Geometry & Topology 20 (2016) 2335–2395

Multisections of Lefschetz fibrations and topology of symplectic 4–manifolds

R İNANÇ BAYKUR
KENTA HAYANO

We initiate a study of positive multisections of Lefschetz fibrations via positive factorizations in framed mapping class groups of surfaces. Using our methods, one can effectively capture various interesting symplectic surfaces in symplectic 4–manifolds as multisections, such as Seiberg–Witten basic classes and exceptional classes, or branched loci of compact Stein surfaces as branched coverings of the 4–ball. Various problems regarding the topology of symplectic 4–manifolds, such as the smooth classification of symplectic Calabi–Yau 4–manifolds, can be translated to combinatorial problems in this manner. After producing special monodromy factorizations of Lefschetz pencils on symplectic Calabi–Yau homotopy K3 and Enriques surfaces, and introducing monodromy substitutions tailored for generating multisections, we obtain several novel applications, allowing us to construct: new counterexamples to Stipsicz's conjecture on fiber sum indecomposable Lefschetz fibrations, nonisomorphic Lefschetz pencils of the same genera on the same new symplectic 4–manifolds, the very first examples of exotic Lefschetz *pencils*, and new exotic embeddings of surfaces.

57M50, 57R17, 57R55, 57R57; 53D35, 20F65, 57R22

1 Introduction

Since the groundbreaking work of Donaldson, it is known that every symplectic 4–manifold admits a symplectic Lefschetz pencil [10], and conversely, every Lefschetz fibration with nonempty critical locus admits a symplectic structure; see Gompf and Stipsicz [25]. On the other hand, Lefschetz pencils/fibrations are determined by their monodromy factorizations, which are prescribed by products of positive Dehn twists isotopic to identity/boundary multitwist on the fiber; see Kas [28] and Matsumoto [40]. These results yield a combinatorial description of symplectic 4–manifolds in terms of ordered tuples of isotopy classes of simple closed curves on an orientable surface. Here we will extend this fundamental approach, by introducing and studying positive factorizations in a *framed mapping class group*, so as to describe symplectic

Published: 15 September 2016 DOI: 10.2140/gt.2016.20.2335

4–manifolds together with various important symplectic surfaces in them in terms of ordered simple closed curves and arcs between marked points on an orientable surface.

Let X be a closed oriented 4–manifold equipped with a Lefschetz fibration $f\colon X \to S^2$. We call an embedded, possibly disconnected surface S in X a *multisection* or *n–section* if $f|_S\colon X \to S^2$ is an n–fold branched cover with only simple branched points. We assume that both Lefschetz critical points and branched points conform to local complex models; that is, we work with *positive* Lefschetz fibrations and *positive* branched points. Precise definitions and the basic background material are given in Section 2 below.

The first main result of our article is the description of multisections and their ambient topology via positive factorizations in framed mapping class groups, given in detail in Theorem 3.6. The framings amount to working with a new mapping class group of a compact oriented surface with marked boundary circles (one marked point on each boundary component), which consists of isotopy classes of orientation-preserving self-diffeomorphisms that are allowed to swap boundary components while matching the marked points. This group is naturally isomorphic to the mapping class group of a *closed* surface with attached vectors at a finite *set* of marked points, so as to frame a tubular neighborhood of the multisection S (where the end points of these vectors, and equivalently the marked points on the boundaries, trace a push-off of S). In Section 3, leading to the proof of this theorem, we introduce the notion of positivity for monodromy factorizations in this more general setting. As we will show, from these positive factorizations, for each multisection S, one can easily read off the degree (ie the number of times S intersects the fibers), topology (number of components and genera of each component of S) and the self-intersection numbers of the components of S. Here is our main theorem of Section 3 (Theorem 3.6), stated for a connected S for simplicity — the reader might want to turn to that section for the description of various mapping classes appearing in the factorizations below:

Theorem 1.1 *A genus-g Lefschetz fibration (X, f) with a self-intersection m connected n–section $S \subset X$ with k branched points away from $\mathrm{Crit}(f)$, and r branched points at Lefschetz singularities corresponding to vanishing cycles $c_1, \dots, c_r$ among $c_1, \dots, c_l$, yields a lift of the monodromy factorization of (X, f) to a factorization*

$$\widetilde{\tau}_{\alpha_k} \cdots \widetilde{\tau}_{\alpha_1} \cdot t_{\widetilde{c_l}} \cdots t_{\widetilde{c_{r+1}}} \cdot \widetilde{t_{c_r}} \cdots \widetilde{t_{c_1}} = t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}$$

in $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$, where $\{u_1, \dots, u_n\}$ is a subset of $\partial\Sigma_g^n$ which covers all the elements of $\pi_0(\partial\Sigma_g^n)$, $\widetilde{\tau}_{\alpha_i}$ is a lift of the half twist along a simple arc α_i between two points in $\{u_1, \dots, u_n\}$ as described in Figure 1, and $\widetilde{t_{c_i}}$ is a lift of the Dehn twist t_{c_i} as described in Figure 3. Here $\widetilde{c_j}$ is a simple closed curve in Σ_g^n which is isotopic *to c_j via the inclusion $i\colon \Sigma_g^n \hookrightarrow \Sigma_g$, and $\{\delta_1, \dots, \delta_n\}$ is a set of simple closed curves parallel to $\partial\Sigma_g^n$, where*

$$g(S) = \tfrac{1}{2}(k+r) - n + 1 \quad \text{and} \quad m = -\left(\sum_{i=1}^{n} a_i\right) + 2k + r.$$

The connectivity of S implies that the collection of $\widetilde{\tau}_{\alpha_i}$ and $\widetilde{t_{c_i}}$ act transitively on the collection of u_j.

Conversely, from any such relation in $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$, subject to the conditions listed above, one can construct a genus-g Lefschetz fibration (X, f) with a connected n–section S of genus $g(S)$ and self-intersection m as above, whose monodromy factorization is given by the image of the factorization on the left hand side under the homomorphism forgetting $\{u_1, \dots, u_n\}$.

Moreover, up to an extended set of Hurwitz moves, there exists a one-to-one correspondence between Lefschetz fibrations with multisections and positive factorizations in the above framed mapping class group. The authors show this in [8] (where the intricate connection between possible choices of $\alpha_i, \widetilde{c_i}$ and a_i are explained in detail).

As observed by Donaldson and Smith [11], any, possibly disconnected, symplectic surface S in (X, ω) can indeed be realized as a multisection of a high enough degree Lefschetz pencil on X, which does not go through any Lefschetz critical points. Our theorem, therefore, extends the combinatorial interpretation of a symplectic 4–manifold, which couples the results of Donaldson and Gompf with the earlier works of Kas and Matsumoto, to that of a symplectic 4–manifold and disjoint symplectic surfaces in it in terms of ordered tuples of interior curves $\widetilde{c}_1, \dots, \widetilde{c}_l$ and arcs $\alpha_1, \dots, \alpha_r$ with end points on marked points $u_1, \dots, u_n$ on distinct boundary components of Σ_g^n (corresponding to the factors $t_{\widetilde{c}_i}$, and $\widetilde{\tau}_{\alpha_j}$, respectively). On the other hand, it was shown by Loi and Piergallini [38] that any compact Stein surface (X, J) can be obtained as a covering of the unit 4–ball D^4 branched along a *braided surface* S, such that the composition $f\colon X \to D^4 \to D^2$ is an allowable Lefschetz fibration, along with the obvious converse result. In this case, we obtain a similar combinatorial description of a Stein surface (X, J) together with the branched locus S in terms of *pairs* of arcs $\alpha_1', \alpha_1'', \dots, \alpha_r', \alpha_r''$, each with end points on the marked points on the same pair of distinct boundary components of Σ_g^n (corresponding to the factors $\widetilde{t_{c_i}}$). Although we note that the latter can be always perturbed (see Remark 3.9) to a factorization which only consists of factors $t_{\widetilde{c}_i}$ and $\widetilde{\tau}_{\alpha_j}$, we present our results in this full generality so as to not only mark the case of compact Stein surfaces, but also because we often find it useful to first produce factorizations containing multisections going through Lefschetz critical points.

We should make two remarks here. First, although the framed mapping class group $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$ has a large set of generators, which for instance involve boundary pushing maps (lifts of point pushing maps), what we have manifested in Theorem 1.1 is that, for the geometric situations discussed above, it suffices to work with boundary twists and usual Dehn twists. Secondly, our framed mapping class group is larger than the mapping class group of a surface with boundary which consists of isotopy classes of self-diffeomorphisms that fix *each boundary component*, the latter being the natural mapping class group to work with when dealing with *n disjoint sections*. The distinction between these two groups is analogous to that of the framed surface braid group versus the framed pure surface braid group, which have appeared in two recent works that are worth mentioning here: Bellingeri and Gervais studied the exact sequences relating these braid groups [9], whereas Massuyeau, Oancea and Salamon used the same groups to describe the monodromy action of the fundamental group on the first homology of the fiber in terms of the Picard–Lefschetz intersection data associated to vanishing cycles of a given Lefschetz fibration [39].

Combining the seminal work of Taubes [53; 54] and Donaldson [10], and following the ideas of Donaldson and Smith [11] mentioned above, a blow-up of any given symplectic 4–manifold X with $b^+(X) > 1$ admits a Lefschetz fibration with respect to which all Seiberg–Witten basic classes are multisections (called *standard surfaces* in [11]), as discussed in the Appendix. Translating this to positive mapping class group factorizations as we prescribed in Section 3, we conclude that symplectic 4–manifolds and their Seiberg–Witten basic classes can be *a priori* represented combinatorially in terms of our positive factorizations. Section 5B contains many examples of Kodaira dimension zero symplectic 4–manifolds, where all the Seiberg–Witten basic classes are represented by a collection of (-1)–multisections (as dictated by the blow-up formula for Seiberg–Witten invariants) of the constructed Lefschetz fibrations on them. In the Appendix, we present Kodaira dimension-1 examples; namely, we carry out a sample calculation of monodromy factorizations of Lefschetz fibrations on the knot-surgered elliptic surfaces which capture all their Seiberg–Witten basic classes.

The remaining Sections 4–7 of the article gather a variety of applications, relying on the constructive converse direction of our main theorem. Each section focuses on a different problem related to the topology of symplectic 4–manifolds and Lefschetz fibrations on them, yet what is in common for all is the essential use of our mapping class group techniques involving multisections. The novelty of ideas and techniques employed in our constructions of examples in Section 5–7 can easily be seen to amount to *recipes* one can employ with the right set of Lefschetz fibrations and multisections in hand, where we will be focusing on producing the *smallest* genus examples of each kind, which are the hardest to obtain in our experience.

In Section 4 we provide an alternate approach to the *smooth* classification of symplectic 4–manifolds of Kodaira dimension zero, ie (blow-ups) of symplectic Calabi–Yau 4–manifolds. The only known examples of Kodaira dimension zero symplectic 4–manifolds are torus bundles over tori, the K3 and the Enriques surfaces, which, up to diffeomorphisms, conjecturally exhaust all the possibilities. Using our work from Section 3 and the following theorem we prove in Section 4 (Theorem 4.1), we translate the problem to a combinatorial one (see Theorem 4.1, Corollary 4.2 and Question 4.3):

Theorem 1.2 *Let (X, f) be a genus-g Lefschetz fibration with $g \geq 2$, and X be neither rational nor ruled. Then, there exists a symplectic form ω on X compatible with f such that (X, ω) is a (blow-up of) a symplectic Calabi–Yau 4–manifold, if and only if there is a disjoint collection of (-1)–spheres that are n_j–sections of (X, f) such that $\sum_j n_j = 2g-2$.*

Motivated by this, we introduce new techniques based on certain symmetries, to lift better understood relations from genus-0 and genus-1 surfaces to higher genus surfaces under involutions, so as to construct explicit monodromy factorizations of Lefschetz pencils on symplectic Calabi–Yau K3 and Enriques surfaces, ie minimal symplectic 4–manifolds of Kodaira dimension zero, homeomorphic to K3 and Enriques surfaces, respectively; see Propositions 4.5 and 4.7.

In Section 5 we turn to an interesting conjecture of Stipsicz on fiber sum indecomposable Lefschetz fibrations, which can be regarded as prime building blocks of any Lefschetz fibration via the fiber sum operation. In [51], having proved the converse statement, Stipsicz conjectured that any fiber sum indecomposable Lefschetz fibration admits a (-1)–sphere section, an affirmative answer to which would allow one to think of any Lefschetz fibration to be obtained from Lefschetz pencils through blow-ups and fiber sums. Curiously, up to date, there was only one known counterexample to this conjecture, which was a genus-2 Lefschetz fibration constructed by Auroux, as observed by Sato in [45]. In Lemma 5.1, we introduce a generalization of the lantern relation involving multisections, which allows us to braid exceptional sections into exceptional multisections of a new Lefschetz fibration obtained by a rational blow-down of the underlying symplectic 4–manifold. Relying on this key lemma, and our special monodromy factorizations of symplectic Calabi–Yau Lefschetz fibrations obtained in Section 4, where one can keep track of *all* exceptional classes and sections, we prove that the above counterexample is not a mere exception (Theorems 5.4 and 5.6):

Theorem 1.3 *There are several genus-3 and genus-2 fiber sum indecomposable Lefschetz fibrations on blow-ups of symplectic Calabi–Yau 4–manifolds which do not admit any (-1) sphere sections.*

Section 6 deals with the *diversity* of Lefschetz pencils/fibrations on a symplectic 4–manifold. Namely, we prove that blow-ups of symplectic Calabi–Yau K3 surfaces can be supported by nonisomorphic Lefschetz pencils of the same genera and same number of base points, which have ambiently homeomorphic fibers. Park and Yun used monodromy groups to construct pairs of nonisomorphic Lefschetz fibrations on knot-surgered elliptic surfaces, which are Kodaira dimension 1 symplectic 4–manifolds [44], and more recently, the first author proved that blow-ups of any symplectic 4–manifold which is not rational or ruled carry an arbitrarily large number of nonisomorphic Lefschetz fibrations [6]. Here we show that, for a certain configuration of Lefschetz vanishing cycles and (-1)–sphere sections, one can perform a pair of monodromy substitutions which amount to rational blow-downs that "mirror" each other's topological effect. These result in Lefschetz fibrations on the *same* symplectic 4–manifold with ambiently homeomorphic fibers. Building on our examples of monodromy factorizations tailored specifically to contain such configurations, we then obtain the following on symplectic 4–manifolds of Kodaira dimension 0 (Theorem 6.2):

Theorem 1.4 *There are pairs of genus-g relatively minimal nonisomorphic Lefschetz pencils (X, f_i), $i = 1, 2$, where g can be taken as small as 3, or arbitrarily large.*

In Section 7 we investigate a natural question: does the *topology* of a Lefschetz pencil (fiber genus, number of separating/nonseparating vanishing cycles and base points) uniquely determine the diffeomorphism type of a symplectic 4–manifold within its homeomorphism class? Here we answer this question in the negative by constructing the first examples of pairwise homeomorphic but not diffeomorphic symplectic 4–manifolds, supported by Lefschetz pencils with the same topology. To the best of our knowledge, the only previously known examples of this type were the Lefschetz *fibrations* of Fintushel and Stern on knot-surgered elliptic surfaces — all of Kodaira dimension 1, again. Here we construct the first examples of such pencils (Theorem 7.1):

Theorem 1.5 *There are genus-3 exotic Lefschetz pencils (X_i, f_i), $i = 0, 1$, with symplectic Kodaira dimension $\kappa(X_i) = i$, where X_i are homeomorphic to $\mathrm{K3} \# \overline{\mathbb{CP}}^2$. Moreover, there are similar examples with arbitrarily high genus and the same topology for the singular fibers on higher blow-ups of homotopy K3 surfaces.*

Lastly, in the same section, we also show that a careful application of the same circle of ideas provides a new way of constructing exotic embeddings of surfaces in 4–manifolds, ie F_i in X, $i = 1, 2$, such that there are ambient homeomorphisms taking one to the other, but there exist no such ambient diffeomorphisms (Theorem 7.4):

Theorem 1.6 *There are exotic embeddings of genus-3 surfaces F_i in a blow-up of a symplectic Calabi–Yau* K3 *surface such that F_i is symplectic with respect to deformation equivalent symplectic forms ω_i on X, for $i = 1, 2$.*

We will finish with noting a further motivation for our study of multisections. The rather explicit description of a 4–manifold obtained via the monodromy of a Lefschetz fibration on it very often allows one to detect various configurations of symplectic surfaces in it; disjoint copies of fibers and sections, as well as matching pairs of Lefschetz vanishing cycles, are a few examples of this sort. Coupled with the nontriviality of Seiberg–Witten invariants on symplectic 4–manifolds, this has been the most essential source of producing new symplectic and smooth 4–manifolds in the past few decades. (See, for instance, Fintushel and Stern [21] for an excellent survey of such construction methods.) A close look at these constructions shows that sections and multisections of such Lefschetz fibrations feature a key role. We therefore expect that the monodromy factorizations, which involve multisections to produce interesting configurations of surfaces (such as the ones we used in our rational blow-downs in Sections 6 and 7), will be useful for building new symplectic and exotic 4–manifolds. We plan to investigate this direction in future work.

Acknowledgements The first author was partially supported by the NSF Grant DMS-1510395, the Simons Foundation Grant 317732 and the ERC Grant LDTBud. The second author was partially supported by JSPS Research Fellowships for Young Scientists (24·993) and JSPS KAKENHI (26800027). We also would like to thank Ersin Çelik for running the code for our calculations to detect the smallest genera exotic pencils.

2 Preliminaries

In this article, we assume that all manifolds are compact, connected, smooth and oriented, and all the maps between them are smooth.

2A Lefschetz fibrations and multisections

Let X and Σ be compact manifolds (possibly with boundary) of dimensions 4 and 2, respectively.

A smooth map $f\colon X \to \Sigma$ is a *Lefschetz fibration* if $\operatorname{Crit} f$ is a discrete set in the interior of X such that for any $p_i \in \operatorname{Crit}(f)$, we can take a complex coordinate (U, φ)

(resp. (V, ψ)) of p_i (resp. $f(p_i)$) compatible with the orientation of X (resp. of Σ) so that

$$\psi \circ f \circ \varphi^{-1}(z_1, z_2) = z_1 z_2.$$

We furthermore assume that for each point $q_i \in C = f(\mathrm{Crit}(f))$, the *singular fiber* $f^{-1}(q_i)$ contains exactly one critical point $p_i \in X$ of f. Any point $p_i \in \mathrm{Crit}\, f$ is called a *Lefschetz singularity*, and if g is the genus of a regular fiber of f, then $f\colon X \to \Sigma$ is called a *genus-g Lefschetz fibration*. Each critical point p_i locally arises from shrinking a simple loop c_i on F, called the *vanishing cycle*. A singular fiber of a Lefschetz fibration is called *reducible* (resp. *irreducible*) if c_i is separating (resp. nonseparating). In particular, if c_i is null-homotopic in F, it gives rise to a (-1)–sphere contained in the singular fiber, which can be blown down preserving the rest of the fibration. We will always work with *relatively minimal* Lefschetz fibrations, which do not contain any (-1)–spheres in the fibers, and our focus will be on *nontrivial* Lefschetz fibrations, which are assumed to have nonempty critical locus.

Given any fibration with only Lefschetz critical points, after a small perturbation one can always guarantee that there is at most one critical point on each fiber, as we built into our definition above. It shall be clear that f restricts to a genus-g surface bundle over $\Sigma \setminus C$. Lastly, an *achiral Lefschetz fibration* is defined similarly as above except that the local coordinate (U, φ) is allowed to be incompatible with the orientation of X.

Lefschetz fibrations arise naturally from pencils, where the domain 4–manifold is closed and the target surface is S^2. A *Lefschetz pencil* on a closed 4–manifold X is a Lefschetz fibration $f\colon X \setminus B \to S^2$, defined on the complement of a *nonempty* discrete set B in X, such that around any point $b_j \in B$, we have that f is locally modeled (again in a manner compatible with orientations) as $(z_1, z_2) \to z_1/z_2$. Blowing up all the points in B, one obtains an honest Lefschetz fibration $\widetilde{f}\colon \widetilde{X} \to S^2$ with $|D|$ distinct (-1)–sphere sections S_j, namely the exceptional spheres of the respective blow-ups. We will often use the short-hand notation (X, f) for a Lefschetz fibration or pencil whenever $\Sigma = S^2$.

Definition 2.1 Let $f\colon X \to \Sigma$ be a Lefschetz fibration and S an embedded surface in X. The surface S is called a *multisection* or *n–section* of f if it satisfies the following conditions:

1. $f|_S$ is an n–fold simple branched covering for some nonnegative integer n;
2. if a branched point $p \in S$ is not in $\mathrm{Crit}\, f$, the induced map $df_p\colon N_pS \to T_{f(p)}\Sigma$ is an orientation-preserving isomorphism, where N_pS is the fiber of the normal bundle of S at p which has the canonical orientation induced by that of X and S;

(3) if a branched point $p \in S$ of $f|_S$ is in $\mathrm{Crit}(f)$, then there are complex coordinates (U,φ) and (V,ψ) as in the definition of a Lefschetz fibration above such that $\varphi(S \cap U)$ is equal to $\{(z,z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$.

Clearly a 1–section is an honest section of a Lefschetz fibration. Note that in both definitions we have given above, there is a *positivity* imposed by requiring the compatibility with orientations in local complex models. In the language of [11] a multisection which is branched away from Lefschetz singularities is called a *standard surface*. As will become clear later, allowing our multisections to be branched at Lefschetz critical points as well (although subject to the local model given above), we will have a more flexible setting which makes is possible to deal with larger families of examples of Lefschetz fibrations with multisections of geometric significance. Lastly, as in the case of achiral Lefschetz fibrations, one can possibly work more generally with multisections that are not necessarily positive by allowing the local models to be incompatible with the orientations.

2B Mapping class groups

As it will become crucial in capturing the local topology of multisections (namely the self-intersections of them in the ambient 4–manifold), we are going to set up mapping class groups relevant to our purposes in a framed fashion.

Let Σ be a compact, oriented and connected surface. In this paper, we regard Σ as the zero-section of the tangent bundle $T\Sigma$. Take subsets $U_i, P \subset T\Sigma$. We define a group $\mathrm{Mod}_P(\Sigma; U_1, \dots, U_n)$ as follows:

$$\mathrm{Mod}_P(\Sigma; U_1, \dots, U_n) = \pi_0(\mathrm{Diff}^+_P(\Sigma; U_1, \dots, U_n)),$$

where

$$\mathrm{Diff}^+_P(\Sigma; U_1, \dots, U_n) = \{T \in \mathrm{Diff}^+(\Sigma) \mid dT|_P = \mathrm{id}|_P,\ dT(U_i) = U_i \text{ for all } i\}.$$

Here we denote by $\mathrm{Diff}^+(\Sigma)$ the group of orientation-preserving self-diffeomorphisms of Σ. For simplicity, when P or $U_1, \dots, U_n$ is the empty set, we drop it from the notation. For example, $\mathrm{Diff}^+(\Sigma; U_1, \dots, U_n) = \mathrm{Diff}^+_\varnothing(\Sigma; U_1, \dots, U_n)$, $\mathrm{Mod}_P(\Sigma) = \mathrm{Mod}_P(\Sigma; \varnothing)$, and so on. Note that $\mathrm{Mod}(\Sigma) = \mathrm{Mod}_\varnothing(\Sigma; \varnothing)$ is the "standard" mapping class group of Σ, which consists of isotopy classes of orientation-preserving self-diffeomorphisms of Σ.

The group structures on all of the above are defined via compositions as maps; ie for $T_1, T_2 \in \mathrm{Diff}^+_P(\Sigma; U_1, \dots, U_n)$, we have $T_1 \cdot T_2 = T_1 \circ T_2$, etc.

2C Monodromy factorizations

Let $h\colon X \to D^2$ be a genus-g Lefschetz fibration and $C = \{p_1, \dots, p_l\} \subset D^2$ the set of critical values of h. We take a regular value $q_0 \in \mathrm{Int}(D^2)$ and an identification $\Sigma_g \cong h^{-1}(q_0)$. For each i we also take a path γ_i in $\mathrm{Int}(D^2)$ connecting q_0 with q_i so that all γ_i are pairwise disjoint except at q_0. We give indices of these paths so that $\gamma_1, \dots, \gamma_l$ appear in this order when we travel around q_0 counterclockwise. Let $a_i\colon S^1 \to D^2 \setminus C$ be a loop obtained by connecting a small circle around q_i oriented counterclockwise using γ_i. The pullback $a_i^* h$ is a Σ_g–bundle over S^1 and we can obtain a self-diffeomorphism by taking a parallel transport of a flow in the total space of $a_i^* h$ transverse to each fiber.

Although a diffeomorphism depends on a choice of a flow, its isotopy class is uniquely determined from the Σ_g–bundle structure. The isotopy class is called a *monodromy* of the bundle $a_i^* h$. Kas [28] proved that the monodromy of $a_i^* h$ is the right-handed Dehn twist along some simple closed curve $c_i \subset \Sigma_g$, which is called a *vanishing cycle* of the Lefschetz singularity p_i in $h^{-1}(q_i)$. Let a be a loop obtained by connecting $a_1, \dots, a_l$ in this order. It is easy to verify that a is homotopic to the boundary ∂D^2 in $D^2 \setminus C$. The product $t_{c_l} \cdots t_{c_1}$ is the monodromy of the bundle $a^* h$, where t_{c_i} denotes a positive (right-handed) Dehn twist along the loop c_i. For a genus-g Lefschetz fibration $f\colon X \to S^2$ over S^2, we take a disk $D \subset S^2$ so that D contains all the critical values of f. The restriction $f|_{f^{-1}(D)}$ is a Lefschetz fibration over the disk. Since the monodromy of $f|_{f^{-1}(\partial D)}$ is trivial, we can obtain the following factorization of the unit element of the mapping class group $\mathrm{Mod}(\Sigma_g)$:

$$t_{c_l} \cdots t_{c_1} = 1,$$

where $c_i \subset \Sigma_g$ is a vanishing cycle of a Lefschetz singularity of f. We call this factorization a *monodromy factorization* associated with f.

In the case of a Lefschetz pencil, recall that blowing up each base point b_j yields a (-1)–sphere section S_j. The section S_j provides a lift of the monodromy representation $\pi_1(S^2 \setminus f(C)) \to \mathrm{Mod}(\Sigma_g)$ to the mapping class group $\mathrm{Mod}_{\{x_j\}}(\Sigma_g)$, where x_j is a marked point on Σ_g. One can then fix a disk neighborhood of this section preserved under the monodromy, and get a lift of the factorization to $\mathrm{Mod}_{\partial \Sigma_g^1}(\Sigma_g^1)$, which equals a power of the boundary parallel Dehn twist. Doing this for each b_j we get a defining word

$$t_{c_l} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_m}$$

in $\mathrm{Mod}_{\partial \Sigma_g^m}(\Sigma_g^m)$, where $m = |B|$, the number of base points, and δ_j are boundary parallel along distinct boundary components of Σ_g^m. The powers of the t_{δ_j} are determined by the self-intersection number -1 of the corresponding exceptional section.

2D Symplectic 4–manifolds and Kodaira dimension

By the ground-breaking work of Donaldson every symplectic 4–manifold (X, ω) admits a *symplectic* Lefschetz pencil whose fibers are symplectic with respect to ω [10]. Conversely, building on a construction of Thurston, Gompf showed that the total space of a Lefschetz fibration with a homologically essential fiber, and in particular the blow-up of any pencil, always admits a *compatible* symplectic form ω, for which the fibers are symplectic. This holds whenever the fiber genus is at least 2, or there are critical points. In this case ω can be chosen so that not only the fibers but also any chosen collection of disjoint sections are symplectic, and moreover, any such two symplectic forms are deformation equivalent [25]. We will use the notation (X, ω, f) to indicate that f is a symplectic Lefschetz pencil/fibration with respect to ω, where any explicitly discussed sections of f will always be assumed to be symplectic with respect to it.

The Kodaira dimension for projective surfaces can be extended to symplectic 4–manifolds. Recall that a symplectic 4–manifold (X, ω) is called *minimal* if it does not contain any embedded symplectic sphere of square -1, and that it can always be blown down to a minimal symplectic 4–manifold $(X_{\min}, \omega')$. Let $K_{X_{\min}}$ be the canonical class of $(X_{\min}, \omega_{\min})$. We can now define the *symplectic Kodaira dimension* of (X, ω), denoted by $\kappa = \kappa(X, \omega)$, as

$$\kappa(X,\omega) = \begin{cases} -\infty & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] < 0 \text{ or } K^2_{X_{\min}} < 0, \\ 0 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] = K^2_{X_{\min}} = 0, \\ 1 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] > 0 \text{ and } K^2_{X_{\min}} = 0, \\ 2 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] > 0 \text{ and } K^2_{X_{\min}} > 0. \end{cases}$$

Importantly, κ is independent of the minimal model $(X_{\min}, \omega_{\min})$ and is a smooth invariant of the 4–manifold X [34].

3 Multisections via mapping class groups

In this section we explain how to capture multisections of Lefschetz fibrations and self-intersections of them in terms of mapping class groups.

3A A preliminary lemma

Let Σ be an oriented surface, $\eta \colon \Sigma \to \Sigma$ an orientation-preserving involution, $S \subset \Sigma$ a union of components of $\partial\Sigma$ and $T \subset \Sigma \setminus V$ a finite set. Suppose that η preserves the sets S and T setwise. We denote the fixed points set of η by $V \subset \Sigma$. We define a subgroup $C_S(\Sigma, T; \eta)$ of $\mathrm{Diff}_S^+(\Sigma; V, T)$ as follows:

$$C_S(\Sigma, T; \eta) = \{\varphi \in \mathrm{Diff}_S^+(\Sigma; V, T) \mid \varphi \circ \eta = \eta \circ \varphi\}.$$

We denote the sets $C_S(\Sigma, \varnothing; \eta)$ and $C_\varnothing(\Sigma, T; \eta)$ by $C_S(\Sigma; \eta)$ and $C(\Sigma, T; \eta)$, respectively. The following lemma will be of key use to us for producing several mapping class relations as well as for proving the main theorem:

Lemma 3.1 *The kernel of the natural map*

$$\eta_*\colon \pi_0(C_S(\Sigma, T; \eta)) \to \mathrm{Mod}_{S/\eta}(\Sigma/\eta; V/\eta, T/\eta),$$

induced by the quotient map $/\eta\colon \Sigma \to \Sigma/\eta$*, is generated by the class* $[\eta]$ *if* η *preserves* S *pointwise, and it is trivial otherwise.*

Proof We first prove the statement under the assumption that η does not have fixed points. Let $[\varphi]$ be a mapping class in $\mathrm{Ker}(\eta_*)$. We denote the self-diffeomorphism of Σ/η induced by φ by $\overline{\varphi}$. There exists an isotopy $H_t\colon \Sigma/\eta \to \Sigma/\eta$ such that $H_0 = \overline{\varphi}$, $H_1 = \mathrm{id}_{\Sigma/\eta}$ and $H_t \in \mathrm{Diff}^+_{S/\eta}(\Sigma/\eta; V/\eta, T/\eta)$ for any t. Since $/\eta\colon \Sigma \to \Sigma/\eta$ is an unbranched covering, there exists a lift $\widetilde{H}_t\colon \Sigma \to \Sigma$ of H_t under $/\eta$ such that $\widetilde{H}_0$ is equal to φ. The map $\widetilde{H}_t$ preserves the set T since $H_t(T/\eta) = T/\eta$. By the uniqueness of a lift under a covering map, the restriction $\widetilde{H}_t|_S$ is the identity map for any t. It is easy to verify that the composition $/\eta \circ \widetilde{H}_t \circ \eta$ is equal to $/\eta \circ \eta \circ \widetilde{H}_t$. The composition $\widetilde{H}_t \circ \eta$ is equal to $\eta \circ \widetilde{H}_t$ since $\widetilde{H}_0 = \varphi$ and φ commutes with η. Thus the map $t \mapsto \widetilde{H}_t$ gives a path in $C_S(\Sigma, T; \eta)$. The map $\widetilde{H}_1$ is equal to either id_Σ or η since $/\eta \circ \widetilde{H}_1$ is equal to $\mathrm{id}_{\Sigma/\eta}$. Hence $[\varphi] \in \pi_0(C_S(\Sigma, T; \eta))$ is represented by either the identity map or η. Since the isotopy class of η is contained in $\pi_0(C_S(\Sigma, T; \eta))$ if and only if η preserves S pointwise, the kernel of η_* is generated by $[\eta]$ if $\eta|_S = \mathrm{id}|_S$ and is trivial otherwise.

We next consider an involution η with fixed points. Let $\nu V \subset \Sigma$ be a neighborhood of V consisting of a disjoint union of disks. The restriction of η on a component of $\partial \nu V$ is the 180–degree rotation (with respect to some local coordinates). For each component of $\partial \nu V$ we take two points in it which are preserved by η setwise, and denote the set of these points by $U \subset \partial \nu V$. The following diagram commutes:

$$\begin{array}{ccc}
\pi_0\big(C_S\big((\Sigma \setminus \mathrm{Int}(\nu V)), T \cup U; \eta\big)\big) & \xrightarrow{\eta_*} & \mathrm{Mod}_{S/\eta}\big((\Sigma \setminus \mathrm{Int}(\nu V))/\eta; (T \cup U)/\eta\big) \\
{\scriptstyle c_1}\big\downarrow & & \big\downarrow{\scriptstyle c_2} \\
\pi_0(C_S(\Sigma, T; \eta)) & \xrightarrow{\eta_*} & \mathrm{Mod}_{S/\eta}(\Sigma/\eta; V/\eta, T/\eta)
\end{array}$$

where the vertical maps are the capping maps. The map C_2 is surjective and the kernel of C_2 is generated by the Dehn twists along simple closed curves parallel to components of $\partial(\nu V/\eta)$. These Dehn twists are the images of the square roots of the Dehn twists along components of $\partial \nu V$, which are contained in $\pi_0(C_S(\Sigma \setminus \mathrm{Int}(\nu V), T \cup U; \eta))$ and interchange two points in U. It is easy to verify that the map C_1 is surjective. Furthermore any square root of a Dehn twist along a curve parallel to a component of $\partial \nu V$ is contained in the kernel of C_1. The kernel of the induced map η_* defined on $\pi_0(C_S(\Sigma \setminus \mathrm{Int}(\nu V), T \cup U; \eta))$ is generated by $[\eta]$ if $\eta|_S = \mathrm{id}|_S$ and is trivial otherwise since the quotient map $/\eta$ on $\Sigma \setminus \mathrm{Int}(\nu V)$ is an unbranched covering. Thus the kernel of the map

$$\eta_*\colon \pi_0(C_S(\Sigma, T; \eta)) \to \mathrm{Mod}_{S/\eta}(\Sigma/\eta; V/\eta, T/\eta)$$

is generated by $[\eta]$ if $\eta|_S = \mathrm{id}|_S$ and is trivial otherwise. This completes the proof of Lemma 3.1. □

3B Local model for the fibration around a regular branched point

Let $f_0\colon \mathbb{C}^2 \to \mathbb{C}$ be the projection onto the first component. We take a subset $S_0 \subset \mathbb{C}^2$ as follows:

$$S_0 = \{(z^2, z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$

The restriction $f_0|_{S_0}$ is a double branched covering branched at the origin.

Lemma 3.2 *Let $f\colon X \to \Sigma$ be a Lefschetz fibration, $S \subset X$ a multisection of f and $p \in S \setminus \mathrm{Crit}(f)$ a branched point of $f|_S$. Then there exist a local coordinate $\Phi\colon U \to \mathbb{C}^2$ of p and a local coordinate $\varphi\colon V \to \mathbb{C}$ of $q = f(p)$ which make the following diagram commute:*

$$\begin{array}{ccc} (U, U \cap S) & \xrightarrow{\Phi} & (\mathbb{C}^2, S_0) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ V & \xrightarrow{\varphi} & \mathbb{C} \end{array}$$

That is, $f|_S$ conforms to a local model of a branched covering map with a simple branched point at p.

Proof Since p is not a critical point of f, there exist local coordinates $\Phi_0\colon U \to \mathbb{C}^2$ and $\varphi_0\colon V \to \mathbb{C}$ of p and $f(p)$, respectively, such that p is mapped to the origin of $\mathbb{C}^2$, and that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\Phi_0"] \arrow[d, "f"'] & \mathbb{C}^2 \arrow[d, "f_0"] \\ V \arrow[r, "\varphi_0"] & \mathbb{C} \end{tikzcd}$$

Without loss of generality, we can assume that the neighborhood U does not contain any branched points of $f|_S$ except p. Then, the intersection $U \cap S$ is diffeomorphic to $\mathbb{C}$, and $\Phi_0(S)$ is described as follows:

$$\Phi_0(S) = \{(s_1(z), s_2(z)) \in \mathbb{C}^2 \mid z \in \mathbb{C}\},$$

where $s_i\colon \mathbb{C} \to \mathbb{C}$ is a smooth function ($i = 1, 2$).

Since p is a branched point of $f|_S$, the map s_1 is a double branched covering branched at the origin. Thus, there exist diffeomorphisms $\widetilde{\varphi_1}\colon \mathbb{C} \to \mathbb{C}$ and $\varphi_1\colon \mathbb{C} \to \mathbb{C}$ which make the following diagram commute:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\widetilde{\varphi_1}"] \arrow[d, "s_1"'] & \mathbb{C} \arrow[d, "(\cdot)^2"] \\ \mathbb{C} \arrow[r, "\varphi_1"] & \mathbb{C} \end{tikzcd}$$

Now, as S is an embedded surface in X, we can assume that the map $z \mapsto (s_1(z), s_2(z))$ is an embedding. In particular, s_2 is locally diffeomorphic at the origin of $\mathbb{C}$. Thus, by replacing the local coordinates with sufficiently small ones if necessary, we can take diffeomorphisms $\widetilde{\varphi_2}\colon \mathbb{C} \to \mathbb{C}$ and $\varphi_2\colon \mathbb{C} \to \mathbb{C}$ which make the following diagram commute:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\widetilde{\varphi_2}"] \arrow[d, "s_2"'] & \mathbb{C} \arrow[d, "\mathrm{id}"] \\ \mathbb{C} \arrow[r, "\varphi_2"] & \mathbb{C} \end{tikzcd}$$

We put $\Phi_1 = \varphi_1 \times \mathrm{id}$ and $\Phi_2 = \mathrm{id} \times \varphi_2$. Now, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\Phi_0"] \arrow[d, "f"'] & \mathbb{C}^2 \arrow[r, "\Phi_1"] \arrow[d, "f_0"'] & \mathbb{C}^2 \arrow[r, "\Phi_2"] \arrow[d, "f_0"'] & \mathbb{C}^2 \arrow[d, "f_0"'] \\ V \arrow[r, "\varphi_0"] & \mathbb{C} \arrow[r, "\varphi_1"] & \mathbb{C} \arrow[r, "\mathrm{id}"] & \mathbb{C} \end{tikzcd}$$

The following equality can be checked easily:

$$\Phi_2 \circ \Phi_1 \circ \Phi_0(S \cap U) = \{(z^2, \widetilde{\varphi_2} \circ \widetilde{\varphi_1}^{-1}(z)) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$

Thus, the diffeomorphisms $\Phi = (\mathrm{id} \times \widetilde{\varphi_1} \circ \widetilde{\varphi_2}^{-1}) \circ \Phi_2 \circ \Phi_1 \circ \Phi_0$ and $\varphi = \varphi_1 \circ \varphi_0$ satisfy the desired condition. This completes the proof of Lemma 3.2. $\square$

Hence we can always make a local coordinate φ in Lemma 3.2 compatible with the orientation of Σ. The branched point $p \in S \setminus \mathrm{Crit}(f)$ of $f|_S$ is positive if and only if a local coordinate Φ of p obtained in Lemma 3.2 is compatible with the orientation of X after making φ compatible with the orientation of Σ.

3C Standard monodromy factorization around a regular branched point

We are now going to study the monodromy factorization around a branched point of a multisection, which will play a key role in the proof of Theorem 3.6 below.

We denote by Σ_g^n an oriented, connected and compact surface of genus g with n boundary components. Let $S_0 \subset \mathbb{C}^2$ be a standard model of a branched point away from Lefschetz singularities as explained in the previous subsection. We denote the subset $\{z \in \mathbb{C} \mid |z| \le k\}$ by B_k. We consider the restriction

$$q = f_0|_{B_1 \times B_2} \colon B_1 \times B_2 \to B_1.$$

The subset $S_0 \cap (B_1 \times B_2)$ is a bisection of q. This bisection, together with an identification $B_2 \cong \Sigma_0^1$, makes the monodromy ϱ_0 of $q|_{q^{-1}(\partial B_1)}$ be contained in the group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{s_1, s_2\})$ where s_1 and s_2 are two points in $q^{-1}(1) \cap S_0$. It is known that this monodromy is equal to the positive half twist along an arc between s_1 and s_2. Let $\varepsilon \in \mathbb{R}$ be a sufficiently small real number and we put $e = 1 - \varepsilon$. We take another subset $S_0' \subset \mathbb{C}^2$ as follows:

$$S_0' = \{(z^2, ez) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$

The subset $S_0' \cap (B_1 \times B_2)$ is also a bisection of q. By using the bisections, we can take a lift $\widetilde{\varrho_0} \in \mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^3; \{u_1, u_2\})$ of the monodromy ϱ_0, where u_1 and u_2 are points in $\partial\Sigma_0^3 \setminus \partial\Sigma_0^1$ which cover the set $\pi_0(\partial\Sigma_0^3 \setminus \partial\Sigma_0^1)$. Note that the group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^3; \{u_1, u_2\})$ is isomorphic to the group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{v_1, v_2\})$, where v_i is a nonzero tangent vector in $T_{s_i}\Sigma_0^1$.

Lemma 3.3 *The mapping class $\widetilde{\varrho_0}$ is represented by the map described in Figure 1.*

Proof The element $\widetilde{\varrho_0}$ is a lift of ϱ_0. Thus the bold arc in Figure 1 should be sent by a representative of $\widetilde{\varrho_0}$ (up to isotopy) as described in the figure. It is sufficient to prove an arc connecting s_1 and s_2 is preserved by some representative of $\widetilde{\varrho_0}$ since the group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^3; \{u_1, u_2\})$ is isomorphic to the group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{v_1, v_2\})$. We denote the arc $\{(1, 1-2t) \in \mathbb{C}^2 \mid t \in [0,1]\}$ by $\gamma \subset q^{-1}(1)$. This arc connects the two

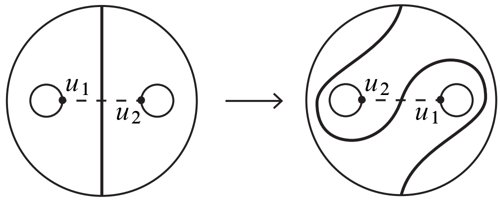

Figure 1: The element $\widetilde{\varrho_0}$ interchanges the points u_1 and u_2, and keeps the dotted arc α between u_1 and u_2.

points in $S_0 \cap q^{-1}(1)$. We take a horizontal distribution $\mathcal{P}$ of $q|_{q^{-1}(\nu\partial B_1)}$ so that it coincides with the following distribution on $\partial B_1 \times B_{3/2}$:

$$\left\langle \frac{\partial}{\partial x_1} + \tfrac{1}{2}x_3\frac{\partial}{\partial x_3} - \tfrac{1}{2}x_4\frac{\partial}{\partial x_4},\ \frac{\partial}{\partial x_2} + \tfrac{1}{2}x_4\frac{\partial}{\partial x_3} + \tfrac{1}{2}x_3\frac{\partial}{\partial x_4} \right\rangle,$$

where (x_1, x_2, x_3, x_4) is a real coordinate determined by the formula

$$(z_1, z_2) = (x_1 + \sqrt{-1}x_2, x_3 + \sqrt{-1}x_4).$$

We define a loop $c\colon [0, 2\pi] \to \partial B_1$ as follows:

$$c(t) = \exp(\sqrt{-1}t).$$

Take a point $t_0 \in [-1, 1]$. It is easy to see that the horizontal lift $\widetilde{c}_{t_0}(t)$ with base point $w = (1, 0, t_0, 0) \in q^{-1}(1)$ is given by

$$\widetilde{c}_{t_0}(t) = \bigl(\cos t, \sin t, t_0 \cos \tfrac{1}{2}t, t_0 \sin \tfrac{1}{2}t\bigr).$$

Thus, the arc γ is preserved by the parallel transport along the curve c with respect to $\mathcal{P}$. Since this parallel transport is a representative of $\widetilde{\varrho_0}$, this completes the proof of Lemma 3.3. □

The two bisections S_0 and S_0' intersect only at the origin, but do not intersect transversely. In order to make the two bisections intersect transversely, we will take a small perturbation of S_0'. We first take a smooth function $\rho\colon \mathbb{R} \to [0, \varepsilon]$ satisfying the following conditions:

(a) $\rho(t) = \rho(-t)$;

(b) $\rho(t) = \varepsilon^2$ for all $t \in [0, \varepsilon/2]$;

(c) $\rho(t) = 0$ for all $t \in [\varepsilon, \infty)$;

(d) $-3\varepsilon < \dfrac{d\rho}{dt}(t) < 0$ for all $t \in [\varepsilon/2, \varepsilon]$.

We define the subset $S'_{0,\rho} \subset \mathbb{C}^2$ as follows:

$$S'_{0,\rho} = \{(z^2, ez + \rho(|z|^2)) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$

The two subsets S_0 and $S'_{0,\rho}$ intersect at $(r_1^2, r_1), (r_2^2, r_2) \in \mathbb{C}^2$, where $r_1, r_2 \in \mathbb{R}$ are the real numbers which satisfy the following conditions:

$$r_1 = \frac{\rho(r_1^2)}{\varepsilon}, \quad r_2 = \frac{\rho(r_2^2)}{2-\varepsilon}.$$

We can see that S_0 intersects $S'_{0,\rho}$ at both points transversely and positively with respect to the standard orientation of $\mathbb{C}^2$.

3D Multisections branching at Lefschetz critical points

We will now study the local model around a branched point of a multisection coinciding with a Lefschetz critical point of the fibration. Such branched points appear *exclusively* in Loi and Piergallini's description of compact Stein surfaces, up to diffeomorphisms, as total spaces of *allowable Lefschetz fibrations* over the 2–disk with bounded fibers, arising as the branched cover of the projection $D^2 \times D^2 \to D^2$ branched along a positive multisection [38] (also see [2]). Such a multisection, along with the fiber, carries the entire information one needs to describe the diffeomorphism type of any compact Stein surface.

We take two points $s_1, s_2 \in \mathrm{Int}(\Sigma_0^2)$. We denote an involution with fixed point set $\{s_1, s_2\}$ by $\iota\colon \Sigma_0^2 \to \Sigma_0^2$. The quotient space Σ_0^2/ι is diffeomorphic to the disk Σ_0^1. Denote the images of s_1 and s_2 under the quotient map $\Sigma_0^2 \to \Sigma_0^2/\iota \cong \Sigma_0^1$ by s'_1 and s'_2, respectively. The group $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{s'_1, s'_2\})$ is an infinite cyclic group. By Lemma 3.1, the following natural map induced by the quotient map is injective:

$$\pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota)) \to \mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{s'_1, s'_2\}).$$

The inclusion map $C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_0^2}(\Sigma_0^2)$ induces the homomorphism

$$i\colon \pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota)) \to \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2) \cong \mathbb{Z}.$$

Since this map is surjective, the group $\pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota))$ is also an infinite cyclic group and the map i is an isomorphism. On the other hand, the inclusion map $C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_0^2}(\Sigma_0^2; \{s_1, s_2\})$ also induces a homomorphism

$$i_*\colon \pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2; \iota)) \to \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2; \{s_1, s_2\}).$$

We denote the forgetful map by

$$F_{s_1,s_2}\colon \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2; \{s_1, s_2\}) \to \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2).$$

Since the composition $F_{s_1,s_2} \circ i_*$ is equal to i and i is isomorphic, the map i_* is injective. Thus, we can regard the group $\pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2;\iota)) \cong \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2)$ as a subgroup of $\mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$. Under this identification, the Dehn twist $t_c \in \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2)$ along the curve parallel to $\partial\Sigma_0^2$, which is the generator of this group, is regarded as an element in $\mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$ described in Figure 2.

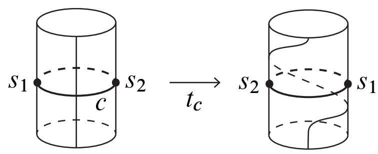

Figure 2: The element t_c interchanges s_1 and s_2.

We denote by $Y \subset \mathbb{C}^2$ the intersection $B_2 \times B_2 \cap f^{-1}(B_1)$, where B_k is the disk $\{z \in \mathbb{C} \mid |z| \leq k\}$ and $f\colon \mathbb{C}^2 \to \mathbb{C}$ is the standard local model of a Lefschetz singularity, that is, f is defined as $f(z_1,z_2) = z_1 z_2$. Let f_0 be the restriction $f|_Y$. Take the standard bisection $\Delta_0 = \{(z,z) \in Y \mid z \in B_1\}$ of f_0. We define the involution $\eta\colon \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

$$\eta(z_1,z_2) = (z_2,z_1).$$

The fixed point set of η is equal to Δ_0. The regular fiber $f_0^{-1}(1)$ is the annulus Σ_0^2. We take an identification $f_0^{-1}(1) \cong \Sigma_0^2$ so that the restriction $\eta|_{f_0^{-1}(1)}$ equals to the involution ι. By taking a horizontal distribution $\mathcal{P}$ of the fibration $f_0|_{Y\setminus\{0\}}$ which is along both Δ_0 and ∂Y, we can regard the monodromy ϱ_0 of ∂B_1 as an element of the group $\mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$, where $\{s_1,s_2\}$ is the intersection $\Delta_0 \cap f_0^{-1}(1)$.

Lemma 3.4 *Under the identification of $f_0^{-1}(1)$ with Σ_0^2 as above, the monodromy ϱ_0 is equal to the Dehn twist $t_c \in \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$.*

Proof We take a horizontal distribution $\mathcal{P}$ so that $\mathcal{P}$ is preserved by η. The monodromy ϱ_0 is contained in the group $\pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^2,\iota)) \subset \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$. Furthermore, using the result in [28], it is easy to see that this monodromy is sent to the Dehn twist $t_c \in \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2)$ by F_{s_1,s_2}. □

We take a disk neighborhood $D_i \subset \Sigma_0^2$ of the point s_i which is preserved by ι. We put $D = D_1 \sqcup D_2$ and fix an identification $\Sigma_0^2 \setminus D \cong \Sigma_0^4$. We also take points $u_i, u_i' \in \partial D_i$ so that $\iota(u_i) = u_i'$. We can define the homomorphism

$$\mathrm{Cap}\colon \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^4;\{u_1,u_2\}) \to \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2;\{s_1,s_2\})$$

by capping Σ_0^4 by D.

We take a sufficiently small number $\varepsilon > 0$ and put $\xi = \exp(\sqrt{-1}\varepsilon)$. We define another bisection Δ_0' of f_0 as follows:

$$\Delta_0' = \{(\xi z, \xi^{-1} z) \in Y \mid z \in B_1\}.$$

Note that Δ_0' intersects Δ_0 at the origin transversely. This bisection, together with the bisection Δ_0, gives a lift $\widetilde{\varrho_0} \in \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^4; \{u_1, u_2\})$ of the monodromy ϱ_0 under the map Cap.

Lemma 3.5 *Under a suitable identification $\Sigma_0^4 \cong f_0^{-1}(1) \setminus \nu\Delta_0$, the monodromy $\widetilde{\varrho_0}$ is represented by the map described in Figure 3.*

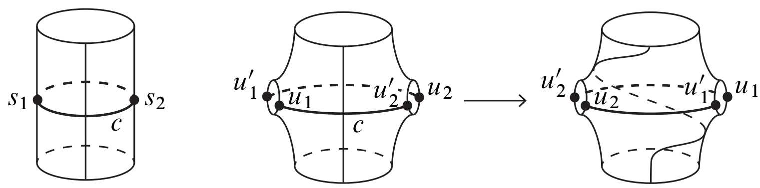

Figure 3: The element $\widetilde{\varrho_0}$ interchanges the points u_1 and u_2.

Proof The map described in Figure 3 is contained in $C_{\partial\Sigma_0^2}(\Sigma_0^4, \{u_1, u_1', u_2, u_2'\}; \iota)$. By the same argument as in the proof of Lemma 3.4, we can assume that the element $\widetilde{\varrho_0}$ is contained in the group $\pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^4, \{u_1, u_1', u_2, u_2'\}; \iota))$. It is easy to see that the following map is a diffeomorphism:

$$\mathbb{C}^2/\eta \to \mathbb{C}^2, \quad [(z_1, z_2)] \mapsto \left(z_1 z_2, \frac{z_1 + z_2}{2}\right).$$

We identify these spaces via this diffeomorphism. The following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}^2, \Delta_0, \Delta_0') & \xrightarrow{/\eta} & (\mathbb{C}^2, S_0, S_0') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow{\mathrm{id}} & \mathbb{C} \end{array}$$

where S_0 and S_0' are the subsets of $\mathbb{C}^2$ defined in the previous section (in this case, e is equal to $\mathrm{Re}(\xi)$). Thus the monodromy $\widetilde{\varrho_0}$ is mapped to the mapping class described in Figure 1 by the map

$$\eta_*\colon \pi_0(C_{\partial\Sigma_0^2}(\Sigma_0^4, \{u_1, u_1', u_2, u_2'\}; \iota)) \to \mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^3; \{u_1, u_2\})$$

induced by $/\eta$. On the other hand, we can see that the mapping class described in Figure 3 is also mapped to that described in Figure 1 by η_*. Since η does not preserve $\partial\Sigma_0^2$ pointwise, the homomorphism η_* is injective by Lemma 3.1. Thus the mapping class in Figure 3 coincides with $\widetilde{\varrho_0}$. $\square$

3E Capturing multisections via mapping class group factorizations

With all the preliminary results we have obtained in the previous subsections, we are now ready to prove the main theorem of this section (giving Theorem 1.1):

Theorem 3.6 *Let $f\colon X \to S^2$ be a genus-g Lefschetz fibration with monodromy factorization*

$$t_{c_l} \cdots t_{c_1} = 1.$$

Let $S \subset X$ be a genus-g surface with self-intersection m, which is an n–section of f with k branched points away from $\mathrm{Crit}(f)$, and r branched points at Lefschetz singularities corresponding to cycles $c_1, \dots, c_r$. Then there exists a lift $\widetilde{c_i} \subset \Sigma_g^n$ of c_i such that the following holds in $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$:

$$\widetilde{\tau}_{\alpha_k} \cdots \widetilde{\tau}_{\alpha_1} \cdot t_{\widetilde{c_l}} \cdots t_{\widetilde{c_{r+1}}} \cdot \widetilde{t_{c_r}} \cdots \widetilde{t_{c_1}} = t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}, \tag{1}$$

where $\{u_1, \dots, u_n\}$ is a subset of $\partial \Sigma_g^n$ which covers all the elements of $\pi_0(\partial \Sigma_g^n)$, $\widetilde{\tau}_{\alpha_i}$ is a lift of the half twist along a simple arc α_i between two points in $\{u_1, \dots, u_n\}$ as described in Figure 1, $\widetilde{t_{c_i}}$ is a lift of the Dehn twist t_{c_i} as described in Figure 3, and $\{\delta_1, \dots, \delta_n\}$ is a set of simple closed curves parallel to $\partial \Sigma_g^n$. Here the arcs for $\widetilde{\tau}_{\alpha_1}, \dots, \widetilde{\tau}_{\alpha_k}$ and the Dehn twist curves for $\widetilde{t_{c_1}}, \dots, \widetilde{t_{c_r}}$ should contain all $u_1, \dots, u_n$, and the integral equalities

$$g(S) = \tfrac{1}{2}(k + r) - n + 1 \quad \text{and} \quad m = -\left(\sum_{i=1}^{n} a_i\right) + 2k + r$$

should hold. The connectivity of S implies that the collection of $\widetilde{\tau}_{\alpha_i}$ and $\widetilde{t_{c_i}}$ act transitively on the collection of u_j. The connectivity of S implies that every u_j is acted on nontrivially by some $\widetilde{\tau}_{\alpha_i}$ or $\widetilde{t_{c_i}}$.

Conversely, for any relation in $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$ of the form (1) and satisfying the conditions above, there exists a genus-g Lefschetz fibration $f\colon X \to S^2$ with a connected n–section $S \subset X$ of genus $\frac{1}{2}(k + r) - n + 1$ and self-intersection $-\left(\sum_{i=1}^{n} a_i\right) + 2k + r$, whose monodromy factorization is given by the image of the factorization on the left hand side of (1) under $i_*\colon \mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\}) \to \mathrm{Mod}(\Sigma_g)$ which is induced by the inclusion $i\colon \Sigma_g^n \hookrightarrow \Sigma_g$.

Note that after relabeling the arcs we choose for the monodromy description of f, we can always assume that the first r cycles are the ones corresponding to those at which S is branched.

The reader might find it illuminating to look at an example before we move on to proving our theorem:

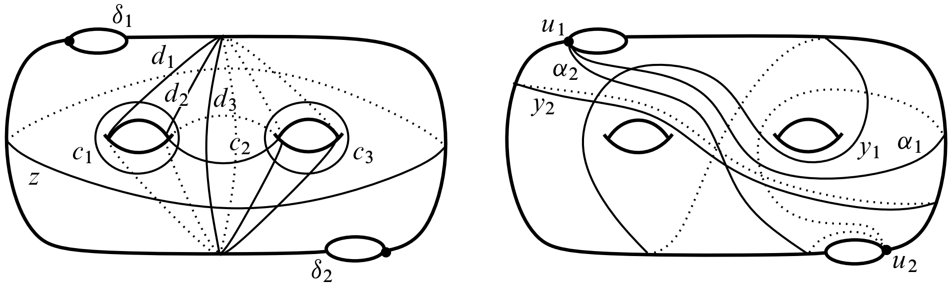

Figure 4: Dehn twist curves $c_1, c_2, c_3, d_1, d_2, d_3, z, y_1, y_2$ and arc twist curves α_1, α_2 in Σ_2^2 with framed boundary

Example 3.7 A monodromy factorization of a genus-2 Lefschetz fibration with a 2–section we will produce in Section 5C is the following:

$$t_{\delta_1}^2 t_{\delta_2}^3 = (t_{d_3} t_{d_2} t_{d_1})^2 t_{y_2} \widetilde{\tau}_{\alpha_2} t_z t_{y_1} \widetilde{\tau}_{\alpha_1} t_{t_{c_1}^{-1} t_{c_3}^{-2}(c_2)} t_{t_{c_3}^{-1}(c_2)} (t_{c_1} t_{c_2} t_{c_3})^2,$$

in $\mathrm{Mod}(\Sigma_2^2; \{u_1, u_2\})$, where the Dehn twist curves c_i, d_i, y_j, z, the arcs α_j for the arc twists, the two boundary components δ_i and the marked points u_i on them are all given in Figure 4. (One can of course conjugate each $\widetilde{\tau}_{\alpha_i}$ all the way to the left to put it into the above "standard form".) Since u_1 and u_2 are connected by τ_{α_i}, this gives a connected 2–section S of genus $(k/2) - n + 1 = 0$ and self-intersection $-\sum_{i=1}^n a_i + 2k = -1$; so it is an exceptional sphere.

Proof of Theorem 3.6 For a given genus-g Lefschetz fibration $f\colon X \to S^2$ with an n–section S and its monodromy factorization $t_{c_l} \cdots t_{c_1} = 1$, let $\gamma_1, \dots, \gamma_l$ be reference paths from a regular value $q_0 \in S^2$ which gives the factorization $t_{c_l} \cdots t_{c_1} = 1$. We take reference paths $\alpha_1, \dots, \alpha_k$ satisfying the following properties:

- α_i connects q_0 with the image of a branched point of S away from $\mathrm{Crit}(f)$;
- $\gamma_1, \dots, \gamma_l, \alpha_1, \dots, \alpha_k$ appear in this order when we go around q_0 counterclockwise.

We take a perturbation S' of S so that the pair (S, S') coincides with either of the pairs (S_0, S_0') or (Δ_0, Δ_0') in a small coordinate neighborhood of each branched point of S. This perturbation gives a lift of monodromies of f to the group $\mathrm{Mod}(\Sigma_g^n; \{u_1, \dots, u_n\})$. By Lemma 3.3, local monodromies obtained from paths α_i are lifts of half twists described in Figure 1. On the other hand, by Lemma 3.5, a local monodromy obtained from a path γ_i ($i \in \{1, \dots, r\}$) is a lift of the Dehn twist t_{c_i} described in Figure 3. Thus we can obtain a factorization in Theorem 3.6.

Using the observation following the proof of Lemma 3.3 and the fact that Δ_0 intersects Δ_0' at the origin transversely, it is easy to verify that this factorization satisfies the condition on the self-intersection number of S.

Conversely, for a given lift of a factorization given in Theorem 3.6, we can prescribe a genus-g Lefschetz fibration $f\colon X \to S^2$ and an n–section S of f with desired conditions by pasting local models given in the present section according to the factorization.

There is a correspondence between a multisection S and that of a graph Γ with vertices corresponding to u_i, and with edges between u_i and $u_{i'}$ corresponding to half twists $\widetilde{\tau}_{\alpha_j}$ or Dehn twists $\widetilde{t_{c_j}}$ in the relation (1) interchanging them. The Euler characteristic of S is then given by $2v - e$, for v the number of vertices and e the number of edges of Γ. Here S is connected if and only if the graph Γ is. In this case, we have $g(S) = \frac{1}{2}(k+r) - n + 1$. □

Per the last paragraph of the proof above, the monodromy factorization in Theorem 3.6 can be generalized to disconnected multisections in a straightforward way — see Section 4A. A sample calculation of a monodromy factorization of a Lefschetz fibration with its multisections is given in the Appendix, and many more examples can be found in Sections 4–6.

Remark 3.8 After a small modification of the proof above we can similarly obtain a monodromy factorization for a *not necessarily positive* multisection, where each negative branched point away from $\mathrm{Crit}(f)$ contributes -2 and each branched point at a negative critical point contributes -1 to the total count of the self-intersection of the multisection.

Remark 3.9 We shall note that, although multisections going through Lefschetz critical points are of particular interest in certain contexts (for instance for allowable Lefschetz fibrations on Stein surfaces), it is in fact always possible to perturb any given multisection of a Lefschetz fibration so as to obtain one which is branched completely away from the Lefschetz critical points. This can be achieved by the following perturbation around each branched point on a Lefschetz singularity:

$$\Delta_\varepsilon = \{(z+\varepsilon, z-\varepsilon) \in \mathbb{C}^2 \mid z \in \mathbb{C}\},$$

where ε is a sufficiently small positive number. In this perturbation, a branched point on a Lefschetz singularity is substituted for a positive branched point. Indeed, we can easily verify the following relation in the group $\mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^4; \{u_1, u_2\})$ (using the Alexander method [15, Proposition 2.8], for example):

$$\widetilde{t_c} = t_\delta^{-1} t_{c'} \tau_\gamma, \tag{2}$$

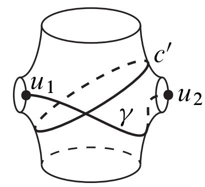

Figure 5: Simple closed curves and paths in Σ_0^4

where δ is a simple closed curve parallel to the boundary component containing u_2, c' is a simple closed curve described in Figure 5 and τ_γ is a lift of a half twist preserving the path γ in Figure 5.

Using Theorem 3.6, we can then make a multisection avoiding Lefschetz singularities by substituting a lift $\widetilde{t_c}$ in a lift of a factorization (1) of the right hand side of (2).

Remark 3.10 (Hurwitz equivalence for Lefschetz fibrations with multisections) It is well-known that for $g \geq 2$, there is a one-to-one correspondence between genus-g Lefschetz fibrations *up to isomorphisms* and monodromy factorizations in $\mathrm{Mod}(\Sigma_g)$ *up to Hurwitz moves and global conjugations*. It is possible to extend this correspondence to our setting, by considering positive factorizations in the framed mapping class group $\mathrm{Mod}(\Sigma_g^n; \{u_1, \ldots, u_n\})$ up to usual Hurwitz moves, global conjugations that are allowed to swap boundary components, and an additional move which compensates for the ambiguity in boundary framings. A detailed study will be given in [8].

4 Lefschetz fibrations on symplectic Calabi–Yau 4–manifolds

Symplectic 4–manifolds of negative Kodaira dimension are classified up to symplectomorphisms, which are precisely the rational and ruled surfaces [34]. The next compelling target has been the symplectic 4–manifolds of Kodaira dimension zero, which are analogues of the Calabi–Yau surfaces that have torsion canonical class [36]. With a slight abuse of language, we will thus call (X, ω) with $\kappa = 0$ a *symplectic Calabi–Yau*, referring to its minimal model having a torsion canonical class. It has been shown by Li, and independently by Bauer [35; 36; 4], that the rational homology type of any *minimal* symplectic Calabi–Yau 4–manifold is that of either a torus bundle over a torus, the K3 surface, or the Enriques surface. In particular, a folklore conjecture states that a symplectic Calabi–Yau with $b_1 = 0$ is diffeomorphic to a (blow-up of) either the Enriques surface or the K3 surface.

With this conjecture in mind, below we will determine the defining properties for a Lefschetz pencil/fibration to be on a (blow-up of) a symplectic Calabi–Yau 4–manifold, essentially relying on Taubes' seminal work [53; 54]. We will then construct two model examples, a genus-3 Lefschetz pencil on a symplectic Calabi–Yau K3 surface, and a genus-2 pencil on a symplectic Calabi–Yau Enriques surface. (That is, these are symplectic Calabi–Yaus *homeomorphic* to K3 and Enriques surfaces, respectively.)

4A Characterizing Lefschetz fibrations on symplectic Calabi–Yaus

Fibers of a symplectic Lefschetz fibration (X,ω,f) are J–holomorphic with respect to any almost complex structure J–compatible with ω. It follows from Taubes' seminal work on the correspondence between Gromov and Seiberg–Witten invariants on symplectic 4–manifolds with $b^+(X)>1$ that exceptional classes e_j in $H_2(X)$ are represented by disjoint J–holomorphic (-1)–spheres S_j [53; 54], and by the work of Li and Liu [37], the same holds even when $b^+(X)=1$, as long as X is not a rational or ruled surface. We then conclude from the positivity of intersections for J–holomorphic curves that each S_j is an n_j–section, intersecting the genus $g\geq 2$ fiber F positively at exactly $n_j=S\cdot F\geq 1$ points. Moreover, in this case, we can use the Seiberg–Witten adjunction inequality to show that $\sum n_j=\bigl(\sum S_j\bigr)\cdot F\leq 2g-2$. Note that this can fail to be true only for rational and ruled surfaces; a Lefschetz pencil on a rational or ruled surface can have more than $2g-2$ base points. We will show that the equality is sharp precisely for Lefschetz fibrations on symplectic Calabi–Yaus.

We can now characterize Lefschetz fibrations on minimal symplectic Calabi–Yau 4–manifolds and their blow-ups, using Sato's work in [46]:

Theorem 4.1 *Let (X,f) be a genus-g Lefschetz fibration with $g\geq 2$, and X be neither rational nor ruled. Then, there exists a symplectic form ω on X compatible with f such that (X,ω) is a (blow-up of) a symplectic Calabi–Yau 4–manifold, if and only if there is a disjoint collection of (-1)–spheres that are n_j–sections of (X,f) such that $\sum_j n_j=2g-2$.*

Note that if X is minimal, then (X,ω,f) can be a symplectic Calabi–Yau only if the fiber genus is 1, by the adjunction formula. We can thus assume that X is not minimal and $g\geq 2$.

Proof In this case, (X,ω) should be an $m\geq 1$ times blow-up of a minimal symplectic Calabi–Yau, so $c_1(X,\omega)$ is Poincaré dual to $\sum_{j=1}^m e_j$, where e_j are the exceptional classes. We have m disjoint n_j–sections S_j, representing the exceptional classes e_j. Then the adjunction formula for the symplectic fiber dictates that

$$\sum_j n_j=\Bigl(\sum S_j\Bigr)\cdot F=K_X\cdot F=2g-2-F^2=2g-2.$$

Conversely, it is shown by Sato [46, Theorem 5.5.] that for a genus $g \geq 2$ Lefschetz fibration on a nonminimal 4–manifold (X, f), where X is not rational or ruled, if the maximal collection of exceptional classes e_j intersect the fiber exactly at $2g-2$ times, then $c_1(X, \omega)$ is Poincaré dual to $\sum e_j$. Although there is an oversight in this observation, which for instance contradicts with the case of Lefschetz fibrations on blow-ups of the Enriques surface (such examples for homotopy Enriques surfaces are given in the later sections), Sato's proof in [46], which is obtained by a thorough analysis of intersections between pseudoholomorphic curves, goes through for a *rational* homology class, ie modulo torsion. With this corrected statement in mind, blowing down all e_j yields a minimal symplectic model for (X, ω) with torsion canonical class. □

We will thus call (X, f) a genus-g *symplectic Calabi–Yau Lefschetz fibration* if X is not rational or ruled, and there is a disjoint collection of (-1)–spheres that are n_j–sections of (X, f) with $\sum_j n_j = 2g-2$. Note that not every symplectic Calabi–Yau Lefschetz fibration is a blow-up of a Lefschetz pencil on a minimal symplectic Calabi–Yau, the examples of which we will provide in Sections 4–6.

Let W be a factorization of the multitwist $t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}$ in $\mathrm{Mod}(\Sigma_g^{2g-2}; \{u_1, \ldots, u_n\})$:

$$\widetilde{\tau}_{\alpha_k} \cdots \widetilde{\tau}_{\alpha_1} \cdot t_{\widetilde{c_l}} \cdots t_{\widetilde{c_1}} = t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}, \tag{3}$$

for $n = 2g-2$. Recall that by Remark 3.9 we can simplify the right-hand side as above so that there are no $\widetilde{t_{c_j}}$. Consider the associated graph $\Gamma = \Gamma_W$ whose vertices correspond to u_i and edges to half twists $\widetilde{\tau}_{\alpha_j}$ interchanging them. After relabeling δ_j if needed, we can assume that the connected components $\Gamma_1, \ldots, \Gamma_s$ of Γ have vertices $\{u_1, \ldots, u_{j_1}\}, \{u_{j_1+1}, \ldots, u_{j_2}\}, \ldots, \{u_{j_{s-1}+1}, \ldots, u_n\}$, respectively, for a subsequence $j_1, \ldots, j_{s-1}$ of $1, \ldots, n$. Let k_t be the corresponding number of $\widetilde{\tau}_{\alpha_j}$ involved in the points u_i in each Γ_t. We impose the following additional conditions:

- $2v_t - e_t = 2$ for each Γ_t, and
- $-\left(\sum_{j=j_{t-1}+1}^{j_t} a_j\right) + 2k_t = -1$ for every $t = 1, \ldots, s$.

Observe that these two conditions translate to each connected component of the corresponding multisection to be a 2–sphere and of self-intersection -1, respectively. Isolated vertices of Γ_W amount to exceptional *sections*, which can be blown down to a pencil.

Let $G(W)$ be the quotient of $\pi_1(\Sigma_g)$ by $N(c_1, \ldots, c_l)$, the subgroup normally generated by c_i, and denote by $b_1(W)$ the first Betti number of G. Let $\sigma(W)$ be the signature of the image of the positive word $t_{c_1} \cdots t_{c_l}$ in $\mathrm{Mod}(\Sigma_g)$ under the boundary capping homomorphism, and $\mathrm{e}(W) = 4 - 4g + l$ be the associated Euler characteristic. We can then set $b^+(W) = \frac{1}{2}(\mathrm{e}(W) + 2b_1(W) - 2 + \sigma(W))$.

We obtain a characterization of monodromy factorizations of symplectic Calabi–Yau Lefschetz fibrations:

Corollary 4.2 *Let W be a factorization in the framed mapping class group for $g \geq 2$, such that either $G(W)$ is not a surface group, or $G(W) = 1$ but $b^+(W) \neq 1$. If the associated graph Γ_W satisfies the properties listed above and, in addition, has at least one isolated vertex, then the reduced word $t_{c_l} \cdots t_{c_1}$ is a monodromy factorization of a symplectic Calabi–Yau Lefschetz pencil. Conversely, on any symplectic Calabi–Yau 4–manifold, one can find a Lefschetz pencil with a monodromy lift like such W.*

As demonstrated by our examples in Section 5, one can also have SCY Lefschetz fibrations with *no* exceptional sections. The first direction of the corollary can be extended to include such examples, too, provided a little care is given to the calculation of $G(W)$ (and thus $b_1(W)$) if no other lifts with pure sections are known.

Motivated by the *conjectural* smooth classification of symplectic Calabi–Yau 4–manifolds, we can formulate a parallel problem for groups that can possibly be fundamental groups of SCYs [22]:

Question 4.3 (symplectic Calabi–Yau groups via mapping class factorizations) For any positive factorization W of the boundary multitwist as in the corollary, is it always the case that $G(W) = 1$, $\mathbb{Z}/2\mathbb{Z}$, or an infrasolvmanifold[1] fundamental group?

A negative answer to this question amounts to the existence of *new* symplectic Calabi–Yaus. Whereas for a positive answer, since it suffices to work with pencils on minimal SCYs, one can restrict to positive factorizations W in $\mathrm{Mod}_{\partial \Sigma_g^{2g-2}}(\Sigma_g^{2g-2})$ with no τ_{α_i} on the left and all $a_i = 1$ on the right side of the Equality (3). Thus, understanding all possible SCY groups is equivalent to understanding $G(W)$, where W runs through all possible positive Dehn twist factorizations of the boundary multitwist $t_{\delta_1} \cdots t_{\delta_{2g-2}}$.

However, constructing mapping class group factorizations of the boundary multitwist in $\mathrm{Mod}(\Sigma_g^{2g-2}; U)$ is a rather challenging task in general. The next two subsections will demonstrate two successful cases: we will construct Lefschetz pencils on symplectic Calabi–Yau K3 and Enriques surfaces with *explicit monodromy factorizations*, respectively. These will serve as sources of various interesting fibrations we will derive from them via surgical operations in the later sections of our paper. What is of key importance

[1] which covers all other known SCYs with $b_1 \neq 0$ [22]

here is the *special configurations of Lefschetz vanishing cycles* in the factorizations we get, and to produce them we will appeal to several symmetries of surfaces and lift better known mapping class relations on spheres or tori with boundaries to higher genera surfaces.

4B A genus-3 pencil on a symplectic Calabi–Yau K3 surface

We now construct an explicit monodromy for a genus-3 Lefschetz fibration with exactly 4 disjoint (-1)–sphere sections on a 4 times blown-up symplectic Calabi–Yau K3 surface, thus a pencil on a symplectic Calabi–Yau 4–manifold homeomorphic to the K3 surface. The equivalent monodromy factorization we derive at the end will be used in further constructions, and is the main motivation for us to go after this particular factorization.

Lemma 4.4 *The following relation holds in* $\mathrm{Mod}_{\partial\Sigma_3^4}(\Sigma_3^4)$*:*

$$(t_{c_1}t_{c_7}t_{c_3}t_{c_5}t_{c_2}t_{c_6}t_{a_1}t_{a_2}t_{b_1}t_{b_2}t_{c_1}t_{c_7}t_{c_3}t_{c_5}t_{b_1}t_{b_2}t_{c_2}t_{c_6})^2 = t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}, \tag{4}$$

where $a_i, b_j, c_k, \delta_l \subset \Sigma_3^4$ *are simple closed curves shown in Figure 6.*

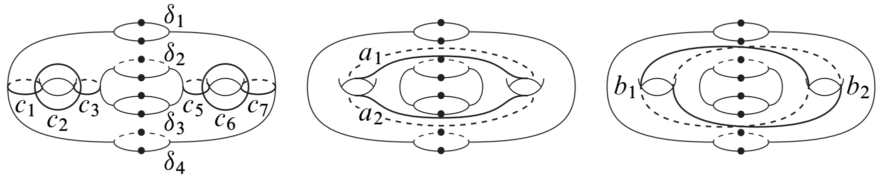

Figure 6: Simple closed curves in Σ_3^4. The curve δ_i is parallel to a boundary component.

Proof Let λ be an involution on Σ_3^4 as shown in Figure 7. The quotient space Σ_3^4/λ is diffeomorphic to the surface Σ_1^4. We describe the images of curves under $/\lambda$ by hatted symbols. Using the relation given in [32, Section 3.4] we obtain the following

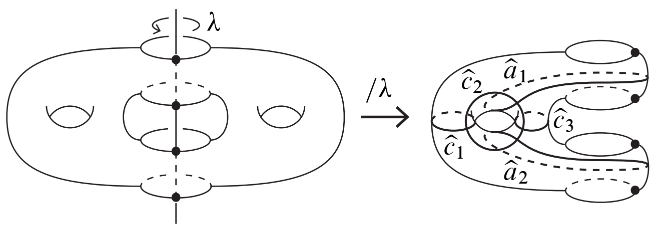

Figure 7: The quotient map by an involution λ

relation in $\mathrm{Mod}_{\partial\Sigma_1^4}(\Sigma_1^4)$:

$$(5)\quad \begin{aligned}(t_{\hat{c}_1}t_{\hat{c}_3}t_{\hat{c}_2}t_{\hat{a}_1}t_{\hat{a}_2}t_{\hat{c}_2})^2 &= t_{\hat{\delta}_1}t_{\hat{\delta}_2}t_{\hat{\delta}_3}t_{\hat{\delta}_4}\\ \Longleftrightarrow (t_{\hat{c}_1}t_{\hat{c}_3}t_{\hat{c}_2}t_{\hat{a}_1}t_{\hat{a}_2}t_{\hat{c}_2}t_{\hat{c}_1}t_{\hat{c}_3}t_{\hat{c}_2}t_{\hat{a}_1}t_{\hat{a}_2}t_{\hat{c}_2})^2 &= t^2_{\hat{\delta}_1}t^2_{\hat{\delta}_2}t^2_{\hat{\delta}_3}t^2_{\hat{\delta}_4}\\ \Longleftrightarrow (t_{\hat{c}_1}t_{\hat{c}_3}t_{\hat{c}_2}t^2_{\hat{a}_1}t^2_{\hat{a}_2}t_{\hat{b}_1}t_{\hat{c}_1}t_{\hat{c}_3}t_{\hat{b}_1}t_{\hat{c}_2})^2 &= t^2_{\hat{\delta}_1}t^2_{\hat{\delta}_2}t^2_{\hat{\delta}_3}t^2_{\hat{\delta}_4},\end{aligned}$$

where the last relation holds since $t_{\hat{b}_1}$ is equal to $(t_{\hat{a}_1}t_{\hat{a}_2})^{-1}t_{\hat{c}_2}t_{\hat{a}_1}t_{\hat{a}_2}$. The quotient map $/\lambda\colon \Sigma_3^4 \to \Sigma_1^4$ induces the following homomorphism:

$$\lambda_*\colon \pi_0(C_{\partial\Sigma_3^4}(\Sigma_3^4;\lambda)) \to \mathrm{Mod}_{\partial\Sigma_1^4}(\Sigma_1^4).$$

By Lemma 3.1 this homomorphism is injective. Furthermore it is not hard to see that the following equalities hold:

$$\begin{aligned} t_{\hat{c}_1} &= \lambda_*(t_{c_1}t_{c_7}), & t_{\hat{c}_2} &= \lambda_*(t_{c_2}t_{c_6}), & t_{\hat{c}_3} &= \lambda_*(t_{c_3}t_{c_5}),\\ t^2_{\hat{a}_i} &= \lambda_*(t_{a_i}), & t_{\hat{b}_1} &= \lambda_*(t_{b_1}t_{b_2}), & t^2_{\hat{\delta}_j} &= \lambda_*(t_{\delta_j}). \end{aligned}$$

Thus we can obtain the relation (4) using a homomorphism

$$\pi_0(C_{\partial\Sigma_3^4}(\Sigma_3^4;\lambda)) \to \mathrm{Mod}_{\partial\Sigma_3^4}(\Sigma_3^4)$$

induced by the inclusion $C_{\partial\Sigma_3^4}(\Sigma_3^4;\lambda) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_3^4}(\Sigma_3^4)$. This completes the proof of Lemma 4.4. □

The relation (4) gives rise to a genus-3 Lefschetz fibration (X,f) over the 2–sphere with four disjoint (-1)–sphere sections.

Proposition 4.5 *(X,f) is a genus-3 symplectic Calabi–Yau Lefschetz fibration, where the minimal model of X is homeomorphic to a* K3 *surface.*

Proof The topological invariants of X can be calculated using the monodromy factorization. The Euler characteristic of X is the easiest to derive from $\mathrm{e}(X) = 4-4g+m = 28$, where $g=3$ is the genus of the fibration and $m=36$ is the number of critical points. On the other hand, as the fibration $f\colon X\to S^2$ has a section, $\pi_1(X)\cong \pi_1(\Sigma_3)/\mathcal{C}$, where $\mathcal{C}$ is the normal subgroup of $\pi_1(\Sigma_3)$ generated by the vanishing cycles of f. The subcollection $c_1, c_2, c_5, c_6, c_7, b_1$ of vanishing cycles of f, taken with base points on the fiber in a straightforward fashion, generates the group $\pi_1(\Sigma_3)$. It follows that $\mathcal{C}=\pi_1(\Sigma_3)$, and in turn, $\pi_1(X)=1$.

The signature calculation is more involved, and will constitute the rest of the proof. Note that if we knew $b^+(X)>1$ at this point (or simply that X was not a rational surface) then Theorem 4.1 would imply that (X,f) is a symplectic Calabi–Yau Lefschetz fibration which is a 4 times blow-up of its minimal model. Since it should then have a minimal model which has the rational type of K3 surface, by Freedman's topological classification, the minimal model of X should be homeomorphic to K3. However, we will be able to derive the simple, yet essential conclusion $b^+(X) > 1$ after coupling our signature calculation to follow with our knowledge of $\mathrm{e}(X)$ and vanishing $b_1(X)$.

Each Dehn twist on the left hand side of (4) corresponds to a Lefschetz singularity of f. In particular there are two pairs of consecutive critical values of f corresponding to the elements $t_{a_1}t_{a_2}$. For each of the pairs we take a disk containing them which is away from the other critical values. We denote these disks by $D_1, D_2 \subset S^2$ and assume that D_1 and D_2 are disjoint. We also take disk neighborhoods $S_1, \ldots, S_{32} \subset S^2$ of the other critical values of f which are mutually disjoint and away from $D_1 \cup D_2$. Let $D_0 \subset S^2$ be a small disk away from all the disks above. We decompose the surface $S^2 \setminus \big(\mathrm{Int}(D_0) \sqcup \bigsqcup_i \mathrm{Int}(S_i) \sqcup \bigsqcup_j \mathrm{Int}(D_j)\big)$ into pants $P_1, \ldots, P_{33}$ (which are surfaces diffeomorphic to Σ_0^3) as follows:

- ∂P_1 contains ∂S_1 and ∂S_2. We denote the circle $\partial P_1 \setminus (\partial S_1 \cup \partial S_2)$ by L_1.
- ∂P_i contains L_{i-1} and ∂S_{i+1} for each $i = 2, \ldots, 31$. We denote the circle $\partial P_i \setminus (L_{i-1} \cup \partial S_{i+1})$ by L_i.
- ∂P_j contains L_{j-1} and ∂D_{j-31} for $j = 32$ and 33. We denote the circle $\partial P_j \setminus (L_{j-1} \cup \partial D_{j-31})$ by L_j. We take the pants P_{33} so that $L_{33} = \partial D_0$.

The signatures of $f^{-1}(D_0)$ and $f^{-1}(S_i)$ are equal to -1 and 0, respectively, which can be easily calculated from the intersection forms of their respective Kirby diagrams, or by invoking the algorithm of Ozbagci in [43]. We can thus deduce the following equality by the Novikov additivity:

$$\begin{aligned}\sigma(X) &= \sigma\Big(f^{-1}\Big(S^2 \setminus \Big(\mathrm{Int}(D_0) \sqcup \bigsqcup_i \mathrm{Int}(S_i) \sqcup \bigsqcup_j \mathrm{Int}(D_j)\Big)\Big)\Big) \\ &\qquad + \sigma(f^{-1}(D_0)) + \sum_{i=1}^{32} \sigma(f^{-1}(S_i)) + \sum_{j=1}^{2} \sigma(f^{-1}(D_j)) \\ &= \sigma\Big(f^{-1}\Big(S^2 \setminus \Big(\bigsqcup_i \mathrm{Int}(S_i) \sqcup \bigsqcup_j \mathrm{Int}(D_j)\Big)\Big)\Big) - 2 \\ &= \sum_{i=1}^{33} \sigma(f^{-1}(P_i)) - 2.\end{aligned}$$

Let ι be an involution of Σ_3 shown in Figure 8. We denote by $\mathcal{H}_3$ the image of the homomorphism $\pi_1(C(\Sigma_3;\iota)) \to \mathrm{Mod}(\Sigma_3)$ induced by the inclusion map

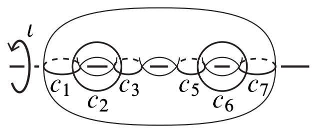

Figure 8: The involution ι

$C(\Sigma_g;\iota) \hookrightarrow \mathrm{Diff}^+(\Sigma_3)$. Let $\phi_3\colon \mathcal{H}_3 \to \mathbb{Z}/7$ be a class function given in [13, Proposition 3.1]. For a simple closed curve $L \subset S^2 \setminus \mathrm{Crit}(f)$ we denote the monodromy along L by $\rho(L)$, which is a conjugacy class of an element in $\mathrm{Mod}(\Sigma_3)$. By the configuration of vanishing cycles of f we can regard $\rho(\partial S_i)$, $\rho(\partial D_j)$ and $\rho(L_k)$ as conjugacy classes in $\mathcal{H}_3$. We obtain the following equality by [42, Satz 1] and [13, Proposition 3.1]:

$$\begin{aligned}\sigma(X) &= \phi_3(\rho(L_1)) - \phi_3(\rho(\partial S_1)) - \phi_3(\rho(\partial S_2)) \\ &\qquad + \sum_{i=2}^{31}\bigl(\phi_3(\rho(L_i)) - \phi_3(\rho(\partial S_{i+1})) - \phi_3(\rho(L_{i-1}))\bigr) \\ &\qquad + \sum_{j=32}^{33}\bigl(\phi_3(\rho(L_j)) - \phi_3(\rho(\partial D_{j-31})) - \phi_3(\rho(L_{j-1}))\bigr) - 2 \\ &= -\sum_{i=1}^{32}\phi_3(\rho(\partial S_i)) - \sum_{j=1}^{2}\phi_3(\rho(\partial D_j)) + \phi_3(\rho(L_{33})) - 2.\end{aligned}$$

Since the curve L_{33} is equal to ∂D_0 the monodromy $\rho(L_{33})$ is trivial. In particular $\phi_3(\rho(L_{33}))$ is equal to 0. The monodromy $\rho(\partial S_i)$ is a Dehn twist along a nonseparating simple closed curve in Σ_3. Thus the value $\phi_3(\rho(\partial S_i))$ is equal to $\frac{4}{7}$ (see [13, Lemma 3.3]). Since the monodromy $\rho(\partial D_1)$ is equal to $\rho(\partial D_2)$, the value $\phi_3(\rho(\partial D_1))$ is equal to $\phi_3(\rho(\partial D_2))$.

The manifold $S^2 \times T^2 \# 8\overline{\mathbb{CP}}^2$ admits a genus-3 Lefschetz fibration f with the following properties (see [31]):

- f has 16 Lefschetz singularities with nonseparating vanishing cycles;
- a monodromy factorization of f contains four pairs of Dehn twists along bounding pairs (a *bounding pair* of curves in a closed surface Σ is a pair of nonseparating simple closed curves such that the complement of the union of them is disconnected);
- the other eight vanishing cycles are preserved by the involution ι.

Since the function ϕ_3 is preserved by conjugation and a bounding pair in Σ_3 is unique up to conjugation, we obtain the following equality (using the same method as above):

$$-8 = \sigma(S^2 \times T^2 \# 8\overline{\mathbb{CP}}^2) = -8 \cdot \tfrac{4}{7} - 4\phi_3(\rho(\partial D_1)) - 4$$
$$\Longrightarrow\ \phi_3(\rho(\partial D_1)) = -\tfrac{1}{4}\bigl(-8 + 4 + \tfrac{32}{7}\bigr) = -\tfrac{1}{7}.$$

As a result we can calculate the signature of X as

$$\sigma(X) = -32 \cdot \tfrac{4}{7} - 2 \cdot \bigl(-\tfrac{1}{7}\bigr) - 2 = -20.$$

We can now use

$$b^+(X) - b^-(X) = \sigma(X) = -20 \quad \text{and} \quad 2 - 2b_1(X) + b^+(X) + b^-(X) = \mathrm{e}(X) = 28,$$

where $b_1(X) = 0$, to conclude that $b^+(X) = 3$. Hence, per our discussion preceding the signature calculation, (X, f) is a symplectic Calabi–Yau Lefschetz fibration, where the minimal model of X is homeomorphic to a K3 surface. □

4C A genus-2 pencil on a symplectic Calabi–Yau Enriques surface

We now construct a genus-2 Lefschetz fibration with exactly two disjoint (-1)–sphere sections on a 2 times blown-up symplectic Calabi–Yau Enriques surface, that is, a genus-2 Lefschetz pencil on a symplectic Calabi–Yau 4–manifold homeomorphic to the Enriques surface.

Lemma 4.6 *The following relation holds in* $\mathrm{Mod}_{\partial\Sigma_2^2}(\Sigma_2^2)$*:*

$$(t_{d_4} t_{d_3} t_{d_2})^2 t_{d_+} t_{d_-} = t_{\delta_1} t_{\delta_2},$$

where $d_*, \delta_i \subset \Sigma_2^2$ *are simple closed curves shown in Figure 9.*

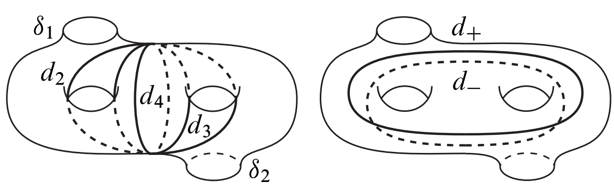

Figure 9: Simple closed curves in Σ_2^2

Proof We regard Σ_2^4 as a subsurface of Σ_2^2 in the obvious way. We take simple closed curves $d_1, \Gamma_i \subset \Sigma_2^4$, a point $u_i \in \partial\Sigma_2^4$ and involutions κ and ι as shown in Figure 10.

Denote a simple closed curve in Σ_2^4 parallel to the component of $\partial\Sigma_2^4$ containing u_i by δ_i and fixed points of κ (resp. ι) by v_i (resp. w_i). We add a hat to any symbols to

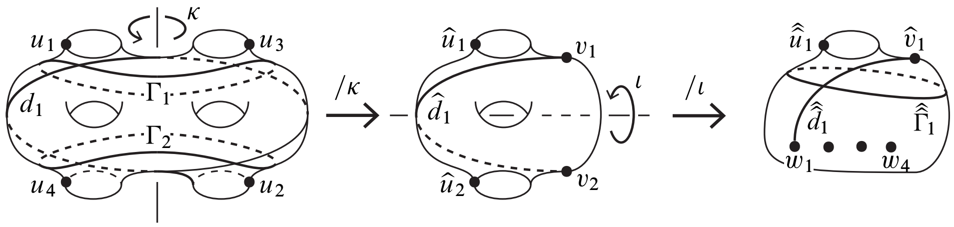

Figure 10: Symmetries on surfaces

describe the image of quotient maps. Let $\alpha_i \subset \Sigma_0^1 \setminus \{w_1, \dots, w_4\}$ be a loop with the base point $\hat{v}_1$ obtained by connecting $\hat{\hat{d}}_i$ with a counterclockwise circle around w_i. The following equality holds in $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1, \{w_1, \dots, w_4\})$:

$$\mathrm{Push}(\alpha_4)\,\mathrm{Push}(\alpha_3)\,\mathrm{Push}(\alpha_2)\,\mathrm{Push}(\alpha_1) = t_{\hat{\hat{\Gamma}}_1}^{-1} t_{\hat{\hat{\delta}}_1}.$$

By Lemma 3.1, the following homomorphism is injective:

$$\iota_*\colon \pi_0(C_{\partial\Sigma_1^2}(\Sigma_1^2;\iota)) \to \mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1;\{w_1, \dots, w_4\}).$$

It is easy to verify (by using Alexander's lemma, for example) that the following equalities hold:

$$\mathrm{Push}(\alpha_i) = \iota_*(\tau_{\hat{d}_i}), \quad t_{\hat{\hat{\Gamma}}_1} = \iota_*(t_{\hat{\Gamma}_1} t_{\hat{\Gamma}_2}), \quad t_{\hat{\hat{\delta}}_1} = \iota_*(t_{\hat{\delta}_1} t_{\hat{\delta}_2}).$$

Thus we obtain the following relation in $\pi_0(C_{\partial\Sigma_1^2}(\Sigma_1^2;\iota))$:

$$\tau_{\hat{d}_4}\tau_{\hat{d}_3}\tau_{\hat{d}_2}\tau_{\hat{d}_1} = t_{\hat{\Gamma}_1}^{-1} t_{\hat{\Gamma}_2}^{-1} t_{\hat{\delta}_1} t_{\hat{\delta}_2}. \tag{6}$$

Using the inclusion $C_{\partial\Sigma_1^2}(\Sigma_1^2;\iota) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_1^2}(\Sigma_1^2;\{v_1, v_2\})$ we obtain the same relation as (6) in $\mathrm{Mod}_{\partial\Sigma_1^2}(\Sigma_1^2;\{v_1, v_2\})$.

The involution κ induces the following homomorphism which is injective by Lemma 3.1:

$$\kappa_*\colon \pi_0(C_{\partial\Sigma_2^4}(\Sigma_2^4;\kappa)) \to \mathrm{Mod}_{\partial\Sigma_1^2}(\Sigma_1^2;\{v_1, v_2\}).$$

It is easy to see that the images of the mapping classes t_{d_i}, t_{Γ_i} and $t_{\delta_i}t_{\delta_{i+2}}$ under the homomorphism κ_* are $\tau_{\hat{d}_i}$, $t_{\hat{\Gamma}_i}^2$ and $t_{\hat{\delta}_i}$, respectively. Thus we obtain the following relation in $\pi_0(C_{\partial\Sigma_2^4}(\Sigma_2^4;\kappa))$:

$$\begin{aligned} (7)\quad (t_{d_4}t_{d_3}t_{d_2}t_{d_1})^2 &= t_{\Gamma_1}^{-1} t_{\Gamma_2}^{-1} t_{\delta_1}^2 t_{\delta_2}^2 t_{\delta_3}^2 t_{\delta_4}^2 \\ \iff (t_{d_4}t_{d_3}t_{d_2})^2((t_{d_4}t_{d_3}t_{d_2})^{-1} t_{d_1} (t_{d_4}t_{d_3}t_{d_2}))t_{d_1} &= t_{\Gamma_1}^{-1} t_{\Gamma_2}^{-1} t_{\delta_1}^2 t_{\delta_2}^2 t_{\delta_3}^2 t_{\delta_4}^2. \end{aligned}$$

We can obtain the same relation as (7) via the inclusion $C_{\partial\Sigma_2^4}(\Sigma_2^4;\kappa) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_2^4}(\Sigma_2^4)$. Let $C\colon \mathrm{Mod}_{\partial\Sigma_2^4}(\Sigma_2^4) \to \mathrm{Mod}_{\partial\Sigma_2^2}(\Sigma_2^2)$ be a homomorphism obtained by capping the components of $\partial\Sigma_2^4$ containing u_3 and u_4 with punctured disks. It is easy to see that the image of the left (resp. right) side of the equality (7) under C is equal to the left (resp. right) side of the equality in the statement. This completes the proof of Lemma 4.6. □

Let $c_2, c_3, c_4 \subset \Sigma_2^2$ be simple closed curves shown in Figure 11.

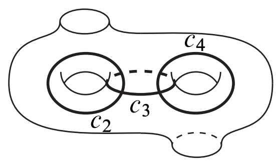

Figure 11: Simple closed curves in Σ_2^2

By the chain relation (see [15, Proposition 4.12]) and Lemma 4.6 we obtain the following relation in $\mathrm{Mod}_{\partial\Sigma_2^2}(\Sigma_2^2)$:

$$(t_{d_4} t_{d_3} t_{d_2})^2 (t_{c_2} t_{c_3} t_{c_4})^4 = t_{\delta_1} t_{\delta_2}.$$

This relation gives rise to a genus-2 Lefschetz fibration $h \colon Y \to S^2$ with two (-1)–sphere sections.

Proposition 4.7 (Y, h) *is a genus-*2 *symplectic Calabi–Yau Lefschetz fibration, where the minimal model of* Y *is homeomorphic to an Enriques surface.*

Proof Since h has 18 Lefschetz singularities, $\mathrm{e}(Y) = 14$. As the genus-2 mapping class group is hyperelliptic, we can also calculate $\sigma(Y)$ easily by using Matsumoto's signature formula [40, Theorem 3.3(2), Proposition 3.6] as

$$\sigma(Y) = 16\left(-\tfrac{3}{5}\right) + 2\left(-\tfrac{1}{5}\right) = -10.$$

The calculation that is more involved this time is that of $\pi_1(Y)$, since the vanishing cycles of (Y, h) do not kill all the generators of $\pi_1(\Sigma_2)$, the fundamental group of the fiber. To calculate $\pi_1(Y)$, let us take oriented based loops α_i and β_i in Σ_2 as shown in Figure 12. We also use the symbols α_i and β_i to represent the homotopy classes of each loop. The fundamental group $\pi_1(\Sigma_2)$ has the presentation $\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\beta_2, \alpha_2] \rangle$. The curves c_2, c_3 and c_4 are homotopic to the

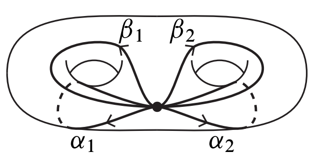

Figure 12: Generators of $\pi_1(\Sigma_2)$

curves $\beta_1, \alpha_1\overline{\alpha_2}$ and β_2, respectively, where $\overline{\alpha_2}$ is the based loop α_2 with the opposite orientation. It is easy to see that the curves d_2, d_3 and d_4 are homotopic to $\alpha_1\overline{\beta_1}\beta_2\alpha_2, \alpha_1\beta_2\alpha_2\overline{\beta_2}$ and $[\alpha_1, \beta_1]$, respectively. Since the fibration h has a section the fundamental group of Y is calculated as follows:

$$\begin{aligned}
&\pi_1(Y)\\
&\quad\cong \pi_1(\Sigma_2)/\langle c_2, c_3, c_4, d_2, d_3, d_4\rangle\\
&\quad\cong \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\beta_2, \alpha_2], \beta_1, \alpha_1\overline{\alpha_2}, \beta_2, \alpha_1\overline{\beta_1}\beta_2\alpha_2, \alpha_1\beta_2\alpha_2\overline{\beta_2}, [\alpha_1, \beta_1]\rangle\\
&\quad\cong \langle \alpha_1, \alpha_2 \mid \alpha_1\overline{\alpha_2}, \alpha_1\alpha_2\rangle\\
&\quad\cong \langle \alpha \mid \alpha^2\rangle \cong \mathbb{Z}/2\mathbb{Z}.
\end{aligned}$$

Now, since Y is not rational or ruled, it is a symplectic Calabi–Yau, and has two disjoint exceptional spheres. Blowing them down, we arrive at the minimal symplectic Y'. Since its signature is not divisible by 16, $w_2(Y')$ does not vanish. It is easily seen that the universal cover $\widetilde{Y}'$ of Y' has a torsion canonical class, and is trivial since $H^2(\widetilde{Y}')$ has no torsion. Since the modulo 2 reduction of the canonical class coincides with the second Stiefel–Whitney class, $w_2(\widetilde{Y}')$ vanishes. Thus, Y' has the same w_2–type [26] as that of the Enriques surface. By [27, Theorem C], we conclude that Y' is homeomorphic to the Enriques surface. □

As we are unable to detect whether the total spaces of our pencils are *diffeomorphic* to the K3 and *the* Enriques surfaces, we finish by highlighting this question:

Question 4.8 Are the symplectic Calabi–Yau manifolds X and Y we have constructed above diffeomorphic to blow-ups of the K3 and the Enriques surfaces, respectively?

5 Fiber sum indecomposability and Stipsicz's conjecture

A common way of constructing new Lefschetz fibrations from given ones is the *fiber sum* operation, defined as follows: Let (X_i, f_i), $i = 1, 2$, be genus-g Lefschetz fibrations with regular fiber F. The *fiber sum* $(X_1, f_1) \#_{F,\Phi} (X_2, f_2)$ is a genus-g Lefschetz fibration obtained by removing a fibered tubular neighborhood of a regular fiber from each (X_i, f_i) and then identifying the resulting boundaries via a fiber-preserving, orientation-reversing diffeomorphism Φ. In terms of monodromy factorizations, this translates to a monodromy factorization with a proper subfactorization of the identity in the mapping class group of F. A Lefschetz fibration (X, f) is called *fiber sum indecomposable* (or *indecomposable* in short) if it cannot be expressed as a fiber sum of any two nontrivial Lefschetz fibrations. These can be regarded as prime building blocks of Lefschetz fibrations.

Stipsicz in [51], and Smith in [49] independently proved that Lefschetz fibrations admitting (-1)–sphere sections are fiber sum indecomposable. Moreover, Stipsicz conjectured in the same work that the converse is also true, that is, every fiber sum indecomposable Lefschetz fibration contains a (-1)–sphere section, an affirmative answer to which would suggest blow-ups of Lefschetz pencils as the elementary building blocks of Lefschetz fibrations through fiber sums. However, Sato showed that a genus-2 Lefschetz fibration constructed by Auroux on a once blown-up minimal symplectic 4–manifold provided a counterexample to this conjecture, by showing that the exceptional class could only be represented by a 2–section of this fibration [45].

Here we will show that Auroux's genus-2 fibration is not a mere exception, by constructing further counterexamples to Stipsicz's conjecture via explicit monodromy factorizations of Lefschetz fibrations with their multisections. To be able to detect how all exceptional classes lie with respect to a Lefschetz fibration (X, f) (and that none is a section), we will work with Lefschetz fibrations on blow-ups of symplectic Calabi–Yau 4–manifolds, which are the perfect fit to our purposes because of two reasons: any symplectic Calabi–Yau (X, f) is fiber sum indecomposable ([7; 55]), and, as we have reviewed earlier, each 2–sphere S_j representing an exceptional class is an s_j–section of (X, f), where the *inequality* $\sum s_j = \left(\sum S_j\right) \cdot F \leq 2g-2$ in this case is sharp. The strategy of our proof then boils down to constructing an explicit monodromy factorization for a symplectic Calabi–Yau (X, f) detecting all exceptional multisections, and then applying monodromy substitutions which turn all exceptional classes to s_j–sections with $s_j \geq 2$ for all j. This monodromy substitution comes from a generalization of the lantern relation to the framed mapping class group, which we present next.

5A A lantern relation for multisections

We generalize the lantern relation to that in the mapping class group with commutative boundary components:

Lemma 5.1 (braiding lantern relation) *Let $a, b, c, d, \delta_1, \delta_2 \subset \Sigma_0^6$ be simple closed curves parallel to boundary components. Denote the boundary components parallel to δ_i by S_i. Let $x \subset \Sigma_0^6$ be a simple closed curve and $y, z \subset \Sigma_0^6$ pairs of arcs, where x, y and z become simple closed curves in the usual lantern relation when we cap off S_1 and S_2 by disks* (see Figure 13). *The following relation holds in $\mathrm{Mod}_{\partial\Sigma_0^6 \setminus (S_1 \sqcup S_2)}(\Sigma_0^6; \{u_1, u_2\})$:*

$$\widetilde{t_z} t_x \widetilde{t_y} = t_a t_b t_c t_d t_{\delta_2}.$$

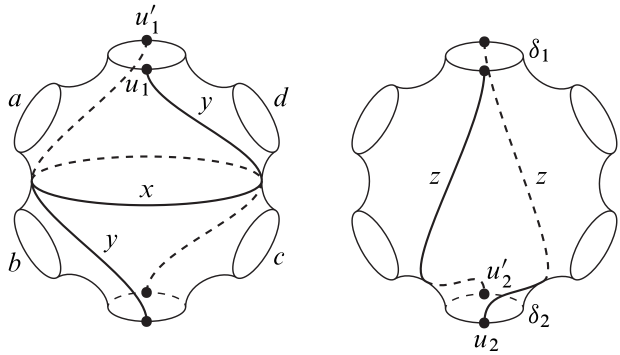

Figure 13: Curves in Σ_0^6

Proof Let η be an involution of Σ_0^6 defined as the 180–degree rotation as shown in Figure 14.

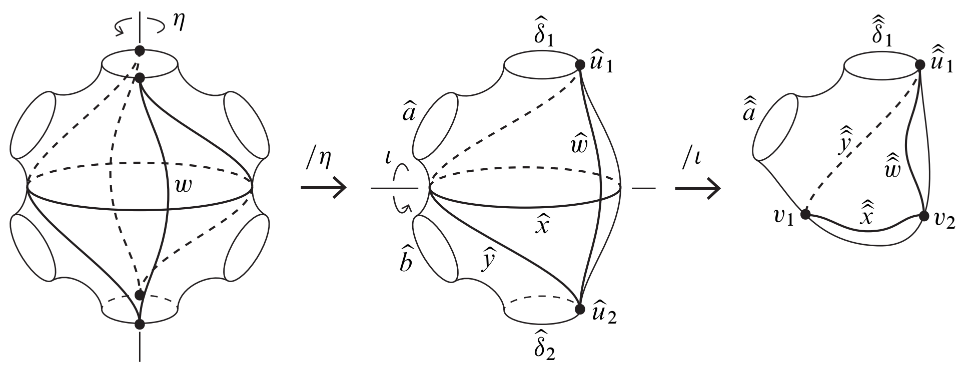

Figure 14: The quotient map $/\eta$: $\Sigma_0^6 \to \Sigma_0^6/\eta \cong \Sigma_0^4$

The quotient space Σ_0^6/η is homeomorphic to the surface Σ_0^4. We take a pair of arcs w in Σ_0^6 as shown in Figure 14. Let $\hat{w} \subset \Sigma_0^4$ be the image of w under the quotient map $/\eta$. We define other symbols in the same way (see Figure 14).

We further take an involution ι on Σ_0^4 as shown in Figure 14. The quotient space Σ_0^4/ι is homeomorphic to the surface Σ_0^2. Denote the image of the fixed points of ι under the quotient map $/\iota$: $\Sigma_0^4 \to \Sigma_0^2$ by v_1, v_2. We use double-hatted symbols to describe images of curves and arcs in Σ_0^4 under the map $/\iota$; see Figure 14. We denote by $\hat{\hat{S}}$ the boundary component of Σ_0^2 parallel to the curve $\hat{\hat{\delta}}_1$. Let Y and W be simple closed curves in Σ_0^2 which bound regular neighborhoods of the unions $\hat{\hat{y}} \cup \hat{\hat{S}}$ and $\hat{\hat{w}} \cup \hat{\hat{S}}$, respectively, and X a simple closed curve which bounds the arc $\hat{\hat{x}}$. By the lantern relation [15, Proposition 5.1], we obtain the following relation in $\mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2; v_1, v_2)$:

$$t_W t_X t_Y = t_{\hat{\hat{a}}} t_{\hat{\hat{\delta}}_1}. \tag{8}$$

Consider the following homomorphism induced by the quotient map $/\iota\colon \Sigma_0^4 \to \Sigma_0^2$:

$$\iota_*\colon \pi_0(C_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\iota)) \to \mathrm{Mod}_{\partial\Sigma_0^2}(\Sigma_0^2; v_1, v_2).$$

By Lemma 3.1 this homomorphism is injective. Since the paths $\hat{y}, \hat{w}$ and a loop $\hat{x}$ are invariant under ι, we can regard $\tau_{\hat{y}}$, $\tau_{\hat{w}}$ and $t_{\hat{x}}$ as elements in $\pi_0(C_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\iota))$, where $\tau_{\hat{y}}$ (resp. $\tau_{\hat{w}}$) is the half twist along $\hat{y}$ (resp. $\hat{w}$) interchanging the boundary components $\hat{S}_1$ and $\hat{S}_2$. It is not hard to see (by using the Alexander method [15, Section 2.3], for example) that the following equalities hold:

$$\iota_*(\tau_{\hat{y}}) = t_Y, \quad \iota_*(\tau_{\hat{w}}) = t_W, \quad \iota_*(t_{\hat{x}}^2) = t_X, \quad \iota_*(t_{\hat{a}}t_{\hat{b}}) = t_{\hat{\hat{a}}}, \quad \iota_*(t_{\hat{\delta}_1}t_{\hat{\delta}_2}) = t_{\hat{\hat{\delta}}_1}.$$

Thus we obtain the following relation in $\pi_0(C_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\iota))$ from the relation (8):

$$\tau_{\hat{w}} t_{\hat{x}}^2 \tau_{\hat{y}} = t_{\hat{a}} t_{\hat{b}} t_{\hat{\delta}_1} t_{\hat{\delta}_2}. \tag{9}$$

Since the inclusion

$$C_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\iota) \hookrightarrow \mathrm{Diff}^+_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\{u_1,u_2\})$$

induces a homomorphism

$$\pi_0(C_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\iota)) \to \mathrm{Mod}_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\{u_1,u_2\}),$$

we can regard the relation (9) as that in $\mathrm{Mod}_{\partial\Sigma_0^4\setminus(\hat{S}_1\sqcup\hat{S}_2)}(\Sigma_0^4;\{u_1,u_2\})$.

We denote by $t_{\delta_i}^{1/2} \in \mathrm{Mod}_{\partial\Sigma_0^6\setminus(S_1\sqcup S_2)}(\Sigma_0^6;\{u_1,u_1',u_2,u_2'\})$ the square root of the Dehn twist along δ_i, which interchanges the points u_i and u_i'. In a way quite similar to that in the previous paragraph, from the relation (9), we obtain the following relation in $\mathrm{Mod}_{\partial\Sigma_0^6\setminus(S_1\sqcup S_2)}(\Sigma_0^6;\{u_1,u_1',u_2,u_2'\})$:

$$\begin{aligned}
\tilde{t}_w t_x \tilde{t}_y = t_a t_b t_c t_d t_{\delta_1}^{1/2} t_{\delta_2}^{1/2} &\iff t_{\delta_2}^{1/2} \tilde{t}_w t_x \tilde{t}_y t_{\delta_1}^{-1/2} = t_a t_b t_c t_d t_{\delta_2} \\
&\iff (t_{\delta_2}^{1/2} \tilde{t}_w t_{\delta_2}^{-1/2}) t_x \tilde{t}_y = t_a t_b t_c t_d t_{\delta_2} \\
&\iff \tilde{t}_z t_x \tilde{t}_y = t_a t_b t_c t_d t_{\delta_2}.
\end{aligned}$$

This completes the proof of Lemma 5.1. □

Remark 5.2 As the braiding lantern relation allows us to perform a local substitution in a monodromy factorization, it can be used to pass from one Lefschetz fibration to another while braiding two given sheets of (multi)sections. If the boundary components δ_1 and δ_2 correspond to two sections S_1 and S_2 of self-intersections s_1 and s_2, the substitution will hand a new Lefschetz fibration with a 2–section S_{12} which is an embedded 2–sphere of self-intersection $s_1 + s_2 + 1$. In particular, if S_1 and S_2

are exceptional classes, so is S_{12}, which will play a crucial role in our applications to follow.

Remark 5.3 Let us sketch a toy example of our braiding lantern substitution: Consider the trivial rational fibration on $S^2 \times S^2$, and blow up one of the fibers 4 times so it now consists of a (-4)–sphere V and 4 exceptional spheres. Let S_1 and S_2 be two disjoint self-intersection 0 sections of this fibration on $S^2 \times S^2 \# 4\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$, each intersecting V once. The braiding lantern substitution along the vanishing cycles and the two boundary components for the sections S_1 and S_2 amounts to rationally blowing down V, and the result will be a new rational Lefschetz fibration on $\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ with 3 vanishing cycles and a 2–sphere bisection S_{12} of self-intersection $+1$. The latter is equivalent to the blow-up of the degree-2 pencil on $\mathbb{CP}^2$, where S_{12} is identified with $\mathbb{CP}^1 \subset \mathbb{CP}^2$.

5B Genus-3 counterexamples to Stipsicz's conjecture

In the previous section, we have obtained the following monodromy factorization for a genus-3 Lefschetz fibration with 4 (-1)–sphere sections on a symplectic Calabi–Yau K3 surface:

$$(t_{c_1} t_{c_7} t_{c_3} t_{c_5} t_{c_2} t_{c_6} t_{a_1} t_{a_2} t_{b_1} t_{b_2} t_{c_1} t_{c_7} t_{c_3} t_{c_5} t_{b_1} t_{b_2} t_{c_2} t_{c_6})^2 = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$$

We will now derive a factorization Hurwitz equivalent to this one. In fact, the motivation behind all of our constructions in that section is indeed to arrive at this next configuration containing various lantern curves.

Denote by $y_1, y_2, z_1, z_2, w_1, w_2$ pairs of arcs shown in Figure 15. By Lemma 5.1, we

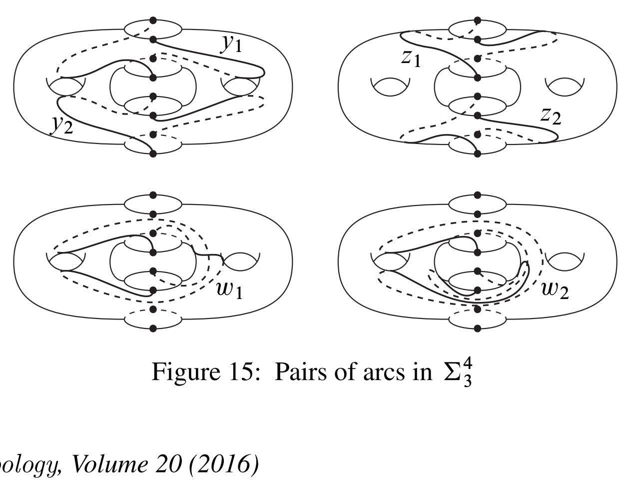

Figure 15: Pairs of arcs in Σ_3^4

obtain the following relations in $\mathrm{Mod}(\Sigma_3^4; \{u_1, u_2, u_3, u_4\})$:

$$\widetilde{t_{z_1}} t_{a_1} \widetilde{t_{y_1}} t_{\delta_1}^{-1} = t_{c_1} t_{c_3} t_{c_5} t_{c_7}, \tag{10}$$

$$\widetilde{t_{z_2}} t_{a_2} \widetilde{t_{y_2}} t_{\delta_4}^{-1} = t_{c_1} t_{c_3} t_{c_5} t_{c_7}, \tag{11}$$

$$\widetilde{t_{w_2}} t_{c_5} \widetilde{t_{w_1}} t_{\delta_3}^{-1} = t_{c_3}^2 t_{a_1} t_{a_2}. \tag{12}$$

The monodromy factorization of f given in Lemma 4.4 can be then changed by elementary transformations to obtain the following:

$$\begin{aligned} t_{\delta_4} t_{\delta_3} t_{\delta_2} t_{\delta_1} = {} & \underbrace{t_{c_1} t_{c_7} t_{c_3} t_{c_5}}_{\text{(a)}} t_{c_2} t_{c_6} t_{a_1} t_{a_2} t_{b_1} t_{b_2} \underbrace{t_{c_1} t_{c_7} t_{c_3} t_{c_5}}_{\text{(b)}} t_{b_1} t_{b_2} t_{c_2} t_{c_6} \\ & \cdot t_{c_1} t_{c_7} t_{c_5} t_{t_{c_3}(c_2)} t_{c_6} \underbrace{t_{c_3} t_{a_1} t_{a_2} t_{c_3}}_{\text{(c)}} t_{b_1} t_{t_{c_3}^{-1}(b_2)} t_{c_1} t_{c_7} t_{c_5} t_{b_1} t_{b_2} t_{c_2} t_{c_6}. \end{aligned} \tag{13}$$

We will denote this fibration as $(X, f) = (X_{(1,1,1,1)}, f_{(1,1,1,1)})$ in regards to the intersection numbers n_j of the exceptional classes with the fiber, which are all honest sections in this case.

Let S_1, S_2, S_3, S_4 be the exceptional sections of (X, f). By Lemma 5.1 we can derive the following new symplectic Calabi–Yau Lefschetz fibrations: The monodromy substitution by the relation (10) at the part (a) gives rise to a new genus-3 Lefschetz fibration $(X_{(2,1,1)}, f_{(2,1,1)})$ with a 2–section S_{12} derived from S_1 and S_2 and the two sections S_3 and S_4 inherited from (X, f). By Theorem 3.6, S_{12} is a 2–sphere with self-intersection equal to -1.

It was shown in [14] that a substitution by the lantern relation corresponds to a *rational blow-down along* C_2. (Basic definitions and properties of rational blow-downs are reviewed in Section 6A.) Since the relation (10) is a lift of a lantern relation in $\mathrm{Mod}(\Sigma_3)$, the substitution applied above corresponds to a rational blow-down along some $C_2 \subset X$. Since b^+ does not change under the rational blow-down operation, we obtain $b^+(X_{(2,1,1)}) = b^+(X) = 3$. In particular, $X_{(2,1,1)}$ is not a rational or a ruled surface. Note that we could also apply the lantern substitution along (b) to arrive at a similar Lefschetz fibration.

We can further apply the substitution by the relation (11) at part (b) to $(X_{(2,1,1)}, f_{(2,1,1)})$. This substitution now gives rise to a genus-3 Lefschetz fibration $(X_{(2,2)}, f_{(2,2)})$ with two (-1)–spheres S_{12} and S_{34} that are 2–sections. Once again, $b^+(X_{(2,2)}) = b^+(X) = 3$. We have, therefore, obtained a fiber sum indecomposable Lefschetz fibration on $X_{(2,2)}$, which is not a rational or ruled surface, where there are no other exceptional classes other than S_{12} and S_{34}, and thus, there are no (-1) sphere sections due to formula $\sum n_j = \left(\sum S_j\right) \cdot F = 4$.

We can apply a substitution by the relation (12) at (c) to arrive at a symplectic Calabi–Yau Lefschetz fibration $(X_{(4)}, f_{(4)})$ with a single exceptional class represented by a sphere 4–section S_{1234}. By the same arguments as above, this is a fiber sum indecomposable fibration without any (-1) sphere sections. Note that we could change the order of substitutions and braid S_1, S_2 and S_3 using simultaneous substitutions along (a) and (b) first into a 3–section S_{123}, and then apply a substitution along (c) to arrive at $(X_{(4)}, f_{(4)})$.

Hence we have obtained a pair of counterexamples to Stipsicz's conjecture:

Theorem 5.4 *The Lefschetz fibrations $(X_{(2,2)}, f_{(2,2)})$ and $(X_{(4)}, f_{(4)})$ are fiber sum indecomposable, but do not admit any (-1) sphere sections.*

Remark 5.5 As shown above, fiber sum indecomposable Lefschetz fibrations without (-1)–sphere sections do appear when the fiber has genus $g > 2$ as well. Furthermore, invoking Endo's signature formula for hyperelliptic Lefschetz fibrations, one can easily observe that these fibrations are not hyperelliptic, as opposed to any genus-2 example one can produce.

5C A new genus-2 counterexample

The following is the monodromy factorization for a genus-2 symplectic Calabi–Yau Lefschetz fibration obtained in the previous section:

$$\begin{aligned} t_{\delta_1} t_{\delta_2} &= (t_{d_4} t_{d_3} t_{d_2})^2 (t_{c_2} t_{c_3} t_{c_4})^4 \\ &= (t_{d_4} t_{d_3} t_{d_2})^2 \underbrace{t_{c_2}^2 t_{c_4}^2}_{\text{(d)}} t_{t_{c_2}^{-1} t_{c_4}^{-2}(c_3)} t_{t_{c_4}^{-1}(c_3)} (t_{c_2} t_{c_3} t_{c_4})^2. \end{aligned}$$

We can apply the braiding lantern substitution at the part (d) above as we applied in the previous subsection. This substitution changes the two exceptional sections into a sphere bisection with self-intersection -1. We denote the resulting fibration by $h_{(2)}\colon Y_{(2)} \to S^2$. Using [24, Lemma 5.1] we can prove that $Y_{(2)}$ is diffeomorphic to the blow-down of $Y_{(1,1)}$ (see also Proposition 6.1). In particular $Y_{(2)}$ is not rational or ruled. Thus, by the same argument as in the previous subsection we obtain the following theorem:

Theorem 5.6 *The Lefschetz fibration $(Y_{(2)}, h_{(2)})$ is fiber sum indecomposable, but does not admit any (-1) sphere sections.*

Tracing the braided lantern curves in the above monodromy substitution, one can verify that the explicit positive factorization of this fibration, along with its exceptional bisection, is the one we have given earlier in Example 3.7.

Remark 5.7 (more examples) Following the same recipe, one can also obtain counterexamples to Stipsicz's conjecture from the already known 2–boundary chain relation on the genus-2 surface:

$$(t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^6 = 1,$$

which prescribes a Lefschetz fibration f on $X = \mathrm{K3} \# 2\overline{\mathbb{CP}}^2$. Through Hurwitz moves, one can easily find a lantern configuration in an equivalent factorization. Since the 4 curves of the lantern configuration yield a symplectic 2–sphere V of self-intersection -4 [23], and since (X, f) is an SCY, we know that there are 2 exceptional sections S_1 and S_2 of f, each hitting V once. In turn, this a priori tells that we have a lift of this factorization where one can apply the braiding lantern relation of Lemma 5.1 to produce a new SCY Lefschetz fibration, which is fiber sum indecomposable but doesn't have any exceptional sections. In fact, Auroux's genus-2 Lefschetz fibration, the first known counterexample to Stipsicz's conjecture [45], can be seen to arise in this way as well; see [8].

Intrinsic to our strategy to argue that we obtain true counterexamples is that they are all symplectic Calabi–Yau Lefschetz fibrations. Although one would expect the answer to be affirmative, a natural question that arises is:

Question 5.8 Are there any fiber sum indecomposable Lefschetz fibrations with no (-1)–sphere sections on symplectic 4–manifolds of nonzero Kodaira dimension?

As we discussed earlier, by Usher's theorem [55], any Lefschetz fibration on a non-minimal symplectic 4–manifold is necessarily fiber sum indecomposable. (A short alternative proof of this particular fact was given in [7] making use of multisections.) To the best of our knowledge, there are no known examples of fiber sum indecomposable — nontrivial — Lefschetz fibrations on *minimal* symplectic 4–manifolds, while there are many examples of fiber sum decomposable ones such as the Lefschetz fibrations on knot-surgered elliptic surfaces [20]. We end with noting this curious question:

Question 5.9 Are there any fiber sum indecomposable Lefschetz fibrations on minimal symplectic 4–manifolds?

6 Rational blow-downs and nonisomorphic Lefschetz fibrations

Two Lefschetz pencils/fibrations (X, f_i) are called *isomorphic* if there are orientation-preserving self-diffeomorphisms of the 4–manifold and the base S^2 which make the two commute. In particular, the fiber genera, as well as the number of base points in the case of pencils, should match. This translates to the two associated monodromy factorizations to be equivalent up to global conjugations and Hurwitz moves. Park and Yun used the latter approach to show that there are pairs of odd genus $g \geq 5$ inequivalent Lefschetz *fibrations* on certain knot-surgered elliptic surfaces [44] of Fintushel and Stern. These are all fiber sum *decomposable*, in particular do not blow down to pencils, and are easily seen to be equivalent via partial conjugations — conjugations applied to a subword of monodromy factorizations. (Another isolated example on $T^2 \times \Sigma_2 \# 9\overline{\mathbb{CP}}{}^2$, as we learn from the same authors, is given by Smith in his thesis.) More recently, the first author constructed arbitrary number of nonisomorphic Lefschetz pencils/fibrations on blow-ups of any symplectic 4–manifold which is not a rational or ruled surface [6].

Here we will introduce a new approach and construct the first examples of nonisomorphic Lefschetz *pencils* with fiber genus as low as 3. To do so, we will employ rational blow-down operations that correspond to certain monodromy substitutions, which we briefly review below. The key ingredient in our arguments will be an observation originally due to Gompf which we review in the next subsection.

6A A mirror rational blow-down operation

The *rational blow-down* operation introduced by Fintushel–Stern [17] is defined as follows: Let $p \geq 2$ and C_p be the smooth 4–manifold obtained by plumbing disk bundles over the 2–sphere according to the linear diagram in Figure 16, where each vertex u_i of the linear diagram represents a disk bundle over 2–sphere with the indicated Euler number.

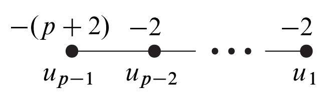

Figure 16: A plumbing diagram representing C_p

The boundary of C_p is the lens space $L(p^2, 1-p)$, which also bounds a rational ball B_p with $\pi_1(B_p) = \mathbb{Z}/p\mathbb{Z}$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. Whenever C_p is embedded in a 4–manifold X, we can thus produce a new 4–manifold

$$X_p = (X \setminus C_p) \cup B_p.$$

Algebraic topological invariants of X_p are easily derived from those of X:

$$b^+(X_p) = b^+(X), \quad \mathrm{e}(X_p) = \mathrm{e}(X) - (p-1), \quad \sigma(X_p) = \sigma(X) + (p-1),$$
$$c_1{}^2(X_p) = c_1{}^2(X) + (p-1).$$

If X and $X \setminus C_p$ are simply connected, then so is X_p.

A *monodromy substitution* is the trading of a subword in a given monodromy factorization by positive Dehn twists with another subword of the same type. We will employ this operation more generally to monodromy factorizations capturing multisections as well; ie not only positive Dehn twists but also positive arc twists for multisections will be allowed in the subwords. As shown by Endo and Gurtas, a particular substitution in a monodromy factorization of a Lefschetz fibration which trades 4 Dehn twists with 3 Dehn twists by the lantern relation on a 2–sphere with 4 holes corresponds to produce a new Lefschetz fibration corresponds to the simplest possible rational blow-down operation for the underlying 4–manifolds: blowing down a C_2 configuration which is a (-4)–sphere formed by the 4 vanishing cycles on the fiber. Notably, the two fibrations can be supported by symplectic structures in a natural way [23].

We will be interested in a special configuration, which can be blown down in 3 different ways: two disjoint (-4)–spheres V_1, V_2 and a (-1)–sphere S intersecting each of the (-4)–sphere positively at one point. One can then blow down S or rationally blow down V_i. As it will become apparent in the proof of Theorem 6.2, we will be interested in monodromy substitutions that correspond to this "mirror" blow-downs of V_1 and V_2 in the presence of a (-1)–sphere section intersecting both. The essential ingredient here is a result of Gompf [24] (also see [12]) quoted below, for which we will sketch a handlebody proof. Thus the 4–manifolds that result from rationally blowing down V_1 or V_2 and S are diffeomorphic, yielding diffeomorphic 4–manifolds for all 3 blow-downs.

Proposition 6.1 [24, Lemma 5.1] *Let V_i be disjoint embedded (-4)–spheres, each intersecting a (-1)–sphere S in X once, let X_i denote the 4–manifold obtained by a rational blow-down of V_i in X, $i = 1, 2$, and X_0 be the one obtained by the blow-down of S. Then the 4–manifolds X_i, $i = 0, 1, 2$, are all diffeomorphic.*

Proof Let X_0 denote the manifold obtained by blowing down S. It is sufficient to prove that each of X_1 and X_2 is diffeomorphic to X_0. We will verify the blow-downs along V_i and S give rise to diffeomorphic 4–manifolds using handlebody diagrams.

Rational blowdown is equivalent to removing a tubular neighborhood of a (-4)–sphere and pasting $\mathbb{CP}^2 \setminus \nu C$, where C is a nonsingular quadratic curve and νC is its tubular neighborhood. Following a procedure for drawing a diagram of the surface complement in [25, Section 6.2], we can obtain a diagram of $\mathbb{CP}^2 \setminus \nu C$ which is shown in Figure 17 (left) (the configuration of C in the diagram of $\mathbb{CP}^2$ was discussed in [1]). By turning the handlebody corresponding to the diagram up side down, we obtain the diagram in Figure 17 (middle). The diagram in Figure 17 (right) can be obtained by blowdown.

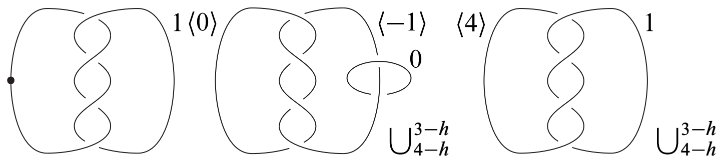

Figure 17: Diagrams of $\mathbb{CP}^2 \setminus \nu C$

Thus attaching $\mathbb{CP}^2 \setminus \nu C$ to the boundary of a tubular neighborhood of a (-4)–sphere is equivalent to attaching handles as shown in Figure 17 (right). We eventually obtain the diagram of the manifold obtained by rationally blowing down a regular neighborhood of $V_i \cup S$ along V_i as shown in Figure 18 (left). (Note that the 0–framed handle in this diagram coincides with the (-1)–sphere S). On the other hand, Figure 18 (right)

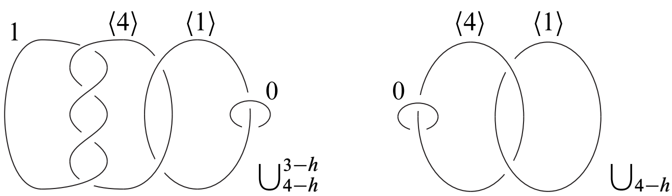

Figure 18: The manifolds obtained by blowing down a neighborhood of $V_i \cup S$

shows a diagram of the manifold resulting from blowing down a regular neighborhood of $V_i \cup S$ along S. These two manifolds are diffeomorphic relative to the boundaries as shown in Figure 19. The diagram in Figure 19 (left) can be obtained by sliding the 0–framed 2–handle in Figure 18 (left) to the knot with the label $\langle 1\rangle$. Sliding the resulting (-1)–framed handle to that with framing 1 yields the diagram in Figure 19 (middle). The resulting 0–framed handle becomes a meridian of the knot with the label $\langle 4\rangle$ after sliding the 1–framed handle to the knot with the label $\langle 4\rangle$, especially we can obtain the diagram in Figure 19 (right). Lastly, sliding the 1–framed handle to the knot with the label $\langle 1\rangle$ and removing the canceling pair yields the diagram in Figure 18 (right). $\square$

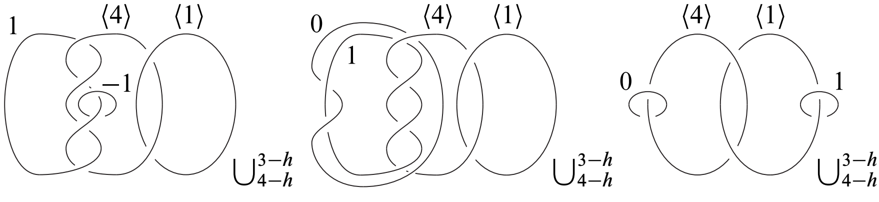

Figure 19: A sequence of handleslides

6B A new construction of nonisomorphic Lefschetz fibrations

Here we prove the main theorem of this section:

Theorem 6.2 *There are pairs of genus-g relatively minimal Lefschetz pencils (X, f_i), $i = 1, 2$, which are nonisomorphic, where g can be taken as small as 3, and arbitrarily large.*

Proof Let $(X_{(2,1,1)}, f_{(2,1,1)})$ be the genus-3 symplectic Calabi–Yau Lefschetz fibration constructed in Section 5B by a monodromy substitution along (a) in (13). Alternatively, we can construct another genus-3 symplectic Calabi–Yau Lefschetz fibration $(X_{(2',1,1)}, f_{(2',1,1)})$ by a monodromy substitution along (c). Note that the exceptional section S_4 descends to both fibrations. The configurations of the lantern curves for these are as in Figure 20. Since the section S_2 intersects both (-4)–

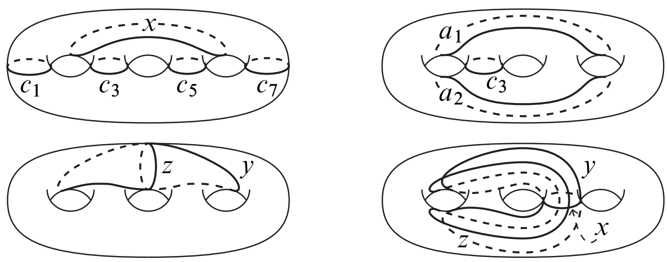

Figure 20: The lantern curves appearing in the substitution at parts (a) (left) and (c) (right)

spheres V_1 and V_2 corresponding to the monodromy substitutions along (a) and (c), respectively, this provides us the configuration of surfaces in $X_{(1,1,1,1)}$ studied in the previous paragraph. As we observed, the 4–manifolds $X_{(2,1,1)}$ and $X_{(2',1,1)}$ are diffeomorphic. Since only one of them contains a separating vanishing cycle, they are nonisomorphic genus-3 Lefschetz pencils on a once blown-up symplectic Calabi–Yau K3 surface.

An infinite family of pairs of nonisomorphic Lefschetz pencils of arbitrarily high fiber genera can be produced by applying the degree doubling construction (as discussed in detail in the next section) simultaneously to both (X, f_i), at each step changing the genus and number of base points by $\widetilde{g} = 2g + m - 1$ and $\widetilde{m} = 4m$. The separating vanishing cycle in (X, f_2), obtained from $(X_{(2',1,1)}, f_{(2',1,1)})$ by blowing down the (-1)–sphere section S_4 splits the fiber to genus-1 and genus-2 components, only one of which is hit by S_4. As seen by the explicit monodromy factorization obtained by Auroux and Katzarkov [3], this separating vanishing cycle will then contribute to a separating vanishing cycle of the resulting pencil. On the other hand, doubling produces no new separating vanishing cycles. Therefore, iterated doubles of (X, f_i), for $i = 1, 2$ at each step will remain to be nonisomorphic, as one will contain a reducible fiber and the other one will not. □

Remark 6.3 (an interlude on smallest possible fiber genera of nonisomorphic Lefschetz fibrations/pencils) Here we would like to make a few remarks on how small the genus of nonisomorphic Lefschetz pencils/fibrations can be, discussed by varying the additional features of such examples that we know of so far.

First of all, if we are after obtaining nonisomorphic Lefschetz fibrations, for $g \leq 1$ this phenomenon does not appear. If we ask for nonisomorphic examples with differing number of reducible fibers as we have in Theorem 6.2, our $g = 3$ fibrations can be seen to be the smallest possible genera representatives of this kind. For, any genus-2 Lefschetz fibration is hyperelliptic, the signature formula of Endo shows that the signature contributions of reducible and irreducible singular fibers are different, and thus their total spaces cannot have the same total space.

As for having nonisomorphic Lefschetz fibrations in general (so both can have the same number of reducible fibers, or even none), the smallest possible genus among Park and Yun's examples of nonisomorphic Lefschetz fibrations is $g = 5$ for a pair of knot-surgered $E(2n)$ with $n = 1$ where the knots are 2–bridge knots of genus 2. This is now improved to $g = 3$ by our examples, leaving out $g = 2$ as the smallest possible genus.

We speculate that $g = 2$ examples with transitive monodromy and without reducible fibers do not exist. Indeed, it is known that any such a Lefschetz fibration is isomorphic to a fiber-sum whose components are either of the two basic Lefschetz fibrations. Furthermore, the number of the two basic fibrations in the fiber-sum decomposition is uniquely determined by the number of Lefschetz singularities in the original fibration (see [48, Corollary 0.2] for details), in particular the isomorphism class of such a fibration is uniquely determined by the number of Lefschetz singularities. This rigidity then implies that the only counterexamples can come from fibrations with reducible fibers or intransitive monodromies. This concludes our interlude.

The examples of nonisomorphic Lefschetz fibrations with topologically isotopic fibers we have constructed here are all on symplectic 4–manifolds of $\kappa = 0$. We hope to address the same question for the remaining Kodaira dimensions in a future work.

On the other hand, given higher genus fibrations with many exceptional sections and several (braiding) lantern factorizations embedded in them (which are hard to produce!), one can turn our strategy of the proof above into a recipe to produce examples of *arbitrarily many* pairwise nonisomorphic Lefschetz fibrations with pairwise ambiently homeomorphic fibers.

Nevertheless, the following question remains open:

Question 6.4 [44] Are there infinitely many nonisomorphic Lefschetz fibrations of the same genera on any symplectic 4–manifold?

7 Exotic Lefschetz pencils and exotic embeddings of surfaces

A pair of 4–manifolds X_i, $i=0,1$, that are pairwise homeomorphic but not diffeomorphic, is commonly called an *exotic pair* of 4–manifolds. Pairs that are both symplectic are particularly interesting in regards to the symplectic botany problem, which asks about the diversity of symplectic structures supported in the same homeomorphism class. Similarly, a pair of Lefschetz pencils/fibrations (X_i, f_i) is called exotic if X_i constitute an exotic pair of symplectic 4–manifolds and f_i have the same fiber genus and the same number of base points. Up to date, the only known examples are some particular families of exotic Lefschetz *fibrations*: for $X_i = E(n)_{K_i}$ knot-surgered elliptic surfaces, it was shown by Fintushel and Stern that for K_i fibered knots with the same genus g but different Alexander polynomials, one obtains genus-$(2g+n-1)$ exotic Lefschetz fibrations (X_i, f_i) — which, however, do not yield pencils. (These are discussed in detail in our Appendix.) However, although every symplectic 4–manifold admits a Lefschetz *pencil* by Donaldson, there are no known exotic pairs of Lefschetz pencils up to date. Our goal in this final section is to present the first examples of this kind.

Theorem 7.1 *There are genus-*3 *exotic Lefschetz pencils* (X_i, f_i), $i=0,1$, *with symplectic Kodaira dimension* $\kappa(X_i)=i$, *where* X_i *are homeomorphic to* $\mathrm{K3}\,\#\,\overline{\mathbb{CP}}{}^2$. *Moreover, there are similar examples with arbitrarily high genus and the same topology for the singular fibers on higher blow-ups of homotopy* $\mathrm{K3}\,\#\,\overline{\mathbb{CP}}{}^2$*s.*

Note that since any symplectic 4–manifold X with $\kappa=-\infty$ is *diffeomorphic* to a rational or ruled surface, and any minimal X with $\kappa=2$ has $c_1^2>0$, this is the best possible result one can obtain for varying the Kodaira dimensions within the same homeomorphism class.

Lastly, in Section 7B, we will show that similar techniques can be employed to produce exotic embeddings of surfaces.

7A Constructing pairs of exotic Lefschetz pencils

Recall the monodromy factorization of the symplectic Calabi–Yau Lefschetz fibration $(X_{(1,1,1,1)}, f_{(1,1,1,1)})$ we have produced in (13). A monodromy substitution at (a) (resp. (b)) amounts to rationally blowing down a (-4)–sphere split off from the regular fiber by the 4 vanishing cycles in (a) (resp. (b)), for which there are two possibilities: one can blow down the (-4)–sphere intersecting the sections S_1 and S_2 or the one intersecting S_3 and S_4. (This choice was implicitly made when producing our earlier examples by each time indicating which sections we were braiding.) Thus, we can blow down two disjoint (-4)–spheres in $(X_{(1,1,1,1)}, f_{(1,1,1,1)})$ *both* intersecting S_1 and S_2 by monodromy substitutions along (a) and (b) to produce a new Lefschetz fibration $(X_{([2],1,1)}, f_{([2],1,1)})$. The Euler characteristic and the signature of $X_{([2],1,1)}$ is easily calculated from those of $X_{(1,1,1,1)}$ under rational and regular blow downs, whereas its fundamental group can be seen to be trivial for instance by observing that the collection of unused vanishing cycles in the above substitutions already contain the loops $c_1, c_2, c_5, c_6, c_7, b_1$ which fully generate the fundamental group of the fiber. We conclude that $X_{([2],1,1)}$ is *homeomorphic* to $\mathrm{K3} \# 2\overline{\mathbb{CP}}^2$.

Using Theorem 3.6 we see that S_1 and S_2 together turn into a self-intersection 0 torus bisection of $f_{([2],1,1)}$, whereas the (-1)–sphere sections S_3 and S_4 descend to sections of the new fibration as well. Since the minimal model of an SCY with $b^+ = 3$ should have the same rational homology as the K3 surface, by Theorem 4.1 this cannot be a symplectic Calabi–Yau Lefschetz fibration. Blowing down S_3, we get a Lefschetz pencil (X_1, f_1). Calculating $c_1^2 = 2\mathrm{e} + 3\sigma = 0$ on the minimal model of X_1 we note that $\kappa(X_1) = 1$.

On the other hand, let (X_0, f_0) be the pencil obtained by blowing down the SCY Lefschetz fibration $(X_{(3,1)}, f_{(3,1)})$ we constructed earlier along the only (-1)–sphere section, so $\kappa(X_0) = 0$. Thus (X_i, f_i) for $i = 0, 1$ is a pair of genus-3 Lefschetz pencils promised in Theorem 7.1.

The only caveat in our construction of these exotic Lefschetz pencils is that (X_1, f_1) has no reducible fibers, whereas (X_0, f_0) has one reducible fiber. Below, we will show that, if we compromise on the smallness of the pencil genus, we can also produce exotic Lefschetz pencils both having only irreducible fibers.

In the arguments to follow, we will need a variant of the degree doubling procedure [50; 3], introduced in [6]. Degree doubling construction produces a new genus-$\widetilde{g}$ symplectic Lefschetz pencil $(X, \omega, \widetilde{f})$ with $\widetilde{m}$ base points from a given genus-g symplectic Lefschetz pencil (X, ω, f) with m base points, where $\widetilde{g} = 2g + m - 1$ and $\widetilde{n} = 4m$. It is described for Donaldson's pencils in Smith's work [50], for pencils obtained via branched coverings of $\mathbb{CP}^2$ by Auroux and Katzarkov in [3], and for arbitrary topological pencils by the first author in [6] based on [50; 3] — which is the one that suits to pencils constructed via monodromy factorizations. We define a *partial double along $m \geq k \geq 1$ points* as the Lefschetz pencil one gets by *first* symplectically blowing up (X, ω, f) at $m-k$ points and then taking the double of the resulting pencil on $(\widetilde{X}, \widetilde{\omega}, \widetilde{f})$, where $\widetilde{X} = X \# (m-k)\overline{\mathbb{CP}}^2$. Moreover, if $(\widetilde{X}, \widetilde{\omega}, \widetilde{f})$ is obtained from (X, ω, f) by a sequence of partial doublings, where in the very last step we in addition blow up all the base points, then both the smooth 4–manifold $\widetilde{X}$ and the genus $\widetilde{g}$ of $\widetilde{f}$ are *uniquely* determined by the initial pencil (X, ω, f) and the ordered tuple of integers $k_1, \dots, k_d$, for each partial doubling along k_j points. We can then blow down the (-1)–sphere sections to produce a pencil.

Following [6], we denote the latter sequence by $D = [k_1, \dots, k_d]$, which is only subject to the condition $4k_j \geq k_{j+1} \geq 1$ for all j. The next lemma is a simple variation of [6, Lemma 3.1] proved in an identical way:

Lemma 7.2 *Let f and f' be genus-g_0 and genus-g_0' Lefschetz pencils on homeomorphic 4–manifolds X and X' with m_0 and m_0' base points, respectively. Two partial doubling sequences*

$$D = [k_1, \dots, k_d] \quad \text{and} \quad D' = [k_1', \dots, k_{d'}']$$

applied to f and f', respectively, result in Lefschetz fibrations on $X \# M\overline{\mathbb{CP}}^2$ and $X' \# M\overline{\mathbb{CP}}^2$ with the same fiber genus g if and only if

$$M = m_0 + 3\sum_{i=1}^{d} k_i = m_0' + 3\sum_{i=1}^{d'} k_i',$$

and

$$g = 2^d g_0 + \sum_{i=1}^{d} 2^{d-i}(k_i - 1) = 2^{d'} g_0' + \sum_{i=1}^{d'} 2^{d'-i}(k_i' - 1).$$

Now let (X, f) be a genus-8 Lefschetz pencil on the K3 surface with 14 base points [50], so $\kappa(X) = 0$. We can then apply Lemma 7.2 to (X, f) and $(X', f') = (X_1, f_1)$ above, using the (very short!) partial doubling sequences

$$D = [1] \quad \text{and} \quad D' = [2, 3]$$

to produce a pair of genus-g Lefschetz fibrations on the topological 4–manifold $\mathrm{K3} \# M\overline{\mathbb{CP}}^2$ with $g = 16$ and $M = 17$. Blowing down the same number of (-1)–sphere sections (and at most 4 of them, as the doubling sequence D results in 4 base points) in both we obtain the desired exotic pair of Lefschetz pencils (X_0, f_0) and

(X_1, f_1) (overriding our earlier picks of (X_i, f_i)) where X_0 now denotes (a blow-up) of $\mathrm{K3}\,\#\,13\overline{\mathbb{CP}}^2$ and X_1 is homeomorphic to it.

Applying further simultaneous doublings to any one of the exotic Lefschetz pencils (X_i, f_i) we produced above give us exotic pairs of pencils of arbitrarily high genera. This completes the proof of Theorem 7.1.

Remark 7.3 We can obtain, after one more substitution along part (c) in the monodromy of $(X_{([2],1,1)}, f_{([2],1,1)})$, another *fibration* $(X_1, f_1) = (X_{([3],1)}, f_{([3],1)})$. Letting $(X_0, f_0) = (X_{(4)}, f_{(4)})$ be the SCY Lefschetz *fibration* produced in Section 5, we then obtain a pair of exotic genus-3 Lefschetz fibrations (X_i, f_i) with $\kappa(X_i) = i$, both of which having one reducible fiber — and thus, with exact same topology.

7B Exotic embeddings of symplectic surfaces

We call two surfaces $F_i \subset X$, $i = 1, 2$, *exotically embedded* in X if there exists an ambient homeomorphism of X taking F_1 to F_2 but there exists no such diffeomorphism. Such symplectic surfaces are harder to produce: for instance, the work of Siebert and Tian shows that up to isotopy there is a unique symplectic surface in the homology class of an algebraic curve of degree ≤ 17 in $\mathbb{CP}^2$ [48]. In contrast, Finashin [16], and H-J Kim [29] (also see [30]) constructed knotted surfaces in $\mathbb{CP}^2$ that are not isotopic to algebraic curves, which can be seen to be not symplectic. The latter rely on a construction method of Fintushel and Stern [18], called (twisted) rim-surgery, and up to date this has been the only way of producing exotic embeddings of surfaces — and curiously, only producing symplectic tori when asked to lie in the same homology class. The purpose of this section is to present a new way of constructing exotically embedded orientable surfaces:

Theorem 7.4 *There is a pair of genus-3 surfaces F_i exotically embedded in a blow-up of a symplectic Calabi–Yau K3 surface such that F_i is symplectic with respect to deformation equivalent symplectic forms ω_i on X, for $i = 1, 2$.*

Proof In Section 5B, we constructed a Lefschetz fibration $(X_{(2,1,1)}, f_{(2,1,1)})$ by a braiding lantern substitution at part (a) of (13). We can now apply another braiding lantern substitution at part (c), which yields to the genus-3 Lefschetz fibration $(X_{(2,2)}, f_{(2,2)})$ with two (-1)–sphere bisections S_{12} and S_{34} we obtained earlier, or at part (b), which yields to a new genus-3 Lefschetz fibration $(X_{(3,1)}, f_{(3,1)})$ with (-1)–sphere 3–section S_{123} and a (-1)–section S_4.

Since $X_{(2,2)}$ and $X_{(3,1)}$ are obtained from $X_{(2,1,1)}$ by rational blow-downs along (-4)–spheres V_1 and V_2 (prescribed by the Lantern curves in parts (c) and (b)) both intersecting the exceptional sphere S_3 at one point, they are diffeomorphic by Proposition 6.1. Let F_1 and F_2 be regular fibers of $f_{(2,2)}$ and $f_{(3,1)}$, respectively.

There exists a pairwise homeomorphism between $(X_{(2,2)}, F_1)$ and $(X_{(3,1)}, F_2)$: To see this, we first observe that the vanishing cycles in the complement of $X_{(2,2)} \setminus F_1$, and respectively of $X_{(3,1)} \setminus F_2$, allows us to easily compute

$$\pi_1(X_{(2,2)} \setminus F_1) = 1 = \pi_1(X_{(3,1)} \setminus F_2).$$

So both homology classes $[F_i]$ are indivisible. Moreover, $F_1 \cdot S_{12} = 2$ but $S_{12}^2 = -1$, whereas $f_{(3,1)}$ has a reducible fiber component R for F_2; hence $F_1 \cdot R = 0$, but $R^2 = -1$. So both $[F_i]$ are not characteristic. Since $b_2 - \sigma \geq 4$ and $\pi_1 = 1$ for $X_{(2,2)} \cong X_{(3,1)}$, by Wall's theorem on automorphisms of the intersection form and Freedman's topological h-cobordism theorem (see for example [47, pages 152–153]), we get a homeomorphism between $X_{(2,2)}$ and $X_{(3,1)}$ matching the homology classes of F_1 and F_2. Finally, viewing the two surfaces in the same manifold under this homeomorphism, we can invoke [52] to find a topological isotopy between them, which yields the desired homeomorphism between the pairs $(X_{(2,2)}, F_1)$ and $(X_{(3,1)}, F_2)$.

On the other hand, since $X_{(2,2)}$ and $X_{(3,1)}$ are symplectic Calabi–Yaus, and thus not rational or ruled, by Li's work in [33], any diffeomorphism between them maps exceptional classes to exceptional classes in the same homology classes. However, in $X_{(2,2)}$, the two exceptional classes S_{12} and S_{34} intersect F_1 both twice, whereas in $X_{(3,1)}$ we have two exceptional classes S_{123} and S_4 intersecting F_2 thrice and once. Hence, there is no pairwise diffeomorphism between $(X_{(2,2)}, F_1)$ and $(X_{(3,1)}, F_2)$.

Now if we let $X = X_{(3,1)}$ and identify F_1 with its image under the diffeomorphism between $X_{(2,2)}$ and $X_{(3,1)}$, we conclude that F_1, F_2 is a pair of exotically embedded surfaces in X. Lastly, to prove our additional claim on the existence of deformation equivalent symplectic forms ω_i on X with respect to which F_i are symplectic, we first perturb the Lefschetz fibration $f_{(2,1,1)}$ so that each quadruple of vanishing cycles appearing in part (c) and (b) of the monodromy factorization lie on the same singular fiber, forming a reducible (-4)–sphere fiber component V_i. We can then equip $(X_{(2,1,1)}, f_{(2,1,1)})$ with a compatible symplectic form with respect to which both V_1 and V_2, and the section S_3, are symplectic. Now, by the work of McDuff and Symington, who showed that Gompf's diffeomorphism we employed here can be interpreted as a symplectic 4–sum operation, the symplectic 4–manifolds we produce by rational blow-downs of V_i are symplectic deformation equivalent [41], with F_i symplectic surfaces in them. We thus obtain the desired symplectic forms on X by pulling-back the latter form on $X_{(2,2)}$. $\square$

Appendix: Seiberg–Witten basic classes of homotopy K3 surfaces via mapping class group factorizations

A1 Seiberg–Witten basic classes of symplectic 4–manifolds

Let X be a symplectic 4–manifold with $b^+(X) > 1$. We further assume that it has an integral symplectic form ω, which can always be achieved by replacing a given form with a multiple of a rational symplectic form approximating it. By Taubes, for a generic almost complex structure J on (X, ω), any Seiberg–Witten basic class $\beta \in H_2(X; \mathbb{Z})$ can be represented by a sum of J–holomorphic curves C_i in X [53; 54]. Moreover, each component of the representative of $\beta = \Sigma_i m_i [C_i]$ is an embedded smooth curve unless it is a torus of self-intersection zero (in which case the image of the curve is still smoothly embedded, but the parametrization is a multiple cover) or a sphere of negative self-intersection. Since J is ω tamed, each C_i is a symplectic surface in (X, ω).

Since the number of basic classes of a 4–manifold is finite, so is the collection of the symplectic surfaces C_i, sums of which represent the basic classes in (X, ω). As noted by Donaldson and Smith [11, Proposition 2.9] replacing ω with a sufficiently high multiple $k\omega$, we can then assume that there exists a symplectic Lefschetz pencil on X for which all C_i are multisections ("standard surfaces" in the language of [11]). By the blow-up formula for Seiberg–Witten classes, we conclude that after passing to a blow-up of X we get a symplectic Lefschetz fibration $f\colon \widetilde{X} \to S^2$ where all basic classes are represented by a collection of symplectic surfaces C_i and the exceptional spheres E_j. Hence, each Seiberg–Witten basic class of $\widetilde{X}$ is represented by a multisection (possibly with several components).

To sum up, combining the works of Taubes and Donaldson, after passing to a blow-up $\widetilde{X}$, one can represent all Seiberg–Witten classes of a symplectic 4–manifold X as multisections with respect to a Lefschetz fibration. We shall note that this is merely an existence result, as the construction of such a Lefschetz fibration is not explicit.

A2 Sample calculation: basic classes of knot-surgered elliptic surfaces

We will now present explicit monodromy factorizations in the framed mapping class group capturing all basic classes of knot-surgered elliptic surfaces as multisections of certain Lefschetz fibrations on them.

Here is a quick review of the knot-surgery construction: Let X be a smooth 4–manifold and $T \subset X$ an embedded torus with self-intersection 0. For a fibered knot $K \subset S^3$, let M_K denote the 3–manifold obtained by 0–surgery along K from S^3, then M_K admits a natural fibration over S^1, where fibers are capped of Seifert surfaces. In turn, $S^1 \times M_K$ is a genus-g symplectic surface bundle over T^2, with g the Seifert genus of K. For μ_K the meridian of K in S^3, we obtain a torus $S^1 \times \mu_K$ as a symplectic section of this bundle. We then define a *knot-surgered* 4*–manifold* X_K as the generalized fiber sum $X_K = X \setminus \nu T \cup_{S^1 \times \mu_K} S^1 \times M_K$, which can be performed symplectically. (When K is not fibered, the same construction — for M_K admitting an S^1–valued Morse function this time — results in a new 4–manifold which is not necessarily symplectic.) Fintushel and Stern [19] introduced this operation and proved that a Laurent polynomial associated with the Seiberg–Witten invariant of X_K is the product of that of X and the symmetrized Alexander polynomial of the knot K for homologically essential T in X. For $X = E(n)$, all basic classes arise as multiples of the image of the elliptic fiber T of X in X_K. Moreover, assuming K is a fibered knot with Seifert genus-g, the knot-surgery 4–manifold $E(n)_K$ admits a genus-$(2g+n-1)$ Lefschetz fibration $(E(n)_K, f_{n,K})$ [20]. It is easy to see that T becomes a bisection (ie a 2–section) of this fibration. Capturing all basic classes of X_K in this case therefore comes to identifying disjoint copies of T via a monodromy factorization of an appropriate lift of $f_{n,K}$ to the framed mapping class group.

Let $A_1, \dots, A_{2n-2}, B_1, \dots, B_{2g+1}, C_1, C_2$ be simple closed curves in Σ_{2g+n-1} as described in Figure 21. We remove two disks D_1 and D_2 from Σ_{2g+n-1} as in Figure 21 and take points u_1 and u_2 on each boundary component of $\Sigma^2_{2g+n-1} = \Sigma_{2g+n-1} \setminus (D_1 \sqcup D_2)$. Let K be a fibered knot with genus-g and $\varphi_K \in \mathrm{Mod}(\Sigma_g)$ a monodromy of K. We decompose Σ_{2g+n-1} into three pieces: the upper Σ_g, the

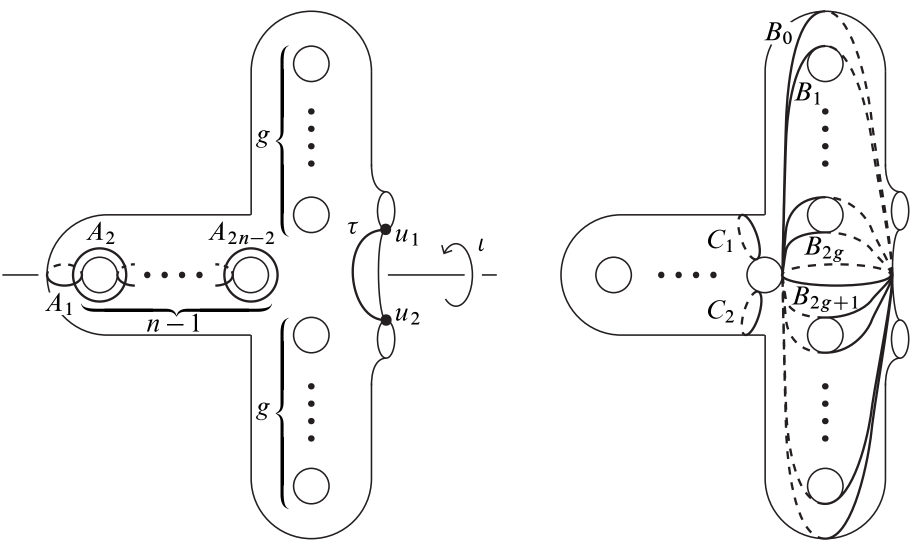

Figure 21: Simple closed curves and a path in Σ^2_{2g+n-1}

lower Σ_g and the central Σ_{n-1} in Figure 21, so that both of the disks D_1 and D_2 are contained in Σ_{n-1}. Let Φ_K be an element $\mathrm{Mod}(\Sigma_{2g+n-1})$ defined as follows:

$$\Phi_K = \varphi_K \,\#\, \mathrm{id} \,\#\, \mathrm{id}\colon \Sigma_g \,\#\, \Sigma_{n-1} \,\#\, \Sigma_g \to \Sigma_g \,\#\, \Sigma_{n-1} \,\#\, \Sigma_g.$$

The genus-$(2g+n-1)$ Lefschetz fibration $f_{n,K}\colon E(n)_K \to S^2$ mentioned above has the following monodromy factorization (see [20]):

$$\eta_{n,g}\eta_{n,g}\Phi_K(\eta_{n,g})\Phi_K(\eta_{n,g}) = 1,$$

where $\eta_{n,g}$ is equal to $t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} t_{B_0} \cdots t_{B_{2g+1}}$ and $\Phi_K(\eta_{n,g})$ is a factorization obtained from $\eta_{n,g}$ by substituting A_i and B_j in $\eta_{n,g}$ for $\Phi_K(A_i)$ and $\Phi_K(B_j)$, respectively.

Proposition A.1 *The following equality holds in* $\mathrm{Mod}(\Sigma_{2g+n-1}^2; \{u_1, u_2\})$*:*

$$t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} t_{B_0} \cdots t_{B_{2g+1}} = t_{\delta_1} t_{\delta_2} \widetilde{\tau}^{-1} \iota,$$

where δ_i is a simple closed curve in Σ_{2g+n-1}^2 parallel to the boundary component containing u_i, $\widetilde{\tau}$ is a lift of a half twist along a path given in Figure 21 as described in Figure 1 and ι is an involution described on the left side of Figure 21.

Proof We cut the surface Σ_{2g+n-1}^2 along the curves C_1 and C_2 to obtain the surface Σ_{2g}^4. Take points $u_3 \in C_1$ and $u_4 \in C_2$. We denote the set $\{u_1, u_2, u_3, u_4\}$ by U and the fixed points of ι in Σ_{2g}^4 by v_1 and v_2. Since the simple closed curve B_i is preserved by ι, we can regard the Dehn twist t_{B_i} as an element in $\pi_0(C(\Sigma_{2g}^4, U; \iota))$ (for the definition of $C(\Sigma_{2g}^4, U; \iota)$, see Section 3A). The quotient space Σ_{2g}^4/ι is homeomorphic to Σ_g^2. The quotient map $/\iota\colon \Sigma_{2g}^4 \to \Sigma_g^2$ induces the following homomorphism:

$$\iota_*\colon \pi_0(C(\Sigma_{2g}^4, U; \iota)) \to \mathrm{Mod}(\Sigma_g^2; \widehat{U}, \{\widehat{v}_1, \widehat{v}_2\}),$$

where $\widehat{U}$ and $\widehat{v}_i$ are the images of U and v_i, respectively, under $/\iota$; see Figure 22. By Lemma 3.1 the kernel of ι_* is generated by the isotopy class of ι. Let $\widehat{B}_i$ be the image of the simple closed curve B_i under $/\iota$. Since the image $\iota_*(t_{B_i})$ is the half twist $\tau_{\widehat{B}_i}$, the following holds in $\mathrm{Mod}(\Sigma_g^2; \widehat{U}, \{v_1, v_2\})$:

$$\iota_*(t_{B_0} \cdots t_{B_{2g+1}}) = \tau_{\widehat{B}_0} \cdots \tau_{\widehat{B}_{2g+1}}. \tag{14}$$

Let κ be an involution of Σ_g^2 as in the middle of Figure 22 and $w_0, \dots, w_{2g+1} \in \Sigma_g^2$ fixed points of κ. Regard the half twist $\tau_{\widehat{B}_i}$ as an element in $\pi_0(C(\Sigma_g^2, \widehat{U}; \kappa), \mathrm{id})$. The quotient space Σ_g^2/κ is homeomorphic to Σ_0^1. Thus the quotient map $/\kappa\colon \Sigma_g^2 \to \Sigma_0^1$ induces the following homomorphism:

$$\kappa_*\colon \pi_0(C(\Sigma_g^2, \widehat{U}; \kappa)) \to \mathrm{Mod}_{\partial \Sigma_0^1}(\Sigma_0^1; \{\widehat{w}_0, \dots, \widehat{w}_{2g+1}\}),$$

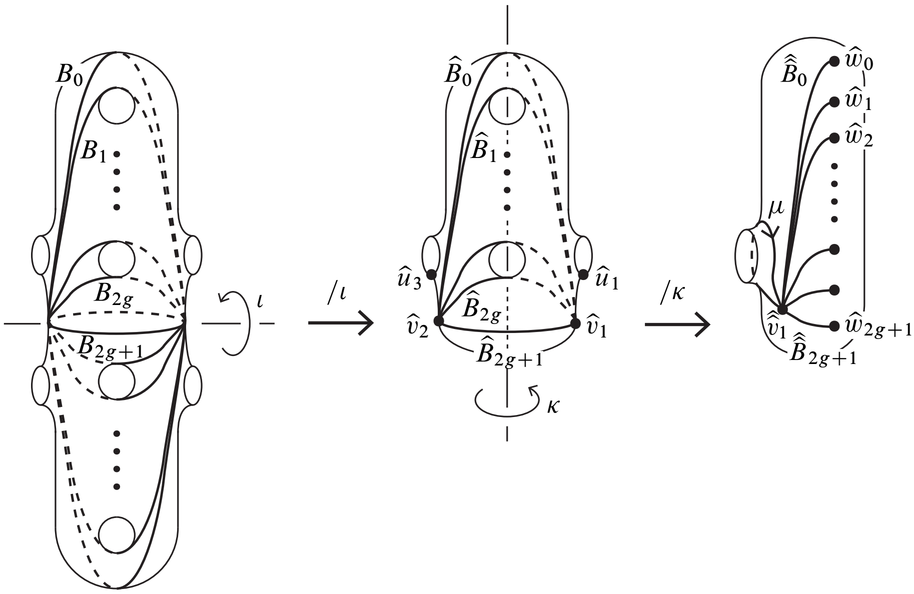

Figure 22: The involutions ι and κ

where $\widehat{w}_i$ is the image $/\kappa(w_i)$. By Lemma 3.1 the kernel of κ_* is generated by the isotopy class of κ. Let $\widehat{\widehat{B}}_i$ be the image of $\widehat{B}_i$ under $/\kappa$ (see Figure 22). We take an oriented loop $\beta_i \subset \Sigma_0^1$ based at $\widehat{v}_1 = /\kappa(v_1)$ by connecting p_0 with a small circle around $\widehat{w}_i$ oriented counterclockwise using $\widehat{\widehat{B}}_i$. The following equation holds in $\mathrm{Mod}_{\partial\Sigma_0^1}(\Sigma_0^1; \{\widehat{w}_0, \dots, \widehat{w}_{2g+1}\})$:

$$\kappa_*(\tau_{\widehat{B}_0} \cdots \tau_{\widehat{B}_{2g+1}}) = \mathrm{Push}(\beta_0) \cdots \mathrm{Push}(\beta_{2g+1}) \tag{15}$$
$$= \mathrm{Push}(\mu),$$

where μ is an oriented based loop described in Figure 22. Combining the equations (14) and (15), we obtain the following relation in $\mathrm{Mod}(\Sigma_{2g}^4; U)$:

$$t_{B_0} \cdots t_{B_{2g+1}} = \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{C_1} t_{C_2} t_{\delta_1} t_{\delta_2} \iota,$$

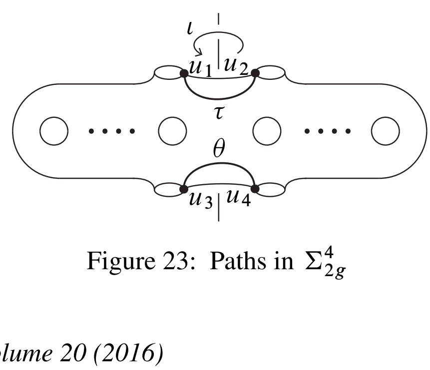

Figure 23: Paths in Σ_{2g}^4

where $\theta \subset \Sigma_{2g}^4$ is a path between u_3 and u_4 described in Figure 23. Thus, we calculate the following product:

$$
\begin{aligned}
&t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} t_{B_0} \cdots t_{B_{2g+1}} \\
&= t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{C_1} t_{C_2} t_{\delta_1} t_{\delta_2} [\iota|_{\Sigma_{2g}^4}] \\
&= t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} (t_{A_{2n-3}} \cdots t_{A_1})^{2n-2} \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{\delta_1} t_{\delta_2} [\iota|_{\Sigma_{2g}^4}] \\
&= t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-3}} (t_{A_{2n-2}} \cdots t_{A_1})(t_{A_{2n-3}} \cdots t_{A_1})^{2n-3} \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{\delta_1} t_{\delta_2} [\iota|_{\Sigma_{2g}^4}] \\
&= t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-4}} (t_{A_{2n-2}} \cdots t_{A_1})^2 (t_{A_{2n-3}} \cdots t_{A_1})^{2n-4} \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{\delta_1} t_{\delta_2} [\iota|_{\Sigma_{2g}^4}] \\
&= \cdots \\
&= (t_{A_{2n-2}} \cdots t_{A_1})^{2n-1} \widetilde{\tau}^{-1} \widetilde{\theta}^{-1} t_{\delta_1} t_{\delta_2} [\iota|_{\Sigma_{2g}^4}].
\end{aligned}
$$

It is easy to verify (using the Alexander method, for example) that the product $(t_{A_{2n-2}} \cdots t_{A_1})^{2n-1} \widetilde{\theta}^{-1} [\iota|_{\Sigma_{2g}^4}]$ is equal to ι in $\mathrm{Mod}(\Sigma_{2g+n-1}^2; \{u_1, u_2\})$. Thus, we obtain

$$t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} t_{B_0} \cdots t_{B_{2g+1}} = \iota \widetilde{\tau}^{-1} t_{\delta_1} t_{\delta_2}.$$

This completes the proof of Proposition A.1. □

We take simple closed curves c_1, c_2, c_3 in Σ_0^{2m+1} and points $u_1, \ldots, u_{2m}$ on the boundary of Σ_0^{2m+1} as described in Figure 24. Let τ_i be a radial arc between u_i and u_{m+i} and U' the set $\{u_1, \ldots, u_{2m}\}$. We denote by $\lambda \in \mathrm{Mod}_{\partial\Sigma_0}(\Sigma_0^{2m+1}; U')$ a mapping class represented by a diffeomorphism which is a positive 180–degree rotation inside of νc_3 and preserves the outside of νc_3, where $\partial\Sigma_0$ is the outermost boundary component of Σ_0^{2m+1} in Figure 24.

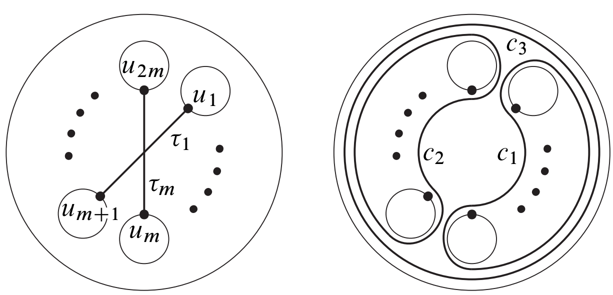

Figure 24: Simple closed curves and arcs in the disk with $2m$ small disks removed

Proposition A.2 *The following equality holds in* $\mathrm{Mod}_{\partial\Sigma_0}(\Sigma_0^{2m+1};U')$*:*

$$t_{c_1}t_{c_2}\lambda^{-1}=t_{\delta_1}\cdots t_{\delta_{2m}}\widetilde{\tau_1}^{-1}\cdots\widetilde{\tau_m}^{-1},$$

where $\widetilde{\tau_i}\in\mathrm{Mod}_{\partial\Sigma_0}(\Sigma_0^{2m+1};U')$ *is a lift of a half twist along* τ_i *as described in Figure 1.*

Proof We denote the involution of Σ_0^{2m+1} given by the 180–degree rotation by $\widetilde{\lambda}$. We regard $\widetilde{\tau_i}$ and $t_{\delta_i}t_{\delta_{m+i}}$ as elements in $\pi_0(C_{\partial\Sigma_0}(\Sigma_0^{2m+1},U';\widetilde{\lambda}))$. The quotient map

$$/\widetilde{\lambda}\colon \Sigma_0^{2m+1}\to\Sigma_0^{2m+1}/\widetilde{\lambda}\cong\Sigma_0^{m+1}$$

induces the following homomorphism:

$$\widetilde{\lambda}_*\colon \pi_0(C_{\partial\Sigma_0}(\Sigma_0^{2m+1},U';\widetilde{\lambda}))\to\mathrm{Mod}_{\partial\Sigma_0}(\Sigma_0^{m+1};\{u_0\},\widehat{U}'),$$

where $u_0\in\Sigma_0^{m+1}$ is the image of the origin of the disk under $/\widetilde{\lambda}$ and $\widehat{U}'=/\lambda(U')$. By Lemma 3.1 the map $\widetilde{\lambda}_*$ is an isomorphism and the image $\widetilde{\lambda}_*(\widetilde{\tau_i}^{-1}t_{\delta_i}t_{\delta_{m+i}})$ is a pushing map along some loop based at u_0. We can easily obtain the equality in Proposition A.2 using these fact together with some equality in $\pi_1(\Sigma_0^{m+1}\setminus\widehat{U}',u_0)$. The details are left to the readers. □

We remove m disks from the disk Σ_0 to obtain $\Sigma_0^{m+1}\subset\Sigma_0$. We obtain the surface Σ_{2g+n-1}^{2m} by attaching two Σ_0^{m+1} to Σ_{2g+n-1}^{2}:

$$\Sigma_{2g+n-1}^{2m}=\Sigma_{2g+n-1}^{2}\cup_{\partial\Sigma_{2g+n-1}^{2}=\partial\Sigma_0\sqcup\partial\Sigma_0}(\Sigma_0^{m+1}\sqcup\Sigma_0^{m+1}).$$

Combining the equalities in Propositions A.1 and A.2, we obtain the following equality in $\mathrm{Mod}(\Sigma_{2g+n-1}^{2m};U')$:

$$\eta_{n,g}\eta_{n,g}\Phi_K(\eta_{n,g})\Phi_K(\eta_{n,g})=t_{\delta_1}^4\cdots t_{\delta_{2m}}^4\widetilde{\tau_1}^{-4}\cdots\widetilde{\tau_m}^{-4}.$$

Eventually, for arbitrarily large m, we can find m disjoint bisections in the Lefschetz fibration $f_{n,K}\colon E(n)_K\to S^2$ each of which has self-intersection 0. Furthermore, each of the bisections has 4 branched points. Thus, all the bisections are tori.

Remark A.3 It is in fact possible to generalize these examples to cover knot-surgered elliptic surfaces which are not symplectic, when the knots used in the construction are not fibered. In this case, following the arguments in [5], we instead obtain a broken Lefschetz fibration on each knot-surgered 4–manifold, where Seiberg–Witten basic classes still appear as a collection of torus bisections.

References

[1] **S Akbulut**, **R Kirby**, *Branched covers of surfaces in 4–manifolds*, Math. Ann. 252 (1979/80) 111–131 MR

[2] **S Akbulut**, **B Ozbagci**, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. 5 (2001) 319–334 MR

[3] **D Auroux**, **L Katzarkov**, *A degree doubling formula for braid monodromies and Lefschetz pencils*, Pure Appl. Math. Q. 4 (2008) 237–318 MR

[4] **S Bauer**, *Almost complex 4–manifolds with vanishing first Chern class*, J. Differential Geom. 79 (2008) 25–32 MR

[5] **R İ Baykur**, *Topology of broken Lefschetz fibrations and near-symplectic four-manifolds*, Pacific J. Math. 240 (2009) 201–230 MR

[6] **R İ Baykur**, *Inequivalent Lefschetz fibrations and surgery equivalence of symplectic 4–manifolds*, J. Symplectic Geom. 14 (2016) 671–686

[7] **R İ Baykur**, *Minimality and fiber sum decompositions of Lefschetz fibrations*, Proc. Amer. Math. Soc. 144 (2016) 2275–2284 MR

[8] **R İ Baykur**, **K Hayano**, *Hurwitz equivalence for Lefschetz fibrations and their multisections*, to appear in Contemp. Math., "Proceedings of the 13th International Workshop on Real and Complex Singularities" (2015) arXiv

[9] **P Bellingeri**, **S Gervais**, *Surface framed braids*, Geom. Dedicata 159 (2012) 51–69 MR

[10] **S K Donaldson**, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. 53 (1999) 205–236 MR

[11] **S Donaldson**, **I Smith**, *Lefschetz pencils and the canonical class for symplectic four-manifolds*, Topology 42 (2003) 743–785 MR

[12] **J G Dorfmeister**, *Kodaira dimension of fiber sums along spheres*, Geom. Dedicata 177 (2015) 1–25 MR

[13] **H Endo**, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. 316 (2000) 237–257 MR

[14] **H Endo**, **Y Z Gurtas**, *Lantern relations and rational blowdowns*, Proc. Amer. Math. Soc. 138 (2010) 1131–1142 MR

[15] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR

[16] **S Finashin**, *Knotting of algebraic curves in* $\mathbb{C}\mathrm{P}^2$, Topology 41 (2002) 47–55 MR

[17] **R Fintushel**, **R J Stern**, *Rational blowdowns of smooth 4–manifolds*, J. Differential Geom. 46 (1997) 181–235 MR

[18] **R Fintushel**, **R J Stern**, *Surfaces in 4–manifolds*, Math. Res. Lett. 4 (1997) 907–914 MR

[19] **R Fintushel**, **R J Stern**, *Knots, links, and* 4*–manifolds*, Invent. Math. 134 (1998) 363–400 MR

[20] **R Fintushel**, **R J Stern**, *Families of simply connected* 4*–manifolds with the same Seiberg–Witten invariants*, Topology 43 (2004) 1449–1467 MR

[21] **R Fintushel**, **R J Stern**, *Six lectures on four* 4*–manifolds*, from: "Low dimensional topology", (T S Mrowka, P S Ozsváth, editors), IAS/Park City Math. Ser. 15, Amer. Math. Soc. (2009) 265–315 MR

[22] **S Friedl**, **S Vidussi**, *On the topology of symplectic Calabi–Yau* 4*–manifolds*, J. Topol. 6 (2013) 945–954 MR

[23] **D Gay**, **T E Mark**, *Convex plumbings and Lefschetz fibrations*, J. Symplectic Geom. 11 (2013) 363–375 MR

[24] **R E Gompf**, *A new construction of symplectic manifolds*, Ann. of Math. 142 (1995) 527–595 MR

[25] **R E Gompf**, **A I Stipsicz**, 4*–manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc. (1999) MR

[26] **I Hambleton**, **M Kreck**, *Smooth structures on algebraic surfaces with cyclic fundamental group*, Invent. Math. 91 (1988) 53–59 MR

[27] **I Hambleton**, **M Kreck**, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. 444 (1993) 79–100 MR

[28] **A Kas**, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. 89 (1980) 89–104 MR

[29] **H J Kim**, *Modifying surfaces in* 4*–manifolds by twist spinning*, Geom. Topol. 10 (2006) 27–56 MR

[30] **H J Kim**, **D Ruberman**, *Topological triviality of smoothly knotted surfaces in* 4*–manifolds*, Trans. Amer. Math. Soc. 360 (2008) 5869–5881 MR

[31] **M Korkmaz**, *Noncomplex smooth* 4*–manifolds with Lefschetz fibrations*, Internat. Math. Res. Notices (2001) 115–128 MR

[32] **M Korkmaz**, **B Ozbagci**, *On sections of elliptic fibrations*, Michigan Math. J. 56 (2008) 77–87 MR

[33] **T-J Li**, *Smoothly embedded spheres in symplectic* 4*–manifolds*, Proc. Amer. Math. Soc. 127 (1999) 609–613 MR

[34] **T-J Li**, *The Kodaira dimension of symplectic* 4*–manifolds*, from: "Floer homology, gauge theory, and low-dimensional topology", (D A Ellwood, P S Ozsváth, A I Stipsicz, Z Szabó, editors), Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 249–261 MR

[35] **T-J Li**, *Quaternionic bundles and Betti numbers of symplectic* 4*–manifolds with Kodaira dimension zero*, Int. Math. Res. Not. 2006 (2006) Art. ID 37385 MR

[36] **T-J Li**, *Symplectic 4–manifolds with Kodaira dimension zero*, J. Differential Geom. 74 (2006) 321–352 MR

[37] **T J Li**, **A Liu**, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. 2 (1995) 453–471 MR

[38] **A Loi**, **R Piergallini**, *Compact Stein surfaces with boundary as branched covers of B^4*, Invent. Math. 143 (2001) 325–348 MR

[39] **G Massuyeau**, **A Oancea**, **D A Salamon**, *Lefschetz fibrations, intersection numbers, and representations of the framed braid group*, Bull. Math. Soc. Sci. Math. Roumanie 56(104) (2013) 435–486 MR

[40] **Y Matsumoto**, *Lefschetz fibrations of genus two—a topological approach*, from: "Topology and Teichmüller spaces", (S Kojima, Y Matsumoto, K Saito, M Seppälä, editors), World Sci. Publ., River Edge, NJ (1996) 123–148 MR

[41] **D McDuff**, **M Symington**, *Associativity properties of the symplectic sum*, Math. Res. Lett. 3 (1996) 591–608 MR

[42] **W Meyer**, *Die Signatur von Flächenbündeln*, Math. Ann. 201 (1973) 239–264 MR

[43] **B Ozbagci**, *Signatures of Lefschetz fibrations*, Pacific J. Math. 202 (2002) 99–118 MR

[44] **J Park**, **K-H Yun**, *Nonisomorphic Lefschetz fibrations on knot surgery 4–manifolds*, Math. Ann. 345 (2009) 581–597 MR

[45] **Y Sato**, *2–spheres of square -1 and the geography of genus–2 Lefschetz fibrations*, J. Math. Sci. Univ. Tokyo 15 (2008) 461–491 MR

[46] **Y Sato**, *Canonical classes and the geography of nonminimal Lefschetz fibrations over S^2*, Pacific J. Math. 262 (2013) 191–226 MR

[47] **A Scorpan**, *The wild world of 4–manifolds*, Amer. Math. Soc. (2005) MR

[48] **B Siebert**, **G Tian**, *On the holomorphicity of genus two Lefschetz fibrations*, Ann. of Math. 161 (2005) 959–1020 MR

[49] **I Smith**, *Geometric monodromy and the hyperbolic disc*, Q. J. Math. 52 (2001) 217–228 MR

[50] **I Smith**, *Lefschetz pencils and divisors in moduli space*, Geom. Topol. 5 (2001) 579–608 MR

[51] **A I Stipsicz**, *Indecomposability of certain Lefschetz fibrations*, Proc. Amer. Math. Soc. 129 (2001) 1499–1502 MR

[52] **N S Sunukjian**, *Surfaces in 4–manifolds: concordance, isotopy, and surgery*, Int. Math. Res. Not. 2015 (2015) 7950–7978 MR

[53] **C H Taubes**, *The Seiberg–Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994) 809–822 MR

[54] **C H Taubes**, $\mathrm{SW} \Rightarrow \mathrm{Gr}$*: from the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996) 845–918 MR

[55] **M Usher**, *Minimality and symplectic sums*, Int. Math. Res. Not. 2006 (2006) Art. ID 49857 MR

Department of Mathematics and Statistics, University of Massachusetts, Lederle Graduate Research Tower, 710 North Pleasant Street, Amherst, MA 01003-9305, United States

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama, Kanagawa 223-8522, Japan

baykur@math.umass.edu, k-hayano@math.keio.ac.jp

http://people.math.umass.edu/~baykur,
http://www.math.keio.ac.jp/~k-hayano/index_en.html

Proposed: Ronald Stern Received: 20 March 2015
Seconded: Ciprian Manolescu, Robion Kirby Revised: 31 August 2015