanding them.

The paper is organized as follows. In Section 2 we study the fundamental group $\pi_1(X-\mathcal{H},a)$, in particular proving Theorem 1.1. The proof relies on van Kampen's theorem. In Section 3 we study $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/P\Gamma,a\big)$, in particular proving Theorem 1.2. The core of that proof is Lemma 3.1, which is more general than needed for Theorem 1.2. The extra generality is needed for our application to $\mathbb{C}\mathbb{H}^{13}$. Section 4 gives background on complex hyperbolic space and the particular hyperplane arrangement referred to in Conjecture 1.3 and Theorems 1.4–1.5. Finally, Section 5 proves Theorem 1.5. Most of the proof consists of tricky calculations verifying the hypothesis of Theorem 1.2 that the basepoint can be moved closer to the various points $p\in H$. In a few cases this is not possible, so we have to do additional work.

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2 Loops in arrangement complements

For the rest of the paper we fix $X=$ one of three spaces, $\mathcal{M}=$ a locally finite set of hyperplanes in X, and $\mathcal{H}=$ their union. The precise assumption on X is that it is complex affine space with its Euclidean metric, or complex hyperbolic space, or the Hermitian symmetric space for $\mathrm{O}(2,n)$. To understand the general machinery in this section and the next, it is enough to think about the affine case. In our application in Section 5 we specialize to the case that X is complex hyperbolic 13–space; for background see Section 4. Most of the other potential applications mentioned in the introduction would use the $\mathrm{O}(2,n)$ case.

We will freely use a few standard properties of X: it is contractible, and its natural Riemannian metric is complete and has nonpositive sectional curvature. By [12, Theorems II.1A.6 and II.4.1], it follows that X is a complete CAT(0) metric space. We will also use the usual notions of complex lines and complex hyperplanes in X, both of which are totally geodesic.

For $b,c\in X$ we write $\overline{bc}$ for the geodesic segment from b to c. Now suppose $b,c\notin\mathcal{H}$. It may happen that $\overline{bc}$ meets $\mathcal{H}$, so we will define a perturbation $\widetilde{bc}$ of $\overline{bc}$ in the obvious way. The notation may be pronounced "b dodge c" or "b detour c". We write $\overline{bc}^{\,\mathbb{C}}$ for the complex line containing $\overline{bc}$. By the local finiteness of $\mathcal{M}$, $\overline{bc}^{\,\mathbb{C}}\cap\mathcal{H}$ is a discrete set. Consider the path gotten from $\overline{bc}$ by using positively oriented semicircular detours in $\overline{bc}^{\,\mathbb{C}}$ around the points of $\overline{bc}\cap\mathcal{H}$ in place of the corresponding segments of $\overline{bc}$.

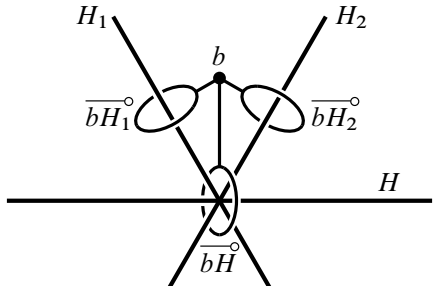

Figure 2

After taking the radius of these detours small enough, the construction makes sense and the resulting homotopy class in $X-\mathcal{H}$ (rel endpoints) is radius-independent. This homotopy class is what we mean by $\overline{bc}$.

At times we will need to speak of the "restriction of $\mathcal{H}$ at p", where p is a point of X. So we write $\mathcal{M}_p$ for the set of hyperplanes in $\mathcal{M}$ that contain p, and $\mathcal{H}_p$ for their union.

Now suppose $b \in X-\mathcal{H}$ and $H \in \mathcal{M}$. We will define a homotopy class $\overline{bH}^\circ$ of loops in $X-\mathcal{H}$ based at b; the notation can be pronounced "b loop H" or "b lasso H". We write p for the point of H nearest b. It exists and is unique by the convexity of H and the nonpositive curvature of X [12, Proposition II.2.4]. Let U be a ball around p that is small enough that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$, and let c be a point of $(\overline{bp} \cap U) - \{p\}$. Consider the circular loop in $\overline{bp}^{\,\mathbb{C}}$ centered at p, based at c, and traveling once around p in the positive direction. It misses $\mathcal{H}$, because under the exponential map $T_pX \to X$ the elements of $\mathcal{M}_p$ correspond to complex hyperplanes in T_pX, while $\overline{bp}^{\,\mathbb{C}}$ corresponds to a complex line. And the line misses the hyperplanes except at 0, because $b \notin \mathcal{H}$. Finally, $\overline{bH}^\circ$ means $\overline{bc}$ followed by this circular loop, followed by reverse$(\overline{bc})$.

Remark 2.1 (Caution in the non-generic case) This definition has some possibly unexpected behavior when b is not generic. For example, take $\mathcal{M}$ to be the A_2 arrangement in $X = \mathbb{C}^2$, let H be one of the three hyperplanes, and take $b \in X-\mathcal{H}$ orthogonal to H. We write H_1 and H_2 for the other two hyperplanes; see Figure 2. It is easy to see that $\overline{bH}^\circ$ encircles *all three hyperplanes*, not just H. Furthermore, this phenomenon cannot be avoided by any procedure that respects symmetry. To explain this we note that $\pi_1(X-\mathcal{H}, b) \cong \mathbb{Z} \times F_2$, where the first factor is generated by $\overline{bH}^\circ$ and the second is free on $\overline{bH_1}^\circ$ and $\overline{bH_2}^\circ$. Let f be the isometry of X that fixes b and acts by negation on H. It exchanges H_1 and H_2. So the action of f on $\pi_1(X-\mathcal{H}, b)$ fixes $\overline{bH}^\circ$ and swaps the other two generators. It follows that the group of fixed points of f in $\pi_1(X-\mathcal{H}, b)$ is just the first factor $\mathbb{Z}$. So any symmetry-respecting definition of $\overline{bH}^\circ$ must give some power of our definition, at least in this example.

The main result of this section, Theorem 2.3, shows that the various $\overrightarrow{bH}$ generate $\pi_1(X-\mathcal{H}, b)$. But for our applications to $\mathbb{CH}^{13}$ in Section 5 it will be useful to formulate the fundamental group with a "fat basepoint" A in place of b, in the sense explained below. This is because we will want to choose our basepoint to be a cusp of a finite-covolume discrete subgroup of $\mathrm{PU}(13,1) = \mathrm{Aut}\,\mathbb{CH}^{13}$. Strictly speaking this is not possible, since a cusp is not a point of $\mathbb{CH}^{13}$. So we will use a closed horoball A centered at that boundary point in place of a basepoint. For purposes of understanding the current section, the reader may take A to be a point.

2.2 Standing assumptions Our assumptions so far are that X is one of three spaces, $\mathcal{M}$ is a locally finite hyperplane arrangement, and $\mathcal{H}$ is the union of the hyperplanes. Henceforth we also assume that A is a nonempty closed convex subset of X, disjoint from $\mathcal{H}$. To avoid some minor technical issues, we assume two more properties, both automatic when A is a point. First, for every $H \in \mathcal{M}$, there is a unique point of A closest to H. (This holds if A is strictly convex, by the argument used for [12, Proposition II.2.4].) Second, some group of isometries of X, preserving $\mathcal{M}$ and A, acts cocompactly on the boundary ∂A. (This holds in our application to $\mathbb{CH}^{13}$ because the stabilizer of a cusp acts cocompactly on any horosphere centered there.) We will think of the open r–neighborhood B_r of A for some $r > 0$ as being "like" an open ball. If A is a point then of course B_r actually is a ball. In any case, the convexity of A implies that of B_r by [12, Corollary II.2.5] and the remark preceding it.

Because $A - \mathcal{H} = A$ is simply connected (even contractible), the fundamental groups of $X-\mathcal{H}$ based at any two points of A are canonically identified. So we write just $\pi_1(X-\mathcal{H}, A)$ for $\pi_1(X-\mathcal{H}, a)$, where a is any point of A. If $c \in X$ then we define $\overline{Ac}$ as $\overline{bc}$, where b is the point of A nearest c. If $c \notin \mathcal{H}$ then we also define $\widehat{Ac}$ as $\widehat{bc}$. Similarly, if $H \in \mathcal{M}$ then we define $\overrightarrow{AH}$ as $\overrightarrow{bH}$, where b is the point of A closest to H. We sometimes write $\overline{b,c}$ and $\widehat{b,c}$ and $\overrightarrow{b,H}$ for $\overline{bc}$ and $\widehat{bc}$ and $\overrightarrow{bH}$, and similarly for $\overline{A,c}$ and $\widehat{A,c}$ and $\overrightarrow{A,H}$.

Theorem 2.3 (π_1 of a ball-like set minus hyperplanes) *Let B_r be the open r–neighborhood of A, where $r \in (0,\infty]$. Then $\pi_1(B_r - \mathcal{H}, A)$ is generated by the loops $\overrightarrow{AH}$ for which $d(A,H) < r$.*

If A is compact, for example $A = \{a\}$, then this gives a finite number of generators for $\pi_1(B_r - \mathcal{H})$. But if A is non-compact then the number of generators may be infinite. This happens in Theorem 1.5, where A is a horoball in $\mathbb{CH}^{13}$.

The rest of the section is devoted to the proof of Theorem 2.3, beginning with two lemmas.

Lemma 2.4 (π_1 of $\mathbb{C}^n$ minus hyperplanes through the origin) *Suppose X is complex Euclidean space, every $H \in \mathcal{M}$ contains the origin 0, and $c \in X - \mathcal{H}$. Write $\frac{1}{2}X$ for the open halfspace of X that contains c and is bounded by the real orthogonal complement to $\overline{c0}$. (In the trivial case $\mathcal{M} = \varnothing$ we also assume $c \neq 0$, so that $\frac{1}{2}X$ is defined.)*

(1) *If c is not orthogonal to any element of $\mathcal{M}$, then $\pi_1(X - \mathcal{H}, c)$ is generated by $\pi_1\big(\frac{1}{2}X - \mathcal{H}, c\big)$.*

(2) *If c is orthogonal to some hyperplane $H \in \mathcal{M}$, then $\pi_1(X - \mathcal{H}, c)$ is generated by $\pi_1\big(\frac{1}{2}X - \mathcal{H}, c\big)$ together with any element of $\pi_1(X - \mathcal{H}, c)$ having linking number ± 1 with H, for example $\overline{cH}^{\circ}$.*

Proof (2) Write H' for the translate of H containing c. Every point of $X - \mathcal{H}$ is a nonzero scalar multiple of a unique point of $H' - \mathcal{H}$. It follows that $X - \mathcal{H}$ is the topological product of $H' - \mathcal{H} \subseteq \frac{1}{2}X - \mathcal{H}$ with $\mathbb{C} - \{0\}$. The map $\pi_1(X - \mathcal{H}, c) \to \mathbb{Z}$ corresponding to the projection to the second factor is the linking number with H. All that remains to prove is that $\overline{cH}^{\circ}$ has linking number 1 with H. In fact more is true: essentially by definition, this loop generates the fundamental group of the factor $\mathbb{C} - \{0\}$.

(1) We define H as the complex hyperplane through 0 that is orthogonal to $\overline{c0}$. We apply the argument from the previous paragraph to $\mathcal{M}' = \mathcal{M} \cup \{H\}$ and $\mathcal{H}' = \mathcal{H} \cup H$. Using $\frac{1}{2}X - \mathcal{H} = \frac{1}{2}X - \mathcal{H}'$ yields

$$(*) \qquad \pi_1(X - \mathcal{H}', c) = \pi_1\big(\tfrac{1}{2}X - \mathcal{H}', c\big) \times \langle\, \overline{cH}^{\circ} \,\rangle = \pi_1\big(\tfrac{1}{2}X - \mathcal{H}, c\big) \times \langle\, \overline{cH}^{\circ} \,\rangle.$$

Our goal is to show that the first factor on the right surjects to $\pi_1(X - \mathcal{H}, c)$. Let γ be any element of $\pi_1(X - \mathcal{H}', c)$ that is freely homotopic to the boundary of a small disk transverse to H at a generic point of H. It dies under the natural map $\pi_1(X - \mathcal{H}', c) \to \pi_1(X - \mathcal{H}, c)$. Because γ has linking number ± 1 with H, the product decomposition $(*)$ shows that every element of $\pi_1(X - \mathcal{H}', c)$ can be written as a power of γ times an element of $\pi_1\big(\frac{1}{2}X - \mathcal{H}, c\big)$. It is standard that $\pi_1(X - \mathcal{H}', c) \to \pi_1(X - \mathcal{H}, c)$ is surjective. (Take any loop in $X - \mathcal{H}$, perturb it to miss H, and then regard it as a loop in $X - \mathcal{H}'$.) Since this map kills γ, it must send the subgroup $\pi_1\big(\frac{1}{2}X - \mathcal{H}, c\big)$ of $\pi_1(X - \mathcal{H}', c)$ surjectively to $\pi_1(X - \mathcal{H}, c)$. □

Lemma 2.5 (π_1 of a ball-like set with a bump, minus hyperplanes) *Suppose $r > 0$, B is the open r–neighborhood of A, and $p \in \partial B$. Assume U is any open ball centered at p, small enough that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$.*

(1) *If no $H \in \mathcal{M}_p$ is orthogonal to $\overline{Ap}$, then $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$.*

(2) *If some $H \in \mathcal{M}_p$ is orthogonal to $\overline{Ap}$, then $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$, together with any loop of the form $\alpha\lambda\alpha^{-1}$, for example $\overline{A\mathring{H}}$. Here α is a path in $B - \mathcal{H}$ from A to a point of $(B \cap U) - \mathcal{H}$ and λ is a loop in $U - \mathcal{H}$, based at that point and having linking number ± 1 with H.*

Proof For uniformity, in case (1) we choose some path α in $B - \mathcal{H}$ beginning in A and ending in $(B \cap U) - \mathcal{H}$. In both cases we write c for the final endpoint of α; without loss of generality we may suppose $c \in \overline{Ap} - \{p\}$. Van Kampen's theorem shows that $\pi_1\big((B \cup U) - \mathcal{H}, c\big)$ is generated by the images of $\pi_1(B - \mathcal{H}, c)$ and $\pi_1(U - \mathcal{H}, c)$. We claim that $\pi_1(U - \mathcal{H}, c)$ is generated by the image of $\pi_1\big((B \cap U) - \mathcal{H}, c\big)$, supplemented in case (2) by λ.

Assuming this, we move the basepoint from c into A along reverse(α). This identifies $\pi_1(B - \mathcal{H}, c)$ with $\pi_1(B - \mathcal{H}, A)$ and identifies λ with $\alpha\lambda\alpha^{-1}$. It follows that $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$, supplemented in case (2) by $\alpha\lambda\alpha^{-1}$. This is the statement of the lemma.

So it suffices to prove the claim. We transfer this to a problem in the tangent space T_pX by the exponential map and its inverse (written log). So we must show that $\pi_1(\log U - \log \mathcal{H}_p, \log c)$ is generated by the image of $\pi_1\big(\log(B \cap U) - \log \mathcal{H}_p, \log c\big)$, supplemented in case (2) by $\log \lambda$. The key to this is that the vertical arrows in the commutative diagram

$$\begin{array}{ccc} \log(B \cap U) - \log \mathcal{H}_p & \longrightarrow & \log U - \log \mathcal{H}_p \\ \downarrow & & \downarrow \\ \frac{1}{2}T_pX - \log \mathcal{H}_p & \longrightarrow & T_pX - \log \mathcal{H}_p \end{array} \tag{2-1}$$

are homotopy equivalences, as we explain next. Here $\frac{1}{2}T_pX$ is as in Lemma 2.4: the open halfspace containing $\log c$ and bounded by the (real) orthogonal complement of $\log(\overline{cp}) = \overline{\log c, 0}$.

The right vertical arrow in (2-1) is a weak homotopy equivalence by a standard scaling argument. Namely, any compact set in T_pX can be homotoped (by scaling) until it lies in $\log U$. Since scaling preserves $\log \mathcal{H}_p$, any compact set in $T_pX - \log \mathcal{H}_p$ can be homotoped into $\log U - \log \mathcal{H}_p$. This shows that $\log U - \log \mathcal{H}_p \to T_pX - \log \mathcal{H}_p$ is a weak homotopy equivalence.

After making a few observations, the same argument will also work for the left vertical arrow. First, we claim that $B \subseteq X$ contains an open ball D with p in its boundary.

To understand the rest of the proof below, one may restrict to the case $A = \{a\}$, when this claim is trivial by taking $D = B$. The reason we use D in the general case is that ∂B need not be C^∞ and we wish to avoid fussing over its degree of differentiability. To construct D, take q to lie in the interior of $\overline{Ap}$, and then take D to be the open $d(q,p)$–ball around q. This lies in B by the triangle inequality.

Second, we claim that $\partial(\log D)$ is tangent to the boundary of $\frac{1}{2}T_pX$. To see this, note that the segment $\overline{qp}$ is orthogonal to ∂D at p, because any sphere in a Riemannian manifold is orthogonal to its radial segments. (This is Gauss' lemma [20, Lemma 3.70].) Since $\partial\big(\frac{1}{2}T_pX\big)$ was defined to be orthogonal at p to $\overline{Ap}$, which contains $\overline{qp}$, it follows that $\partial(\log D)$ is tangent to the boundary of $\frac{1}{2}T_pX$.

Third, we claim that $\log B$ lies in $\frac{1}{2}T_pX$. Otherwise, since it is open, $\log B$ would contain some $x \in T_pX - \overline{\frac{1}{2}T_pX}$. Because $\partial(\log D)$ is tangent to $\partial\big(\frac{1}{2}T_pX\big)$, continuing the line segment $\overline{x0}$ through 0 yields some element y of $\log D \subseteq \log B$. Since $\overline{xy}$ is a segment in T_pX passing through the origin, its image $\exp(\overline{xy})$ under the exponential map is a geodesic of X. Now, B contains $\exp(\overline{xy})$ by convexity, so it contains p, which is a contradiction.

We can now adapt the scaling argument. Having proven $\log B \subseteq \frac{1}{2}T_pX$, it now makes sense to speak of the left vertical arrow in (2-1), and we will show that it is a weak homotopy equivalence. Because $\partial(\log D)$ is tangent to $\partial\big(\frac{1}{2}T_pX\big)$, any compact set in $\frac{1}{2}T_pX$ may be scaled down until it lies in $\log D$, hence $\log B$. By further scaling, we may shrink it into $\log(B \cap U)$. Since scaling preserves $\log \mathcal{H}_p$, we have shown that any compact subset of $\frac{1}{2}T_pX - \log \mathcal{H}_p$ may be homotoped by scaling until it lies in $\log(B \cap U) - \log \mathcal{H}_p$. It follows that $\log(B \cap U) - \log \mathcal{H}_p \to \frac{1}{2}T_pX - \log \mathcal{H}_p$ is a weak homotopy equivalence, as claimed.

We have shown that the vertical arrows in (2-1) are weak homotopy equivalences. It follows from Whitehead's theorem that they are homotopy equivalences. (One could avoid quoting this theorem by refining the scaling arguments.) To prove the theorem it now suffices to show that $\pi_1(T_pX - \log \mathcal{H}_p, \log c)$ is generated by the image of $\pi_1\big(\frac{1}{2}T_pX - \log \mathcal{H}_p, \log c\big)$, supplemented in case (2) by $\log \lambda$. This is just Lemma 2.4, completing the proof. □

Proof of Theorem 2.3 Let R be the set of $r \in (0,\infty]$ for which the conclusion of the theorem holds. Recall our assumption from 2.2 that some group of isometries of X acts cocompactly on ∂A while preserving $\mathcal{M}$ and A. This implies that the distances $d(A,H)$ are bounded away from 0 as H varies over $\mathcal{M}$. Therefore $B_r \cap \mathcal{H} = \varnothing$ for all sufficiently small r. It follows that R contains all small enough r. We will show below that if $r \in R - \{\infty\}$ then $[r, r+\delta) \subseteq R$ for some $\delta > 0$. Also, we obviously

have $B_r = \bigcup_{q<r} B_q$ for any $r \in (0,\infty]$. Therefore $(0,r) \subseteq R$ implies $(0,r] \subseteq R$. The connectedness of $(0,\infty]$ then implies $R = (0,\infty]$, proving the theorem.

So fix $r \in R - \{\infty\}$; we will exhibit $\delta > 0$ such that $[r, r+\delta) \subseteq R$. Since $r \in R$ we know that $\pi_1(B_r - \mathcal{H}, A)$ is generated by the loops $\overline{AH}^{\circ}$ for which $d(A,H) < r$. We abbreviate B_r to B and define S as the "sphere" ∂B. For each $p \in S$ there is an open ball U_p centered at p such that $U_p \cap \mathcal{H} = U_p \cap \mathcal{H}_p$. The cocompact action on ∂A we used in the previous paragraph is also cocompact on S. Also, we may choose the balls U_p so that the set of all of them is preserved by this action. It follows that there exists $\delta > 0$ such that $B_{r+\delta}$ is covered by B and all the U_p.

To prove $[r, r+\delta) \subseteq R$, suppose we are given some $r' \in (r, r+\delta)$, and write B' for $B_{r'}$. Since B' is covered by B and the U_p, every mirror that meets B' either meets B or is tangent to S. So we must prove that $\pi_1(B' - \mathcal{H}, A)$ is generated by $\pi_1(B - \mathcal{H}, A)$ and the $\overline{AH}^{\circ}$ with $H \in \mathcal{M}$ tangent to S. Lemma 2.5 says that $\pi_1\big((B \cup U_p) - \mathcal{H}, A\big)$ is generated by $\pi_1(B - \mathcal{H}, A)$, supplemented by $\overline{AH}^{\circ}$ if p is the point of tangency of S with some $H \in \mathcal{M}$.

For $p \in S$ we define $V_p = (B \cup U_p) \cap B'$. It is easy to see that the inclusion $V_p - \mathcal{H} \to (B \cup U_p) - \mathcal{H}$ is a homotopy equivalence. (Retract points of $U_p - B'$ along geodesics toward p.) So $\pi_1(V_p - \mathcal{H}, A)$ is generated by $\pi_1(B - \mathcal{H}, A)$, supplemented by $\overline{AH}^{\circ}$ if p is the point of tangency of S with some $H \in \mathcal{M}$. Because $B' = \bigcup_{p \in S} V_p$, repeatedly using van Kampen's theorem shows that $\pi_1(B' - \mathcal{H}, A)$ is generated by the images therein of all the $\pi_1(V_p - \mathcal{H}, A)$, finishing the proof.

This use of van Kampen's theorem requires checking that every set gotten from the V_p by repeated unions and intersections is connected. To help verify this, we call a subset Y of X star-shaped (around A) if it contains A and the geodesics $\overline{Ay}$ for all $y \in Y$. The lemma below shows that each $B \cup U_p$ is star-shaped. Intersecting with B' preserves star-shapedness and yields V_p. Since unions and intersections of star-shaped sets are again star-shaped, our repeated application of van Kampen's theorem is legitimate. □

Lemma 2.6 *In the notation of the previous proof, $B \cup U_p$ is star-shaped around A.*

Proof We must show that $y \in U_p$ implies $\overline{Ay} \subseteq B \cup U_p$. It suffices to prove $\overline{yz} \subseteq U_p$, where z is the point of ∂B closest to y. (We remarked above that the convexity of A implies that of B, and the uniqueness of z then follows from [12, Proposition II.2.4].) Note that $\overline{zp}$ lies in the closure of B, since z and p do and B (hence its closure) is convex.

Consider the triangle p, y, z in X and a comparison triangle, meaning a triangle p', y', z' in $\mathbb{R}^2$ with the same edge lengths. We write θ for the angle between $\overline{zy}$

and $\overline{zp}$ at z, and similarly for θ'. Since X is a CAT(0) metric space we have $\theta' \geq \theta$ by [12, Proposition II.3.1]. And we have $\theta \geq \frac{\pi}{2}$, because otherwise one could find a path from y to A of length $< d(y, A)$. (A point of $\overline{zp}$ very near p lies in B by convexity, hence has distance $\leq r$ to A. It is also closer to y than z is, by the inequality $\theta < \frac{\pi}{2}$ and the CAT(0) inequality.) Therefore θ' is the largest angle of the comparison triangle, so $\overline{p'y'}$ is its longest edge. Since the two triangles have the same edge lengths, $\overline{pz}$ is shorter than $\overline{py}$, so $z \in U_p$. Since U_p is a ball, all of $\overline{yz}$ lies in U_p. □

3 Loops in quotients of arrangement complements

We continue using the notation X, $\mathcal{M}$ and $\mathcal{H}$ from the previous section. We also suppose a group $\mathrm{P}\Gamma$ acts isometrically and properly discontinuously on X, preserving $\mathcal{H}$. Our goal is to understand the orbifold fundamental group of $(X - \mathcal{H})/\mathrm{P}\Gamma$. We use the following definition from [31] and [10]; more general formulations exist [36; 26].

Fixing a basepoint $a \in X - \mathcal{H}$, consider the set of pairs (γ, g) where $g \in \mathrm{P}\Gamma$ and γ is a path in $X - \mathcal{H}$ from a to $g(a)$. We regard one such pair as equivalent to another one (γ', g') if $g = g'$ and γ and γ' are homotopic in $X - \mathcal{H}$, rel endpoints. The orbifold fundamental group $G_a := \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/\mathrm{P}\Gamma, a\big)$ means the set of equivalence classes. The group operation is $(\gamma, g) \cdot (\gamma', g') = (\gamma \text{ followed by } g \circ \gamma', gg')$. Projection of (γ, g) to g defines a homomorphism $G_a \to \mathrm{P}\Gamma$. It is surjective because $X - \mathcal{H}$ is connected. The kernel is obviously $\pi_1(X - \mathcal{H}, a)$, yielding the exact sequence

$$1 \to \pi_1(X - \mathcal{H}, a) \to G_a \to \mathrm{P}\Gamma \to 1. \tag{3-1}$$

Although we don't need it, we remark that if a has trivial $\mathrm{P}\Gamma$ stabilizer then there is a simpler $\mathrm{P}\Gamma$–invariant description of the orbifold fundamental group. Writing o for the orbit of a, we define $G_o := \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/\mathrm{P}\Gamma, o\big)$ as the set of $\mathrm{P}\Gamma$–orbits on the homotopy classes (rel endpoints) of paths in $X - \mathcal{H}$ that begin and end in o. The $\mathrm{P}\Gamma$–action is the obvious one: $g \in \mathrm{P}\Gamma$ sends a path γ to $g \circ \gamma$. To define $\gamma\gamma'$, where $\gamma, \gamma' \in G_o$, one translates γ' so that it begins where γ ends, and then composes paths in the usual way. Well-definedness of multiplication, and the identification with the definition of G_a, uses the fact that every path starting in o has a unique translate starting at a.

A complex reflection means a finite-order isometry of X whose fixed-point set is a complex hyperplane, called its mirror. In our applications, $\mathrm{P}\Gamma$ is generated by complex reflections whose mirrors are hyperplanes in $\mathcal{M}$. This leads to certain natural elements of the orbifold fundamental group: for $H \in \mathcal{M}$ we next define a loop $\mu_{a,H} \in G_a$ which is a fractional power of $\overline{aH}^{\circ}$. (The meridians of Conjecture 1.3 and Theorems 1.4–1.5

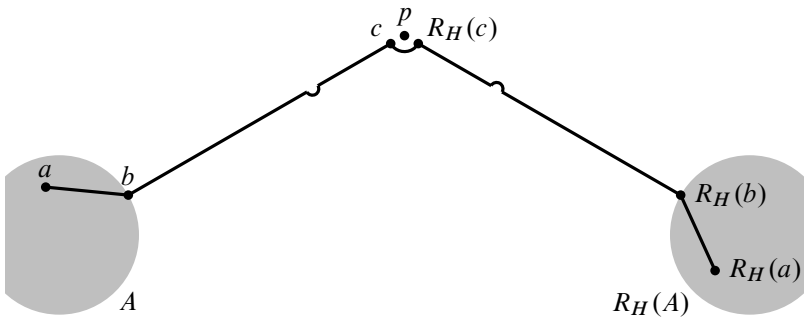

Figure 3: The path $\mu_{a,A,H}$ goes from left to right. Here R_H is the complex reflection of order 3, acting by counterclockwise rotation by $\frac{2\pi}{3}$. The hyperplane H is not shown except for its point p closest to A. The small semicircles indicate that the path from b to c may detour around some points of $\mathcal{H}$.

are special cases of these loops.) Write p for the point of H closest to a and n_H for the order of the cyclic group generated by the complex reflections in $P\Gamma$ with mirror H. Write R_H for the isometry of X that fixes H pointwise and acts on its normal bundle by $\exp(2\pi i/n_H)$. This is an element of $P\Gamma$, and is either a complex reflection or the identity map. The latter holds when $n_H = 1$, ie H is not the mirror of any reflection in $P\Gamma$.

Recall that the definition of $\overline{aH}^{\circ}$ involved a point c of $\overline{ap}$ very near p, and a circular loop in $\overline{ap}^{\,\mathbb{C}}$ centered at p and based at c. We define $\mu_{a,H}$ as $\widehat{ac}$ followed by the first $(1/n_H)^{\text{th}}$ of this loop (going from c to $R_H(c)$), followed by $R_H\big(\text{reverse}(\widehat{ac})\big)$. (One can see such a path in Figure 3, although the notation there is intended for a more elaborate situation considered below. The portion of the path in the figure that goes from b to $R_H(b)$ is $\mu_{b,H}$.) This is a path from a to $R_H(a)$, so the pair $(\mu_{a,H}, R_H)$ is an element of the orbifold fundamental group G_a. Using the definition of multiplication, the first component of $(\mu_{a,H}, R_H)^{n_H}$ is the path gotten by following $\mu_{a,H}$, then $R_H(\mu_{a,H})$, then $R_H^2(\mu_{a,H}), \dots$ and finally $R_H^{n_H-1}(\mu_{a,H})$. It is easy to see that this path is homotopic to $\overline{aH}^{\circ}$. So we have $(\mu_{a,H}, R_H)^{n_H} = \overline{aH}^{\circ}$.

At this point we have defined everything in the statement of Theorem 1.2. But before proving it we will adapt our construction to accommodate the "fat basepoints" of the previous section. This is necessary for our application to $\mathbb{CH}^{13}$. So we fix A as in Section 2, and assume it contains our basepoint a. We will use A as the base "point" when discussing $\pi_1(X - \mathcal{H})$, and a as the basepoint when discussing $\pi_1^{\text{orb}}\big((X - \mathcal{H})/P\Gamma\big)$. In particular, the left term of (3-1) could also be written $\pi_1(X - \mathcal{H}, A)$. The analogue of $\mu_{a,H}$ is defined as follows, in terms of the point b

of A that is closest to H. We define $\mu_{a,A,H}$ to be $\overline{ab}$ followed by $\mu_{b,H}$ followed by $R_H(\overline{ba})$. See Figure 3 for a picture. The argument from the previous paragraph goes through and shows that $(\mu_{a,A,H}, R_H)^{n_H} = \overline{AH}^{\circ} \in \pi_1(X - \mathcal{H}, A)$.

In applications one typically has some distinguished set of $\mu_{a,H}$ or $\mu_{a,A,H}$ in mind and wants to prove that they generate G_a. Theorem 1.2 in the introduction is a result of this sort, and the rest of the section is devoted to proving it. The following lemma is really the inductive step in the proof, so the reader might prefer to read the theorem's proof first. Also, Theorem 1.2 uses only case (1) of the lemma; the other cases are for our application to $\mathbb{CH}^{13}$ in Section 5.

Lemma 3.1 *Suppose $\mathcal{C} \subseteq \mathcal{M}$ is the set of hyperplanes closest to A, and let G be the subgroup of $G_a = \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/\mathrm{P}\Gamma, a\big)$ generated by the $(\mu_{a,A,C}, R_C)$ with $C \in \mathcal{C}$. Suppose $H \in \mathcal{M}$, write p for the closest point of H to A, r for $d(A,p)$, and B for the open r–neighborhood of A. Suppose G contains $\pi_1(B - \mathcal{H}, A)$ and that there exists a complex reflection $R \in \mathrm{P}\Gamma$ with mirror in $\mathcal{C}$, such that one of the following holds:*

(1) *R moves A closer to p.*

(2) *R moves A closer to H, and no farther from p.*

(3) *There exists an open ball U around p such that*

$$U \cap \mathcal{H} = U \cap \mathcal{H}_p, \quad B \cap R(B) \cap U \neq \varnothing, \quad \text{and} \quad R(B) \cap U \cap H \neq \varnothing.$$

Then G contains $\overline{AH}^{\circ}$.

Proof In every case we have $R(B) \cap H \neq \varnothing$, so $R^{-1}(H)$ is closer to A than H is. Therefore $H \notin \mathcal{C}$, or, in other words, the hyperplanes in $\mathcal{C}$ lie at distance $< r$ from A. We will prove the lemma under hypothesis (3), and then show that the other two cases follow. We hope Figure 4 helps the reader. First we introduce the various objects pictured. As we did above, we write b for the point of A closest to H. Under our identification of $\pi_1(X - \mathcal{H}, A)$ with $\pi_1(X - \mathcal{H}, a)$, the loop $\overline{AH}^{\circ}$ corresponds to $\overline{ab}$ followed by $\overline{bH}^{\circ}$ followed by $\overline{ba}$.

The complex reflection R equals R_C^i for some $C \in \mathcal{C}$. The point marked C in the figure represents the point of C nearest to A. It lies inside B by the previous paragraph's remark that the elements of $\mathcal{C}$ are closer to A than H is. It also lies in $R(B)$, since R fixes C pointwise.

Consider the first component of $(\mu_{a,A,C}, R_C)^i$, ie its "path" part. After a homotopy it may be regarded as a path μ_1 in $B - \mathcal{H}$ from a to a point $c \in (B \cap R(B)) - \mathcal{H}$,

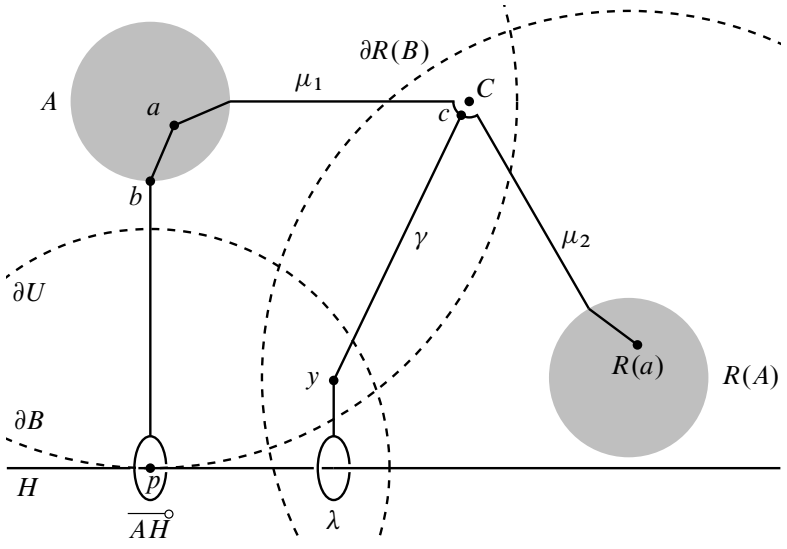

Figure 4: Illustration for the proof of Lemma 3.1.

followed by a path μ_2 in $R(B)-\mathcal{H}$ from c to $R(a)$. These paths are marked in the figure. So $(\mu_{a,A,C}, R_C)^i = (\mu_1\mu_2, R)$ in G_a.

The hypothesis that $B\cap R(B)\cap U \neq \varnothing$ is exactly what we need for some point y to lie in this intersection, as drawn. The connectedness of $U\cap R(B)$ and the hypothesis that H meets $U\cap R(B)$ are exactly what we need to construct a loop λ in $(R(B)\cap U)-\mathcal{H}$, based at y, with linking number 1 with H. Finally, the connectedness of $B\cap R(B)$ allows us to construct a path γ in $(B\cap R(B))-\mathcal{H}$ from c to y. This finishes the construction of the objects in the figure.

Our goal is to prove that G contains $\overline{AH}^{\circ}$. Lemma 2.5 shows that this loop lies in the subgroup of $\pi_1(X-\mathcal{H}, a)$ generated by $\pi_1(B-\mathcal{H}, a)$ and $\mu_1\gamma\lambda\gamma^{-1}\mu_1^{-1}$. This uses our hypothesis $U\cap\mathcal{H} = U\cap\mathcal{H}_p$. Since we assumed G contains the image of $\pi_1(B-\mathcal{H}, a)$, it suffices to show that G contains $\mu_1\gamma\lambda\gamma^{-1}\mu_1^{-1}$, or equivalently the homotopic loop $(\mu_1\mu_2)(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2)(\mu_2^{-1}\mu_1^{-1})$.

An element of the orbifold fundamental group G_a is really a pair, so we must prove $\big((\mu_1\mu_2)(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2)(\mu_2^{-1}\mu_1^{-1}), 1\big)\in G$. One checks that this equals

$$(\mu_1\mu_2, R)\cdot\big(R^{-1}(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2), 1\big)\cdot\big(R^{-1}(\mu_2^{-1}\mu_1^{-1}), R^{-1}\big).$$

The last term is the inverse of the first, which G contains by definition. So it suffices to show that the middle term lies in G, which is easy: the loop $\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2$ lies in $R(B)-\mathcal{H}$, so its image under R^{-1} lies in $B-\mathcal{H}$. This finishes case (3).

Next we claim that (1) implies (3). Take U to be any ball around p with $U \cap \mathcal{H} = U \cap \mathcal{H}_p$. Then the remaining hypotheses of (3) follow immediately from $p \in R(B)$, which is a restatement of (1).

Finally we claim that (2) implies (3). By the previous paragraph it suffices to treat the case that $p \in \partial R(B)$. Take U to be any ball around p with $U \cap \mathcal{H} = U \cap \mathcal{H}_p$. The hypothesis $d(R(A), H) < r$ says that H is not orthogonal to $\overline{R(A), p}$. It follows that $R(B)$ contains elements of H arbitrarily close to p, so $U \cap R(B) \cap H \neq \varnothing$. Similarly, $d(R(A), H) < r$ implies the non-tangency of ∂B and $\partial R(B)$ at p. From this it follows that $B \cap R(B)$ has elements arbitrarily close to p, hence in U. This finishes the proof. □

Proof of Theorem 1.2 We will apply Lemma 3.1 with $A = \{a\}$, noting that $\mu_{a,A,H} = \mu_{a,H}$ for all $H \in \mathcal{M}$. Write G for the subgroup of G_a generated by the $(\mu_{a,C}, R_C)$. By the exact sequence (3-1) and the assumed surjectivity $G \to \mathrm{P}\Gamma$, it suffices to show that G contains $\pi_1(X - \mathcal{H}, a)$. By Theorem 1.1 it suffices to show that it contains every $\overline{aH}^{\circ}$. We do this by induction on $d(a, H)$.

The base case is $H \in \mathcal{C}$, for which we use the fact that $\overline{aH}^{\circ}$ is a power of $(\mu_{a,H}, R_H)$. So suppose $H \in \mathcal{M} - \mathcal{C}$ and set $r := d(a, H)$. We may assume, by Theorem 2.3 and the inductive hypothesis, that G contains $\pi_1(B - \mathcal{H}, a)$, where B is the open r–neighborhood of a. Then case (1) of Lemma 3.1 shows that G also contains $\overline{aH}^{\circ}$, completing the inductive step. □

4 A monstrous(?) hyperplane arrangement

In this section we give background information on the conjecturally-monstrous hyperplane arrangement in $\mathbb{CH}^{13}$ which is the subject of Conjecture 1.3 and Theorems 1.4 and 1.5. For more information, see [4; 9; 10; 5; 23; 25].

We write $\mathbb{C}^{n,1}$ for a complex vector space equipped with a Hermitian form $\langle \cdot \mid \cdot \rangle$ of signature $(n, 1)$, assumed linear in its first argument and antilinear in its second. The *norm* v^2 of a vector v means $\langle v \mid v \rangle$. Complex hyperbolic space $\mathbb{CH}^n$ means the set of negative-definite 1–dimensional subspaces. If $V, W \in \mathbb{CH}^n$ are represented by vectors v, w then their hyperbolic distance is

$$d(V, W) = \cosh^{-1} \sqrt{|\langle v \mid w \rangle|^2 / v^2 w^2}. \tag{4-1}$$

If s is a vector of positive norm, then $s^\perp \subseteq \mathbb{C}^{n,1}$ defines a hyperplane in $\mathbb{CH}^n$, also written $s^\perp$, and

$$d(V, s^\perp) = \sinh^{-1} \sqrt{-|\langle v \mid s \rangle|^2 / v^2 s^2}. \tag{4-2}$$

These formulas are from [21], up to an unimportant factor of 2.

A null vector means a nonzero vector of norm 0. If v is one then it represents a point V of the boundary $\partial\mathbb{CH}^n$. For any vector w of non-zero norm we define the *height* of w with respect to v by

$$\mathrm{ht}_v(w) := -\big|\langle v \mid w\rangle\big|^2/w^2. \tag{4-3}$$

This function is invariant under rescaling w, so it descends to a function on $\mathbb{CH}^n$, which is positive. The horosphere centered at V, of height h with respect to v, means the set of $p \in \mathbb{CH}^n$ with $\mathrm{ht}_v(p) = h$. We define open and closed horoballs the same way, replacing $=$ by $<$ and $\leq$. (More abstractly, one can define horospheres as the orbits of the unipotent radical of the $\mathrm{PU}(n,1)$–stabilizer of V.)

We think of V as the center of these horospheres and horoballs and h as a sort of generalized radius, even though strictly speaking the distance from any point of $\mathbb{CH}^n$ to V is infinite. In particular, if $p, p' \in \mathbb{CH}^n$ then we say that p is closer to V than p' is if $\mathrm{ht}_v(p) < \mathrm{ht}_v(p')$. To see that this notion depends on V rather than v, one checks that replacing v by a nonzero scalar multiple of itself does not affect this inequality. (It multiplies both sides by the same positive number.) Another way to think about this, at least for points outside some fixed closed horoball A centered at V, is to regard "closer to V" as alternate language for "closer to A". In any case, in our application there will be a canonical choice for v, up to roots of unity.

Next we will describe the hyperplane arrangement appearing in Conjecture 1.3 and Theorems 1.4–1.5. We write ω for a primitive cube root of unity and define the Eisenstein integers $\mathcal{E}$ as $\mathbb{Z}[\omega]$. The Eisenstein integer $\omega - \bar{\omega} = \sqrt{-3}$ is so important that it has its own name θ. An $\mathcal{E}$–*lattice* means a free $\mathcal{E}$–module L equipped with a Hermitian form taking values in $\mathcal{E} \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-3})$, denoted $\langle \cdot \mid \cdot \rangle$. Sometimes we think of lattice elements as column vectors and $\langle \cdot \mid \cdot \rangle$ as specified by a matrix M equal to the transpose of its complex conjugate. Then $\langle v \mid w\rangle = v^T M \bar{w}$.

In Definitions 4.1–4.2 we will describe two $\mathcal{E}$–lattices, from [5] and [1], respectively. Each has signature $(13,1)$ and is equal to θ times its dual lattice. By [9] there is only one lattice with these properties, so we may regard them as two different descriptions of the same lattice L. We will not actually use this uniqueness in this paper, and the first description of L is presented only to make precise the statement of Theorem 1.4. The second one is what we will use for the heavy calculations in the next section.

The definitions of $\mathrm{P}\Gamma$ and $\mathcal{M}$, and some other language we will use, are independent of the model. We regard $L \otimes_{\mathcal{E}} \mathbb{C}$ as a copy of $\mathbb{C}^{13,1}$, and take Γ to be the group of $\mathcal{E}$–linear automorphisms of L that preserve the inner product. As usual, $\mathrm{P}\Gamma$ means

the quotient by its subgroup of scalars. A *root* means a norm-3 lattice vector, the hyperplane arrangement $\mathcal{M}$ consists of the orthogonal complements in $\mathbb{CH}^{13}$ of the roots, and $\mathcal{H}$ means the union of these hyperplanes. The subject of Conjecture 1.3 is the orbifold fundamental group of $(\mathbb{CH}^{13}-\mathcal{H})/P\Gamma$.

The first author showed in [1] that all roots are equivalent under Γ. Their special role, and the name "root", arises as follows. First let $s\in\mathbb{C}^{13,1}$ be any vector of positive norm. Then the linear map

$$x\mapsto x-(1-\omega)\frac{\langle x\mid s\rangle}{s^2}s$$

is an isometry of $\langle\cdot\mid\cdot\rangle$, called the *$\omega$–reflection in s* and denoted R_s. Replacing ω by $\bar\omega$ gives the $\bar\omega$–reflection in s, which is the inverse of R_s. They are called triflections, because they are complex reflections of order 3. To see that R_s is a complex reflection (in particular an isometry) one checks that it fixes $s^\perp$ pointwise and multiplies s by ω.

In the special case that s is a root, R_s preserves L because of a conspiracy among the coefficients. First, the factor $(1-\omega)$ is a unit multiple of $\theta=\sqrt{-3}$. Second, for any lattice vector x, $\langle x\mid s\rangle$ is divisible by θ, since all inner products in L are. (This is what it means for L to lie in θ times its dual lattice.) Together these two factors of θ cancel the term $s^2=3$ in the denominator, up to a unit. So $R_s(x)$ is an $\mathcal{E}$–linear combination of x and s, hence lies in L. When one has a reflection in mind (real or complex), it is customary to call a vector orthogonal to its fixed-point set a root. When one also has a lattice in mind, one usually fixes the scale of a root by requiring it to be a primitive lattice vector. This is why we call norm-3 vectors roots. One can show that no other elements of $P\Gamma$ act on $\mathbb{CH}^{13}$ by complex reflections. (The analogous result for unimodular $\mathcal{E}$–lattices is contained in [2, Lemmas 8.1–8.2]; for the current case one uses the fact that L is equal to θ times its dual lattice, rather than merely lying in it.) So $\mathcal{M}$ is exactly the set of mirrors of the complex reflections in $P\Gamma$, making $\pi_1^{\mathrm{orb}}\big((\mathbb{CH}^{13}-\mathcal{H})/P\Gamma\big)$ a braid-like group in the sense of this paper.

Definition 4.1 Our first description of L is the "$\mathbb{P}^2(\mathbb{F}_3)$ model" from [5], or in preliminary form from the proof of [9, Proposition 6.1]. As mentioned above, we include it to give precise meaning to Theorem 1.4, and we will not refer to it later. We start with the diagonal inner product matrix $[-1;1,\dots,1]$ on $\mathbb{C}^{13,1}$, and regard the last 13 coordinates as being indexed by the 13 points of $\mathbb{P}^2(\mathbb{F}_3)$. The "point roots" are the vectors of the form $(0;\theta,0^{12})$ with the θ in any of the last 13 positions. The "line roots" are the vectors of the form $(1;1,1,1,1,0,\dots,0)$, with 1 in positions corresponding to the points of a line in $\mathbb{P}^2(\mathbb{F}_3)$. L is defined as the span of these 26 roots. This construction obviously has $\mathrm{PGL}_3(\mathbb{F}_3)$ symmetry, and it also has less-obvious

symmetries exchanging the point roots and line roots (up to scalars). This yields a subgroup $\mathrm{PGL}_3(\mathbb{F}_3)\rtimes\mathbb{Z}/2$ of $\mathrm{P}\Gamma$ that acts transitively on these 26 roots. We take τ in Theorem 1.4 to be the unique point of $\mathbb{CH}^{13}$ invariant under this group; its coordinates are $(4+\sqrt{3};1,\dots,1)$. Among other results, it was shown in [9, Proposition 6.1] that the hyperplanes in $\mathcal{M}$ that are closest to τ are exactly the mirrors of the point and line roots. We have now described concretely all the objects in Theorem 1.4.

Definition 4.2 Now we give the "Leech model" of L from [1], and make concrete the objects in Theorem 1.5. We will use this model for the rest of the paper. We define L as the $\mathcal{E}$–lattice $\Lambda\oplus\left(\begin{smallmatrix}0&\bar\theta\\ \theta&0\end{smallmatrix}\right)$, where Λ is the complex Leech lattice at the smallest scale at which all inner products lie in $\mathcal{E}$. The complex Leech lattice is studied in detail in [38]. At our scale it has minimal norm 6, all inner products are divisible by θ, and Λ is equal to θ times its dual lattice. It is called the complex Leech lattice because its underlying real lattice is a scaled copy of the usual (real) Leech lattice described in [14]. The properties of Λ that we will use are that its automorphism group is transitive on its vectors of norms 6 and 9, and that its covering radius is $\sqrt{3}$. The transitivity is proven in [38]. The meaning of the covering radius is that closed balls of that radius, centered at lattice points, exactly cover Euclidean space. It is $\sqrt{3}$ because the real Leech lattice, at its own natural scale, has minimal norm 4 and (by [15]) covering radius $\sqrt{2}$.

We will write vectors of L in the form $(x;y,z)$, where $x\in\Lambda$ and $y,z\in\mathcal{E}$.

Conceptually, the "basepoint" for the description of generators for $\pi_1^{\mathrm{orb}}\big((\mathbb{CH}^{13}-\mathcal{H})/\mathrm{P}\Gamma\big)$ in Theorem 1.5 is the cusp of $\mathrm{P}\Gamma$ represented by the null vector $\rho:=(0;0,1)$. As explained in the introduction, really this is a shorthand for choosing a "fat basepoint": a closed horoball A centered at ρ that misses $\mathcal{H}$. The following lemma shows that such a horoball exists. More precisely, it identifies the largest open horoball centered at ρ that misses $\mathcal{H}$; we may take A to be any closed horoball inside it. We also fix a basepoint $a\in A$, so now the paths $\mu_{a,A,H}$ in Theorem 1.5 have been defined.

Lemma 4.3 *The open horoball $\{p\in\mathbb{CH}^{13}\mid \mathrm{ht}_\rho(p)<1\}$ is disjoint from $\mathcal{H}$, and the mirrors that meet its boundary are the orthogonal complements of the roots l that satisfy $|\langle\rho\,|\,l\rangle|^2=3$.*

Proof The special property of ρ we need is that it is orthogonal to no roots. This is clear because $\rho^\perp=\Lambda\oplus\langle\rho\rangle$ has no vectors of norm 3. Now, if l is a root then the point of the mirror of l nearest to ρ is represented by the vector projection of ρ to $l^\perp$, namely $p=\rho-\frac{1}{3}\langle\rho\,|\,l\rangle l$. One computes $\mathrm{ht}_\rho(p)=\frac{1}{3}|\langle\rho\,|\,l\rangle|^2$. This is at least 1, with equality just if $|\langle\rho\,|\,l\rangle|^2=3$. □

Lemma 4.3 also identifies the mirrors closest to ρ, namely the orthogonal complements of the roots l with $\langle\rho\,|\,l\rangle$ equal to a unit multiple of θ. After scaling l by a unit we may suppose

$$l = \big(\lambda; 1, \theta\big(\tfrac{1}{6}(\lambda^2-3)+\nu_l\big)\big), \tag{4-4}$$

where $\lambda\in\Lambda$ and ν_l is purely imaginary and chosen so that the last coordinate lies in $\mathcal{E}$. The set of possibilities for ν_l is $\frac{1}{\theta}\big(\frac{1}{2}+\mathbb{Z}\big)$ if 6 divides λ^2 and $\frac{1}{\theta}\mathbb{Z}$ otherwise. Despite its elaborate form, the last coordinate is well-suited for the calculations required in next section. We call these roots the *Leech roots* (hence the notation l), their mirrors the *Leech mirrors* and the meridians $(\mu_{a,A,l^\perp}, R_l)$ the *Leech meridians*. We have now made concrete all the objects in Theorem 1.5.

We remark that there is a 25–dimensional integral Heisenberg group in $\mathrm{P}\Gamma$ that acts simply transitively on the Leech roots. It consists of the "translations" from the proof of Lemma 5.1. Conceptually, this is a simpler way to index the Leech roots than by the pairs λ, ν_l, but in the end it is equivalent. Also, these translations act cocompactly on ∂A, verifying the technical condition we required on A in paragraph 2.2.

Early in the section we explained how one can meaningfully say that one point of $\mathbb{CH}^{13}$ is closer to a point of $\partial\mathbb{CH}^{13}$ than another point of $\mathbb{CH}^{13}$ is. We will also need to be able to compare the "distance" from a point $p\in\mathbb{CH}^{13}$ to two different cusps $V, V'\in\partial\mathbb{CH}^{13}$ of $\mathrm{P}\Gamma$. Being cusps, they can be represented by lattice vectors v, v', which we may choose to be primitive. Then v, v' are well-defined up to multiplication by sixth roots of unity, and the corresponding height functions ht_v, $\mathrm{ht}_{v'}$ are independent of these factors. So we will say that p is closer to V than to V' if $\mathrm{ht}_v(p) < \mathrm{ht}_{v'}(p)$. Note that this construction depends on the fact that V, V' can be represented by lattice vectors; it does not make sense for general points of $\partial\mathbb{CH}^{13}$. In our applications, v and v' will always be ρ and a translate of ρ.

5 The Leech meridians generate

The purpose of this section is to prove Theorem 1.5, showing that the Leech meridians generate the orbifold fundamental group $G_a := \pi_1^{\mathrm{orb}}\big((\mathbb{CH}^{13}-\mathcal{H})/\mathrm{P}\Gamma, a\big)$, where $\mathcal{M}, \mathcal{H}, \mathrm{P}\Gamma, A, a$ and the Leech meridians $(\mu_{a,A,l^\perp}, R_l)$ are defined in the previous section. We write G for the subgroup of G_a they generate, and we must prove that G is all of G_a.

We begin with an overview of the proof, which follows that of Theorem 1.2. It amounts to showing that the mirror of any non-Leech root s satisfies one of the hypotheses (1)–(3) of Lemma 3.1. It turns out (Lemma 5.2) that if $|\langle\rho\,|\,s\rangle|^2 > 21$ then the simplest

hypothesis (1) holds. If $|\langle\rho \mid s\rangle|^2 > 9$ then the same method shows that the next simplest hypothesis (2) holds (Lemmas 5.2 and 5.3). For the case $|\langle\rho \mid s\rangle|^2 = 9$ we enumerate the orbits of roots $(?;\theta,?)$ under the Γ–stabilizer of ρ (Lemma 5.1). There are three orbits, satisfying hypotheses (1), (2) and (3) of Lemma 3.1, respectively. The last orbit is especially troublesome (Lemma 5.4). The proof of Theorem 1.5 is then a wrapper around these results.

The following description of vectors in $L\otimes\mathbb{C}$ is very important in our computations. We will use it constantly, often specializing to the case of roots. Every vector $s\in(L\otimes\mathbb{C})-\rho^\perp$ can be written uniquely in the form

$$s=\Big(\sigma;m,\frac{\theta}{\overline{m}}\Big(\frac{\sigma^2-N}{6}+\nu\Big)\Big), \tag{5-1}$$

where $\sigma\in\Lambda\otimes\mathbb{C}$, $m\in\mathbb{C}-\{0\}$, N is the norm s^2, and ν is purely imaginary. Restricting the first coordinate to Λ and the others to $\mathcal{E}$ gives the elements of $L-\rho^\perp$. Further restricting N to 3 gives the roots of L, and finally restricting m to 1 gives the Leech roots from (4-4). For vectors of any fixed negative (resp. positive) norm, the larger the absolute value of the middle coordinate m, the further from ρ lie the corresponding points (resp. hyperplanes) in $\mathbb{C}\mathbb{H}^{13}$.

One should think of s from (5-1) as being associated to the vector σ/m in the positive-definite Hermitian vector space $\Lambda\otimes_{\mathcal{E}}\mathbb{C}$. By this we mean that the most important part of $\langle s\mid s'\rangle$ is governed by the relative positions of σ/m and σ'/m' in complex Euclidean space. Namely, by writing out $\langle s\mid s'\rangle$, completing the square and patiently rearranging, one can check

$$\begin{aligned}\langle s\mid s'\rangle = m\overline{m}'\Big[\frac{1}{2}\Big(\frac{N'}{|m'|^2}+\frac{N}{|m|^2}-\Big(\frac{\sigma}{m}-\frac{\sigma'}{m'}\Big)^2\Big)\\ +\operatorname{Im}\Big\langle\frac{\sigma}{m}\;\Big|\;\frac{\sigma'}{m'}\Big\rangle+3\Big(\frac{\nu'}{|m'|^2}-\frac{\nu}{|m|^2}\Big)\Big].\end{aligned} \tag{5-2}$$

Caution We are using the convention that the imaginary part of a complex number is imaginary; for example $\operatorname{Im}\theta$ is θ rather than $\sqrt{3}$.

In the rest of this section, "s" will only be used for roots. Our first step is to classify the roots whose mirrors are next-closest to ρ, after the Leech mirrors.

Lemma 5.1 *Suppose λ_6,λ_9 are fixed vectors in Λ with norms 6 and 9. Then under the Γ–stabilizer of ρ, every root with $m=\theta$ is equivalent to $(0;\theta,-\omega)$ or $(\lambda_6;\theta,\omega)$ or $(\lambda_9;\theta,-1)$.*

Proof The Γ–stabilizer of ρ contains the Heisenberg group of "translations"

$$T_{\lambda,z}\colon \begin{aligned}(l;0,0) &\mapsto (l;0,\overline{\theta}^{-1}\langle l \mid \lambda\rangle),\\ (0;1,0) &\mapsto \big(\lambda;1,\theta^{-1}\big(z-\tfrac{1}{2}\lambda^2\big)\big),\\ (0;0,1) &\mapsto (0;0,1),\end{aligned}$$

where $\lambda \in \Lambda$ and $z \in \operatorname{Im}\mathbb{C}$ are such that $z - \frac{1}{2}\lambda^2 \in \theta\mathcal{E}$. Suppose $s \in L$ has the form (5-1) with $N = 3$ and $m = \theta$. Applying $T_{\lambda,z}$ to s changes the first coordinate by $\theta\lambda$. By [38, page 153], every element of Λ is congruent modulo $\theta\Lambda$ to a vector of norm 0, 6 or 9, so we may suppose σ has one of these norms. Since $\operatorname{Aut}\Lambda$ fixes ρ and acts transitively on the vectors of each of these norms [38, page 155], we may suppose $s = 0$, λ_6 or λ_9. That is, s is one of

$$\big(0;\theta,\tfrac{1}{2}-\nu\big), \quad \big(\lambda_6;\theta,-\tfrac{1}{2}-\nu\big), \quad (\lambda_9;\theta,-1-\nu).$$

In each of the three cases, the possibilities for ν differ by the elements of $\operatorname{Im}\mathcal{E}$. Applying $T_{0,z}$ ($z \in \operatorname{Im}\mathcal{E}$) adds z to the third coordinate of s. Therefore we may take $\nu = \frac{1}{2}\theta$, $\frac{1}{2}\overline{\theta}$ and 0 in the three cases, yielding the roots in the statement of the lemma. (These three roots are inequivalent under the Γ–stabilizer of ρ, but we don't need this.) □

Lemma 5.2 *Suppose s is the root $(0;\theta,-\omega)$ or a root as in (5-1) with $|m| = 2$ or $|m| > \sqrt{7}$, and define p as the point of $s^\perp$ nearest ρ. Then there is a triflection in a Leech root that moves ρ closer to p.*

Proof This proof grew from simpler arguments used for [1, Theorem 4.1] and [9, Proposition 4.2].

We have $p = \rho - \frac{1}{3}\langle\rho \mid s\rangle s = \rho + \frac{1}{\theta}\overline{m}s$. We want to choose a Leech root l, and $\zeta = \omega^{\pm 1}$, such that the ζ–reflection in l (call it R) moves ρ closer to p. This is equivalent to $\langle p \mid R(\rho)\rangle$ being smaller in absolute value than $\langle p \mid \rho\rangle$. We will write down these inner products explicitly and then choose l and ζ appropriately. Direct calculation gives $\langle p \mid \rho\rangle = -|m|^2$. Also,

$$R(\rho) = \rho - (1-\zeta)\frac{\langle\rho \mid l\rangle}{\langle l \mid l\rangle}l = \rho + \frac{1-\zeta}{\theta}l.$$

It turns out that the necessary estimates on $\langle p \mid R(\rho)\rangle$ are best expressed in terms of the following parameter:

$$y := \frac{\theta}{|m|^2}\langle p \mid l\rangle = \frac{\theta}{|m|^2}\Big\langle \rho + \frac{\overline{m}}{\theta}s \;\Big|\; l\Big\rangle = -\frac{3}{|m|^2} + \frac{1}{m}\langle s \mid l\rangle \tag{5-3}$$

$$\in -\frac{3}{|m|^2} + \frac{1}{m}\theta\mathcal{E}. \tag{5-4}$$

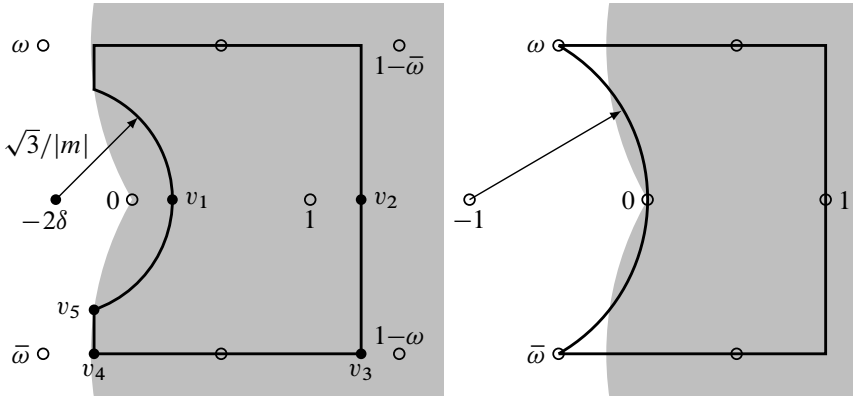

Figure 5: See the proof of Lemma 5.2. V is the union of the gray (open) disks, which have radius $\sqrt{3}$ and centers $1-\omega^{\pm 1}$. We seek a Leech root l so that y lies in this region. U is the closed region bounded by the solid line, and is where we can arrange for y to be. U varies with $|m|$; we have drawn the case $|m|=\sqrt{7}$, when v_5 is on the boundary of V, and the case $|m|=\sqrt{3}$, when v_4 and v_5 coalesce at $\bar{\omega}$. Hollow circles indicate Eisenstein integers.

First one works out

$$\left|\frac{\langle p \mid R(\rho)\rangle}{\langle p \mid \rho\rangle}\right| = \left|\frac{\langle p \mid R(\rho)\rangle}{|m|^2}\right| = \left|\tfrac{1}{3}(1-\bar{\zeta})y - 1\right|. \tag{5-5}$$

Our goal is to choose l and ζ so that this is less than 1. This is equivalent to

$$|y-(1-\zeta)| < \sqrt{3}.$$

Because the possibilities for ζ are $\omega^{\pm 1}$, this amounts to being able to choose l so that y lies in the union V of the open balls in $\mathbb{C}$ of radius $\sqrt{3}$ around the points $1-\omega$ and $1-\bar{\omega}$. So our goal is to choose l such that y lies in the shaded region in Figure 5.

Now we examine how our choice of l affects y. Writing l as in (4-4), choosing it amounts to choosing $\lambda \in \Lambda$, and then choosing $\nu_l \in \operatorname{Im}\mathbb{C}$ subject to the condition that the last coordinate of (4-4) is in $\mathcal{E}$. Specializing (5-2) to the case that s has norm $N=3$ and s' is the Leech root l gives

$$\langle s \mid l\rangle = m\Big[\frac{3}{2|m|^2} + \frac{3}{2} - \frac{1}{2}\Big(\frac{\sigma}{m}-\lambda\Big)^2 + \operatorname{Im}\Big\langle \frac{\sigma}{m} \;\Big|\; \lambda\Big\rangle + 3\Big(\nu_l - \frac{\nu}{|m|^2}\Big)\Big].$$

Plugging this into formula (5-3) gives

$$y = -\frac{3}{2|m|^2} + \frac{3}{2} - \frac{1}{2}\Big(\frac{\sigma}{m}-\lambda\Big)^2 + \operatorname{Im}\Big\langle \frac{\sigma}{m} \;\Big|\; \lambda\Big\rangle + 3\Big(\nu_l - \frac{\nu}{|m|^2}\Big). \tag{5-6}$$

The covering radius of a lattice in Euclidean space is defined as the smallest number such that the closed balls of that radius, centered at lattice points, cover Euclidean space. The covering radius of Λ is $\sqrt{3}$, because the underlying real lattice has norms equal to $\frac{3}{2}$ times those of the real Leech lattice, whose covering radius is $\sqrt{2}$ by [15]. Therefore we may take λ so that $0 \le (\sigma/m - \lambda)^2 \le 3$. It follows that the real part of (5-6) lies in $\left[-\delta, \frac{3}{2}-\delta\right]$ where $\delta := 3/2|m|^2$.

Next we choose v_l. The only constraint on it is that the last component of $l = (\lambda; 1, ?)$ must lie in $\mathcal{E}$. As mentioned after (4-4), this amounts to $v_l \in \frac{1}{\theta}\left(\frac{1}{2}+\mathbb{Z}\right)$ if λ^2 is divisible by 6, and $v_l \in \frac{1}{\theta}\mathbb{Z}$ otherwise. In either case, referring to (5-6) shows that changing our choice of v_l allows us to change y by any rational integer multiple of θ. So we may take $\operatorname{Im} y \in \left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$. After these choices we have

$$\operatorname{Re} y \in \left[-\delta, \tfrac{3}{2}-\delta\right] \quad \text{and} \quad \operatorname{Im} y \in \left[-\tfrac{\theta}{2}, \tfrac{\theta}{2}\right]. \tag{5-7}$$

Now we can derive additional information about y. We have $y \neq -2\delta$ since -2δ is not in the rectangle (5-7). Since $y \in -2\delta + (\theta/m)\mathcal{E}$ by (5-4), y lies at distance $\ge \sqrt{3}/|m|$ from -2δ. We define U as the closed rectangle (5-7) in $\mathbb{C}$, minus the open $(\sqrt{3}/|m|)$–disk around -2δ. We have shown that we may choose a Leech root l such that $y \in U$. We have indicated U in outline in Figure 5 for $|m| = \sqrt{7}$ or $\sqrt{3}$, and in Figure 6 for $|m| = 2$. As $|m|$ increases, the rectangle moves to the right, approaching $\left[0, \frac{3}{2}\right] \times \left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$, the center of the removed disk approaches zero, and its radius approaches zero more slowly than the center does.

Now suppose $|m|^2 > 7$. We claim $U \subseteq V$. Assuming this for the moment, we may choose l such that y is in U, hence V, which allows us to choose $\zeta = \omega^{\pm 1}$ so that ζ–reflection in l moves ρ closer to p. This finishes the proof. To prove the claim it will suffice to show that the lower half of U lies in the open $\sqrt{3}$–ball around $1-\omega$. Obviously it suffices to check this for the points marked $v_1, \dots, v_5$ in Figure 5. These are

$$v_1 = -2\delta + \sqrt{3}/|m|, \quad v_2 = \tfrac{3}{2}-\delta, \quad v_3 = \tfrac{3}{2}-\delta-\tfrac{1}{2}i\sqrt{3}, \quad v_4 = -\delta-\tfrac{1}{2}i\sqrt{3}$$

and

$$v_5 = -\delta - i\sqrt{3/|m|^2 - 9/(4|m|^4)}.$$

Using $|m|^2 > 7$, one can check that each of these lies at distance $< \sqrt{3}$ from $1-\omega$. This finishes the proof of the $|m|^2 > 7$ case. (If $|m|^2 = 7$ then v_5 lies on the boundary of V. If $|m|^2 = 4$ then v_4 and v_5 are outside the boundary; see Figure 6. If $|m|^2 = 3$ then $v_1 = 0$ is on the boundary and $v_4 = v_5 = \bar{\omega}$ is outside it; see the second part of Figure 5.)

Next we treat the special case $s = (0; \theta, -\omega)$. Choosing $\lambda = 0$ gives $\operatorname{Re} y = 1$ by (5-6). Then choosing v_l as above, so that $\operatorname{Im} y$ lies in $\left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$, yields $y \in V$. So we can move ρ closer to p just as in the $|m|^2 > 7$ case.

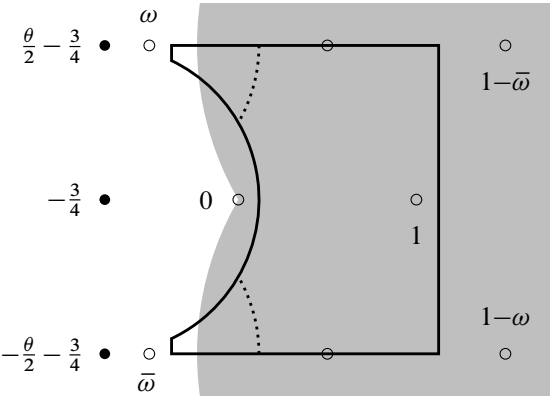

Figure 6: The analogue of Figure 5 for the special case $|m| = 2$ in the proof of Lemma 5.2. The proof shows that y lies in U (bounded by the solid path) but outside two open disks (indicated by the dotted arcs), hence in V (the shaded region).

Finally, we suppose $|m| = 2$; we may take $m = 2$ by multiplying s by a unit. Recall that after we proved that y lies in the rectangle (5-7), we could use (5-4) to show that y lies outside the open disk used in the definition of U. For $m = 2$ the argument shows more. Since $\delta = \frac{3}{8}$ when $|m| = 2$, (5-4) shows $y \in -\frac{3}{4} + \frac{\theta}{2}\mathcal{E}$. Since both $-\frac{3}{4} \pm \frac{\theta}{2}$ lie in $-\frac{3}{4} + \frac{\theta}{2}\mathcal{E}$ but not in the rectangle (5-7), y lies at distance $\geq \frac{1}{2}\sqrt{3}$ from each of them, just as it lies at distance $\geq \frac{1}{2}\sqrt{3}$ from $-\frac{3}{4}$. It is easy to check that U, minus the open $\frac{1}{2}\sqrt{3}$–balls around $-\frac{3}{4} \pm \frac{\theta}{2}$, lies in V; see Figure 6. Therefore $y \in V$, finishing the proof as before. (One can consider analogues of these extra disks for any m. They are unnecessary if $|m|^2 > 7$, and turn out to be useless if $|m|^2 = 3$ or 7.) □

Lemma 5.3 *Suppose s is the root $(\lambda_6; \theta, \omega)$ or a root as in (5-1) with $|m| = \sqrt{7}$, and define p as the point of $s^\perp$ nearest ρ. Then there is a triflection in a Leech root that either moves ρ closer to p, or else moves ρ closer to $s^\perp$ while preserving the distance between ρ and p.*

Proof Suppose first $|m| = \sqrt{7}$. Then the proof of Lemma 5.2 goes through unless y is v_5 in Figure 5, or its complex conjugate. So suppose $y = v_5$ or $\overline{v}_5$, and take $\zeta = \omega$ or $\bar{\omega}$, respectively. The argument in the proof of Lemma 5.2, that R moves ρ closer to p, fails because $|y - (1 - \zeta)|$ equals $\sqrt{3}$ rather than being strictly smaller. But it does show that $R(\rho)$ is exactly as far from p as ρ is. This is one of our claims, and what remains to show is that R moves ρ closer to $s^\perp$.

To do this we first solve (5-3) for $\langle s \mid l\rangle$ in terms of y, obtaining $\langle s \mid l\rangle = (3+|m|^2 y)/\overline{m}$. Then one works out

$$\left|\frac{\langle s \mid R(\rho)\rangle}{\langle s \mid \rho\rangle}\right| = \left|\frac{\langle s \mid \rho\rangle - \frac{1}{\theta}(1-\overline{\zeta})\langle s \mid l\rangle}{\langle s \mid \rho\rangle}\right| = \left|1 - \tfrac{1}{3}(1-\overline{\zeta})\left(\frac{3}{|m|^2}+y\right)\right|.$$

We want this to be less than 1. By copying the argument following (5-5), this is equivalent to $y+3/|m|^2$ lying in the open $\sqrt{3}$–disk around $1-\zeta$. This is obvious from the figure because $y+3/|m|^2$ is $\frac{3}{7}$ to the right of $y=v_5$ or $\overline{v}_5$. This finishes the $|m|=\sqrt{7}$ case.

The case $s=(\lambda_6;\theta,\omega)$ is similar. In this case U appears in Figure 5. Writing l as in (4-4) with $\lambda=0$ leads to $\operatorname{Re} y = 0$, so either $y \in V$ (so the proof of Lemma 5.2 applies) or else $y = 0 \in \partial V$. In this case the argument for $|m|=\sqrt{7}$ shows that $R(\rho)$ is just as close to p as ρ is, and that R moves ρ closer to $s^\perp$. □

Lemma 5.4 *Let $s=(\lambda_9;\theta,-1)$, define p as the point of $s^\perp$ nearest ρ, and B as the open horoball centered at ρ, whose bounding horosphere is tangent to $s^\perp$ at p. Then there exists an open ball U around p with $U\cap\mathcal{H} = U\cap\mathcal{H}_p$, and a triflection R in one of the Leech mirrors, such that $B\cap R(B)\cap U\neq\varnothing$ and $R(B)\cap U\cap s^\perp\neq\varnothing$.*

Proof Since we are verifying hypothesis (3) of Lemma 3.1, we will use that lemma's notation H for $s^\perp$. By definition,

$$p = \rho - \tfrac{1}{3}\langle\rho \mid s\rangle s = (-\lambda_9;\overline{\theta},2).$$

This has norm -3 and lies in L. One computes $\operatorname{ht}_\rho(p)=3$, so B is the height-3 open horoball around ρ. We take U to have radius $\sinh^{-1}\sqrt{1/3}$. To check that $U\cap\mathcal{H}=U\cap\mathcal{H}_p$, consider a root s' not orthogonal to p. Then $|\langle p \mid s'\rangle|\geq\sqrt{3}$ since $p\in L$, so

$$d(p,s'^\perp) = \sinh^{-1}\sqrt{-|\langle p \mid s'\rangle|^2/p^2 s'^2} \geq \sinh^{-1}\sqrt{1/3},$$

as desired.

Next we choose R to be the ω–reflection in the Leech root $l=(0;1,-\omega)$. (We found l by applying the proof of Lemma 5.2 as well as we could. That is, we choose l so that y in that proof equals the lower left corner $\overline{\omega}$ of the second part of Figure 5.) This yields $R(\rho)=(0;\overline{\omega},0)$. We must verify $R(B)\cap U\cap H\neq\varnothing$ and $B\cap R(B)\cap U\neq\varnothing$.

Our strategy for showing $R(B)\cap U\cap H\neq\varnothing$ is to begin by defining p' as the projection of $R(\rho)$ to H, which happens to lie outside U. Then we parametrize $\overline{p'p}\subseteq H$, find the point x where it crosses ∂U, and check that $x\in R(B)$. So $x\in R(B)\cap H\cap\partial U$. Therefore a point of $\overline{p'p}$, slightly closer to p than x is, lies in $R(B)\cap H\cap U$, showing that this intersection is nonempty.

Here are the details. Computation gives $p' = (\overline{\omega}\lambda_9/\theta; 2\overline{\omega}, -\overline{\omega}/\theta)$, of norm -1. One checks $\langle p' \mid p\rangle = 2\overline{\omega}\,\overline{\theta}$, so

$$d(p, p') = \cosh^{-1} 2 > \sinh^{-1}\sqrt{1/3}$$

and p' lies outside U, as claimed. Also, $-\omega\theta p'$ and p have negative inner product. Therefore $\overline{p'p} - \{p\}$ is parametrized by $x_t = -\omega\theta p' + tp$ with $t \in [0, \infty)$. One computes $\langle x_t \mid p\rangle = -3t - 6$ and $x_t^2 = -3t^2 - 12t - 3$, yielding

$$d(x_t, p) = \cosh^{-1}\sqrt{\frac{|\langle x_t \mid p\rangle|^2}{x_t^2 p^2}} = \cosh^{-1}\sqrt{\frac{t^2+4t+4}{t^2+4t+1}}.$$

Now, x_t lies in ∂U just when this equals $\sinh^{-1}\sqrt{1/3}$, yielding a quadratic equation for t. There is just one nonnegative solution, namely $t = 2\sqrt{3} - 2$. So $x = x_{2\sqrt{3}-2}$. Then one computes $\langle R(\rho) \mid x\rangle = \overline{\omega}\,\overline{\theta}(4\sqrt{3} - 3)$, so

$$\mathrm{ht}_{R(\rho)}(x) = -\frac{|\langle R(\rho) \mid x\rangle|^2}{x^2} = -\frac{3(57 - 24\sqrt{3}\,)}{-27} < 3.$$

That is, $x \in R(B)$ as desired.

Our strategy for $B \cap R(B) \cap U \neq \varnothing$ is similar. We parametrize the geodesic $\overline{x\rho}$, where x is the point found in the previous paragraph, find the point y where it crosses ∂B, and check that y lies in $R(B)$ and U. Here are the details. Computation shows $\langle x \mid \rho\rangle = -6\sqrt{3} < 0$, so $\overline{x\rho} - \{\rho\}$ is parametrized by $y_u = x + u\rho$ with $u \in [0, \infty)$. Further computation shows $\langle y_u \mid \rho\rangle = -6\sqrt{3}$ and $y_u^2 = -27 - 12u\sqrt{3}$, so $\mathrm{ht}_\rho(y_u) = 36/(9 + 4u\sqrt{3})$. Setting this equal to 3 yields $u = \sqrt{3}/4$, so $y = y_{\sqrt{3}/4}$. Now one checks that $\mathrm{ht}_{R(\rho)}(y) < 3$, so that $y \in R(B)$. A similar calculation proves $y \in U$. (In fact this last calculation can be omitted, because y, p are the projections to ∂B of the two points x, p outside B, but not both in ∂B. Projection to a closed horoball decreases the distance between two points, if at least one of them is outside it. Therefore $d(y, p) < d(x, p) = \sinh^{-1}\sqrt{1/3}$.) □

Proof of Theorem 1.5 We will mimic the proof of Theorem 1.2 (see the end of Section 3), using Lemmas 5.2–5.4 in place of the "moves a closer to p" hypothesis of that theorem. Write G for the subgroup of $G_a = \pi_1^{\mathrm{orb}}\big((\mathbb{CH}^{13} - \mathcal{H})/\mathrm{P}\Gamma, a\big)$ generated by the Leech meridians, ie the pairs $(\mu_{a,A,l^\perp}, R_l)$ with l a Leech root. We must show that G is all of G_a. It is known (see [9], or [5] for a later proof) that the R_l generate $\mathrm{P}\Gamma$. By the exact sequence (3-1), it therefore suffices to show that G contains $\pi_1(\mathbb{CH}^{13} - \mathcal{H}, a)$. By Theorem 2.3 it suffices to show that G contains every $\overline{AH}^\circ$, with H varying over $\mathcal{M}$. We do this by induction on the distance from H to ρ, or, properly speaking, on $|\langle \rho \mid s\rangle|$, where s is a root with $H = s^\perp$. The base case is when s is a Leech root, ie $|\langle \rho \mid s\rangle| = \sqrt{3}$, and we just observe $\overline{AH}^\circ = (\mu_{a,A,H}, R_s)^3$.

Now suppose s is a root but not a Leech root, $H = s^\perp$, p is the point of H closest to ρ, and B is the open horoball centered at ρ and tangent to H at p. We may assume by induction that G contains every $\overline{A, s'^\perp}^\circ$ with s' a root satisfying $|\langle\rho\,|\,s'\rangle| < |\langle\rho\,|\,s\rangle|$. It follows from Theorem 2.3 that G contains $\pi_1(B - \mathcal{H}, a)$.

The smallest possible value of $|\langle\rho\,|\,s\rangle|$ for a non-Leech root s is 3, occurring when $|m| = \sqrt{3}$ in (5-1). In the case $s = (0; \theta, -\omega)$ (resp. $(\lambda_6; \theta, \omega)$, $(\lambda_9; \theta, -1)$), hypothesis (1) (resp. (2), (3)) of Lemma 3.1 is satisfied, by Lemma 5.2 (resp. Lemma 5.3, Lemma 5.4). If s is any root with $\langle\rho\,|\,s\rangle = 3$ then it is equivalent to one of these examples under the Γ–stabilizer of ρ, by Lemma 5.1. Therefore Lemma 3.1 applies to $s^\perp$ for every root s with $\langle\rho\,|\,s\rangle = 3$. It follows that G contains the corresponding loops $\overline{AH}^\circ$. If $|\langle\rho\,|\,s\rangle| = 3$ then scaling s by a unit reduces to the $\langle\rho\,|\,s\rangle = 3$ case.

The next possible value of $|\langle\rho\,|\,s\rangle|$ is $2\sqrt{3}$, occurring when $|m| = 2$ in (5-1). In this case Lemma 5.2 verifies hypothesis (1) of Lemma 3.1, which tells us that G contains $\overline{AH}^\circ$. The next possible value of $|\langle\rho\,|\,s\rangle|$ is $\sqrt{21}$, occurring when $|m| = \sqrt{7}$. In this case Lemma 5.3 verifies hypothesis (2) of Lemma 3.1, which tells us that G contains $\overline{AH}^\circ$. The general step of the induction is the same. If $|\langle\rho\,|\,s\rangle|$ is larger than $\sqrt{21}$, then $|m|$ is larger than $\sqrt{7}$, so Lemma 5.2 verifies hypothesis (1) of Lemma 3.1. This tells us that G contains $\overline{AH}^\circ$, completing the inductive step. □

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