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Towards a dynamical interpretation of Hamiltonian spectral invariants on surfaces

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Inspired by Le Calvez's theory of transverse foliations for dynamical systems on surfaces, we introduce a dynamical invariant, denoted by $\mathcal{N}$, for Hamiltonians on any surface other than the sphere. When the surface is the plane or is closed and aspherical, we prove that on the set of autonomous Hamiltonians this invariant coincides with the spectral invariants constructed by Viterbo on the plane and Schwarz on closed and aspherical surfaces.

Along the way, we obtain several results of independent interest: we show that a *formal* spectral invariant, satisfying a minimal set of axioms, must coincide with $\mathcal{N}$ on autonomous Hamiltonians, thus establishing a certain uniqueness result for spectral invariants; we obtain a "max formula" for spectral invariants on aspherical manifolds; we give a very simple description of the Entov–Polterovich partial quasi-state on aspherical surfaces, and we characterize the heavy and super-heavy subsets of such surfaces.

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1 Introduction

Let (M,ω) denote an aspherical symplectic manifold. Recall that being aspherical means $\omega|_{\pi_2} = c_1|_{\pi_2} = 0$, where c_1 is the first Chern class of M. We allow M to be either the euclidean space $\mathbb{R}^{2n}$ with its standard symplectic structure or a closed and connected symplectic manifold. As a consequence of the theory of spectral invariants, one can associate to every smooth Hamiltonian H a real number $c(H)$ referred to as the spectral invariant of H. This number is, roughly speaking, the action level at which the fundamental class $[M]$ appears in the Floer homology of the Hamiltonian H.[1] These invariants were introduced by Viterbo [46] for $M = \mathbb{R}^{2n}$ using generating function theory and by Schwarz [40] for closed aspherical symplectic manifolds using

[1] Similarly, one can associate spectral invariants to other homology classes of M as well. The focus of this article is on the invariant associated to the fundamental class.

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Hamiltonian Floer theory.[2] Spectral invariants have had many important and interesting applications in symplectic topology and dynamical systems; see for example Entov and Polterovich [7; 8], and Ginzburg [13]. A recently discovered application which has largely motivated this article is a simple solution to the displaced disks problem of Béguin, Crovisier and Le Roux: using the spectral invariant c one can show that arbitrarily C^0–small area-preserving homeomorphisms of a closed surface cannot displace disks of a given area; see Seyfaddini [41] and Dore–Hanlon [5].

One drawback of the spectral invariant c is the complexity of its construction which relies on the difficult machinery of Floer theory. As a consequence, despite its widespread use, c can only be computed in a handful of scenarios where the Floer theoretic picture is simple enough.

Motivated by the resolution of the displaced disks problem, we introduce a new invariant $\mathcal{N}$ on aspherical surfaces which, like c, associates a real number to every Hamiltonian. The construction of $\mathcal{N}$ is purely dynamical and is far more elementary than that of c. We then prove that $\mathcal{N}$ and c coincide on autonomous Hamiltonians.

An intriguing aspect of this work is that, beyond spectrality, the obvious properties of $\mathcal{N}$ are quite different from the known properties of c. Indeed, $\mathcal{N}$ is computable in practice for autonomous Hamiltonians. Furthermore, one can easily see that it satisfies a certain maximum formula which was not known for c. On the other hand, $\mathcal{N}$ does not *a priori* seem to share the continuity properties of c (see Definition 3 below). Proving that c and $\mathcal{N}$ coincide consists of two main components which are perhaps of their own independent interest: First, we prove that c satisfies the same max formula as $\mathcal{N}$. Second, we show that a "formal" spectral invariant satisfying a minimal set of axioms must coincide with $\mathcal{N}$ on autonomous Hamiltonians. This establishes a certain uniqueness result for spectral invariants which would be interesting to pursue in more general settings. See Theorem 4.

As a byproduct of our work, we obtain a very simple description of the Entov–Polterovich partial quasi-state on closed aspherical surfaces using which we characterize heavy and super-heavy subsets of these surfaces.

An inspirational factor in writing this article has been our hope of better understanding the link between Hamiltonian Floer theory and Le Calvez's theory of transverse foliations for dynamical systems on surfaces [24; 25]. In a sense, as far as surfaces are concerned, the two theories appear to be equivalent: much of what can be done via one theory can also be achieved via the other. As examples of this phenomenon, one could

[2] In [29], Oh extended Schwarz's work to arbitrary closed symplectic manifolds. See Frauenfelder and Schlenk [12] and Lanzat [23] for extensions to other types of symplectic manifolds.

point to proofs of the Arnol'd conjecture and recent articles by Bramham [2; 3] and Le Calvez [26]. A prominent missing link from this hypothetical equivalence is the spectral invariant c which to this date has had no analogue in Le Calvez's theory. The introduction of $\mathcal{N}$ in this article is an attempt to recover spectral invariants, and the solution to the displaced disks problem, via the techniques of transverse foliations. Of course, whether $\mathcal{N}$ coincides with c on all Hamiltonians, and not just the autonomous ones, is a glaring open question which we hope to answer in the future.

Acknowledgments This work began after the crucial insight by Patrice Le Calvez, following a talk by the third author on the solution of the displaced disks problem, that the spectral norm γ could be equal to the quantity

$$\inf\Big\{\Big(\sup_{x\in X}\mathcal{A}_H(x)-\inf_{x\in X}\mathcal{A}_H(x)\Big):X\text{ maximal unlinked set for }\phi_H^1\Big\}$$

(see below for the definitions). This formula is still a conjecture. In addition to this seminal proposal, Patrice's theory of equivariant Brouwer foliations is both a powerful tool and an exciting motivation to understand the link between unlinked sets and spectral invariants. We also owe to him the suggestion that the existence of maximal unlinked sets for diffeomorphisms could be proved using Handel's lemma. We warmly thank him for all this!

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1.1 The invariant $\mathcal{N}$

We work with a symplectic surface Σ which is either the plane $\mathbb{R}^2$ or a closed surface other than the sphere. Let $S^1=\mathbb{R}/\mathbb{Z}$, and denote by $C^\infty(S^1\times\Sigma)$ the set of compactly supported, 1–periodic in time, Hamiltonians on Σ, and by $C^\infty(\Sigma)$ the set of compactly supported autonomous Hamiltonians. Let $H \in C^\infty(S^1 \times \Sigma)$. Our sign convention is that the Hamiltonian vector field X_H is induced by H via $\omega(X_H^t, \cdot) = -dH_t$ for all t. Integrating the time-dependent vector field X_H^t yields a Hamiltonian isotopy $(\phi_H^t)_{t\in[0,1]}$. Recall that a diffeomorphism is said to be nondegenerate if its graph intersects the diagonal transversely. A Hamiltonian is called nondegenerate if the time-1 map of its flow is nondegenerate.

The main ingredient in the definition of $\mathcal{N}$ is the notion of *unlinked sets*. We consider fixed points of ϕ_H^1 whose trajectories under the Hamiltonian isotopy are contractible in Σ; these are referred to as contractible fixed points. An *unlinked set* is a set X of contractible fixed points of ϕ_H^1 for which there exists an isotopy $(f_t)_{t\in[0,1]}$, $f_0 = \mathrm{Id}$, $f_1 = \phi_H^1$, such that every point of X is fixed by every f_t. Unlinked sets play a crucial role in Le Calvez's theory of transverse foliations for dynamical systems on surfaces. We will say an unlinked set X is *negative* if for every x in X, the direction of every tangent vector at x is either fixed or is turned in the negative direction by the isotopy $(f_t)_{t\in[0,1]}$. We denote by $\mathrm{mnus}(H)$ the family of negative unlinked sets that are maximal with respect to the inclusion among negative unlinked sets. Finally, the invariant $\mathcal{N}$ is defined by the formula

$$\mathcal{N}(H) = \inf_{X\in \mathrm{mnus}(H)} \sup_{x\in X} \mathcal{A}_H(x),$$

where $\mathcal{A}_H(x)$ denotes the symplectic action (see Section 2 for details).

An interesting aspect of the invariant $\mathcal{N}$ is that it is defined directly for all smooth Hamiltonians while spectral invariants are first constructed for nondegenerate Hamiltonians and then extended to all Hamiltonians by a limiting process. Regarding computational issues, note that for a generic Hamiltonian function H, the map ϕ_H^1 has a finite number of fixed points. For a finite set X of contractible fixed points, the unlinkedness is equivalent to the triviality of the braid $(\phi_H^t X)_{t\in[0,1]}$ (see Section 2). Then the value of $\mathcal{N}$ depends only on the total braid associated to the set of all contractible fixed points, colored with the value of the action at each contractible fixed point. In particular, the types of braids generated by autonomous Hamiltonian functions are very constrained, and we provide a recursive formula that makes explicit computations easy (see Propositions 28 and 39 below).

In the following theorem, the function $c\colon C^\infty(S^1\times\Sigma) \to \mathbb{R}$ denotes either the spectral invariant defined via generating function theory (when $\Sigma = \mathbb{R}^2$) or the spectral invariant defined via Hamiltonian Floer theory; these spectral invariants are defined in Sections 5.1.1 and 5.2.1, respectively. Here is the main result of this article.

Theorem 1 *$c(H) = \mathcal{N}(H)$ for every autonomous $H \in C^\infty(\Sigma)$.*

This theorem immediately gives rise to the following question:

Question Is it true that $c(H) = \mathcal{N}(H)$ for all $H \in C^\infty(S^1 \times \Sigma)$?

1.2 Max formulas and formal spectral invariants

We now outline the two main components of the proof of Theorem 1.

Max formula In Section 5, we prove max formulas for the spectral invariant c which hold on higher dimensional symplectic manifolds as well as surfaces. Here, we will state a simplified version of these max formulas and refer to Theorems 44 and 45 in Section 5.2 for the more general statements.

Below, the function $c\colon C^\infty(S^1 \times M) \to \mathbb{R}$ denotes either the spectral invariant defined via generating function theory, when $M = \mathbb{R}^{2n}$, or the one defined via Hamiltonian Floer theory on closed aspherical manifolds. We denote by $U_1, \dots, U_N$ disjoint open subsets of M each of which is symplectomorphic to a euclidean ball.

Theorem 2 *Suppose that $H_1, \dots, H_N$ are Hamiltonians whose supports are contained, respectively, in the symplectic balls $U_1, \dots, U_N$. Then*

$$c(H_1 + \cdots + H_N) = \max\{c(H_1), \dots, c(H_N)\}.$$

Our proof of the above theorem relies crucially on the assumption that M is aspherical. Indeed, we have constructed an example, which will appear in a future article, proving that the above max formula does not hold on $\mathbb{S}^2$.

The above theorem and its more general version, Theorem 45, relate to questions which arise from the recent work of Polterovich on Poisson bracket invariants of coverings [33]; see also Question 1 in [42]. We will not delve into this topic as it goes beyond the intended scope of this article.

Formal spectral invariants Although the following definition makes sense on any symplectic manifold we will restrict our attention here to the case of a surface Σ which is either the plane $\mathbb{R}^2$ or is closed and aspherical.

Definition 3 A function $c\colon C^\infty(S^1 \times \Sigma) \to \mathbb{R}$ is a *formal* spectral invariant if it satisfies the following four axioms:

(1) **Spectrality** $c(H) \in \operatorname{spec}(H)$ for all $H \in C^\infty(S^1 \times \Sigma)$ where $\operatorname{spec}(H)$, the *spectrum* of H, is the set of critical values of the Hamiltonian action, that is, the set of actions of fixed points of ϕ^1_H.

(2) **Nontriviality** There exists a topological disk $D \subset \Sigma$ and H supported in D such that $c(H) \neq 0$.

(3) **Continuity** The function c is continuous with respect to the C^∞ topology on $C^\infty(S^1 \times M)$.

(4) **Max formula** If $H_i \in C^\infty(S^1 \times M)$ are supported in pairwise disjoint discs, then $c(H_1 + \cdots + H_N) = \max\{c(H_1), \dots, c(H_N)\}$.

The fact that the invariant $\mathcal{N}$ satisfies the spectrality and nontriviality axioms is an immediate consequence of its definition. It is also not difficult to check that $\mathcal{N}$ satisfies the max formula. However, we do not know if $\mathcal{N}$ satisfies the continuity axiom and thus we do not know if $\mathcal{N}$ is a formal spectral invariant. The two spectral invariants constructed by Viterbo and Schwarz satisfy a long list of well known properties which include the above spectrality, nontriviality and continuity axioms. It is a consequence of Theorem 2 that these two spectral invariants are indeed formal. Theorem 1 is now an immediate consequence of the following theorem.

Theorem 4 *Let $c\colon C^\infty(S^1 \times \Sigma) \to \mathbb{R}$ denote a formal spectral invariant. Then, $c(H) = \mathcal{N}(H)$ for every $H \in C^\infty(\Sigma)$.*

See Section 1.4 for an overview of the proof of the above theorem. An interesting feature of Theorem 4 is that it establishes a partial uniqueness result for spectral invariants which relies only on the above four axioms. As mentioned earlier the spectral invariants constructed via Floer and generating functions theories satisfy many properties. We will prove in Section 3 that formal spectral invariants share some of the same properties, such as Lipschitz continuity, monotonicity, conjugation invariance, and the energy-capacity inequality. We do not know if formal spectral invariants satisfy the triangle inequality, or the property that $c(H)$ is attained by an orbit of Conley–Zehnder index $2n$. However, it is a consequence of Theorems 1 and 4 that at the level of autonomous Hamiltonians the triangle inequality and the index property are satisfied by formal spectral invariants. It would be interesting to see if this can be extended to nonautonomous Hamiltonians or higher dimensional manifolds.

1.3 Further consequences

A simple description of the Entov–Polterovich partial quasi-state In this portion of the paper, Σ denotes a closed surface other than $\mathbb{S}^2$. Take c to be any formal spectral invariant on $C^\infty(S^1 \times \Sigma)$ and define

$$\zeta(H) = \lim_{k \to \infty} \frac{1}{k} c(kH) \tag{1}$$

for an autonomous function H. The functional ζ was introduced by Entov and Polterovich in [8] and it is referred to as a partial symplectic quasi-state. Partial and genuine symplectic quasi-states have been constructed on a large class of symplectic manifolds; see [6] for a survey of the subject. It is well-known that the quasi-state on $\mathbb{S}^2$ admits a very simple description [7]. We will now give a simple description of ζ on aspherical surfaces.

Let H be a Morse function on Σ and suppose that $s \in \Sigma$ is a saddle point of H. Note that the connected component of s in $H^{-1}(H(s))$ is a "pinched" loop. We will call s an *essential* saddle if this pinched loop is not contractible in Σ. In Section 4.2.3, we will prove the following theorem.

Theorem 5 *For any Morse function H on Σ, $\zeta(H)$ is the maximum of H over all of its essential saddles. More generally, for any continuous function $H\colon \Sigma \to \mathbb{R}$,*

$$\zeta(H) = \inf\{h_0 : H^{-1}(h_0, +\infty) \text{ is contractible in } \Sigma\}. \tag{2}$$

The quantity on the right-hand side of Formula (2) has already appeared in the literature in a different (but related) context: it was introduced by Polterovich and Siburg in [34] to study the asymptotic behavior of Hofer's metric on open surfaces with infinite area.

A rather surprising consequence of the above result is that the functional ζ which is constructed via symplectic techniques, namely Floer theory, is in fact invariant under the action of all diffeomorphisms, ie, $\zeta(f \circ \phi) = \zeta(f)$ for any diffeomorphism ϕ. Building on the works of Py [35; 36], Zapolsky and Rosenberg constructed genuine (and not partial) quasi-states on the torus [48] and surfaces of genus higher than one [37]. Many other examples of quasi-states on higher genus surfaces were then provided by Zapolsky [47]. Like ζ, these quasi-states can be described by simple formulas which are different than the formula for ζ. The quasi-state on the torus is only invariant under the action of symplectomorphisms while the other ones are invariant under the action of all diffeomorphisms, like ζ.

The above theorem has some interesting corollaries. In [9], Entov and Polterovich introduced the notions of heaviness and super-heaviness. A closed subset $X \subset \Sigma$ is called *heavy* if $\zeta(H) \geqslant \inf(H|_X)$ for every function H. A closed subset X is called *super-heavy* if $\zeta(H) \leqslant \sup(H|_X)$ for every function H.[3] In [22], Kawasaki proves that the union of a longitude and a meridian in the torus T^2 is super-heavy.[4] Using the

[3] Although it is not obvious from the definition, every super-heavy set is necessarily heavy; see Entov and Polterovich [9].

[4] We have been informed by Kawasaki that he is able to generalize the methods of [22] to recover Proposition 6.

above theorem we generalize Kawasaki's result and give the following characterization of heavy and super-heavy subsets of closed aspherical surfaces; see Section 4.2.3 for the proof.

Proposition 6 *Let $X \subset \Sigma$ be a closed subset. Then*

(1) *X is heavy if and only if X is not included in a disk,*

(2) *X is super-heavy if and only if any closed curve included in its complement is contractible in Σ.*

Since the product of two super-heavy sets is super-heavy, by [9, Theorem 1.5], the above result can be used to construct new examples of (strongly) nondisplaceable sets. We refer the reader to [22] for a sample of such nondisplaceability results.

Dispersion free quasi-states A symplectic quasi-state ζ is said to be dispersion free if $\zeta(H^2) = \zeta(H)^2$ for any function H. It is known that the Entov–Polterovich quasi-state on $\mathbb{S}^2$ is dispersion free. The functional $\zeta\colon C^\infty(\Sigma) \to \mathbb{R}$, defined above, is not dispersion free. Of course, this is not surprising since one can find H, using Theorem 5, for example, such that $\zeta(H) \neq -\zeta(-H)$. However, an interesting consequence of Theorem 5 is ζ is dispersion free on the set of positive or negative functions and more generally

$$\zeta(H^2) = \max\{\zeta(H)^2, \zeta(-H)^2\}.$$

See [6, Question 3.4] and [9, Question 8.5] for a discussion on this topic.

Nonclosed surfaces It is not difficult to see that the invariant $\mathcal{N}$ can be defined for compactly supported Hamiltonians on any surface. Indeed, the definition does not rely on Σ being closed. Following the work of Frauenfelder and Schlenk [12] (see also [23] by Lanzat) one can construct a formal spectral invariant c on compact surfaces with boundary. We expect that the equality $c = \mathcal{N}$ continues to hold, for autonomous Hamiltonians, in this setting.

In [23], Lanzat constructs (partial) quasi-states on a class of nonclosed symplectic manifolds which includes compact surfaces with boundary. Now, given a formal spectral invariant c on a nonclosed surface one can define the functional ζ via Equation (1). We expect that ζ will continue to satisfy Equation (2). Furthermore, we anticipate that the proof of Theorem 5 can be adapted to show that the partial quasi-state constructed by Lanzat coincides with ζ. Lastly, note that one could directly define ζ on any aspherical surface (closed or not) via Equation (2). It can be checked that ζ (defined via Equation (2)) is a partial quasi-state in the sense of Lanzat [23].

1.4 An overview of the proof of Theorem 4

The strategy for proving that $c = \mathcal{N}$ for autonomous Hamiltonians consists of three main steps. First, in Section 4.1, we prove it for Morse functions on the plane. This is achieved by proving that $\mathcal{N}$ and c satisfy the same recursive relation.

The second main step is carried out in Section 4.2 where we prove the equality for Morse functions on closed surfaces. This is done by relating the values of both $\mathcal{N}$ and c to their values on the plane.

Finally, in Section 4.3, we complete the proof by perturbing a general Hamiltonian to a carefully chosen nearby Morse Hamiltonian; the nontriviality of this final step stems from the fact that we do not know if $\mathcal{N}(H)$ depends continuously on H.

The most difficult step is (perhaps) the proof of Proposition 29 which establishes the aforementioned recursive formula for c. Essentially, the argument consists in considering a continuous deformation from the zero Hamiltonian to H and following the value of c using the continuity axiom and a careful analysis of the deformation of the spectrum. An important simplifying factor here is that, having obtained the recursive formula for $\mathcal{N}$, we already know what it is that we are searching for.

Following the value of c during deformations is facilitated by the tools developed in Section 3. In particular, we prove that every formal spectral invariant c is monotone and Lipschitz continuous with respect to H, and satisfies the energy-capacity inequality: the value of c for functions supported on a disk is bounded by the area of the disk.

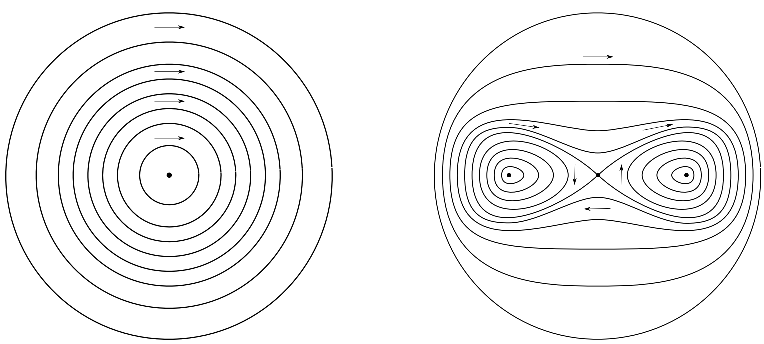

Figure 1: Two simple examples of Morse Hamiltonians on the plane: a "single mountain" and a "double mountain"

To get a taste for the real work, we shall consider here the two simplest scenarios; see Figure 1. We focus on a nonnegative Morse function H on the plane. The first and easiest scenario is that of a function without any saddle point; the graph of such a function looks like a "single mountain". Let us call trivial the fixed points lying outside the support of H. Then every two nontrivial fixed points of ϕ_H^1 are linked, and the definition of $\mathcal{N}$ implies that it coincides with the minimum value, say a, of the actions of its nontrivial fixed points. By spectrality, the value of c cannot be less than a. On the other hand, we can bound H from above by a function G which still has a as the minimal positive action, and whose other action values are larger than the area of its support. By the energy-capacity inequality $c(G)$ must be equal to a, and by monotonicity we get $c(H) \leqslant c(G) = a$, as wanted.

In the second simplest scenario, H has a single saddle point s, and is larger than $H(s)$ on the two disks T_0, T_1 bounded by the level set of s. In this case the graph of H looks like a "double mountain". Again the list of all maximal negative unlinked sets is easy to establish. Mnus's are of two kinds: in addition to the set of trivial fixed points which is contained in every mnus, the first kind consists of a single fixed point of ϕ_H^1 whose orbit surrounds the saddle point, and the second kind consists of the saddle together with one fixed point in each of the two disks T_0, T_1. Denoting by a_0, a_1, b the minimal positive values of the action inside T_0, T_1, and outside the saddle level, respectively, the definition of $\mathcal{N}$ yields

$$\mathcal{N}(H) = \min(b, \max(a_0, a_1)).$$

Now we try to prove that $c(H) = \mathcal{N}(H)$. Proving the upper bound $c(H) \leqslant \mathcal{N}(H)$ is not much more difficult than in the first scenario (but it does rely on the max formula). The lower bound is the most delicate step of the proof, and goes as follows. First, we consider the case when the value of c is attained outside the saddle level set. Here the definition of b gives $c(H) \geqslant b \geqslant \mathcal{N}(H)$, which lets us conclude the equality in this case. In the remaining case we have $c(H) \neq b$; since by the upper bound $c(H) \leqslant b$, we get $c(H) < b$. Now let us write $H = F + H_{T_0} + H_{T_1}$, where F equals the constant value $H(s)$ on $T_0 \cup T_1$, and H_{T_0} and H_{T_1} are supported respectively on T_0 and T_1. In this outline we will pretend that these are smooth functions; note that H_{T_0} and H_{T_1} have no saddle points and hence they are both "single mountains." By a careful analysis of the action values, we construct a deformation H_σ from $H_0 = H$ to $H_1 = H_{T_0} + H_{T_1}$ during which

- the part of the action spectrum corresponding to orbits in $T_0 \cup T_1$, which we will refer to as the "inside" spectrum, decreases at the constant speed $v = H(s)$,
- the remainder of the spectrum, which we will refer to as the "outside" spectrum, does not decrease faster than v.

Now the crucial point is that $c(H_0) < b$, whereas the "outside" spectrum for H_0 is no smaller than b. Thus in the bifurcation diagram $\sigma \mapsto \mathrm{spec}(H_\sigma)$, the connected component of $c(H_0)$ is disjoint from the connected components of the "outside" spectrum, and this component is a single line with slope $-H(s)$ (this will be clear in Figure 10 in Section 4.1.2). By continuity, we get that $c(H_0) = c(H_1) + H(s)$. Then the max formula and the "single mountain" scenario give

$$c(H_1) = \max(c(H_{T_0}), c(H_{T_1})) = \max(a_0 - H(s), a_1 - H(s)).$$

We conclude that $c(H) = \max(a_0, a_1) \geqslant \mathcal{N}(H)$, as wanted.

Organization of the paper

In Section 2 we give the precise definition of the invariant $\mathcal{N}$ and discuss some of its properties. In Section 3 we establish those properties of formal spectral invariants which will be used later on in the paper. Section 4 is devoted to the proof of the main theorem, namely that every formal spectral invariant is equal to $\mathcal{N}$ on the set of autonomous Hamiltonians. The "max formulas", which show that the Viterbo and Schwarz spectral invariants are indeed formal, are proved in Section 5. Finally, a fundamental characterization of unlinked sets, a key ingredient in the definition of $\mathcal{N}$, is proved in the appendix.

2 Preliminaries: definition of $\mathcal{N}$

In this section we introduce the notions of unlinked sets, rotation number of a fixed point, and Hamiltonian action that lead to the definition of our invariant $\mathcal{N}$. Many of the definitions and results of this section hold for general surface diffeomorphisms, not just Hamiltonian diffeomorphisms, and thus in Sections 2.1 and 2.2 we work in this more general context.

2.1 Unlinked sets

Definition and characterizations We consider an orientable surface Σ which may be noncompact but has empty boundary. We denote by $\mathrm{Diff}_0(\Sigma)$ the group of diffeomorphisms which are the time-1 maps of compactly supported isotopies. Let $(\phi^t)_{t\in[0,1]}$ be a compactly supported isotopy in Σ, and denote its time-1 map ϕ^1 by ϕ. A *contractible fixed point* for the isotopy is a fixed point x of ϕ whose trajectory under $(\phi^t)_{t\in[0,1]}$ is a contractible loop in Σ. If in addition $\phi^t(x) = x$ for every $t \in [0,1]$, we say that the isotopy *fixes* x.

Definition 7 A set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is *unlinked* if there exists another isotopy I whose time-1 map is ϕ, which is homotopic to $(\phi^t)_{t\in[0,1]}$ as a path in $\mathrm{Diff}_0(\Sigma)$ with fixed endpoints, and that fixes every point of X.

Note than when $\mathrm{Diff}_0(\Sigma)$ is simply connected, the notion of unlinkedness depends only on ϕ^1. This includes the case when Σ is the disk, the plane or any closed orientable surface except the sphere and the torus [15]. Likewise, on the torus, since $\mathrm{Ham}(\mathbb{T}^2)$ is simply connected [32, Section 7.2], it depends only on ϕ^1 if we restrict ourselves to Hamiltonian isotopies.[5]

The basic result on unlinked sets is the following.

Theorem 8 *A set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if every finite subset of X is unlinked.*

An important corollary of this theorem is the existence of unlinked sets that are maximal for inclusion (Corollary 63). In this paper, we will use the theorem to prove the existence of maximal *negative* unlinked sets (see Corollary 15 below). Theorem 8 as well as Proposition 9 below, are proved in the appendix. Note that the existence of maximal unlinked sets for homeomorphisms is discussed in [20].

Theorem 8 is complemented by a geometric characterization of unlinkedness for finite sets, which we describe now. Let X be a finite set of contractible fixed points. A *geometric pure braid* (based on X) is a map $b\colon X\times[0,1]\to\Sigma$ such that $b(x,0)=b(x,1)$ for every x in X, and $x\mapsto b(x,t)$ is injective for every t. The isotopy $(\phi^t)_{t\in[0,1]}$ generates the geometric pure braid

$$b_{X,(\phi^t)}=(x,t)\mapsto\phi^t(x).$$

We will say that this geometric braid *represents the trivial braid* if there exists a continuous map $B\colon X\times[0,1]\times[0,1]\to\Sigma$ such that $B(\cdot,\cdot,0)$ is the constant braid $(x,t)\mapsto x$, $B(\cdot,\cdot,1)=b_{X,(\phi^t)}$, and $B(\cdot,\cdot,s)$ is a geometric braid for every s.

Proposition 9 *A finite set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if the geometric braid $b_{X,(\phi^t)}$ represents the trivial braid.*

If x is a contractible fixed point, then the geometric pure braid $b_{\{x\},(\phi^t)}$ clearly represents the trivial braid. As a consequence of the proposition, the set $\{x\}$ is unlinked. For a more interesting example, let us consider a pair $\{x,y\}$ of contractible distinct fixed points in $\Sigma=\mathbb{R}^2$. One can define the *linking number* $\ell(x,y)$ as the degree of the circle map

$$t\mapsto\frac{\phi^t(x)-\phi^t(y)}{\|\phi^t(x)-\phi^t(y)\|}.$$

Then the pair $\{x,y\}$ is unlinked if and only if $\ell(x,y)=0$.

[5] The space $\mathrm{Diff}_0(\Sigma)$ is also most probably simply connected when Σ is any noncompact surface, but we have no reference for this fact.

Unlinked subsets of disks Another consequence of the above results concerns the following situation. Assume that Σ is not the sphere, and that our isotopy $(\phi^t)_{t\in[0,1]}$ fixes every point in some neighborhood of the boundary ∂D of some open disk D in Σ. Let X be a set of contractible fixed points of ϕ that is included in D. We will say that X is *unlinked in* D if there is an isotopy in $\mathrm{Diff}_0(D)$ whose time-1 map is $\phi_{|D}$ that fixes every point of X.

Corollary 10 *In this situation, X is unlinked if and only if it is unlinked in D.*

Proof By Theorem 8 it suffices to consider the case when X is finite. If X is unlinked in D, then the isotopy in D given by the definition may be glued with the restriction of (ϕ^t) outside D to provide an isotopy in Σ which fixes every point of X, and we get that X is unlinked. Now assume X is unlinked. This means that the geometric braid $b_{X,(\phi^t)}$ may be deformed into the trivial braid in Σ. The deformation starts with a braid included in D and ends with the trivial braid in D, but the braid may go out of D during the deformation. Let us consider the situation in the universal cover $\widetilde{\Sigma}$. We lift D to a disk $\widetilde{D}$, and let $\widetilde{X}$ be the pre-image of X in $\widetilde{D}$. The braid $b_{X,(\phi^t)}$ lifts to a braid $\widetilde{b}$ based on $\widetilde{X}$ in $\widetilde{D}$, and the deformation of $b_{X,(\phi^t)}$ to the trivial braid lifts to a deformation of $\widetilde{b}$ to the trivial braid in $\widetilde{\Sigma}$. Since the universal cover $\widetilde{\Sigma}$ is contractible, it is easy to modify the deformation so that it takes place entirely in $\widetilde{D}$. Now we project this new deformation down to D, and we see that the braid is trivial in D. Finally we apply Proposition 9 in D to get that X is unlinked in D. □

Remark 11 It can easily be seen that the above corollary still holds if D is replaced with any compact incompressible subsurface of Σ. We do not use the corollary in this generality.

Unlinked sets for autonomous systems In this paragraph we assume again that Σ is not the sphere. We call an isotopy *autonomous* if it is the flow of a time-independent vector field. Let $(\phi^t)_{t\in[0,1]}$ be an autonomous isotopy, and x a contractible fixed point of $\phi=\phi^1$ which is not fixed by the isotopy. Then the trajectory of x is a simple closed curve which bounds a unique disk, we denote this disk by $D(x)$.

Corollary 12 *Let X be a set of contractible fixed points of the autonomous isotopy $(\phi^t)_{t\in[0,1]}$. Then X is unlinked if and only if $X\cap D(x)=\{x\}$ for every point x of X which is not fixed by the isotopy.*

Proof First assume y is a point in $X\cap D(x)$ distinct from x. In the universal cover of Σ, the lifts of x and y in some lift of $D(x)$ have a nonzero linking number, and thus they are linked. An argument similar to the proof of Corollary 10 shows that $\{x, y\}$ is linked in Σ. This proves the direct implication.

The reverse implication goes as follows. Assume that for every point x in X which is not fixed by the isotopy, $X \cap D(x) = \{x\}$, and let us prove that X is unlinked. According to Theorem 8 and Proposition 9, it suffices to prove that every geometric braid generated by a finite subset X' of X represents the trivial braid. For a point x in X' which is not fixed by the isotopy, the single-strand braid generated by $\{x\}$ represents the trivial braid, and we can choose the map B deforming the braid so that it is supported in $D(x)$. Due to the hypothesis on X, all these deformations do not interfere, and together they give rise to a deformation of the braid $b_{X',(\phi^t)}$ into the trivial braid (the strands corresponding to the fixed points of the isotopy stay still during the deformation). $\square$

2.2 Rotation number and negative unlinked sets

Definitions For simplicity we restrict ourselves to a surface Σ which is either the plane or a closed surface which is not the sphere. Consider a contractible fixed point x for an isotopy $(\phi^t)_{t\in[0,1]}$ as before. Since x is a contractible fixed point, there exists a "capping disk", ie, a smooth map $u\colon \mathbb{D}^2 \to \Sigma$ from the unit disk $\mathbb{D}^2$ whose restriction to the unit circle is (a parametrization of) the trajectory $t \mapsto \phi^t(x)$. Since $\mathbb{D}^2$ is contractible, the pullback of $T\Sigma$ under u may be identified with the trivial bundle $\mathbb{D}^2 \times \mathbb{R}^2$. Given a unit vector v in $\mathbb{R}^2 \simeq \{x\} \times \mathbb{R}^2 \simeq T_x\Sigma$, the pullback of the path $t \mapsto (\phi^t(x), D_{\phi^t(x)}\phi^t \cdot v)$ is a path (t, v_t) in $\mathbb{D}^2 \times (\mathbb{R}^2 \setminus \{0\})$. The map

$$(t, v) \mapsto \frac{v_t}{\|v_t\|}$$

is an isotopy in the circle. We call the rotation number of this isotopy the *rotation number of* x and denote it by $\rho(x)$, which is a real number defined as follows. We lift the isotopy to an isotopy $(F_t)_{t\in[0,1]}$ of $\mathbb{R}$, whose time-1 map F_1 is a homeomorphism of the line that commutes with the translation $s \mapsto s+1$; the rotation number of the isotopy is, by definition, the translation number of F_1,

$$\lim_{n\to+\infty} \frac{1}{n}(F_1^n(s) - s)$$

for any $s \in \mathbb{R}$ (see, for example [21]).

Lemma 13 *The rotation number $\rho(x)$ depends only on x and ϕ^1.*

Here is a sketch of the proof. Notice that the rotation number of a circle homeomorphism is well defined as a real number modulo one. Thus, modulo one, $\rho(x)$ depends only on

x and ϕ^1. From this we first deduce that $\rho(x)$ does not depend on the trivialization of the tangent bundle over u. Then, since $\pi_2(\Sigma) = 0$, we conclude that it does not depend on the choice of the capping disk u either. Likewise, we see that it depends only on the homotopy class of $(\phi^t)_{t\in[0,1]}$ as a path of diffeomorphisms. If Σ is not the torus then $\mathrm{Diff}_0(\Sigma)$ is simply connected and we are done. It remains to take care of the torus. First note that on any surface, since according to Proposition 9 the set $\{x\}$ is unlinked, the homotopy class of $(\phi^t)_{t\in[0,1]}$ contains an isotopy $I = (f_t)_{t\in[0,1]}$ that fixes the point x. Thus we can use the isotopy I and the trivial capping to define $\rho(x)$, and we see that $\rho(x)$ equals the rotation number of the action of the differential of this isotopy on the unit tangent bundle at x. Finally, when Σ is the torus, we can conclude since the subgroup of elements of $\mathrm{Diff}_0(\Sigma)$ fixing x is simply connected [15, Théorème 2 and Proposition 2].

In the Hamiltonian context the rotation number may be generalized to higher dimensions, and is called the mean index, see for example [39; 14].

Definition 14 We say that an unlinked set X is *negative* if $\rho(x) \leqslant 0$ for every $x \in X$. We say that a negative unlinked set X is *maximal* if there is no negative unlinked set X' strictly containing X.

Note that the rotation number, and hence being negatively unlinked, is invariant under conjugation in the group $\mathrm{Diff}_0(\Sigma)$. Theorem 8 has the following important consequence.

Corollary 15 *Every negative unlinked set is contained in a maximal negative unlinked set. Furthermore, the closure of a negative unlinked set is still a negative unlinked set, and maximal negative unlinked sets are closed.*

Proof For the first part, we provide a short argument relying on Zorn's Lemma (a more constructive proof may be obtained by adapting the proof of Corollary 63). It suffices to consider a family $\mathcal{F}$ of negative unlinked sets which is totally ordered by inclusion, and check that $\mathcal{F}$ has an upper bound. Consider the union X of all elements of $\mathcal{F}$. Theorem 8 implies that X is unlinked, and clearly every point of X has nonpositive rotation number. Thus X is an upper bound for $\mathcal{F}$. This proves the first sentence. For the second sentence, consider an unlinked set X, and let $I = (f_t)_{t\in[0,1]}$ be an isotopy fixing every point of X. Then I also fixes every point of $\overline{X}$, which shows that $\overline{X}$ is unlinked. Furthermore, at every point $x \in \overline{X} \setminus X$ the differential $D_x f_t$ has a fixed vector v that does not depend on t; thus the rotation number $\rho(x)$ vanishes. This proves that $\overline{X}$ is a negative unlinked set. The closedness of maximal negative unlinked sets follows immediately. □

2.2.1 Negative unlinked sets for Hamiltonian isotopies We now describe some properties that are specific to Hamiltonian systems. Let Σ be equipped with a symplectic form ω, consider a time-dependent Hamiltonian function $H \in C^\infty(S^1 \times \Sigma)$, and the corresponding Hamiltonian isotopy $(\phi_H^t)_{t\in[0,1]}$.

Lemma 16 *There exists a negative contractible fixed point.*

Proof In the case when Σ is not compact, every point outside the support of ϕ is a negative contractible fixed point.

In the case when Σ is a compact surface, a negative contractible fixed point is provided by P Le Calvez's proof of the Arnol'd conjecture. Let us recall the outline of the proof (see [25] for more details). According to Corollary 63 in the appendix, there exists a maximal unlinked set X for ϕ. Since every accumulation point of X has zero rotation number, we may assume that X is finite. Le Calvez's Brouwer foliated equivariant theorem provides an oriented foliation on $\Sigma \setminus X$ which is "homotopically transverse" to the flow $(\phi_H^t)_{t\in[0,1]}$, which means that every trajectory of the flow is homotopic in $\Sigma \setminus X$, with fixed endpoints, to a curve which is positively transverse to the foliation (in other words, "every leaf is pushed towards its right"). Such a foliation is "gradient-like": in particular, for every leaf L, there exists two distinct points $\alpha(L), \omega(L)$ in X such that the closure of L equals $L \cup \{\alpha(L), \omega(L)\}$. By transversality, the point $\alpha(L)$ has nonpositive rotation number, and the point $\omega(L)$ has nonnegative rotation number. □

Remark 17 (relation with the Conley–Zehnder index) When ϕ_H^1 is nondegenerate, its 1–periodic orbits can be indexed by the well known Conley–Zehnder index μ_{CZ} which takes values in the integers. Many conventions are used for normalizing μ_{CZ}. Our convention is as follows: Suppose that $H\colon \Sigma \to \mathbb{R}$ is a C^2–small Morse function. We normalize the Conley–Zehnder index so that for every critical point p of H,

$$\mu_{\mathrm{CZ}}(p) = i_{\mathrm{Morse}}(p),$$

where $i_{\mathrm{Morse}}(p)$ is the Morse index of p. To be specific, in this case $\mu_{\mathrm{CZ}}(p)$ is equal to 2 if p is a local maximum of H, 1 if it is a saddle point and 0 if it is a local minimum. Note that in the first case the rotation number $\rho(x)$ belongs to $(-1,0)$, it vanishes in the second case, and in the last case it belongs to $(0,1)$. In general, the Conley–Zehnder index and the rotation number are related by the following formula: if p is any contractible fixed point of a nondegenerate ϕ_H^1, then:

- If $\mu_{\mathrm{CZ}}(p)$ is odd, then $\rho(p) = \frac{1}{2}\big(-\mu_{\mathrm{CZ}}(p)+1\big)$.
- If $\mu_{\mathrm{CZ}}(p)$ is even, then $\rho(p) \in \big(-\frac{1}{2}\mu_{\mathrm{CZ}}(p), \frac{1}{2}\big(-\mu_{\mathrm{CZ}}(p)+2\big)\big)$.

2.3 Action functional

The definitions of this section are valid on every symplectic manifold (M^{2n}, ω) which is symplectically aspherical, ie, $\langle \omega, \pi_2(M) \rangle = 0$. Given a (time-dependent) Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$, the *action functional* is the function $\mathcal{A}_H$ defined on the space of contractible loops in M by the formula

$$\mathcal{A}_H(x) = \int_0^1 H(t, x(t))\, dt - \int_{\mathbb{D}^2} u^* \omega,$$

where u is a capping disk of the loop x, ie, a map $u\colon \mathbb{D}^2 \to S$ such that $u|_{\partial \mathbb{D}^2} = x$. In other words, the term $\int_{\mathbb{D}^2} u^* \omega$ is the algebraic area enclosed by x. Since the manifold is assumed symplectically aspherical, this term does not depend on the choice of the capping disk. Moreover, if one only allows *mean normalized* Hamiltonians, ie, Hamiltonians which are normalized by the condition $\int_0^1 H(t, x) \omega^n = 0$ for all $t \in [0, 1]$, then the value of the action on periodic orbits does not depend on the choice of the generating Hamiltonian but only on the time-1 map ϕ_H^1. This means that the action functional is well defined for fixed points of Hamiltonian diffeomorphisms. If x, y are two points that are fixed under the Hamiltonian flow, then $\mathcal{A}_H(y) - \mathcal{A}_H(x)$ can be geometrically interpreted as the quantity of area flowing through any curve joining x to y under the isotopy $(\phi_H^t)_{t \in [0,1]}$.

The most important feature of the Hamiltonian action is that its critical points are exactly the contractible 1–periodic orbits of the Hamiltonian flow ϕ_H^t (by this we mean the periodic orbits whose period divides 1). The set of critical values of the action, ie, values on 1–periodic orbits, is called the *spectrum* of the Hamiltonian H and is denoted $\mathrm{spec}(H)$. It has Lebesgue measure zero. See Section 2.5 for an example of computation.

2.4 Definition of $\mathcal{N}$

For simplicity, again we restrict ourselves to a surface Σ which is either the plane $\mathbb{R}^2$, the interior of a closed disk in the plane, or a closed surface which is not the sphere, although everything works on any surface Σ for which the inclusion of $\mathrm{Ham}(\Sigma)$ into $\mathrm{Diff}_0(\Sigma)$ is trivial at the level of the fundamental groups (see the footnote above).

Let us consider a time-dependent Hamiltonian function $H \in C^\infty(S^1 \times \Sigma)$, the corresponding Hamiltonian isotopy $(\phi_H^t)_{t \in [0,1]}$, and $\phi = \phi^1$. Remember that the notions of unlinkedness and rotation number depend only on ϕ^1 and not on the isotopy.

For short we write mnus for "maximal negative unlinked set", and we denote the family of mnus's by $\mathrm{mnus}(\phi)$ or $\mathrm{mnus}(H)$. According to Corollary 15, there exists at least one mnus. Furthermore, since by Lemma 16 there exists a negative contractible fixed point, every mnus is nonempty. Hence the following definition is valid.

Definition 18 $$\mathcal{N}(H) = \inf_{X \in \mathrm{mnus}(\phi)} \sup_{x \in X} \mathcal{A}_H(x).$$

If $\Sigma = \mathbb{R}^2$ then $\mathcal{A}_H(x) = \mathcal{A}_G(x)$ for every (compactly supported) Hamiltonian function G whose time-1 map is ϕ. If Σ is a closed surface then the same equality holds if $\int_\Sigma G\,d\omega = \int_\Sigma H\,d\omega$. In particular we may give the following definition.

Definition 19 Let $\mathcal{N}(\phi)$ be $\mathcal{N}(H)$ where H is any Hamiltonian function whose time-1 map is ϕ, normalized by the condition $\int_\Sigma H\,d\omega = 0$ in the case Σ is a closed surface.

Note that $\mathcal{N}$ is invariant under conjugation by symplectic diffeomorphisms.

2.5 Example: radial Hamiltonians

In this subsection, we illustrate the notions introduced above on a basic but fundamental example. We will make intensive use of this example in the proof of Theorem 4. Let $H \in C^\infty(\mathbb{R}^2)$ be a smooth autonomous Hamiltonian on the plane, that only depends on the distance to the origin. It will be convenient to write H in the form

$$H(x, y) = f(\pi(x^2 + y^2)) \quad \text{for all } x, y \in \mathbb{R}$$

for some function $f\colon [0, +\infty) \to \mathbb{R}$.

Fixed points The Hamiltonian vector field is given by

$$X_H = \big(-2\pi y f'(\pi(x^2 + y^2)), 2\pi x f'(\pi(x^2 + y^2))\big)$$

and we see that the time-1 map of the flow restricted to the circle of radius r is the rotation by $2\pi f'(\pi r^2)$. Thus, the fixed points of ϕ_H^1 are, besides the origin, the points of $\mathbb{R}^2$ whose distance to the origin r is such that $f'(\pi r^2)$ is an integer.

Rotation numbers Let (x, y) be such a point, then denote $s = \pi(x^2 + y^2)$ and set $k = f'(s)$. The orbit of H makes exactly k oriented turns along the circle centered at the origin and passing through (x, y). The linearized flow of H along the orbit, ie, the linear map $D\phi_H^t(x, y)$, acts on a vector $\vec{v}$ tangent to the circle as the rotation by angle $2\pi k t$, thus $\rho(x, y) = k$. Therefore the fixed points with nonpositive rotation number correspond to values of s where f is nonincreasing. Note that the rotation number of the origin is $f'(0)$.

Mnus's Let p_1, p_2 be two distinct fixed points of ϕ_H^1. To fix ideas, assume that p_2 is no closer to the origin than p_1. Then the linking number $l(p_1, p_2)$ equals the rotation number $\rho(p_2)$. This immediately leads to the following complete description of the mnus's. Let X denote the set of critical points of H. For every point $p = (x, y)$ such that $f'(\pi(x^2 + y^2))$ is a negative integer, let X_p denote the union of $\{p\}$ and of the critical points of H farther than p from the origin. The sets X_p are mnus's. If $f'(0) \leqslant 0$ then X is a mnus, in the opposite case $X \setminus \{0\}$ is a mnus (note that, by the intermediate value theorem, in this case this last set is not included in any of the X_p).

Reading the Hamiltonian action on diagrams The Hamiltonian action of these fixed points is given by

$$\mathcal{A}_H(x, y) = f(s) - sk = f(s) - sf'(s).$$

It corresponds to the intersection of the vertical axis $\{0\} \times \mathbb{R}$ with the tangent to the graph of f at the point $(s, f(s))$, see Figure 2. The action can also be seen on the graph of $-f'$ (the negative of the rotation number). With the above notation, $\mathcal{A}_H(x, y) = -(ks + \int_s^{+\infty} f'(\sigma)\, d\sigma)$. This corresponds to the gray area in Figure 3.

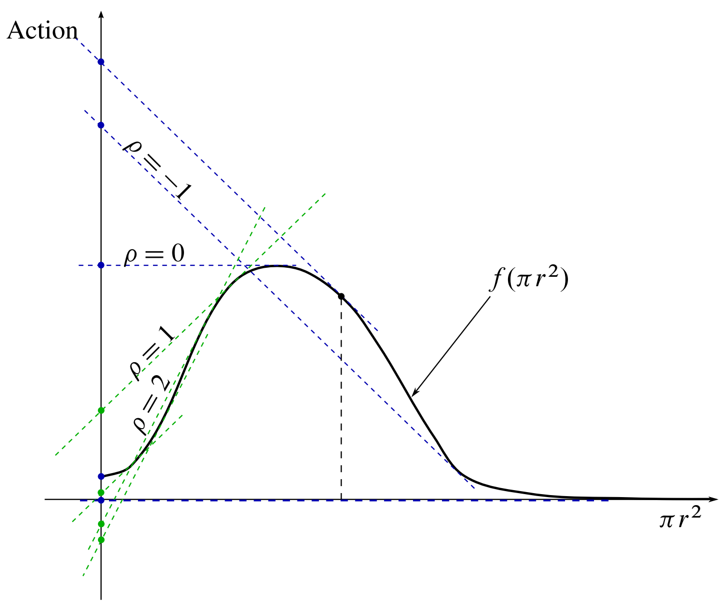

Figure 2: The dotted lines are the tangents to the graph of f with integer slope. Their tangency points correspond to the fixed points of ϕ_H^1. The intersections of these lines with the vertical axis (represented by thick dots) give the action. The points with nonpositive rotation numbers are in blue.

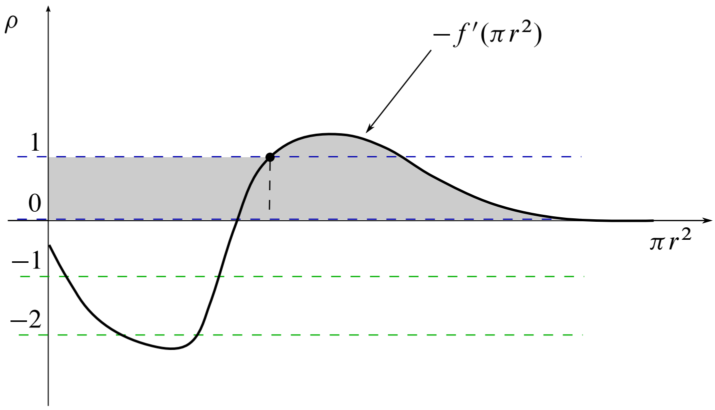

Figure 3: The fixed points correspond to intersections of the graph $\rho = -f'$ with the horizontal lines "ρ = integer constant". The action of the thick black dot is the area of the gray region. This thick black dot corresponds to the thick black dot on Figure 2.

Computing $\mathcal{N}$ First assume that the function $f\colon [0,+\infty) \to \mathbb{R}$ is decreasing and has nonvanishing derivative on $(0, r_0)$, where $[0, r_0]$ is the support of f. Let Y be the complement in the plane of the open disk with radius r_0. The mnus's are the sets of the form $\{x\} \cup Y$ where x is any fixed point not in Y. Finally we get

$$\mathcal{N}(H) = \min_x \mathcal{A}_H(x), \tag{3}$$

where the minimum runs on all fixed points of ϕ_H^1 that are not in Y. With the interpretation of the action explained above, we see that it is a positive number, attained at a periodic orbit of period exactly one.

Another case when $\mathcal{N}$ is easy to compute is when f takes only nonpositive values. Indeed, remember that the set of all critical points of H, taking out the origin in case $f'(0) > 0$ is a mnus. Since every critical point has a nonpositive action, we see that $\mathcal{N}(H) = 0$. There does not seem to be any easy formula in the case of a general radial Hamiltonian.

2.6 Max formula for $\mathcal{N}$

Here again we assume that Σ is the plane or a closed aspherical surface.

Lemma 20 (max formula for $\mathcal{N}$) *Suppose that $H_1, \dots, H_N \in C^\infty(S^1 \times \Sigma, \mathbb{R})$ are Hamiltonian functions whose supports are contained in pairwise disjoint open disks $U_1, \dots, U_N$. Then*

$$\mathcal{N}(H_1 + \dots + H_N) = \max\{\mathcal{N}(H_1), \dots, \mathcal{N}(H_N)\}.$$

Proof By an easy induction, the proof boils down to the case $N = 2$. Let Y_i denote the complement of the disk U_i in Σ, and Y be the complement of $U_1 \cup U_2$. The crucial remark is the following: *the unlinked sets (resp. negative unlinked sets) of $H_1 + H_2$ are the sets of the form*

$$Y' \cup X_1 \cup X_2,$$

where

- *Y' is included in Y,*
- *X_i, $i = 1, 2$ is included in U_i,*
- *$X_i \cup Y_i$ is an unlinked set (resp. negative unlinked set) of H_i.*

The mnus's of $H_1 + H_2$ have the same form, where $Y' = Y$ and $X_i \cup Y_i$ is a mnus of H_i.

The proof of this remark is a consequence of Corollary 10.

We first check that unlinked sets correspond. If X_1 is a subset of U_1 which is unlinked for H_1, then by Corollary 10 it is unlinked for H_1 in U_1; this provides us with some isotopy which is compactly supported in U_1. If likewise X_2 is unlinked for H_2 in U_2 we get a second isotopy, and we can glue the two isotopies with the identity on Y into an isotopy on Σ, yielding that $X_1 \cup X_2 \cup Y$ is unlinked for $H_1 + H_2$. For the converse implication, whose proof is similar to the above, let X be unlinked for $H_1 + H_2$. Then the set $X_i = X \cap U_i$ is also unlinked, thus by Corollary 10 it is unlinked in U_i for $H_1 + H_2$, but this is exactly the same thing as being unlinked in U_i for H_i. Then obviously $X_i \cup Y_i$ in unlinked for H_i, and we get $X = Y' \cup X_1 \cup X_2$ as wanted. The correspondence between negative unlinked sets and mnus's follow immediately. $\square$

3 Preliminaries: properties of formal spectral invariants

The main goal of this section is to establish certain properties of formal spectral invariants which will be used later on in the paper. Throughout the section c denotes a formal spectral invariant in the sense of Definition 3. In Section 3.1, we present those properties of c which are standard in the sense that they are known to hold for the Floer and generating-function theoretic spectral invariants. In Section 3.2, we introduce the symplectic contraction principle which provides a powerful tool in the study of spectral invariants on aspherical manifolds.

3.1 The standard properties of c

Properties 1–6 listed below are among the standard properties which are known to hold for the Floer and generating-function theoretic spectral invariants; see [46; 40; 29], for example. The proofs we give in this section for the first six properties are similar to those presented in [46]. It is interesting to observe that the proofs of the first five properties rely solely on spectrality and continuity of formal spectral invariants. The last two properties, which prove that $c(H)$ is positive for a large class of Hamiltonians, rely on the max formula. Lastly, we should mention that one standard property of Floer and generating function theoretic spectral invariants which we have not been able to prove is the triangle inequality.

(1) Symplectic invariance $c(H) = c(H \circ \psi)$ for all $H \in C^\infty(S^1 \times \Sigma)$ and for all $\psi \in \mathrm{Symp}_0$, where Symp_0 denotes the path component of the identity in $\mathrm{Symp}(\Sigma, \omega)$, the group of symplectomorphisms of (Σ, ω).

Proof It is a classical fact that $\mathrm{spec}(H \circ \phi) = \mathrm{spec}(H)$ for any symplectomorphism ϕ. Let ψ_s denote a path in Symp_0 such that $\psi_0 = \mathrm{Id}$ and $\psi_1 = \psi$. Now, the continuous function $s \mapsto c(H \circ \psi_s)$ takes values in the measure zero set $\mathrm{spec}(H)$ and hence it must be constant. □

(2) Shift $c(H + r) = c(H) + \int_0^1 r(t)\, dt$, where $r\colon S^1 \to \mathbb{R}$ is a function of time.

Proof For $s \in [0, 1]$ let $H_s = H + sr$. Note that $\mathrm{spec}(H_s) = \mathrm{spec}(H) + s\int_0^1 r(t)\, dt$. By the continuity and spectrality axioms, the function $s \mapsto c(H_s) - s\int_0^1 r(t)\, dt$ is continuous and takes values in the measure-zero set $\mathrm{spec}(H)$ and so it must be constant. The shift property follows immediately. □

(3) Monotonicity $c(H) \leqslant c(G)$ if $H \leqslant G$.

Proof See the proof of Lipschitz continuity. □

(4) Lipschitz continuity $\displaystyle\int_0^1 \min_{x\in M}(H_t - G_t)\, dt \leqslant c(H) - c(G) \leqslant \int_0^1 \max_{x\in M}(H_t - G_t)\, dt.$

Proof We will simultaneously prove monotonicity and Lipschitz continuity. The continuity axiom implies that it is sufficient to prove these properties in the special case where both H and G are nondegenerate. For nondegenerate Hamiltonians both of these properties follow from Lemma 21, stated below: take $F_s = G + s(H - G)$ and note that $\partial F_s/\partial s = H - G$. If F_s is an admissible family in the sense of Lemma 21 then both results follow immediately. If F_s is not admissible then we can perturb it by a C^2–small amount and obtain an admissible family such that $\partial F_s/\partial s \approx H - G$. We leave the details of this to the reader.

We will now state and prove Lemma 21. Consider a 1–parameter family of time dependent Hamiltonians $H_s(t, x)$, $s \in [0, 1]$ which depends smoothly on s. We call H_s *admissible* if there exists a finite (possibly empty) set of points $\{s_1, \dots, s_k\} \subset (0, 1)$ such that:

(1) The set of fixed points of $\phi^1_{H_s}$ is finite for all $s \in [0,1]$.

(2) For all $s \in [0,1] \setminus \{s_1, \dots, s_k\}$, the Hamiltonian H_s is nondegenerate and no two fixed points of $\phi^1_{H_s}$ have the same action.

A generic (in the sense of Baire) 1–parameter family of Hamiltonians is admissible.

Lemma 21 *Let $H_s(t,x)$, $s \in [0,1]$ denote an* admissible *family of Hamiltonians. The function $s \mapsto c(H_s)$ is differentiable at every $s \in [0,1]$ except the finite set of points $\{s_1, \dots, s_k\}$ where H_s is degenerate, and furthermore*

$$\int_0^1 \min_{x \in M} \frac{\partial H_s}{\partial s}(t,x)\,dt \leqslant \frac{d}{ds} c(H_s) \leqslant \int_0^1 \max_{x \in M} \frac{\partial H_s}{\partial s}(t,x)\,dt.$$

Proof of Lemma 21 Let I_k denote the open interval (s_k, s_{k+1}) and consider $s \in I_k$. There exists a 1–periodic orbit x_s of $\phi^1_{H_s}$ such that $c(H_s) = \mathcal{A}_{H_s}(x_s)$. The admissibility condition implies that the fixed point x_s varies smoothly on the entire interval I_k; indeed no bifurcations take place in this interval. Furthermore, since c is continuous it must be the case that $c(H_s) = \mathcal{A}_{H_s}(x_s)$. This implies that in fact c is smooth in the interval I_k hence we can differentiate: we will use the symbol x_s to denote the 1–periodic orbit associated to the fixed point x_s.

$$\frac{d}{ds} c(H_s) = \frac{d}{ds} A_{H_s}(x_s) = \frac{\partial}{\partial r}\mathcal{A}_{H_r}(x_s) + \frac{\partial}{\partial r}\mathcal{A}_{H_s}(x_r).$$

Now, $\partial(\mathcal{A}_{H_s}(x_r))/\partial r = 0$ because x_s is a critical point of $\mathcal{A}_{H_s}$. A simple computation yields

$$\frac{\partial}{\partial r}\mathcal{A}_{H_r}(x_s) = \int_0^1 \frac{\partial H_s}{\partial s}(t, x_s(t))\,dt.$$

The result follows immediately. This concludes the proof of Lipschitz continuity. □

Remark 22 Observe that the Lipschitz continuity property of c allows us to extend c to all continuous functions.

(5) Energy-capacity inequality Let K, H be two Hamiltonians such that ϕ^1_K displaces the support of H. Then, $|c(H)| \leqslant \int_0^1 (\max_{x \in M} K_t - \min_{x \in M} K_t)\,dt$.

Proof For each $s \in [0,1]$, consider the Hamiltonian

$$F_s(t,x) = sH(st,x) + K(t, (\phi^{st}_H)^{-1}(x)).$$

The time-1 map of the flow of F_s is given by $\phi^s_H \circ \phi^1_K$. Using the fact that ϕ^1_K displaces the support of H one can prove that the fixed points of $\phi^s_H \circ \phi^1_K$ are precisely the fixed points of ϕ^1_K and furthermore for each fixed point x we have $\mathcal{A}_{F_s}(x) = \mathcal{A}_K(x)$.

Hence, $\mathrm{spec}(F_s) = \mathrm{spec}(K)$. It then follows that the continuous function $s \mapsto c(F_s)$ is constant and thus,

$$c(K) = c\big(H(t,x) + K(t, (\phi_H^t)^{-1}(x))\big).$$

This, combined with the Lipschitz continuity of c, yields

$$c(H) - c(K) \leqslant \int_0^1 \max_{x\in M}\big(-K_t(\phi_H^t)^{-1}(x)\big)\,dt = -\int_0^1 \min_{x\in M} K_t\,dt,$$

and thus $c(H) \leqslant c(K) - \int_0^1 \min_{x\in M} K_t\,dt$. Using the Lipschitz continuity again we obtain $c(H) \leqslant \int_0^1 (\max_{x\in M} K_t - \min_{x\in M} K_t)\,dt$. Similarly, one can prove that $-\int_0^1 (\max_{x\in M} K_t - \min_{x\in M} K_t)\,dt \leqslant c(H)$. □

We do not use the following property in this article. However, we state it as it is one of the standard properties of the Floer and generating function theoretic spectral invariants.

(6) Path independence Suppose that $\phi_H^1 = \phi_G^1$. If $\Sigma = \mathbb{R}^2$ then $c(H) = c(G)$. In the case where $\Sigma \neq \mathbb{R}^2$ then $c(H) = c(G)$ provided we assume additionally that $\int H_t\omega^2 = 0 = \int G_t\omega^2$ for each $t \in \mathbb{S}^1$.

Proof Since $\mathrm{Ham}(\Sigma)$ is simply connected there exists a path of Hamiltonians F_s such that $F_0 = H$, $F_1 = G$ and $\phi_{F_s}^1 = \phi_H^1 = \phi_G^1$. It follows from our assumptions that $\mathrm{spec}(F_s) = \mathrm{spec}(H) = \mathrm{spec}(G)$ and hence the function $s \mapsto c(F_s)$ is constant. □

(7) Positivity If H is supported in a disk, then $c(H) \geqslant 0$.

Proof This follows readily from the max formula applied to H and 0 which gives: $c(H) = \max(c(H), 0)$. □

(8) Nondegeneracy If $H \neq 0$ and $H \geqslant 0$, then $c(H) > 0$.

Proof One can find a small disk D, a short time interval $[t_0, t_1] \subset (0,1) \subset S^1$, and a positive constant m such that $H(t,x) \geqslant m$ for all $(t,x) \in [t_0,t_1] \times D$. By Lemma 23, there exists a Hamiltonian F such that

- F is supported in D,
- $F(t,x) < \frac{1}{2}(t_1 - t_0)m$ for each x in the interior of D,
- $c(F) > 0$.

We will show that $c(H) \geqslant c(F)$. Let $\alpha\colon S^1 \to S^1$ denote a smooth reparametrization of S^1 such that

- $\alpha(t) = 0$ for all $t \in [0, t_0]$,
- $\alpha'(t) < 2/(t_1 - t_0)$,
- $\alpha(t) = 1$ for all $t \in [t_1, 1]$.

Set $G(t,x) = \alpha'(t)F(\alpha(t),x)$. The flow of F is given by $\phi_F^t(x) = \phi_f^{\alpha(t)}(x)$ and so by the path independence property $c(G) = c(F)$. On the other, $G(t,x) \leqslant H(t,x)$ and hence, by monotonicity, $c(G) \leqslant c(H)$. It remains to prove the following lemma.

Lemma 23 *For any disk $D \subset \Sigma$ and any positive constant ε, there exists a Hamiltonian F such that $F \leqslant \varepsilon$, and the support of F is contained in D and $c(F) > 0$.*

The proof of this lemma uses the "symplectic contraction" principle which is described in Section 3.2. We postpone the proof to the end of that section. □

3.2 The symplectic contraction principle

In this section we introduce the "symplectic contraction" technique which describes the effect of the flow of a Liouville vector field on a formal spectral invariant c. This technique has been used by Polterovich in [33]. Throughout this section we will work on a general aspherical symplectic manifold M and we suppose that $c\colon C^\infty(S^1 \times M) \to \mathbb{R}$ is any function satisfying the four axioms of Definition 3. The reason for working in this generality is that the symplectic contraction technique is used in our proof of Theorem 45 which holds for aspherical manifolds of higher dimensions.

Throughout this article, by the word "domain" we mean an open subset of M whose boundary components are smooth. We should caution the reader that we will be interested in domains which may have several connected components. Recall that a domain $U \subset M$ is said to be *Liouville domain* if the closure of U admits a vector field ξ which is transverse to the boundary ∂U and satisfies $L_\xi \omega = \omega$, where L is the Lie derivative. The vector field ξ is referred to as the Liouville vector field of the domain U. Note that the Liouville vector field ξ necessarily points outward along ∂U and therefore the flow $A_t\colon U \to U$ of ξ is defined for all $t \leqslant 0$. This flow "contracts" the symplectic form ω: $A_t^*\omega = e^t\omega$. Recall also that $U \subset M$ (not necessarily Liouville) is called incompressible if the map $i_*\colon \pi_1(U) \to \pi_1(M)$, induced by the inclusion $i\colon U \to M$, is injective.

Let U denote an *incompressible Liouville* domain in M and let $F\colon S^1 \times M \to \mathbb{R}$ be a Hamiltonian supported in U. For each fixed $s \leqslant 0$ consider the Hamiltonian

$$F_s(t,x) := \begin{cases} e^s F(t, A_s^{-1}(x)) & \text{if } x \in A_s(U), \\ 0 & \text{if } x \notin A_s(U). \end{cases}$$

It can be checked that the Hamiltonian flow of F_s is given by

$$\phi^t_{F_s}(x) := \begin{cases} A_s \phi^t_F A_s^{-1}(x) & \text{if } x \in A_s(U), \\ x & \text{if } x \notin A_s(U). \end{cases}$$

It follows that there exists a $1-1$ correspondence between the 1–periodic orbits of F and F_s. Indeed, if $x(t)$ is a 1–periodic orbit of F then $x_s := A_s(x(t))$ is a 1–periodic orbit of F_s. Next, we claim that x is contractible if and only if x_s is and furthermore $\mathcal{A}_{F_s}(x_s) = e^s \mathcal{A}_F(x)$. Since U is incompressible we can pick a capping disk D contained in U for the orbit x. Then, $A_s(D)$ is a capping disk for x_s. Now, we compute

$$\begin{aligned} \mathcal{A}_{F_s}(x_s) &= \int_0^1 e^s F\big(t, A_s^{-1}(x_s(t))\big) - \int_{A_s(D)} \omega \\ &= \int_0^1 e^s F\big(t, (x(t))\big) - \int_D e^s \omega \quad = \quad e^s \mathcal{A}_F(x). \end{aligned}$$

It follows that

$$\operatorname{spec}(F_s) = e^s \operatorname{spec}(F). \tag{4}$$

Using the spectrality and continuity properties of spectral invariants we conclude that

$$c(F_s) = e^s c(F). \tag{5}$$

We have symplectically contracted the Hamiltonian F.

We end this section with a proof of Lemma 23.

Proof of Lemma 23 By the nontriviality axiom there exists a disk D_0 and a Hamiltonian H supported in D_0 such that $c(H) \neq 0$. By the max formula $c(H)$ is necessarily positive. Note that a disk is a Liouville domain and so we can apply the symplectic contraction principle. Let H_s denote a symplectic contraction of H as described above. Picking s to be sufficiently negative yields $|H_s| \leqslant \varepsilon$. Observe that H_s is supported in the disk $A_s(D_0)$ whose area is $e^s \operatorname{Area}(D_0)$. Hence, by picking s to be sufficiently negative we can ensure that the area of the support of H_s is smaller than the area of the disk D and so we can find a Hamiltonian diffeomorphism ψ which maps the support of H_s into D. Set $F = H_s \circ \psi$. The Hamiltonian F is supported in D, is bounded above by ε and, using the symplectic invariance property and the symplectic contraction principle, we see that $c(F) = c(H_s) = e^s c(H) > 0$. This completes the proof of nondegeneracy of c. □

4 Proof of Theorem 4

As mentioned in the introduction, Theorem 1 is an immediate consequence of Theorems 2 and 4. The main goal of this section is to prove Theorem 4. This is done in three stages. In Sections 4.1 and 4.2, we prove the theorem for Morse functions on the plane and closed surfaces of positive genus, respectively. In Section 4.3, we explain how one can pass from Morse functions to general autonomous Hamiltonians.

Along the way, we obtain several results which may be of independent interest as they describe algorithms for computing formal spectral invariants and the invariant $\mathcal{N}$ on autonomous Hamiltonians. For example, Propositions 28 and 29 provide recursive formulas for computing $\mathcal{N}$ and c on the plane. Propositions 39 and 40 are quite surprising as they demonstrate that computing $\mathcal{N}$ and c on closed surfaces can easily be reduced to computations on the plane! In Section 4.2.3, we use Proposition 40 to prove Theorem 5 and Proposition 6 on the Entov–Polterovich quasi-state.

4.1 Theorem 4 for Morse functions on the plane

In this section, we prove the equality $c = \mathcal{N}$ for Morse functions on the plane. Throughout this section, we call a function $H\colon \mathbb{R}^2 \to \mathbb{R}$ a *Morse function* if its support is a closed topological disk and it admits finitely many critical points in the interior of its support, all of which are nondegenerate and correspond to distinct values of H.

Proving that $c = \mathcal{N}$ for such functions is done in two steps. We first establish a recursive formula for $\mathcal{N}$; see Proposition 28 in Section 4.1.1. We then show that this relation is also satisfied by c; see Proposition 29 in Section 4.1.2.

4.1.1 A recursive formula for $\mathcal{N}$ The main goal of this section is to present and prove a recursive formula for $\mathcal{N}$; this formula appears in Proposition 28. Giving a precise statement of this formula will require some preparation. Let $H\colon \mathbb{R}^2 \to \mathbb{R}$ be a Morse function. Assume H admits at least one saddle point. For a saddle point s of H, we consider the level set $H^{-1}(H(s))$ and let $C(s)$ be the connected component of s in this set; $C(s)$ is the union of the stable and the unstable manifold of s for the flow (ϕ_H^t), and it is homeomorphic to a bouquet of two circles (see Figure 5 below). Let D be one of the two bounded connected components of $\mathbb{R}^2 \setminus C(s)$. The function $H_D = H|_{\overline{D}} - H(s)$ vanishes on the boundary of D. We would like to relate $\mathcal{N}(H)$ to $\mathcal{N}(H_D)$. However, H_D is not smooth. To circumvent this problem we will introduce an appropriate class $\overline{H}_D$ of smoothings of H_D. Then the recursive formula in Proposition 28 will express $\mathcal{N}(H)$ in terms of $\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1})$, where T_0, T_1 are the two bounded connected components of the complement of $C(s_0)$ for

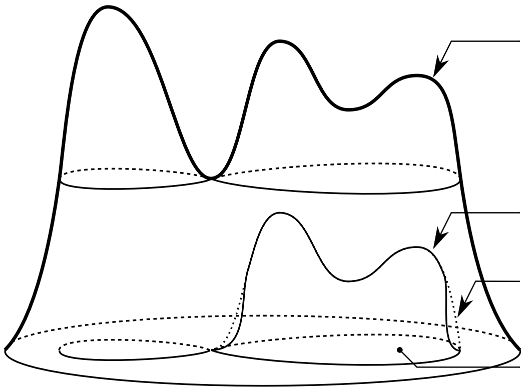

Figure 4: Graphs of $H, H_D, \overline{H}_D$

the outermost saddle point s_0 of H (see Notation 27). We will keep the following notation throughout Section 4.

Notation 24 (see Figure 4) Assume that $H|_D > H(s)$ near the boundary of D (we leave it to the reader to adapt the notation in the opposite case). We denote by $\mathcal{E}_D$ the set of all functions $\overline{H}_D$ of the form

$$\overline{H}_D(x) = \begin{cases} 0 & \text{if } x \notin D, \\ \rho \circ H(x) - H(s) & \text{if } x \in D \setminus D', \\ H(x) - H(s) & \text{if } x \in D', \end{cases}$$

where:

- $D' \subsetneq D$ is an open disk which contains all the 1–periodic orbits of H in D and such that for some constant $h > H(s)$, $H|_{\partial D'} = h$.
- ρ: $(H(s), h) \to [H(s), h)$ is a smooth function such that for some $\varepsilon > 0$,
  - $\rho(t) = H(s)$ for all $t \in (H(s), H(s) + \varepsilon]$,
  - $\rho(t) = t$ for all $t \in [h - \varepsilon, h)$,
  - $0 < \rho'(t) < \tau$ for all $t \in (H(s) + \varepsilon, h - \varepsilon)$, where $\tau > 1$ denotes the smallest period of orbits of H in $D \setminus D'$.

In the sequel, the notation $\overline{H}_D$ will be used for any function in $\mathcal{E}_D$. The relevant properties of the functions $\overline{H}_D$ are summarized in the next lemma, whose proof is straightforward.

Lemma 25 *Every Hamiltonian $\overline{H}_D \in \mathcal{E}_D$ is smooth and possesses the following properties:*

(1) *The support of* $\overline{H}_D$ *is a disk* D'' *included in* D.

(2) $\overline{H}_D = H - H(s)$ *on a closed disk included in the interior of* D'', *which contains all the fixed points of both* ϕ_H^1 *and* $\phi_{\overline{H}_D}^1$ *that are contained in the interior of* D''.

(3) $\mathcal{A}_{\overline{H}_D}(x) = \mathcal{A}_H(x) - H(s)$ *for every such fixed point* x *of* ϕ_H^1 *in* D, *so that, in particular, all the elements in* $\mathcal{E}_D$ *have the same spectrum.*

Moreover, the set $\mathcal{E}_D$ *is convex, and the continuous function* H_D *that coincides with* $H - H(s)$ *on* D *and vanishes elsewhere, belongs to its* C^0*–closure.*

Remark 26 The continuity and spectrality properties imply that the spectral invariant c is constant on the set $\mathcal{E}_D$. Moreover this constant value is $c(H_D)$.

Before giving a precise statement of the recursive formula promised at the beginning of this section, we need to introduce a new set of notation that will follow us throughout the proof.

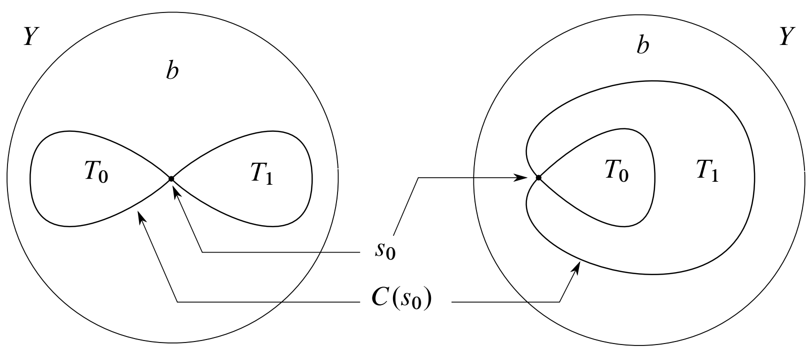

Figure 5: Notation; $Y, s_0, C(s_0), b, T_0, T_1$: the two cases

Notation 27 (see Figure 5) Let H be a Morse function which admits at least one saddle point.

(1) We denote by $\mathrm{supp}(H)$ the support of H in $\mathbb{R}^2$, and by Y the unbounded component of the closure of the complement of $\mathrm{supp}(H)$.

(2) There exists a saddle point of H, which we denote by s_0, such that the interior of the outer component of $\mathrm{supp}(H) \setminus C(s_0)$ contains no critical point of H. Here is a brief argument as to why this outermost saddle s_0 must exist. For every saddle s choose a nearby periodic orbit surrounding $C(s)$, and remove the (open) disk bounded by this orbit. Likewise for every local maximum or minimum remove a small open disk bounded by a periodic orbit. We are left

with a set A which is a disk with a certain number of holes, foliated by level sets of H, containing no critical point of H. According to the Poincaré–Hopf theorem, the Euler characteristic of A is zero, thus A is an annulus, which means there was only one hole after all. Hence, there was either only one critical point (local maximum or minimum), or there was an outermost saddle.

(3) We denote by b, T_0, T_1 the three connected components of $\mathrm{int}(\mathrm{supp}(H)) \setminus C(s_0)$, b being the outer one; here $\mathrm{int}(\mathrm{supp}(H))$ stands for the interior of $\mathrm{supp}(H)$. Note that b contains no critical point of H.

(4) If moreover $H > 0$ on b, we set

$$\mathcal{N}_b = \min\{\mathcal{A}_H(x) : x \text{ fixed point of } \phi_H^1 \text{ in } b\}.$$

In the case where H has no saddle, we set b to be the interior of the support of H and define $\mathcal{N}_b$ by the same formula when $H > 0$ on b. Note that when H is positive on b, the orbits inside b turn in the negative direction and hence $\rho(x) \leqslant 0$ for every fixed point $x \in b$.

The above construction may be applied to T_0, T_1 giving rise to two sets $\mathcal{E}_0$ and $\mathcal{E}_1$ of functions $\overline{H}_{T_0}$ and $\overline{H}_{T_1}$. We are now ready to state our recursive formula.

Proposition 28 *If H has no saddle points, then*

$$\mathcal{N}(H) = \begin{cases} 0 & \text{if } H|_b < 0, \\ \mathcal{N}_b & \text{if } H|_b > 0. \end{cases}$$

If H has at least one saddle, then

$$\mathcal{N}(H) = \begin{cases} \max(0, H(s_0) + \max(\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1}))) & \text{if } H|_b < 0, \\ \min(\mathcal{N}_b, H(s_0) + \max(\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1}))) & \text{if } H|_b > 0. \end{cases}$$

Proof We first assume that H has no saddle point. In this case H has only one critical point p not in Y, which is either a maximum or a minimum. The complement of $Y \cup \{p\}$ is foliated by invariant closed curves surrounding p, and it is well known that there exists a compactly supported symplectic diffeomorphism Ψ such that $H \circ \Psi$ is a radial function as in Section 2.5. We have already computed the value of $\mathcal{N}$ in this case; see Equation (3).

Let us now assume that H has at least one saddle and is negative on b. In that case, we need to prove that

$$\mathcal{N}(H) = \max\big(0, H(s_0) + \max(\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1}))\big). \tag{6}$$

Note that two fixed points of ϕ_H^1 that lie respectively in T_0 and T_1 are always unlinked. Such fixed points are also unlinked with the critical point s_0 and all the points in Y.

Moreover, every orbit in b rotates in the positive direction and hence the base b contains no negative fixed point. It follows that the maximal negative unlinked sets (mnus's) of H are exactly the sets of fixed points that are of the form $X = Y \cup \{s_0\} \cup X_0 \cup X_1$, where X_0 is a set that is maximal for inclusion among the negative unlinked sets of ϕ_H^1 that are included in T_0, and likewise for X_1; for short we say that X_0 and X_1 are mnus's for the restrictions $H|_{T_0}$ and $H|_{T_1}$. As a consequence,

$$\mathcal{N}(H) = \max\Big(0, H(s_0), \inf_{X_0} \sup_{x \in X_0} \mathcal{A}_H(x), \inf_{X_1} \sup_{x \in X_0} \mathcal{A}_H(x)\Big), \tag{7}$$

where infima are taken over the mnus's X_0 of $H|_{T_0}$ and the mnus's X_1 of $H|_{T_1}$. For $i = 1, 2$, the Hamiltonian $\overline{H}_{T_i}$ has been built so that the mnus's of $\overline{H}_{T_i}$ are of the form $\overline{X}_i = Y_i \cup X_i$ where X_i is a mnus of $H|_{T_i}$ and Y_i is the complement of the support of $\overline{H}_{T_i}$. We can compute the maximum of the action on such a set:

$$\begin{aligned} \sup_{x \in \overline{X}_i} \mathcal{A}_{\overline{H}_{T_i}}(x) &= \max\Big(0, \sup_{x \in X_i} \mathcal{A}_{\overline{H}_{T_i}}(x)\Big) \\ &= \max\Big(H(s_0), \sup_{x \in X_i} \mathcal{A}_H(x)\Big) - H(s_0). \end{aligned} \tag{8}$$

We then deduce (6) from (7) and (8).

We now assume that H is positive on b; recall that this means every fixed point in b is a negative fixed point. A nontrivial orbit in b is linked with any other fixed point of H that it encloses. Therefore, the mnus's of H are of two possible types:

(A) $X = Y \cup \{x\}$ where x is a fixed point of ϕ_H^1 in b.

(B) $X = Y \cup \{s_0\} \cup X_0 \cup X_1$ where X_0 is a mnus of $H|_{T_0}$ and X_1 is a mnus of $H|_{T_1}$.

Thus,

$$\mathcal{N}(H) = \min\Big(\inf_{X \text{ of type A}} \sup_{x \in X} \mathcal{A}_H(x), \inf_{X \text{ of type B}} \sup_{x \in X} \mathcal{A}_H(x)\Big).$$

The same argument as in the case $H|_b < 0$ gives

$$\begin{aligned} \inf_{X \text{ of type B}} \sup_{x \in X} \mathcal{A}_H(x) &= \max(0, H(s_0) + \mathcal{N}(\overline{H}_{T_0}), H(s_0) + \mathcal{N}(\overline{H}_{T_1})) \\ &= H(s_0) + \max(\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1})). \end{aligned}$$

The last equality follows from the fact that $H(s_0) > 0$. Note that since $H > 0$ on b, the nontrivial orbits in b enclose disks with negative area and hence have positive actions.

Therefore, for a mnus of the form $Y \cup \{x\}$, where x is a fixed point of ϕ_H^1 in b, the maximum of the action is precisely the action of x. Thus,

$$\inf_{X \text{ of type A}} \sup_{x \in X} \mathcal{A}_H(x) = \mathcal{N}_b,$$

and we get the equality $\mathcal{N}(H) = \min\big(\mathcal{N}_b, H(s_0) + \max(\mathcal{N}(\overline{H}_{T_0}), \mathcal{N}(\overline{H}_{T_1}))\big)$, as we wished. □

4.1.2 Proof of $c = \mathcal{N}$ Let c be a formal spectral invariant on the plane $\mathbb{R}^2$. In this section we prove that $c = \mathcal{N}$ for all Morse functions on the plane. The main step towards this will be to prove the following proposition.

Proposition 29 *If H has no saddle points, then*

$$c(H) = \begin{cases} 0 & \text{if } H|_b < 0, \\ \mathcal{N}_b & \text{if } H|_b > 0. \end{cases}$$

If H has at least one saddle, then

$$c(H) = \begin{cases} \max(0, H(s_0) + \max(c(\overline{H}_{T_0}), c(\overline{H}_{T_1}))) & \text{if } H|_b < 0, \\ \min(\mathcal{N}_b, H(s_0) + \max(c(\overline{H}_{T_0}), c(\overline{H}_{T_1}))) & \text{if } H|_b > 0. \end{cases}$$

Before giving the proof of this proposition, we explain how to deduce from it that $c = \mathcal{N}$ for Morse functions on the plane.

Proof of $c = \mathcal{N}$ for Morse functions on the plane We argue by induction on the number of saddles of H. First, it follows immediately from Propositions 28 and 29 that $\mathcal{N}$ and c coincide on functions having no saddle points. Then, assume that $c = \mathcal{N}$ for all Morse functions having at most k saddle points and let H be a Morse function with $k+1$ saddle points. Then, $\overline{H}_{T_0}$ and $\overline{H}_{T_1}$ both have at most k saddle points, hence $c(\overline{H}_{T_0}) = \mathcal{N}(\overline{H}_{T_0})$ and $c(\overline{H}_{T_1}) = \mathcal{N}(\overline{H}_{T_1})$. Now using Propositions 28 and 29 again, we deduce $c(H) = \mathcal{N}(H)$. □

We now turn to the proof of Proposition 29. We note once and for all that since $\overline{H}_{T_0}$ and $\overline{H}_{T_1}$ are supported on disjoint disks, the max formula applies. Thus in the case when H has at least one saddle, the formula we wish to prove reduces to

$$c(H) = \begin{cases} \max(0, H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})) & \text{if } H|_b < 0, \\ \min(\mathcal{N}_b, H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})) & \text{if } H|_b > 0. \end{cases}$$

Proof The proof will be split into the two cases $H|_b < 0$ and $H|_b > 0$.

Case 1 $H < 0$ on b.

First assume that H admits no saddle point. Then, $H \leqslant 0$ hence $c(H) \leqslant 0$ by monotonicity. On the other hand, we have $c(H) \geqslant 0$ by positivity. Thus $c(H) = 0$ as claimed.

We now assume that H has a saddle point, so that we can use Notation 24 and let $\varepsilon > 0$. Then, it is possible to find at a C^0–distance less than ε from H a function that can be written as a sum $F + \overline{H}_{T_0} + \overline{H}_{T_1}$, where F is a smooth nonpositive function with exactly two critical values: 0 with critical locus $F^{-1}(0) = Y$, and $H(s_0)$ with critical locus $F^{-1}(H(s_0)) = \overline{T_0 \cup T_1}$. The Lipschitz property of c yields

$$|c(H) - c(F + \overline{H}_{T_0} + \overline{H}_{T_1})| \leqslant \varepsilon.$$

We will prove that

$$c(F + \overline{H}_{T_0} + \overline{H}_{T_1}) = \max(0, H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})). \tag{9}$$

By taking arbitrarily small ε, it follows immediately that the formula of the proposition holds in this case.

We consider the 1–parameter family of functions $\sigma \mapsto K_\sigma = \sigma F + \overline{H}_{T_0} + \overline{H}_{T_1}$. For $\sigma \in [0, 1]$, the spectrum of K_σ is given by:

$$\operatorname{spec}(K_\sigma) = \operatorname{spec}(\sigma F) \cup (\sigma H(s_0) + \operatorname{spec}(\overline{H}_{T_0} + \overline{H}_{T_1})).$$

Note that $\operatorname{spec}(\sigma F)$ only contains nonpositive values. Therefore, by the positivity property of c, we know that $c(K_\sigma)$ is either 0 or belongs to $\sigma H(s_0) + \operatorname{spec}(\overline{H}_{T_0} + \overline{H}_{T_1})$. For $\sigma = 0$ we have $c(K_0) = c(\overline{H}_{T_0} + \overline{H}_{T_1}) \geqslant 0$. The continuity of c imposes that as long as $\sigma H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1}) \geqslant 0$, one has $c(K_\sigma) = \sigma H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})$, and $c(K_\sigma) = 0$ in the opposite case. In particular for $\sigma = 1$, we get (9) (see Figure 6).

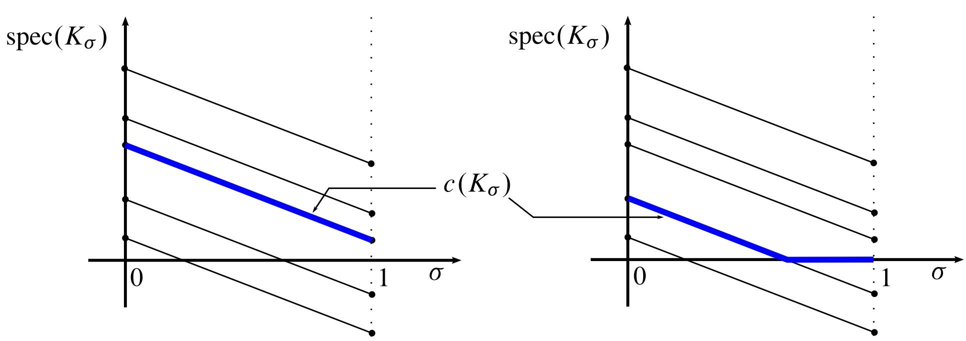

Figure 6: Bifurcation diagram for the spectrum of the deformation K_σ: the case $H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1}) > 0$ (left) and the opposite case (right)

Case 2 $H > 0$ on b.

This case is much more complicated than the previous one and we will divide its proof into several claims. We will first prove, in Claims 30 and 32, that

$$c(H) \leqslant \min(\mathcal{N}_b, H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})).$$

Claim 30 *Assume that H is positive on b. Then*

$$c(H) \leqslant \mathcal{N}_b.$$

The proof of this claim will require us to compute c explicitly for a class of very simple functions. This is the content of the following lemma.

Lemma 31 (see Figure 7) *Define a radial function H by $H(x,y) = f(\pi(x^2+y^2))$ for some function $f\colon [0,+\infty) \to [0,+\infty)$ satisfying the following conditions for some $A > 0$:*

- *$f(a) = 0$ for all $a \geqslant A$.*
- *$f(0) > A$ and $f'(0) = 0$.*
- *f'' vanishes at a unique point a_0 in $(0, A)$.*

Then $c(H) = f(a_1) + a_1$ where a_1 is the unique real number for which $f'(a_1) = -1$ and $f''(a_1) > 0$.

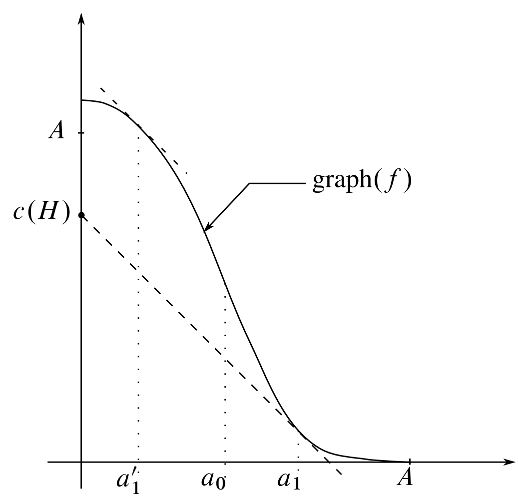

Figure 7: Graph of f satisfying the assumptions of Lemma 31: a_1' and a_1 are the only points where $f' = -1$

Proof First note that it follows from the assumption that $f' < 0$ on $(0, A)$. Moreover, f' decreases between 0 and a_0 and then increases between a_0 and A. Thus f' attains its minimum at a_0 and $f'(a_0) < -1$. As explained in Section 2.5, the fixed points of ϕ_H^1 correspond to the values of a for which, $f'(a)$ is an integer. We see that in our case, for each integer $f'(a_0) < k < 0$, we have either zero or two possibilities that we denote $a'_k < a_k$. We have also seen in Section 2.5 how to compute the action of such fixed points. In particular, all points have nonnegative action, and the critical point 0 has action $> A$. A crucial remark for our purpose is that the point a_1 corresponds to the strict minimum of all nonzero actions of the fixed points of ϕ_H^1.

Let us first study the case where our function satisfies $f' > -2$. The spectrum of H is made up of four values corresponding to the actions of 0, a'_1, a_1 and the points outside the support. The spectral invariant c cannot be reached outside the support by nondegeneracy. Moreover, the action of a'_1 is larger than that of 0 and the action of 0 is larger than A. Now, the area of the support of H is less than A and thus by the energy-capacity inequality the spectral invariant cannot be reached at any of these two points and therefore is reached at a_1 as claimed.

Let us now turn to the general case where f' is not assumed larger than -2. Let $\tilde{f}$ be a function satisfying the assumptions of the lemma and with $\tilde{f}' > -2$. We leave to the reader to check that there exists a continuous path between f and $\tilde{f}$ within the functions satisfying the assumptions of the lemma. We consider the bifurcation diagram of spectra obtained from this deformation. For $\tilde{f}$, the spectral invariant is reached at the point a_1 (which moves along the deformation but never disappears). Now it follows from the remark made above that the path in the bifurcation diagram associated to a_1 has no bifurcation, and therefore that the spectral invariant for f is reached at a_1. □

We are now ready to prove the claim.

Proof of Claim 30 Let x_0 be a fixed point of ϕ_H^1 in b for which $\mathcal{N}_b = \mathcal{A}_H(x_0)$. We need to prove that $c(H) \leqslant \mathcal{A}_H(x_0)$. Denote by α_0 the area enclosed by the orbit of x_0. If $\alpha_0 = 0$, which means that x_0 is the unique critical point of H, then $\mathcal{A}_H(x_0) = \max(H) \geqslant c(H)$. Assume now that $\alpha_0 > 0$.

By conjugating with an area-preserving diffeomorphism and using symplectic invariance of c, we can assume that the 1–periodic orbits of H in the base b are all included in an annulus $b' = \{(x, y) \in \mathbb{R}^2 : \alpha < \pi(x^2 + y^2) < \alpha'\} \subset b$ having the same outer boundary as b, and that on this annulus b', H has the form of Section 2.5, ie, $H(x, y) = f(\pi(x^2 + y^2))$, for all $(x, y) \in b'$, for some smooth decreasing function $f\colon (\alpha, \alpha') \to \mathbb{R}$. Note that for $\pi(x^2 + y^2) \geqslant \alpha'$, one has $H(x, y) = 0$. Also note that $f'(\alpha_0) = -1$. To see this, assume that we have $f'(\alpha_0) < -1$, and consider the smallest value $\alpha_1 > \alpha_0$ for which $f'(\alpha_1) = -1$. Then, we see easily by considering the diagram of Figure 3 that the action value associated to α_1 is smaller than that of α_0. This would then contradict the definition of α_0.

We now choose a radial Hamiltonian $H_1 \geqslant H$ given by $H_1(x,y) = f_1(\pi(x^2+y^2))$, for a function $f_1\colon [0,+\infty) \to [0,+\infty)$ satisfying the assumptions of Lemma 31 and the following additional properties (see Figure 8): $f_1(\alpha_0) = f(\alpha_0)$, $f_1'(\alpha_0) = f'(\alpha_0) = -1$ and $f''(\alpha_0) > 0$. By Lemma 31, $c(H_1) = \mathcal{A}_H(x_0)$. As a consequence, we obtain $c(H) \leqslant \mathcal{A}_H(x_0)$ using monotonicity. □

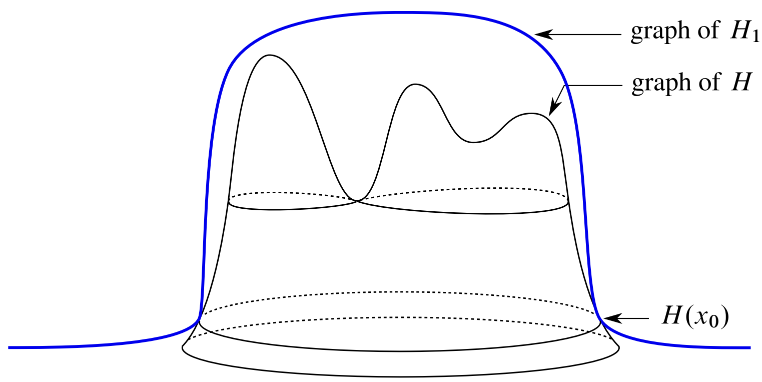

Figure 8: (proof of Claim 30) Construction of $H_1 \geqslant H$ with $c(H_1) = \mathcal{N}_b$

Claim 32 *If H has at least one saddle and if $H > 0$ on b, then*

$$c(H) \leqslant H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1}).$$

Proof Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth compactly supported nonnegative function that equals $H(s_0)$ on the support of H, has only 0 and $H(s_0)$ as critical values, and whose flow has no nontrivial 1–periodic orbit (see Figure 9). By construction $H \leqslant F + H_{T_0} + H_{T_1}$, where for $i = 1, 2$, H_{T_i} is the continuous function that coincides with $H - H(s_0)$ on T_i and vanishes elsewhere. Hence $c(H) \leqslant c(F + H_{T_0} + H_{T_1})$. Let $\varepsilon > 0$. According to Lemma 25, the functions $\overline{H}_{T_0}$, $\overline{H}_{T_1}$ can be chosen so that their C^0–distance to respectively H_{T_0} and H_{T_1} is arbitrarily small. The continuity of spectral invariants gives

$$c(H) \leqslant c(F + \overline{H}_{T_0} + \overline{H}_{T_1}) + \varepsilon.$$

By the Lipschitz property we get

$$\begin{aligned} c(H) &\leqslant c(\overline{H}_{T_0} + \overline{H}_{T_1}) + \max F + \varepsilon \\ &= H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1}) + \varepsilon. \end{aligned}$$

Now according to Remark 26, the values of $c(\overline{H}_{T_0})$ and $c(\overline{H}_{T_1})$ are independent of

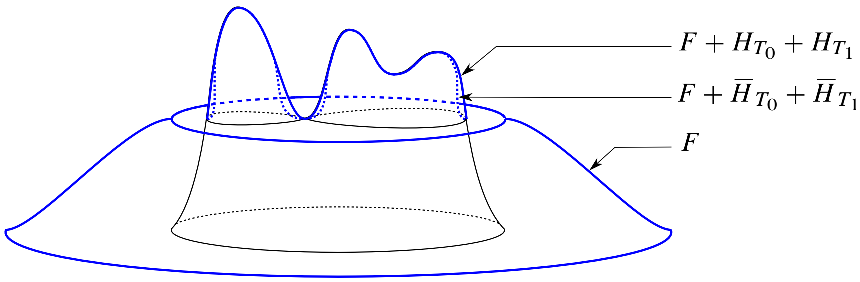

Figure 9: (proof of Claim 32) Construction of $F + H_{T_0} + H_{T_1} \geqslant H$ with $c(F + H_{T_0} + H_{T_1}) \leqslant H(s_0) + c(H_{T_0} + H_{T_1})$

the choices of $\overline{H}_{T_0}$ and $\overline{H}_{T_1}$. This means that ε can be made arbitrarily small and concludes the proof. □

By Claims 30 and 32, we have established upper bounds required for Proposition 29. We now turn to the proof of the lower bounds. The next claim achieves the case of Hamiltonians without any saddle point.

Claim 33 *Assume the function H is Morse, nonnegative and has no saddle point. Then $c(H) = \mathcal{N}_b$.*

Proof By nondegeneracy, we have $c(H) > 0$. Thus, $c(H)$ is the action of a point in the interior of the support of H, hence, by definition, cannot be smaller than $\mathcal{N}_b$. By Claim 30, we get $c(H) = \mathcal{N}_b$. □

End of the proof of Proposition 29 It remains to establish that

$$c(H) \geqslant \min(\mathcal{N}_b, H(s_0) + c(\overline{H}_{T_0} + \overline{H}_{T_1})). \tag{10}$$

First assume that $c(H)$ is the action of a fixed point in b. Then, $c(H) \geqslant \mathcal{N}_b$, by definition of $\mathcal{N}_b$. By Claim 30, we get $c(H) = \mathcal{N}_b$ which implies (10).

Assume now that $c(H)$ is not attained on b. Then, by Claim 30, $c(H) < \mathcal{N}_b$. Similarly to the argument used in Case 1, for all $\varepsilon > 0$, we can find at C^0–distance less than ε from H a Hamiltonian of the form $F + \overline{H}_{T_0} + \overline{H}_{T_1}$, where F is a smooth nonnegative function, with only two critical values: 0, attained on Y, and $h = H(s_0)$ attained on a neighborhood of $\overline{T_0 \cup T_1}$. We choose F close enough to H on b so it has no nontrivial 1–periodic orbit with action in $(0, \mathcal{N}_b)$. Since $|c(H) - c(F + \overline{H}_{T_0} + \overline{H}_{T_1})| \leqslant \varepsilon$, we have for ε small enough $c(F + \overline{H}_{T_0} + \overline{H}_{T_1}) < \mathcal{N}_b$. This implies in particular that $c(F + \overline{H}_{T_0} + \overline{H}_{T_1})$ is attained in $T_0 \cup T_1$. Similarly to Case 1, we will consider a deformation of the form $K_\sigma = F_\sigma + \overline{H}_{T_0} + \overline{H}_{T_1}$, with F_0 arbitrarily close to F and $F_1 = 0$ to prove that $c(F_0 + \overline{H}_{T_0} + \overline{H}_{T_1}) = h + c(\overline{H}_{T_0} + \overline{H}_{T_1})$, which in turn implies the same equality for $c(H)$. Nevertheless, we will have to be slightly more careful in the way we construct it.

Claim 34 *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth nonnegative function, with only two critical values: 0 attained on the complement of an open disk D, and $h > 0$ attained on a smaller closed disk $D' \subset D$. Then, arbitrarily C^1–close to F, there exists a function $\widetilde{F}$ that coincides with F on $D' \cup (\mathbb{R}^2 \setminus D)$, and a 1–parameter family of smooth functions $(F_\sigma)_{\sigma \in [0,1]}$ with $F_0 = \widetilde{F}$, $F_1 = 0$ and the following two properties:*

1. *F_σ has only two critical values, 0 and $\max F_\sigma = (1-\sigma)h$.*
2. *Every Lipschitz function δ defined on an interval $I \subset [0,1]$ such that $\delta(\sigma) \in \operatorname{spec}(F_\sigma)$ for all $\sigma \in I$, satisfies $\delta'(\sigma) \geqslant -h$ almost everywhere.*

Remark 35 The second property, which may appear rather strange at first glance, simply states that all the curves in the bifurcation diagram of the deformation $(F_\sigma)_{\sigma \in [0,1]}$ have slope $\geqslant -h$.

We assume this claim for the time being and postpone its proof to the end of this section. As explained above, we let $K_\sigma = F_\sigma + \overline{H}_{T_0} + \overline{H}_{T_1}$, where F_σ is a 1–parameter family as provided by Claim 34. Of course, in our settings, the disks D' and D of Claim 34 are respectively a neighborhood of $\overline{T_0 \cup T_1}$ and the interior of the support of H. For all $\sigma \in [0,1]$, the spectrum of K_σ is given by

$$\operatorname{spec}(K_\sigma) = \operatorname{spec}(F_\sigma) \cup \big((1-\sigma)h + \operatorname{spec}(\overline{H}_{T_0} + \overline{H}_{T_1})\big).$$

The bifurcation diagram $\bigcup_{\sigma \in [0,1]} \{\sigma\} \times \operatorname{spec}(K_\sigma)$ is the union of the horizontal line corresponding to the action 0, parallel lines with slope $-h$ that correspond to the subset $(1-\sigma)h + \operatorname{spec}(\overline{H}_{T_0} + \overline{H}_{T_1})$ and pieces of curves corresponding to the actions of the nontrivial 1–periodic orbits of F_σ (see Figure 10).

These pieces of curves never decrease faster than $-h$, as follows from Property 2 of Claim 34. Moreover at $\sigma = 0$ these curves are all above the value $\mathcal{N}_b$. Thus, no curve in the bifurcation diagram that starts from a value $> \mathcal{N}_b$ crosses a line of slope $-h$ with initial value $< \mathcal{N}_b$. Since $c(K_0)$ is smaller than $\mathcal{N}_b$ and belongs

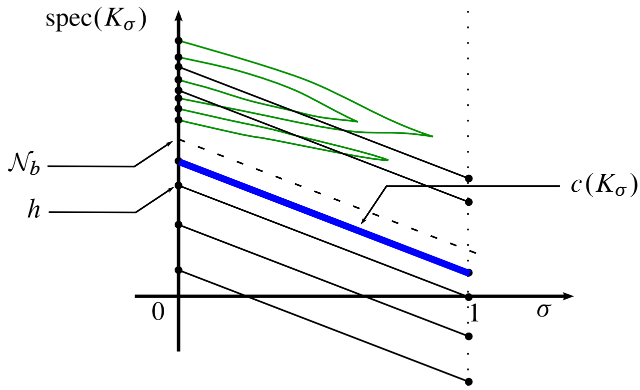

Figure 10: The bifurcation diagram of the deformation K_σ

to the spectrum, $c(K_\sigma)$ remains on the line $(\sigma, c(K_0) - \sigma h)_{\sigma \in [0,1]}$, until it reaches the value 0. After that point, if it exists, the positivity of c implies that it remains constant equal to zero. Now, since H is positive on b and s_0 is a nondegenerate saddle, one of the two functions $\overline{H}_{T_0}$ and $\overline{H}_{T_1}$ must be positive near the boundary of its support. Thus, it is a consequence of the max formula and Lemma 37 below that $c(K_1) = c(\overline{H}_{T_0} + \overline{H}_{T_1}) = \max(c(\overline{H}_{T_0}), c(\overline{H}_{T_1}))$ is positive. This implies $c(K_0) = h + c(K_1) = h + c(\overline{H}_{T_0} + \overline{H}_{T_1})$ and we see that the proof of Proposition 29 is achieved up to Claim 34 and Lemma 37 below. □

Remark 36 A consequence of Proposition 29 and the positivity of c is that in the case $c(H) < \mathcal{N}_b$, which we were just considering, we have the inequality $h \leqslant \mathcal{N}_b$. Since $\operatorname{spec}(F_1) = \{0\}$, this implies in particular that all the curves in the bifurcation diagram corresponding to the nontrivial periodic orbits of F_σ that start at $\sigma = 0$ must die at some point. Moreover, it will follow from the proof of Claim 34 that F_σ can be constructed so that no birth occurs in its bifurcation diagram. This is illustrated in Figure 10.

Lemma 37 *If H is positive on b, then $c(H) > 0$.*

Proof Up to conjugation by an area-preserving diffeomorphism, H is larger than a smooth radial Hamiltonian of the form of Section 2.5: $F_1(x,y) = f_1(\pi(x^2+y^2))$, for all $(x,y) \in \mathbb{R}^2$, with $f_1\colon [0,+\infty) \to \mathbb{R}$ having a simple profile: for some real numbers $0 < a_0 < a_1 < a_2$, it is strictly increasing on $[0,a_1]$, strictly decreasing on $[a_1,a_2]$, $f_1(a_0) = 0$ and f_1 vanishes on $[a_2,+\infty)$ (note that $f_1(0)$ may be negative). By monotonicity, we only have to verify that $c(F_1) > 0$ to prove $c(H) > 0$.

Let F_0 be a smooth nonnegative approximation of the function $\max(0, F_1)$. By nondegeneracy, $c(F_0) > 0$. Now, let F_σ be a smooth decreasing deformation from F_0

to F_1. We may also assume that all the functions F_σ are radial, hence of the form $F_\sigma(x,y) = f_\sigma(\pi(x^2+y^2))$ and that all the functions f_σ are increasing on $[0,a_0]$ and coincide with f_1 on $[a_0,+\infty)$. All the orbits of F_σ located in the circle of area a_0 have negative action. Thus, the nonnegative part of the spectrum remains unchanged along the deformation. As a consequence, using spectrality and continuity we obtain $c(F_1) = c(F_0) > 0$, and thus $c(H) > 0$. □

It only remains to construct the deformation of Claim 34.

Proof of Claim 34 Up to conjugation with an area-preserving diffeomorphism, we may assume that $D \setminus D'$ is given in coordinates by $\{(x,y) \in \mathbb{R}^2 : \alpha < \pi(x^2+y^2) < \beta\}$ and F is a radial Hamiltonian, as in Section 2.5: for all $(x,y) \in \mathbb{R}^2$, $F(x,y) = f(\pi(x^2+y^2))$, where $f = h$ on $[0,\alpha]$, f decreases on (α,β) and $f = 0$ on $[\beta,+\infty)$. We denote $g = -f'$. We will construct the deformation F_σ as radial functions

$$F_\sigma(x,y) = \int_{\pi(x^2+y^2)}^{+\infty} g_\sigma(u)\,du,$$

where g_σ will be a deformation such that $g_1 = \tilde{g}$ where $\tilde{g}$ is a function arbitrarily close to g, $g_0 = 0$ and g_σ vanishes on $[0,\alpha]$ and $[\beta,+\infty)$ for all σ. Recall from Section 2.5 that the spectrum of F_σ is calculated by considering the points where g_σ is an integer.

We first perturb g so that the set of $s \in (\alpha,\beta)$ such that $g(s)$ is an integer is finite. Then, we let $\tilde{g}$ be a smooth C^0–perturbation of g obtained by flattening g in a small neighborhood of all the points s where $g(s)$ is an integer (see Figure 11). The function $\tilde{F}$ is then defined as

$$\tilde{F}(x,y) = \int_{\pi(x^2+y^2)}^{+\infty} \tilde{g}(u)\,du.$$

To construct the deformation g_σ from $\tilde{g}$ to 0, we first introduce the following set of

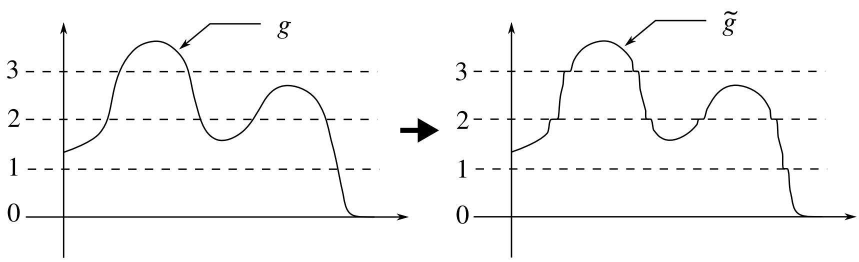

Figure 11: The deformation from g to $\tilde{g}$

notation. Let N be the integer part of $\max \widetilde{g}$. For all integers $k = 0, 1, \dots, N+1$, we set the truncated functions $\gamma_k = \min(g, k)$ and $\delta_k = \gamma_{k+1} - \gamma_k$. Clearly, $\gamma_{N+1} = \widetilde{g}$ and $\gamma_0 = 0$, hence $\widetilde{g} = \sum_{k=0}^{N} \delta_k$. The effect of the perturbation $\widetilde{g}$ is that each function γ_k, δ_k is smooth whereas an analogous definition for g would only yield continuous functions. We also set

$$h_k = \int_0^{+\infty} \delta_k(u)\, du,$$

so that $h = \sum_{k=0}^{N} h_k$. Finally, let $\tau_k = (h_k + \cdots + h_N)/h$. In particular, $0 = \tau_{N+1} < \tau_N < \cdots < \tau_1 < \tau_0 = 1$.

We can now define the deformation

$$g_\sigma(s) = \gamma_k(s) + \frac{h}{h_k}(\tau_k - \sigma)\delta_k(s), \tag{11}$$

for all $k = 0, \dots, N$, $\sigma \in [\tau_{k+1}, \tau_k)$, and $s \in [0, +\infty)$ (see Figure 12). Let us check that this deformation suits our needs.

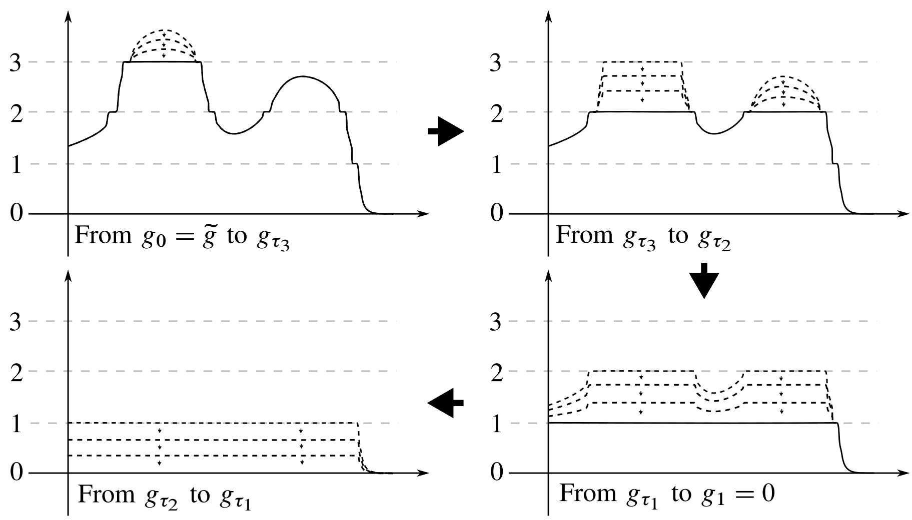

Figure 12: The deformation from $\widetilde{g}$ to 0 via g_{τ_3}, g_{τ_2} and g_{τ_1}

First, note that $\sigma \mapsto g_\sigma$ is continuous on $[0, 1]$ in the C^0–topology. This follows from the fact that when σ evolves from τ_{k+1} to τ_k, the factor $(h/h_k)(\tau_k - \sigma)$ evolves from 1 to 0, and so g_σ evolves from γ_{k+1} to γ_k. As a consequence of this continuity, the Property (1) of Claim 34 can be checked by considering separately each interval of deformation (τ_{k+1}, τ_k). The maximum of F_σ is the total integral $\int_0^{+\infty} g_\sigma(u)\, du$. By

Equation (11), its rate of decrease on the interval (τ_{k+1}, τ_k) is

$$\frac{h}{h_k} \int_0^{+\infty} \delta_k(u)\, du = h.$$

This proves the first property.

As with the first one, the second property in Claim 34 only needs to be established on each interval (τ_{k+1}, τ_k). As we already recalled, it follows from Section 2.5 that the spectrum of F_σ can be computed by only considering the points where g_σ is an integer ℓ. It turns out that along each interval (τ_{k+1}, τ_k) and for each integer ℓ, the set of these points remains unchanged. Moreover, for each such point s, the action is obtained as the area of the shaded region in Figure 3. This area has two parts, a rectangle part whose area is ℓs and an integral part whose area is $\int_s^{+\infty} g_\sigma(u)\, du$. Along the deformation interval (τ_{k+1}, τ_k), the rectangle part of the area remains constant, whereas the integral part decreases at the rate

$$\frac{h}{h_k} \int_s^{+\infty} \delta_k(u)\, du \leqslant h.$$

As a consequence, over the interval (τ_{k+1}, τ_k), the action spectrum of F_σ is a finite union of nonincreasing smooth curves whose slopes are never smaller than $-h$. Property (2) of Claim 34 follows. □

4.2 Theorem 4 for Morse functions on closed surfaces of genus $\geqslant 1$

In this section, we prove the equality $c = \mathcal{N}$ for Morse functions on closed surfaces of positive genus. This is done in two steps. We first establish a formula which reduces the problem of computing $\mathcal{N}$ to computations for Hamiltonians supported in disks; see Proposition 39 in Section 4.2.1. We then show that this formula is also satisfied by c; see Proposition 40 in Section 4.2.2.

4.2.1 A formula for $\mathcal{N}$ Let Σ denote a closed surface of positive genus and consider a Morse function $H\colon \Sigma \to \mathbb{R}$. The goal of this section is to present a formula which reduces computing $\mathcal{N}(H)$ to computing $\mathcal{N}$ on the restriction of ϕ_H^1 to a collection of invariant disks; see Proposition 39 below.

Let s be a saddle point of H and denote by $C(s)$ the connected component of $H^{-1}(H(s))$ which contains s. Note that $C(s)$ is a circle pinched at s. Equivalently, we can view $C(s)$ as a union of two circles $C_0(s), C_1(s)$ whose intersection is $\{s\}$. We will say that the saddle s is *essential* if at least one of these two circles is not contractible in Σ. The following proposition describes a decomposition of the surface Σ obtained by cutting it along the pinched circles of essential saddles. We postpone the proof to the end of this section.

Proposition 38 (see Figure 13) *Let Σ' be the open and disconnected surface obtained from Σ by removing $C(s)$ for each essential saddle s of H. Let S denote a connected component of Σ'. Then:*

(1) *S is either a disk or a cylinder.*

(2) *If S is a cylinder then ϕ_H^1 has no contractible fixed point in S.*

(3) *The map $i_*\colon \pi_1(S) \to \pi_1(\Sigma)$ induced by inclusion is injective.*

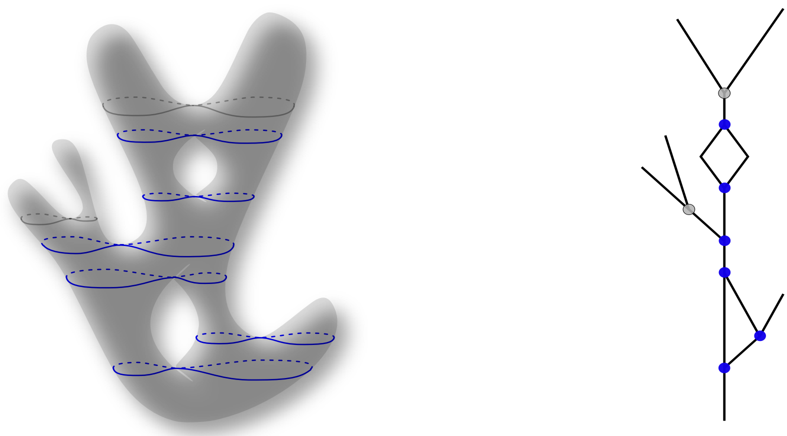

Figure 13: A typical Hamiltonian function (the z coordinate) on a genus two surface, with six essential saddles which decompose the surface into four disks and seven essential annuli. On the right, the corresponding Reeb graph, whose vertices are the critical points and whose edges are the connected component of the complement of the union of the $C(s)$. The "free ends" of the Reeb graph correspond to the components of Σ' that are disks. Essential saddles correspond to vertices which belong to the "core graph", the subgraph obtained by removing the free ends.

Define $\mathcal{D}$ to be the set of all the disks obtained via the above decomposition of Σ. Note that H is constant on the boundary of each of these disks. For every disk $D \in \mathcal{D}$ let $\overline{H}_D \in \mathcal{E}_D$ be an appropriate smoothing of $H|_D - H(\partial D)$ defined exactly as in Notation 24. We can now present the main result of this section.

Proposition 39 $\mathcal{N}(H) = \max\{H(\partial D) + \mathcal{N}(\overline{H}_D) : D \in \mathcal{D}\}.$

Proof Let $\mathcal{S}$ denote the set of critical points of H that do not belong to the union of the open disks $D \in \mathcal{D}$. According to Proposition 38, this is exactly the set of essential saddles. Using Corollary 12 and Proposition 38, we get the following description. The mnus's for H are the sets of the form

$$X = \mathcal{S} \cup \bigcup_{D \in \mathcal{D}} X_D, \tag{12}$$

where X_D is a subset of D which is negative, unlinked, and maximal for inclusion among the negative unlinked subsets of D. Similarly, according to the definition of $\overline{H}_D$, the mnus's for $\phi^1_{\overline{H}_D}$ are the sets

$$X_D \cup \nu_D,$$

where ν_D is the connected component containing $\Sigma \setminus D$ in the set $\overline{H}_D^{-1}(0)$, and X_D is as above. Now the properties of $\overline{H}_D$, as expressed in Lemma 25, imply that

$$\sup_{x \in X_D \cup \{s_D\}} \mathcal{A}_H(x) = H(\partial D) + \sup_{x \in X_D \cup \nu_D} \mathcal{A}_{\overline{H}_D}(x),$$

where, for each $D \in \mathcal{D}$, the point s_D is the unique saddle point of H in ∂D. From Equation (12) we deduce that

$$\mathcal{N}(H) = \max\bigl(\{H(\partial D) + \mathcal{N}(\overline{H}_D) : D \in \mathcal{D}\} \cup \{H(s) : s \in \mathcal{S}\}\bigr).$$

We would like to get rid of the last term of the union. Let $s \in \mathcal{S}$ be an essential saddle. Consider first the case where s does not belong to the boundary of any disk $D \in \mathcal{D}$ (in the example of Figure 13, $\mathcal{S}$ contains 2 such elements). According to Proposition 38, s is in the boundary of three essential annuli, and the second point of the proposition implies that there exists at least another essential saddle s' on the other boundary of one of the three annuli such that $H(s) < H(s')$. In the opposite case when s belongs to the boundary of some $D \in \mathcal{D}$, note that $H(s) \leqslant H(\partial D) + \mathcal{N}(\overline{H}_D)$, indeed $H(s) = H(\partial D)$ and $\mathcal{N}(\overline{H}_D) \geqslant 0$. From these considerations it follows that in the last formula for $\mathcal{N}(H)$, the maximum is always attained in the first term of the union, and we get Proposition 39. □

Proof of Proposition 38 Let S be a component of Σ'. Denote by $\chi(S)$, the Euler characteristic of S. Recall that

$$\chi(S) = 2 - 2g(S) - N_b(S),$$

where g is the genus of S and N_b denotes the number of boundary components of S. Since S is a surface with boundary we see immediately that $\chi(S) \leqslant 1$. We will suppose for the rest of the proof that S is not a disk which implies that $\chi(S) \leqslant 0$.

If s is a saddle point in S then it is not essential, and each of the loops of $C(s)$ bounds a disk in Σ. Let D be a disk bounding one of the two loops of $C(s)$, say $C_0(s)$. We claim that D is included in S. Indeed, otherwise, the interior of D meets the boundary of S, and hence it intersects $C(s')$ for an essential saddle s'. But the boundary of D, ie, $C_0(s)$, is contained in the interior of S and hence it does not meet $C(s')$. By a connectedness argument, $C(s')$ is entirely included in D and hence s' is not an essential saddle; this is a contradiction.

Now let S' be the surface obtained from S by removing a neighborhood of the disks bounding the two loops of $C(s)$ for each saddle s in S. We have obtained S' from S by removing a number of disks from S and hence $\chi(S') \leqslant \chi(S)$. Now, let $N_{\max}(S')$, $N_{\min}(S')$, $N_{\mathrm{sad}}(S')$ denote number of maxima, minima, and saddles of H inside S'. By the Poincaré–Hopf theorem,

$$\chi(S') = N_{\max}(S') - N_{\mathrm{sad}}(S') + N_{\min}(S').$$

The function H has no saddles in S' and so $N_{\mathrm{sad}}(S') = 0$. Therefore, $\chi(S') \geqslant 0$. Since $\chi(S) \leqslant 0$, we see immediately that $\chi(S') = \chi(S) = 0$. We conclude that S is a cylinder and that there are no saddles of H in S. Another application of the Poincaré–Hopf theorem implies that H has, in fact, no critical point inside S. Of course, this implies that the time-1 map ϕ_H^1 has no contractible fixed point in S as the disk bounding a 1–periodic orbit would necessarily contain a critical point of H.

It remains to prove that $i_*\colon \pi_1(S) \to \pi_1(\Sigma)$ is injective. Note that $\pi_1(S)$ is generated by either one of the boundary components of S and so it is sufficient to prove that these two loops are not contractible in Σ. These boundary components, say C, C', are loops associated, as described earlier, to two essential saddles s, s', respectively. We will first show that s, s' are distinct: indeed, if $s = s'$ then H takes the same value on the two boundary components of S and this would force H to have a critical point inside S. Next, for a contradiction suppose that C is contractible in Σ. Let D be a disk bounding C. Then, $D' = D \cup S$ is a disk bounding C'. The disk D' meets the pinched circle $C(s)$ but the boundary of D', ie, C', is disjoint from $C(s)$. We see that $C(s)$ is entirely contained in D', which contradicts the fact that the saddle s is essential. This completes the proof. □

4.2.2 Proof of $c = \mathcal{N}$ The main step here is to prove that c is determined by its value on functions supported on the disks delimited by essential saddles, in the same way as $\mathcal{N}$. We use the notation of Proposition 39.

Proposition 40 $c(H) = \max\{H(\partial D) + c(\overline{H}_D) : D \in \mathcal{D}\}.$

Proof It will be convenient to assume that H is positive. This can be assumed without loss of generality thanks to the shift property of formal spectral invariants.

We first claim that the only contractible periodic orbits of H (of any length) inside $S = \Sigma \setminus \bigcup_{D\in\mathcal{D}} D$ are the essential saddles of H. Indeed, according to Proposition 38, all the critical points of H in the surface S are essential saddles, and if we remove from S the sets $C(s)$ for all essential saddles s, we are left with a collection of disjoint cylinders which contain no critical points of H. If S_0 is such a cylinder, it is foliated by periodic orbits of the Hamiltonian parallel to the boundary curves of S_0. It follows from the third point of Proposition 38 that S_0 contains no contractible periodic orbit of H.

One can find a Hamiltonian H' of the form $H' = F + \sum_{D\in\mathcal{D}} \overline{H}_D$ arbitrarily close to H, where $F\colon \Sigma \to \mathbb{R}$ is a smooth positive function which is constant and equal to $H(\partial D)$ on each of the disks $D \in \mathcal{D}$ and which coincides with H on all of S except near the boundary of S. We will prove that

$$c(H') = \max\{H(\partial D) + c(\overline{H}_D) : D \in \mathcal{D}\}. \tag{13}$$

By continuity of c, the result then follows.

We will now use the symplectic contraction principle to build a deformation of H' as follows. Note that we cannot treat the disks $D \in \mathcal{D}$ as Liouville domains as the boundary of $D \in \mathcal{D}$ could be a pinched circle. For each $D \in \mathcal{D}$, we choose an open disk ν_D with smooth boundary whose closure is contained in D and which contains the support of $\overline{H}_D$. It is clear that $\bigcup \nu_D$ is a Liouville domain. Let ξ be a Liouville vector field for $\bigcup \nu_D$. For $s \in (-\infty, 0]$, denote by $A_s\colon \bigcup \nu_D \to \bigcup \nu_D$ the negative flow of ξ and set

$$H'_s(x) = \begin{cases} e^s F(x) & \text{if } x \in \Sigma \setminus A_s(\bigcup \nu_D), \\ e^s H(\partial D) + e^s \overline{H}_D(A_s^{-1}(x)) & \text{if } x \in A_s(\nu_D),\ D \in \mathcal{D}. \end{cases}$$

Note that $H'_0 = H'$. Moreover, since (by the discussion in the first paragraph of the proof) F has no nontrivial contractible periodic orbits in $S = \Sigma \setminus \bigcup D$, the spectrum of H'_s satisfies: $\operatorname{spec}(H'_s) = e^s \operatorname{spec}(H')$. The spectrality and continuity of spectral invariants thus yield $c(H'_s) = e^s c(H')$ for all $s \in (-\infty, 0]$. Similarly, $c(\overline{H}_{D,s}) = e^s c(\overline{H}_D)$ where $\overline{H}_{D,s}$ is defined by

$$\overline{H}_{D,s}(x) = \begin{cases} 0 & \text{if } x \in \Sigma \setminus A_s(\bigcup \nu_D), \\ e^s \overline{H}_D(A_s^{-1}(x)) & \text{if } x \in A_s(\nu_D),\ D \in \mathcal{D}. \end{cases}$$

Therefore, (13) will be proved if we prove that the following equality holds for some, and hence all, $s \in (-\infty, 0]$:

$$c(H'_s) = \max\{e^s H(\partial D) + c(\overline{H}_{D,s}) : D \in \mathcal{D}\}. \tag{14}$$

We will prove this in two steps.

Step I We prove that $c(H'_s) \geqslant \max\{e^s H(\partial D) + c(\overline{H}_{D,s}) : D \in \mathcal{D}\}$, for all $s \in (-\infty, 0]$.

Since F is positive, there exists a function $G \leqslant F$, supported in $\bigcup D$, that coincides with $H(\partial D)$ on each of the sets ν_D and with no critical points other than those outside its support and those in $\bigcup \nu_D$. Pick s close enough to $-\infty$ so that $e^s G$ has no nontrivial periodic orbit of length 1. By monotonicity,

$$c(H'_s) = c\Bigl(e^s F + \sum_{D \in \mathcal{D}} \overline{H}_{D,s}\Bigr) \geqslant c\Bigl(e^s G + \sum_{D \in \mathcal{D}} \overline{H}_{D,s}\Bigr).$$

Since G is supported in the union of the disks $D \in \mathcal{D}$ we may apply the max formula of Definition 3. If G_D stands for the component of G supported in D, we get:

$$c\Bigl(e^s G + \sum_{D \in \mathcal{D}} \overline{H}_{D,s}\Bigr) = \max\{c(e^s G_D + \overline{H}_{D,s}) : D \in \mathcal{D}\}.$$

Now we claim that $c(e^s G_D + \overline{H}_{D,s}) = e^s H(\partial D) + c(\overline{H}_{D,s})$. Together with the previous inequality it yields $c(H'_s) \geqslant \max\{e^s H(\partial D) + c(\overline{H}_{D,s}) : D \in \mathcal{D}\}$ for s sufficiently close to $-\infty$ which of course implies that the inequality holds for all $s \in (-\infty, 0]$.

To prove our claim we need to distinguish between two cases. To simplify the notation, we name the functions involved by $h = \overline{H}_{D,s}$, let $g = e^s G_D$ and let κ be the real number $e^s H(\partial D)$, so that our claimed equality is now

$$c(g + h) = \kappa + c(h). \tag{15}$$

To summarize the settings, $g + h \geqslant 0$, g has no nontrivial 1–periodic orbits and has only two critical values 0 and κ, the critical locus $g^{-1}(\kappa)$ contains the open set ν_D which contains the support of h.

First case $(c(h) > 0)$ In that case, we consider the deformation $(K_u)_{u \in [0,1]}$ defined by $K_u = ug + h$. The spectrum of K_u is by construction the union of $\{0\}$ and a shifted part $u\kappa + \mathrm{spec}(h)$. By monotonicity, $c(K_u)$ increases with u. Moreover, since $c(K_0) = c(h) > 0$, we conclude that $c(K_u)$ never vanishes along the deformation and hence belongs to the shifted part of the spectrum. By continuity, it follows that $c(K_u) = u\kappa + c(K_0)$ for all u. Taking $u = 1$, we get exactly Equation (15).

Second case $(c(h) = 0)$ In that case we can find arbitrarily C^2–close to h a function $\widetilde{h}$ satisfying $c(\widetilde{h}) > 0$. Indeed, take f to be a C^2–small nonnegative bump function whose support is included in a disk contained in ν_D that does not intersect a disk containing the support of h. Let $\widetilde{h} = f + h$. By the max formula and the nondegeneracy property, $c(\widetilde{h}) = \max(c(f), c(h)) = c(f) > 0$. Now we may apply the first case to $\widetilde{h}$ to obtain $c(g + \widetilde{h}) = \kappa + c(\widetilde{h})$. Equation (15) then follows by continuity of c.

Step II We prove $c(H'_s) = \max\{e^s H(\partial D) + c(\overline{H}_{D,s}) : D \in \mathcal{D}\}$, for all $s \in (-\infty, 0]$. Let G be as in Step I. Once again, we pick s close enough to $-\infty$ so that $e^s G$ has no nontrivial periodic orbit of length 1.

We will now show that

$$c(H'_s) = c\Big(e^s F + \sum_{D \in \mathcal{D}} \overline{H}_{D,s}\Big) = c\Big(e^s G + \sum_{D \in \mathcal{D}} \overline{H}_{D,s}\Big).$$

To simplify the notation let $f = e^s F$, $g = e^s G$, $h_D = \overline{H}_{D,s}$. We want to show that

$$c\Big(f + \sum_D h_D) = c(g + \sum_D h_D\Big).$$

Consider the deformation $K_u = ug + (1-u) f + \sum_D h_D$, where $u \in [0, 1]$.

Note that

- on ν_D we have $g = f = e^s H(\partial D)$, hence $K_u = e^s H(\partial D) + h_D$,
- on $D \setminus \nu_D$, f is still constant, hence $K_u = (1-u) e^s H(\partial D) + ug$,
- on S, $K_u = (1-u) f$ since $g = \sum h_D = 0$.

Since ug and $(1-u) f$ have no 1–periodic orbits except their critical points, we get

$$\mathrm{spec}(K_u) = (1-u)\,\mathrm{spec}(f) \cup \bigcup_D (e^s H(\partial D) + \mathrm{spec}(h_D)).$$

Note that $K_0 = f + \sum_D h_D$ and $K_1 = g + \sum_D h_D$.

Claim 41 $$c(K_0) \geqslant \max(\mathrm{spec}(f)).$$

Proof It follows from Step I that $c(f + \sum_D h_D) \geqslant \max\{e^s H(\partial D) : D \in \mathcal{D}\}$, since by positivity $c(\overline{H}_{D,s}) \geqslant 0$. As for the other values in $\mathrm{spec}(f)$, they are all smaller than $\max\{e^s H(\partial D) : D \in \mathcal{D}\}$. This is because, by Proposition 38, the Morse function $H|_S$, where $S = \Sigma \setminus \bigcup \nu_D$, has no local maxima in the interior of S and hence it must attain its maximum on a boundary component of the surface S. □

We will now use the above claim to finish the proof of Step II.

First, assume that $c(K_0) > \max(\operatorname{spec}(f))$. It follows from the above description of $\operatorname{spec}(K_u)$ that the bifurcation diagram $\bigcup_{u\in[0,1]}\{u\} \times \operatorname{spec}(K_u)$ consists of straight lines with slope 0 corresponding to elements of the form $(e^s H(\partial D) + \operatorname{spec}(h_D))$ and decreasing lines corresponding to elements of $(1-u)\operatorname{spec}(f)$. It follows from the above claim that the decreasing lines in $(1-u)\operatorname{spec}(f)$ never intersect the line with slope zero corresponding to $c(K_0)$. Hence, by continuity of c, $c(K_u) = c(K_0)$ for all $u \in [0,1]$. In particular, $c(K_1) = c(K_0)$.

Next, suppose that $c(K_0) = \max(\operatorname{spec}(f))$. As in the last paragraph of Step I, by making a C^2–small perturbation we can ensure that $c(\overline{H}_D) > 0$ for each $\overline{H}_D$ and therefore $\max\{H(\partial D) + c(\overline{H}_D) : D \in \mathcal{D}\} > \max\{H(\partial D) : D \in \mathcal{D}\}$. It follows from Step I that in fact $c(H'_s) > \max\{e^s H(\partial D) : D \in \mathcal{D}\}$. Hence, we may in fact assume that $c(K_0) > \max(\operatorname{spec}(f))$. □

Proof of Theorem 4 for Morse functions on higher genus surfaces Let H be a Morse function on Σ. Cut Σ along all essential saddles as described in Section 4.2.1 and let $\mathcal{D}$ be the family of disks obtained. For every disk D, the function $\overline{H}_D$ has only nondegenerate critical points in the interior of its support. Since we have already proved Theorem 4 for Morse functions on the plane, Lemma 42 below implies that $c(\overline{H}_D) = \mathcal{N}(\overline{H}_D)$.

As an immediate consequence of Propositions 40 and 39 we get $c(H) = \mathcal{N}(H)$. □

The following lemma compares the invariant $\mathcal{N}\colon C^\infty(S^1 \times \mathbb{R}^2) \to \mathbb{R}$ with its sibling $\mathcal{N}\colon C^\infty(S^1 \times \Sigma) \to \mathbb{R}$. We will denote the first one by $\mathcal{N}_{\mathbb{R}^2}$ and the latter by $\mathcal{N}_\Sigma$.

Lemma 42 *Let $\iota\colon a\mathbb{D}^2 \to \Sigma$ be an area-preserving embedding of the standard disk of area a into Σ. Let $D = \iota(a\mathbb{D}^2)$ be its image.*

- *For every function H with support in $a\mathbb{D}^2$, $\mathcal{N}_\Sigma(H') = \mathcal{N}_{\mathbb{R}^2}(H)$, where $H' = H \circ \iota^{-1}$ on D and $H' = 0$ elsewhere.*
- *Let $c\colon C^\infty(S^1 \times \Sigma) \to \mathbb{R}$ be a formal spectral invariant. Then, the map $\iota^* c\colon C^\infty(S^1 \times \mathbb{R}^2) \to \mathbb{R}$ defined for every function H supported in $a\mathbb{D}^2$ by $\iota^* c(H) = c(H')$, extends to a formal spectral invariant on $\mathbb{R}^2$.*

Proof The first part of the lemma follows immediately from Corollary 10. Let us now turn to the second part of the lemma. Denote $\bar{c} = \iota^* c$. Since the spectrum of H is the same as the spectrum of H', it is clear that $\bar{c}$ defines a formal spectral invariant on the set of Hamiltonians supported in $a\mathbb{D}^2$. To extend $\bar{c}$ to every function on $\mathbb{R}^2$ we use the symplectic contraction principle. Let ξ be the standard Liouville vector field on $\mathbb{R}^2$ and denote by A_s, where $s \in \mathbb{R}$, the time s map of its flows.

Given a Hamiltonian $F\colon S^1 \times \mathbb{R}^2 \to \mathbb{R}$, we pick $s \leqslant 0$ such that $A_s(\operatorname{supp}(F)) \subset a\mathbb{D}^2$ and define $F_s(t,x) := e^s F(t, A_s^{-1}(x))$. Now, F_s is supported in $a\mathbb{D}^2$ and so we can now define $\bar{c}(F)$ by

$$\bar{c}(F) := e^{-s}\bar{c}(F_s).$$

One can easily check that this defines a formal spectral invariant on $\mathbb{R}^2$. □

4.2.3 Byproduct: partial quasi-states, heavy and super-heavy sets Since they are consequences of the tools developed in the preceding sections 4.2.1 and 4.2.2, we now give the proofs of Theorem 5 and Proposition 6.

Proof of Theorem 5 First note that every disk in Σ can be symplectically contracted to a displaceable disk. It follows from the energy-capacity inequality and the symplectic contraction principle that for every disk D there is a uniform bound C_D on the value of c on functions supported in D.

Now let H be a Morse function on Σ and let $\mathcal{D}$ be the family of disks obtained by cutting along essential saddles as in Section 4.2.1. Note that for all integers k, the function kH yields to the same decomposition. Thus we may apply Proposition 40 to kH:

$$c(kH) = \max\{kH(\partial D) + c((\overline{kH})_D) : D \in \mathcal{D}\}.$$

Since $0 \leqslant c((\overline{kH})_D) \leqslant C_D$, we deduce that $\zeta(H) = \max\{H(\partial D) : D \in \mathcal{D}\}$. Since the maximum of H over its essential saddles is nothing but the maximum of $H(\partial D)$ for $D \in \mathcal{D}$, this concludes the proof of the theorem for Morse functions.

Now for any continuous function H on Σ we consider the quantity

$$\eta(H) = \inf\bigl\{h_0 : H^{-1}(h_0, +\infty) \text{ is contractible in } \Sigma\bigr\}.$$

We leave it to the reader to check that η depends continuously on H when the space of continuous functions is equipped with the sup norm. Besides, ζ is 1–Lipschitz for the sup norm. Proposition 38 implies that if H is Morse then $\zeta(H) = \eta(H)$. Since Morse functions are dense in the space of continuous functions, we conclude that $\zeta = \eta$. □

Proof of Proposition 6 We begin with the first part of the proposition. Assume that a closed subset $X \subset \Sigma$ is not included in an open disk. Since ζ is continuous, it is sufficient to verify the definition of heaviness, or super heaviness, for Morse functions. Thus, let H be a Morse function on Σ and denote $C := \inf(H|_X)$. Consider the decomposition described in Proposition 38 associated to H. If X intersects the level set of an essential saddle, then $\zeta(H) \geqslant C$ by Theorem 5. Otherwise X is included in the surface Σ' of Proposition 38. Since it is not included in a disk, X meets one of the cylinders, say S, which form the connected components of Σ'. Since the restriction of H to S has no critical point, it attains its maximum on one of the boundary components of S, hence this maximum is the value of H at an essential saddle; on the other hand, it is larger than the minimum of H on X, hence larger than C. Using Theorem 5 again, we get $\zeta(H) \geqslant C$.

Conversely assume that X is included in an open disk D. Then, by the argument used in the proof of Theorem 5, there exists a constant C_D uniformly bounding the values of ζ on functions supported in D. Thus, ζ vanishes on all functions supported in D. Taking a smooth function supported in D with value C on X, we see that X cannot be heavy.

Next, we prove the second half of the proposition. Let H be a Morse function and denote $C := \sup(H|_X)$. Assume that the complement of X admits no closed noncontractible curves. By Theorem 5, showing $\zeta(H) \leqslant C$ reduces to showing that X meets the level set of all essential saddles of H. But this is immediate since X meets all noncontractible curves.

Conversely, assume that the complement of X contains a curve Y which is noncontractible. Then, by the first part of the proposition Y is heavy. Moreover, it is disjoint from X. It is easy to see from the definition that every super-heavy set must intersect every heavy set. We conclude that X is not super-heavy. □

4.3 From Morse functions to any autonomous Hamiltonian

In this section, we finish the proof of Theorem 4. In Sections 4.1.2 and 4.2.2, we proved it for Morse functions. We will deduce Theorem 4 from this particular case. The formal spectral invariant c depends continuously on H; thus the deduction would be immediate if $\mathcal{N}$ shared the same property. While this continuity property is still unknown, we will see that every H can be approximated by some particular Morse function H' for which we can prove that $\mathcal{N}(H')$ is close to $\mathcal{N}(H)$.

Topology of fixed and periodic orbits Let $H\colon \Sigma \to \mathbb{R}$ be a smooth function. We decompose the set of contractible fixed points as a disjoint union

$$\mathrm{Fix}_c(\phi_H^1) = \mathrm{Per}_c(H) \sqcup \mathrm{Crit}_{\mathrm{iso}}(H) \sqcup \mathrm{Crit}_{\mathrm{acc}}(H),$$

where $\mathrm{Per}_c(H)$ is the set of contractible fixed points of ϕ_H^1 that are not critical points of H, $\mathrm{Crit}_{\mathrm{iso}}(H)$ the subset of isolated points in $\mathrm{Crit}(H)$ and $\mathrm{Crit}_{\mathrm{acc}}(H)$ its subset of nonisolated points.

Since H is smooth, the closure of $\mathrm{Per}_c(H)$ does not meet $\mathrm{Crit}_{\mathrm{acc}}(H)$. Indeed, let x be accumulated by critical points. Then the second differential $d^2H(x)$ is degenerate. Up to replacing H by $H \circ A$ where A is a Hamiltonian diffeomorphism for which x is a saddle point with a big dilatation in the degenerate direction, we may assume that $\|d^2H(x)\|$ is arbitrarily small. Choose a neighborhood V of x on which $\|d^2H\|$ is still small. Then on the one hand, by continuity of the flow, every 1–periodic orbit starting close enough to x is included in V; on the other hand, by a standard argument, V does not contain any 1–periodic orbit (see for example [1, Proposition 6.5.1]).

Let U be the complement of the closure of $\mathrm{Per}_c(H)$. As a consequence, only a finite number of connected components of U intersect $\mathrm{Crit}(H)$, and each of these connected components is the interior of a compact manifold with boundary, the boundary being made of a finite number of contractible 1–periodic orbits. We denote by $U_1, \dots, U_\ell$ the connected components that meet the set $\mathrm{Crit}(H)$. Each U_i is invariant by the flow.

We will now use the above description to build a Morse perturbation H' such that $\mathcal{N}(H)$ is close to $\mathcal{N}(H')$.

Perturbation lemma For each index i, we set

$$X(U_i) = \{x \in \mathrm{Crit}(H) \cap U_i : \rho(x) \leqslant 0\}.$$

and choose a function G_i compactly supported in U_i as follows. If $X(U_i)$ is empty, we let $G_i = 0$. Otherwise, let $x_i \in X(U_i)$ be such that $H(x_i) = \max_{x \in X(U_i)} H(x)$. Again if $d^2H(x_i)$ is negative definite then let $G_i = 0$. In the remaining case, define G_i so that its maximum is attained at x_i, such that the Hessian $d^2G_i(x_i)$ is negative definite. Note that x_i is still a fixed point for the time-1 map of the flow associated to $H + G_i$, and its rotation number is strictly negative. Moreover, we choose G_i to be C^2–small enough that $H + G_i$ has no nontrivial periodic orbit in U_i.

Now, for each U_i, choose F_i compactly supported in a neighborhood of the set of degenerate critical points of H in U_i, which is C^2–small and such that the Hamiltonian

$$H' = H + \sum_{i=1}^{l} (G_i + F_i)$$

is a Morse function. Finally, let $X'(U_i) = \{x \in \mathrm{Crit}(H') \cap U_i : \rho(x) \leqslant 0\}$.

Lemma 43 *If each F_i is small enough in the C^2 topology, then*

(1) $\mathrm{Per}_c(H') = \mathrm{Per}_c(H)$,

(2) *for each U_i, the set $X(U_i)$ is empty if and only if the set $X'(U_i)$ is empty,*

(3) $\max_{X'(U_i)} H'$ *is close to* $\max_{X(U_i)} H$,

(4) $\mathcal{N}(H')$ *is close to* $\mathcal{N}(H)$.

Proof The three first properties are easily obtained. Indeed, by choosing the F_i small enough in the C^2 sense, we first ensure that H' has the same nontrivial 1–periodic orbits as H. This gives property (1).

If $X(U_i)$ is empty then it contains no degenerate critical point of H, we get $H' = H$ on U_i and thus $X'(U_i)$ is also empty. If it is not empty, then the C^2–smallness of F_i implies that the rotation number of the point x_i remains negative. Property (2) follows. Finally, property (3) is an immediate consequence of property (2) and the C^0–smallness of the F_i.

To prove property (4), we establish a bijective correspondence between mnus's of H and mnus's of H'. The structure of unlinked sets for autonomous systems is described by Corollary 12. As a consequence of this description, the mnus's are the sets of the following form: a certain (finite) collection $Y \subset \mathrm{Per}_c(H)$ and all the critical points in the complement of the union of the disks $D(y)$ bounded by the 1–periodic orbits of points y in Y. In particular, the mnus's of H are all of the form

$$X = Z \cup \bigcup_{i=1}^{l} X_i,$$

where Z is a subset of the closure of $\mathrm{Per}_c(H)$ and each X_i is either $\varnothing$ or $X(U_i)$. The mnus's of H' have a similar description. To every mnus X of H as above, we associate a set

$$\Psi(X) = Z \cup \bigcup_{i=1}^{l} X'_i,$$

where for every $i \in \{1, \dots, l\}$, $X'_i = \varnothing$ if $X_i = \varnothing$, and $X'_i = X'(U_i)$ if $X_i = X(U_i)$. It follows from property (2) that the map Ψ is a bijection between mnus's of H and mnus's of H'. Moreover, property (3) implies that the maximum of the action of H over X is close to the maximum of the action of H' over $\Psi(X)$. Taking the minimum over all mnus's we get property (4). □

End of the proof of Theorem 4 Let H be a smooth function on Σ. Then, according to Lemma 43, we can find arbitrarily close to H a Morse function H' such that $\mathcal{N}(H')$ is close to $\mathcal{N}(H)$. On the other hand, the continuity of spectral invariants also implies that $c(H')$ is close to $c(H)$. Since we proved that $c = N$ for all Morse functions, we obtain that $c(H)$ is arbitrarily close to $\mathcal{N}(H)$. Thus $c(H) = \mathcal{N}(H)$. □

5 Max formulas for spectral invariants of Schwarz and Viterbo

The main goal of this section is to prove that the spectral invariants constructed by Viterbo on $\mathbb{R}^{2n}$ and by Schwarz on closed aspherical manifolds satisfy certain max formulas. It is an immediate consequence of these max formulas that the spectral invariants of Viterbo and Schwarz are both formal spectral invariants in the sense of Definition 3. As mentioned in the introduction, these max formulas are of independent interest and have consequences that go beyond the scope of this paper. For this reason, in this section of the paper we no longer restrict ourselves to two dimensional symplectic manifolds.

The max formula on $\mathbb{R}^{2n}$ Following Viterbo's notation, we will denote by c_+ and c_- the two spectral invariants constructed by him in [46]. We will recall their construction, which is based on generating functions, in Section 5.1.1.

We will say that N subsets $A_1, \dots, A_N$ in $\mathbb{R}^{2n}$ are *symplectically separated* if the minimum over all indices $1 \leqslant i < j \leqslant N$ of the euclidean distance between $\psi(A_i)$ and $\psi(A_j)$ can be made arbitrarily large for some symplectic diffeomorphism ψ. For example, two disjoint convex sets are always symplectically separated. In Section 5.1.2 will prove the following statement.

Theorem 44 *If $H_1, \dots, H_N$ are compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ whose supports are symplectically separated, then*

$$c_+(H_1 + \dots + H_N) = \max\{c_+(H_1), \dots, c_+(H_N)\},$$
$$c_-(H_1 + \dots + H_N) = \min\{c_-(H_1), \dots, c_-(H_N)\}.$$

The proof of this theorem is by induction. For $N = 2$, the idea is that when both supports are far enough from each other (which can be achieved by a suitable symplectic diffeomorphism), then it becomes possible to build a generating function of $H_1 + H_2$ that coincides with a generating function of H_1 on some open set surrounding the support of H_1 and with a generating function of H_2 on some open set surrounding the support of H_2. Then an argument based on the Mayer–Vietoris long exact sequence, applied to the sublevels of the generating functions, allows us to compare the different spectral invariants. The details will be carried out in Section 5.1.2.

The max formula on closed and aspherical symplectic manifolds Let c denote the spectral invariant constructed by Schwarz on a closed and aspherical symplectic manifold M. We will recall the construction of c in Section 5.2.1.

Recall the definition of an incompressible Liouville domain from Section 3.2. In Section 5.2.2 will prove the following max formula for Hamiltonians whose supports are contained in a disjoint union of incompressible Liouville domains.

Theorem 45 *Suppose that $F_1, \dots, F_N$ are Hamiltonians whose supports are contained, respectively, in pairwise disjoint incompressible Liouville domains $U_1, \dots, U_N$. Then,*

$$c(F_1 + \cdots + F_N) = \max\{c(F_1), \dots, c(F_N)\}.$$

Here is an overview of our strategy for proving the above theorem. The idea is to symplectically contract each of the F_i, as described in Section 3.2, to obtain functions $F_{i,s}$. Equation (5) implies that it is sufficient to prove the max formula for the $F_{i,s}$. Next we study the Floer trajectories of (an appropriate perturbation of) $F_{1,s} + \cdots + F_{N,s}$. An application of Lemma 49 will provide us with a positive constant $\varepsilon > 0$ such that any Floer trajectory which travels between distinct U_i and U_j has energy greater than ε. On the other hand, by picking s to be sufficiently negative we can ensure, using Equation (4), that the spectrum of $F_{1,s} + \cdots + F_{N,s}$ is contained in $\left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right)$ and hence any Floer trajectory traveling between distinct U_i and U_j has action less than $\frac{\varepsilon}{2}$. Using these ideas, in Lemmas 51 and 52, we conclude that there exist no such Floer trajectories. This drastically simplifies the Floer homological picture and allows us to fully describe the relations among the various Floer cycles representing the fundamental class $[M]$; see Lemma 53. We carry out the details of this strategy in Section 5.2.2.

5.1 The max formula on $\mathbb{R}^{2n}$

In this section, we establish the max formula for the spectral invariant c_+ introduced by Viterbo in [46] using generating functions. Let us quickly remind the reader of its construction.

5.1.1 Generating functions and the construction of c_+ Given a Lagrangian submanifold L in a cotangent bundle T^*M of a closed manifold M, a generating function quadratic at infinity (or gfqi) for L is a function $S\colon M \times \mathbb{R}^N \to \mathbb{R}$ for some integer N such that L admits the description

$$L = \{(x, p) \in T^*M : \exists \xi \in \mathbb{R}^N \text{ such that } \partial_\xi S(x, \xi) = 0 \text{ and } \partial_x S(x, \xi) = p\},$$

and moreover S coincides with a quadratic form Q at infinity, ie, there exists a compact set $K \subset M \times \mathbb{R}^N$ and a nondegenerate quadratic form Q on $\mathbb{R}^N$ such that $S(x, \xi) = Q(\xi)$ for every $(x, \xi) \notin K$. According to a theorem of Laudenbach and

Sikorav [43; 4], every Lagrangian submanifold which is Hamiltonian isotopic to the zero section admits a gfqi.

Hamiltonian diffeomorphisms of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ can also be represented by generating functions via the following construction. Let $\phi \in \mathrm{Ham}_c(\mathbb{R}^{2n})$, and denote by Γ_ϕ its graph, which is a Lagrangian submanifold of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$. Given a symplectic diffeomorphism $\Psi\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$, mapping the diagonal to the zero section, the Lagrangian $\Psi(\Gamma_\phi)$ is Hamiltonian isotopic to the zero section and therefore admits a gfqi $S\colon \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$. This function can be extended to $\mathbb{S}^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$ by setting $S(\infty, \xi) = Q(\xi)$ for all $\xi \in \mathbb{R}^N$. We continue to denote this extension by S and refer to it as a gfqi for ϕ.

Spectral invariants are defined as follows. Let us denote by e and μ the generators of the cohomology groups $H^0(\mathbb{S}^{2n})$ and $H^{2n}(\mathbb{S}^{2n})$ (with coefficients in a field $\mathbb{F}$). Given a function F, we denote by $F^\lambda = \{x : F(x) \leqslant \lambda\}$ its λ sublevel set. Moreover, the notation "$F^{-\infty}, F^{\infty}$" will mean "F^λ for λ close to $-\infty, \infty$", respectively. Let d stand for the dimension of the negative space of the quadratic form Q. Recall that $H^k(Q^{+\infty}, Q^{-\infty}) = \{0\}$ for every integer $k \neq d$ and $H^d(Q^{+\infty}, Q^{-\infty}) = \mathbb{F}$. For every real number λ there is a group homomorphism $i_\lambda\colon H^*(\mathbb{S}^{2n}) \to H^{*+d}(S^\lambda, s^{-\infty})$ which is the composition of the following natural maps:

$$\begin{aligned} H^*(\mathbb{S}^{2n}) &\simeq H^*(\mathbb{S}^{2n}) \otimes H^*(Q^{+\infty}, Q^{-\infty}) \\ &\simeq H^*(\mathbb{S}^{2n} \times Q^{+\infty}, \mathbb{S}^{2n} \times Q^{-\infty}) = H^*(S^{+\infty}, S^{-\infty}) \\ &\to H^*(S^\lambda, S^{-\infty}). \end{aligned}$$

Note that for a class $\alpha \in H^*(\mathbb{S}^{2n})$ of degree k, $i_\lambda(\alpha)$ has degree $k + d$. It follows from the Viterbo–Théret uniqueness theorem [46; 44] that the following definition does not depend on the choice of the gfqi.

Definition 46 (Viterbo [46])

$$\begin{aligned} c_-(\phi) &= \inf\{\lambda : i_\lambda(e) \neq 0\}, \\ c_+(\phi) &= \inf\{\lambda : i_\lambda(\mu) \neq 0\}. \end{aligned}$$

The two invariants are related by the duality formula $c_+(\phi) = -c_-(\phi^{-1})$ for every $\phi \in \mathrm{Ham}_c(\mathbb{R}^{2n})$ and satisfy the inequalities $c_- \leqslant 0 \leqslant c_+$. It is known that the invariant c_+ satisfies all the axioms of Theorem 4 except for the max formula which will be established below.

We define these spectral invariants for a compactly supported Hamiltonian H by setting

$$c_+(H) = c_+(\phi_H^1), \quad c_-(H) = c_-(\phi_H^1).$$

5.1.2 Proof of the max formula on $\mathbb{R}^{2n}$

Proof of Theorem 44 First note that by an easy induction argument the general case follows from the particular case where $N=2$. Next, remark that by the duality formula, the max formula for c_+ is equivalent to the min formula for c_-. We will prove the min formula for c_-.

We will use the notation ϕ_1, ϕ_2 for the time-1 maps of H_1 and H_2. Let

$$S_1\colon \mathbb{S}^{2n}\times\mathbb{R}^{N_1}\to\mathbb{R} \quad\text{and}\quad S_2\colon \mathbb{S}^{2n}\times\mathbb{R}^{N_2}\to\mathbb{R}$$

be generating functions quadratic at infinity for ϕ_1 and ϕ_2. It follows from the proof of the existence of generating functions that S_1 and S_2 can be chosen so that they have the same number of extra parameters, ie, $N_1=N_2=:N$ and they coincide at infinity with the same quadratic form $Q\colon \mathbb{R}^N\to\mathbb{R}$. Indeed, if we refer for instance to the proof given in [4], the quadratic form obtained when one constructs a gfqi for a diffeomorphism ϕ can be chosen to depend only on the number of diffeomorphisms which are C^1–close to the identity used to decompose ϕ. Moreover, using the fact that the supports are symplectically separated, we can conjugate ϕ_1 and ϕ_2 by an appropriate symplectic diffeomorphism ψ to ensure that we are in the following situation (recall that $c_\pm$ is conjugation invariant): there exist open sets U_1 and U_2 in $\mathbb{S}^{2n}$ such that

- $U_1\cup U_2=\mathbb{S}^{2n}$,
- U_1 and U_2 are contractible and their intersection is connected,
- $S_1(x,v)=Q(v)$ for all $(x,v)\in U_2\times\mathbb{R}^N$,
- $S_2(x,v)=Q(v)$ for all $(x,v)\in U_1\times\mathbb{R}^N$.

In particular, S_1 and S_2 coincide with Q on $(U_1\cap U_2)\times\mathbb{R}^N$.

Let $\phi=\phi_1\circ\phi_2=\phi^1_{H_1+H_2}$. It follows from the assumptions above that the Lagrangian $\Psi(\Gamma_\phi)$ then coincides with $\Psi(\Gamma_{\phi_1})$ on T^*U_1 and with $\Psi(\Gamma_{\phi_2})$ on T^*U_2. Therefore, the function $S\colon \mathbb{S}^{2n}\times\mathbb{R}^N\to\mathbb{R}$ defined by

$$S(x,v)=\begin{cases} S_1(x,v) & \text{if } x\in U_1,\\ S_2(x,v) & \text{if } x\in U_2,\end{cases}$$

is a generating function for ϕ.

Let $\lambda<0$ be a negative real number. For $i=1,2$, we consider the following (commutative) diagram of inclusions of pairs

$$\begin{tikzcd}
(S^\lambda, S^{-\infty}) \arrow[d] & (S_i^\lambda\cap(U_i\times\mathbb{R}^N), S_i^{-\infty}\cap(U_i\times\mathbb{R}^N)) \arrow[l] \arrow[d] \\
(S^{+\infty}, S^{-\infty}) & (S_i^\lambda, S_i^{-\infty}) \arrow[l]
\end{tikzcd}$$

Note that $(S_i^\lambda \cap (U_i \times \mathbb{R}^N), S_i^{-\infty} \cap (U_i \times \mathbb{R}^N)) = (S^\lambda \cap (U_i \times \mathbb{R}^N), S^{-\infty} \cap (U_i \times \mathbb{R}^N))$, which gives the top horizontal arrow. We denote by A_i the cohomology group

$$A_i = H^d(S_i^\lambda \cap (U_i \times \mathbb{R}^N), S_i^{-\infty} \cap (U_i \times \mathbb{R}^N)).$$

The above diagram induces the following commutative diagram:

$$\begin{tikzcd} H^d(S^\lambda, S^{-\infty}) \arrow[r] & A_i \\ H^0(\mathbb{S}^{2n}) \arrow[u] \arrow[r] & H^d(S_i^\lambda, S_i^{-\infty}) \arrow[u] \end{tikzcd}$$

We now prove that the right vertical map $H^d(S_i^\lambda, S_i^{-\infty}) \to A_i$ is injective. We prove it for $i = 1$, the case $i = 2$ being similar. Consider the Mayer–Vietoris sequence for the covering $\{S_1^\lambda \cap (U_1 \times \mathbb{R}^N), S_1^\lambda \cap (U_2 \times \mathbb{R}^N)\}$ of S_1^λ. It provides in particular an exact sequence

$$C \to H^d(S_1^\lambda, S_1^{-\infty}) \to A_1 \oplus B,$$

where

$$\begin{aligned} B &= H^d\big(S_1^\lambda \cap (U_2 \times \mathbb{R}^N), S_1^{-\infty} \cap (U_2 \times \mathbb{R}^N)\big) \\ &= H^d(U_2 \times Q^\lambda, U_2 \times Q^{-\infty}) \\ &= \{0\}, \end{aligned}$$

where the last equality holds since $\lambda < 0$ and hence $U_2 \times Q^\lambda$ retracts onto $U_2 \times Q^{-\infty}$, and

$$\begin{aligned} C &= H^{d-1}\big(S_1^\lambda \cap ((U_1 \cap U_2) \times \mathbb{R}^N), S_1^{-\infty} \cap ((U_1 \cap U_2) \times \mathbb{R}^N)\big) \\ &= H^{d-1}\big((U_1 \cap U_2) \times Q^\lambda, (U_1 \cap U_2) \times Q^{-\infty}\big) \\ &= \{0\}. \end{aligned}$$

Thus, $H^d(S_1^\lambda, S_1^{-\infty}) \to A_1$ is injective.

We then consider the "direct sum" diagram:

$$\begin{tikzcd} H^d(S^\lambda, S^{-\infty}) \arrow[r] & A_1 \oplus A_2 \\ H^0(\mathbb{S}^{2n}) \arrow[u] \arrow[r] & H^d(S_1^\lambda, S_1^{-\infty}) \oplus H^d(S_2^\lambda, S_2^{-\infty}) \arrow[u] \end{tikzcd} \tag{16}$$

We have seen that the right vertical arrow is injective. Let us now show that the top horizontal arrow is also injective. This follows again from a Mayer–Vietoris sequence, the same as before but with S instead of S_1:

$$C \to H^d(S^\lambda, S^{-\infty}) \to A_1 \oplus A_2,$$

where

$$C = H^{d-1}(S^\lambda \cap ((U_1 \cap U_2) \times \mathbb{R}^N), S^{-\infty} \cap ((U_1 \cap U_2) \times \mathbb{R}^N)) = \{0\},$$

as above.

We can now conclude. In the diagram (16), the top horizontal arrow and the right vertical arrow are both injective. Therefore, for all $\lambda < 0$ the image of a generator e of $H^0(\mathbb{S}^{2n})$ by the bottom horizontal arrow is zero if and only if its image by the left vertical arrow is zero. Since $c_- \leqslant 0$, this implies the min formula for c_-. By duality, the max formula for c_+ follows. □

5.2 The max formula on closed and aspherical symplectic manifolds

In this section, we establish the max formula for the spectral invariant c introduced by Schwarz in [40] using Hamiltonian Floer theory. Let us quickly remind the reader of its construction.

5.2.1 Hamiltonian Floer theory and spectral invariants In this section, we review the necessary preliminaries on Hamiltonian Floer theory and spectral invariants. We refer the reader to Section 2 for preliminaries, and our conventions on the action functional and the Conley–Zehnder index. Throughout the section, (M, ω) will denote a closed, connected and aspherical symplectic manifold. The closed symplectic manifolds we are interested in in this paper, ie, closed surfaces other than $\mathbb{S}^2$, are all aspherical. Floer homology was first introduced in the setting of aspherical manifolds by Floer [10]. The standard reference for Floer theory in the settings of this section is [38]. For further information on the subject we invite the reader to consult [28; 1].

Although spectral invariants are defined for degenerate and even continuous Hamiltonians, Hamiltonian Floer homology can only be defined for nondegenerate Hamiltonians and therefore throughout the rest of this section we suppose that all Hamiltonians are nondegenerate. The Floer complex of (nondegenerate) H is defined as the $\mathbb{Z}_2$–vector space spanned by $\mathrm{Crit}(\mathcal{A}_H)$, the set of critical points of the action functional. Recall that $\mathrm{Crit}(\mathcal{A}_H)$ is the set of contractible 1–periodic orbits of ϕ_H^t. This complex is graded by the Conley–Zehnder index.

Floer's differential is defined by counting perturbed pseudo-holomorphic cylinders: pick a time-periodic family of ω–compatible almost complex structures J_t and consider maps $u\colon \mathbb{R} \times S^1 \to M$ satisfying Floer's equation

$$\partial_s u + J_t(u)(\partial_t u - X_H^t(u)) = 0. \tag{17}$$

The set of Floer trajectories between two critical points of $\mathcal{A}_H$, x_- and x_+, is defined as

$$\widehat{\mathcal{M}}(x_-, x_+; H, J) = \big\{u\colon \mathbb{R} \times S^1 \to M : u \text{ satisfies (17)} \\ \text{and } u(\pm\infty, t) = x_\pm(t) \text{ for all } t\big\},$$

where the limits $u(\pm\infty, t)$ are uniform in t. Note the above set admits an $\mathbb{R}$–action by reparametrization $s \mapsto s+\tau$. The moduli space of Floer trajectories between x_- and x_+, denoted by $\mathcal{M}(x_-, x_+; H, J)$, is the quotient $\widehat{\mathcal{M}}(x_-, x_+; H, J)/\mathbb{R}$.

A solution u of (17) is said to be *regular* if the linearization of the operator

$$u \mapsto \partial_s u + J_t(u)(\partial_t u - X_H^t(u))$$

is surjective at u. The almost complex structure J is said to be *regular* if every $u \in \widehat{\mathcal{M}}(x_-, x_+; H, J)$, for any x_-, x_+, is regular. Regularity of J implies that the above moduli spaces are all smooth finite dimensional manifolds, and if $x_- \neq x_+$, the dimension of $\widehat{\mathcal{M}}(x_-, x_+; H, J)$ is $\mu_{CZ}(x_-) - \mu_{CZ}(x_+) - 1$. A suitably generic choice of J is regular in the following sense: The set of regular J, denoted by $\mathcal{J}_{\mathrm{reg}}(H)$, is of second category in the set of all compatible almost complex structures. If $\mu_{CZ}(x_-) - \mu_{CZ}(x_+) = 1$, the moduli space is compact and hence finite. This allows us to define the Floer boundary map $\partial\colon \mathrm{CF}_*(H) \to \mathrm{CF}_{*-1}(H)$. For a generator x_- we define $\partial(x_-)$ by

$$\partial(x_-) = \sum_{x_+} \#\mathcal{M}(x_-, x_+; H, J) \cdot x_+$$

where the sum is taken over all 1–periodic orbits x_+ such that $\mu_{CZ}(x_-) - \mu_{CZ}(x_+) = 1$ and # denotes the mod 2 cardinality of $\mathcal{M}(x_-, x_+; H, J)$. The above definition is extended to the entire chain complex by linearity.

It is well-known that $\partial^2 = 0$ and thus ∂ defines a differential on $\mathrm{CF}_*(H)$. The Floer homology of (H, J), denoted by $\mathrm{HF}_*(H, J)$, is the homology of the complex $(\mathrm{CF}_*(H), \partial)$.

In the course of the proof of Theorem 45, we will appeal to the following observation about the structure of $\mathcal{J}_{\mathrm{reg}}(H)$.

Remark 47 Suppose that H is a nondegenerate Hamiltonian and let W denote an open subset of M containing all the 1–periodic orbits of the flow of H. Fix an almost complex structure J_0 on M. One can find a regular almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(H)$ such that $J = J_0$ on the complement of W.

This fact, which was explained to us by A Oancea, follows easily from the content of the proof of transversality presented in [11, Theorem 5.1].

Invariance of Floer homology Although the Floer complex depends on (H, J), the Floer homology groups are independent of this auxiliary data. Indeed, there exist morphisms

$$\Psi_{H_0}^{H_1}\colon \mathrm{CF}(H_0) \to \mathrm{CF}(H_1)$$

inducing isomorphisms in homology which are called continuation morphisms. (To keep the notation light we have eliminated the almost complex structures from our notation.) We now describe the morphism $\Psi_{H_0}^{H_1}$. Pick $J_i \in \mathcal{J}_{\mathrm{reg}}(H_i)$ and take a homotopy, denoted by (H_s, J_s), from (H_0, J_0) to (H_1, J_1) such that

$$(H_s, J_s) = \begin{cases} (H_0, J_0) & \text{if } s \leqslant 0, \\ (H_1, J_1) & \text{if } s \geqslant 1. \end{cases}$$

Consider maps $u\colon \mathbb{R}\times\mathbb{S}^1 \to M$ solving an s–dependent version of Floer's Equation (17):

$$\partial_s u + J_{s,t}(u)(\partial_t u - X_H^{(s,t)}(u)) = 0 \quad \text{for all } (s,t) \in \mathbb{R} \times S^1. \tag{18}$$

For 1–periodic orbits $x_0 \in \mathrm{Crit}(\mathcal{A}_{H_0})$, $x_1 \in \mathrm{Crit}(\mathcal{A}_{H_1})$, define the moduli space

$$\mathcal{M}(x_0, x_1; H_s, J_s) = \big\{u\colon \mathbb{R} \times S^1 \to M : u(-\infty, t) = x_0(t), \\ u(+\infty, t) = x_1(t), \text{ and } u \text{ satisfies (18)}\big\}.$$

A Floer trajectory $u \in \mathcal{M}(x_0, x_1; H_s, J_s)$ is said to be *regular* if the linearization of the operator

$$u \mapsto \partial_s u + J_{s,t}(u)(\partial_t u - X_H^{(s,t)}(u))$$

is onto at u. The homotopy (H_s, J_s) is said to be regular if every $u \in \mathcal{M}(x_0, x_1; H_s, J_s)$, for any x_0, x_1, is regular. Regularity of (H_s, J_s) implies that the above moduli spaces are smooth finite dimensional manifolds of dimension $\mu_{CZ}(x_0) - \mu_{CZ}(x_1)$. A suitably generic choice of (H_s, J_s) is indeed regular. When the moduli space is zero–dimensional it is compact and hence finite. Thus, we can define

$$\Psi_{H_0}^{H_1}(x_0) = \sum_{x_1} \#\mathcal{M}(x_0, x_1; H_s, J_s) \cdot x_1 \tag{19}$$

where the sum is taken over all $x_1 \in \mathrm{Crit}(\mathcal{A}_{H_1})$ such that $\mu_{CZ}(x_0) = \mu_{CZ}(x_1)$ and # is used to denote mod 2 cardinality. The morphism $\Psi_{H_0}^{H_1}$ is then extended by linearity to all of $\mathrm{CF}_*(H_0)$. It can be shown that continuation morphisms are chain maps and thus, descend to homology; we will continue to denote the maps induced on homology by the same notation. The induced map on homology does not depend on the choice of the homotopy (H_s, J_s). Furthermore, at the homology level, continuation maps satisfy the following composition rule:

$$\Psi_{H_0}^{H_0} = \mathrm{Id} \quad \text{and} \quad \Psi_{H_1}^{H_2} \circ \Psi_{H_0}^{H_1} = \Psi_{H_0}^{H_2}. \tag{20}$$

We see that $\Psi_{H_0}^{H_1}$ gives an isomorphism between $\mathrm{HF}_*(H_0, J_0)$ and $\mathrm{HF}_*(H_1, J_1)$.

Lastly, if H is taken to be a C^2–small Morse function then the Floer homology of H coincides with its Morse homology. It follows from the above that for any regular pair $\mathrm{HF}_*(H, J) = H_*(M)$.

Invariance of Floer homology can also be established via the PSS morphism [31],

$$\Phi_H\colon H_*(M) \to \mathrm{HF}_*(H, J),$$

which gives a direct isomorphism between Morse homology and Floer homology. Below, we will use the fact that such isomorphism exists to construct spectral invariants but we will not recall the construction of the PSS isomorphism.

The following observation, which is analogous to Remark 47, will be used in the course of the proof of Theorem 45.

Remark 48 Suppose that H_0, H_1 are nondegenerate Hamiltonians and $J_i \in \mathcal{J}_{\mathrm{reg}}(H_i)$ are regular almost complex structures. Let (H_s, J_s) be any homotopy, as described above, from (H_0, J_0) to (H_1, J_1). Let W denote an open subset of M which intersects the image of every *nonregular* Floer trajectory u given by Equation (18). One can find a regular homotopy (H_s', J_s') from (H_0, J_0) to (H_1, J_1) such that $H_s' = H_s$ and $J_s' = J_s$ on the complement of W.

This fact, which was explained to us by Oancea, follows easily from the content of the proof of Theorem 5.1 of [11].

Spectral invariants Let $u\colon \mathbb{R} \times S^1 \to M$ denote a Floer trajectory solving either one of Equations (17)–(18). The energy of u is defined as

$$E(u) := \int_{\mathbb{R}\times[0,1]} \|\partial_s u\|^2 \, ds\, dt, \tag{21}$$

where $\|\cdot\|$ is the norm associated to the metric $\omega(\cdot, J\cdot)$. Clearly, $E(u) \geqslant 0$.

It follows from a standard computation that if u is a Floer trajectory contributing to the boundary map, ie $u \in \widehat{\mathcal{M}}(x_-, x_+; H, J)$, then

$$\mathcal{A}_H(x_-) - \mathcal{A}_H(x_+) = E(u). \tag{22}$$

Thus action decreases along Floer trajectories. Now let $a \in \mathbb{R}$ be a regular value of the action functional, ie, $a \notin \mathrm{spec}(H)$. It follows from this observation that if we denote by $\mathrm{CF}^a_*(H)$ the $\mathbb{Z}_2$–vector space generated by 1–periodic orbits of action $< a$, then $\mathrm{CF}^a_*(H)$ is a subcomplex of $\mathrm{CF}_*(H)$ whose homology will be denoted by $\mathrm{HF}^a_*(H, J)$. Let $i^a\colon \mathrm{HF}^a_*(H, J) \to \mathrm{HF}_*(H, J)$ be the map induced on homology by the inclusion. Let $[M] \in H_*(M)$ denote the fundamental class [6] of M and define the spectral invariant of H to be the number

$$c(H) = \inf\{a \in \mathbb{R} : \Phi_H([M]) \in \mathrm{im}(i^a)\}. \tag{23}$$

Roughly speaking, this is the minimal action required to see the fundamental class $[M]$ in $\mathrm{HF}_*(H, J)$. Thus far we have defined $c(H)$ for nondegenerate H. One can show that spectral invariants of two nondegenerate Hamiltonians H, G satisfy the Lipschitz estimate from the Lipschitz continuity property in Section 3. This estimate allows us to extend $c(\cdot)$ continuously to all smooth (in fact continuous) Hamiltonians.

The spectral invariant constructed in this section satisfies the spectrality and continuity axioms from Definition 3 and all the properties discussed in Section 3; for proofs we refer the reader to [29; 30; 40]. Below we prove that c is indeed a formal spectral invariant, in the sense of Definition 3, by showing that it satisfies the max formula.

5.2.2 Proof of the max formula on closed aspherical manifolds Our proof of Theorem 45 relies on the following preliminary fact.

Energy estimates for Floer trajectories The following lemma is a slight reformulation of Proposition 3.2 of [17]. We will not provide a proof as it follows quite easily from Hein's argument. A similar result appears in [45, Lemma 2.3]. Recall that $E(u)$ denotes the energy of a Floer trajectory as defined by Equation (21).

Lemma 49 *Let V denote an open subset of M with (at least) two distinct smooth boundary components W_1, W_2. Consider a Hamiltonian H which is autonomous in V and whose time-1 map ϕ^1_H has no fixed points in V. Furthermore, assume that W_1 and W_2 are contained in two distinct level sets of H. There exists a constant $\varepsilon(V, H|_V, J|_V) > 0$, depending on the domain V and the restrictions of the Hamiltonian H and the almost complex structure J to the domain V with the following property: if $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ satisfies Floer's Equation* (17) *and the image of u intersects W_1 and W_2, then*

$$E(u) \geqslant \varepsilon.$$

[6] Spectral invariants can be defined for Morse homology classes other than $[M]$ however, we have not introduced spectral invariants in full generality since we will only be dealing with the spectral invariants associated to $[M]$.

Proof of Theorem 45 Observe that it is sufficient to prove the theorem under the assumption that each U_i is connected; we will make this assumption from this point onward.

We first choose an auxiliary connected incompressible Liouville domain U_0 that does not intersect any of the U_i. For every $i=0,\dots,N$, let ξ_i denote a Liouville vector field of U_i. We construct shells $V_0,\dots,V_N$ near the boundary of the domains $U_0,\dots,U_N$ as follows: a tubular neighborhood of the boundaries ∂U_i can be identified, via a diffeomorphism, with $(-\delta,\delta)\times\partial U_i$ such that $(-\delta,0)\times\partial U_i$ is contained inside U_i. Set $V_i=(0,\delta)\times\partial U_i$. Observe that, since we are not supposing ∂U_i is connected each shell V_i might in fact be a union of connected shells.

Take δ from the previous paragraph to be small enough such that $(-\delta,0)\times\partial U_i$ does not intersect the support of F_i. Pick an autonomous Hamiltonian H such that:

(1) $H=0$ on U_0, $H=0$ on $U_i\setminus(-\delta,0)\times\partial U_i$ for $i=1,\dots,N$, and $H<0$ on the rest of M. Hence, H vanishes on the supports of all F_i.

(2) H has no critical points in $[-\delta,\delta]\times\partial U_i$, $i=0,\dots,N$.

(3) For each $i=0,\dots,N$, the sets ∂U_i and $\{\delta\}\times\partial U_i$ are contained in distinct level sets of H.

(4) In the interior of its support, H is Morse and has no local maxima.

(5) In the interior of its support, we pick H sufficiently C^2–small to ensure that the only 1–periodic orbits of H are its critical points and that the Morse indices of these critical points, by our conventions, coincide with their Conley–Zehnder indices.

Fix an almost complex structure J on M. Suppose that u is a Floer trajectory, solving Floer's Equation (17) for any Hamiltonian and almost complex structure which coincide with H and J on the shells $V_0,\dots,V_N$. By applying Lemma 49, we obtain $\varepsilon>0$ such that if the image of u crosses[7] one of the shells $V_0,\dots,V_N$ then

$$E(u)\geqslant 4\varepsilon. \tag{24}$$

Next, we symplectically contract each of the F_i to obtain $F_{1,s},\dots,F_{N,s}$ such that for each $i\in\{1,\dots,N\}$ we have

$$\operatorname{spec}(F_{i,s})\subset\left(-\tfrac{1}{2}\varepsilon,\tfrac{1}{2}\varepsilon\right)\quad\text{and}\quad\|F_{i,s}(t,\cdot)\|_\infty\leqslant\tfrac{1}{2}\varepsilon.$$

[7] To be more precise, by saying that the image of u crosses one of the shells $V_0,\dots,V_N$ we mean that there exists i such that the image of u intersects $U_i\setminus V_i$ and $M\setminus U_i$.

By Equation (5), $c(F_{i,s}) = e^s c(F_i)$ and $c(F_{1,s} + \cdots + F_{N,s}) = e^s c(F_1 + \cdots + F_N)$. Hence, it is sufficient to prove the max formula for the $F_{i,s}$. To simplify our notation, we will continue to denote the newly obtained Hamiltonians $F_{i,s}$ by F_i.

Define $F_{N+1} = F_1 + \cdots + F_N$. We will need the following lemma to prove the max formula. We postpone its proof to the end of this section.

Lemma 50 $\quad c(F_i + H) = c(F_i)$ *for* $i = 1, \dots, N+1$.

Next, pick an autonomous Morse Hamiltonian G_0 which is a C^2–small perturbation of H, which coincides with H outside of $U_0, \dots, U_N$ and which has precisely $N+1$ maximum points $p_0 \in U_0,\ p_1 \in U_1, \dots, p_N \in U_N$. Then define $G_i = G_0 + F_i$ for $i = 1, \dots, N+1$. For any indices i, j denote

$$\operatorname{spec}(G_i; U_j) = \{\mathcal{A}_{G_i}(x) : x \in \operatorname{Crit}(\mathcal{A}_{G_i}) \text{ and } x \text{ contained in } U_j\}.$$

Recall that $\operatorname{spec}(F_{i,s}) \subset \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. Therefore, taking G_0 to be sufficiently C^2–close to H, and thus sufficiently C^2–close to 0 on $U_0 \cup \cdots \cup U_N$, we can guarantee that

$$\operatorname{spec}(G_i; U_j) \subset (-\varepsilon, \varepsilon), \tag{25}$$

for all $i \in \{0, \dots, N+1\}$ and $j \in \{0, \dots, N+1\}$. Furthermore, since $\|F_{i,s}(t, \cdot)\|_\infty \leqslant \frac{\varepsilon}{2}$ for all $t \in [0, 1]$ and the Hamiltonians G_i all coincide with H outside of the U_i we can also guarantee that

$$\|G_i(t, \cdot) - G_j(t, \cdot)\|_\infty \leqslant \varepsilon \quad \text{for all } t \in [0, 1]. \tag{26}$$

Lastly, by replacing $F_1, \dots, F_{N+1}$ with C^2–nearby Hamiltonians we may assume that $G_1, \dots, G_{N+1}$ are nondegenerate as well.

By Remark 47 we can pick almost complex structures $J_i \in \mathcal{J}_{\mathrm{reg}}(G_i)$ such that on the shells $V_0, \dots, V_N$ each J_i coincides with the almost complex structure J introduced above to obtain the estimate (24). By doing so, and noting that the G_i coincide with H on the shells $V_0, \dots, V_N$, we can ensure that the estimate

$$E(u) > 4\varepsilon$$

holds for any Floer trajectory u of the Hamiltonians G_i, solving Equation (17), which crosses any of the shells $V_0, \dots, V_N$.

In the course of this proof we will also need to use the estimate (24) for Floer trajectories of the various continuation morphisms $\Psi_{G_i}^{G_j} \colon \mathrm{CF}_*(G_i) \to \mathrm{CF}_*(G_j)$, for any $i, j \in \{0, \dots, N+1\}$. To define these morphisms, we must make a specific choice of a homotopy from (G_i, J_i) to (G_j, J_j). Using Remark 48, we can select a regular homotopy (G_s^{ij}, J_s^{ij}) from (G_i, J_i) to (G_j, J_j) such that on the shells $V_0, \dots, V_N$, the almost complex structure J, introduced above, coincides with the almost complex structures J_s^{ij}, and the Hamiltonians G_s^{ij} coincide with the linear homotopy $(1-\beta(s))G_i + \beta(s)G_j = G_i + \beta(s)(G_j - G_i)$, where $\beta\colon \mathbb{R} \to [0,1]$ is a smooth nondecreasing function such that $\beta(s) = 0$ for $s \leqslant 0$ and $\beta(s) = 1$ for $s \geqslant 1$. Note that for each s we have $G_s^{ij} = H$ on the shells $V_0, \dots, V_N$. Once again, it follows that the estimate

$$E(u) > 4\varepsilon$$

holds for every Floer trajectory u, solving Equation (18) for G_s, which crosses some of the shells $V_0, \dots, V_N$. We will now use this estimate to prove the following lemma which will be used repeatedly.

Lemma 51 *Let $x_0 \in \mathrm{Crit}(\mathcal{A}_{G_i}), x_1 \in \mathrm{Crit}(\mathcal{A}_{G_j})$ be of Conley–Zehnder index $2n$. Consider solutions u of (18) contributing to the continuation morphism*

$$\Psi_{G_i}^{G_j}\colon \mathrm{CF}_*(G_i) \to \mathrm{CF}_*(G_j).$$

If there exists u such that $u(-\infty, t) = x_0(t)$ and $u(\infty, t) = x_1(t)$, then there exists $k \in \{0, \dots, N\}$ such that both of x_0, x_1 are contained in U_k. Furthermore, the entire image of the Floer trajectory u is contained in $\overline{U}_k \cup V_k$.

Proof Note that all the 1–periodic orbits of the G_i with Conley–Zehnder index $2n$ are contained in the U_i. For a contradiction suppose that x_0, x_1 are not contained in the same U_i. The Floer trajectory u would have to cross at least one of the shells V_i and hence must have energy greater than 4ε; see (24). On the other hand, a standard computation in Floer theory (see for example [40, Lemma 2.12]) yields

$$E(u) \leqslant \mathcal{A}_{G_i}(x_0) - \mathcal{A}_{G_j}(x_1) + \int_0^1 \|G_i(t,\cdot) - G_j(t,\cdot)\|_\infty \, dt + \varepsilon, \tag{27}$$

where $\|\cdot\|$ denotes the L^∞ norm on functions. The terms from the right-hand side of the above inequality are all smaller than ε by Equations (25) and (26). Therefore, the right-hand side gives an upper bound of approximately 3ε for $E(u)$ contradicting the lower bound of 4ε for $E(u)$.

We see from the above that the Floer trajectory u cannot cross any of the shells V_i. Hence, the entire image of u must be contained in $\overline{U}_k \cup V_k$ for some k. □

We will also need a variation of the above lemma for the Floer boundary maps $\partial\colon \mathrm{CF}_*(G_i) \to \mathrm{CF}_*(G_i)$. We will not give a proof of this lemma as it is similar to, and in fact simpler than, the proof of Lemma 51.

Lemma 52 *Let $x_0, x_1 \in \mathrm{Crit}(\mathcal{A}_{G_i})$ such that x_0, x_1 are contained in $U_0 \cup \cdots \cup U_N$. Consider solutions u of (17) contributing to the boundary map $\partial\colon \mathrm{CF}_*(G_i) \to \mathrm{CF}_*(G_i)$. If there exists u such that $u(-\infty, t) = x_0(t)$ and $u(\infty, t) = x_1(t)$, then there exists $k \in \{0, \dots, N\}$ such that both of x_0, x_1 are contained in U_k. Furthermore, the entire image of the Floer trajectory u is contained in $\overline{U}_k \cup V_k$.*

For $i = 0, \dots, N+1$, denote by $[M]_{G_i} \in \mathrm{HF}_*(G_i, J_i)$ the element of $\mathrm{HF}_*(G_i, J_i)$ representing the fundamental class of M. The max formula is an immediate corollary of the following lemma which describes the relations among the Floer cycles which represent these fundamental classes.

Lemma 53 *The Floer homology classes $[M]_{G_i}$ have the following forms:*

1. *$[M]_{G_0}$ is represented uniquely by $p_0 + \cdots + p_N$.*
2. *For all $i = 1, \dots, N$, any representative of the fundamental class $[M]_{G_i}$ is of the form $C_i + \sum_{j \in \{0, \dots, N\}, j \neq i} p_j$, where C_i is a nontrivial sum of 1–periodic orbits of G_i each of which is contained in the region U_i.*
3. *Any representative of the fundamental class $[M]_{G_{N+1}}$ is of the form $p_0 + C_1 + \cdots + C_N$, where each C_i is a nontrivial sum of 1–periodic orbits of G_i each of which is contained in the region U_i.*

Furthermore, if $C_i + \sum_{j \in \{0, \dots, N\}, j \neq i} p_j$ is a representative of $[M]_{G_i}$ for $i = 1, \dots, N$ then, $p_0 + C_1 + \cdots + C_N$ represents $[M]_{G_{N+1}}$. Conversely, if $p_0 + C_1 + \cdots + C_N$ is a representative of $[M]_{G_{N+1}}$ then, $C_i + \sum_{j \in \{0, \dots, N\}, j \neq i} p_j$ represents $[M]_{G_i}$ for $i = 1, \dots, N$.

The max formula is an easy consequence of the above lemmas. Since all the points p_i have action almost zero, it follows immediately from Lemma 53 that $c(G_{N+1})$ is almost equal to $\max\{c(G_1), \dots, c(G_N)\}$. Since we can choose G_i to be arbitrarily C^2–close to $H + F_i$, we conclude that $c(H + F_{N+1})$ is almost equal to

$$\max\{c(H + F_1), \dots, c(H + F_N)\}.$$

By Lemma 50, $c(H + F_i) = c(F_i)$ and thus, the F_i satisfy the max formula.

To finish the proof of Theorem 45, it remains to prove Lemmas 50 and 53.

Proof of Lemma 53 Since G_0 is C^2–small its Floer and Morse complexes coincide. Now in Morse homology the fundamental class is uniquely represented by the sum of all maxima and hence $[M]_{G_0} = [p_0 + \cdots + p_N]$. To see this, one can think of the isomorphism between Morse homology and cellular homology, induced by the map that associates to a critical point its unstable manifold. In cellular homology, the fundamental class is uniquely represented by the sum of all cells of top dimension. Thus, the fundamental class in Morse homology has to be represented by the sum of all the critical points of maximal Morse index, that is of all maxima.

Next we will prove the second assertion with regards to the form of $[M]_{G_i}$. For simplicity, we write the proof for $i=1$ and $N=2$. The argument in the general case is similar except that the notation is heavier. All of the 1–periodic orbits of G_1 with Conley–Zehnder index $2n$ are contained in the interior of $U_0 \cup U_1 \cup U_2$. Thus, any representative of $[M]_{G_1}$ is of the form $C_1 + \lambda p_0 + \mu p_2$ where C_1 is a sum of 1–periodic orbits in U_1 and $\lambda, \mu \in \mathbb{Z}_2$. We must prove that C_1 is nontrivial, $\lambda \neq 0$ and $\mu \neq 0$.

Consider the continuation morphism $\Psi_{G_1}^{G_0}\colon \mathrm{CF}_*(G_1) \to \mathrm{CF}_*(G_0)$ as is defined by Equation (19). Since $[M]_{G_0}$ is uniquely represented by $p_0 + p_1 + p_2$ it must be the case that

$$p_0 + p_1 + p_2 = \Psi_{G_1}^{G_0}(C_1 + \lambda p_0 + \mu p_2) = \Psi_{G_1}^{G_0}(C_1) + \Psi_{G_1}^{G_0}(\lambda p_0) + \Psi_{G_1}^{G_0}(\mu p_2).$$

Lemma 51 implies that $\Psi_{G_1}^{G_0}(C_1) = p_1$, $\Psi_{G_0}^{G_1}(\lambda p_0) = p_0$ and $\Psi_{G_1}^{G_0}(\mu p_2) = p_2$. In particular, $C_1 \neq 0$, $\lambda \neq 0$ and $\mu \neq 0$.

The proof of the third assertion, about $[M]_{G_3}$ is very similar to the above and hence we will omit it.

Lastly, we prove the final assertion. Again, to lighten the notation we only show the argument in the case $N=2$. In order to compare different continuation maps we will pick the homotopies used in defining these maps such that they satisfy the following compatibility condition:

Claim 54 *There exist regular homotopies (G_s^{ij}, J_s^{ij}), from (G_i, J_i) to (G_j, J_j), such that for all $0 \leqslant i, j, i', j' \leqslant 3$,*

$$(G_s^{ij}, J_s^{ij}) = (G_s^{i'j'}, J_s^{i'j'})$$

on each open set $\overline{U}_k \cup V_k$ where $G_i = G_{i'}$ and $G_j = G_{j'}$.

Postponing the proof of the above claim to the end of this section, we will now proceed with proving the last assertion in the statement of Lemma 53. First, suppose that $[M]_{G_3} = [p_0 + C_1 + C_2]$. We must show that $[M]_{G_1} = [C_1 + p_0 + p_2]$ and $[M]_{G_2} = [C_2 + p_0 + p_1]$. We will begin by proving the following claim about the continuation morphism $\Psi_{G_3}^{G_3}\colon \mathrm{CF}_*(G_3) \to \mathrm{CF}_*(G_3)$.

Claim 55 $\Psi_{G_3}^{G_3}(C_i) = C_i + B_i$ *for* $i = 1, 2$, *where* B_i *is in the image of the Floer boundary map* $\partial\colon \mathrm{CF}_*(G_3) \to \mathrm{CF}_*(G_3)$.

Proof Recall that $\Psi_{G_3}^{G_3}$ induces the identity map on homology; see (20). In particular, this implies that $\Psi_{G_3}^{G_3}(p_0+C_1+C_2) = p_0+C_1+C_2+B$, where B is a boundary term. First suppose that B has a nontrivial p_0 contribution. This would entail the existence of a Floer boundary trajectory u, solving Equation (17) for the Hamiltonian G_3, such that $u(\infty, t) = p_0$. Now, $u(-\infty, t)$ would have to be a 1–periodic orbit of Conley–Zehnder index $2n+1$. Since all such 1–periodic orbits are contained in the open sets U_i, we conclude using Lemma 52 that $u(-\infty, t)$ is contained in U_0. But $G_3|_{U_0} = G_0|_{U_0}$ and G_0 is a C^2–small Hamiltonian and hence it has no 1–periodic orbit of Conley–Zehnder index $2n + 1$. We see that B cannot have a nontrivial p_0 contribution. Since all the remaining 1–periodic orbits of G_3 with Conley–Zehnder index $2n$ are contained in $U_1 \cup U_2$ it follows that $B = B_1 + B_2$ where B_i is a sum of 1–periodic orbits contained in U_i. Applying Lemma 51, we conclude that $\Psi_{G_3}^{G_3}(C_i) = C_i + B_i$.

It remains to show that each B_i is a boundary term. We know that there exists $D \in \mathrm{CF}_{2n+1}(G_3)$ such that $\partial D = B$. Observe that all of the 1–periodic orbits of G_3 with Conley–Zehnder index greater than $2n$ are contained in $U_1 \cup U_2$: this is because outside of $U_1 \cup U_2$ the Hamiltonian G_3 coincides with H which is Morse and sufficiently C^2–small; see the fifth property in the list of properties of H. It follows that we can write $D = D_1 + D_2$ with D_i being a sum of 1–periodic orbits contained in U_i. Finally, applying Lemma 52 we conclude that $\partial(D_i) = B_i$. □

We will next show that $[M]_{G_1} = [p_0 + C_1 + p_2]$. This will be achieved by proving that the continuation morphism

$$\Psi_{G_3}^{G_1}\colon \mathrm{CF}_*(G_3) \to \mathrm{CF}_*(G_1)$$

satisfies

$$\Psi_{G_3}^{G_1}(C_1) = C_1 + B_1, \quad \Psi_{G_3}^{G_1}(p_0) = p_0 \quad \text{and} \quad \Psi_{G_3}^{G_1}(C_2) = p_2,$$

where B_1 is a boundary term. Since

$$\Psi_{G_3}^{G_1}(p_0 + C_1 + C_2) = \Psi_{G_3}^{G_1}(p_0) + \Psi_{G_3}^{G_1}(C_1) + \Psi_{G_3}^{G_1}(C_2)$$

is a representative for $[M]_{G_1}$, by the second assertion it is of the form $p_0 + C_1' + p_2$, where C_1' is a sum of 1–periodic orbits contained in U_1. By Lemma 51, this can only occur if

$$\Psi_{G_3}^{G_1}(C_1) = C_1', \quad \Psi_{G_3}^{G_1}(p_0) = p_0 \quad \text{and} \quad \Psi_{G_3}^{G_1}(C_2) = p_2.$$

We must now show that

$$\Psi_{G_3}^{G_1}(C_1) = C_1 + B_1.$$

We will apply the latter part of Lemma 51: since C_1 and $\Psi_{G_3}^{G_1}(C_1) = C_1'$ are both contained in U_1, it must be the case that all the Floer trajectories contributing to $\Psi_{G_3}^{G_1}(C_1)$ are contained in the set $\bar{U}_1 \cup V_1$. Note that

$$G_1|_{\bar{U}_1 \cup V_1} = G_3|_{\bar{U}_1 \cup V_1}$$

(indeed, they both coincide with $G_0|_{\bar{U}_1 \cup V_1} + F_1$) and hence

$$(G_s^{33}, J_s^{33}) = (G_s^{31}, J_s^{31})$$

by the compatibility requirement of Claim 54. It can easily be checked that this implies that

$$\Psi_{G_3}^{G_1}(C_1) = \Psi_{G_3}^{G_3}(C_1).$$

But Claim 55 tells us that

$$\Psi_{G_3}^{G_3}(C_1) = C_1 + B_1,$$

where B_1 is a boundary term.

Similarly, one can prove that $[M]_{G_2} = [p_0 + p_1 + C_2]$ by showing that the continuation morphism $\Psi_{G_3}^{G_2}\colon \mathrm{CF}_*(G_3) \to \mathrm{CF}_*(G_2)$ satisfies

$$\Psi_{G_3}^{G_2}(p_0) = p_0, \quad \Psi_{G_3}^{G_2}(C_1) = p_1, \quad \Psi_{G_3}^{G_2}(C_2) = C_2 + B_2,$$

where B_2 is a boundary term.

It only remains to prove that if $[M]_{G_1} = [p_0 + C_1 + p_2]$ and $[M]_{G_2} = [p_0 + p_1 + C_2]$ then $[M]_{G_3} = [p_0 + C_1 + C_2]$. We will use the following claim which is analogous to Claim 55. Its proof is similar to the proof of Claim 55 and hence will be omitted.

Claim 56 $\Psi_{G_i}^{G_i}(C_i) = C_i + B_i$ *for* $i = 1, 2$, *where* B_i *is in the image of the Floer boundary map* $\partial\colon \mathrm{CF}_*(G_i) \to \mathrm{CF}_*(G_i)$.

Clearly,

$$\Psi_{G_1}^{G_3}(p_0 + C_1 + p_2) = \Psi_{G_1}^{G_3}(p_0) + \Psi_{G_1}^{G_3}(C_1) + \Psi_{G_1}^{G_3}(p_2).$$

As was done in the previous paragraph, by appealing to the latter part of Lemma 51 and observing that $G_1|_{\bar{U}_1 \cup V_1} = G_3|_{\bar{U}_1 \cup V_1}$, which by Claim 54 implies $(G_s^{13}, J_s^{13}) = (G_s^{11}, J_s^{11})$, one proves that

$$\Psi_{G_1}^{G_3}(C_1) = \Psi_{G_1}^{G_1}(C_1) = C_1 + B_1,$$

where B_1 is a boundary term and the last equality follows from Claim 56. Similarly, by appealing to the latter part of Lemma 51 and observing that $G_1|_{\bar{U}_2 \cup V_2} = G_0|_{\bar{U}_2 \cup V_2}$, we conclude that

$$\Psi_{G_1}^{G_3}(p_2) = \Psi_{G_0}^{G_3}(p_2).$$

Lastly, using similar arguments, we check that

$$\Psi^{G_3}_{G_1}(p_0) = \Psi^{G_3}_{G_0}(p_0).$$

We conclude from the above discussion that

$$\begin{aligned}[M]_{G_3} = [\Psi^{G_3}_{G_1}(p_0 + C_1 + p_2)] &= [\Psi^{G_3}_{G_0}(p_0) + C_1 + B_1 + \Psi^{G_3}_{G_0}(p_2)] \\ &= [\Psi^{G_3}_{G_0}(p_0) + C_1 + \Psi^{G_3}_{G_0}(p_2)].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}[M]_{G_3} = [\Psi^{G_3}_{G_2}(p_0 + p_1 + C_2)] &= [\Psi^{G_3}_{G_0}(p_0) + \Psi^{G_3}_{G_0}(p_1) + C_2 + B_2] \\ &= [\Psi^{G_3}_{G_0}(p_0) + \Psi^{G_3}_{G_0}(p_1) + C_2].\end{aligned}$$

Comparing the above, we get

$$[\Psi^{G_3}_{G_0}(p_0) + C_1 + \Psi^{G_3}_{G_0}(p_2)] = [\Psi^{G_3}_{G_0}(p_0) + \Psi^{G_3}_{G_0}(p_1) + C_2].$$

Rearranging and simplifying the terms in the above equality we obtain

$$[\Psi^{G_3}_{G_0}(p_0) + C_1 + C_2] = [\Psi^{G_3}_{G_0}(p_0 + p_1 + p_2)].$$

Lastly, we appeal to the second assertion of Lemma 53 to conclude that $\Psi^{G_3}_{G_0}(p_0) = p_0$ and hence obtain

$$[p_0 + C_1 + C_2] = [\Psi^{G_3}_{G_0}(p_0 + p_1 + p_2)].$$

The right-hand side is clearly a representative for $[M]_{G_3}$; thus so is $[p_0 + C_1 + C_2]$. □

Proof of Lemma 50 Without loss of generality, we may assume that every 1–periodic orbit of F_i which is contained in the interior of its support is nondegenerate. Indeed, this can be achieved by making a C^2–small perturbation of F_i in the interior of its support. This in particular implies that every 1–periodic orbit of F_i with nonzero action is nondegenerate.

Consider the 1–parameter family of Hamiltonians $sH + F_i$, where $s \in [0, 1]$. The Hamiltonians sH and F_i have disjoint supports and hence

$$\operatorname{spec}(sH + F_i) = \operatorname{spec}(sH) \cup \operatorname{spec}(F_i).$$

Since $\operatorname{spec}(F_i)$ is a set of measure zero and $c(sH + F_i)$ is a continuous function of s, Lemma 50 follows immediately from the following claim.

Claim 57 $\quad c(sH + F_i) \in \operatorname{spec}(F_i)$ *for all* $s \in [0, 1]$.

To prove the above claim we will use Lemma 9 of [19], which states the following: Let H denote a possibly degenerate Hamiltonian and let

$$A = \{z \in \mathrm{Crit}(\mathcal{A}_H) : c(H) = \mathcal{A}_H(z)\}.$$

Suppose that every 1–periodic orbit in A is nondegenerate. Then there exists $z \in A$ such that $\mu_{CZ}(z) = 2n$.

Observe that every 1–periodic orbit of $sH + F_i$ with nonzero action is nondegenerate: this is because 1–periodic orbits with nonzero action are contained either in the interior of the support of sH or the interior of the support of F_i. Both sH and F_i are nondegenerate in the interior of their supports.

In order to obtain a contradiction, suppose that the claim does not hold and hence there exists $s_0 \in [0, 1]$ such that $c(s_0 H + F_i) \in \mathrm{spec}(s_0 H) \backslash \mathrm{spec}(F_i)$. Note $c(s_0 H + F_i) \neq 0$ because $0 \in \mathrm{spec}(F_i)$. We see that $\{z \in \mathrm{Crit}(\mathcal{A}_{s_0 H + F_i}) : c(s_0 H + F_i) = \mathcal{A}_{s_0 H + F_i}(z)\}$ is a subset of the nondegenerate critical points of H. Because H is C^2–small and Morse in the interior of its support the Conley–Zehnder index of these points coincides with their Morse index and, by construction, none of these critical points have Morse index $2n$. This contradicts Lemma 9 of [19], since every 1–periodic orbit of $sH + F_i$ with nonzero action is nondegenerate. □

We now present the proof of Claim 54.

Proof of Claim 54 By taking $J \in \bigcap_{i=0}^{3} \mathcal{J}_{\mathrm{reg}}(G_i)$ we may assume that $J_i = J_j$. Fix a regular homotopy (G_s^{00}, J_s^{00}) from (G_0, J_0) to itself. We will construct the required homotopies in four stages as outlined below.

Step 1 Constructing regular homotopies (G_s^{i0}, J_s^{i0}) First, considering the cases $i = 1, 2$, we pick any homotopy (G_s^{i0}, J_s^{i0}) from (G_i, J_i) to (G_0, J_0) such that $(G_s^{i0}, J_s^{i0}) = (G_s^{00}, J_s^{00})$ on $\bar{U}_k \cup V_k$ for $k \neq i$. Now, this homotopy is not necessarily regular. However, we claim that it can be perturbed to a regular homotopy by changing it in the complement of the open sets $\bar{U}_k \cup V_k$ for $k \neq i$. Indeed, let u be a solution to the Floer equation which defines the continuation morphism $\Psi_{G_0}^{G_i}$; see Equation (18). If the image of u is contained in one of the open sets $\bar{U}_k \cup V_k$, $k \neq i$, then u is regular because $(G_s^{i0}, J_s^{i0}) = (G_s^{00}, J_s^{00})$ on $\bar{U}_k \cup V_k$, $k \neq i$ and so u solves the corresponding Floer equation for the regular homotopy (G_s^{00}, J_s^{00}). As for the remaining Floer trajectories, whose images are not contained entirely in one of the sets $\bar{U}_k \cup V_k$, $k \neq i$, we can ensure their regularity by perturbing the homotopy (G_s^{i0}, J_s^{i0}) outside the open sets $\bar{U}_k \cup V_k$, $k \neq i$; see Remark 48.

In the case where $i = 3$, we pick any homotopy from (G_3, J_3) to (G_0, J_0) such that $(G_s^{30}, J_s^{30}) = (G_s^{k0}, J_s^{k0})$ on each open set $\bar{U}_k \cup V_k$. By an argument similar to the one described in the previous paragraph we can ensure this homotopy is regular by only perturbing it in the complement of $\bigcup_{k=0}^{3} \bar{U}_k \cup V_k$.

Step 2 Constructing regular homotopies (G_s^{0j}, J_s^{0j}) This is similar to the previous step. First, for any $j \neq 3$ we pick a regular homotopy (G_s^{0j}, J_s^{0j}) from (G_0, J_0) to (G_j, J_j) such that $(G_s^{0j}, J_s^{0j}) = (G_s^{00}, J_s^{00})$ on $\bar{U}_k \cup V_k$ for $k \neq j$. Next, in the case where $i = 3$, we pick a regular homotopy from (G_0, J_0) to (G_3, J_3) such that $(G_s^{03}, J_s^{03}) = (G_s^{0k}, J_s^{0k})$ on each open set $\bar{U}_k \cup V_k$. It is clear that the homotopies we have picked so far satisfy the required compatibility condition.

Step 3 Constructing regular homotopies (G_s^{ii}, J_s^{ii}) First suppose that $i \neq 3$. Pick a regular homotopy from (G_i, J_i) to itself such that $(G_s^{ii}, J_s^{ii}) = (G_s^{00}, J_s^{00})$ outside of $\bar{U}_i \cup V_i$. In the case where $i = 3$, we pick a regular homotopy from (G_3, J_3) to (G_3, J_3) such that $(G_s^{33}, J_s^{33}) = (G_s^{kk}, J_s^{kk})$ on each open set $\bar{U}_k \cup V_k$. Once again it is clear that the homotopies we have picked so far satisfy the required compatibility condition.

Step 4 Constructing regular homotopies (G_s^{ij}, J_s^{ij})**, where** $i \neq j$ First, we consider the case where $i, j \neq 3$. Pick any homotopy from (G_i, J_i) to (G_j, J_j) such that

$$(G_s^{ij}, J_s^{ij}) = \begin{cases} (G_s^{i0}, J_s^{i0}) & \text{on } \bar{U}_i \cup V_i, \\ (G_s^{0j}, J_s^{0j}) & \text{on } \bar{U}_j \cup V_j, \\ (G_s^{00}, J_s^{00}) & \text{on } \bar{U}_k \cup V_k,\ k \neq i, j. \end{cases}$$

If $i = 3$, pick a homotopy from (G_3, J_3) to (G_j, J_j) such that

$$(G_s^{3j}, J_s^{3j}) = \begin{cases} (G_s^{jj}, J_s^{jj}) & \text{on } \bar{U}_j \cup V_j, \\ (G_s^{30}, J_s^{30}) & \text{on } \bar{U}_k \cup V_k,\ k \neq j. \end{cases}$$

If $j = 3$, pick a homotopy from (G_i, J_i) to (G_3, J_3) such that

$$(G_s^{i3}, J_s^{i3}) = \begin{cases} (G_s^{ii}, J_s^{ii}) & \text{on } \bar{U}_i \cup V_i, \\ (G_s^{03}, J_s^{03}) & \text{on } \bar{U}_k \cup V_k,\ k \neq i. \end{cases}$$

Using an argument similar to the one we used in Step 1, we can ensure that each of the above homotopies is regular by perturbing it in the complement of the open sets $\bigcup_{k=0}^{2} \bar{U}_k \cup V_k$. It can easily be checked that the homotopies we have picked satisfy the required compatibility condition. □

Appendix: Existence of maximal unlinked sets

In this appendix, we prove Theorem 8 and Proposition 9 about unlinked sets. Throughout the appendix we consider a compactly supported isotopy $(\phi^t)_{t\in[0,1]}$ on an orientable surface Σ, and denote its time-1 map ϕ^1 by ϕ.

A.1 Finite unlinked sets

This section contains the proof of Proposition 9 which characterizes finite unlinked sets as finite sets whose associated geometric braid represents the trivial braid. The crucial point in the proof is the following Lemma.

Lemma 58 *Let X be a finite subset of Σ of cardinality n. Then the map $f \mapsto f(X)$ from $\mathrm{Diff}_0(\Sigma)$ to the space $\Xi(\Sigma, n)$ of n–tuples of distinct points in Σ, is a fiber bundle.*

Proof Using local charts the proof of the lemma reduces to the following easy fact. Let $\mathbb{D}^2$ denote the open unit disk, and 0 be some point in $\mathbb{D}^2$. There exists a continuous map $x \mapsto \gamma_x$ from $\mathbb{D}^2$ to the space Diff($\mathbb{D}^2$) of diffeomorphisms of $\mathbb{D}^2$ with compact support, such that γ_0 is the identity, and for every point x of $\mathbb{D}^2$, $\gamma_x(0) = x$. There are many ways to construct γ, one possibility is to use Hamiltonian functions: then γ_x is a Hamiltonian diffeomorphism of the disk (in particular, the lemma also holds when $\mathrm{Diff}_0(\Sigma)$ is replaced by $\mathrm{Ham}(\Sigma)$). □

Proof of Proposition 9 The direct implication is straightforward. For the reverse one, consider a finite set X of contractible fixed points for $(\phi^t)_{t\in[0,1]}$. The geometric braid $b_{X,(\phi^t)}$ is a loop based at X in the space $\Xi(\Sigma, n)$. Assume that $b_{X,(\phi^t)}$ represents the trivial braid. This means that it is a loop homotopic to the constant loop. A fiber bundle is a Serre fibration, that is, it has the "homotopy lifting property" for disks. Thus, according to the lemma, the homotopy from $b_{X,(\phi^t)}$ to the constant loop may be lifted to a homotopy, with endpoints fixed, between the isotopy $(\phi^t)_{t\in[0,1]}$ and an isotopy I which lifts the constant loop in $\Xi(\Sigma, n)$, ie, which fixes every point of X. In other words, the set X is unlinked. □

An unlinked set X is *maximal* if there is no unlinked set X' strictly containing X.

Corollary 59 *Let X be an unlinked set, and I an isotopy that fixes every point of X and whose time-1 map is ϕ. If Σ is the sphere, assume furthermore that X does not contain exactly two elements. Then X is maximal if and only if for every fixed point x of ϕ which is not in X, the trajectory of x under I is not contractible in $\Sigma \setminus X$.*

Proof The direct implication may be proved by an argument similar to the proof of Proposition 9. The proof of the converse goes as follows. The case when X is empty directly follows from the proposition. Let X be a nonempty unlinked set, I be an isotopy fixing every point of X, x a point outside X, and J an isotopy fixing $X \cup \{x\}$ and such that I and J are homotopic as paths in $\mathrm{Diff}_0(\Sigma)$. We want to prove that the trajectory of x under I is contractible in $\Sigma \setminus X$. Let α be the class of this trajectory in $\pi_1(\Sigma \setminus X, x)$. Then α commutes with every element β in $\pi_1(\Sigma \setminus X, x)$: indeed, the map $(s,t) \mapsto f_t(b(s))$, where (f_t) is the concatenation of I with J^{-1} and b is a loop in the class β, may be seen as a homotopy between $\alpha\beta\alpha^{-1}\beta^{-1}$ and the trivial loop. We conclude that α is the trivial loop when the center of $\pi_1(\Sigma \setminus X, x)$ is trivial. Since X is nonempty and we have excluded the case when Σ is the sphere and X contains exactly two elements, this covers every case except when Σ is the plane and X is a single element. This last case may be solved by using the following fact: the space of compactly supported diffeomorphisms of the plane fixing a given point is contractible. □

A.2 Infinite unlinked sets

This section contains the proof of Theorem 8: a set X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if every finite subset of X is unlinked. Note that the direct implication is immediate. For the converse, the key to the proof will be an argument, due to Michael Handel, showing that any surface diffeomorphism is isotopic to the identity in some neighborhood of its fixed point set.

Let $A(\phi)$ denote the set of accumulation points of the set of fixed points of ϕ. Then if $A(\phi) = \Sigma$ the theorem is obvious, thus we may assume $A(\phi) \neq \Sigma$. We consider an open neighborhood V of $A(\phi)$ which is not Σ. The surface V, being noncompact, may be endowed with a flat Riemannian metric, ie, a metric which is locally isometric to the euclidean plane. Given a relatively compact subset $V' \subset V$, every close enough pair of points x, y in V' may be joined by a unique geodesic segment of length $d(x,y)$. We denote by $\gamma_{x,y}\colon [0,1] \to \Sigma$ the parametrization of this segment with constant speed.

Lemma 60 (Handel [16, Lemma 4.1]) *For every open set V containing $A(\phi)$, there exists some open set V', $A(\phi) \subset V' \subset V$, such that the "straight" homotopy*

$$f_t(x) = \gamma_{x,\phi(x)}(t)$$

is one-to-one on $V' \cup \mathrm{Fix}(\phi)$.

Proof In local charts isometric to the plane, $\gamma_{x,\phi(x)}(t)$ reads $(1-t)x + t\phi(x)$. The differential of f_t at the point x is $(1-t)\mathrm{Id} + tD\phi(x)$. At every point of $A(\phi)$ the differential of ϕ has a fixed vector, and thus (since ϕ is orientation-preserving) no negative eigenvalue. If V' is close enough to $A(\phi)$ then $D\phi$ has no negative eigenvalue on V' either. The inverse function theorem implies that ϕ is one-to-one on V'. For more details we refer to [16]. □

Note that every germ of an orientation-preserving diffeomorphism at some fixed point x is locally isotopic to the identity (up to composing with a rotation $D\phi_x$ has no negative eigenvalue, and then one can use again the straight line isotopy). By extension of isotopies (see for example [18, Chapter 8]), we get the following corollary.

Corollary 61 *There exists an isotopy $\Delta = (f_t)_{t\in[0,1]}$ from $f_0 = \phi$ to a diffeomorphism f_1, and an open neighborhood U of the set of contractible fixed points of $(\phi^t)_{t\in[0,1]}$, such that*

- *the isotopy Δ fixes every contractible fixed point of $(\phi^t)_{t\in[0,1]}$,*
- *f_1 is the identity on U.*

Let I denote the isotopy which is the concatenation of $(\phi^t)_{t\in[0,1]}$ and Δ, whose time-1 map is f_1. Note that a family X of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ is unlinked if and only if it is unlinked for I. We may assume that the closure of U is a compact subsurface of Σ (with boundary). We denote by $\{U_i : i \in \pi_0(U)\}$ the connected components of U; the set $\pi_0(U)$ is finite, and we may assume that each U_i contains some contractible fixed point of $(\phi^t)_{t\in[0,1]}$; then every point in U_i is a contractible fixed point for the isotopy I. Also note that since f_1 is the identity on U_i, all the points in U_i have the same rotation number ρ_i for I, which is an integer. Let x, y be two distinct points in the same U_i, and join them by an arc γ. In the universal cover $\widetilde{\Sigma} \simeq \mathbb{R}^2$, let $\tilde{x}, \tilde{y}$ be two lifts of x, y which are the endpoints of some lift $\tilde{\gamma}$ of γ. Then the linking number $\ell(\tilde{x}, \tilde{y})$ for the isotopy that lifts I is also equal to ρ_i. As a consequence, if ρ_i is not zero, then every pair $\{x, y\}$ of contractible fixed points of $(\phi^t)_{t\in[0,1]}$ included in some U_i is linked.

Let X be a family of contractible fixed points of $(\phi^t)_{t\in[0,1]}$. We choose a subset of X by selecting at most two points of X in each U_i. More precisely, we denote by X_U any subset of X with the following property: for every $i \in \pi_0(U)$,

- if the set $X \cap U_i$ has zero or one element, then $X_U \cap U_i = X \cap U_i$;
- if the set $X \cap U_i$ contains more than one element, then $X_U \cap U_i$ has exactly two elements.

Note that the set X_U is finite. Thus Theorem 8 is an immediate consequence of the following lemma.

Lemma 62 *The set X is unlinked for $(\phi^t)_{t\in[0,1]}$ if and only if the set X_U is unlinked for $(\phi^t)_{t\in[0,1]}$.*

Before proving the lemma, we will deduce the existence of maximal unlinked sets. The following corollary is not used in this text.

Corollary 63 *Every unlinked set is included in a maximal unlinked set.*

Proof of the corollary The corollary follows from Theorem 8 by using Zorn's lemma, as in the proof of Corollary 15. Here is a more constructive argument. Let X be an unlinked set, and X_U be as before. Note that X_U is unlinked since it is a subset of X. Let Y be a set which contains X_U, which is unlinked, which contains at most two points in each $U_i \in \pi_0(U)$, and which is maximal for inclusion among such sets. The cardinality of such a set is clearly less than twice the cardinality of $\pi_0(U)$ which is finite: thus the existence of Y is immediate. Let X' be obtained from Y by adding to it, for every $U_i \in \pi_0(U)$ which contains two points of Y, all the contractible fixed points of $(\phi^t)_{t\in[0,1]}$ which are included in U_i. Note that Y satisfies the required properties for the set X'_U. Thus according to the above lemma, since $X'_U = Y$ is unlinked, the set X' is unlinked. If x is a contractible fixed point which is not in X', then it follows from the lemma and the maximality of Y that $X' \cup \{x\}$ is not unlinked. Thus X' is a maximally unlinked set containing X, as wanted. □

Proof of the lemma The direct implication is immediate. Let X, X_U be as before and assume that X_U is unlinked for $(\phi^t)_{t\in[0,1]}$. Then it is also unlinked for I. Let $J_0 = (g_t)_{t\in[0,1]}$ be an isotopy from the identity to f_1 fixing every point of X_U. We want to modify J_0 to an isotopy from the identity to f_1 that fixes every point of X. This will prove that X is unlinked for I, and thus also for $(\phi^t)_{t\in[0,1]}$.

Call *arotational* the values of i for which U_i contains two elements of X_U. Since X_U is unlinked, by the considerations following Corollary 61, the rotation number ρ_i vanishes for arotational i's. Using this property, it is not difficult to modify the isotopy J_0 to an isotopy $J_1 = (h_t)_{t\in[0,1]}$ which still fixes every point of X_U, and such that, for the arotational indices i, the differential $Dh_t(x)$ is the identity for every t at both points x of $X_U \cap U_i$.

We want to further modify J_1 so that it fixes every point in the arotational U_i. For this we will use the following basic results on embeddings, which are proved below. Let S be a surface with boundary, S' a connected subsurface in the interior of S, and x_0 a point in the interior of S'. We denote by $\mathrm{Diff}_c(S; x_0)$ the space of diffeomorphisms of S which are compactly supported in the interior of S, that fixes x_0, and whose differential at x_0 is the identity. Likewise, let $E(S', S; x_0)$ be the space of embeddings of S' into the interior of S that fixes x_0 and whose differential at x_0 is the identity.

Proposition 64 (1) *Every connected component of* $E(S', S; x_0)$ *is simply connected.*

(2) *The restriction map from* $\mathrm{Diff}_c(S; x_0)$ *to* $E(S', S; x_0)$ *is a fiber bundle.*

Let i_1 be some arotational index, and choose some x_1 in $X_{U_{i_1}}$. Since $h_1 = f_1$ fixes every point in U_{i_1}, the family $\ell = (h_{t|U_{i_1}})_{t\in[0,1]}$ is a loop in $E(U_{i_1}, \Sigma; x_1)$ based at the inclusion map $e\colon U_{i_1} \subset \Sigma$. The first assertion of the proposition tells us that this loop ℓ is contractible in $E(U_{i_1}, \Sigma; x_1)$: let $(\ell_s)_{s\in[0,1]}$ be a deformation of loops with fixed base-point e from $\ell_1 = \ell$ to the trivial loop. According to second assertion of the proposition, this deformation may be lifted to a deformation, with fixed endpoints Id and h_1, from the isotopy $J_1 = (h_t)$ to a new isotopy J_2 which is a lift of the trivial loop in $E(U_{i_1}, \Sigma; x_1)$, which means that J_2 fixes every point of U_{i_1}. We now consider a second arotational index i_2, and apply the proposition with $S = \Sigma \setminus U_{i_1}$, and S' equal to the closure of U_{i_2}. This yields a new isotopy J_3 that fixes every point of $U_{i_1} \cup U_{i_2}$. We go on until we get an isotopy J from Id to f_1 which fixes every point of every arotational U_i. This isotopy fixes every point of X, as wanted. □

Proof of Proposition 64 We begin with the second assertion. When no base point is given, the fact that the restriction map is a fiber bundle is due to Palais. The base point case that we need follows immediately from the following result, due to Cerf: *For every embedding f from some compact manifold V into some manifold M, there is a neighborhood U of f in the space of embeddings, and a continuous map ξ from U to the space of diffeomorphisms of M with compact support, such that for every g in U,* $g = \xi(g) \circ f$. For references and details we refer to the very short paper of Lima [27].

Now to prove the first assertion, let γ be a loop in the space $E(S', S; x_0)$. We deform γ into a trivial loop by successively using the following three ingredients. Details are left to the reader (again, [15] is a good reference). The first ingredient allows us to deform γ into a loop γ_1 that fixes one vector tangent to each boundary component of S'. The second ingredient allows us to further deform γ_1 into a loop γ_2 that fixes each point of the boundary of S'. The last ingredient shows that γ_2 is contractible in $E(S', S; x_0)$.

Ingredient 1 Let $x_1, \dots, x_k$ be distinct points of S', distinct from x_0, and choose for each i a nonzero vector v_i tangent to S' at x_i (if the points are on the boundary of S' then the vectors are tangent to the boundary). Consider the natural map $\Psi_{(x_i, v_i)}$ from $E(S', S; x_0)$ to the space Ξ_k of k–tuples of nonzero vectors over distinct points in $S' \setminus \{x_0\}$, obtained by taking the images of the (x_i, v_i). This map is a fiber bundle. Furthermore, the image of every loop $\gamma = (f_t)$ in $E(S', S; x_0)$ is contractible in Ξ_k.

The key observation for this last property is the following. By definition, $f_t(x_0) = x_0$ and $Df_t(x_0) = \mathrm{Id}$ for every t. Thus if x_1' is some point close enough to x_0 then the loop $t \mapsto f_t(x_1')$ will be included in a small disk D_1 not containing x_0, and furthermore for any nonzero vector v_1' at x_1' the loop $\Psi_{(x_1',v_1')}\gamma\colon t \mapsto Df_t(x_1') \cdot v_1'$ will be close to the constant vector, and thus contractible in the complement of the zero section in the tangent bundle of D_1. To make use of this observation, we move the points $x_1, \dots, x_k$ into points $x_1', \dots, x_k'$ with x_1' close to x_0 as above, x_2' much closer to x_0 than x_1' so that it is included in a disk D_2 disjoint from D_1 and x_0, and so on. These moves induce a deformation of the loop $\Psi_{(x_i,v_i)}\gamma$ into the loop $\Psi_{(x_i',v_i')}\gamma$. According to the observation, this new loop is contractible.

Ingredient 2 Choose one point x_i on each boundary component of S', and a vector v_i tangent to $\partial S'$ at x_i. Let $E(S', S; x_0, (x_1, v_1), \dots, (x_k, v_k))$ denote the subspace of $E(S', S; x_0)$ that fixes all the x_i and v_i.

With obvious notation, the restriction map

$$E(S', S; x_0, (x_1, v_1), \dots, (x_k, v_k)) \mapsto E(\partial S', S; x_0, (x_1, v_1), \dots, (x_k, v_k))$$

is a fiber bundle. Furthermore, each connected component of the base of the fibration is contractible. This last property follows from [15, Théorème 4], which considers the case of the embedding of a single circle in S, by induction on the number of boundary components.

Ingredient 3 Every connected component of the space $\mathrm{Diff}(S'; \partial S', x_0)$ of diffeomorphisms of S' that are the identity on the boundary and tangent to the identity at x_0 has trivial homotopy groups. Indeed, this space is the fiber of the restriction map from $\mathrm{Diff}(S'; x_0)$ to $\mathrm{Diff}(\partial S')$. This is a fiber bundle. On the one hand, the space of orientation-preserving diffeomorphisms of the circle has the homotopy type of $\mathrm{SO}(2)$, and thus the connected component of the identity in the base of the fibration has trivial homotopy groups of order $\geqslant 2$. On the other hand, by [15, Théorème 2], the total space of the fibration is contractible. The triviality of the homotopy group of the fiber is now a consequence of the exact sequence of the fibration. □

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