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Some examples of repetitive, nonrectifiable Delone sets

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Burago and Kleiner and, independently, McMullen, gave examples of Delone sets (that is, subsets of Euclidean space that are discrete and separated in a uniform way) that are non-bi-Lipschitz equivalent to the standard lattice. We refine their methods of construction via a discretization technique, thus giving the first examples of Delone sets as above that are also repetitive, in the sense that a translated copy of each patch appears in every large enough ball.

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A Delone set in $\mathbb{R}^d$ is a subset $\mathcal{D}$ that is separated and relatively dense in a uniform way. This means that there exist positive constants ρ, ϱ such that for all $x \neq y$ in $\mathcal{D}$ one has $d(x,y) \geq \rho$, and for all $z \in \mathbb{R}^d$ there is $z' \in \mathcal{D}$ such that $d(z,z') \leq \varrho$. Such a set is said to be *repetitive* if there is a function $R\colon \mathbb{N} \to \mathbb{N}$ so that for every pair of balls B_r and B_R of radius r and $R = R(r)$, respectively, we have that $B_R \cap \mathcal{D}$ contains a translated copy of $B_r \cap \mathcal{D}$.

Besides this pure abstract definition, these sets are relevant in mathematical physics as models of solid materials, especially after the spectacular discovery of quasicrystals in the early eighties by Shechtman, Blech, Gratias and Cahn [13].

A Delone set $\mathcal{D} \subset \mathbb{R}^d$ is said to be *rectifiable* if it is bi-Lipschitz equivalent to $\mathbb{Z}^d$. This means that there exists a bijection $f\colon \mathcal{D} \to \mathbb{Z}^d$ such that, for some constant $L \geq 1$ and all x, y in $\mathcal{D}$,

$$\frac{\|x-y\|}{L} \leq \|f(x)-f(y)\| \leq L\|x-y\|.$$

The question of the existence of *nonrectifiable* Delone sets in $\mathbb{R}^d$, for $d \geq 2$, was raised (with a geometric group-theoretic motivation) by Gromov [8] and (with an ergodic-theoretic motivation) by Furstenberg (see Burago and Kleiner [4]; see also Haynes, Kelly and Weiss [9]). This was solved in the affirmative by Burago and Kleiner [3] and, independently, by McMullen [12]. Later, Burago and Kleiner [4] gave a criterion for a Delone set of the plane to be rectifiable. This was extended to larger dimensions by Aliste-Prieto, Coronel and Gambaudo [2], who applied it to show that Delone sets that are *linearly repetitive*, ie those for which the *repetitivity function* R can be taken

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linear in r, are always rectifiable. This includes, for instance, the (set of vertices of the) Penrose tiling; see Solomon [14]. They left open the question of the existence of (nonlinearly) repetitive Delone sets that are nonrectifiable. The aim of this work is to answer this in the affirmative in a very strong way.

Repetitivity has a quite transparent geometric meaning. However, it is also relevant from the dynamical viewpoint. Indeed, it is straightforward to verify that this condition is equivalent to the minimality of the translation action of $\mathbb{R}^d$ on the closure of the orbit of the Delone set (endowed with an appropriate Gromov–Hausdorff metric or the Chabauty topology). In this direction, our construction can be further refined to obtain not only minimality but also unique ergodicity, which is a much stronger property in this setting. Indeed, a result of Solomyak [15] roughly states that, in case of repetitivity, the latter condition is equivalent to not only that each patch of the set appears in every large enough ball, but also the number of occurrences converges (as the radius of the ball goes to infinity, independently of the center) to a certain frequency.

Main Theorem *For each $d \geq 2$, there exists a subset of $\mathbb{Z}^d$ that is a repetitive, nonrectifiable Delone set for which the $\mathbb{R}^d$–action on the closure of its orbit is uniquely ergodic.*

As in Burago and Kleiner [3], in order to avoid technical difficulties mostly concerning notation, we will carry out the explicit construction just for the case $d = 2$. (The general case proceeds analogously.) We strongly use the main idea of [3], though we need to proceed more carefully to get a Delone subset of $\mathbb{Z}^2$ (this is the easy part; compare Garber [7] and Magazinov [11]), to guarantee repetitivity (this is much more tricky), and finally to ensure unique ergodicity (this is the most technical issue). To do this, we develop discrete analogues of the arguments of [3] that are of independent interest, thus giving a proof of the main result of [3] that is completely combinatorial (ie without passing to continuous models and/or approximating them by discrete ones). In this view, computations involving Jacobians become elementary counting arguments, whereas area estimates become density bounds for certain sets. An important advantage of this approach is that it allows giving explicit estimates (and not only existential results) all along the text. In particular, a backtracking of the proof estimates reveals a quite striking fact: given any unbounded function R', there is a repetitive, nonrectifiable Delone set for which the repetitivity function R satisfies $R(r_k) \leq R'(r_k)$ along an infinite sequence of radii $r_k \to \infty$. Our method also gives estimates for the rate of growth of the sequence r_k provided R' grows faster than linearly. This is in contrast to the aforementioned result of Aliste-Prieto, Coronel and Gambaudo [2], according to which we cannot have $R(r) \preceq r$ for a nonrectifiable, repetitive Delone set. Actually, in our examples, linear repetitivity clearly arises as an obstruction for a Delone set to be nonrectifiable. Indeed, along the construction, we need to perform modifications that ensure nonrectifiability but that, after rescaling, become negligible in density. However, in case of linear repetitivity, the density of points where these modifications should be performed persists under scale changes.

The method of construction is still flexible in many ways. In order to illustrate this, recall that by a standard application of the ergodic decomposition, the set of invariant probability measures of an $\mathbb{R}^d$–action is a *Choquet simplex* (that is, a compact, convex, metrizable subset of a locally convex real vector space such that every point therein is the mean with respect to a unique probability measure supported on its subset of extreme points). In the last section of this paper, we show (the $d = 2$ case of) the next extension of our main result (the case of larger dimension d is straightforward and left to the reader).

Main Theorem (extended) *For each $d \geq 2$ and any Choquet simplex $\mathcal{K}$, there exists a subset of $\mathbb{Z}^d$ that is a repetitive, nonrectifiable Delone set for which the $\mathbb{R}^d$–action on the closure of its orbit has a set of invariant probability measures isomorphic to $\mathcal{K}$.*

I Nonexpansiveness implies coarse differentiability

As usual, for a real number A, we denote its integer part by $[A]$. Given two real numbers $A \leq B$, we denote by $[\![A, B]\!]$ the set of integers n such that $A \leq n \leq B$. Given positive integers M, N, we let $R_{M,N} := [\![0, 2MN]\!] \times [\![0, M]\!]$. Given $k \in [\![1, 2N]\!]$ and a positive integer P dividing M, let S_k^P be the subset of $R_{M,N}$ formed by the points of the form

$$x_{i,j}^k := \left((k-1)M + i\,\frac{M}{P}, j\,\frac{M}{P}\right), \tag{1}$$

where i, j lie in $[\![0, P]\!]$. By some abuse of notation, (1) will still be used for $i = P + 1$ (yet $x_{P+1,j}^k$ does not belong to S_k^P). Notice that S_k^P also depends on M and N, but this dependence (which will be clear in each context) is suppressed just to avoid overloading the notation.

To simplify, we will only work with Delone subsets $\mathcal{D}$ of $\mathbb{Z}^2$ satisfying what we call the $2\mathbb{Z}$*–property*: all points (m, n) with an even m do belong to $\mathcal{D}$. In particular, we will consider domino tilings of the plane made only of the pieces 1-1 and 1-0. More generally, we say that a subset $\mathcal{D} \subset [\![A, B]\!] \times [\![A', B']\!]$ satisfies the $2\mathbb{Z}$*–property* if all points $(m, n) \in [\![A, B]\!] \times [\![A', B']\!]$ with an even m do belong to $\mathcal{D}$.

There is a little technical problem that arises when considering maps defined on strict subsets of either $\mathbb{Z}^2$ or $R_{M,N}$. To overcome this, we introduce a general construction.

Namely, given either a Delone set $\mathcal{D} \subset \mathbb{Z}^2$ or a subset $\mathcal{D} \subset R_{M,N}$ satisfying the $2\mathbb{Z}$–property in each case, for every function $f\colon \mathcal{D} \to \mathbb{Z}^2$ we define its extension $\widehat{f}$ to either $\mathbb{Z}^2$ or $R_{M,N}$ taking values in $\frac{1}{2}\mathbb{Z}^2$ by letting

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D}, \\ f\big(x + (1,0)\big) - \big(\frac{1}{2}, 0\big) & \text{if } x \notin \mathcal{D}. \end{cases}$$

The proof of the next lemma is straightforward and we leave it to the reader.

Lemma 1 *If $f\colon \mathcal{D} \to \mathbb{Z}^2$ is L–bi-Lipschitz, then $\widehat{f}$ is a $6L$–bi-Lipschitz map.*

The technical key of the construction is given by the next lemma:

Lemma 2 *Given $L \geq 1$, a positive $\tau < 1$ and an integer $P \geq 1$, there exist $\lambda > 0$ and positive integers M_0, N_0 such that the following holds: Given a multiple $M \geq M_0$ of P and a subset $\mathcal{D} \subset R_{M,N}$, with $N \geq N_0$, satisfying the $2\mathbb{Z}$–property, let $f\colon \mathcal{D} \to \mathbb{Z}^2$ be an L–bi-Lipschitz map, and denote $v^f_{M,N} := f(2MN, 0) - f(0,0)$. Assume that for all points of the form $x^k_{i,j}$ above that do belong to $\mathcal{D}$,*

$$\frac{\| f(x^k_{i+1,j}) - f(x^k_{i,j}) \|}{M/P} \leq (1+\lambda) \frac{\| v^f_{M,N} \|}{2MN} \tag{2}$$

$$\left(\textit{resp. } \frac{\| f(x^k_{i+1,j} + (1,0)) - f(x^k_{i,j}) \|}{1 + M/P} \leq (1+\lambda) \frac{\| v^f_{M,N} \|}{2MN} \right), \tag{3}$$

provided $x^k_{i+1,j}$ lies in $\mathcal{D}$ (resp. does not lie in $\mathcal{D}$). Then there is a k_* in $[\![1, 2N-1]\!]$ such that for all $x^{k_*}_{i,j}$ in $S^P_{k_*}$,

$$\frac{\langle \widehat{f}(x^{k_*}_{i,j} + (M,0)) - \widehat{f}(x^{k_*}_{i,j}), v^f_{M,N} \rangle}{M} \geq (1-\tau) \frac{\| v^f_{M,N} \|^2}{2MN}. \tag{4}$$

Proof We will deal with $\widehat{f}$ instead of f. Accordingly, we denote $\widehat{L} := 6L$. Notice that in case $x^k_{i+1,j}$ does not belong to $\mathcal{D}$, we still have

$$\begin{aligned} \frac{\| \widehat{f}(x^k_{i+1,j} - \widehat{f}(x^k_{i,j}) \|}{M/P} &\leq \frac{\| f(x^k_{i+1,j} + (1,0)) - f(x^k_{i,j}) \| + \frac{1}{2}}{M/P} \\ &\leq \frac{(1+\lambda) \| v^f_{M,N} \|}{2MN} \frac{(1 + M/P)}{M/P} + \frac{P}{2M}. \end{aligned}$$

Thus,

$$\frac{\| \widehat{f}(x^k_{i+1,j}) - \widehat{f}(x^k_{i,j}) \|}{M/P} \leq (1+2\lambda) \frac{\| v^f_{M,N} \|}{2MN}, \tag{5}$$

where the last inequality holds provided

$$\frac{P}{2M} \leq \frac{\lambda \|v^f_{M,N}\|}{4MN} \quad \text{and} \quad \left(1+\frac{P}{M}\right)(1+\lambda) \leq 1+\frac{3\lambda}{2},$$

which is always the case for $M \geq \max\{(LP)/\lambda, 2P(1+\lambda)/\lambda\}$.

Assume no square S^P_{k*} satisfies the required property. A direct application of the pigeonhole principle then shows that there is a "height" $j_* \in [\![0, P]\!]$ such that at least $r := [(2N-1)/(2(P+1))]$ squares $S_{k_1}, \dots, S_{k_r}$ contain points $x^{k_1}_{i_1,j_*}, \dots, x^{k_r}_{i_r,j_*}$, respectively, satisfying the reverse inequality to (4) and such that all the indices k_s have the same parity and are at most $2N-1$.

Notice that $v^f_{M,N}$ equals

$$\hat{f}(0,j_*) - \hat{f}(0,0) + \sum_{s=1}^{r} (\hat{f}(x^{k_s}_{i_s,j_*}) - \hat{f}(x^{1+k_s}_{i_s,j_*})) + \hat{f}(2MN,0) - \hat{f}(2MN,j_*)$$

$$+\left[\hat{f}(2MN,j_*) - \hat{f}(x^{k_r}_{i_r,j_*}) + \sum_{s=2}^{r} \hat{f}(x^{k_s}_{i_s,j_*}) - \hat{f}(x^{1+k_{s-1}}_{i_{s-1},j_*}) + \hat{f}(x^{1+k_1}_{i_1,j_*}) - \hat{f}(0,j_*)\right].$$

The (nonnormalized) projections over $v^f_{M,N}$ of the expression in brackets can be estimated using the hypothesis: it is smaller than or equal to

$$\left(2MN - M\left[\frac{2N-1}{2(P+1)}\right]\right)(1+2\lambda)\frac{\|v^f_{M,N}\|^2}{2MN}.$$

Therefore, by the choice of the points $x^{k_s}_{i_s,j_*}$, the value of $\|v^f_{M,N}\|^2$ is bounded from above by

$$\langle \hat{f}(0,j_*) - \hat{f}(0,0), v^f_{M,N}\rangle + (1-\tau)\frac{2N-1}{2(P+1)}\frac{\|v^f_{M,N}\|^2}{2N}$$

$$+\langle \hat{f}(2MN,0) - \hat{f}(2MN,j_*), v^f_{M,N}\rangle + \left(2MN - M\left[\frac{2N-1}{2(P+1)}\right]\right)(1+2\lambda)\frac{\|v^f_{M,N}\|^2}{2MN}.$$

Since $\hat{f}$ is $\hat{L}$–Lipschitz, we finally conclude that

$$\|v^f_{M,N}\|^2 \leq 2\hat{L}M\|v^f_{M,N}\| + \left(1 - \frac{1}{2N}\left[\frac{2N-1}{2(P+1)}\right]\right)(1+2\lambda)\|v^f_{M,N}\|^2 + (1-\tau)\frac{2N-1}{2(P+1)}\frac{\|v^f_{M,N}\|^2}{2N}.$$

Thus we get

$$\|v^f_{M,N}\|\left(-2\lambda + \frac{(1+2\lambda)}{2N}\left[\frac{2N-1}{2(P+1)}\right] - \frac{(1-\tau)}{2N}\frac{2N-1}{2(P+1)}\right) < 2\hat{L}M,$$

hence

$$\|v^f_{M,N}\|\left(-2\lambda + \frac{(1+2\lambda)(2N-2P-2)}{4N(P+1)} - \frac{1-\tau}{2(P+1)}\right) < 2\hat{L}M.$$

For $N > 2(P+1)/\tau$, we have

$$\frac{N\tau - P - 1}{2(2NP+N+P+1)} > \frac{\tau}{12P},$$

thus for $\lambda \le \tau/(12P)$, we obtain

$$-2\lambda + \frac{(1+2\lambda)(2N-2P-2)}{4N(P+1)} - \frac{1-\tau}{2(P+1)} > 0.$$

The bi-Lipschitz condition of f then yields

$$\frac{2NM}{L}\left(-2\lambda + \frac{(1+2\lambda)(2N-2P-2)}{4N(P+1)} - \frac{1-\tau}{2(P+1)}\right) < 2\hat{L}M.$$

However, one easily checks that given M, this is impossible for

$$\lambda \le \frac{\tau}{12P}, \quad N \ge N_0 := 1 + \left[\tfrac{1}{\tau}(L\hat{L} + 2P + 2 + \tau)\right].$$

This finishes the proof for $M \ge M_0 := (L+4)P/\lambda$. □

In analogy to the terminology introduced in [3], a square $S^P_{k_*}$ satisfying the conclusion of the preceding lemma (ie condition (4)) will be said to be (M, N, τ, f)–regular.

Lemma 3 *Given $L \ge 1$, $\varepsilon > 0$ and an integer $P \ge 1$, there exists a positive $\tau < 1$ such that the following holds: Let $M \ge M_0 := (L+4)P/\lambda$ be a multiple of P, where $\lambda := \tau/(12P)$. Suppose $f\colon \mathcal{D} \to \mathbb{Z}^2$ is an L–bi-Lipschitz map such that for each $x^k_{i,j} \in S^P_k$, either (2) or (3) holds according to the case. Then for every $x^{k_*}_{i,j}$ belonging to an (M, N, τ, f)–regular square $S^P_{k_*}$, one has*

$$\left\| \frac{\hat{f}(x^{k_*+1}_{i,j}) - \hat{f}(x^{k_*}_{i,j})}{M} - \frac{v^f_{M,N}}{2MN} \right\| \le \varepsilon. \tag{6}$$

Proof Again, we denote $\hat{L} := 6L$. Given $x^{k_*}_{i,j} \in S^P_{k_*}$, let us write

$$\hat{f}(x^{k_*+1}_{i,j}) - \hat{f}(x^{k_*}_{i,j}) = \alpha^{k_*}_{i,j} v^f_{M,N} + \beta^{k_*}_{i,j} v^{\perp}_{M,N}$$

for certain reals $\alpha^{k_*}_{i,j}$ and $\beta^{k_*}_{i,j}$, where $v^{\perp}_{M,N}$ is a unit vector orthogonal to $v^f_{M,N}$. On the one hand, by (4),

$$\alpha^{k_*}_{i,j} \frac{\|v^f_{M,N}\|^2}{M} \ge (1-\tau)\frac{\|v^f_{M,N}\|^2}{2MN},$$

hence

$$\alpha^{k_*}_{i,j} \geq \frac{1-\tau}{2N}. \tag{7}$$

On the other hand, since $M \geq M_0$, using P times (5) and the triangle inequality, we obtain

$$(\alpha^{k_*}_{i,j})^2 \|v^f_{M,N}\|^2 \leq (\alpha^{k_*}_{i,j})^2 \|v^f_{M,N}\|^2 + (\beta^{k_*}_{i,j})^2 \leq \left((1+2\lambda) \frac{\|v^f_{M,N}\|}{2N} \right)^2.$$

Therefore,

$$|\alpha^{k_*}_{i,j}| \leq \frac{1+2\lambda}{2N} \leq \frac{1+\tau}{2N}.$$

Similarly, using (2), (7) and the previous estimate, we obtain

$$\left(\frac{(1-\tau)^2 \|v^f_{M,N}\|^2}{4N^2} + (\beta^{k_*}_{i,j})^2 \right) \leq (\alpha^{k_*}_{i,j})^2 \|v^f_{M,N}\|^2 + (\beta^{k_*}_{i,j})^2 \leq (1+2\lambda)^2 \frac{\|v^f_{M,N}\|^2}{4N^2},$$

which yields

$$(\beta^{k_*}_{i,j})^2 \leq \frac{\|v^f_{M,N}\|^2}{4N^2} ((1+\tau)^2 - (1-\tau)^2) = \frac{\tau \|v^f_{M,N}\|^2}{N^2}.$$

As a consequence,

$$\begin{aligned}
\left\| \frac{\hat{f}(x^{k+1}_{i,j}) - \hat{f}(x^k_{i,j})}{M} - \frac{v^f_{M,N}}{2MN} \right\| &= \left\| \frac{\alpha^k_{i,j}}{M} v^f_{M,N} + \frac{\beta^k_{i,j}}{M} v^{\perp}_{M,N} - \frac{v^f_{M,N}}{2MN} \right\| \\
&\leq \|v^f_{M,N}\| \left| \frac{\alpha^k_{i,j}}{M} - \frac{1}{2MN} \right| + \frac{|\beta^k_{i,j}|}{M} \\
&\leq \|v^f_{M,N}\| \frac{\tau}{2MN} + \|v^f_{M,N}\| \frac{\sqrt{\tau}}{MN} \\
&\leq 2MNL \left(\frac{\tau}{2MN} + \frac{\sqrt{\tau}}{MN} \right) \leq 3L\sqrt{\tau} = \varepsilon,
\end{aligned}$$

where the last equality holds for $\tau := \varepsilon^2/(9L^2)$. □

Below we put together the two preceding lemmas into a single statement.

Proposition 4 *Given $L \geq 1$, a positive $\varepsilon < 1$ and a positive integer P, there exist $\lambda > 0$ and positive integers M_0, N_0 such that the following holds: Given a subset $\mathcal{D} \subset R_{M,N}$ satisfying the $2\mathbb{Z}$–property, with $M \geq M_0$ a multiple of P and $N \geq N_0$, let $f\colon \mathcal{D} \to \mathbb{Z}^2$ be an L–bi-Lipschitz map. Assume that for every point of the form $x^k_{i,j}$ that belongs* *to $\mathcal{D}$,*

$$\frac{\|f(x^k_{i+1,j}) - f(x^k_{i,j})\|}{M/P} \leq (1+\lambda)\frac{\|v^f_{M,N}\|}{2MN}$$

$$\left(\text{resp. } \frac{\|f(x^k_{i+1,j} + (1,0)) - f(x^k_{i,j})\|}{1+M/P} \leq (1+\lambda)\frac{\|v^f_{M,N}\|}{2MN}\right),$$

provided $x^k_{i+1,j}$ lies in $\mathcal{D}$ (resp. does not lie in $\mathcal{D}$).

Then there is a subset

$$S = S^P_k := \left\{\left((k-1)M + i\frac{M}{P}, j\frac{M}{P}\right) : 0 \leq i \leq P, 0 \leq j \leq P\right\}$$

such that every $x \in S$ satisfies

$$\left\|\frac{\hat{f}(x+(M,0)) - \hat{f}(x)}{M} - \frac{f(2MN,0) - f(0,0)}{2MN}\right\| \leq \varepsilon.$$

Remark 5 The estimates and definitions given along the proofs of Lemmas 2 and 3 show that, given $L \geq 1$, a positive constant $\varepsilon < 1$ and a positive integer P, the conclusion of Proposition 4 holds for

$$\lambda \leq \frac{\varepsilon^2}{108PL^2}, \quad M_0 \geq \frac{108P^2L^2(L+4)}{\varepsilon^2}, \quad N_0 \geq 2 + \frac{216L^2P(3L^2+P+1)}{\varepsilon^2}.$$

II Coarse differentiability forces densities to be close

Let $f\colon \mathcal{D} \to \mathbb{Z}^2$ be an L–bi-Lipschitz map defined on a Delone set $\mathcal{D} \subset \mathbb{Z}^2$ satisfying the $2\mathbb{Z}$–property. Fix an integer $P \geq 1$, and let S be a square of the form $S^P_k \subset R_{M,N}$, where M is a multiple of P. We let γ^* be the curve obtained by connecting (using line segments) points in $\hat{f}(\partial S)$ coming from consecutive points in ∂S. The curve γ^* is closed though not necessarily simple. However, the curve γ^* contains the simple curve $\gamma = \gamma_S$ obtained by "deleting short loops". Notice that the bi-Lipschitz property of $\hat{f}$ easily implies that each loop has length at most $2\hat{L}^3M/P$. Therefore, if $P \geq 4\hat{L}^4$, then γ has length at least

$$\frac{M\sqrt{2}}{\hat{L}} - \frac{4\hat{L}^3M}{P} \geq 4(\sqrt{2}-1)\hat{L}^3 > 0.$$

In particular, it is well defined. We denote by $\mathrm{int}(\gamma)$ the closed, bounded region of the plane determined by γ. We let

$$\hat{S} := \{((k-1)M + i, j) : i, j \text{ in } [\![0, M-1]\!]\}.$$

This corresponds to the set of all points with integer coordinates in the region (square) bounded by the points of S_k^P, except for those in the upper and the right sides of the square. We call such a subset the *lower-left corner* of the corresponding square.

Given $\varepsilon > -1$, we let $(1+\varepsilon)\widehat{S}$ be the set of all points with integer coordinates lying in the square having the same center as S though side of length $(1+\varepsilon)M$. We also denote by $\mathcal{S}_1$ the unit square in $\mathbb{R}^2$, and by $(1+\varepsilon)\mathcal{S}_1$ the corresponding homothetic copy.

Lemma 6 *Given $L \geq 1$ and $\varepsilon > 0$, there exists P_0 such that the following holds: If $f\colon \mathcal{D} \to \mathbb{Z}^2$ is L–bi-Lipschitz and $P \geq P_0$, then*

(i) *no point of $\widehat{f}(\mathbb{Z}^2 \setminus (1+\varepsilon)\widehat{S}))$ lies in* $\operatorname{int}(\gamma)$*;*

(ii) *all points in $\widehat{f}((1-\varepsilon)\widehat{S})$ are contained in* $\operatorname{int}(\gamma)$*.*

This lemma can be easily shown by contradiction just by renormalizing and passing to the limit (along a subsequence) using a variation of the Arzela–Ascoli theorem. Indeed, such an argument provides not only a limit homeomorphism F from the unit square $\mathcal{S}_1$ but also

in case (i), a point in the exterior of $(1+\varepsilon)\mathcal{S}_1$ which is mapped by F inside $F(\mathcal{S}_1)$;

in case (ii), a point in $(1-\varepsilon)\mathcal{S}_1$ which is mapped by F into a point outside $F(\mathcal{S}_1)$.

In each case, this is certainly impossible, since F is a homeomorphism.

Despite this simple argument, it is better to give a slightly more involved proof that yields a quantitative estimate for P_0 in terms of L and ε.

Proof of Lemma 6 We claim that the lemma holds for $P_0 := \max\{4\widehat{L}^4, 3\widehat{L}^2/\varepsilon\}$.

For (i), let $x \in \mathbb{Z}^2 \setminus \widehat{S}$ be a point that is mapped by $\widehat{f}$ inside $\operatorname{int}(\gamma)$ and lies at a maximal distance to $\widehat{S}$ among these points. (Notice that, by the bi-Lipschitz property and the Delone condition, only finitely many points map into $\operatorname{int}(\gamma)$.) We claim that $\operatorname{dist}(\widehat{f}(x),\gamma) \leq \widehat{L}$. Otherwise, the closed ball of center $\widehat{f}(x)$ and radius $\widehat{L}$ would be contained in $\operatorname{int}(\gamma)$. This ball contains the image under $\widehat{f}$ of the points $x-(1,0)$, $x+(1,0)$, $x-(0,1)$, $x+(0,1)$. However, among these points, at least one is at a greater distance to $\widehat{S}$ than x, which contradicts the choice of x.

Now, it is obvious from the construction that every point in γ lies at distance $\leq \widehat{L}M/P$ from some point of the form $\widehat{f}(y)$, where $y \in \partial S$. Therefore,

$$\frac{\|x-y\|}{\widehat{L}} \leq \|\widehat{f}(x)-\widehat{f}(y)\| \leq \widehat{L} + \frac{\widehat{L}M}{P} = \widehat{L}\left(1+\frac{M}{P}\right),$$

hence

$$\operatorname{dist}(x, \partial \widehat{S}) \leq 1 + \|x - y\| \leq \widehat{L}^2 \left(2 + \frac{M}{P}\right) \leq \frac{3\widehat{L}^2 M}{P} \leq \varepsilon M,$$

where the last inequality holds provided $P \geq P_0$.

The proof of (ii) proceeds analogously, dealing with f^{-1} instead of f. □

Remark 7 It is an open problem whether every bi-Lipschitz map defined on a Delone subset of the plane can be extended to a bi-Lipschitz homeomorphism of the whole plane (see [1, Question 4.14.(ii)]). Certainly, having an affirmative answer for (a quantitative version of) this question would yield another proof of the preceding lemma. The estimates given above are, however, enough for our purposes.

The next elementary lemma will be needed when comparing cardinalities of points enclosed by curves each of which is an almost translated copy of the other one.

Lemma 8 *If γ is a rectifiable curve in $\mathbb{R}^2$ of* $\operatorname{length}(\gamma) \geq 4$ *and* $1 \leq T \leq \frac{1}{4}\operatorname{length}(\gamma)$, *then*

$$|\{x \in \mathbb{Z}^2 : d(x, \gamma) \leq T\}| \leq 25T \operatorname{length}(\gamma).$$

Proof Let $x_1, \dots, x_k$ be points in γ such that every $x \in \gamma$ has distance $\leq T$ to at least one of the points x_i. Notice that we can take such a $k \in \mathbb{N}$ satisfying

$$k \leq 2 + \frac{\operatorname{length}(\gamma)}{2T} \leq \frac{\operatorname{length}(\gamma)}{T}.$$

If $x \in \mathbb{Z}^2$ satisfies $d(x, \gamma) \leq T$, then $\|x - x_i\| \leq 2T$ holds for some $1 \leq i \leq k$. Therefore,

$$\{x \in \mathbb{Z}^2 : d(x, \gamma) \leq T\} \subseteq \bigcup_{i=1}^{k} \{x \in \mathbb{Z}^2 : \|x - x_i\| \leq 2T\}.$$

Thus,

$$|\{x \in \mathbb{Z}^2 : d(x, \gamma) \leq T\}| \leq k(4T + 1)^2 \leq \frac{\operatorname{length}(\gamma)}{T}(5T)^2 = 25T \operatorname{length}(\gamma),$$

which finishes the proof. □

We can now give the main argument involving local densities of points of $\mathcal{D}$ via comparison along the images.

Proposition 9 *Given $L \geq 1$ and $1 \geq d > d' > 0$, there exist a positive $\varepsilon < 1$ and integers P_1, M_1 such that the following holds: Let $\mathcal{D}$ be a Delone set satisfying the $2\mathbb{Z}$–property, and let $f\colon \mathcal{D} \to \mathbb{Z}^2$ be surjective. Assume that for $P \geq P_1$, $N \geq 1$ and $M \geq M_1$ a multiple of P, some square $S := S_k^P \subset R_{M,N}$, with $1 \leq k < 2M$, is such that every $x \in S$ satisfies (6), and denote $S' := S_{k+1}^P$. If $\widehat{S} \cap \mathcal{D}$ contains at least dM^2 (resp. at most $d'M^2$) points and $\widehat{S}' \cap \mathcal{D}$ contains at most $d'M^2$ (resp. at least dM^2) points, then f cannot be L–bi-Lipschitz.*

Proof We will show that the claim holds for all positive $\varepsilon < (d-d')/(20(2+5L))$, $M_1 \geq \max\{2\widehat{L}, 1/\varepsilon\}$ and $P_1 = P_0$, where P_0 is given by Lemma 6. To do this, we will suppose that $|\widehat{S} \cap \mathcal{D}| \geq dM^2$ and $|\widehat{S}' \cap \mathcal{D}| \leq d'M^2$, the other case being analogous.

We proceed by contradiction. Assuming that f is L–bi-Lipschitz, we use Lemma 6. By (ii), for $\gamma := \gamma_S$, the set $f(\mathcal{D} \cap (1-\varepsilon)\widehat{S}) \subset \mathbb{Z}^2$ contains $\geq dM^2 - 4(\varepsilon M + 1)^2$ points, all lying in $\operatorname{int}(\gamma)$:

$$|\operatorname{int}(\gamma) \cap \mathbb{Z}^2| \geq dM^2 - 16\varepsilon M^2.$$

By (i) and the surjectivity of $f\colon \mathcal{D} \to \mathbb{Z}$, for $\gamma' := \gamma_{S'}$, the set $\operatorname{int}(\gamma') \cap \mathbb{Z}^2$ is contained in $f(\mathcal{D} \cap (1+\varepsilon)\widehat{S}')$, hence its cardinality is bounded from above by the quantity $d'M^2 + 4\varepsilon M(M+1) + 4(\varepsilon M + 1)^2$:

$$|\operatorname{int}(\gamma') \cap \mathbb{Z}^2| \leq d'M^2 + 24\varepsilon M^2.$$

We claim that points of $\operatorname{int}(\gamma)$ must lie in $\operatorname{int}(\gamma')$ after translation by $v_{M,N}/(2N)$, except perhaps for those which are moved into points that are εM–close to γ'. Indeed, γ (hence $\operatorname{int}(\gamma)$) is determined by the image $\widehat{f}(\partial S)$, hence by points of the form $\widehat{f}(x_{i,j}^k)$ for which (6) holds. Obviously, similar arguments apply to γ'.

We next claim that we may use the preceding lemma to conclude that the number of points that move into points εM–close to γ' is at most

$$25\varepsilon M \operatorname{length}(\gamma') \leq 100\varepsilon L M^2.$$

Indeed, the choices of P and M yield

$$\operatorname{length}(\gamma') \geq 2M/\widehat{L} > 4 \quad \text{and} \quad \varepsilon M \leq \tfrac{1}{4} \leq \tfrac{1}{4}\operatorname{length}(\gamma'),$$

thus fulfilling the hypothesis of Lemma 8.

The preceding estimates force

$$dM^2 - 16\varepsilon M^2 - 100\varepsilon L M^2 \leq d'M^2 + 24\varepsilon M^2,$$

that is,

$$d \leq d' + (40 + 100L)\varepsilon.$$

However, this is impossible due to the choice of ε. □

We next put together Propositions 4 and 9 into a single one.

Proposition 10 *Given $L \geq 1$ and $1 \geq d > d' > 0$, there exist $\lambda > 0$ and positive integers M_*, N_*, P_* such that the following holds: Let $\mathcal{D}$ be a Delone set satisfying the $2\mathbb{Z}$–property, and let $f\colon \mathcal{D} \to \mathbb{Z}^2$ be L–bi-Lipschitz and surjective. Assume that for $M \geq M_*$ and $N \geq N_*$, with M a multiple of P_*, there are two consecutive squares $S_k^{P_*}$ and $S_{k+1}^{P_*}$ of $R_{M,N}$ such that the lower-left corner of one of them contains at least dM^2 points of $\mathcal{D}$, and the lower-left corner of the other one has no more than $d'M^2$ points of $\mathcal{D}$. Then there must exist a point $x \in \mathcal{D} \cap R_{M,N}$ of the form $x_{i,j}^k$ such that either*

$$\frac{\|f(x+(M/P,0))-f(x)\|}{M/P} \geq (1+\lambda)\frac{\|f(2MN,0)-f(0,0)\|}{2MN}$$

if $x+(M/P,0)$ belongs to $\mathcal{D}$, or

$$\frac{\|f(x+(1+M/P,0))-f(x)\|}{1+M/P} \geq (1+\lambda)\frac{\|f(2MN,0)-f(0,0)\|}{2MN}$$

otherwise.

Roughly, the preceding proposition says that if a Delone set $\mathcal{D}$ with the $2\mathbb{Z}$–property maps onto $\mathbb{Z}^2$ by an L–bi-Lipschitz map f, then variations of the local density of $\mathcal{D}$ force the Lipschitz constant of f to increase when passing from a certain scale to a smaller one. By inductive application of this argument, we will contradict the Lipschitz condition of f for appropriately constructed Delone sets.

Remark 11 The estimates of Remark 5 and those given in Lemma 6 and Proposition 9 show that, given $L \geq 1$ and $1 \geq d > d' > 0$, the conclusion of Proposition 10 holds for

$$\lambda \leq \frac{(d-d')^3}{10^{10}L^7}, \quad M_* \geq \frac{10^{15}L^{11}}{(d-d')^4}, \quad N_* \geq \frac{10^{10}L^{10}}{(d-d')^4}.$$

III Construction of the nonrectifiable, repetitive Delone set

We start by introducing a general recipe for constructing repetitive Delone subsets of $\mathbb{Z}^2$.

Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^2$ satisfying the following properties:

(F1) $(0,0) \in F_n \subseteq F_{n+1}$, for every $n \geq 1$.

(F2) $\mathbb{Z}^2 = \bigcup_{n \geq 1} F_n$.

(F3) For every $n \geq 1$, the set F_{n+1} is a disjoint union of translated copies of F_n.

The last condition yields a finite subset $\Gamma_n \subset F_{n+1}$ such that

$$F_{n+1} = \bigcup_{v \in \Gamma_n} (F_n + v).$$

Assume that for each $n \geq 1$, there exist $k_n \geq 1$ and a family of *patches* $\mathcal{P}_{n,1}, \dots, \mathcal{P}_{n,k_n}$ in $\{0,1\}^{F_n}$ such that:

(F4) $\mathcal{P}_{n+1,k}|_{v+F_n} - v$ is in $\{\mathcal{P}_{n,1}, \dots, \mathcal{P}_{n,k_n}\}$ for all $v \in \Gamma_n$ and all $k \in [\![1, k_{n+1}]\!]$.

(F5) For every $j \in [\![1, k_n]\!]$ and $k \in [\![1, k_{n+1}]\!]$, one has $\mathcal{P}_{n+1,k}|_{v+F_n} = \mathcal{P}_{n,j}$ for a certain $v \in \Gamma_n$.

(F6) $\mathcal{P}_{n+1,1}|_{F_n} = \mathcal{P}_{n,1}$.

By properties **(F1)**, **(F2)** and **(F6)** above, the intersection

$$\bigcap_{n \geq 1} \left\{ D \in \{0,1\}^{\mathbb{Z}^2} : D|_{F_n} = \mathcal{P}_{n,1} \right\}$$

consists of a single point, which can be viewed as a subset $\mathcal{D}$ of $\mathbb{Z}^2$.

Lemma 12 *The set $\mathcal{D}$ is a repetitive Delone set.*

Proof Fix $r > 0$. Since $\mathcal{D}$ is a subset of $\mathbb{Z}^2$, only finitely many patches $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ of diameter $2r$ appear (up to translation) in $\mathcal{D}$. Let $n \geq 1$ be such that the restriction of $\mathcal{D}$ to F_n (ie $\mathcal{P}_{n,1}$) contains (translated copies of) all of the patches $\mathcal{Q}_1, \dots, \mathcal{Q}_m$. Property **(F5)** above ensures that for a large enough $R > 0$, every ball of radius R in $\mathcal{D}$ contains a translated copy of the patch $\mathcal{P}_{n,1}$, hence a copy of each patch $\mathcal{Q}_1, \dots, \mathcal{Q}_m$. Thus, every ball of radius r appears in each ball of radius R. □

In order to implement the strategy above, we need to specify our building blocks (ie the patches along the construction). These will be constructed starting from two data, namely:

- A constant $L \geq 1$ (which will play the role of the Lipschitz constant to discard).
- Two square patches $\mathcal{Q}_1, \mathcal{Q}_2$ in $\mathbb{Z}^2$ that have equal and even length-side but contain different number of points. We let d_i be the density of points in the lower-left corner of $\mathcal{Q}_i$, the notation being such that $d_2 > d_1$. We also assume that both patches contain all boundary points and satisfy the $2\mathbb{Z}$–property when placed centered at the origin.

Given these data, fix d_1', d_2' such that $d_2 > d_2' > d_1' > d_1$. Let λ, M_*, N_*, P_* be the constants provided by Proposition 10 for L, $d := d_2'$ and $d' := d_1'$. Fix an integer $\ell \geq 1$ such that

$$\frac{(1+\lambda)^\ell}{L} > L. \tag{8}$$

Using the elementary inequality $(1+\lambda)^\ell \geq 1 + \lambda\ell$, one easily checks that this holds for

$$\ell \geq \frac{L^2}{\lambda}. \tag{9}$$

Let $2M$ be the side length of the patches $\mathcal{Q}_i^0 := \mathcal{Q}_i$ for $i \in \{1,2\}$. We view these patches as subsets of $[\![-M, M]\!] \times [\![-M, M]\!]$, that is, centered at the origin. We start by constructing new patches $\mathcal{Q}_1^1, \mathcal{Q}_2^1$ as follows (see Figure 1):

- Fix an odd positive integer m so that $2mP_*M \geq M_*$, and form a square (centered at the origin) of $(mP_*)^2$ copies of $\mathcal{Q}_1$, matching left sides to right sides and lower sides to upper sides.
- Next, match to the right a square block consisting of $(mP_*)^2$ copies of $\mathcal{Q}_2$. After this, match to the right a square block consisting of $(mP_*)^2$ copies of $\mathcal{Q}_1$. Proceed similarly up to having matched N blocks made of pieces $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in an alternating manner, where the integer $N \geq N_*$ is to be fixed below.
- Proceed similarly to the left of the centered-at-the-origin block made of pieces $\mathcal{Q}_1$. In this way, we form a rectangle of sides $2mP_*M(2N+1)$ and $2mP_*M$, filled by alternating blocks of copies of $\mathcal{Q}_1$ and $\mathcal{Q}_2$.
- To complete $\mathcal{Q}_1^1$, fill up the whole square of side $2mP_*M(2N+1)$ centered at the origin by matching copies of $\mathcal{Q}_1$ at all places, except for those in the lower rectangle of sides $2mP_*M(2N+1)$ and $2mP_*M$, where we match the rectangle constructed above. (We emphasize that all matchings are made as above, that is, by identifying left to right sides, and lower to upper sides.)
- Finally, to construct $\mathcal{Q}_2^1$, proceed similarly as for $\mathcal{Q}_1^1$, switching the roles of $\mathcal{Q}_1$ and $\mathcal{Q}_2$.
- The integer N is taken $\geq N_*$ and such that the density of points in the lower-left corner of $\mathcal{Q}_1^1$ (resp. $\mathcal{Q}_2^1$) is $< d_1'$ (resp. $> d_2'$). One can easily check that this holds for N satisfying

$$N \geq 2\max\left\{\frac{N_*}{2}, \frac{1}{d_2 - d_2'}, \frac{1}{d_1' - d_1}\right\}. \tag{10}$$

Next, we repeat the procedure, but starting with the patches $\mathcal{Q}_1^1, \mathcal{Q}_2^1$, keeping the same constants L, d_1', d_2'. We thus get new patches $\mathcal{Q}_1^2, \mathcal{Q}_2^2$ of densities less than d_1' and greater than d_2', respectively, to which we may apply the construction again. If we repeat this procedure ℓ times, we obtain new patches, that we denote by $\mathcal{Q}_1^{\text{new}}$ and $\mathcal{Q}_2^{\text{new}}$ (and that have densities less than d_1' and greater than d_2', respectively).

Lemma 13 *Let $\mathcal{D}$ be a Delone subset of $\mathbb{Z}^2$ satisfying the $2\mathbb{Z}$–property. If $\mathcal{D}$ contains translated copies of either $\mathcal{Q}_1^{\text{new}}$ or $\mathcal{Q}_2^{\text{new}}$ as building blocks as above, then $\mathcal{D}$ cannot be mapped onto $\mathbb{Z}^2$ by an L–bi-Lipschitz map.*

Proof We call the *expansion of points x, y under a map f* the expression

$$\frac{\|f(x)-f(y)\|}{\|x-y\|}.$$

By Proposition 10, if f is an L–bi-Lipschitz surjective map $\mathcal{D} \to \mathbb{Z}^2$, the expansion of the endpoints of the lower side of $\mathcal{Q}_i^\ell$ is at most $1/(1+\lambda)$ times the expansion of the endpoints of the lower side of some square made of mP_* copies of $\mathcal{Q}_j^{\ell-1}$, where $m = m_\ell$. By the triangle inequality, the latter is larger than or equal to the expansion of the endpoints of some of the patches $\mathcal{Q}_j^{\ell-1}$ placed at the lower side of this square.

By the construction, the preceding argument yields that the expansion above is no more than $1/(1+\lambda)$ times the expansion of the endpoints of the lower side of a certain square $\mathcal{Q}_{j'}^{\ell-2}$. Continuing this way, in ℓ steps, we get two pairs of points such that the expansion for one pair is at least $(1+\lambda)^\ell$ times that of the other pair. Now, as f is L–bi-Lipschitz, both expansions are $\le L$ and $\ge 1/L$. This is in contradiction to (8). □

It is now easy to construct a nonrectifiable, repetitive Delone set. Indeed, let (L_n) be a sequence of numbers ≥ 1 going to infinity. Start with the square patches $\mathcal{Q}_{1,1}$ and $\mathcal{Q}_{1,2}$ illustrated in Figure 2.

Next, proceed inductively: given the patches $\mathcal{Q}_{n,1} =: \mathcal{Q}_1$ and $\mathcal{Q}_{n,2} =: \mathcal{Q}_2$, we let $\mathcal{Q}_{n+1,1} := \mathcal{Q}_1^{\text{new}}$ and $\mathcal{Q}_{n+1,2} := \mathcal{Q}_2^{\text{new}}$, where we have implemented the preceding procedure to obtain new patches for the constant L_n. This construction fits into that of Lemma 12, except that the patches $\mathcal{P}_{n,1}, \mathcal{P}_{n,2}$ that are involved do not correspond to $\mathcal{Q}_{n,1}, \mathcal{Q}_{n,2}$, respectively, but to the lower-left corners of these. (This occurs because the matchings above were made by identifying left to right sides, and lower to upper sides.) Hence, we have a repetitive Delone set $\mathcal{D}$ containing copies of $\mathcal{Q}_{n,1}$ and $\mathcal{Q}_{n,2}$, for each $n \ge 1$. By Lemma 13, $\mathcal{D}$ cannot be L_n–bi-Lipschitz equivalent to $\mathbb{Z}^2$ for any $n \ge 1$. Since $L_n \to \infty$, the set $\mathcal{D}$ is not bi-Lipschitz equivalent to $\mathbb{Z}^2$.

Remark 14 Clearly, the properties of being repetitive and nonrectifiable are not only valid for $\mathcal{D}$ but also for all points in the closure of its orbit under the translation action.

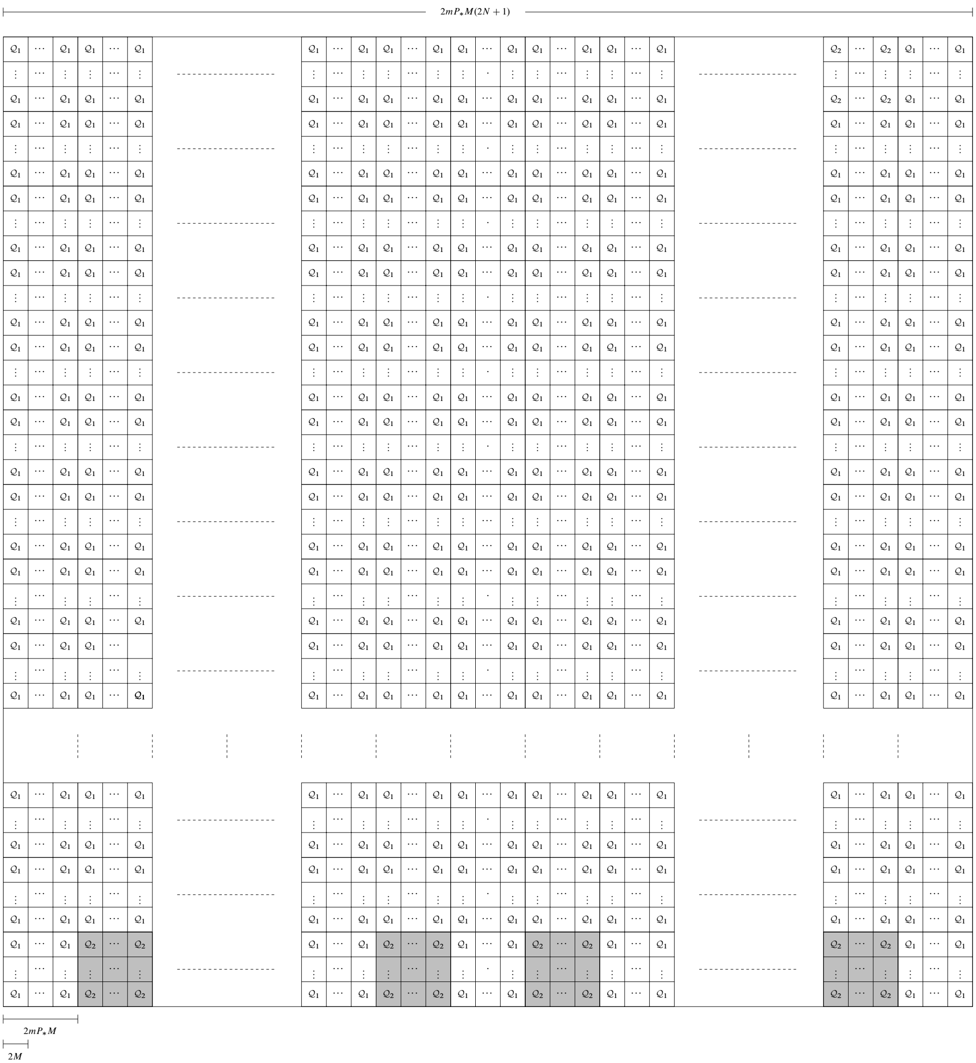


Figure 1: Building $\mathcal{Q}_1^1$ starting with $\mathcal{Q}_1^0 = \mathcal{Q}_1$ and $\mathcal{Q}_2^0 = \mathcal{Q}_2$

We end this section with a brief discussion concerning the lack of linear repetitivity of our examples. Roughly, this amounts to saying that the ratio of the side length of the new squares $\mathcal{Q}_1^{\mathrm{new}}, \mathcal{Q}_2^{\mathrm{new}}$ compared to that of the original ones $\mathcal{Q}_1, \mathcal{Q}_2$ appearing along the construction tends to infinity at least along a subsequence. In our construction, this essentially comes from the condition (see Remark 11 and estimate (10)):

$$N \geq N_* \geq \frac{10^{10} L^{10}}{(d - d')^4}.$$

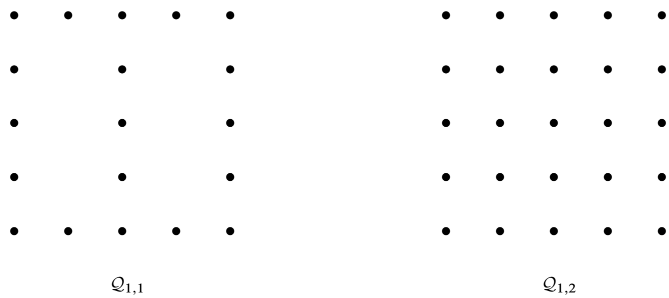


Figure 2: The initial patches $\mathcal{Q}_{1,1}$ and $\mathcal{Q}_{1,2}$

Despite this, given an unbounded function $R'\colon (0,\infty) \to (0,\infty)$, we can artificially introduce steps in which the parameter N does not satisfy (10) but just $N = 1$. Doing this infinitely many times, we obtain an infinite sequence of radii r_k for which the repetitivity function R satisfies $R(r_k) \leq R'(r_k)$. Notice that the resulting Delone set is still nonrectifiable, as an easy application of the triangle inequality shows that these steps do not obstruct the steps along which (10) is satisfied and that yield a contradiction to rectifiability. It is quite surprising that, actually, the choice $N = 1$ allows requiring R' just to be larger than some universal constant for infinitely many values, not necessarily being unbounded.

It is more interesting trying to obtain explicit estimates on the growth of the sequence r_k provided R' has some nice behavior, for instance, if it grows faster than linearly. If we pay attention only to Remark 11, then this requires r_k to be of the order of a product $C^k L_k \cdots L_2 L_1$ for some universal constant $C > 1$ (where L_n is the sequence of Lipschitz constants to be discarded so that $L_n \to \infty$) provided $R'(r_k)$ is larger than $C L_k^{10} r_k$. Indeed, the value of the denominator $(d - d')$ can be bounded from below by a universal positive constant all along the construction. (Notice that (9) does not alter this issue.)

Nevertheless, there is another condition, namely (10), which is more restrictive. Indeed, the corresponding expressions $(d_1' - d_1)$ and $(d_2 - d_2')$ that do appear in the denominators cannot be bounded from below by an universal constant. They can, however, be bounded from below by a sequence of positive numbers with finite sum smaller than 1, as for instance $1/cn^{1+\alpha}$ for an appropriate constant c. This allows controlling the value of the repetitivity function R along a sequence of radii having the order of $c L_k \cdots L_1 (k!)^{1+\alpha}$, where $L_n \to \infty$.

In any case, we think that many steps of our construction can be improved. In this direction, it is very tempting thinking that, given a function R' growing faster than linearly, a nonrectifiable Delone set exists so that $R(r) \leq R'(r)$ holds for all large enough r. However, it may happen that linear repetitiveness is not the optimal condition to ensure rectifiability, and that some finite moment condition on the function R still should imply this. We do not see, however, any potential application of these seemingly hard questions.

IV Combining patches to get unique ergodicity

As we already mentioned, for a repetitive Delone set, unique ergodicity is equivalent to all patches appearing in the tiling having a well-defined asymptotic density. This is closely related to [15, Theorem 3.3], but there is an alternate way to see this. Namely, since the Delone sets that we consider are subsets of $\mathbb{Z}^2$, we can use Wiener's unique ergodicity criterion for $\mathbb{Z}^d$–subshifts (see for example [10]). That is, the $\mathbb{Z}^2$–action on the closure of the orbit of $\mathcal{D}$ is uniquely ergodic if and only if for every $\mathcal{D}'$ in this orbit closure and every patch $\mathcal{Q}$ of $\mathcal{D}'$, the limit

$$\lim_{n\to\infty} \frac{\text{number of occurrences of } \mathcal{Q} \text{ in } \mathcal{D}'|_{[-n,n]^d}}{(2n+1)^d}$$

exists and is independent of $\mathcal{D}'$. Moreover, by the proof of [15, Theorem 3.3], this condition needs to be checked only for a single $\mathcal{D}'$, say for $\mathcal{D}$. We claim that in the schema of Lemma 12, this is the case whenever all asymptotic densities of occurrences of the patches $\mathcal{P}_{m,i}$ as blocks in $\mathcal{P}_{n,i}$, with $n \to \infty$, are equal to $\frac{1}{2}$.

Lemma 15 *Assume that all asymptotic densities of occurrences of the patches $\mathcal{P}_{m,i}$ as blocks in $\mathcal{P}_{n,i}$, with $n \to \infty$, are equal to $\frac{1}{2}$. Then the limit*

$$\lim_{n\to\infty} \frac{\text{number of occurrences of } \mathcal{Q} \text{ in } \mathcal{D}|_{[-n,n]^d}}{(2n+1)^d} \tag{11}$$

exists for every patch $\mathcal{Q}$ appearing in $\mathcal{D}$.

Proof First, an easy application of a Whitney-like decomposition shows that the limit (11) exists if and only if the limit

$$\lim_{n\to\infty} \frac{\text{number of occurrences of } \mathcal{Q} \text{ in } \mathcal{P}_{n,j}}{|\mathcal{P}_{n,j}|} \tag{12}$$

exists and is independent of j. To show that the last condition holds, for each $m \geq 1$ and $j \in \{1,2\}$, denote

$$d_{m,j} := \frac{\text{number of occurrences of } \mathcal{Q} \text{ in } \mathcal{P}_{m,j}}{|\mathcal{P}_{m,j}|}.$$

Additionally, denote by $d_{i,j}^{m\to n}$ the density at which the patch $\mathcal{P}_{m,i}$ appears as a block of $\mathcal{P}_{n,j}$. Let ℓ be the side length of $\mathcal{Q}$, and assume that m is large enough so that ℓ is smaller than the side length of each $\mathcal{P}_{m,j}$. If we divide a given square $\mathcal{P}_{n,j}$ into the blocks $\mathcal{P}_{m,1}$ and $\mathcal{P}_{m,2}$, we have

$$d_{m,1}d_{1,1}^{m\to n} + d_{m,2}d_{2,1}^{m\to n} \le d_{n,1} \le d_{m,1}d_{1,1}^{m\to n} + d_{m,2}d_{2,1}^{m\to n} + \frac{2\ell}{\text{side length of } \mathcal{P}_{m,j}}$$

and

$$d_{m,1}d_{1,2}^{m\to n} + d_{m,2}d_{2,2}^{m\to n} \le d_{n,2} \le d_{m,1}d_{1,2}^{m\to n} + d_{m,2}d_{2,2}^{m\to n} + \frac{2\ell}{\text{side length of } \mathcal{P}_{m,j}}.$$

Indeed, the left-side inequalities are obvious, while in the right-side expressions, the extra term appears because of the possibility that a copy of $\mathcal{Q}_{m,j}$ overlaps with two different blocks (in either the left-to-right or bottom-to-top direction).

By hypothesis, for all fixed m and each i, j in $\{1,2\}$, the value of $d_{i,j}^{m\to n}$ converges to $\frac{1}{2}$ as $n \to \infty$. It thus follows from the inequalities above that given $\varepsilon > 0$, there exist integers m and N such that for all $n \ge N$,

$$\begin{aligned} \tfrac{1}{2}(d_{m,1} + d_{m,2}) - \varepsilon \le d_{n,1} \le \tfrac{1}{2}(d_{m,1} + d_{m,2}) + \varepsilon, \\ \tfrac{1}{2}(d_{m,1} + d_{m,2}) - \varepsilon \le d_{n,2} \le \tfrac{1}{2}(d_{m,1} + d_{m,2}) + \varepsilon. \end{aligned}$$

In particular, there exists a sequence of integers n_k such that for each k and for all $n \ge n_{k+1}$,

$$\begin{aligned} \tfrac{1}{2}(d_{n_k,1} + d_{n_k,2}) - \tfrac{1}{k} \le d_{n,1} \le \tfrac{1}{2}(d_{n_k,1} + d_{n_k,2}) + \tfrac{1}{k}, \\ \tfrac{1}{2}(d_{n_k,1} + d_{n_k,2}) - \tfrac{1}{k} \le d_{n,2} \le \tfrac{1}{2}(d_{n_k,1} + d_{n_k,2}) + \tfrac{1}{k}. \end{aligned} \tag{13}$$

As a consequence, both $(d_{n,1})$ and $(d_{n,2})$ are Cauchy sequences, hence they converge to certain limits d_1 and d_2, respectively. Letting $k \to \infty$ in (13) along $n = n_{k+1}$, we obtain

$$\tfrac{1}{2}(d_1 + d_2) \le d_1 \le \tfrac{1}{2}(d_1 + d_2),$$

hence $d_1 = d_2$, as desired. □

In order to guarantee the hypothesis of the preceding lemma and hence prove unique ergodicity of the translation action on the orbit closure of $\mathcal{D}$, we will need to crucially modify the preceding construction. As above, we will only use two types of patches at each step, and we will start with the same (lower-left corners of the) patches illustrated in Figure 2. Therefore, the density of points in the resulting Delone set will be equal to

$$\tfrac{1}{2} \cdot \tfrac{16}{16} + \tfrac{1}{2} \cdot \tfrac{10}{16} = \tfrac{13}{16}.$$

We begin by introducing the *transition matrices*

$$\mathcal{A}^{n\to n+1} = (d_{i,j}^{n\to n+1}),$$

where, as before, $d_{i,j}^{n\to n+1}$ stands for the density at which the patch $\mathcal{P}_{n,i}$ appears in $\mathcal{P}_{n+1,j}$, with i, j in $\{1,2\}$. If we let

$$\mathcal{A}^{m\to n} = \mathcal{A}^{m\to m+1}\mathcal{A}^{m+1\to m+2}\cdots\mathcal{A}^{n-1\to n}$$

and denote by $d_{i,j}^{m\to n}$ the entries of $\mathcal{A}^{m\to n}$, then $d_{i,j}^{m\to n}$ represents, as before, the density at which the patch $\mathcal{P}_{m,i}$ appears in $\mathcal{P}_{n,j}$. In particular, if d_i is the density of points in the starting patch $\mathcal{P}_{1,i}$, where $i\in\{1,2\}$, then the density of points in $\mathcal{P}_{n,i}$ equals

$$d_{n,i} := d_0\cdot d_{0,i}^{1\to n} + d_1\cdot d_{1,i}^{1\to n}.$$

To simplify, we will only work with transition matrices of the form

$$\mathcal{A}^{n\to n+1} = \tfrac{1}{2}V + \delta_n W, \tag{14}$$

where

$$V = \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix} \quad\text{and}\quad W = \begin{pmatrix} 1 & -1\\ -1 & +1\end{pmatrix}.$$

To deal with these matrices, we will strongly use the identity

$$\big(\tfrac{1}{2}V+\alpha W\big)\big(\tfrac{1}{2}V+\beta W\big) = \tfrac{1}{2}V + 2\alpha\beta W. \tag{15}$$

This shows, in particular, that for α,β between 0 and $\frac{1}{2}$, the $\|\cdot\|_\infty$ distance between

$$\big(\tfrac{1}{2}V+\alpha W\big)\big(\tfrac{1}{2}V+\beta W\big) \quad\text{and}\quad \tfrac{1}{2}V$$

is less than or equal to 2β times the $\|\cdot\|_\infty$ distance between

$$\tfrac{1}{2}V+\alpha W \quad\text{and}\quad \tfrac{1}{2}V.$$

As before, we start the construction with the (lower-left corners of the) patches illustrated in Figure 2. Next, we proceed by induction: assuming that we have constructed the patches $\mathcal{Q}_{n,1} =: \mathcal{Q}_1$ and $\mathcal{Q}_{n,2} =: \mathcal{Q}_2$, we let $\mathcal{Q}'_{n+1,1} := \mathcal{Q}_1^{\mathrm{new}}$ and $\mathcal{Q}'_{n+1,2} := \mathcal{Q}_2^{\mathrm{new}}$, where we have implemented the construction of new patches of the preceding section for the constant $L_n := n$ and

$$d'_2 := d_{n,2} - \tfrac{1}{3}(d_{n,2}-d_{n,1}), \quad d'_1 = d_{n,1} + \tfrac{1}{3}(d_{n,2}-d_{n,1}).$$

By construction, this procedure consists of a certain number $\ell = \ell_n$ of intermediate steps along which all transition matrices are of the form (14). In particular, by the previous discussion, we did not lose any amount of closeness to the desired limit matrix $\frac{1}{2}V$ along this construction.

Next, to construct $\mathcal{Q}_{n+1,1}$ and $\mathcal{Q}_{n+1,2}$, we mix (and match) $\mathcal{Q}'_{n+1,1}$ and $\mathcal{Q}'_{n+1,2}$ appropriately, as shown in Figure 3.

$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,2}$
$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,1}$
$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,2}$

$\mathcal{Q}_{n+1,1}$

$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,1}$
$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,2}$
$\mathcal{Q}'_{n+1,1}$	$\mathcal{Q}'_{n+1,2}$	$\mathcal{Q}'_{n+1,1}$

$\mathcal{Q}_{n+1,2}$

Figure 3: Building $\mathcal{Q}_{n+1,1}$ and $\mathcal{Q}_{n+1,2}$ starting with $\mathcal{Q}'_{n+1,1}$ and $\mathcal{Q}'_{n+1,2}$

Letting $\mathcal{P}_{n+1,i}$ be the lower-left corner of $\mathcal{Q}_{n+1,i}$, with $i \in \{1,2\}$, we have that the density of $\mathcal{P}'_{n+1,1}$ inside $\mathcal{P}_{n+1,1}$ (resp. $\mathcal{P}_{n+1,2}$) equals $\frac{5}{9} = \frac{1}{2} + \frac{1}{18}$ $\left(\text{resp. } \frac{4}{9} = \frac{1}{2} - \frac{1}{18}\right)$. Similarly, we have that the density of $\mathcal{P}'_{n+1,2}$ inside $\mathcal{P}_{n+1,1}$ (resp. $\mathcal{P}_{n+1,2}$) is equal to $\frac{4}{9} = \frac{1}{2} - \frac{1}{18}$ $\left(\text{resp. } \frac{5}{9} = \frac{1}{2} + \frac{1}{18}\right)$.

By the construction, the transition matrix from the patch $\mathcal{P}_{n,i}$ (hence from any $\mathcal{P}_{m,i}$, with $m \leq n$) to each $\mathcal{P}_{n+1,j}$ is of the form (14). In particular, we have $d_{n,2} > d_{n,1}$ for all n. Moreover, because of (15), the $\|\cdot\|_\infty$ distance between any transition matrix $\mathcal{M}^{m \to n+1}$ and $\frac{1}{2}V$ is less than or equal to $\frac{1}{9}$ times the $\|\cdot\|_\infty$ distance between the transition matrix $\mathcal{M}^{m \to n}$ and $\frac{1}{2}V$. Letting n go to infinity, this yields the desired convergence.

V Prescribing the (shape of the) set of invariant probability measures

There are many ways to realize arbitrary Choquet simplices, one of which is given by the next lemma. For the statement, given positive integers k, q, we let $\triangle(k,q)$ be the convex hull of the set of vectors $e_1/q, \ldots, e_k/q$, where $\{e_1, \ldots, e_k\}$ stands for the canonical orthonormal basis of $\mathbb{R}^k$.

Lemma 16 *Let $\mathcal{K}$ be a Choquet simplex, let (q_n) be an increasing sequence of positive integers such that each q_n divides q_{n+1}, and let (r_n) be a sequence of positive integers satisfying $r_n\sqrt{q_n} < \sqrt{q_{n+1}}$. Then there exists a sequence (A_n) of $k_n \times k_{n+1}$ matrices with positive integer entries such that, passing to a subsequence of (q_n) if necessary (as well as to the corresponding subsequence of (r_n)), we have:*

(K1) $k_1 = \max\{3, d\}$ *if $\mathcal{K}$ has dimension d, and $k_1 = 3$ if $\mathcal{K}$ has infinite dimension.*

(K2) $k_n \geq 3$, *for all n.*

(K3) $A_n(1, j) = 1$, *for every* $j \in [\![1, k_{n+1}]\!]$.

(K4) $\sum_{i=1}^{k_n} A_n(i, j) = q_{n+1}/q_n$, *for every* $j \in [\![1, k_{n+1}]\!]$.

(K5) $\min\{A_n(i, j) : 2 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\} \geq k_{n+1}$.

(K6) $\min\{A_n(i, j) : 2 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\} \geq r_n\sqrt{q_{n+1}}$.

(K7) *$\mathcal{K}$ is affine homeomorphic to the inverse limit*

$$\varprojlim_n(\triangle(k_n, q_n), A_n) := \left\{(u_n) \in \prod_{n\geq 1} \triangle(k_n, p_n) : A_n(u_{n+1}) = u_n,\ \textit{for all}\ n\right\}.$$

Proof By [6, Lemmas 9 and 13], there exists a sequence (B_ℓ) of $k_\ell \times k_{\ell+1}$ matrices with positive integer entries such that $k_\ell \geq 2$ for all ℓ, satisfying **(K4)**, **(K7)** and

$$k_{\ell+1} \leq \min\{B_\ell(i, j) : 1 \leq i \leq k_\ell, 1 \leq j \leq k_{\ell+1}\}.$$

Next, notice that since all matrix entries are ≥ 1, using **(K3)** we easily obtain by induction that for every $m > m'$, all $i \in [\![1, k_{m'}]\!]$ and all $j \in [\![1, k_{m+1}]\!]$,

$$B_{m'} \cdots B_m(i, j) \geq \frac{q_{m+1}}{q_{m'+1}}.$$

Let $\ell_1 := 1$, and given ℓ_n, define ℓ_{n+1} so that $q_{\ell_{n+1}} > (1 + q_{\ell_n+1})^2 r_{\ell-n}^2$. Then, the matrices $\widetilde{A}_n := B_{\ell_n} \cdots B_{\ell_{n+1}-1}$ satisfy **(K4)**, **(K7)** and

$$\min\{\widetilde{A}_n(i, j) : 1 \leq i \leq k_{\ell_n}, 1 \leq j \leq k_{\ell_{n+1}}\} \geq \max\{k_{\ell_{n+1}}, r_{\ell_n}\sqrt{q_{\ell_{n+1}}}\}.$$

Finally, defining A_n as the $(k_{\ell_n} + 1) \times (k_{\ell_{n+1}} + 1)$ matrix with columns

$$(A_n(\cdot, 1)) = (A_n(\cdot, 2)) = \begin{pmatrix} 1 \\ \widetilde{A}_n(1, 1) - 1 \\ \widetilde{A}_n(2, 1) \\ \vdots \\ \widetilde{A}_n(k_{\ell_n}, 1) \end{pmatrix} \quad \text{and} \quad (A_n(\cdot, k + 1)) = \begin{pmatrix} 1 \\ \widetilde{A}_n(1, k) - 1 \\ \widetilde{A}_n(2, k) \\ \vdots \\ \widetilde{A}_n(k_{\ell_n}, k) \end{pmatrix},$$

where $2 \leq k \leq k_{\ell_n}$, we have that all properties **(K2)**, **(K3)**, **(K4)**, **(K5)** and **(K6)** are satisfied with respect to the subsequences $(q_{\ell_n}), (r_{\ell_n})$. By [6, Lemmas 1 and 2], property **(K7)** is also satisfied. Finally, property **(K1)** follows from [6, Lemma 9] and the proof of [6, Lemma 13] (this is independent of the choice of (q_n)). □

In all that follows, we will assume that $\mathcal{K}$ is not reduced to a singleton. In other words, we will search for the construction of a translation action over the orbit of a nonrectifiable Delone set that is not uniquely ergodic, the uniquely ergodic case having been settled in the previous section.

Lemma 17 *With the notation above, assume that $\mathcal{K}$ is not reduced to a singleton. Then there exist positive integers $m' \geq m$ and $i_0 \in [\![1, k_m]\!]$ as well as real numbers $\bar{d} > \bar{d}'$ in $]0,1[$ such that for every $n \geq m'$, there exist j_{n+1}, j'_{n+1} in $[\![1, k_{n+1}]\!]$ satisfying*

$$\frac{A_m \cdots A_n(i_0, j_{n+1})}{q_{n+1}} \geq \bar{d} \quad \text{and} \quad \frac{A_m \cdots A_n(i_0, j'_{n+1})}{q_{n+1}} \leq \bar{d}'.$$

Proof Since $\mathcal{K}$ has at least two extreme points, there exist $(u_n), (v_n)$ in the inverse limit $\lim_{\leftarrow n}(\triangle(k_n, q_n), A_n)$ such that for some positive integers m and i in $[\![1, k_m]\!]$, the i^{th}–coordinates $u_{m,i}$ and $v_{m,i}$ of u_m and v_m, respectively, are different. For each $n > m$, we set

$$\alpha_n = \max\left\{ \left| \frac{A_m \cdots A_n(i,r)}{q_{n+1}} - \frac{A_m \cdots A_n(i,s)}{q_{n+1}} \right| : r, s \text{ in } [\![1, k_{n+1}]\!] \right\}.$$

Suppose for a contradiction that there exists a subsequence (α_{n_ℓ}) converging to zero. Then for every $j \in [\![1, k_{n_\ell+1}]\!]$, there exists $\delta_{\ell,j} \in [-\alpha_{n_\ell}, \alpha_{n_\ell}]$ such that

$$\frac{A_m \cdots A_{n_\ell}(i,j)}{q_{n_\ell+1}} = \frac{A_m \cdots A_{n_\ell}(i,1)}{q_{n_\ell+1}} + \delta_{\ell,j}.$$

Therefore,

$$\begin{aligned} u_{m,i} &= \sum_{j=1}^{k_{n_\ell+1}} \frac{A_m \cdots A_{n_\ell}(i,j)}{q_{n_\ell+1}} q_{n_\ell+1} u_{n_\ell,j} \\ &= \sum_{j=1}^{k_{n_\ell+1}} \left(\frac{A_m \cdots A_{n_\ell}(i,1)}{q_{n_\ell+1}} + \delta_{\ell,j} \right) q_{n_\ell+1} u_{n_\ell,j} \\ &= \frac{A_m \cdots A_{n_\ell}(i,1)}{q_{n_\ell+1}} + \sum_{j=1}^{k_{n_\ell+1}} \delta_{\ell,j} q_{n_\ell+1} u_{n_\ell,j} \end{aligned}$$

and

$$v_{m,i} = \frac{A_m \cdots A_{n_\ell}(i,1)}{q_{n_\ell+1}} + \sum_{j=1}^{k_{n_\ell+1}} \delta_{\ell,j} q_{n_\ell+1} v_{n_\ell,j}.$$

Thus we get

$$|u_{m,i} - v_{m,i}| \leq \sum_{j=1}^{k_{n_\ell+1}} |\delta_{\ell,j}| q_{n_\ell+1}(u_{n_\ell,j} + v_{n_\ell,j}) \leq 2\alpha_{n_\ell},$$

which contradicts the fact that $u_{m,i} \neq v_{m,i}$. □

Let $\mathcal{K}$ be a Choquet simplex not reduced to a singleton, and let (p_n) be a sequence of positive integers such that $p_1 = \max\{4, d\}$ for d–dimensional $\mathcal{K}$, $p_1 = 4$ for infinite-dimensional $\mathcal{K}$, and such that for every $n \geq 1$, one has $p_{n+1} = 2(l_n + 1)p_n$ for an integer $l_n \geq 1$. Let (A_n) be a sequence of $k_n \times k_{n+1}$ matrices with positive integer entries verifying the properties of Lemma 16 with respect to $q_n := p_n^2$. Let $m' \geq m$, i_0 in $[\![1, k_m]\!]$, $d > d'$ in $]0,1[$ and j_{n+1}, j'_{n+1} in $[\![1, k_{n+1}]\!]$ be as in Lemma 17, where $n \geq m'$. Observe that we can (and we will) assume that $m = 1$ and that both j_{n+1} and j'_{n+1} are greater than or equal to 2 (the latter assumption is possible because the first two columns of each matrix A_n are equal). Let (r_n) be a sequence of positive integers such that $r_n p_n < p_{n+1}$, for all n.

We set $F_1 := [\![0, p_1 - 1]\!]^2$, and for $n \geq 1$, we let

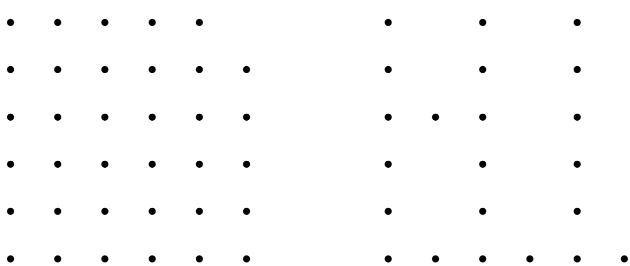
$$F_{n+1} := \bigcup_{v \in [\![-l_n - 1, l_n]\!]^2} (F_n + p_n v).$$

Next, we define the patch

$$\mathcal{P}_{1,i_0} := F_1 \setminus \{(p_1 - 1, p_1 - 1)\}.$$

For $k \in [\![1, k_1]\!] \setminus \{i_0\}$, the patch $\mathcal{P}_{1,k}$ is defined as (see Figure 4)

$$\mathcal{P}_{1,k} := \{(i,j) \in F_1 : i \text{ is even}\} \cup \{(i,j) \in F_1 : j = 0\} \cup \{(1,k)\}.$$



$\mathcal{P}_{1,i_0}$ $\qquad$ $\mathcal{P}_{1,k}$ for $k = 3 \neq i_0$

Figure 4: The patches $\mathcal{P}_{1,k}$ for $p_1 = 6$, $k_1 \geq 3$ and $i_0 \neq 3$

We next proceed to define patches $\mathcal{P}_{2,1},\dots,\mathcal{P}_{2,k_2}$ in $\{0,1\}^{F_2}$ satisfying:

- $\mathcal{P}_{2,j}\cap(F_1+(l_1p_1,l_1p_1))=\mathcal{P}_{1,1}$, for each $j\in[\![1,k_2]\!]$ (that is, the upper-right corner of each $\mathcal{P}_{2,j}$ is a copy of $\mathcal{P}_{1,1}$).
- $\mathcal{P}_{2,j}\cap(F_1+vp_1)$ belongs to $\{\mathcal{P}_{1,1},\dots,\mathcal{P}_{1,k_1}\}$, for every $j\in[\![1,k_2]\!]$ and for all $v\in[\![-l_1-1,l_1]\!]^2$.
- For all $i\in[\![1,k_1]\!]$ and all $j\in[\![1,k_2]\!]$, the number of vectors $v\in[\![-l_1-1,l_1]\!]^2$ such that $\mathcal{P}_{2,j}\cap(F_1+vp_1)=\mathcal{P}_{1,i}$ equals $A_1(i,j)$.

In order to check that it is possible to obtain k_2 different patches satisfying these three properties, just observe that the number of different ways to define a single patch $\mathcal{P}_{2,k}$ satisfying all of them equals

$$\frac{\left(\sum_{i=2}^{k_1}A_1(i,k)\right)!}{A_1(2,k)!\cdots A_1(k_1,k)!}\geq\min\{A_1(i,j):2\leq i\leq k_1,1\leq j\leq k_2\}\geq k_2.$$

Now, suppose that for $n\geq 2$, we have defined a collection $\mathcal{P}_{n,1},\dots,\mathcal{P}_{n,k_n}$ of different patches in $\{0,1\}^{F_n}$. We will next proceed to define k_{n+1} different patches $\mathcal{P}_{n+1,1},\dots,\mathcal{P}_{n+1,k_{n+1}}$ in $\{0,1\}^{F_{n+1}}$ such that for all $k\in[\![1,k_{n+1}]\!]$, the following properties are satisfied (see Figure 5):

(P1) $\mathcal{P}_{n+1,k}\cap(F_n+(l_np_n,l_np_n))=\mathcal{P}_{n,1}$.

(P2) For all $s\in[\![-l_n-1,l_n]\!]$ and $r\in[\![-l_n-1,-l_n+r_n-2]\!]$, it holds that

$$\mathcal{P}_{n+1,k}\cap(F_n+(sp_n,rp_n))=\begin{cases}\mathcal{P}_{n,j_n} & \text{if } [sp_{n+1}/r_n] \text{ is even},\\ \mathcal{P}_{n,j_n'} & \text{if } [sp_{n+1}/r_n] \text{ is odd}.\end{cases}$$

(P3) $\mathcal{P}_{n+1,k}\cap(F_n+vp_n)$ belongs to $\{\mathcal{P}_{n,1},\dots,\mathcal{P}_{n,k_n}\}$, for every $v\in[\![-l_n-1,l_n]\!]^2$.

(P4) The number of $v\in[\![-l_n-1,l_n]\!]^2$ such that $\mathcal{P}_{n+1,k}\cap(F_n+vp_n)=\mathcal{P}_{n,i}$ is equal to $A_n(i,k)$.

Notice that properties **(P1)** and **(P2)** completely determine how to fill $(p_{n+1}/p_n)r_n+1$ translated copies of F_n. We thus need to fill, in different ways, the remaining (free) $p_{n+1}^2/p_n^2-(p_{n+1}/p_n)r_n-1$ translated copies of F_n in a way that **(P4)** is satisfied. To do this, notice that if p_{n+1} is sufficiently large, namely

$$p_{n+1}>\frac{(k_n-1)p_n^2}{k_n-2}\left(\frac{r_n}{p_n}+1\right), \tag{16}$$

Figure 5: Building the patches $\mathcal{P}_{n+1,k}$: the white part must be filled according to the rules **(P3)** and **(P4)**, and the dashed lines indicate that we do not overlap patches as in the previous sections.

then

$$(k_n-2)\frac{p_{n+1}^2}{p_n^2} > (k_n-1)\frac{p_{n+1}r_n}{p_n} + (k_n-1)p_{n+1},$$

which implies

$$\begin{aligned}(k_n-1)\left(\frac{p_{n+1}^2}{p_n^2} - \frac{p_{n+1}r_n}{p_n} - 1\right) &> (k_n-1)\left(\frac{p_{n+1}^2}{p_n^2} - \frac{p_{n+1}r_n}{p_n} - p_{n+1}\right)\\ &> \frac{p_{n+1}^2}{p_n^2} > \sum_{i=2}^{k_n} A_n(i,j)\\ &\geq (k_n-1)\min\{A_n(i,j) : 1 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\}.\end{aligned}$$

Using this and **(K5)**, we obtain

$$\frac{p_{n+1}^2}{p_n^2} - \frac{p_{n+1}r_n}{p_n} - 1 > k_{n+1}.$$

Next, we notice that among the free translated copies of F_n, the number of those that have to be filled by copies of $\mathcal{P}_{n,j_n}$ (resp. $\mathcal{P}_{n,j'_n}$) equals

$$A_n(j_n,k) - \frac{r_n p_{n+1}}{2p_n} \geq r_n p_{n+1} - r_n \frac{p_{n+1}}{2p_n} > 0$$
$$\left(\text{resp. } A_n(j'_n,k) - \frac{r_n p_{n+1}}{2p_n} \geq r_n p_{n+1} - r_n \frac{p_{n+1}}{2p_n} > 0\right).$$

This easily allows producing patches $\mathcal{P}_{n+1,1},\dots,\mathcal{P}_{n+1,k_{n+1}}$ that do satisfy **(P4)** and differ from each other in the places where we put some patches $\mathcal{P}_{n,j_n}$, $\mathcal{P}_{n,j'_n}$ in a fixed family of k_n free translated copies of F_n.

Having defined all patches $\mathcal{P}_{i,j}$, let us now consider the family of sets

$$X_n := \{D \subseteq \mathbb{Z}^2 : D \cap (F_n + v) \in \{\mathcal{P}_{n,1},\dots,\mathcal{P}_{n,k_n}\}, \text{ for every } v \in p_n\mathbb{Z}^2\}.$$

It is clear that (X_n) is a nested sequence of nonempty compact sets, hence their intersection is nonempty. Moreover, every element in this intersection is a Delone set that satisfies the $2\mathbb{Z}$–property. Fix such a set $\mathcal{D}$, and let X be the closure of its orbit with respect to the translation action of $\mathbb{Z}^2$ (equivalently, of $\mathbb{R}^2$). For $n \geq 1$ and for $k \in [\![1,k_n]\!]$, we set

$$C_{n,k} := \{D \in X : D \cap F_n = \mathcal{P}_{n,k}\}.$$

By the construction,

$$U_n := \{C_{n,k} + v : 1 \leq k \leq k_n, v \in F_n\}$$

is a clopen covering of X. We claim that it is actually a partition of X. To show this, let us first consider the case of U_1. For all $D \in X_1$ and all $v \in p_1\mathbb{Z}^2$, the intersection $D \cap (F_1 + v)$ belongs to $\{\mathcal{P}_{1,1},\dots,\mathcal{P}_{1,k_1}\}$. If two atoms of U_1, say $C_{1,k} + v$ and $C_{1,k'} + v'$, have nonempty intersection, then letting $u := v - v'$, we have that $C_{1,k} + u$ intersects $C_{1,k'}$. Then, by looking at all possible intersections and having in mind the geometry of the patches $\mathcal{P}_{1,k}$, one is easily convinced that u must belong to $p_1\mathbb{Z}^2$. Since both v and v' lie in F_1, this implies that $u = 0$, hence $v = v'$, and finally $k = k'$. The proof for (U_n) works by induction. Assuming that U_{n-1} is a partition, a similar argument applies taking into account that the unique position in which $\mathcal{P}_{n-1,1}$ appears in each patch $\mathcal{P}_{n,k}$ is the upper-right corner.

Next, let μ be an invariant probability measure for the translation action of $\mathbb{Z}^2$ on X. We claim that the vectors of the μ–measures, namely

$$\mu_n := (\mu(C_{n,1}),\dots,\mu(C_{n,k_n})),$$

satisfy $\mu_n^T = A_n(\mu_{n+1}^T)$, for every $n \geq 1$. Indeed, we have

$$\begin{aligned}\mu(C_{n,i}) &= \mu\Big(\bigcup_{k=1}^{k_{n+1}} \{C_{n+1,k}+v : v \in F_{n+1}, C_{n+1,k}+v \subseteq C_{n,i}\}\Big)\\ &= \sum_{k=1}^{k_{n+1}} |\{v \in F_{n+1} : C_{n+1,k}+v \subseteq C_{n,i}\}| \cdot \mu(C_{n+1,k})\\ &= \sum_{k=1}^{k_{n+1}} A_n(i,k) \cdot \mu(C_{n+1,k}),\end{aligned}$$

which shows our claim.

We can thus consider the sequence (μ_n) as a point in $\lim_{\leftarrow n}(\triangle(k_n, p_n^2), A_n)$. Notice that the function $\mu \mapsto (\mu_n)$ from the set of invariant probability measures into the space $\lim_{\leftarrow n}(\triangle(k_n, p_n^2), A_n)$ is affine. We claim that it is a bijection. Indeed, on the one hand, given (u_n) in $\lim_{\leftarrow n}(\triangle(k_n, p_n^2), A_n)$, we may produce a probability measure μ on X by letting $\mu(C_{n,k}+v) = u_n(k)$, for every $k \in [\![1, k_n]\!]$ and all $v \in F_n$. It is the not hard to check that μ is invariant under the translation action (see [5, Lemma 5]), thus showing the surjectivity of the map. On the other hand, to check that it is injective, consider the set

$$X^* := \bigcup_{w \in \mathbb{Z}^2} \bigcap_{n \geq 1} \bigcup_{k=1}^{k_n} \bigcup_{v \in F_n \setminus F_n - w} (C_{n,k}+v).$$

This set contains all points of X (if any) that are not separated by the partitions (U_n). Indeed, if D, D' are two such points, then for each $n \geq 1$ they belong to the same atom $C_{n,i_n}+v_n$ in U_n. If D, D' are different, then there is $w \in \mathbb{Z}^2$ contained only in one of them. Thus, $D+w$ and $D+w'$ differ at the origin, and therefore $C_{n,i_n}+v_n+w$ cannot be an atom of U_n. This implies that $v_n+w \notin F_n$, that is $v_n \in F_n \setminus F_n - w$, which shows our claim.

Using the fact that (F_n) is a Følner sequence, one can easily check that $\mu(X^*) = 0$ for every invariant probability measure μ. Indeed, for all $n \geq 1$ and all fixed $w \in \mathbb{Z}^2$,

$$\begin{aligned}\mu\Big(\bigcup_{k=1}^{k_n} \bigcup_{v \in F_n \setminus F_n - w} (C_{n,k}+v)\Big) &= \sum_{k=1}^{k_n} |F_n \setminus F_n - w| \cdot \mu(C_{n,k})\\ &= |F_n \setminus F_n - w| \cdot \sum_{k=1}^{k_n} \mu(C_{n,k}) = \frac{|F_n \setminus F_n - w|}{|F_n|} \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

where the last equality holds since U_n is a partition of X. Thus, any given clopen set C can be written as the union $C_1 \cup C_2$, where C_1 is a (countable) union of atoms of (U_n) and C_2 is a subset of X^*. This shows that any probability measure μ on X that is invariant under the translation action of $\mathbb{Z}^2$ is completely determined by the sequence (μ_n), thus showing the desired injectivity.

We can now finish our construction. To do this, we consider the sequence (p_n) defined by $p_1 := \max\{4, d\}$ if $\mathcal{K}$ is d–dimensional, $p_1 := 4$ if $\mathcal{K}$ is infinite-dimensional, and $p_{n+1} := 2n!(p_n)^2$, for all $n \geq 1$. (This definition ensures property (16).) Then we let $r_n := n!$, and we realize $\mathcal{K}$ as an inverse limit $\lim_{\leftarrow n}(\triangle(k_n, q_n), A_n)$, where $q_n := p_n^2$. Next, we perform the preceding construction for this realization. We thus obtain a Delone set $\mathcal{D}$ satisfying the $2\mathbb{Z}$–property and such that the set of invariant probability measures for the $\mathbb{Z}^2$–action on the closure of its orbit is affine isomorphic to $\mathcal{K}$. It remains to show that $\mathcal{D}$ is nonrectifiable. To do this, we will need the next lemma:

Lemma 18 *There exist $d > d'$ in $]0,1[$ such that for every $n > m'$,*

$$|\mathcal{P}_{n,j_n}| \geq p_n^2 d > p_n^2 d' \geq |\mathcal{P}_{n,j_n'}|.$$

Proof First notice that $|\mathcal{P}_{1,i_0}| = p_1^2 - 1$ and that for every $k \in [\![1, k_1]\!] \setminus \{i_0\}$,

$$|\mathcal{P}_{1,k}| = \tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1.$$

Thus for every $n \geq 1$ and $k \in [\![1, k_n]\!]$, we have

$$\begin{aligned} |\mathcal{P}_{n,k}| &= A_1 \cdots A_{n-1}(i_0, k)\big(p_1^2 - 1\big) + \left(\frac{p_n^2}{p_1^2} - A_1 \cdots A_{n-1}(i_0, k)\right)\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big) \\ &= A_1 \cdots A_{n-1}(i_0, k)\big(\tfrac{1}{2} p_1^2 - \tfrac{1}{2} p_1 - 2\big) + \frac{p_n^2}{p_1^2}\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big). \end{aligned}$$

By Lemma 17, for every $n > m'$,

$$\begin{aligned} |\mathcal{P}_{n,j_n}| &\geq \bar{d}\, p_n^2\big(\tfrac{1}{2} p_1^2 - \tfrac{1}{2} p_1 - 2\big) + \frac{p_n^2}{p_1^2}\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big) \\ &> \bar{d}'\, p_n^2\big(\tfrac{1}{2} p_1^2 - \tfrac{1}{2} p_1 - 2\big) + \frac{p_n^2}{p_1^2}\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big) \\ &\geq A_1 \cdots A_{n-1}(i_0, j_n')\big(\tfrac{1}{2} p_1^2 - \tfrac{1}{2} p_1 - 2\big) + \frac{p_n^2}{p_1^2}\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big) \\ &= |\mathcal{P}_{n,j_n'}|. \end{aligned}$$

Thus, letting

$$d := \bar{d}\big(\tfrac{1}{2} p_1^2 - \tfrac{1}{2} p_1 - 2\big) + \frac{1}{p_1^2}\big(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} p_1 + 1\big)$$

and

$$d' := \overline{d}'\big(\tfrac{1}{2}p_1^2 - \tfrac{1}{2}p_1 - 2\big) + \frac{1}{p_1^2}\big(\tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_1 + 1\big),$$

we get the desired property. □

To conclude, we write $p_{n+1} = 2p_n(n!p_n)$ and in reference to Proposition 10, we identify $n!p_n$ with M (which is a multiple of $P_* p_n$ for any prescribed P_* provided n is large enough) and p_n with N. Then, an application of Proposition 10 along the lines of the proof of Lemma 13 allows showing that $\mathcal{D}$ is not L–bi-Lipschitz equivalent to $\mathbb{Z}^2$ for any prescribed L, hence nonrectifiable.

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