Geometry & Topology 20 (2016) 1213–1255

On type-preserving representations of the four-punctured sphere group

TIAN YANG

We give counterexamples to a question of Bowditch that asks whether a nonelementary type-preserving representation $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2;\mathbb{R})$ of a punctured surface group that sends every nonperipheral simple closed curve to a hyperbolic element must ρ be Fuchsian. The counterexamples come from relative Euler class ± 1 representations of the four-punctured sphere group. We also show that the mapping class group action on each nonextremal component of the character space of type-preserving representations of the four-punctured sphere group is ergodic, which confirms a conjecture of Goldman for this case. The main tool we use are Kashaev and Penner's lengths coordinates of the decorated character spaces.

57M05

To William Goldman on the occasion of his sixtieth birthday

1 Introduction

Let Σ_g be an oriented closed surface of genus $g \geqslant 2$. The $\mathrm{PSL}(2,\mathbb{R})$–representation space $\mathcal{R}(\Sigma_g)$ is the space of group homomorphisms $\rho\colon \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ from the fundamental group of Σ_g into $\mathrm{PSL}(2,\mathbb{R})$, endowed with the compact open topology. The Euler class $e(\rho)$ of ρ is the Euler class of the associated S^1–bundle on Σ_g, which satisfies the Milnor–Wood inequality $2-2g \leqslant e(\rho) \leqslant 2g-2$. In [9], Goldman proved that equality holds if and only if ρ is Fuchsian, ie discrete and faithful; and in [11], he proved that the connected components of $\mathcal{R}(\Sigma_g)$ are indexed by the Euler classes. That is, for each integer k with $|k| \leqslant 2g-2$, representations of Euler class k exist and form a connected component of $\mathcal{R}(\Sigma_g)$. The Lie group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathcal{R}(\Sigma_g)$ by conjugation, and the quotient space

$$\mathcal{M}(\Sigma_g) = \mathcal{R}(\Sigma_g)/\,\mathrm{PSL}(2,\mathbb{R})$$

is the character space of Σ_g. Since the Euler classes are preserved by conjugation, the connected components of $\mathcal{M}(\Sigma_g)$ are also indexed by the Euler classes, ie

$$\mathcal{M}(\Sigma_g) = \coprod_{k=2-2g}^{2g-2} \mathcal{M}_k(\Sigma_g),$$

Published: 28 April 2016 DOI: 10.2140/gt.2016.20.1213

where $\mathcal{M}_k(\Sigma_g)$ is the space of conjugacy classes of representations of Euler class k. By the results of Goldman [9; 11], the extremal components $\mathcal{M}_{\pm(2-2g)}(\Sigma_g)$ are respectively identified with the Teichmüller space of Σ_g and that of Σ_g endowed with the opposite orientation.

The mapping class group $\mathrm{Mod}(\Sigma_g)$ of Σ_g is the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ_g. By the Dehn–Nielsen theorem, $\mathrm{Mod}(\Sigma_g)$ is naturally isomorphic to the group of positive outer-automorphisms $\mathrm{Out}^+(\pi_1(\Sigma_g))$, which acts on $\mathcal{M}(\Sigma_g)$ and preserves the Euler classes. Therefore $\mathrm{Mod}(\Sigma_g)$ acts on each connected component of $\mathcal{M}(\Sigma_g)$. It is known (see Fricke [8]) that the $\mathrm{Mod}(\Sigma_g)$–action is properly discontinuous on the extremal components $\mathcal{M}_{\pm(2-2g)}(\Sigma_g)$, ie the Teichmüller spaces, and the quotients are the Riemann moduli spaces of complex structures on Σ_g. On the nonextremal components $\mathcal{M}_k(\Sigma_g)$, $|k| < 2g-2$, Goldman conjectured in [14] that the $\mathrm{Mod}(\Sigma_g)$–action is ergodic with respect to the measure induced by the Goldman symplectic form [10].

Closely related to Goldman's conjecture is a question of Bowditch [4, Question C] that asks whether for each nonelementary and nonextremal (ie non-Fuchsian) representation ρ in $\mathcal{R}(\Sigma_g)$, there exists a simple closed curve γ on Σ_g such that $\rho([\gamma])$ is an elliptic or a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$. Recall that a representation is nonelementary if its image is Zariski-dense in $\mathrm{PSL}(2,\mathbb{R})$. Recently, Marché and Wolff [21] showed that an affirmative answer to Bowditch's question implies that Goldman's conjecture is true. More precisely, they show that for $(g,k) \neq (2,0)$, $\mathrm{Mod}(\Sigma_g)$ acts ergodically on the subset of $\mathcal{M}_k(\Sigma_g)$ consisting of representations that send some simple closed curve on Σ_g to an elliptic or parabolic element. Therefore, when $(g,k) \neq (2,0)$, the $\mathrm{Mod}(\Sigma_g)$–action on $\mathcal{M}_k(\Sigma_g)$ is ergodic if and only if the subset above has full measure in $\mathcal{M}_k(\Sigma_g)$. In the same work, they answer Bowditch's question affirmatively for the genus-2 surface Σ_2, implying the ergodicity of the $\mathrm{Mod}(\Sigma_2)$–action on $\mathcal{M}_{\pm 1}(\Sigma_2)$. For the action on the component $\mathcal{M}_0(\Sigma_2)$, they find two $\mathrm{Mod}(\Sigma_2)$–invariant open subsets due to the existence of the hyperelliptic involution, and show that on each of them the $\mathrm{Mod}(\Sigma_2)$–action is ergodic. For higher-genus surfaces Σ_g, $g \geqslant 3$, Souto recently gave an affirmative answer to Bowditch's question for the Euler class 0 representations (personal communication), proving the ergodicity of the $\mathrm{Mod}(\Sigma_g)$–action on $\mathcal{M}_0(\Sigma_g)$. For $g \geqslant 3$ and $k \neq 0$, both Bowditch's question and Goldman's conjecture are still open.

Bowditch's question was originally asked for the type-preserving representations of punctured surface groups. Recall that a punctured surface $\Sigma_{g,n}$ of genus g with n punctures is a closed surface Σ_g with n points removed. Throughout this paper, we required that the Euler characteristic of $\Sigma_{g,n}$ be negative. A *peripheral element* of $\pi_1(\Sigma_{g,n})$ is an element that is represented by a curve freely homotopic to a circle that goes around a single puncture of $\Sigma_{g,n}$. A representation $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ is called *type-preserving* if it sends every peripheral element of $\pi_1(\Sigma_{g,n})$ to a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$. Bowditch [4, Question C] asks whether a nonelementary type-preserving representation of a punctured surface group that sends every nonperipheral simple closed curve to a hyperbolic element must be Fuchsian.

The main result of this paper gives counterexamples to this question. To state the result, we recall that there is a notion of a relative Euler class $e(\rho)$ of a type-preserving representation ρ that satisfies the Milnor–Wood inequality

$$|e(\rho)| \leqslant 2g-2+n,$$

and equality holds if and only if ρ is Fuchsian (see [9; 11] and also Proposition A.1).

Theorem 1.1 *There are uncountably many nonelementary type-preserving representations $\rho\colon \pi_1(\Sigma_{0,4}) \to \mathrm{PSL}(2,\mathbb{R})$ with relative Euler class $e(\rho) = \pm 1$ that send every nonperipheral simple closed curve to a hyperbolic element. In particular, these representations are not Fuchsian.*

Our method is to use Penner's lengths coordinates for the decorated character space defined by Kashaev [17]. Briefly speaking, decorated character space of a punctured surface is the space of conjugacy classes of *decorated representations*, namely, nonelementary type-preserving representations together with an assignment of *horocycles* to the punctures. The lengths coordinates of a decorated representation depend on the choice of an ideal triangulation of the surface, and consist of the *λ–lengths* of the edges determined by the horocycles, and of the *signs* of the ideal triangles determined by the representation. The decorated Teichmüller space is a connected component of the decorated character space, and the restriction of the lengths coordinates to this component coincides with Penner's lengths coordinates. (See Kashaev [17; 18] or Section 2 for more details.) A key ingredient in the proof is Equation (3-1) of the traces of closed curves in the lengths coordinates, found by Sun and the author. With a careful choice of an ideal triangulation of the four-punctured sphere, called a *tetrahedral triangulation*, we show that the traces of three *distinguished simple closed curves* are greater than 2 in the absolute value if and only if the λ–lengths of edges in this triangulation satisfy certain antitriangular inequalities. We then show that each simple closed curve is distinguished in some tetrahedral triangulation, and all tetrahedral triangulations are related by a sequence of moves, called the *simultaneous diagonal switches*. By the change of λ–lengths formula (Proposition 2.3), we show that the antitriangular inequalities are preserved by the simultaneous diagonal switches. Therefore, if the three distinguished simple closed curves are hyperbolic, then all the simple closed curves are hyperbolic. Finally, we show that there are uncountably many choices of the λ–lengths that satisfy the antitriangular inequalities.

A consequence of Equation (3-1) is Theorem 3.5, which says that each non-Fuchsian type-preserving representation is dominated by a Fuchsian one, in the sense that the traces of the simple closed curves of the former representation are less than or equal to those of the later in the absolute value. This is a counterpart of the result of Gueritaud, Kassel and Wolff [15] and Deroin and Tholozan [6], where they consider dominance of closed surface group representations.

Using the same technique, we also give an affirmative answer to Bowditch's question for the relative Euler class 0 type-preserving representations of the four-punctured sphere group.

Theorem 1.2 *Every nonelementary type-preserving representation $\rho\colon \pi_1(\Sigma_{0,4}) \to \mathrm{PSL}(2,\mathbb{R})$ with relative Euler class $e(\rho) = 0$ sends some nonperipheral simple closed curve to an elliptic or parabolic element.*

In contrast with the connected components of the character space of a closed surface, those of a punctured surface are more subtle to describe. For $\Sigma_{g,n}$ with $n \neq 0$, denote by $\mathcal{M}_k(\Sigma_{g,n})$ be the space of conjugacy classes of type-preserving representations with relative Euler class k. As explained in Kashaev [17], $\mathcal{M}_k(\Sigma_{g,n})$ can be either empty or nonconnected. For example, $\mathcal{M}_0(\Sigma_{0,3}) = \mathcal{M}_0(\Sigma_{1,1}) = \varnothing$. The nonconnectedness of $\mathcal{M}_k(\Sigma_{g,n})$ comes from the existence of two distinct conjugacy classes of parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. More precisely, each parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ is up to $\pm I$ conjugate to an upper-triangular matrix with trace 2, and its conjugacy class is distinguished by whether the sign of the nonzero off diagonal element is positive or negative. Therefore, two type-preserving representations of the same relative Euler class which respectively send the same peripheral element into different conjugacy classes of parabolic elements cannot be in the same connected components. Throughout this paper, we respectively call the two conjugacy class of parabolic elements the *positive* and the *negative* conjugacy classes. For a type-preserving representation $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$, we say that the *sign* of a puncture v is *positive*, denoted by $s(v) = 1$, if ρ sends a peripheral element around this puncture into the positive conjugacy class of parabolic elements, and is *negative*, denoted by $s(v) = -1$, otherwise. For $s \in \{\pm 1\}^n$, we denote by $\mathcal{M}_k^s(\Sigma_{g,n})$ the space of conjugacy classes of type-preserving representations with relative Euler class k and signs of the punctures s. It is conjectured in [17] that each $\mathcal{M}_k^s(\Sigma_{g,n})$, if nonempty, is connected. The result confirms this for the four-punctured sphere.

Theorem 1.3 *Let* $s \in \{\pm 1\}^4$.

(1) $\mathcal{M}_0^s(\Sigma_{0,4})$ *is nonempty if and only if* s *contains exactly two* -1 *and two* 1.

(2) $\mathcal{M}_1^s(\Sigma_{0,4})$ *is nonempty if and only if* s *contains at most one* -1.

(3) $\mathcal{M}_{-1}^s(\Sigma_{0,4})$ *is nonempty if and only if* s *contains at most one* 1.

(4) *All the nonempty spaces above are connected.*

As a consequence of Theorem 1.3, $\mathcal{M}_0(\Sigma_{0,4})$ has six connected components and each of $\mathcal{M}_{\pm 1}(\Sigma_{0,4})$ has five connected components. The main tool we use in the proof are still the lengths coordinates; we hope the technique can be used for the other punctured surfaces.

The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ of a punctured surface $\Sigma_{g,n}$ is the group of relative isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_{g,n}$ that fix the punctures. By the Dehn–Nielsen theorem, $\mathrm{Mod}(\Sigma_{g,n})$ is isomorphic to the group of positive outer-automorphisms $\mathrm{Out}^+(\pi_1(\Sigma_{g,n}))$ that preserve the cyclic subgroups of $\pi_1(\Sigma_{g,n})$ generated by the peripheral elements, and hence acts on $\mathcal{M}(\Sigma_{g,n})$ and preserve the relative Euler classes and the signs of the punctures. Therefore, for any integer k with $|k| \leqslant 2g-2+n$ and for any $s \in \{\pm 1\}^n$, $\mathrm{Mod}(\Sigma_{g,n})$ acts on $\mathcal{M}_k^s(\Sigma_{g,n})$. For the four-punctured sphere, we have the following.

Theorem 1.4 *The* $\mathrm{Mod}(\Sigma_{0,4})$*–action on each nonextremal connected component of* $\mathcal{M}(\Sigma_{0,4})$ *is ergodic.*

By Marché and Wolff [21], it is not surprising that the $\mathrm{Mod}(\Sigma_{0,4})$–action is ergodic on the connected components of $\mathcal{M}(\Sigma_{0,4})$ where Bowditch's question has an affirmative answer. A new and unexpected phenomenon Theorem 1.4 reveals here is that, for punctured surfaces, the action of the mapping class group can still be ergodic when the answer to Bowditch's question is negative. Evidenced by Theorem 1.4, we make the following conjecture.

Conjecture 1.5 *The* $\mathrm{Mod}(\Sigma_{g,n})$*–action is ergodic on each nonextremal connected component of* $\mathcal{M}(\Sigma_{g,n})$.

The paper is organized as follows. In Section 2, we recall Kashaev's decorated character spaces and the lengths coordinates, in Section 3, we obtain a formula for the traces of closed curves in the lengths coordinates, and in Section 4, we introduce tetrahedral triangulations, distinguished simple closed curves and simultaneous diagonal switches. Then we prove Theorems 1.1, 1.2, 1.3 and 1.4 in Sections 5, 6 and 7. The anonymous referee pointed out that the results concerning representations of relative Euler class ± 1 can be deduced more directly from the results of Goldman in [13]. We present the referee's argument in Appendix B for interested readers.

Acknowledgments The author is grateful to the referee for bringing his attention to the relationship of this work with a previous work of Goldman, and for several valuable suggestions to improve the writing of this paper. The author also would like to thank Steven Kerckhoff, Feng Luo and Maryam Mirzakhani for helpful discussions and suggestions, Ser Peow Tan, Maxime Wolff, Sara Moloni, Frédéric Palesi and Zhe Sun for discussions and their interest, Julien Roger for bringing his attention to Kashaev's work on the decorated character variety, Ronggang Shi for answering his questions on ergodic theory and Yang Zhou for writing a Python program for testing some of the author's ideas.

The author is supported by NSF grant DMS-1405066.

2 Decorated character spaces

We recall the decorated character spaces and the lengths coordinates in this section. The readers are recommended to read Kashaev's papers [17; 18] for the original approach and for more details.

Let $\Sigma_{g,n}$ be a punctured surface of genus g with n punctures, and let $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ be a nonelementary type-preserving representation. A *pseudodeveloping map* D_ρ of ρ is a piecewise smooth ρ–equivariant map from the universal cover of $\Sigma_{g,n}$ to the hyperbolic plane $\mathbb{H}^2$. By [9], ρ is the holonomy representation of D_ρ. Let ω be the hyperbolic area form of $\mathbb{H}^2$. Since D_ρ is ρ–equivariant, the pull-back 2–form $(D_\rho)^*\omega$ descends to $\Sigma_{g,n}$. The relative Euler class $e(\rho)$ of ρ can be calculated as

$$e(\rho) = \frac{1}{2\pi}\int_{\Sigma_{g,n}} (D_\rho)^*\omega.$$

An ideal arc α on $\Sigma_{g,n}$ is an arc connecting two (possibly the same) punctures. The image $D_\rho(\widetilde{\alpha})$ of a lift $\widetilde{\alpha}$ of α is an arc in $\mathbb{H}^2$ connecting two (possibly the same) points on $\partial\mathbb{H}^2$, each of which is the fixed point of the ρ–image of certain peripheral element of $\pi_1(\Sigma_{g,n})$. We call α *ρ–admissible* if the two end points of $D_\rho(\widetilde{\alpha})$ are distinct. It is easy to see that α being ρ–admissible is independent of the choice of the lift $\widetilde{\alpha}$ and the pseudodeveloping map D_ρ. An ideal triangulation $\mathcal{T}$ of $\Sigma_{g,n}$ consists of a set of disjoint ideal arcs, called the edges, whose complement is a disjoint union of triangles, called the ideal triangles. We call $\mathcal{T}$ *ρ–admissible* if all the edges of $\mathcal{T}$ are ρ–admissible. If ρ' is conjugate to ρ, then it is easy to see that $\mathcal{T}$ is ρ'–admissible if and only if it is ρ–admissible.

Theorem 2.1 (Kashaev [17]) *For each ideal triangulation $\mathcal{T}$, the set*

$$\mathcal{M}_{\mathcal{T}}(\Sigma_{g,n}) = \{[\rho] \in \mathcal{M}(\Sigma_{g,n}) \mid \mathcal{T} \text{ is } \rho\text{–admissible}\}$$

is open and dense in $\mathcal{M}(\Sigma_{g,n})$, and there exist finitely many ideal triangulations $\mathcal{T}_i$, $i = 1, \dots, m$, such that

$$\mathcal{M}(\Sigma_{g,n}) = \bigcup_{i=1}^{m} \mathcal{M}_{\mathcal{T}_i}(\Sigma_{g,n}).$$

Let $\Sigma_{g,n}$ and ρ be as above. A *decoration* of ρ is an assignment of horocycles centered at the fixed points of the ρ–image of the peripheral elements of $\pi(\Sigma_{g,n})$, one for each, which is invariant under the $\rho(\pi_1(\Sigma_{g,n}))$–action. In the case that the fixed points of the ρ–image of two peripheral elements coincide, which may happen only when ρ is non-Fuchsian, we do not require the corresponding assigned horocycles to be the same. If d is a decoration of ρ and g is an element of $\mathrm{PSL}(2,\mathbb{R})$, then the g–image of the horocycles in d form a decoration $g \cdot d$ of the conjugation $g\rho g^{-1}$ of ρ. We call a pair (ρ, d) a *decorated representation*, and call two decorated representations (ρ, d) and (ρ', d') *equivalent* if $\rho' = g\rho g^{-1}$ and $d' = g \cdot d$ for some $g \in \mathrm{PSL}(2,\mathbb{R})$. The *decorated character space* of $\Sigma_{g,n}$, denoted by $\mathcal{M}^d(\Sigma_{g,n})$, is the space of equivalence classes of decorated representations. In [17], Kashaev shows that the projection $\pi\colon \mathcal{M}^d(\Sigma_{g,n}) \to \mathcal{M}(\Sigma_{g,n})$ defined by $\pi([(\rho, d)]) = [\rho]$ is a principal $\mathbb{R}^V_{>0}$–bundle, where V is the set of punctures of $\Sigma_{g,n}$, and the preimage of the extremal components are isomorphic as principal $\mathbb{R}^V_{>0}$–bundles to Penner's decorated Teichmüller space [23].

Fixing a ρ–admissible ideal triangulation $\mathcal{T}$ and a pseudodeveloping map D_ρ, the lengths coordinates of (ρ, d) consists of the following two parts: the λ–lengths of the edges and the signs of the ideal triangles. The λ–length of an edge e of $\mathcal{T}$ is defined as follows. Since e is ρ–admissible, for any lift $\tilde{e}$ of e to the universal cover the image $D_\rho(\tilde{e})$ connects to distinct points u_1 and u_2 on $\partial\mathbb{H}^2$. The decoration d assigns two horocycles H_1 and H_2 respectively centered at u_1 and u_2. Let $l(e)$ be the signed hyperbolic distance between the two horocycles, ie $l(e) > 0$ if H_1 and H_2 are disjoint and $l(e) \leqslant 0$ if otherwise. Then the *λ–length* of e in the decorated representation (ρ, d) is defined by

$$\lambda(e) = \exp\frac{l(e)}{2}.$$

The sign of an ideal triangle t of $\mathcal{T}$ is defined as follows. Let v_1, v_2 and v_3 be the vertices of t so that the orientation on t determined by the cyclic order $v_1 \mapsto v_2 \mapsto v_3 \mapsto v_1$ coincides with the one induced from the orientation of $\Sigma_{g,n}$. Let $\tilde{t}$ be a lift of t to the universal cover, and let $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{v}_3$ be the vertices of $\tilde{t}$ so that $\tilde{v}_i$ is a lift of v_i, $i = 1, 2, 3$. Since $\mathcal{T}$ is ρ–admissible, the points $D_\rho(\tilde{v}_1)$, $D_\rho(\tilde{v}_2)$ and $D_\rho(\tilde{v}_3)$ are distinct on $\partial\mathbb{H}^2$, and hence determine a hyperbolic ideal triangle Δ in $\mathbb{H}^2$ with them as the ideal vertices. The *sign* of t is positive, denoted by $\epsilon(t)=1$, if the orientation on Δ determined by the cyclic order $D_\rho(\tilde{v}_1)\mapsto D_\rho(\tilde{v}_2)\mapsto D_\rho(\tilde{v}_3)\mapsto D_\rho(\tilde{v}_1)$ coincides with the one induced from the orientation of $\mathbb{H}^2$. Otherwise, the sign of t is negative, and is denoted by $\epsilon(t)=-1$. From the construction, it is easy to see that the λ–lengths $\lambda(e)$ and the signs $\epsilon(t)$ depend only on the equivalence class of (ρ,d). Let T be the set of ideal triangles of $\mathcal{T}$. Then the integral of the pull-back form $(D_\rho)^*\omega$ over $\Sigma_{g,n}$ equals $\sum_{t\in T}\epsilon(t)\pi$, and the relative Euler class of ρ can be calculated as

$$e(\rho)=\frac{1}{2}\sum_{t\in T}\epsilon(t). \tag{2-1}$$

Let V be the set of punctures of $\Sigma_{g,n}$ and let E be the set of edges of $\mathcal{T}$. Then there is a principal $\mathbb{R}^V_{>0}$–bundle structure on $\mathbb{R}^E_{>0}$ defined as follows. For $\mu\in\mathbb{R}^V_{>0}$ and $\lambda\in\mathbb{R}^E_{>0}$, we define $\mu\cdot\lambda\in\mathbb{R}^E_{>0}$ by $(\mu\cdot\lambda)(e)=\mu(v_1)\lambda(e)\mu(v_2)$, where v_1 and v_2 are the punctures connected by the edge e.

Theorem 2.2 (Kashaev) *Let $\pi\colon\mathcal{M}^d(\Sigma_{g,n})\to\mathcal{M}(\Sigma_{g,n})$ be the principal $\mathbb{R}^V_{>0}$–bundle, and let $\mathcal{M}^d_{\mathcal{T}}(\Sigma_{g,n})$ the preimage of $\mathcal{M}_{\mathcal{T}}(\Sigma_{g,n})$. Then*

$$\mathcal{M}^d_{\mathcal{T}}(\Sigma_{g,n})=\coprod_{\epsilon\in\{\pm1\}^T}\mathcal{R}(\mathcal{T},\epsilon),$$

where each $\mathcal{R}(\mathcal{T},\epsilon)$ is isomorphic as a principal $\mathbb{R}^V_{>0}$–bundle to an open subset of $\mathbb{R}^E_{>0}$. The isomorphism is given by the λ–lengths, and the image of $\mathcal{R}(\mathcal{T},\epsilon)$ is the complement of the zeros of certain rational function coming from the image of the peripheral elements not being the identity matrix.

On $\mathcal{M}(\Sigma_{g,n})$ we have the Goldman symplectic form ω_{WP} which restricts to the Weil–Petersson symplectic form on the Teichmüller component [10]. By [17; 18], for each ideal triangulation $\mathcal{T}$, the pull-back of ω_{WP} to $\mathcal{M}^d_{\mathcal{T}}(\Sigma_{g,n})$ is expressed in the λ–lengths by

$$\pi^*\omega_{\mathrm{WP}}=\sum_{t\in T}\left(\frac{d\lambda(e_1)\wedge d\lambda(e_2)}{\lambda(e_1)\lambda(e_2)}+\frac{d\lambda(e_2)\wedge d\lambda(e_3)}{\lambda(e_2)\lambda(e_3)}+\frac{d\lambda(e_3)\wedge d\lambda(e_1)}{\lambda(e_3)\lambda(e_1)}\right), \tag{2-2}$$

where e_1, e_2 and e_3 are the edges of the ideal triangle t in the cyclic order induced from the orientation of $\Sigma_{g,n}$. This formula was first obtained by Penner [24] for the decorated Teichmüller space. From (2-2), it is easy to see that the measure on each $\mathcal{R}(\mathcal{T},\epsilon)$ induced by $\pi^*\omega_{\mathrm{WP}}$ is in the measure class of the pull-back of the Lebesgue measure on $\mathbb{R}^E_{>0}$.

A diagonal switch at an edge e of $\mathcal{T}$ replaces the edge e by the other diagonal of the quadrilateral formed by the union of the two ideal triangles adjacent to e. By [16], any ideal triangulation can be obtained from another by doing a finite sequence of diagonal switches. Let (ρ, d) be a decorated representation and let $\mathcal{T}$ be a ρ–admissible ideal triangulation of $\Sigma_{g,n}$. If $\mathcal{T}'$ is the ideal triangulation of $\Sigma_{g,n}$ obtained from $\mathcal{T}$ by doing a diagonal switch at an edge e, then the ρ–admissibility of $\mathcal{T}'$ and the lengths coordinates of (ρ, d) in $\mathcal{T}'$ are determined as follows. Let t_1 and t_2 be the two ideal triangles of $\mathcal{T}$ adjacent to e, let e' be the new edge of $\mathcal{T}'$ and let t_1' and t_2' be the two ideal triangles in $\mathcal{T}'$ adjacent to e'. We respectively name the edges of the quadrilateral $e_1, \dots, e_4$ in the way that e_1 is adjacent to t_1 and t_1', e_2 is adjacent to t_1 and t_2', e_3 is adjacent to t_2 and t_2' and e_4 is adjacent to t_2 and t_1'. Then e_1 is opposite to e_3, and e_2 is opposite to e_4 in the quadrilateral.

Proposition 2.3 (Kashaev) (1) *If the signs $\epsilon(t_1) = \epsilon(t_2)$, then $\mathcal{T}'$ is ρ–admissible. In this case,*

$$\epsilon(t_1') = \epsilon(t_2') = \epsilon(t_1) \quad \textit{and} \quad \lambda(e') = \frac{\lambda(e_1)\lambda(e_3) + \lambda(e_2)\lambda(e_4)}{\lambda(e)},$$

and the signs of the common ideal triangles and the λ–lengths of the common edges of $\mathcal{T}$ and $\mathcal{T}'$ do not change.

(2) *If $\epsilon(t_1) \neq \epsilon(t_2)$, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(e_1)\lambda(e_3) \neq \lambda(e_2)\lambda(e_4)$. In this case,*

(a) *if $\lambda(e_1)\lambda(e_3) < \lambda(e_2)\lambda(e_4)$, then*

$$\epsilon(t_1') = \epsilon(t_1), \quad \epsilon(t_2') = \epsilon(t_2) \quad \textit{and} \quad \lambda(e') = \frac{\lambda(e_2)\lambda(e_4) - \lambda(e_1)\lambda(e_3)}{\lambda(e)},$$

(b) *If $\lambda(e_2)\lambda(e_4) < \lambda(e_1)\lambda(e_3)$, then*

$$\epsilon(t_1') = \epsilon(t_2), \quad \epsilon(t_2') = \epsilon(t_1) \quad \textit{and} \quad \lambda(e') = \frac{\lambda(e_1)\lambda(e_3) - \lambda(e_2)\lambda(e_4)}{\lambda(e)},$$

and the signs of the common ideal triangles and the λ–lengths of the common edges of $\mathcal{T}$ and $\mathcal{T}'$ do not change,

The rule for the signs in (2)(a) and (2)(b) is that the signs of the ideal triangles adjacent to the shorter edges do not change. This can be seen as follows. If, for example, $\epsilon(t_1) = -1$, $\epsilon(t_2) = 1$ and $\lambda(e_1)\lambda(e_3) < \lambda(e_2)\lambda(e_4)$, then the hyperbolic ideal triangle Δ_1 determined by t_1 is negatively oriented, the hyperbolic ideal triangle Δ_2 determined by t_2 is positively oriented, and the geodesic arcs a_2 and a_4 determined by e_2 and e_4 intersect. See Figure 1. As a consequence, the hyperbolic ideal triangle Δ_1' determined by t_1' is negatively oriented and the hyperbolic ideal triangle Δ_2' determined by t_2' is positively oriented, hence $\epsilon(t_1')=-1$ and $\epsilon(t_2')=1$. The λ–lengths of e' follow from Penner's Ptolemy relation [23] that $\lambda(e_2)\lambda(e_4)=\lambda(e_1)\lambda(e_3)+\lambda(e)\lambda(e')$. The other cases can be verified similarly.

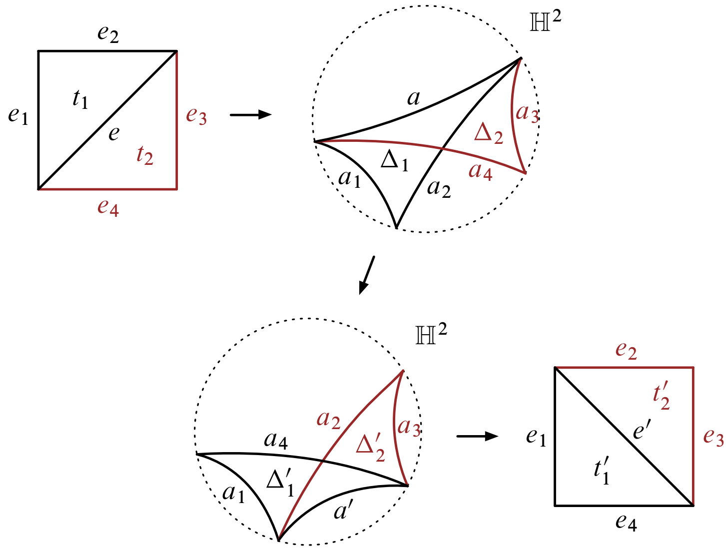


Figure 1

By Theorems 2.1 and 2.2, the family of open subsets $\{\mathcal{R}(\mathcal{T},\epsilon)\}$, where $\mathcal{T}$ goes over all the ideal triangulations of $\Sigma_{g,n}$ together with the λ–lengths functions $\{\lambda\colon \mathcal{R}(\mathcal{T},\epsilon)\to\mathbb{R}^E_{>0}\}$ form a coordinate system of $\mathcal{M}^d(\Sigma_{g,n})$, and the transition functions are given by Proposition 2.3.

Theorem 2.4 (Kashaev) *For each relative Euler class k, let $\mathcal{M}^d_k(\Sigma_{g,n})$ be the preimage of $\mathcal{M}_k(\Sigma_{g,n})$ under the projection $\pi\colon \mathcal{M}^d(\Sigma_{g,n})\to\mathcal{M}(\Sigma_{g,n})$. Then*

$$\mathcal{M}^d_k(\Sigma_{g,n})=\bigcup_{\mathcal{T}}\coprod_{\epsilon}\mathcal{R}(\mathcal{T},\epsilon),$$

where the union is over all the ideal triangulations $\mathcal{T}$ and the disjoint union is over all $\epsilon\in\{\pm1\}^T$ such that $\sum_{t\in T}\epsilon(t)=2k$. Moreover, the $\mathcal{M}^d_k(\Sigma_{g,n})$ are principal $\mathbb{R}^V_{>0}$–bundles, and are disjoint for different k.

3 A trace formula for closed curves

Throughout this section, we let $\mathcal{T}$ be an ideal triangulation of $\Sigma_{g,n}$, and let E and T respectively be the set of edges and ideal triangles of $\mathcal{T}$. Given the lengths coordinates

$(\lambda, \epsilon) \in \mathbb{R}_{>0}^{E} \times \{\pm 1\}^{T}$, the type-preserving representation $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ can be reconstructed up to conjugation as follows. Suppose e is an edge of $\mathcal{T}$, and t_1 and t_2 are the two ideal triangles adjacent to e. Let e_1 and e_2 be the other two edges of t_1 and let e_3 and e_4 be the other two edges of t_2 so that the cyclic orders $e \mapsto e_1 \mapsto e_2 \mapsto e$ and $e \mapsto e_3 \mapsto e_4 \mapsto e$ coincide with the one induced from the orientation of $\Sigma_{g,n}$. Define the quantity $X(e) \in \mathbb{R}_{>0}$ by

$$X(e) = \frac{\lambda(e_2)\lambda(e_4)}{\lambda(e_1)\lambda(e_3)}.$$

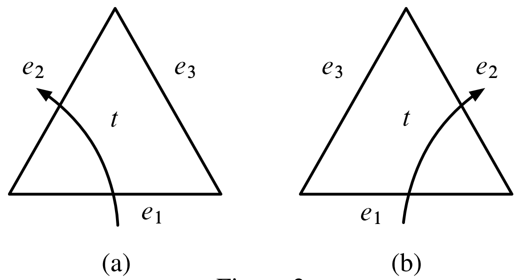


Figure 2

Note that if ρ is discrete and faithful, then $X(e)$ is the shear parameter of the corresponding hyperbolic structure at e. (See [1].) It is well known that each immersed closed curve on $\Sigma_{g,n}$ is homotopic to a normal one that transversely intersects each ideal triangle in simple arcs that connect different edges of the triangle. Let γ be an immersed oriented closed normal curve on $\Sigma_{g,n}$. For each edge e intersecting γ, define

$$S(e) = \begin{bmatrix} X(e)^{1/2} & 0 \\ 0 & X(e)^{-1/2} \end{bmatrix}.$$

For each ideal triangle t intersecting γ, define

$$R(t) = \begin{bmatrix} 1 & \epsilon(t) \\ 0 & 1 \end{bmatrix}$$

if γ makes a left turn in t (Figure 2(a)), and define

$$R(t) = \begin{bmatrix} 1 & 0 \\ \epsilon(t) & 1 \end{bmatrix}$$

if γ makes a right turn in t (Figure 2(b)).

Lemma 3.1 *Let $e_{i_1}, \dots, e_{i_m}$ be the edges and let $t_{j_1}, \dots, t_{j_m}$ be the ideal triangles of $\mathcal{T}$ intersecting γ in the cyclic order induced by the orientation of γ so that e_{i_k} is the* *common edge of $t_{j_{k-1}}$ and t_{j_k} for each $k \in \{1, \dots, m\}$. Then up to a conjugation by an element of* $\mathrm{PSL}(2, \mathbb{R})$,

$$\rho([\gamma]) = \pm S(e_{i_1}) R(t_{j_1}) S(e_{i_2}) R(t_{j_2}) \cdots S(e_{i_m}) R(t_{j_m}).$$

Proof The proof is parallel to that of Lemma 3 in [3]. The idea is to keep track of the image of the unit tangent vector $\frac{\partial}{\partial y}$ at $i \in \mathbb{H}^2$ under $\rho([\gamma])$. The contributions of each edge e and of each ideal triangle t intersecting γ to $\rho([\gamma])$ are respectively $\pm S(e)$ and $\pm R(t)$. See also [2, Exercise 8.5–8.7 and 10.14]. $\square$

For each puncture v of $\Sigma_{g,n}$, let γ_v be the simple closed curve going counterclockwise around v once. By Lemma 3.1, the image of γ_v is up to conjugation

$$\rho([\gamma_v]) = \pm \begin{bmatrix} 1 & \psi_{v,\epsilon}(\lambda) \\ 0 & 1 \end{bmatrix},$$

where $\psi_{v,\epsilon}$ is a rational function of λ depending on ϵ. Therefore, ρ is type-preserving if and only if $\psi_{v,\epsilon}(\lambda) \neq 0$ for all punctures v. The following proposition gives a more precise description of this rational function in Theorem 2.2.

Proposition 3.2 (Kashaev) *Let $(\lambda, \epsilon) \in \mathbb{R}^E_{>0} \times \{\pm 1\}^T$, let V be the set of punctures of $\Sigma_{g,n}$ and let ψ_ϵ be the rational function defined by*

$$\psi_\epsilon = \prod_{v \in V} \psi_{v,\epsilon}.$$

Then (λ, ϵ) defines a type-preserving representation if and only if $\psi_\epsilon(\lambda) \neq 0$.

The following theorem provides a more direct way to calculate the absolute values of the traces of closed curves using the λ–lengths, which was first found by Sun and the author. We include a proof here for the reader's convenience. For each ideal triangle t intersecting γ, let e_1 be the edge of t at which γ enters, let e_2 be the edge of t at which γ leaves and let e_3 be the other edge of t. See Figure 2. Define

$$M(t) = \begin{bmatrix} \lambda(e_1) & \epsilon(t)\lambda(e_3) \\ 0 & \lambda(e_2) \end{bmatrix}$$

if γ makes a left turn in t, and define

$$M(t) = \begin{bmatrix} \lambda(e_2) & 0 \\ \epsilon(t)\lambda(e_3) & \lambda(e_1) \end{bmatrix}$$

if γ makes a right turn in t.

Theorem 3.3 *For an immersed closed normal curve γ on $\Sigma_{g,n}$, let $e_{i_1},\dots,e_{i_m}$ be the edges and let $t_{j_1},\dots,t_{j_m}$ be the ideal triangles of $\mathcal{T}$ intersecting γ in the cyclic order following the orientation of γ so that e_{i_k} is the common edge of $t_{j_{k-1}}$ and t_{j_k} for each $k\in\{1,\dots,m\}$. Then*

$$|\operatorname{tr}\rho([\gamma])| = \frac{|\operatorname{tr}(M(t_{j_1})\cdots M(t_{j_m}))|}{\lambda(e_{i_1})\cdots\lambda(e_{i_m})}. \tag{3-1}$$

Proof For each ideal triangle t and an edge e of t, let e' and e'' be the other two edges of t so that the cyclic order $e\mapsto e'\mapsto e''\mapsto e$ coincides with the one induced by the orientation of $\Sigma_{g,n}$. Define the matrix

$$S(t,e)=\begin{bmatrix}\sqrt{\lambda(e'')/\lambda(e')} & 0\\ 0 & \sqrt{\lambda(e')/\lambda(e'')}\end{bmatrix}.$$

Then

$$S(e_{i_k}) = S(t_{j_{k-1}}, e_{i_k})S(t_{j_k}, e_{i_k}) \tag{3-2}$$

for each $k\in\{1,\dots,m\}$, where as a convention $t_{j_{1-1}}=t_{j_m}$. A case by case calculation shows that

$$S(t_{j_k}, e_{i_k})R(t_{j_k})S(t_{j_k}, e_{i_{k+1}}) = \frac{M(t_{j_k})}{\sqrt{\lambda(e_{i_k})\lambda(e_{i_{k+1}})}} \tag{3-3}$$

for each $k\in\{1,\dots,m\}$, where as a convention $e_{i_{m+1}}=e_{i_1}$. By Lemma 3.1, (3-2), (3-3) and the fact that $\operatorname{tr}(AB)=\operatorname{tr}(BA)$ for any two matrices A and B, we have

$$\begin{aligned}|\operatorname{tr}\rho([\gamma])| &= |\operatorname{tr}(S(e_{i_1})R(t_{j_1})\cdots S(e_{i_m})R(t_{j_m}))|\\ &= |\operatorname{tr}(S(t_{j_1},e_{i_1})R(t_{j_1})S(t_{j_1},e_{i_2})\cdots S(t_{j_m},e_{i_m})R(t_{j_m})S(t_{j_m},e_{i_1}))|\\ &= \frac{|\operatorname{tr}(M(t_{j_1})\cdots M(t_{j_m}))|}{\lambda(e_{i_1})\cdots\lambda(e_{i_m})}.\end{aligned}$$

□

Remark 3.4 Equation (3-1) was first obtained by Roger and Yang [25] for decorated hyperbolic surfaces, ie discrete and faithful decorated representations, using the skein relations, where their formula includes both the traces of closed geodesics and the λ–lengths of geodesics arcs connecting the punctures. It is interesting to know whether there is a similar formula for the λ–lengths of arcs for the non-Fuchsian decorated representations.

As a consequence of Theorem 3.3, we have the following theorem.

Theorem 3.5 (1) *If $\rho\colon \Sigma_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$ is a non-Fuchsian type-preserving representation, there exists a Fuchsian type-preserving representation ρ' such that*

$$|\mathrm{tr}\,\rho([\gamma])| \leqslant |\mathrm{tr}\,\rho'([\gamma])|$$

for each $[\gamma] \in \pi_1(\Sigma_{g,n})$, and the strict equality holds for at least one γ.

(2) *Conversely, for almost every Fuchsian type-preserving representation $\rho'\colon \Sigma_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$ and for each k with $|k| < 2g-2+n$ and $\mathcal{M}_k(\Sigma_{g,n}) \neq \varnothing$, there exists a type-preserving representation ρ with $e(\rho) = k$ such that*

$$|\mathrm{tr}\,\rho([\gamma])| \leqslant |\mathrm{tr}\,\rho'([\gamma])|$$

for each $[\gamma] \in \pi_1(\Sigma_{g,n})$, and the strict equality holds for at least one γ.

Proof For (1), by Theorem 2.1, there exists a ρ–admissible ideal triangulation $\mathcal{T}$. Choose arbitrarily a decoration d of ρ, and let (ρ', d') be the decorated representation that has the same λ–lengths of (ρ, d) and has the positive signs for all the ideal triangles. By (2-1) and Goldman's result in [9], ρ' is Fuchsian. Applying Equation (3-1) to $|\mathrm{tr}\,\rho([\gamma])|$ and $|\mathrm{tr}\,\rho'([\gamma])|$, we see that they have the same summands with different coefficients ± 1, and the coefficients for the later are all positive. Since each summand is a product of the λ–lengths, which is positive, the inequality follows. Since ρ is non-Fuchsian, by (2-1), there must be an ideal triangle t that has negative sign in (ρ, d). Therefore, if γ intersects t, then some of the summands in the expression of $|\mathrm{tr}\,\rho([\gamma])|$ has negative coefficients, and the inequality for γ is strict.

For (2), choose arbitrarily an ideal triangulation $\mathcal{T}$ of $\Sigma_{g,n}$, and let T be the set of ideal triangles of $\mathcal{T}$. By Theorems 2.1, 2.2 and 2.4, if $\mathcal{M}_k(\Sigma_{g,n}) \neq \varnothing$, then there exists $\epsilon \in \{\pm 1\}^T$ such that $\sum_{t\in T} \epsilon(t) = 2k$ and the subset $\mathcal{R}(\mathcal{T}, \epsilon)$ is homeomorphic via the lengths coordinates to a full measure open subset $\Omega(\mathcal{T}, \epsilon)$ of $\mathbb{R}^E_{>0}$. For each $\lambda \in \Omega(\mathcal{T}, \epsilon)$, let (ρ, d) be the decorated representation determined by (λ, ϵ). Then $e(\rho) = k$. On the other hand, $\mathbb{R}^E_{>0}$ is identified with the decorated Teichmüller space via the lengths coordinates, hence λ determines a Fuchsian type-preserving representation ρ'. By the same argument in (1), the inequality holds for ρ and ρ', and is strict for γ intersecting the ideal triangles t with $\epsilon(t) = -1$. □

Remark 3.6 It would be very interesting to know if Theorem 3.5(2) holds for every Fuchsian type-preserving representation. This amounts to asking whether

$$\bigcup_{\mathcal{T},\epsilon} \Omega(\mathcal{T}, \epsilon) = \mathbb{R}^E_{>0},$$

where the union is over all the ideal triangulations $\mathcal{T}$ of $\Sigma_{g,n}$ and all ϵ that give the right relative Euler class.

4 Tetrahedral triangulations

A *tetrahedral triangulation* of the four-punctured sphere is an ideal triangulation of $\Sigma_{0,4}$ that is combinatorially equivalent to the boundary of an Euclidean tetrahedron (Figure 3(a)). A pair of edges of a tetrahedral triangulation are called opposite if they are opposite edges of the tetrahedron. Let $v_1, \dots, v_4$ be the four punctures of $\Sigma_{0,4}$. In the rest of this paper, for each tetrahedral triangulation $\mathcal{T}$, we will let t_i be the unique ideal triangle of $\mathcal{T}$ disjoint from the puncture v_i and let e_{ij} be the unique edge of $\mathcal{T}$ connecting the punctures v_i and v_j. We respectively denote by x the pair of opposite edges $\{e_{12}, e_{34}\}$, by y the pair $\{e_{13}, e_{24}\}$ and by z the pair $\{e_{14}, e_{23}\}$. See Figure 3(b).

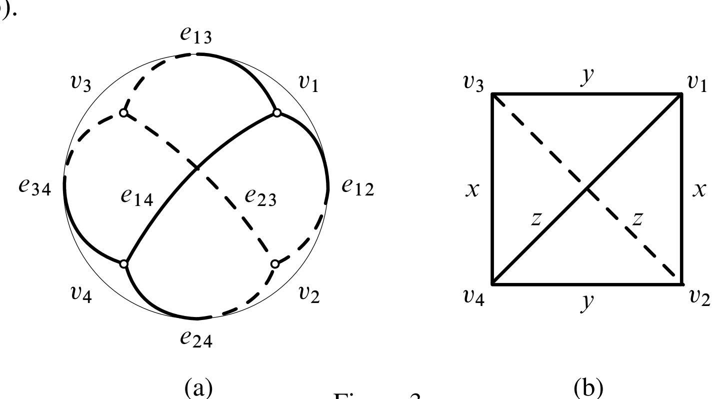


Figure 3

A nonperipheral simple closed curve on $\Sigma_{0,4}$ is *distinguished* in a tetrahedral triangulation $\mathcal{T}$ if it is disjoint from a pair of opposite edges of $\mathcal{T}$ and intersects each of the other four edges at exactly one point. In each tetrahedral triangulation, there are exactly three distinguished simple closed curves. We respectively denote by X, Y and Z the distinguished simple closed curves disjoint from the pair of opposite edges x, y and z. See Figure 4.

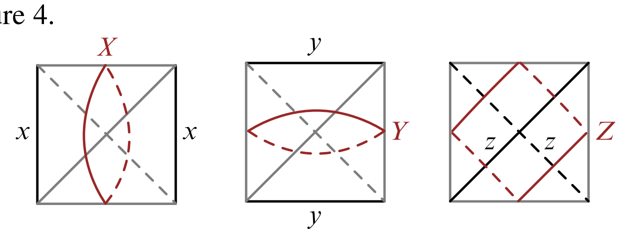


Figure 4

The curves X, Y and Z mutually intersect at exactly two points. On the other hand, for each triple of simple closed curves that mutually intersect at two points, there is a unique tetrahedral triangulation in which these three curves are distinguished. In particular, each nonperipheral simple closed curve on $\Sigma_{0,4}$ is distinguished in some tetrahedral triangulation. Note that being the X–, Y– or Z–curve is independent of the choice of the tetrahedral triangulation, since, for example, the curve X always separates $\{v_1, v_2\}$ from $\{v_3, v_4\}$. In the rest of this paper, we will call a simple closed curve an X– (resp. Y– or Z–) *curve* if it is disjoint from the pair of opposite edges x (resp. y or z) of some tetrahedral triangulation. In this way, we get a tricoloring of the set of nonperipheral simple closed curves on $\Sigma_{0,4}$.

A *simultaneous diagonal switch* at a pair of opposite edges of $\mathcal{T}$ is an operation that simultaneously does diagonal switches at this pair of edges. See Figure 5(a). Denote respectively by S_x, S_y and S_z the simultaneous diagonal switches at the pair of opposite edges x, y and z. Then S_x (reps. S_y and S_z) changes the X– (resp. Y– and Z–) curve and leaves the other two distinguished simple closed curves unchanged. See Figure 5(b).

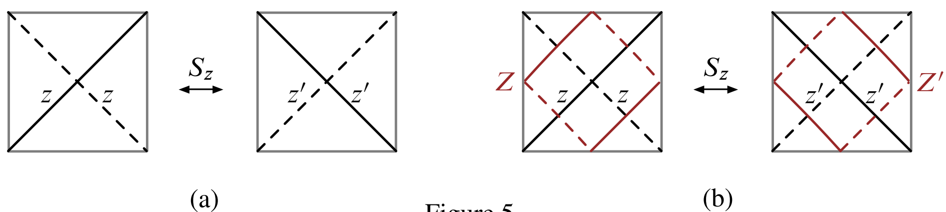
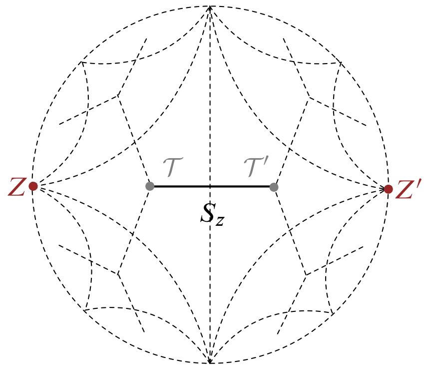


Figure 5

The relationship between tetrahedral triangulations, simultaneous diagonal switches and nonperipheral simple closed curves can be described by (the dual of) the Farey diagram. Recall that the Farey diagram $\mathcal{F}$ is an ideal triangulation of $\mathbb{H}^2$ whose vertices are the extended rational numbers $\mathbb{Q} \cup \{\infty\} \subset \partial\mathbb{H}^2$, and the dual Ferey diagram $\mathcal{F}^*$ is a countably infinite trivalent tree properly embedded in $\mathbb{H}^2$. Each vertex of $\mathcal{F}$ corresponds to a nonperipheral simple closed curve on $\Sigma_{0,4}$, each edge of $\mathcal{F}$ connects two vertices corresponding to two simple closed curves that intersect at exactly two points and each ideal triangle of $\mathcal{F}$ corresponds to a triple of simple closed curves mutually intersecting at two points. (See [19].) Therefore, each vertex of the dual graph $\mathcal{F}^*$ corresponds to a tetrahedral triangulation of $\Sigma_{0,4}$, each edge of $\mathcal{F}^*$ corresponds to a simultaneous diagonal switch and each connected component of $\mathbb{H}^2 \setminus \mathcal{F}^*$ corresponds to a nonperipheral simple closed curves on $\Sigma_{0,4}$. See Figure 6. Since $\mathcal{F}^*$ is connected, any tetrahedral triangulation can be obtained from another by doing a finitely sequence of simultaneous diagonal switches.

We close up this section by showing the relationship between simultaneous diagonal switches and the mapping classes of $\Sigma_{0,4}$.



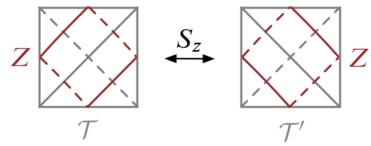


Figure 6

Proposition 4.1 *A composition of an even number of simultaneous diagonal switches determines an element of* $\mathrm{Mod}(\Sigma_{0,4})$*. Conversely, any element of* $\mathrm{Mod}(\Sigma_{0,4})$ *is determined by a composition of an even number of simultaneous diagonal switches.*

Proof Let $\mathcal{T}$ be a tetrahedra triangulation of $\Sigma_{0,4}$. We write $\phi = S'S$ if ϕ is the self-diffeomorphism of $\Sigma_{0,4}$ such that the tetrahedral triangulation $\phi(\mathcal{T})$ is obtained from $\mathcal{T}$ by doing the simultaneous diagonal switch S followed by the simultaneous diagonal switch S'. Then $D_X = S_z S_y$ and $D_Y = S_x S_z$. See Figure 7.

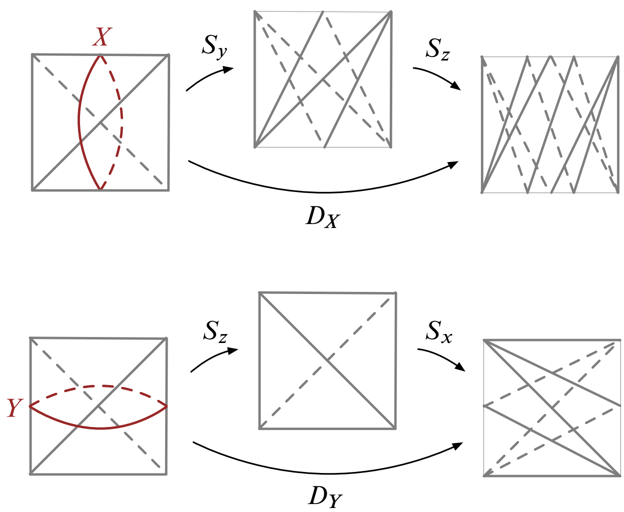


Figure 7

Similarly, we have $S_y S_x = D_Z$, $S_y S_z = D_X^{-1}$, $S_x S_z = D_Y^{-1}$ and $S_x S_y = D_X^{-1}$. Thus, any composition of an even number of simultaneous diagonal switches determines an element of $\mathrm{Mod}(\Sigma_{0,4})$.

For the converse statement, we define a cyclic order on the set $\{x, y, z\}$ of pairs of opposite edges of $\mathcal{T}$ as follows. Since each puncture v of $\Sigma_{0,4}$ is adjacent to three edges e, e' and e'' with $e \in x$, $e' \in y$ and $e'' \in z$, the orientation of $\Sigma_{0,4}$ induces a cyclic order on the set $\{e, e', e''\}$ around v, inducing a cyclic order on the set $\{x, y, z\}$. It is easy to check that this cyclic order is independent of the choose of v, hence is well defined. We call the *sign* of a tetrahedral triangulation $\mathcal{T}$ positive if the cyclic order $x \mapsto y \mapsto z \mapsto x$ coincides with the one induced from the orientation, and negative if otherwise. It easy to see that a simultaneous diagonal switch changes the sign of $\mathcal{T}$, and an orientation preserving self-diffeomorphism of $\Sigma_{0,4}$ preserves the sign of $\mathcal{T}$. Since the dual Farey diagram $\mathcal{F}^*$ is a connected tree, for any self-diffeomorphism ϕ of $\Sigma_{0,4}$, up to redundancy there is a unique path of $\mathcal{F}^*$ connecting the vertices $\mathcal{T}$ and $\phi(\mathcal{T})$. Since $\mathcal{T}$ and $\phi(\mathcal{T})$ have the same sign, the path consists of an even number of edges, corresponding to an even number of simultaneous diagonal switches $S_1, \dots, S_{2m}$. Then $\phi = \phi_k \circ \cdots \circ \phi_1$, where $\phi_k = S_{2k} S_{2k-1}$. □

5 Bowditch's question

Let ρ be a type-preserving representation of $\pi_1(\Sigma_{0,4})$ and let d be a decoration of ρ. Suppose $\mathcal{T}$ is a ρ–admissible tetrahedral triangulation of $\Sigma_{0,4}$, E and T respectively are the sets of edges and ideal triangles of $\mathcal{T}$, and $(\lambda, \epsilon) \in \mathbb{R}^E_{>0} \times \{\pm 1\}^T$ are the lengths coordinates of $[(\rho, d)] \in \mathcal{M}^d_{\pm 1}(\Sigma_{0,4})$. Let $v_1, \dots, v_4$ be the punters of $\Sigma_{0,4}$, let t_i be the ideal triangle of $\mathcal{T}$ disjoint from v_i and let e_{ij} be the edge of $\mathcal{T}$ connecting the punctures v_i and v_j. Define the quantities $\lambda(x) = \lambda(e_{12})\lambda(e_{34})$, $\lambda(y) = \lambda(e_{13})\lambda(e_{24})$ and $\lambda(z) = \lambda(e_{14})\lambda(e_{23})$. The quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ will play a central role in the rest of this paper.

5.1 A proof of Theorem 1.1

Suppose $e(\rho) = 1$. Then by (2-1), there is exactly one ideal triangle, say t_1, such that $\epsilon(t_1) = -1$ and $\epsilon(t_i) = 1$ for $i \neq 1$. As a direct consequence of Lemma 3.1 and Theorem 3.3, we have the following lemmas.

Lemma 5.1 *Let γ_i be the simple closed curve going counterclockwise around the puncture v_i once. Then up to conjugation, the ρ–image of the peripheral element $[\gamma_1] \in \pi_1(\Sigma_{0,4})$ is*

$$\pm \begin{bmatrix} 1 & \lambda(x) + \lambda(y) + \lambda(z) \\ 0 & 1 \end{bmatrix},$$

and the ρ–image of the other peripheral elements $[\gamma_2]$, $[\gamma_3]$ and $[\gamma_4]$ are respectively

$$\pm\begin{bmatrix} 1 & \lambda(y)+\lambda(z)-\lambda(x) \\ 0 & 1 \end{bmatrix},$$
$$\pm\begin{bmatrix} 1 & \lambda(x)+\lambda(z)-\lambda(y) \\ 0 & 1 \end{bmatrix},$$
$$\pm\begin{bmatrix} 1 & \lambda(x)+\lambda(y)-\lambda(z) \\ 0 & 1 \end{bmatrix}.$$

Lemma 5.2 (1) *The absolute values of the traces of the distinguished simple closed curves X, Y and Z of $\mathcal{T}$ are*

$$|\mathrm{tr}\,\rho([X])| = \frac{|\lambda(y)^2+\lambda(z)^2-\lambda(x)^2|}{\lambda(y)\lambda(z)},$$
$$|\mathrm{tr}\,\rho([Y])| = \frac{|\lambda(x)^2+\lambda(z)^2-\lambda(y)^2|}{\lambda(x)\lambda(z)}, \tag{5-1}$$
$$|\mathrm{tr}\,\rho([Z])| = \frac{|\lambda(x)^2+\lambda(y)^2-\lambda(z)^2|}{\lambda(x)\lambda(y)}.$$

(2) *The right-hand sides of the equations in* (5-1) *are strictly greater than 2 if and only if $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ satisfy one of the inequalities*

$$\lambda(x) > \lambda(y)+\lambda(z),$$
$$\lambda(y) > \lambda(x)+\lambda(z), \tag{5-2}$$
$$\lambda(z) > \lambda(x)+\lambda(y).$$

Note that reversing the directions of the inequalities in (5-2), we get the triangular inequality. The idea of the proof of (2) is that if we regard the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ as the edge lengths of a Euclidean triangle, then the right-hand sides of (5-1) are twice of the cosine of the corresponding inner angles. The next lemma shows the rule of the change of the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ under a simultaneous diagonal switch.

Lemma 5.3 *Suppose $\mathcal{T}'$ is a tetrahedral triangulation of $\Sigma_{0,4}$. If $\mathcal{T}'$ is ρ–admissible, then let λ' be the λ–lengths of (ρ, d) in $\mathcal{T}'$, and let $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ be the corresponding quantities.*

(1) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_x, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(y) \neq \lambda(z)$. When $\mathcal{T}'$ is ρ–admissible, $\lambda'(y)=\lambda(y)$, $\lambda'(z)=\lambda(z)$ and*

$$\lambda'(x) = \frac{|\lambda(y)^2-\lambda(z)^2|}{\lambda(x)}.$$

(2) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_y, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(x)\neq\lambda(z)$. When $\mathcal{T}'$ is ρ–admissible, $\lambda'(x)=\lambda(x)$, $\lambda'(z)=\lambda(z)$ and*

$$\lambda'(y)=\frac{|\lambda(z)^2-\lambda(x)^2|}{\lambda(y)}.$$

(3) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_z, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(x)\neq\lambda(y)$. When $\mathcal{T}'$ is ρ–admissible, $\lambda'(x)=\lambda(x)$, $\lambda'(y)=\lambda(y)$ and*

$$\lambda'(z)=\frac{|\lambda(x)^2-\lambda(y)^2|}{\lambda(z)}.$$

Proof For (1), we have that the edge e_{12} is adjacent to the ideal triangle t_3 and t_4 with $\epsilon(t_3)=\epsilon(t_4)$ and e_{34} is adjacent to the ideal triangles t_1 and t_2 with $\epsilon(t_1)\neq\epsilon(t_2)$. Let e'_{34} and e'_{12} respectively be the edges of $\mathcal{T}'$ obtained from diagonal switches at e_{12} and e_{34}, ie e'_{12} is the edge of $\mathcal{T}'$ connecting the punctures v_1 and v_2 and e'_{34} is the edge of $\mathcal{T}'$ connecting the punctures v_3 and v_4. By Proposition 2.3, $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(e_{13})\lambda(e_{24})\neq\lambda(e_{14})\lambda(e_{23})$, ie $\lambda(y)\neq\lambda(z)$. By Proposition 2.3 again, if $\mathcal{T}'$ is ρ–admissible, then $\lambda'(e_{ij})=\lambda(e_{ij})$ for $\{i,j\}\neq\{1,2\}$ or $\{3,4\}$, and

$$\lambda'(e'_{12})=\frac{|\lambda(e_{13})\lambda(e_{24})-\lambda(e_{14})\lambda(e_{23})|}{\lambda(e_{12})},$$
$$\lambda'(e'_{34})=\frac{\lambda(e_{13})\lambda(e_{24})+\lambda(e_{14})\lambda(e_{23})}{\lambda(e_{34})}.$$

Therefore, $\lambda'(y)=\lambda(y)$, $\lambda'(z)=\lambda(z)$ and

$$\lambda'(x)=\lambda'(e'_{12})\lambda'(e'_{34})=\frac{|\lambda(y)^2-\lambda(z)^2|}{\lambda(x)}.$$

The proofs of (2) and (3) are the similar. $\square$

A consequence of Lemma 5.3 is that the inequalities in (5-2) are persevered by simultaneous diagonal switches.

Lemma 5.4 *Suppose $\mathcal{T}'$ is a ρ–admissible tetrahedral triangulation of $\Sigma_{0,4}$ obtained from $\mathcal{T}$ by doing a simultaneous diagonal switch. Let λ' be the λ–lengths of (ρ,d) in $\mathcal{T}'$, and let $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ be the corresponding quantities. Then $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ satisfy one of the inequalities in* (5-2) *if and only if $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ do.*

Proof Without lost of generality, we assume that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_x. If $\lambda(x)>\lambda(y)+\lambda(z)$, then by Lemma 5.3,

$$\lambda'(x)=\frac{|\lambda(y)^2-\lambda(z)^2|}{\lambda(x)}<\frac{|\lambda(y)^2-\lambda(z)^2|}{\lambda(y)+\lambda(z)}=|\lambda(y)-\lambda(z)|=|\lambda'(y)-\lambda'(z)|.$$

Therefore, either $\lambda'(y) > \lambda'(x) + \lambda'(z)$ or $\lambda'(z) > \lambda'(x) + \lambda'(y)$. On the other hand, if either $\lambda(y) > \lambda(x) + \lambda(z)$ or $\lambda(z) > \lambda(x) + \lambda(y)$, ie $\lambda(x) < |\lambda(y) - \lambda(z)|$, then by Lemma 5.3,

$$\lambda'(x) = \frac{|\lambda(y)^2 - \lambda(z)^2|}{\lambda(x)} > \frac{|\lambda(y)^2 - \lambda(z)^2|}{|\lambda(y) - \lambda(z)|} = \lambda(y) + \lambda(z) = \lambda'(y) + \lambda'(z). \qquad \square$$

Another consequence of Lemma 5.3 is the following.

Proposition 5.5 *There are uncountably many $[\rho] \in \mathcal{M}_{\pm 1}(\Sigma_{0,4})$ such that all the tetrahedral triangulations of $\Sigma_{0,4}$ are ρ–admissible.*

Proof Suppose ρ is a type-preserving representation of $\pi_1(\Sigma_{0,4})$ with $e(\rho) = 1$, and d is a decoration of ρ. Let $\mathcal{T}$ be a ρ–admissible tetrahedral triangulation of $\Sigma_{0,4}$ and let (λ, ϵ) be the lengths coordinates of (ρ, d) in $\mathcal{T}$. Recall that there is a one-to-one correspondence between the tetrahedral triangulations of $\Sigma_{0,4}$ and the vertices of the dual Farey diagram $\mathcal{F}^*$, which is a countably infinity tree. Therefore, for each tetrahedral triangulation $\mathcal{T}'$, there is up to redundancy a unique path in $\mathcal{F}^*$ connecting $\mathcal{T}$ and $\mathcal{T}'$, which corresponds to a sequence $\{S_i\}_{i=1}^n$ of simultaneous diagonal switches. Let $\mathcal{T}_0 = \mathcal{T}$, and for each $i \in \{1, \dots, n\}$, let $\mathcal{T}_i$ be the tetrahedral triangulation obtained from $\mathcal{T}_{i-1}$ by doing S_i. Suppose $\mathcal{T}_i$ is ρ–admissible for some $i \in \{1, \dots, n\}$, and suppose λ_i is the λ–lengths of (ρ, d) in $\mathcal{T}_i$. Then by Lemma 5.3, $\mathcal{T}_{i+1}$ is ρ–admissible if and only if the Laurent polynomial

$$\frac{\lambda_i(y)^2 - \lambda_i(z)^2}{\lambda_i(x)} \neq 0.$$

Inducting on i shows that $\mathcal{T}' = \mathcal{T}_n$ is ρ–admissible if and only if a certain Laurent polynomial $L_{\mathcal{T}'}(\lambda(x), \lambda(y), \lambda(z)) \neq 0$. The set of zeros $Z_{\mathcal{T}'}$ of $L_{\mathcal{T}'}$ is a Zariski-closed proper subset of $\mathbb{R}^3_{>0}$. In particular, it has Lebesgue measure 0. Since $\mathcal{F}^*$ is a countably infinite tree, there are in total countably many tetrahedral triangulations $\mathcal{T}$ of $\Sigma_{0,4}$, and hence $m(\bigcup_{\mathcal{T}} Z_{\mathcal{T}}) = 0$. Therefore, the set $\mathcal{C} = \mathbb{R}^3_{>0} \setminus \bigcup_{\mathcal{T}} Z_{\mathcal{T}}$ has full measure in $\mathbb{R}^3_{>0}$. In particular, $\mathcal{C}$ contains uncountable many points.

Now each $(a, b, c) \in \mathcal{C}$ with $a + b \neq c$, $a + c \neq b$ and $b + c \neq a$ determines a type-preserving representation ρ as follows. Take a tetrahedral triangulation $\mathcal{T}$ of $\Sigma_{0,4}$, and let E and T respectively be the set of edges and ideal triangles of $\mathcal{T}$. Choose $\epsilon \in \{\pm 1\}^T$ so that $\sum_{t \in T} \epsilon(t) = 2$, and define $\lambda \in \mathbb{R}^E_{>0}$ by $\lambda(e_{12}) = \lambda(e_{34}) = a^{1/2}$, $\lambda(e_{13}) = \lambda(e_{24}) = b^{1/2}$ and $\lambda(e_{14}) = \lambda(e_{23}) = c^{1/2}$. Then $\lambda(x) = a$, $\lambda(y) = b$ and $\lambda(z)) = c$. By Theorem 2.2, Proposition 3.2 and Lemma 5.1, (λ, ϵ) determines a decorated representation (ρ, d) up to conjugation. In particular, by Lemma 5.1, ρ is type-preserving. By (2-1), the relative Euler class $e(\rho) = 1$. Finally, since

$(\lambda(x), \lambda(y), \lambda(z)) \in \mathcal{C}$, the Laurent polynomial $L_{\mathcal{T}'}(\lambda(x), \lambda(y), \lambda(z)) \neq 0$ for all tetrahedral triangulation $\mathcal{T}'$. As a consequence, all the tetrahedral triangulations are ρ–admissible.

By symmetry, there are also uncountably many type-preserving representations ρ with $e(\rho) = -1$ such that all the tetrahedral triangulations of $\Sigma_{0,4}$ are ρ–admissible. □

Proof of Theorem 1.1 Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$ and let $\mathcal{C}$ be the full-measure subset of $\mathbb{R}^3_{>0}$ constructed in the proof of Proposition 5.5. Then each $(a, b, c) \in \mathcal{C}$ satisfying one of the following identities $a > b + c$, $b > a + c$ or $c > a + b$ determines a decorated representation (ρ, d) with $e(\rho) = \pm 1$ such that all the tetrahedral triangulations of $\Sigma_{0,4}$ are ρ–admissible. Since elementary representation have relative Euler class 0 and $e(\rho) = \pm 1$, ρ is nonelementary. For each tetrahedral triangulation $\mathcal{T}'$, let λ' be the λ–lengths of (ρ, d) in $\mathcal{T}'$. Since $\mathcal{T}'$ can by obtained from $\mathcal{T}$ by doing a sequence of simultaneous diagonal switches, by Lemma 5.4, the quantities $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ satisfy one of the inequalities in (5-2). By Lemma 5.2, the traces of the distinguished simple closed curves X, Y and Z in $\mathcal{T}'$ are strictly greater than 2 in the absolute value. Since each simple closed curve γ is distinguished in some tetrahedral triangulation $\mathcal{T}'$, we have $|\mathrm{tr}\,\rho([\gamma])| > 2$. □

5.2 A proof of Theorem 1.2

Suppose $e(\rho) = 0$. Then by (2-1), there are exactly two ideal triangles that have the positive sign and two that have the negative sign. Without loss of generality, we assume that $\epsilon(t_1) = \epsilon(t_2) = -1$ and $\epsilon(t_3) = \epsilon(t_4) = 1$. Note that under this assumption, the edges e_{12} and e_{34} in the pair x are adjacent to ideal triangles that have the same sign, and as will be seen later, the X–curves will play a different role than the Y– and Z–curves do. As a direct consequence of Lemma 3.1, we have:

Lemma 5.6 *Let γ_i be the simple closed going counterclockwise around the puncture v_i. Then up to conjugation, the ρ–image of the peripheral elements $[\gamma_1]$ and $[\gamma_2]$ of $\pi_1(\Sigma_{0,4})$ are*

$$\pm \begin{bmatrix} 1 & \lambda(y) + \lambda(z) - \lambda(x) \\ 0 & 1 \end{bmatrix},$$

and the ρ–image of the other two peripheral elements $[\gamma_3]$ and $[\gamma_4]$ are

$$\pm \begin{bmatrix} 1 & \lambda(x) - \lambda(y) - \lambda(z) \\ 0 & 1 \end{bmatrix}.$$

Lemma 5.7 (1) *The absolute values of the traces of the distinguished simple closed curves X, Y and Z of $\mathcal{T}$ are*

$$
\begin{aligned}
|\mathrm{tr}\,\rho([X])| &= \frac{|\lambda(x)^2+\lambda(y)^2+\lambda(z)^2-2\lambda(x)\lambda(y)-2\lambda(x)\lambda(z)|}{\lambda(y)\lambda(z)},\\
|\mathrm{tr}\,\rho([Y])| &= \frac{\lambda(x)^2+\lambda(y)^2+\lambda(z)^2+2\lambda(y)\lambda(z)-2\lambda(x)\lambda(y)}{\lambda(x)\lambda(z)},\\
|\mathrm{tr}\,\rho([Z])| &= \frac{\lambda(x)^2+\lambda(y)^2+\lambda(z)^2+2\lambda(y)\lambda(z)-2\lambda(x)\lambda(z)}{\lambda(x)\lambda(y)}.
\end{aligned}
\tag{5-3}
$$

(2) *The right-hand sides of the last two equations in* (5-3) *are always strictly greater than* 2, *whereas the right-hand side of the first equation is less than or equal to* 2 *if and only if* $\lambda(x)$, $\lambda(y)$ *and* $\lambda(z)$ *satisfy the inequalities*

$$
\begin{aligned}
\sqrt{\lambda(x)} &\leqslant \sqrt{\lambda(y)}+\sqrt{\lambda(z)},\\
\sqrt{\lambda(y)} &\leqslant \sqrt{\lambda(x)}+\sqrt{\lambda(z)},\\
\sqrt{\lambda(z)} &\leqslant \sqrt{\lambda(x)}+\sqrt{\lambda(y)}.
\end{aligned}
\tag{5-4}
$$

Proof (1) is a direct consequence of Theorem 3.3. For (2), since ρ is type-preserving, by Theorem 2.2, Proposition 3.2 and Lemma 5.6, $\lambda(x)-\lambda(y)-\lambda(z)\neq 0$. Therefore, the right-hand side of the second equation of (5-3) equals

$$\frac{(\lambda(x)-\lambda(y)-\lambda(z))^2}{\lambda(x)\lambda(z)}+2>2,$$

and the right-hand side of the third equation equals

$$\frac{(\lambda(x)-\lambda(y)-\lambda(z))^2}{\lambda(x)\lambda(y)}+2>2.$$

In the case that $\lambda(x)^2+\lambda(y)^2+\lambda(z)^2-2\lambda(x)\lambda(y)-2\lambda(x)\lambda(z)\geqslant 0$, the right-hand side of the first equation of (5-3) equals

$$\frac{(\lambda(x)-\lambda(y)-\lambda(z))^2}{\lambda(y)\lambda(z)}-2>-2.$$

The quantity also equals

$$
\begin{aligned}
&2-(\lambda(x)^{1/2}+\lambda(y)^{1/2}+\lambda(z)^{1/2})(\lambda(x)^{1/2}+\lambda(y)^{1/2}-\lambda(z)^{1/2})\\
&\qquad\times(\lambda(x)^{1/2}+\lambda(z)^{1/2}-\lambda(y)^{1/2})(\lambda(y)^{1/2}+\lambda(z)^{1/2}-\lambda(x)^{1/2})/(\lambda(y)\lambda(z)),
\end{aligned}
$$

which is less than or equal to 2 if and only if the equalities in (5-4) are satisfied. For the case that $\lambda(x)^2+\lambda(y)^2+\lambda(z)^2-2\lambda(x)\lambda(y)-2\lambda(x)\lambda(z)\leqslant 0$, the proof is similar. □

The next lemma shows the rule of the change of the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ under a simultaneous diagonal switch.

Lemma 5.8 *Suppose $\mathcal{T}'$ is a tetrahedral triangulation of $\Sigma_{0,4}$. If $\mathcal{T}'$ is ρ–admissible, then let λ' be the λ–lengths of (ρ,d) in $\mathcal{T}'$, and let $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ be the corresponding quantities.*

(1) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_x, then $\mathcal{T}'$ is ρ–admissible. In this case, $\lambda'(y)=\lambda(y)$, $\lambda'(z)=\lambda(z)$ and*
$$\lambda'(x)=\frac{(\lambda(y)+\lambda(z))^2}{\lambda(x)}.$$

(2) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_y, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(x)\neq\lambda(z)$. In the case that $\mathcal{T}'$ is ρ–admissible, $\lambda'(x)=\lambda(x)$, $\lambda'(z)=\lambda(z)$ and*
$$\lambda'(y)=\frac{(\lambda(z)-\lambda(x))^2}{\lambda(y)}.$$

(3) *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by doing S_z, then $\mathcal{T}'$ is ρ–admissible if and only if $\lambda(x)\neq\lambda(z)$. In the case that $\mathcal{T}$ is ρ–admissible, $\lambda'(x)=\lambda(x)$, $\lambda'(y)=\lambda(y)$ and*
$$\lambda'(z)=\frac{(\lambda(x)-\lambda(y))^2}{\lambda(z)}.$$

Proof This is a consequence of Proposition 2.3, and the proof is similar to that of Lemma 5.3. □

Proof of Theorem 1.2 Let ρ be a type-preserving representation of $\pi_1(\Sigma_{0,4})$ with relative Euler class $e(\rho)=0$, and choose arbitrarily a decoration d of ρ. Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$. If $\mathcal{T}$ is not ρ–admissible, then there is an edge e of $\mathcal{T}$ that is not ρ–admissible, and the element of $\pi_1(\Sigma_{0,4})$ represented by the distinguished simple closed curve in $\mathcal{T}$ disjoint from e is sent by ρ to a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$. If $\mathcal{T}$ is ρ–admissible, then we let (λ,ϵ) be the lengths coordinates of (ρ,d) in $\mathcal{T}$. If the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ satisfy the inequalities in (5-4), then by Lemma 5.7, the element of $\pi_1(\Sigma_{0,4})$ represented by one of the distinguished simple closed curves X, Y and Z is sent by ρ to either an elliptic or a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$. Therefore, to prove the theorem, it suffices to find a tetrahedral triangulation $\mathcal{T}'$ of $\Sigma_{0,4}$ such that either $\mathcal{T}'$ is not ρ–admissible or $\mathcal{T}'$ is ρ–admissible with the quantities $\lambda'(x)$, $\lambda'(y)$ and $\lambda'(z)$ satisfying the inequalities in (5-4). Our strategy of finding $\mathcal{T}'$ is to construct a sequence of tetrahedral triangulations $\{\mathcal{T}_n\}_{n=1}^N$ with $\mathcal{T}_N=\mathcal{T}'$ by the following algorithm:

Trace reduction algorithm Let $\mathcal{T}_0 = \mathcal{T}$ and suppose that $\mathcal{T}_n$ is obtained. If $\mathcal{T}_n$ is not ρ–admissible, then we stop. If $\mathcal{T}_n$ is ρ–admissible, then we let (λ_n, ϵ_n) be the lengths coordinates of (ρ, d) in $\mathcal{T}_n$. If $\lambda_n(x)$, $\lambda_n(y)$ and $\lambda_n(z)$ satisfy the inequalities in (5-4), then we stop. If otherwise, then there is a unique maximum among $\lambda_n(x)$, $\lambda_n(y)$ and $\lambda_n(z)$, since other wise the inequalities (5-4) are satisfied. Suppose $\{e_{ij}, e_{kl}\}$ is the pair of opposite edges of $\mathcal{T}_n$ such that $\lambda(e_{ij})\lambda(e_{kl})$ equals the maximum of $\lambda_n(x)$, $\lambda_n(y)$ and $\lambda_n(z)$. Then we let $\mathcal{T}_{n+1}$ be the tetrahedral triangulation obtained from $\mathcal{T}_n$ by doing a simultaneous diagonal switch at e_{ij} and e_{kl}.

By Lemma 5.9 below, the algorithm stops at some $\mathcal{T}_N$. □

Lemma 5.9 *The trace reduction algorithm stops in finitely many steps.*

Proof For each n, let t_i be the ideal triangle of $\mathcal{T}_n$ disjoint from the puncture v_i, and let e_{ij} be the edge of $\mathcal{T}_n$ connecting the punctures v_i and v_j. Without loss of generality, we assume in $\mathcal{T}$ that $\epsilon(t_1) = \epsilon(t_2) = -1$ and $\epsilon(t_3) = \epsilon(t_4) = 1$. Then by Proposition 2.3, $\epsilon_n(t_1) = \epsilon_n(t_2)$ and $\epsilon_n(t_3) = \epsilon_n(t_4)$ for each $\mathcal{T}_n$. For each n, we let

$$a_n = \frac{\lambda_n(x)}{\lambda_n(x) + \lambda_n(y) + \lambda(z)},$$
$$b_n = \frac{\lambda_n(y)}{\lambda_n(x) + \lambda_n(y) + \lambda_n(z)},$$
$$c_n = \frac{\lambda_n(z)}{\lambda_n(x) + \lambda_n(y) + \lambda_n(z)},$$

and let

$$k_n = \max\{\sqrt{a_n} - \sqrt{b_n} - \sqrt{c_n},\ \sqrt{b_n} - \sqrt{a_n} - \sqrt{c_n},\ \sqrt{c_n} - \sqrt{a_n} - \sqrt{b_n}\}.$$

Then $\lambda_n(x)$, $\lambda_n(y)$ and $\lambda_n(z)$ satisfy the inequalities in (5-4) if and only if $k_n \leqslant 0$.

Assume that the sequence $\{\mathcal{T}_n\}$ is infinite, ie $k_n > 0$ for all $n > 0$. Then we will find a contradiction by the following three steps. In Step I we show that k_n is decreasing in n by considering two mutually complementary cases, where in one of them (Case 1) the gap $k_n - k_{n+1}$ is bounded below by the minimum of a_n, b_n and c_n. In Step II we show that there must be a infinite subsequence $\{\mathcal{T}_{n_i}\}$ of $\{\mathcal{T}_n\}$ such that each $\mathcal{T}_{n_i}$ is of Case 1 of Step I, and in Step III we show that for i large enough, $\min\{a_{n_i}, b_{n_i}, c_{n_i}\}$ is increasing. The three steps together imply that $k_n < 0$ for some n large enough, which is a contradiction.

Step I We show that k_n is decreasing in n. There are the following two cases to verify.

Case 1 $\sqrt{a_n}-\sqrt{b_n}-\sqrt{c_n}>0$. In this case, by Lemma 5.8,

$$(a_{n+1},b_{n+1},c_{n+1})=\left(b_n+c_n,\frac{a_nb_n}{b_n+c_n},\frac{a_nc_n}{b_n+c_n}\right). \tag{5-5}$$

Without loss of generality, we assume that $b_n>c_n$. Then b_{n+1} is the largest among a_{n+1}, b_{n+1} and c_{n+1}. By a direct calculation and that $\sqrt{a_n}>\sqrt{b_n}+\sqrt{c_n}$, we have

$$k_n-k_{n+1}=(\sqrt{a_n}-\sqrt{b_n}-\sqrt{c_n})-(\sqrt{b_{n+1}}-\sqrt{a_{n+1}}-\sqrt{c_{n+1}})>\frac{2c_n}{\sqrt{b_n+c_n}}>0.$$

Moreover, since $a_n+b_n+c_n=1$ and $a_n>0$, we have $\sqrt{b_n+c_n}<1$, and hence $k_n-k_{n+1}>2c_n$. Therefore, we have

$$k_n-k_{n+1}>2\min\{a_n,b_n,c_n\}. \tag{5-6}$$

Case 2 One of $\sqrt{b_n}-\sqrt{a_n}-\sqrt{c_n}$ and $\sqrt{c_n}-\sqrt{a_n}-\sqrt{b_n}$ is strictly greater than 0. In this case, we without loss of generality assume that $\sqrt{b_n}-\sqrt{a_n}-\sqrt{c_n}>0$. Then by Lemma 5.8,

$$(a_{n+1},b_{n+1},c_{n+1})=\left(\frac{a_nb_n}{a_nb_n+b_nc_n+(a_n-c_n)^2},\frac{(a_n-c_n)^2}{a_nb_n+b_nc_n+(a_n-c_n)^2},\frac{b_nc_n}{a_nb_n+b_nc_n+(a_n-c_n)^2}\right). \tag{5-7}$$

Without loss of generality, we assume that $a_n>c_n$. Then a_{n+1} is the largest among a_{n+1}, b_{n+1} and c_{n+1}. By a direct calculation and that $b_n=1-a_n-c_n$, we have

$$\frac{k_{n+1}}{k_n}=\frac{\sqrt{a_{n+1}}-\sqrt{b_{n+1}}-\sqrt{c_{n+1}}}{\sqrt{b_n}-\sqrt{a_n}-\sqrt{c_n}}=\sqrt{\frac{a_n+c_n-2\sqrt{a_nc_n}}{a_n+c_n-4a_nc_n}}.$$

From $\sqrt{b_n}>\sqrt{a_n}+\sqrt{c_n}$ and $a_n+b_n+c_n=1$, we have $a_n<\frac{1}{2}$, $c_n<\frac{1}{2}$, and hence $2\sqrt{a_nc_n}>4a_nc_n$. As a consequence, $k_{n+1}/k_n<1$.

Step II We show that there is an infinite subsequence $\{\mathcal{T}_{n_i}\}$ of $\{\mathcal{T}_n\}$ such that $(a_{n_i},b_{n_i},c_{n_i})$ is in Case 1 of Step I. We use contradiction. For each (a_n,b_n,c_n) in Case 2 of Step I, let $A_n=\max\{\lambda_n(y),\lambda_n(z)\}$ and let $B_n=\min\{\lambda_n(y),\lambda_n(z)\}$. Then $\sqrt{A_n}>\sqrt{B_n}+\sqrt{\lambda_n(x)}$. By Lemma 5.8, $(a_{n+1},b_{n+1},c_{n+1})$ is in Case 1 of Step I if and only if $\lambda_n(x)>B_n$. Now suppose that there is an $m\in\mathbb{N}$ such that (a_m,b_m,c_m) is in Case 2 of Step I and $B_n>\lambda_n(x)$ for all $n\geqslant m$. Then by Lemma 5.8, we have $\lambda_{n+1}(x) = \lambda_n(x)$ and

$$\sqrt{B_{n+1}} = \frac{B_n - \lambda_n(x)}{\sqrt{A_n}} < \frac{B_n - \lambda_n(x)}{\sqrt{B_n} + \sqrt{\lambda_n(x)}} = \sqrt{B_n} - \sqrt{\lambda_n(x)}$$

for $n \geqslant m$. By induction, $\lambda_n(x) = \lambda_m(x)$ and $\sqrt{B_n} < \sqrt{B_m} - (n-m)\sqrt{\lambda_m(x)}$ for all $n > m$, which is impossible.

Step III We show that for i large enough, $\min\{a_{n_i}, b_{n_i}, c_{n_i}\}$ is increasing. In Figure 8 below, we let $\Delta = \{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b+c = 1\}$, and for each k let C_k be the intersection of Δ with the set

$$\{(a,b,c) \in \mathbb{R}^3_{>0} \mid \max\{\sqrt{a} - \sqrt{b} - \sqrt{c},\ \sqrt{b} - \sqrt{a} - \sqrt{c},\ \sqrt{c} - \sqrt{a} - \sqrt{b}\} = k\}.$$

A direct calculation shows that the C_k are parts of the concentric circles centered at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with radii increasing in k, and that C_0 is the inscribed circle of Δ. In Figure 8(a), let Q be the intersection of Δ and the set $\{(a,b,c) \in \mathbb{R}^3_{>0} \mid (b+c)^2 = ac\}$. Then Q is a quadratic curve in Δ going through the points $(1,0,0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$. Let the line segment P be the intersection of Δ and the plane $\{(a,b,c) \in \mathbb{R}^3 \mid a = c\}$. Then by (5-5), if (a_n, b_n, c_n) is on Q with $b_n > c_n$, then $(a_{n+1}, b_{n+1}, c_{n+1})$ is on P. Denote by H the line segment connecting $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$, and by L the line segment connecting $(1,0,0)$ and $(0,1,0)$. Let D be the region of Δ bounded by Q, H and L, and let E be the region in Δ bounded by P, H and L. In Figure 8(b), let p be the intersection of Q and C_{k_0}, let ϵ be the third coordinate of p, let L_ϵ be the intersection of Δ and the plane $\{(a,b,c) \in \mathbb{R}^3 \mid c = \epsilon\}$, and let F be the region in Δ bounded by C_{k_0}, L_ϵ, H and L. Note that F is a subset of D.

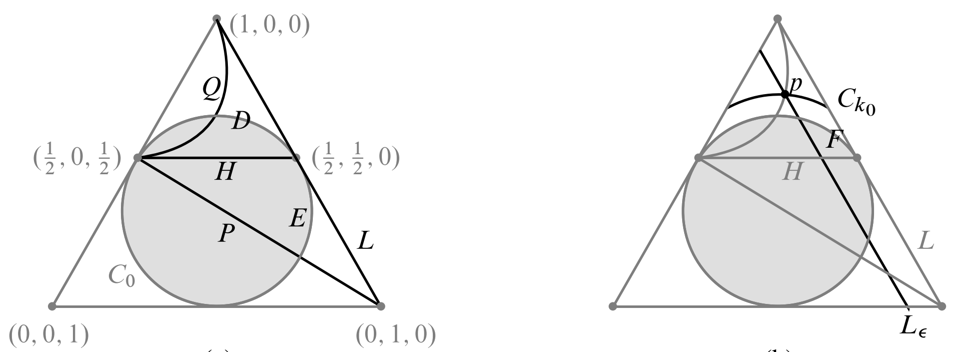


Figure 8

Consider the infinite subsequence $\{\mathcal{T}_{n_i}\}$ guaranteed by Step II such that $(a_{n_i}, b_{n_i}, c_{n_i})$ is in Case 1 of Step I. By (5-6), there exists an i_0 such that $\min\{b_{n_i}, c_{n_i}\} < \epsilon$ for all $i > i_0$, since otherwise $k_{n_i+1} < 0$ for i large enough, and the algorithm stops. Without loss of generality, we assume that $b_{n_{i_0}} > c_{n_{i_0}}$, and we have the following two claims.

Claim 1 If $(a_n,b_n,c_n)\in D$ and $b_n>c_n$, then $(a_{n+1},b_{n+1},c_{n+1})\in E$, $b_{n+1}>c_{n+1}$ and

$$\min\{a_{n+1},b_{n+1},c_{n+1}\}>\min\{a_n,b_n,c_n\}.$$

Indeed, in this case, $c_n=\min\{a_n,b_n,c_n\}$. By (5-5), $(a_{n+1},b_{n+1},c_{n+1})\in E$ and $c_{n+1}>c_n$. Furthermore, we have $b_{n+1}>c_{n+1}$, since otherwise $(a_{n+1},b_{n+1},c_{n+1})$ would be in the disk bounded by the circle C_0, ie $k_{n+1}<0$ and the algorithm stops. Therefore, $c_{n+1}=\min\{a_{n+1},b_{n+1},c_{n+1}\}$ and

$$\min\{a_{n+1},b_{n+1},c_{n+1}\}>\min\{a_n,b_n,c_n\}.$$

Claim 2 For $n>n_0$, if $(a_n,b_n,c_n)\in E$ and $b_n>c_n$, then $(a_{n+1},b_{n+1},c_{n+1})\in D$, $b_{n+1}>c_{n+1}$ and

$$\min\{a_{n+1},b_{n+1},c_{n+1}\}>\min\{a_n,b_n,c_n\}.$$

Indeed, in this case, $c_n=\min\{a_n,b_n,c_n\}$. By (5-7), $(a_{n+1},b_{n+1},c_{n+1})$ is in the triangle above H, $b_{n+1}>c_{n+1}$ and $c_{n+1}>c_n$. Therefore, $c_{n+1}=\min\{a_{n+1},b_{n+1},c_{n+1}\}$ and $\min\{a_{n+1},b_{n+1},c_{n+1}\}>\min\{a_n,b_n,c_n\}$. Furthermore, since $n>n_{i_0}$, we have $c_{n+1}<\epsilon$, and by Step I, we have $k_{n+1}<k_0$. As a consequence, $(a_{n+1},b_{n+1},c_{n+1})\in F\subset D$. Since the intersection $(\frac{2}{3},\frac{1}{6},\frac{1}{6})$ of the quadratic curve Q and the circle C_0 lies on the line determined by $b=c$, F lies on the right half of Δ, and hence $b_{n+1}>c_{n+1}$.

Since $k_{n_{i_0}}<k_0$ and by assumption $b_{n_{i_0}}>c_{n_{i_0}}$ and $c_{n_{i_0}}<\epsilon$, we have $(a_{n_{i_0}},b_{n_{i_0}},c_{n_{i_0}})\in F\subset D$. By induction and Claims 1 and 2, we have for all $m\geqslant 0$ that

$$(a_{n_{i_0}+2m},b_{n_{i_0}+2m},c_{n_{i_0}+2m})\in D$$

with $b_{n_{i_0}+2m}>c_{n_{i_0}+2m}$ and

$$(a_{n_{i_0}+2m+1},b_{n_{i_0}+2m+1},c_{n_{i_0}+2m+1})\in E$$

with $b_{n_{i_0}+2m+1}>c_{n_{i_0}+2m+1}$, and hence for $n>n_{i_0}$ have

$$\min\{a_{n+1},b_{n+1},c_{n+1}\}>\min\{a_n,b_n,c_n\}. \qquad \square$$

Similar to the relative Euler class ± 1 case, we have this:

Proposition 5.10 *There are uncountably many $[\rho]\in\mathcal{M}_0(\Sigma_{0,4})$ such that all the tetrahedral triangulations of $\Sigma_{0,4}$ are ρ–admissible. For each such ρ, there is a simple closed curve γ on $\Sigma_{0,4}$ such that $\rho([\gamma])$ is an elliptic element in* $\mathrm{PSL}(2,\mathbb{R})$.

Proof Since the functions in Lemma 5.8 are rational in $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$, the argument in the proof of Proposition 5.5 applies here and proves the first part. The second part is a result of the trace reduction algorithm. $\square$

6 Connected components of $\mathcal{M}(\Sigma_{0,4})$

We describe the connected component of the character space $\mathcal{M}(\Sigma_{0,4})$ in this section. Recall that for a quadruple s of positive and negative signs, $\mathcal{M}_k^s(\Sigma_{0,4})$ is the space of conjugacy classes of type-preserving representations with relative Euler class k and signs of the punctures s. Let $V=\{v_1,\dots,v_4\}$ be the set of punctures of $\Sigma_{0,4}$. Then Theorem 1.3 is equivalent to the following theorem.

Theorem 6.1 (1) *For $\{i,j,k,l\}=\{1,2,3,4\}$, let $s_{ij}\in\{\pm1\}^V$ be defined by $s_{ij}(v_i)=s_{ij}(v_j)=-1$ and $s_{ij}(v_k)=s_{ij}(v_l)=+1$. Then*

$$\mathcal{M}_0(\Sigma_{0,4})=\coprod_{\{i,j\}\subset\{1,\dots,4\}}\mathcal{M}_0^{s_{ij}}(\Sigma_{0,4}).$$

(2) *For $i\in\{1,\dots,4\}$, let $s_i\in\{\pm1\}^V$ be defined by $s_i(v_i)=-1$ and $s_i(v_j)=+1$ for $j\neq i$, and let $s_+\in\{\pm1\}^V$ be defined by $s_+(v_i)=1$ for all $i\in\{1,\dots,4\}$. Then*

$$\mathcal{M}_1(\Sigma_{0,4})=\coprod_{i=1}^{4}\mathcal{M}_1^{s_i}(\Sigma_{0,4})\sqcup\mathcal{M}_1^{s_+}(\Sigma_{0,4}).$$

(3) *For $i\in\{1,\dots,4\}$, let $s_{-i}\in\{\pm1\}^V$ be defined by $s_{-i}(v_i)=+1$ and $s_{-i}(v_j)=-1$ for $j\neq i$, and let $s_-\in\{\pm1\}^V$ be defined by $s_-(v_i)=-1$ for all $i\in\{1,\dots,4\}$. Then*

$$\mathcal{M}_{-1}(\Sigma_{0,4})=\coprod_{i=1}^{4}\mathcal{M}_{-1}^{s_{-i}}(\Sigma_{0,4})\sqcup\mathcal{M}_{-1}^{s_-}(\Sigma_{0,4}).$$

(4) *$\mathcal{M}_0^{s_{ij}}(\Sigma_{0,4})$, $\mathcal{M}_1^{s_i}(\Sigma_{0,4})$, $\mathcal{M}_1^{s_+}(\Sigma_{0,4})$, $\mathcal{M}_{-1}^{s_{-i}}(\Sigma_{0,4})$ and $\mathcal{M}_{-1}^{s_-}(\Sigma_{0,4})$ are connected.*

Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$. Recall that $\mathcal{M}_\mathcal{T}(\Sigma_{0,4})$ is the space of conjugacy classes of type-preserving representations ρ such that $\mathcal{T}$ is ρ–admissible. By Theorem 2.1, $\mathcal{M}_\mathcal{T}(\Sigma_{0,4})$ is a dense and open subset of $\mathcal{M}(\Sigma_{0,4})$. Let E and T respectively be the sets of edges and ideal triangles of $\mathcal{T}$, let $t_i\in T$ be the ideal triangle disjoint from the puncture v_i and let $e_{ij}\in E$ be the edge connecting the punctures v_i and v_j. For $\lambda\in\mathbb{R}^E_{>0}$, let $\lambda(x)=\lambda(e_{12})\lambda(e_{34})$, $\lambda(y)=\lambda(e_{13})\lambda(e_{24})$ and $\lambda(z)=\lambda(e_{14})\lambda(e_{23})$. We first show that the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ parametrize the components of $\mathcal{M}_\mathcal{T}(\Sigma_{0,4})$.

Lemma 6.2 *Let $\mathbb{R}^E_{>0}$ be with the principal $\mathbb{R}^V_{>0}$–bundle structure given by*

$$(\mu\cdot\lambda)(e_{ij})=\mu(v_i)\lambda(e_{ij})\mu(v_j),$$

and let $\mathbb{R}^3_{>0}$ be with the principal $\mathbb{R}_{>0}$–bundle structure defined by

$$r \cdot (a,b,c) = (ra, rb, rc).$$

Then the map $\phi\colon \mathbb{R}^E_{>0} \to \mathbb{R}^3_{>0}$ given by $(\lambda(e_{12}),\dots,\lambda(e_{34})) \mapsto (\lambda(x),\lambda(y),\lambda(z))$ induces a diffeomorphism $\phi^*\colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} \to \mathbb{R}^3_{>0}/\mathbb{R}_{>0}$.

Proof Since $\phi(\mu\cdot\lambda) = \prod_{i=1}^4 \mu(v_i)\cdot\phi(\lambda)$, ϕ^* is well defined, and since

$$\phi(a^{1/2}, b^{1/2}, c^{1/2}, c^{1/2}, b^{1/2}, a^{1/2}) = (a,b,c)$$

for all $(a,b,c) \in \mathbb{R}^3_{>0}$, ϕ^* is surjective. For the injectivity, we suppose that $\phi(\lambda') = r\cdot\phi(\lambda)$. Let

$$\nu_i(\lambda) = \prod_{j\neq i} \lambda(e_{ij})^2 \prod_{j,k\neq i} \lambda(e_{jk})$$

and let $\mu(v_i) = r^{1/2}\nu_i(\lambda')^{1/6}/\nu_i(\lambda)^{1/6}$. Then

$$\lambda'(e_{ij}) = \mu(v_i)\lambda(e_{ij})\mu(v_j).$$

Therefore, ϕ^* is injective. The differentiability of ϕ^* and $(\phi^*)^{-1}$ follows from the definition of ϕ. □

As a consequence of Theorem 2.2, and Lemmas 5.1, 5.6 and 6.2, we have:

Corollary 6.3 *Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$ with the set of ideal triangles T. Then*

$$\mathcal{M}_{\mathcal{T}}(\Sigma_{0,4}) \cong \coprod_{\epsilon\in\{\pm1\}^T} \Delta(\mathcal{T},\epsilon),$$

where each $\Delta(\mathcal{T},\epsilon)$ is a subset of $\Delta = \{(a,b,c)\in\mathbb{R}^3_{>0} \mid a+b+c=1\}$ defined as follows.

(1) *For $i\in\{1,\dots,4\}$, let $\epsilon_i \in \{\pm1\}^T$ be given by $\epsilon_i(t_i) = -1$ and $\epsilon_i(t_j) = 1$ for $j\neq i$, and let $\epsilon_{-i}\in\{\pm1\}^T$ be given by $\epsilon_{-i}(t_i) = 1$ and $\epsilon_{-i}(t_j) = -1$ for $j \neq i$. Then*

$$\Delta(\mathcal{T},\epsilon_i) = \Delta(\mathcal{T},\epsilon_{-i}) = \{(a,b,c)\in\Delta \mid a\neq b+c,\ b\neq a+c \text{ and } c\neq a+b\}.$$

(2) *For $\{i,j,k,l\} = \{1,\dots,4\}$, let $\epsilon_{ij}\in\{\pm1\}^T$ be given by $\epsilon_{ij}(t_i) = \epsilon_{ij}(t_j) = -1$ and $\epsilon_{ij}(t_k) = \epsilon_{ij}(t_l) = 1$. Then*

$$\Delta(\mathcal{T},\epsilon_{12}) = \Delta(\mathcal{T},\epsilon_{34}) = \{(a,b,c)\in\Delta \mid a\neq b+c\},$$
$$\Delta(\mathcal{T},\epsilon_{13}) = \Delta(\mathcal{T},\epsilon_{24}) = \{(a,b,c)\in\Delta \mid b\neq a+c\},$$
$$\Delta(\mathcal{T},\epsilon_{14}) = \Delta(\mathcal{T},\epsilon_{23}) = \{(a,b,c)\in\Delta \mid c\neq a+b\}.$$

Proof of Theorem 6.1 Suppose s is a quadruple of positive and negative signs and $k \in \{1, 0, -1\}$. Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$. Since $\mathcal{M}_{\mathcal{T}}(\Sigma_{0,4})$ is dense and open in $\mathcal{M}(\Sigma_{0,4})$, $\mathcal{M}_k^s(\Sigma_{0,4}) \neq \varnothing$ if and only if $\mathcal{M}_k^s(\Sigma_{0,4}) \cap \mathcal{M}_{\mathcal{T}}(\Sigma_{0,4}) \neq \varnothing$. For (1), by Lemma 5.6, the only possibility for $\mathcal{M}_{\mathcal{T}}(\Sigma_{0,4}) \cap \mathcal{M}_0^s(\Sigma_{0,4}) \neq \varnothing$ is that $s = s_{ij}$ for some $\{i, j\} \subset \{1, \dots, 4\}$. For (2), by Lemma 5.1, the only possibility for $\mathcal{M}_{\mathcal{T}}(\Sigma_{0,4}) \cap \mathcal{M}_1^s(\Sigma_{0,4}) \neq \varnothing$ is that either $s = s_+$, in which case $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ satisfy the triangular inequality, or $s = s_i$ for some $i \in \{1, \dots, 4\}$, in which case $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$ satisfy one of the inequalities in (5-2). The proof of (3) is parallel to that of (2).

For (4), by symmetry, it suffices to prove the connectedness of $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$, $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$ and $\mathcal{M}_1^{s_+}(\Sigma_{0,4})$. We consider the following subsets of Δ. Let $\Delta_x(\mathcal{T}, \epsilon)$ (resp. $\Delta_y(\mathcal{T}, \epsilon)$ and $\Delta_z(\mathcal{T}, \epsilon)$) be the set of points $(a, b, c) \in \Delta$ such that $a > b + c$ (resp. $b > a + c$ and $c > a + b$), let $\Delta_x^c(\mathcal{T}, \epsilon)$ (resp. $\Delta_y^c(\mathcal{T}, \epsilon)$ and $\Delta_z^c(\mathcal{T}, \epsilon)$) be the set of points $(a, b, c) \in \Delta$ such that $a < b + c$ (resp. $b < a + c$ and $c < a + b$) and let $\Delta^c(\mathcal{T}, \epsilon) = \Delta_x^c(\mathcal{T}, \epsilon) \cap \Delta_y^c(\mathcal{T}, \epsilon) \cap \Delta_z^c(\mathcal{T}, \epsilon)$. See Figure 9. By Theorem 2.2, Lemma 5.1, Lemma 5.6 and Corollary 6.3, we have via the lengths coordinates that:

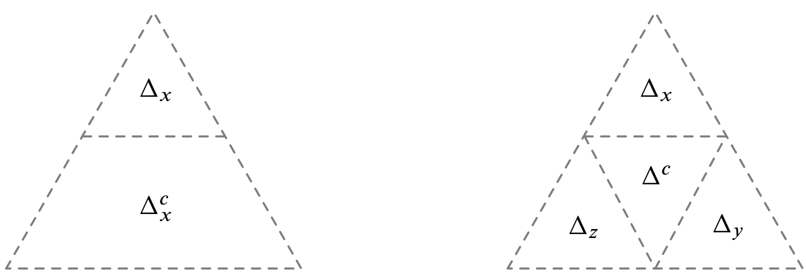


Figure 9

(1) $\Delta_x(\mathcal{T}, \epsilon_{12}) \sqcup \Delta_x^c(\mathcal{T}, \epsilon_{34})$ is diffeomorphic to a dense open subset of $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$.

(2) $\Delta_x(\mathcal{T}, \epsilon_2) \sqcup \Delta_y(\mathcal{T}, \epsilon_3) \sqcup \Delta_z(\mathcal{T}, \epsilon_4)$ is diffeomorphic to a dense open subset of $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$.

(3) $\coprod_{i=1}^4 \Delta^c(\mathcal{T}, \epsilon_i)$ is diffeomorphic to a dense open subset of $\mathcal{M}_1^{s_+}(\Sigma_{0,4})$.

In the rest of the proof, we let $\mathcal{T}'$ be the tetrahedral triangulation of $\Sigma_{0,4}$ obtained from $\mathcal{T}$ by doing a simultaneous diagonal switch S_z, let T' be the set of ideal triangles of $\mathcal{T}'$, and let ϵ_i' and $\epsilon_{ij}' \in \{\pm 1\}^{T'}$ be sign assignments defined in the same way as ϵ_i and ϵ_{ij}. For $(a, b, c) \in \mathbb{R}_{>0}^3$, we let

$$[a, b, c] \doteq \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right) \in \Delta.$$

For the connectedness of $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$, since both $\Delta_x(\mathcal{T},\epsilon_{12})$ and $\Delta_x^c(\mathcal{T},\epsilon_{34})$ are connected, it suffices to choose two points p and q respectively in $\Delta_x(\mathcal{T},\epsilon_{12})$ and $\Delta_x^c(\mathcal{T},\epsilon_{34})$ and find a path in $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$ connecting p and q. Now let $p=(a,b,c)\in\Delta_x(\mathcal{T},\epsilon_{12})$ and let $q=(a',b',c')\in\Delta_x^c(\mathcal{T},\epsilon_{34})$ with $a'>b'$. By Proposition 2.3 and Lemma 5.8, p corresponds to the point $p'=[a,b,(a-b)^2/c]\in\Delta_x(\mathcal{T}',\epsilon'_{12})$ and q corresponds to the point $q'=[a',b',(a'-b')^2/c']\in\Delta_x(\mathcal{T}',\epsilon'_{12})$. Since $\Delta_x(\mathcal{T}',\epsilon'_{12})$ is connected, there is a path in $\Delta_x(\mathcal{T}',\epsilon'_{12})$ connecting p' and q', giving a path in $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$ connecting p and q.

For the connectedness of $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$, we let $p=(a,b,c)\in\Delta_x(\mathcal{T},\epsilon_2)$ and let $q=(a',b',c')\in\Delta_y(\mathcal{T},\epsilon_3)$. By Proposition 2.3, Lemma 5.3 and Lemma 5.4, p corresponds to the point $p'=[a,b,|a^2-b^2|/c]\in\Delta_z(\mathcal{T}',\epsilon'_4)$ and q corresponds to the point $q'=[a',b',|a'^2-b'^2|/c']\in\Delta_z(\mathcal{T}',\epsilon'_4)$. Since $\Delta_z(\mathcal{T}',\epsilon'_4)$ is connected, there is a path in $\Delta_z(\mathcal{T}',\epsilon'_4)$ connecting p' and q', giving a path in $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$ connecting p and q. Similarly, any pair of points $q\in\Delta_y(\mathcal{T},\epsilon_3)$ and $r\in\Delta_z(\mathcal{T},\epsilon_4)$ and any pair of points $p\in\Delta_y(\mathcal{T},\epsilon_2)$ and $r\in\Delta_z(\mathcal{T},\epsilon_4)$ can respectively be connected by paths in $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$. Therefore, $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$ is connected.

Finally, for the connectedness of $\mathcal{M}_1^{s+}(\Sigma_{0,4})$, we let $p=(a,b,c)\in\Delta^c(\mathcal{T},\epsilon_2)$ with $a>b$ and let $q=(a',b',c')\in\Delta^c(\mathcal{T},\epsilon_3)$ with $b'>a'$. By Proposition 2.3, Lemma 5.3 and Lemma 5.4, p corresponds to the point $p'=[a,b,|a^2-b^2|/c]\in\Delta^c(\mathcal{T}',\epsilon'_4)$ and q corresponds to the point $q'=[a',b',|a'^2-b'^2|/c']\in\Delta^c(\mathcal{T}',\epsilon'_4)$. Since $\Delta^c(\mathcal{T}',\epsilon'_4)$ is connected, there is a path in $\Delta^c(\mathcal{T}',\epsilon'_4)$ connecting p' and q', giving a path in $\mathcal{M}_1^{s+}(\Sigma_{0,4})$ connecting p and q. Similarly, all the other pieces can be connected by paths in $\mathcal{M}_1^{s+}(\Sigma_{0,4})$, and $\mathcal{M}_1^{s+}(\Sigma_{0,4})$ is connected. □

7 Ergodicity of the Mod($\Sigma_{0,4}$)–action

The goal of this section is to prove the ergodicity of the Mod($\Sigma_{0,4}$)–action on the nonextremal connected components of $\mathcal{M}(\Sigma_{0,4})$. To use the techniques we used in the previous sections, we need to understand the measure on $\mathcal{M}(\Sigma_{0,4})$ induced by the Goldman symplectic 2–form in terms of the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$. Let $\mathcal{T}$ be a tetrahedral triangulation of $\Sigma_{0,4}$, and let T be the set of ideal triangles of $\mathcal{T}$. For each $\epsilon\in\{\pm1\}^T$, let $\Delta(\mathcal{T},\epsilon)$ be the subset of $\mathbb{R}^3_{>0}$ defined in Corollary 6.3. Then by Equation (2-2), Lemma 6.2, Corollary 6.3 and a direct calculation, we have the following proposition.

Proposition 7.1 *For each $\epsilon\in\{\pm1\}^T$, the 2–form*

$$\omega=\frac{d\lambda(x)\wedge d\lambda(y)}{\lambda(x)\lambda(y)}+\frac{d\lambda(y)\wedge d\lambda(z)}{\lambda(y)\lambda(z)}+\frac{d\lambda(z)\wedge d\lambda(x)}{\lambda(z)\lambda(x)}$$

on $\Delta(\mathcal{T},\epsilon)$ corresponds to the Goldman symplectic 2–form ω_{WP} on $\mathcal{M}(\Sigma_{0,4})$, and the measure induced by ω is in the same measure class of the Lebesgue measure on $\Delta(\mathcal{T},\epsilon)$.

As a consequence, we have:

Proposition 7.2 *For $k\in\{-1,0,1\}$, the set $\Omega_k(\Sigma_{0,4})$ consisting of conjugacy classes of type-preserving representations ρ with the relative Euler class $e(\rho)=k$ such that all the tetrahedra triangulation of $\Sigma_{0,4}$ are ρ–admissible is a full measure subset of $\mathcal{M}_k(\Sigma_{0,4})$, and is invariant under the* $\mathrm{Mod}(\Sigma_{0,4})$*–action.*

Proof By the proof of Proposition 5.5, $\mathcal{M}_1(\Sigma_{0,4})\setminus\Omega_1(\Sigma_{0,4})$ is a countable union of Lebesgue measure zero subsets, hence is of Lebesgue measure zero. Then by Proposition 7.1, $\mathcal{M}_1(\Sigma_{0,4})\setminus\Omega_1(\Sigma_{0,4})$ is a null set in the measure induced by the Goldman symplectic 2–form. By the similar argument, $\Omega_0(\Sigma_{0,4})$ and $\Omega_{-1}(\Sigma_{0,4})$ are respectively full measure subsets of $\mathcal{M}_0(\Sigma_{0,4})$ and $\mathcal{M}_{-1}(\Sigma_{0,4})$. By Proposition 4.1, since all the simultaneous diagonal switches act on each $\Omega_k(\Sigma_{0,4})$, so does $\mathrm{Mod}(\Sigma_{0,4})$. $\square$

Remark 7.3 Since $\Omega_1(\Sigma_{0,4})$ is dense in $\mathcal{M}_1(\Sigma_{0,4})$ and since any representation in $\Omega_1(\Sigma_{0,4})\cap\coprod_{i=1}^4\mathcal{M}_1^{s_i}(\Sigma_{0,4})$ sends each simple closed curve to a hyperbolic element, by continuity, every representation in $\coprod_{i=1}^4\mathcal{M}_1^{s_i}(\Sigma_{0,4})$ sends each simple closed curve to either a hyperbolic or a parabolic element. In [5], Delgado explicitly constructed a family $\{\rho_t\}$ of representations in $\mathcal{M}_1(\Sigma_{0,4})$ that send every simple closed curve to either a hyperbolic or a parabolic element, and for each ρ_t, at least one simple closed curve is sent to a parabolic element. Therefore, the representations $\{\rho_t\}$ are in the measure-zero subset $\mathcal{M}_1(\Sigma_{0,4})\setminus\Omega_1(\Sigma_{0,4})$.

Let $V=\{v_1,\dots,v_4\}$ be the set of punctures of $\Sigma_{0,4}$. For $k\in\{-1,0,1\}$ and $s\in\{\pm1\}^V$, let $\Omega_k^s(\Sigma_{0,4})=\Omega_k(\Sigma_{0,4})\cap\mathcal{M}_k^s(\Sigma_{0,4})$. By Theorem 6.1 and Proposition 7.2, Theorem 1.4 follows from the following theorem.

Theorem 7.4 (1) *The* $\mathrm{Mod}(\Sigma_{0,4})$*–action on $\Omega_0^{s_{ij}}(\Sigma_{0,4})$ is ergodic for each choice of $\{i,j\}\subset\{1,\dots,4\}$.*

(2) *The* $\mathrm{Mod}(\Sigma_{0,4})$*–action on $\Omega_1^{s_+}(\Sigma_{0,4})$ and $\Omega_{-1}^{s_-}(\Sigma_{0,4})$ is ergodic.*

(3) *The* $\mathrm{Mod}(\Sigma_{0,4})$*–action on $\Omega_1^{s_i}(\Sigma_{0,4})$ and $\Omega_{-1}^{s_{-i}}(\Sigma_{0,4})$ is ergodic for each $i\in\{1,\dots,4\}$.*

Remark 7.5 Maloni, Palesi and Tan [20] were the first to learn of the ergodicity of the $\mathrm{Mod}(\Sigma_{0,4})$–action on the components of $\mathcal{M}_{\pm1}(\Sigma_{0,4})$ using the Markoff triple technique.

7.1 A proof of Theorem 7.4(1)

By symmetry, it suffices to prove that the $\mathrm{Mod}(\Sigma_{0,4})$–action on $\Omega_0^{s_{12}}(\Sigma_{0,4})$ is ergodic. Let

$$\Delta = \{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b+c=1\} \quad \text{and} \quad \Delta_x = \{(a,b,c) \in \Delta \mid a \neq b+c\}.$$

By Theorem 2.2, Lemma 6.2 and Corollary 6.3, given a tetrahedral triangulation of $\Sigma_{0,4}$, Δ_x is diffeomorphic to a dense and open subset of $\mathcal{M}_0^{s_{12}}(\Sigma_{0,4})$, where the diffeomorphism is given by the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$.

Consider the embedding of $i\colon \Delta \to \mathbb{R}^3_{>0}$ defined by $i((a,b,c)) = (1, b/a, c/a)$. Then

$$i(\Delta_x) = \{(1,b,c) \in \mathbb{R}^3_{>0} \mid b+c \neq 1\}.$$

Let Ω_X be the subset of $i(\Delta_x)$ consisting of the elements coming from $\Omega_0^{s_{12}}(\Sigma_{0,4})$. As an immediate consequences of Lemma 5.8, the simultaneous diagonal switches S_y and S_z act on Ω_X by

$$S_y((1,b,c)) = \left(1, \frac{(1-c)^2}{b}, c\right) \quad \text{and} \quad S_z((1,b,c)) = \left(1, b, \frac{(1-b)^2}{c}\right).$$

Lemma 7.6 *Let $\langle D_X\rangle$ be the cyclic subgroup of* $\mathrm{Mod}(\Sigma_{0,4})$ *generated by the Dehn twist D_X along the distinguished simple closed curve X. Then for every $k \in (-2,2)$, the ellipse*

$$E_k = \{(1,b,c) \in \Omega_X \mid (b+c-1)^2 = (k+2)bc\}$$

is invariant under the action of $\langle D_X\rangle$, and for almost every $k \in (-2,2)$, the action of $\langle D_X\rangle$ on E_k is ergodic.

Proof A direct calculation shows that E_k is invariant under the actions of S_y and S_z for all $k \in (-2,2)$. Recall that $D_X = S_z S_y$. Therefore, the ellipse E_k is invariant under the $\langle D_X\rangle$–action. By Lemma 5.8, the action of S_y and S_z respectively move a point p on E_k vertically and horizontally. As show in Figure 10, the an affine transformation of $\mathbb{R}^2$ sending the ellipse E_k to a circle C_k sends the vertical and the horizontal lines in $\mathbb{R}^2$ respectively to two family of parallel lines in C_k. As a consequence, for each point p on C_k, the angle $\angle pp'D_X(p)$ is a constant $\theta_k/2$ depending only on k, and the center angle $\angle pOD_X(p) = 2\angle pp'D_X(p) = \theta_k$. Therefore, D_X acts on C_k by a rotation of angle θ_k. Since θ_k is an irrational multiple of 2π for almost every $k \in (-2,2)$, the action of $\langle D_X\rangle$ is ergodic. $\square$

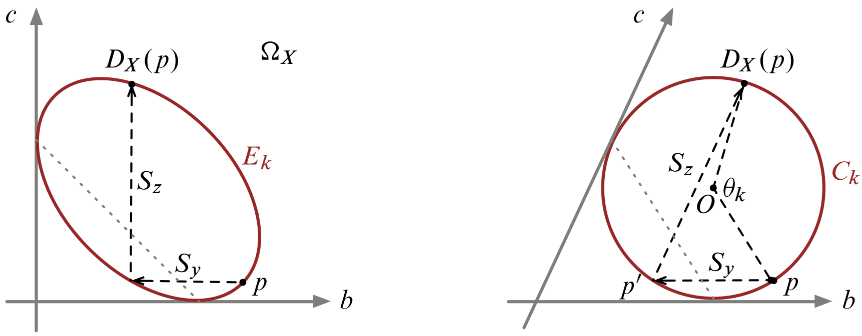


Figure 10

Lemma 7.7 *Let $D_Y D_Z$ be the self-diffeomorphism of $\Sigma_{0,4}$ given by the Dehn twist D_Z along the distinguished curve Z in $\mathcal{T}$ followed by the Dehn twist D_Y along the distinguished curve Y in $D_Z(\mathcal{T})$, and let $\langle D_Y D_Z\rangle$ be the cyclic subgroup of* $\mathrm{Mod}(\Sigma_{0,4})$ *generated by $D_Y D_Z$. Then for every $k \in (-2,2)$, the quartic curve*

$$Q_k = \{(1,b,c) \in \Omega_X \mid (b+c)^2(b+c-1)^2 = (k+2)bc\}$$

is invariant under the action of $\langle D_Y D_Z\rangle$, and for almost every $k \in (-2,2)$, the action of $\langle D_Y D_Z\rangle$ on Q_k is ergodic.

Proof A direct calculation shows that Q_k is the S_x–image of E_k. Since $D_Y D_Z = S_x S_z S_y S_x = S_x D_X S_x$, the map $S_z\colon E_k \to Q_k$ is $\mathbb{Z}$–equivariant, where $1 \in \mathbb{Z}$ acts on E_k by D_X and acts on Q_k by $D_Y D_Z$. By Lemma 7.6, $\langle D_Y D_Z\rangle$ acts on Q_k for every $k \in (-2,2)$, and the action is ergodic for almost every $k \in (-2,2)$. $\square$

Proof of Theorem 7.4(1) We show that every $\mathrm{Mod}(\Sigma_{0,4})$–invariant measurable function $F\colon \Omega_X \to \mathbb{R}$ is almost everywhere a constant. Consider the following region

$$R = \{(1,b,c) \in \Omega_X \mid (b+c-1)^2 < 4bc,\ b+c < 1\}$$

in Ω_X inclosed by the parabola $P = \{(1,b,c) \in \Omega_X \mid (b+c-1)^2 = 4bc\}$ and the line segment $L = \{(1,b,c) \in \Omega_X \mid b+c = 1\}$. We claim that each point $p = (1, b_0, c_0)$ in R is an intersection an ellipse E_{k_1} and a quintic curve Q_{k_2} for some k_1, $k_2 \in (-2,2)$. Indeed, we can let

$$k_1 = \frac{(b_0+c_0-1)^2}{b_0c_0} - 2 \quad \text{and} \quad k_2 = \frac{(b_0+c_0)^2(b_0+c_0-1)^2}{b_0c_0} - 2.$$

Since $(b_0+c_0-1)^2 < 4b_0c_0$, $k_1 \in (-2,2)$, and since $b_0 + c_0 < 1$, $k_2 \in (-2,2)$. A direct calculation shows that the intersection of E_{k_1} and Q_{k_2} is transverse at p, ie the gradients $\nabla E_{k_1}(p)$ and $\nabla Q_{k_2}(p)$ span the tangent space of Ω_X at p. Then by Lemma 7.6 and Lemma 7.7, the restriction of F to R is almost everywhere a constant. For $p \in \Omega_X$, let $O(p)$ be the $\mathrm{Mod}(\Sigma_{0,4})$–orbit of p. To show that the F is almost everywhere a constant in Ω_X, it suffices to show that $O(p) \cap R \neq \varnothing$ for almost every p in Ω_X. Let R' be the region in Ω_X enclosed by parabola P, ie $R' = \{(1,b,c) \in \Omega_X \mid (b+c-1)^2 < 4bc\}$. Then R' is foliated by the ellipses $\{E_k\}$. We note that the parabola P is the i–image of the inscribe circle C_0 of Δ. Then by the trace reduction algorithm, Lemma 5.9 and Proposition 7.2, for almost every p in Ω_X, there is a composition ϕ of finitely many, say m, simultaneous diagonal switches such that $\phi(p) \in R'$. By Proposition 4.1, if m is even, then $\phi \in \mathrm{Mod}(\Sigma_{0,4})$ and $O(p) \cap R' \neq \varnothing$; and if m is odd, then $\phi' = S_y\phi \in \mathrm{Mod}(\Sigma_{0,4})$. Since S_y keeps invariant an ellipse $E_k \subset R'$ passing through $\phi(p)$, $\phi'(p) = S_y\phi(p) \in E_k \subset R'$, and hence $O(p) \cap R' \neq \varnothing$. Finally, by Lemma 7.6, for almost every p in R', there is n such that $D_X^n(p) \in E_k \cap R \subset R$. □

7.2 A proof of Theorem 7.4(2)

By symmetry, it suffices to prove the ergodicity of the $\mathrm{Mod}(\Sigma_{0,4})$–action on Ω_1^{s+}. The strategy is to find two transversely intersecting families of curves $\{E_{X,k}\}$ and $\{E_{Y,k}\}$ foliating $\mathcal{M}_1^{s+}(\Sigma_{0,4})$ such that the $\langle D_X\rangle$ -action on almost every $E_{X,k}$ and the $\langle D_Y\rangle$–action on almost every $E_{Y,k}$ is ergodic. Let $\mathcal{T}$ be an tetrahedral triangulation of $\Sigma_{0,4}$, and let T be the set of ideal triangles of $\mathcal{T}$. Let $\Delta = \{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b+c = 1\}$, and for $\epsilon \in \{\pm 1\}^T$, let $\Delta^c(\mathcal{T},\epsilon) = \{(a,b,c) \in \Delta \mid a < b+c, b < a+c, c < b+a\}$. By Theorem 2.2, Lemma 5.1 and Corollary 6.3, $\coprod_{i=1}^4 \Delta^c(\mathcal{T},\epsilon_i)$ is diffeomorphic to a dense and open subset of $\mathcal{M}_1^{s+}(\Sigma_{0,4})$, where the diffeomorphism is given by the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$.

We define the embedding $i_X\colon \coprod_{i=1}^4 \Delta^c(\mathcal{T},\epsilon_i) \to \mathbb{R}^3$ by

$$i_X((a,b,c)) = \begin{cases} (1, b/a, c/a) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T},\epsilon_1), \\ (1, -b/a, -c/a) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T},\epsilon_2), \\ (1, -b/a, c/a) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T},\epsilon_3), \\ (1, b/a, -c/a) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T},\epsilon_4). \end{cases}$$

For $i \in \{1,\dots,4\}$, we let $\Omega_{X,i}$ be the subset of $i_X(\Delta^c(\mathcal{T},\epsilon_i))$ consisting of the elements coming from $\Omega_1^{s+}(\Sigma_{0,4})$, and let $\Omega_X = \coprod_{i=1}^4 \Omega_{X,i}$. (See Figure 11.)

Lemma 7.8 *For every $k \in (-2,2)$, the ellipse*

$$E_{X,k} = \{(1,b,c) \in \Omega_X \mid b^2 + c^2 - 1 = kbc\}$$

is invariant under the action of $\langle D_X \rangle$, *and for almost every* $k \in (-2, 2)$, *the action of* $\langle D_X \rangle$ *on* $E_{X,k}$ *is ergodic.*

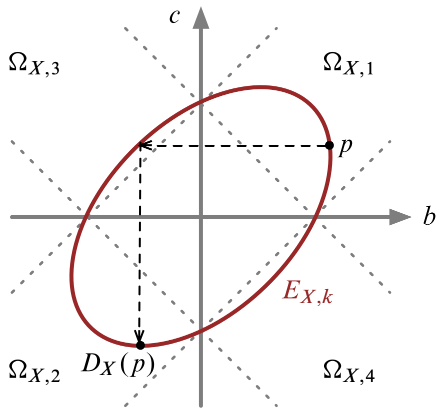


Figure 11

Proof For $(x, y, z) \in \mathbb{R}^3$, let $|(a, b, c)| = (|a|, |b|, |c|)$. By Lemma 5.3, the action of S_y and S_z on Ω_X satisfies

$$|S_y((1, b, c))| = \left(1, \left|\frac{c^2 - 1}{b}\right|, |c|\right) \quad \text{and} \quad |S_z((1, b, c))| = \left(1, |b|, \left|\frac{b^2 - 1}{c}\right|\right).$$

Therefore, we have

$$|D_X((1, b, c))| = \left(1, \left|\frac{c^2 - 1}{b}\right|, \left|\frac{\left(\frac{c^2-1}{b}\right)^2 - 1}{c}\right|\right).$$

We claim that

$$D_X((1, b, c)) = \left(1, \frac{c^2 - 1}{b}, \frac{\left(\frac{c^2-1}{b}\right)^2 - 1}{c}\right). \tag{7-1}$$

If (7-1) is true, then a direct calculation shows that $D_X((1, b, c))$ is on $E_{X,k}$.

To verify (7-1), we let $\mathcal{T}'$ be the tetrahedral triangulation obtained from $\mathcal{T}$ by doing S_y, and let $\mathcal{T}''$ be the tetrahedral triangulation obtained from $\mathcal{T}'$ by doing S_z. Let ϵ' and ϵ'' respectively be the signs of ρ assigned to the ideal triangles of $\mathcal{T}'$ and $\mathcal{T}''$. In $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$, we denote uniformly by t_i the ideal triangle disjoint from the puncture v_i. If $p = (1, b, c) \in \Omega_{X,1}$, ie $i_X^{-1}(p) \in \Delta^c(\mathcal{T}, \epsilon_1)$, then we consider the following cases.

Case 1 $c > 1$ and $(c^2-1)/b > 1$. In this case, since $\epsilon_1(t_1) = -1$ and $c > 1$, we have by Proposition 2.3 that $\epsilon'(t_2) = -1$. Since $(c^2-1)/b > 1$, by Proposition 2.3 again, $\epsilon''(t_1) = -1$. Therefore, $D_X(i_X^{-1}(p)) \in \Delta^c(\mathcal{T}'', \epsilon_1)$, and (7-1) follows.

Case 2 $c > 1$ and $(c^2-1)/b < 1$. In this case, by Proposition 2.3, $\epsilon'(t_2) = -1$ and $\epsilon''(t_4) = -1$. Therefore, $D_X(i_X^{-1}(p)) \in \Delta^c(\mathcal{T}'', \epsilon_4)$, and (7-1) follows.

Case 3 $c < 1$ and $(c^2-1)/b > 1$. In this case, by Proposition 2.3, $\epsilon'(t_4) = -1$ and $\epsilon''(t_3) = -1$. Therefore, $D_X(i_X^{-1}(p)) \in \Delta^c(\mathcal{T}'', \epsilon_3)$, and (7-1) follows.

Case 4 $c < 1$ and $(c^2-1)/b < 1$. In this case, by Proposition 2.3, $\epsilon'(t_4) = -1$ and $\epsilon''(t_2) = -1$. Therefore, $D_X(i_X^{-1}(p)) \in \Delta^c(\mathcal{T}'', \epsilon_2)$, and (7-1) follows.

The verification of (7-1) for p in $\Omega_{X,2}$, $\Omega_{X,3}$ and $\Omega_{X,4}$ is similar, and is left to the readers.

By (7-1), the action of D_X on $E_{X,k}$ is a horizontal translation followed by a vertical translation. See Figure 11. By doing a suitable affine transform, the $\langle D_X\rangle$–action is a rotation of an angle θ_k on a circle, where θ_k is an irrational multiple of 2π for almost every k. Therefore, for almost every $k \in (-2,2)$, the $\langle D_X\rangle$–action on $E_{X,k}$ is ergodic. □

Consider the embedding $i_Y\colon \coprod_{i=1}^4 \Delta^c(\mathcal{T}, \epsilon_i) \to \mathbb{R}^3$ by

$$i_Y((a,b,c)) = \begin{cases} (a/b, 1, c/b) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T}, \epsilon_1), \\ (-a/b, 1, c/b) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T}, \epsilon_2), \\ (-a/b, 1, -c/b) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T}, \epsilon_3), \\ (a/b, 1, -c/b) & \text{if } (a,b,c) \in \Delta^c(\mathcal{T}, \epsilon_4). \end{cases}$$

For $i \in \{1, \dots, 4\}$, we let $\Omega_{Y,i}$ be the subset of $i_Y(\Delta^c(\mathcal{T}, \epsilon_i))$ consisting of the elements coming from $\Omega_1^{s+}(\Sigma_{0,4})$, and let $\Omega_Y = \coprod_{i=1}^4 \Omega_{Y,i}$. Then we have the following lemma whose proof is similar to that of Lemma 7.8.

Lemma 7.9 *For every $k \in (-2,2)$, the ellipse*

$$E_{Y,k} = \{(a,1,c) \in \Omega_Y \mid a^2 + c^2 - 1 = kac\}$$

is invariant under the action of $\langle D_Y\rangle$, and for almost every $k \in (-2,2)$, the action of $\langle D_Y\rangle$ on $E_{Y,k}$ is ergodic.

Proof of Theorem 7.4(2) A direct calculation shows that the two family of curves $\{i_X^{-1}(E_{X,k})\}$ and $\{i_Y^{-1}(E_{Y,k})\}$ transversely intersect. Then by Lemmas 7.8 and 7.9, the $\mathrm{Mod}(\Sigma_{0,4})$–action on $\mathcal{M}_1^{s+}(\Sigma_{0,4})$ is ergodic. □

7.3 A proof of Theorem 7.4(3)

By symmetry, it suffices to prove the ergodicity of the $\mathrm{Mod}(\Sigma_{0,4})$–action on $\Omega_1^{s_1}(\Sigma_{0,4})$. We let $\Delta_x = \{(a,b,c) \in \Delta \mid a > b + c\}$, $\Delta_y = \{(a,b,c) \in \Delta \mid b > a + c\}$ and $\Delta_z = \{(a,b,c) \in \Delta \mid c > a + b\}$. By Theorem 2.2, Lemma 6.2 and Corollary 6.3, given a tetrahedral triangulation of $\Sigma_{0,4}$, $\Delta_x \sqcup \Delta_y \sqcup \Delta_z$ is diffeomorphic to a dense and open subset of $\mathcal{M}_1^{s_1}(\Sigma_{0,4})$, where the diffeomorphism is given by the quantities $\lambda(x)$, $\lambda(y)$ and $\lambda(z)$.

Let

$$R = \{(s,t) \in \mathbb{R}^2 \mid s \neq 0, t \neq 0, s + t \neq 0\},$$

and consider the two-fold covering map $\psi\colon R \to \Delta_x \sqcup \Delta_y \sqcup \Delta_z$ defined by

$$\psi((s,t)) = [\sinh|s|, \sinh|t|, \sinh|s+t|],$$

where

$$[a,b,c] \doteq \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right).$$

Let Ω be the subset of $\Delta_x \sqcup \Delta_y \sqcup \Delta_z$ consisting of the elements coming from $\Omega_1^{s_1}(\Sigma_{0,t})$, and let $\Omega' = \psi^{-1}(\Omega)$. Then by Proposition 7.2, Ω is invariant under the $\mathrm{Mod}(\Sigma_{0,4})$–action. Recall that $\mathrm{Mod}(\Sigma_{0,4})$ is isomorphic to a free group F_2 of rank two generated by the Dehn twists D_X and D_Y. (See [7].) It is well known that F_2 is isomorphic to the quotient group $\Gamma(2)/\pm I$, where

$$\Gamma(2) = \left\{ A \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; A \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod 2 \right\}$$

is the mod-2 congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and the matrices $\left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ correspond to the generators of F_2. This induces a group homomorphism $\pi\colon \Gamma(2) \to \mathrm{Mod}(\Sigma_{0,4})$ defined by

$$\pi\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) = D_X \quad \text{and} \quad \pi\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) = D_Y.$$

By Lemma 5.3 and a direct calculation, $\psi\colon \Omega' \to \Omega$ is π–equivariant, ie $\psi \circ A = \pi(A) \circ \psi$ for all $A \in \Gamma(2)$, where the $\Gamma(2)$–action on Ω' is the standard linear action. By Moore's Ergodicity theorem [22], the $\Gamma(2)$–action on $\mathbb{R}^2$, hence on Ω', is ergodic. Therefore, the $\mathrm{Mod}(\Sigma_{0,4})$–action on Ω is ergodic. □

Appendix A: Equivalence of extremal and Fuchsian representations

Proposition A.1 below is a consequence of Goldman [11, Theorem D] and is stated without proof in [4]. The purpose of this appendix is to include a proof of it for the reader's interest, where the argument is from a discussion with F Palesi and M Wolff.

Proposition A.1 *A type-preserving representation* $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ *is extremal, ie* $|e(\rho)| = 2g-2+n$*, if and only if* ρ *is Fuchsian.*

Proof By Goldman [11, Theorem D], a representation ρ is extremal if and only if ρ is Fuchsian and the quotient $\mathbb{H}^3/\rho(\pi_1(\Sigma_{g,n}))$ is homeomorphic to $\Sigma_{g,n}$. Therefore, to prove the proposition, it suffices to rule out the possibility that ρ is nonmaximum, Fuchsian and $\mathbb{H}^3/\rho(\pi_1(\Sigma_{g,n})) = \Sigma_{g',n'} \ncong \Sigma_{g,,n,}$, which we will do using a contradiction.

Now since $\mathbb{H}^3/\rho(\pi_1(\Sigma_{g,n})) = \Sigma_{g',n'}$, there is an isomorphism

$$\phi\colon \pi_1(\Sigma_{g',n'}) \to \rho(\pi_1(\Sigma_{g,n}));$$

and since ρ is type-preserving, $\phi(\pi_1(\Sigma_{g',n'})) = \rho(\pi_1(\Sigma_{g,n}))$ contains n parabolic elements from the primitive peripheral elements of $\pi_1(\Sigma_{g,n})$. On the other hand, since the only possible parabolic elements of a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{R})$ are from the peripheral elements, the composition

$$\phi^{-1}\circ\rho\colon \pi_1(\Sigma_{g,n}) \to \pi_1(\Sigma_{g',n'})$$

must send the primitive peripheral elements of $\pi_1(\Sigma_{g,n})$ to the primitive peripheral elements of $\pi_1(\Sigma_{g',n'})$. This is impossible when $n > n'$, since $\pi_1(\Sigma_{g',n'})$ has only n' primitive peripheral elements. For the case $n < n'$, we recall the fact that in the first homology $H_1(\Sigma,\mathbb{R})$ of a punctured surface Σ, the full set of vectors represented by the primitive peripheral elements of $\pi_1(\Sigma)$ are linearly dependent, but the vectors in any proper subset of it are linearly independent. Therefore, the induced isomorphism

$$(\phi^{-1}\circ\rho)^*\colon H_1(\Sigma_{g,n};\mathbb{R}) \to H_1(\Sigma_{g',n'};\mathbb{R})$$

sends a set of linearly dependent vectors represented by the primitive peripheral elements of $\pi_1(\Sigma_{g,n})$ to a set of linearly indecent vectors, which is a contradiction. □

Appendix B: Relationship with Goldman's work on the one-punctured torus

In this appendix, we show that the results concerning representations of relative Euler class ± 1 in this paper can also be seen, and more straightforwardly, as consequences of some previous results of Goldman [13, Chapter 4], where the argument presented here is due to the anonymous referee.

In [13], Goldman considers $\mathrm{SL}(2,\mathbb{R})$–representations of the one-puncture torus group $\pi(\Sigma_{1,1})$, which is the free group of two generators $\langle X, Y\rangle$. The character space $\mathcal{M}_{\mathrm{red}}(\Sigma_{1,1})$ of reducible representations $\rho\colon \pi(\Sigma_{1,1}) \to \mathrm{SL}(2,\mathbb{R})$ satisfy $\mathrm{tr}(\rho[X,Y]) = 2$, and hence could be described by the equation

$$x^2+y^2+z^2-xyz-4=0, \tag{B-1}$$

where $x=\mathrm{tr}(\rho(X))$, $y=\mathrm{tr}(\rho(Y))$ and $z=\mathrm{tr}(\rho(XY))$. On the other hand, the fundamental group of the four puncture sphere $\pi_1(\Sigma_{0,4}) \cong \langle A,B,C,D \mid ABCD\rangle$, where the generators are the four primitive peripheral elements corresponding to the four punctures. If $\rho\colon \pi_1(\Sigma_{0,4}) \to \mathrm{PSL}(2,\mathbb{R})$ is type-preserving, then $|\mathrm{tr}(\rho(A))| = |\mathrm{tr}(\rho(B))| = |\mathrm{tr}(\rho(C))| = |\mathrm{tr}(\rho(D))| = 2$; and if $e(\rho)=\pm 1$, then one can lift ρ to a representation $\widetilde{\rho}\colon \pi_1(\Sigma_{0,4}) \to \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{tr}(\widetilde{\rho}(A))\,\mathrm{tr}(\widetilde{\rho}(B))\,\mathrm{tr}(\widetilde{\rho}(C))\,\mathrm{tr}(\widetilde{\rho}(D)) < 0$. Hence the character spaces $\mathcal{M}_{\pm 1}(\Sigma_{0,4})$ can be described by the equation

$$x^2+y^2+z^2+xyz-4=0, \tag{B-2}$$

where $x=\mathrm{tr}(\rho(AB))$, $y=\mathrm{tr}(\rho(BC))$ and $z=\mathrm{tr}(\rho(CA))$. (See [12; 20] for more details.) Comparing (B-1) and (B-2), it is clear that

$$\mathcal{M}_{\mathrm{red}}(\Sigma_{1,1}) \cong \mathcal{M}_{\pm 1}(\Sigma_{0,4}).$$

Moreover, the mapping class group actions are commensurable and the variables x, y, z correspond in each case to the traces of simple closed curves on the surface, hence all the results known for $\mathcal{M}_{\mathrm{red}}(\Sigma_{1,1})$ can be translated to the results on $\mathcal{M}_{\pm 1}(\Sigma_{0,4})$.

To be more precise, by [13, Chapter 4], $\mathcal{M}_{\mathrm{red}}(\Sigma_{1,1})$ has five connected components, one of which is compact corresponding to $\mathcal{M}^{s}_{\pm 1}(\Sigma_{0,4})$ and four of which are noncompact corresponding to $\mathcal{M}^{s_i}_{\pm 1}(\Sigma_{0,4})$. A full measure subset of the characters in the noncompact components have all coordinates x, y, z strictly greater than 2 in absolute value. Each coordinate corresponds to the trace of the image of a simple closed curve. Starting from a representation in one of these components and using the transitivity of the mapping class group action on the set of simple closed curves, one gets that every simple closed curve is sent to an hyperbolic element. Therefore, a full-measure subset of representations in the noncompact components are counterexamples to Bowditch's question. Finally, the ergodicity of the $\mathrm{PSL}(2,\mathbb{Z})$–action on the noncompact components is already proved in [13, Chapter 4], implying the $\mathrm{Mod}(\Sigma_{0,4})$–action on $\mathcal{M}^{s_i}_{\pm 1}(\Sigma_{0,4})$.

References

[1] **F Bonahon**, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. 5 (1996) 233–297 MR1413855

[2] **F Bonahon**, *Low-dimensional geometry: From Euclidean surfaces to hyperbolic knots*, Student Mathematical Library 49, Amer. Math. Soc. (2009) MR2522946

[3] **F Bonahon**, **H Wong**, *Quantum traces for representations of surface groups in* $\mathrm{SL}_2(\mathbb{C})$, Geom. Topol. 15 (2011) 1569–1615 MR2851072

[4] **B H Bowditch**, *Markoff triples and quasi-Fuchsian groups*, Proc. London Math. Soc. 77 (1998) 697–736 MR1643429

[5] **R Delgado**, *Density properties of Euler characteristic-*2 *surface group,* $\mathrm{PSL}(2,\mathbb{R})$ *character varieties*, PhD thesis, University of Maryland (2009) MR2713261 Available at `http://search.proquest.com/docview/304923259`

[6] **B Deroin**, **N Tholozan**, *Dominating surface group representations by Fuchsian ones*, preprint (2013) `arXiv:1311.2919`

[7] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR2850125

[8] **R Fricke**, **F Klein**, *Vorlesungen über die Theorie der automorphen Funktionen, Vol. I*, Johnson Reprint, New York (1965)

[9] **W M Goldman**, *Discontinuous groups and the Euler class*, PhD thesis, Univ. California, Berkeley (1980) MR2630832 Available at `http://search.proquest.com/docview/302998874`

[10] **W M Goldman**, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. 54 (1984) 200–225 MR762512

[11] **W M Goldman**, *Topological components of spaces of representations*, Invent. Math. 93 (1988) 557–607 MR952283

[12] **W M Goldman**, *Ergodic theory on moduli spaces*, Ann. of Math. 146 (1997) 475–507 MR1491446

[13] **W M Goldman**, *The modular group action on real* $\mathrm{SL}(2)$*–characters of a one-holed torus*, Geom. Topol. 7 (2003) 443–486 MR2026539

[14] **W M Goldman**, *Mapping class group dynamics on surface group representations*, from: "Problems on mapping class groups and related topics", (B Farb, editor), Proc. Sympos. Pure Math. 74, Amer. Math. Soc. (2006) 189–214 MR2264541

[15] **F Guéritaud**, **F Kassel**, **M Wolff**, *Compact anti-de Sitter* 3*–manifolds and folded hyperbolic structures on surfaces*, preprint (2013) `arXiv:1311.2804`

[16] **J L Harer**, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. 84 (1986) 157–176 MR830043

[17] **R M Kashaev**, *Coordinates for the moduli space of flat* $\mathrm{PSL}(2,\mathbb{R})$*–connections*, Math. Res. Lett. 12 (2005) 23–36 MR2122727

[18] **R M Kashaev**, *On quantum moduli space of flat* $\mathrm{PSL}_2(\mathbb{R})$*–connections on a punctured surface*, from: "Handbook of Teichmüller theory, Vol. I", (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc., Zürich (2007) 761–782 MR2349684

[19] **L Keen**, **C Series**, *The Riley slice of Schottky space*, Proc. London Math. Soc. 69 (1994) 72–90 MR1272421

[20] **S Maloni**, **F Palesi**, **S P Tan**, *On the character varieties of the four-holed sphere* to appear in Groups, Geometry, and Dynamics

[21] **J Marché**, **M Wolff**, *The modular action on* $\mathrm{PSL}_2(\mathbb{R})$*–characters in genus* 2, Duke Math. J. 165 (2016) 371–412 MR3457677

[22] **C C Moore**, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math. 88 (1966) 154–178 MR0193188

[23] **R C Penner**, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. 113 (1987) 299–339 MR919235

[24] **R C Penner**, *Weil–Petersson volumes*, J. Differential Geom. 35 (1992) 559–608 MR1163449

[25] **J Roger**, **T Yang**, *The skein algebra of arcs and links and the decorated Teichmüller space*, J. Differential Geom. 96 (2014) 95–140 MR3161387

Department of Mathematics, Stanford University
Stanford, CA 94305, USA

`yangtian@math.stanford.edu`

Proposed: David Gabai
Seconded: Benson Farb, Gang Tian

Received: 16 February 2015
Revised: 14 May 2015