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Unified quantum invariants for integral homology spheres associated with simple Lie algebras

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For each finite-dimensional, simple, complex Lie algebra $\mathfrak{g}$ and each root of unity ξ (with some mild restriction on the order) one can define the Witten–Reshetikhin–Turaev (WRT) quantum invariant $\tau_M^{\mathfrak{g}}(\xi) \in \mathbb{C}$ of oriented 3–manifolds M. We construct an invariant J_M of *integral homology spheres* M, with values in $\widehat{\mathbb{Z}[q]}$, the cyclotomic completion of the polynomial ring $\mathbb{Z}[q]$, such that the evaluation of J_M at each root of unity gives the WRT quantum invariant of M at that root of unity. This result generalizes the case $\mathfrak{g} = \mathrm{sl}_2$ proved by Habiro. It follows that J_M unifies all the quantum invariants of M associated with $\mathfrak{g}$ and represents the quantum invariants as a kind of "analytic function" defined on the set of roots of unity. For example, $\tau_M(\xi)$ for all roots of unity are determined by a "Taylor expansion" at any root of unity, and also by the values at infinitely many roots of unity of prime power orders. It follows that WRT quantum invariants $\tau_M(\xi)$ for all roots of unity are determined by the Ohtsuki series, which can be regarded as the Taylor expansion at $q = 1$, and hence by the Lê–Murakami–Ohtsuki invariant. Another consequence is that the WRT quantum invariants $\tau_M^{\mathfrak{g}}(\xi)$ are algebraic integers. The construction of the invariant J_M is done on the level of quantum group, and does not involve any finite-dimensional representation, unlike the definition of the WRT quantum invariant. Thus, our construction gives a unified, "representation-free" definition of the quantum invariants of integral homology spheres.

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1 Introduction

The main goal of the paper is to construct an invariant $J_M^{\mathfrak{g}}$ of integral homology spheres M associated to each finite-dimensional simple Lie algebra $\mathfrak{g}$, which unifies the Witten–Reshetikhin–Turaev (WRT) quantum invariants at various roots of unity. The invariant $J_M^{\mathfrak{g}}$ takes values in the completion $\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((1-q)(1-q^2)\cdots(1-q^n))$ of the polynomial ring $\mathbb{Z}[q]$, which may be regarded as a ring of analytic functions on roots of unity. This invariant unifies the quantum invariants at various roots of unity in the sense that, for each root of unity ξ, the evaluation $\mathrm{ev}_\xi(J_M^{\mathfrak{g}})$ at $q = \xi$ of $J_M^{\mathfrak{g}}$ is equal to the WRT quantum invariant $\tau_M^{\mathfrak{g}}(\xi)$ of M at ξ whenever $\tau_M^{\mathfrak{g}}(\xi)$ is defined. This invariant is a generalization of the sl_2 case constructed by Habiro [26].

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1A The WRT invariant

Witten [84], using not mathematically rigorous path integrals in quantum field theory, gave a physics interpretation of the Jones polynomial [31] and predicted the existence of 3–manifold invariants associated to every simple Lie algebra and certain integers called levels. Using the quantum group $U_q(\mathrm{sl}_2)$ at roots of unity, Reshetikhin and Turaev [73] gave a rigorous construction of 3–manifold invariants, which are believed to coincide with the Witten invariants. These invariants are called the Witten–Reshetikhin–Turaev (WRT) quantum invariants. Later the machinery of quantum groups helped to generalize the WRT invariant $\tau_M^{\mathfrak{g}}(\xi)$ to the case when $\mathfrak{g}$ is an arbitrary simple Lie algebra and ξ is a root of unity.

In this paper we will focus on the quantum invariants of an integral homology 3–sphere, ie a closed oriented 3–manifold M such that $H_*(M,\mathbb{Z}) = H_*(S^3,\mathbb{Z})$.

Let $\mathcal{Z} \subset \mathbb{C}$ denote the set of all roots of unity. For each simple Lie algebra $\mathfrak{g}$, there is a subset $\mathcal{Z}_{\mathfrak{g}} \subset \mathcal{Z}$ and the $\mathfrak{g}$–WRT invariant of an integral homology sphere M gives a function

$$\tau_M^{\mathfrak{g}} \colon \mathcal{Z}_{\mathfrak{g}} \to \mathbb{C}. \tag{1}$$

(We recall the definition of $\tau_M^{\mathfrak{g}}(\xi)$ in Section 8. The definition of $\tau_M^{\mathfrak{g}}(\xi)$ for closed 3–manifolds involves the choice of a certain root of ξ, but it turns out that for integral homology spheres this choice is irrelevant.)

We are interested in the behavior of the WRT function (1) associated to each Lie algebra $\mathfrak{g}$. It is natural to raise the following questions:

- Is it possible to extend the domain of the map $\tau_M^{\mathfrak{g}}$ to $\mathcal{Z}$ in a natural way?
- How strongly are the values at different roots of unity ξ, $\xi' \in \mathcal{Z}_{\mathfrak{g}}$ related?
- Is there some restriction on the range of the function? In particular, is $\tau_M^{\mathfrak{g}}(\xi)$ an algebraic integer for all $\mathfrak{g}$ and ξ?
- How are the quantum invariants related to finite-type invariants of 3–manifolds (see Ohtsuki [66], Habiro [20] and Goussarov [19])? In particular, is there any relation between the quantum invariants and the Lê–Murakami–Ohtsuki invariant [51]?

1B The ring $\widehat{\mathbb{Z}[q]}$ of analytic functions on roots of unity

Define a completion $\widehat{\mathbb{Z}[q]}$ of the polynomial ring $\mathbb{Z}[q]$ by

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q;q)_n),$$

where, as usual,

$$(x;q)_n := \prod_{j=1}^{n}(1-xq^{j-1}).$$

The ring $\widehat{\mathbb{Z}[q]}$ may be regarded as the ring of "analytic functions defined on the set $\mathcal{Z}$ of roots of unity"; see Habiro [22; 26]. This statement is justified by the following facts. For more details, see [26, Section 1.2].

For a root of unity $\xi \in \mathcal{Z}$ of order r, we have $(\xi;\xi)_n = 0$ for $n \geq r$. Hence the evaluation map

$$\mathrm{ev}_\xi\colon \mathbb{Z}[q] \to \mathbb{Z}[\xi], \quad f(q) \mapsto f(\xi),$$

induces a ring homomorphism

$$\mathrm{ev}_\xi\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\xi].$$

We write $f(\xi) = \mathrm{ev}_\xi(f(q))$.

Each element $f(q) \in \widehat{\mathbb{Z}[q]}$ defines a function from $\mathcal{Z}$ to $\mathbb{C}$. Thus we have a ring homomorphism

$$\mathrm{ev}\colon \widehat{\mathbb{Z}[q]} \to \mathbb{C}^{\mathcal{Z}} \tag{2}$$

defined by $\mathrm{ev}(f(q))(\xi) = \mathrm{ev}_\xi(f(q))$. This homomorphism is injective [22], ie $f(q)$ is determined by the values $f(\xi)$ for $\xi \in \mathcal{Z}$. Therefore, we may regard $f(q)$ as a function on the set $\mathcal{Z}$.

In fact, a function $f(q) \in \widehat{\mathbb{Z}[q]}$ can be determined by values on a subset $\mathcal{Z}'$ of $\mathcal{Z}$ if $\mathcal{Z}'$ has a limit point $\xi_0 \in \mathcal{Z}$ with respect to a certain topology of $\mathcal{Z}$; see [22, Theorem 6.3]. In this topology, an element $\xi \in \mathcal{Z}$ is a limit point of a subset $\mathcal{Z}' \subset \mathcal{Z}$ if and only if there are infinitely many $\xi' \in \mathcal{Z}'$ such that the orders (as roots of unity) of $\xi'\xi^{-1}$ are prime powers. For example, each $f(q) \in \widehat{\mathbb{Z}[q]}$ is determined by the values at infinitely many roots of unity of prime orders.

For $\xi \in \mathcal{Z}$, there is a ring homomorphism

$$T_\xi\colon \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\xi][\![q-\xi]\!]$$

induced by the inclusion $\mathbb{Z}[q] \subset \mathbb{Z}[\xi][q]$, since, for $n \geq 0$, the element $(q;q)_{n\,\mathrm{ord}(\xi)}$ is divisible by $(q-\xi)^n$ in $\mathbb{Z}[\xi][q]$. The image $T_\xi(f(q))$ of $f(q) \in \widehat{\mathbb{Z}[q]}$ may be regarded as the "Taylor expansion" of $f(q)$ at ξ. The homomorphism T_ξ is injective [22, Theorem 5.2]. Hence, a function $f(q) \in \widehat{\mathbb{Z}[q]}$ is determined by its Taylor expansion at a point $\xi \in \mathcal{Z}$. Injectivity of T_ξ implies that $\widehat{\mathbb{Z}[q]}$ is an integral domain.

The above-explained properties of $\widehat{\mathbb{Z}[q]}$ depend on the ground ring $\mathbb{Z}$ of integers in an essential way. In fact, the similar completion $\widehat{\mathbb{Q}[q]} = \varprojlim_n \mathbb{Q}[q]/((q;q)_n)$ is radically different. For example, $\widehat{\mathbb{Q}[q]}$ is not an integral domain and the Taylor expansion map $T_\xi\colon \widehat{\mathbb{Q}[q]} \to \mathbb{Q}[\xi][\![q-\xi]\!]$ is surjective but not injective; see [22, Section 7.5].

Recently, Manin [59] and Marcolli [60] have promoted the ring $\widehat{\mathbb{Z}[q]}$ as a candidate for the ring of analytic functions on the nonexistent "field of one element".

1C Main result and consequences

The following is the main result of the present paper:

Theorem 1.1 *For each simple Lie algebra $\mathfrak{g}$, there is a unique invariant $J_M = J_M^{\mathfrak{g}}$ in $\widehat{\mathbb{Z}[q]}$ of an integral homology sphere M such that for all $\xi \in \mathcal{Z}_{\mathfrak{g}}$ we have*

$$\mathrm{ev}_\xi(J_M) = \tau_M^{\mathfrak{g}}(\xi).$$

Theorem 1.1 is proved in Section 8T. It follows from Theorems 2.25, 4.9, 7.3, and 8.1.

The case $\mathfrak{g} = \mathrm{sl}_2$ of Theorem 1.1 was announced in Habiro [21] and proved in [26]. For $\mathfrak{g} = \mathrm{sl}_2$, the invariant J_M has been generalized to invariants of rational homology spheres with values in modifications of $\widehat{\mathbb{Z}[q]}$ in Beliakova, Blanchet and Lê [5], Lê [50], Beliakova and Lê [8] and Beliakova, Bühler and Lê [6].

Theorem 1.1 implies that for integral homology 3–spheres, $\tau_M^{\mathfrak{g}}(\xi)$ does not depend on the choice of a root of ξ that is used in the definition of $\tau_M^{\mathfrak{g}}(\xi)$.

We list here a few consequences of Theorem 1.1. For the results stated without proof and with the sl_2 case proved in [26], the proof is the same as the proof of the corresponding result in [26].

1C1 Analytic continuation of $\tau_M^{\mathfrak{g}}$ to all roots of unity Even if a root of unity $\xi \in \mathcal{Z}$ is not contained in $\mathcal{Z}_{\mathfrak{g}}$, the domain of definition of the WRT function $\tau_M^{\mathfrak{g}}$, we have a well-defined value $\mathrm{ev}_\xi(J_M) \in \mathbb{Z}[\xi]$. By the uniqueness of J_M, it would be natural to *define* the $\mathfrak{g}$–WRT invariant $\tau_M^{\mathfrak{g}}(\xi)$ at $\xi \in \mathcal{Z} \setminus \mathcal{Z}_{\mathfrak{g}}$ as $\mathrm{ev}_\xi(J_M)$. We may regard it as an analytic continuation of $\tau_M^{\mathfrak{g}}\colon \mathcal{Z}_{\mathfrak{g}} \to \mathbb{C}$.

The specializations $\mathrm{ev}_\xi(J_M)$ are compatible also with the projective version of the $\mathfrak{g}$–WRT invariant

$$\tau_M^{P\mathfrak{g}}\colon \mathcal{Z}_{P\mathfrak{g}} \to \mathbb{C}, \tag{3}$$

where $\mathcal{Z}_{P\mathfrak{g}}$ is another subset of $\mathcal{Z}$. See Section 8.

Proposition 1.2 *For an integral homology sphere M and for $\xi \in \mathcal{Z}_{P_{\mathfrak{g}}}$, we have*

$$\mathrm{ev}_\xi(J_M) = \tau_M^{P\mathfrak{g}}(\xi).$$

As a consequence, for $\xi \in \mathcal{Z}_{\mathfrak{g}} \cap \mathcal{Z}_{P_{\mathfrak{g}}}$ we have

$$\tau_M^{\mathfrak{g}}(\xi) = \tau_M^{P\mathfrak{g}}(\xi). \tag{4}$$

Remark 1.3 For a closed 3–manifold M which is not necessarily an integral homology sphere, we do not have (4), but for some values of ξ we have identities of the form

$$\tau_M^{\mathfrak{g}}(\xi) = \tau_M^{P\mathfrak{g}}(\xi)\tilde{\tau}_M^{\mathfrak{g}}(\xi),$$

where $\tilde{\tau}_M^{\mathfrak{g}}(\xi)$ is an invariant of M satisfying $\tilde{\tau}_M^{\mathfrak{g}}(\xi) = 1$ for M an integral homology sphere. For details, see eg Blanchet [9], Kirby and Melvin [40], Kohno and Takata [42] and Lê [49].

1C2 Integrality of quantum invariants An immediate consequence of Theorem 1.1 is the following integrality result:

Corollary 1.4 *For any integral homology sphere M and $\xi \in \mathcal{Z}_{\mathfrak{g}}$, we have $\tau_M^{\mathfrak{g}}(\xi) \in \mathbb{Z}[\xi]$. In particular, $\tau_M^{\mathfrak{g}}(\xi)$ is an algebraic integer.*

Here we list related integrality results for quantum invariants for closed 3–manifolds, which are not necessarily integral homology spheres.

H Murakami [63] (see also Masbaum and Roberts [61]) proved that the Psl_2–WRT invariant — also known as the quantum SO(3) invariant (see Kirby and Melvin [40]) — of a closed 3–manifold at $\xi \in \mathcal{Z}$ of *prime order* is contained in $\mathbb{Z}[\xi]$. This result, for roots of unity of prime orders, has been generalized to sl_n by Masbaum and Wenzl [62] and independently by Takata and Yokota [80], and to all simple Lie algebras by Lê [49].

The case of roots of *nonprime orders*, conjectured by R Lawrence [45] in the sl_2 case, was developed later. The case $\mathfrak{g} = \mathrm{sl}_2$ of Corollary 1.4 was obtained by Habiro [26]. Beliakova, Chen and Lê [7] proved that $\tau_M^{\mathrm{sl}_2}(\xi)$ (which depends on a fourth root of ξ) is an algebraic integer for any root of unity ξ. For general Lie algebras, however, the proof in [7] does not work. Corollary 1.4 is the first integrality result for general Lie algebras in the case of nonprime orders.

1C3 Relationships between quantum invariants at different roots of unity One can obtain from Theorem 1.1 results about the values of the WRT invariants more refined than integrality.

Let $\mathbb{Q}^{\mathrm{ab}} \subset \mathbb{C}$ denote the maximal abelian extension of $\mathbb{Q}$, which is the smallest extension of $\mathbb{Q}$ containing $\mathcal{Z}$. The image of the WRT function $\tau_M^{\mathfrak{g}}$ is contained in the integer ring $\mathcal{O}(\mathbb{Q}^{\mathrm{ab}})$ of $\mathbb{Q}^{\mathrm{ab}}$, which is the subring of $\mathbb{Q}^{\mathrm{ab}}$ generated by $\mathcal{Z}$. Note that an automorphism $\alpha \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ maps each root of unity ξ to a root of unity $\alpha(\xi)$ of the same order as ξ. There is a canonical isomorphism

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \mathrm{Aut}_{\mathrm{Grp}}(\mathcal{Z}),$$

which maps $\alpha \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ to its restriction to $\mathcal{Z}$. Here $\mathrm{Aut}_{\mathrm{Grp}}(\mathcal{Z})$ is the group of automorphisms of $\mathcal{Z}$, considered as a subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$.

Proposition 1.5 *For every integral homology sphere M, the $\mathfrak{g}$–WRT function*

$$\tau_M^{\mathfrak{g}} \colon \mathcal{Z} \to \mathbb{Q}^{\mathrm{ab}}$$

is Galois-equivariant, in the sense that, for each automorphism $\alpha \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$,

$$\tau_M^{\mathfrak{g}}(\alpha(\xi)) = \alpha(\tau_M^{\mathfrak{g}}(\xi)).$$

The sl_2 case of Proposition 1.5 is mentioned in Habiro [26].

Proposition 1.6 (see [26] for $\mathfrak{g} = \mathrm{sl}_2$) *We have $\mathrm{ev}_1(J_M) = 1$ for every integral homology sphere M.*

Proposition 1.6 is proved in Section 8U.

Proposition 1.7 (see [26] for $\mathfrak{g} = \mathrm{sl}_2$) *For $\xi, \xi' \in \mathcal{Z}$ with $\mathrm{ord}(\xi'\xi^{-1})$ a prime power, we have*

$$\tau_M^{\mathfrak{g}}(\xi) \equiv \tau_M^{\mathfrak{g}}(\xi') \pmod{\xi' - \xi}$$

in $\mathbb{Z}[\xi, \xi']$.

Proposition 1.7 holds also when $\mathrm{ord}(\xi'\xi^{-1})$ is not a prime power, but in this case the statement is trivial, since $\xi' - \xi$ is a unit in $\mathbb{Z}[\xi, \xi']$.

Corollary 1.8 *For every integral homology sphere M and every root of unity $\xi \in \mathcal{Z}$ of prime power order, we have*

$$\tau_M^{\mathfrak{g}}(\xi) - 1 \in (1 - \xi)\mathbb{Z}[\xi].$$

Consequently, we have $\tau_M^{\mathfrak{g}}(\xi) \neq 0$.

For $\mathfrak{g} = \mathrm{sl}_2$, a refined version of Corollary 1.8 is given in [26, Corollary 12.10].

1C4 Integrality of the Ohtsuki series When M is a rational homology sphere, Ohtsuki [67] extracted a power series invariant, $\tau_\infty^{\mathrm{sl}_2}(M) \in \mathbb{Q}[\![q-1]\!]$, from the values of $\tau_M^{\mathrm{Psl}_2}(\xi)$ at roots of unity of prime orders. The Ohtsuki series is characterized by certain congruence relations modulo odd primes. The existence of the Ohtsuki series invariant for other Lie algebras was proved in Lê [48; 49]; see also Rozansky [75].

The Ohtsuki series $\tau_\infty^{\mathfrak{g}}(M) \in \mathbb{Q}[\![q-1]\!]$ and the unified WRT invariant J_M are related as follows:

Proposition 1.9 (see [26] for $\mathfrak{g} = \mathrm{sl}_2$) *For every integral homology sphere M, we have*

$$\tau_\infty^{\mathfrak{g}}(M) = T_1(J_M) \in \mathbb{Z}[\![q-1]\!].$$

In other words, the Ohtsuki series is equal to the Taylor expansion of the unified WRT invariant at $q = 1$. Moreover, all the coefficients in the Ohtsuki series are integers.

The fact $\tau_\infty^{\mathfrak{g}}(M) \in \mathbb{Z}[\![q-1]\!]$ for $\mathfrak{g} = \mathrm{sl}_2$ was conjectured by Lawrence [45] and first proved by Rozansky [76]. Here we have general results for all simple Lie algebras.

1C5 Relation to the Lê–Murakami–Ohtsuki invariant The LMO invariant [51] is a counterpart of the Kontsevich integral for homology 3–spheres; it is a universal invariant for finite-type invariants of integral homology 3–spheres; see Lê [46]. The LMO invariant $\tau^{\mathrm{LMO}}(M)$ of a closed 3–manifold takes values in an algebra $\mathcal{A}(\varnothing)$ of the so-called Jacobi diagrams, which are certain types of trivalent graphs. For each simple Lie algebra $\mathfrak{g}$, there is a ring homomorphism (the weight map)

$$W_{\mathfrak{g}}\colon \mathcal{A}(\varnothing) \to \mathbb{Q}[\![h]\!].$$

It was proved in Kuriya, Lê and Ohtsuki [43] that

$$W_{\mathfrak{g}}(\tau^{\mathrm{LMO}}(M)) = \tau_\infty^{\mathfrak{g}}(M)|_{q=e^h}.$$

Hence, we have the following:

Corollary 1.10 (see [26] for $\mathfrak{g} = \mathrm{sl}_2$) *For an integral homology 3–sphere M, the LMO invariant totally determines the WRT invariant $\tau_M^{\mathfrak{g}}(\xi)$ for every simple Lie algebra and every root of unity $\xi \in \mathcal{Z}_{\mathfrak{g}}$.*

It is still an open question whether the LMO invariant determines the WRT invariant for *rational* homology 3–spheres.

1C6 Determination of the quantum invariants

Corollary 1.11 (see [26] for $\mathfrak{g} = \mathrm{sl}_2$) *For an integral homology 3–sphere, J_M is determined by the WRT function $\tau_M^{\mathfrak{g}}$. (Thus J_M and $\tau_M^{\mathfrak{g}}$ have the same strength in distinguishing two integral homology 3–spheres.) Moreover, both J_M and $\tau_M^{\mathfrak{g}}$ are determined by the values of $\tau_M^{\mathfrak{g}}(\xi)$ for $\xi \in \mathcal{Z}'$, where $\mathcal{Z}' \subset \mathcal{Z}$ is any infinite subset with at least one limit point in $\mathcal{Z}$* (in the sense explained in Section 1B).

For example, the value of $\tau_M^{\mathfrak{g}}(\xi)$ at any root of unity ξ is determined by the values $\tau_M^{\mathfrak{g}}(\xi_k)$ at $\xi_k = \exp(2\pi i/2^k)$ for infinitely many integers $k \geq 0$.

1D Formal construction of the unified invariant

Here we outline the proof of Theorem 1.1. Since we are not able to directly generalize the proof of the case $\mathfrak{g} = \mathrm{sl}_2$ in [26], we use another approach, which involves deep results in quantized enveloping algebras (quantum groups). The conceptual definition of the unified invariant presented here is also different.

1D1 First step: construction of J_M The first step is to construct an invariant $J_M \in \widehat{\mathbb{Z}[q]}$ using the quantum group $U_h(\mathfrak{g})$ of $\mathfrak{g}$. Here we use neither the definition of $\tau_M^{\mathfrak{g}}(\xi)$ nor the quantum link invariants associated to finite-dimensional representations of $U_h(\mathfrak{g})$. Instead, we use the universal quantum invariant of bottom tangles and the *full twist forms*, which are partially defined functionals $\mathcal{T}_\pm$ on the quantum group $U_h(\mathfrak{g})$ and play the role of ±1–framed surgery on link components.

Every integral homology 3–sphere M can be obtained as the result S^3_L of surgery on S^3 along an algebraically split link L with framing ±1 on each component. Here a link is said to be algebraically split if the linking number between any two distinct components is 0. Surgery on two algebraically split, ±1 framing links L and L' gives the orientation-preserving homeomorphic integral homology 3–spheres if and only if L and L' are related by a sequence of Hoste moves (see Figure 7); see Habiro [24]. Hence, in order to construct an invariant of integral homology 3–spheres, it suffices to construct an invariant of algebraically split, ±1 framing links which is invariant under the Hoste moves.

To construct such a link invariant, we use the universal quantum invariant of bottom tangles associated to the quantum group $U_h(\mathfrak{g})$. Here a bottom tangle is a tangle in a cube consisting of arc components whose endpoints are on the bottom square in such a way that the two endpoints of each component are placed side by side (see Section 2F). For an n–component bottom tangle T, the universal $\mathfrak{g}$ quantum invariant

$J_T = J_T^{\mathfrak{g}}$ of T is defined by using the universal R–matrix and the ribbon element for the ribbon Hopf algebra structure of $\boldsymbol{U}_h(\mathfrak{g})$, and takes values in the n–fold completed tensor power $\boldsymbol{U}_h(\mathfrak{g})^{\hat{\otimes} n}$.

The invariant $J_M \in \widehat{\mathbb{Z}[q]}$ is defined as follows. As above, let L be an n–component algebraically split framed link with framings $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, and assume that $S^3_L \cong M$. Let T be an n–component bottom tangle whose closure is isotopic to L, where the framings of T are switched to 0. Define

$$J_M := (\mathcal{T}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{T}_{\varepsilon_n})(J_T). \tag{5}$$

Here $\mathcal{T}_{\pm}\colon \boldsymbol{U}_h(\mathfrak{g}) \dashrightarrow \mathbb{C}[\![h]\!]$ are *partial maps* (ie maps defined on a submodule of $\boldsymbol{U}_h$) defined formally by

$$\mathcal{T}_{\pm}(x) = \langle x, \boldsymbol{r}^{\pm 1} \rangle,$$

where $\boldsymbol{r} \in \boldsymbol{U}_h(\mathfrak{g})$ is the ribbon element, and

$$\langle\,\cdot\,,\cdot\,\rangle\colon \boldsymbol{U}_h(\mathfrak{g}) \mathbin{\hat{\otimes}} \boldsymbol{U}_h(\mathfrak{g}) \dashrightarrow \mathbb{C}[\![h]\!]$$

is the quantum Killing form, which is a partial map. The tensor product $\mathcal{T}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{T}_{\varepsilon_n}$ is not well defined on the whole $\boldsymbol{U}_h(\mathfrak{g})^{\hat{\otimes} n}$, but is well defined on a $\widehat{\mathbb{Z}[q]}$–submodule $\widetilde{\mathcal{K}}_n \subset \boldsymbol{U}_h(\mathfrak{g})^{\hat{\otimes} n}$ and we have a $\widehat{\mathbb{Z}[q]}$–module homomorphism

$$\mathcal{T}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{T}_{\varepsilon_n}\colon \widetilde{\mathcal{K}}_n \to \widehat{\mathbb{Z}[q]}.$$

Here we regard $\widehat{\mathbb{Z}[q]}$ as a subring of $\mathbb{C}[\![h]\!]$ by setting $q = \exp h$. The module $\widetilde{\mathcal{K}}_n$ contains J_T for all n–component, algebraically split 0–framed links T. We will prove that J_M as defined in (5) does not depend on the choice of T and is invariant under the Hoste moves. Hence $J_M \in \widehat{\mathbb{Z}[q]}$ is an invariant of an integral homology sphere.

One step in the construction of J_M is to construct a certain integral form of the quantum group $\boldsymbol{U}_h(\mathfrak{g})$ which is sandwiched between the Lusztig integral form and the De Concini–Procesi integral form.

1D2 Second step: specialization to the WRT invariant at roots of unity The next step is to prove the specialization property $\mathrm{ev}_\xi(J_M) = \tau_M^{\mathfrak{g}}(\xi)$ for each $\xi \in \mathcal{Z}_{\mathfrak{g}}$. Once we have proved this identity, uniqueness of J_M follows, since every element of $\widehat{\mathbb{Z}[q]}$ is determined by the values at infinitely many $\xi \in \mathcal{Z}$ of prime power order; see Section 1B.

Organization of the paper In Section 2 we give a general construction of an invariant of *integral homology* 3*–spheres* from what we call a *core subalgebra* of a ribbon Hopf algebra. Section 3 introduces the quantized enveloping algebra $\boldsymbol{U}_h(\mathfrak{g})$ and its subalgebras. In Section 4 we construct a core subalgebra of the ribbon Hopf algebra $\boldsymbol{U}_{\sqrt{h}}$, which is $\boldsymbol{U}_h$ with a slightly bigger ground ring. From the core subalgebra we get the invariant J_M of integral homology 3–spheres. In Section 5 we construct an integral version of the core algebra. Section 6 a (generically noncommutative) grading of the quantum group is introduced. In Section 7 we prove that $J_M \in \widehat{\mathbb{Z}[q]}$. In Section 8 we show that the WRT invariant can be recovered from J_M, proving the main results. In appendices we give an independent proof of a duality result of Drinfel'd and Gavarini and provide proofs of a couple of technical results used in the main body of the paper.

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2 Invariants of integral homology 3–spheres derived from ribbon Hopf algebras

In this section, we give the part of the proofs of our main results that can be stated without giving the details of the structure of the quantized enveloping algebra $\boldsymbol{U}_h = \boldsymbol{U}_h(\mathfrak{g})$. We introduce the notion of a *core subalgebra* of a ribbon Hopf algebra and show that every core subalgebra gives rise to an invariant of *integral homology* 3*–spheres*.

2A Modules over $\mathbb{C}[\![h]\!]$

Let $\mathbb{C}[\![h]\!]$ be the ring of formal power $\mathbb{C}$–series in the variable h.

Note that $\mathbb{C}[\![h]\!]$ is a local ring, with maximal ideal $(h) = h\mathbb{C}[\![h]\!]$. An element $x = \sum x_k h^k \in \mathbb{C}[\![h]\!]$ is invertible if and only if the constant term x_0 is nonzero.

2A1 h–adic topology, separation and completeness Let V be a $\mathbb{C}[\![h]\!]$–module. Then V is equipped with the h–adic topology given by the filtration $h^k V$, $k \geq 0$. Any $\mathbb{C}[\![h]\!]$–module homomorphism $f \colon V \to W$ is automatically continuous. In general, the h–adic topology of a $\mathbb{C}[\![h]\!]$–submodule W of a $\mathbb{C}[\![h]\!]$–module V is different from the topology of W induced by the h–adic topology of V.

Suppose I is an index set. Let V^I be the set of all collections $(x_i)_{i\in I}$, $x_i \in V$. We say a collection $(x_i)_{i\in I} \in V^I$ is 0*–convergent* in V if, for every positive integer k, $x_i \in h^k V$ except for a finite number of $i \in I$. In this case, the sum $\sum_{i\in I} x_i$ is convergent in the h–adic topology of V. If I is finite, then any collection $(x_i)_{i\in I}$ is 0–convergent.

The *h–adic completion* $\widehat{V}$ of V is defined by

$$\widehat{V} = \varprojlim_k V/h^k V.$$

A $\mathbb{C}[\![h]\!]$–module V is *separated* if the natural map $V \to \widehat{V}$ is injective, which is equivalent to $\bigcap_k h^k V = \{0\}$. If V is separated, we identify V with the image of the embedding $V \hookrightarrow \widehat{V}$.

A $\mathbb{C}[\![h]\!]$–module V is *complete* if the natural map $V \to \widehat{V}$ is surjective.

For a $\mathbb{C}[\![h]\!]$–submodule W of a completed $\mathbb{C}[\![h]\!]$–module V, the *topological completion of W in V* is the image of $\widehat{W}$ under the natural map $\widehat{W} \to \widehat{V} = V$. One should not confuse the topological completion of W and the topological closure of W, the latter being the smallest closed (in the h–adic topology) subset containing W. See Example 2.2.

2A2 Topologically free modules For a vector space A over $\mathbb{C}$, let $A[\![h]\!]$ denote the $\mathbb{C}[\![h]\!]$–module of formal power series $\sum_{n\geq 0} a_n h^n$, $a_n \in A$. Then $A[\![h]\!]$ is naturally isomorphic to the h–adic completion of $A \otimes_{\mathbb{C}} \mathbb{C}[\![h]\!]$.

A $\mathbb{C}[\![h]\!]$–module V is said to be *topologically free* if V is isomorphic to $A[\![h]\!]$ for some vector space A. A *topological basis* of V is the image by an isomorphism $A[\![h]\!] \cong V$ of a basis of $A \subset A[\![h]\!]$. The cardinality of a topological basis of V is called the *topological rank* of V.

It is known that a $\mathbb{C}[\![h]\!]$–module is topologically free if and only if it is separated, complete, and torsion-free; see eg [35, Proposition XVI.2.4].

Let I be a set. Let $\mathbb{C}[\![h]\!]^I = \prod_{i\in I} \mathbb{C}[\![h]\!]$ be the set of all collections $(x_i)_{i\in I}$, $x_i \in \mathbb{C}[\![h]\!]$. Let $(\mathbb{C}[\![h]\!]^I)_0 \subset \mathbb{C}[\![h]\!]^I$ be the $\mathbb{C}[\![h]\!]$–submodule consisting of the 0–convergent collections. Then $(\mathbb{C}[\![h]\!]^I)_0 \cong (\mathbb{C}I)[\![h]\!]$ is topologically free, where $\mathbb{C}I$ is the vector space generated by I.

Note that $\mathbb{C}[\![h]\!]^I$ is also topologically free. In fact, we have a $\mathbb{C}[\![h]\!]$–module isomorphism $\mathbb{C}[\![h]\!]^I \cong \mathbb{C}^I[\![h]\!]$. If I is infinite, then the topological rank of $\mathbb{C}[\![h]\!]^I$ is uncountable.

For $j \in I$, define a collection $\delta_j = ((\delta_j)_i)_{i\in I} \in (\mathbb{C}[\![h]\!]^I)_0$ by

$$(\delta_j)_i = \delta_{j,i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{6}$$

Suppose V is a topologically free $\mathbb{C}[\![h]\!]$–module with isomorphism $f\colon (\mathbb{C}[\![h]\!]^I)_0 \to V$. Let $e(i) = f(\delta_i) \in V$. For $x \in V$, the collection $(x_i)_{i\in I} = f^{-1}(x)$ is called the *coordinates* of x in the topological basis $\{e(i)\}$. We then have

$$x = \sum_{i\in I} x_i e(i), \tag{7}$$

where the sum on the right-hand side converges to x in the h–adic topology of V.

2A3 Formal series modules A $\mathbb{C}[\![h]\!]$–module V is a *formal series* $\mathbb{C}[\![h]\!]$*–module* if there is a $\mathbb{C}[\![h]\!]$–module isomorphism $f\colon \mathbb{C}[\![h]\!]^I \to V$ for a *countable* set I.

Remark 2.1 Besides the h–adic topology, another natural topology on $\mathbb{C}[\![h]\!]^I = \prod_{i\in I} \mathbb{C}[\![h]\!]$ is the *product topology*. (Recall that the product topology of $\prod_{i\in I} \mathbb{C}[\![h]\!]$ is the coarsest topology with all the projections $p_i\colon \prod_{i\in I} \mathbb{C}[\![h]\!] \to \mathbb{C}[\![h]\!]$ being continuous.)

Suppose V is a formal series module, with an isomorphism $f : \mathbb{C}[\![h]\!]^I \to V$. Let $e(i) = f(\delta_i)$, where δ_i is defined as in (6). The set $\{e(i) \mid i \in I\}$ is called a *formal basis* of V.

For $x \in V$ the collection $f^{-1}(x) \in \mathbb{C}[\![h]\!]^I$ is called the *coordinates* of x in the formal basis $\{e(i) \mid i \in I\}$. Unlike the case of topological bases, in general the sum $\sum_{i\in I} x_i e(i)$ does not converge in the h–adic topology of V (but does converge to x in the product topology). However, it is often the case that V is a $\mathbb{C}[\![h]\!]$–submodule of a bigger $\mathbb{C}[\![h]\!]$–module V' in which $\{e(i) \mid i \in I\}$ is 0–convergent. Then the sum $\sum_{i\in I} x_i e(i)$, though not convergent in the h–adic topology of V, does converge (to x) in the h–adic topology of V'.

The following example is important for us:

Example 2.2 Suppose V is a topologically free $\mathbb{C}[\![h]\!]$–module with a countable topological basis $\{e(i) \mid i \in I\}$. Assume that $a\colon I \to \mathbb{C}[\![h]\!]$ is a function such that $a(i) \neq 0$ for every $i \in I$ and $(a(i))_{i\in I}$ is 0–convergent. Let $V(a)$ be the topological completion in V of the $\mathbb{C}[\![h]\!]$–span of $\{a(i)e(i) \mid i \in I\}$. Then $V(a)$ is topologically free with $\{a(i)e(i) \mid i \in I\}$ as a topological basis.

The submodule $V(a)$ is not closed in the h–adic topology of V. The closure $\overline{V(a)}$ of $V(a)$ in the h–adic topology is a formal series $\mathbb{C}[\![h]\!]$–module, with an isomorphism

$$f\colon \mathbb{C}[\![h]\!]^I \to V, \quad \delta_i \mapsto a(i)e(i).$$

The topology of $\overline{V(a)}$ induced by the h–adic topology of V is the product topology.

If $x \in V(a)$, then we have a unique presentation

$$x = \sum_{i \in I} x_i (a(i) e(i)) \tag{8}$$

with $(x_i)_{i \in I} \in (\mathbb{C}[\![h]\!]^I)_0$.

If $x \in \overline{V(a)}$, then x also has a unique presentation (8), with $(x_i)_{i \in I} \in \mathbb{C}[\![h]\!]^I$.

2A4 Completed tensor products For two complete $\mathbb{C}[\![h]\!]$–modules V and V', the *completed tensor product* $V \hat{\otimes} V'$ of V and V' is the h–adic completion of $V \otimes V'$, ie

$$V \hat{\otimes} V' = \varprojlim_n (V \otimes V')/h^n(V \otimes V').$$

Suppose both V and V' are topologically free with topological bases $\{b(i) \mid i \in I\}$ and $\{b'(j) \mid j \in J\}$, respectively. Then $V \hat{\otimes} V'$ is topologically free with a topological basis naturally identified with $\{b(i) \otimes b'(j) \mid i \in I,\ j \in J\}$.

Proposition 2.3 *Suppose W_1, V_1, W_2 and V_2 are topologically free $\mathbb{C}[\![h]\!]$–modules, where W_j is a submodule of V_j for $j = 1, 2$.*

Then the natural maps $W_1 \otimes W_2 \to V_1 \otimes V_2$ and $W_1 \hat{\otimes} W_2 \to V_1 \hat{\otimes} V_2$ are injective.

Proof The map $W_1 \otimes W_2 \to V_1 \otimes V_2$ is the composition of two maps $W_1 \otimes W_2 \to W_1 \otimes V_2$ and $W_1 \otimes V_2 \to V_1 \otimes V_2$. This reduces the proposition to the case $W_2 = V_2$, which we will assume.

Let $\iota\colon W_1 \hookrightarrow V_1$ be the inclusion map. We need to show that $\iota \otimes \mathrm{id}\colon W_1 \otimes V_2 \to V_1 \otimes V_2$ and $\iota \hat{\otimes} \mathrm{id}\colon W_1 \hat{\otimes} V_2 \to V_1 \hat{\otimes} V_2$ are injective.

Since $W_1 \otimes V_2$ is separated, we can consider $W_1 \otimes V_2$ as a submodule of $W_1 \hat{\otimes} V_2$. Then $\iota \otimes \mathrm{id}$ is the restriction of $\iota \hat{\otimes} \mathrm{id}$. Thus, it is enough to show that $\iota \hat{\otimes} \mathrm{id}$ is injective.

Suppose $x \in W_1 \hat{\otimes} V_2$ with $(\iota \hat{\otimes} \mathrm{id})(x) = 0$. We have to show that $x = 0$.

Let $\{b(i) \mid i \in I\}$ be a topological basis of V_2. Using a topological basis of W_1 one sees that x has a unique presentation

$$x = \sum_{i \in I} x_i \otimes b(i) \tag{9}$$

with $x_i \in W_1$, and the collection $(x_i)_{i \in I}$ is 0–convergent in V_1. Then we have

$$0 = (\iota \hat{\otimes} \mathrm{id})(x) = \sum_{i \in I} \iota(x_i) \otimes b(i) \in V_1 \hat{\otimes} V_2.$$

The uniqueness of the presentation of the form (9) for elements in $V_1 \hat{\otimes} V_2$ implies that $\iota(x_i) = 0$ for every $i \in I$. Because ι is injective, we have $x_i = 0$ for every i. This means $x = 0$, and hence $\iota \hat{\otimes} \mathrm{id}$ is injective. □

2B Topological ribbon Hopf algebra

In this paper, by a *topological Hopf algebra* $\mathscr{H} = (\mathscr{H}, \boldsymbol{\mu}, \boldsymbol{\eta}, \Delta, \boldsymbol{\epsilon}, S)$ we mean a topologically free $\mathbb{C}[\![h]\!]$–module $\mathscr{H}$ of countable topological rank, together with $\mathbb{C}[\![h]\!]$–module homomorphisms

$$\boldsymbol{\mu}\colon \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H} \to \mathscr{H}, \quad \boldsymbol{\eta}\colon \mathbb{C}[\![h]\!] \to \mathscr{H}, \quad \Delta\colon \mathscr{H} \to \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H},$$
$$\boldsymbol{\epsilon}\colon \mathscr{H} \to \mathbb{C}[\![h]\!], \quad S\colon \mathscr{H} \to \mathscr{H},$$

which are the multiplication, unit, comultiplication, counit and antipode of $\mathscr{H}$, respectively, satisfying the usual axioms of a Hopf algebra. For simplicity, we include invertibility of the antipode in the axioms of Hopf algebra. We denote $\eta(1)$ by $1 \in \mathscr{H}$.

Note that $\mathscr{H}$ is a $\mathbb{C}[\![h]\!]$–algebra in the usual (noncomplete) sense, although $\mathscr{H}$ is not a $\mathbb{C}[\![h]\!]$–coalgebra in general. A (left) $\mathscr{H}$–module V (in the usual sense) is said to be *topologically free* if V is topologically free as a $\mathbb{C}[\![h]\!]$–module. In that case, by continuity the left action $\mathscr{H} \otimes V \to V$ induces a $\mathbb{C}[\![h]\!]$–module homomorphism

$$\mathscr{H} \mathbin{\hat{\otimes}} V \to V.$$

For details on topological Hopf algebras and topologically free modules, see eg [35, Section XVI.4].

Let $\boldsymbol{\mu}^{[n]}\colon \mathscr{H}^{\hat{\otimes} n} \to \mathscr{H}$ and $\Delta^{[n]}\colon \mathscr{H} \to \mathscr{H}^{\hat{\otimes} n}$ be respectively the multiproduct and the multicoproduct defined by

$$\boldsymbol{\mu}^{[n]} = \boldsymbol{\mu}(\mathrm{id} \mathbin{\hat{\otimes}} \boldsymbol{\mu}) \cdots (\mathrm{id}^{\hat{\otimes}(n-3)} \mathbin{\hat{\otimes}} \boldsymbol{\mu})(\mathrm{id}^{\hat{\otimes}(n-2)} \mathbin{\hat{\otimes}} \boldsymbol{\mu}),$$
$$\Delta^{[n]} = (\mathrm{id}^{\hat{\otimes}(n-2)} \mathbin{\hat{\otimes}} \Delta)(\mathrm{id}^{\hat{\otimes}(n-3)} \mathbin{\hat{\otimes}} \Delta) \cdots (\mathrm{id} \mathbin{\hat{\otimes}} \Delta)\Delta,$$

with the convention that $\Delta^{[1]} = \boldsymbol{\mu}^{[1]} = \mathrm{id}$, $\Delta^{[0]} = \boldsymbol{\epsilon}$, and $\boldsymbol{\mu}^{[0]} = \boldsymbol{\eta}$.

A *universal R–matrix* [17] for $\mathscr{H}$ is an invertible element $\mathcal{R} = \sum \alpha \otimes \beta \in \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H}$ satisfying

$$\mathcal{R}\Delta(x)\mathcal{R}^{-1} = \sum x_{(2)} \otimes x_{(1)} \quad \text{for } x \in \mathscr{H},$$
$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ (Sweedler's notation), $\mathcal{R}_{12} = \sum \alpha \otimes \beta \otimes 1$, $\mathcal{R}_{13} = \sum \alpha \otimes 1 \otimes \beta$, and $\mathcal{R}_{23} = \sum 1 \otimes \alpha \otimes \beta$. A Hopf algebra with a universal R–matrix is called a *quasitriangular Hopf algebra*. The universal R–matrix satisfies

$$\mathcal{R}^{-1} = (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S^{-1})(\mathcal{R}), \quad (\boldsymbol{\epsilon} \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \boldsymbol{\epsilon})(\mathcal{R}) = 1,$$
$$(S \otimes S)(\mathcal{R}) = \mathcal{R}. \tag{10}$$

A quasitriangular Hopf algebra $(\mathscr{H}, \mathcal{R})$ is called a *ribbon Hopf algebra* [72] if it is equipped with a *ribbon element*, which is defined to be an invertible, central element $\boldsymbol{r} \in \mathscr{H}$ satisfying

$$\boldsymbol{r}^2 = \boldsymbol{u} S(\boldsymbol{u}), \quad S(\boldsymbol{r}) = \boldsymbol{r}, \quad \boldsymbol{\epsilon}(\boldsymbol{r}) = 1, \quad \Delta(\boldsymbol{r}) = (\boldsymbol{r} \otimes \boldsymbol{r})(\mathcal{R}_{21}\mathcal{R})^{-1},$$

where $\boldsymbol{u} = \sum S(\beta)\alpha \in \mathscr{H}$ and $\mathcal{R}_{21} = \sum \beta \otimes \alpha \in \mathscr{H}^{\hat{\otimes} 2}$.

The element $\boldsymbol{g} := \boldsymbol{u}\boldsymbol{r}^{-1} \in \mathscr{H}$, called the *balanced element*, satisfies

$$\Delta(\boldsymbol{g}) = \boldsymbol{g} \otimes \boldsymbol{g}, \qquad S(\boldsymbol{g}) = \boldsymbol{g}^{-1}, \qquad \boldsymbol{g} x \boldsymbol{g}^{-1} = S^2(x) \quad \text{for } x \in \mathscr{H}.$$

See [35; 68; 82] for more details on quasitriangular and ribbon Hopf algebras.

2C Topologically free $\mathscr{H}$–modules

The ground ring $\mathbb{C}[\![h]\!]$ is considered as a topologically free $\mathscr{H}$–module, called the *trivial module*, by the action of the counit:

$$a \cdot x = \boldsymbol{\epsilon}(a)x.$$

Suppose V and W are topologically free $\mathscr{H}$–modules. Then $V \hat{\otimes} W$ has the structure of an $\mathscr{H} \hat{\otimes} \mathscr{H}$–module, given by

$$(a \otimes b) \cdot (x \otimes y) = (a \cdot x) \otimes (b \cdot y).$$

Using the comultiplication, $V \hat{\otimes} W$ has an $\mathscr{H}$–module structure given by

$$a \cdot (x \otimes y) := \Delta(a) \cdot (x \otimes y) = \sum a_{(1)} x \otimes a_{(2)} y.$$

An element $x \in V$ is called *invariant* (or *$\mathscr{H}$–invariant*) if, for every $a \in \mathscr{H}$,

$$a \cdot x = \boldsymbol{\epsilon}(a)x.$$

The set of invariant elements of V is denoted by V^{inv}. The following is standard and well-known:

Proposition 2.4 *Suppose that V and W are topologically free $\mathscr{H}$–modules and $f\colon V \hat{\otimes} W \to \mathbb{C}[\![h]\!]$ is a $\mathbb{C}[\![h]\!]$–module homomorphism.*

(a) *An element $x \in V \hat{\otimes} W$ is invariant if and only if, for every $a \in \mathscr{H}$,*

$$(S(a) \otimes 1) \cdot x = (1 \otimes a) \cdot x. \tag{11}$$

(b) *Dually, f is an $\mathscr{H}$–module homomorphism if and only if, for every $a \in \mathscr{H}$ and $x \in V \hat{\otimes} W$,*

$$f[(a \otimes 1) \cdot (x)] = f[(1 \otimes S(a)) \cdot (x)].$$

(c) *Suppose f is an $\mathscr{H}$–module homomorphism and $x \in V$ is invariant. Then the $\mathbb{C}[\![h]\!]$–module homomorphism*

$$f_x \colon W \to \mathbb{C}[\![h]\!], \quad y \mapsto f(x \otimes y),$$

is an $\mathscr{H}$–module homomorphism.

(d) *Suppose $g \colon V \to \mathbb{C}[\![h]\!]$ is an $\mathscr{H}$–module homomorphism. Then, for every i with $1 \le i \le n$,*

$$(\mathrm{id}^{\hat{\otimes}(i-1)} \mathbin{\hat{\otimes}} g \mathbin{\hat{\otimes}} \mathrm{id}^{\hat{\otimes}(n-i)})((V^{\hat{\otimes}n})^{\mathrm{inv}}) \subset (V^{\hat{\otimes}(n-1)})^{\mathrm{inv}}.$$

Proof (a) Suppose one has (11). Let $a \in \mathscr{H}$ with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. Assume $x = \sum x' \otimes x''$. By definition,

$$\begin{aligned} a \cdot x &= \sum (a_{(1)} \otimes a_{(2)}) \cdot (x' \otimes x'') = \sum (a_{(1)} \otimes 1)(1 \otimes a_{(2)}) \cdot (x' \otimes x'') \\ &= \sum (a_{(1)} \otimes 1) \cdot (x' \otimes a_{(2)} \cdot x'') \\ &= \sum (a_{(1)} \otimes 1)(S(a_{(2)}) \cdot x' \otimes x'') \\ &= \sum (a_{(1)} S(a_{(2)}) \otimes 1) \cdot (x' \otimes x'') = \boldsymbol{\epsilon}(a) x, \end{aligned}$$

which shows that x is invariant.

Conversely, suppose x is invariant. From the axioms of a Hopf algebra,

$$1 \otimes a = \sum (S(a_{(1)}) \otimes 1) \Delta(a_{(2)}).$$

Applying both sides to x, we have

$$\begin{aligned} (1 \otimes a) \cdot x &= \sum (S(a_{(1)}) \otimes 1) \cdot (a_{(2)} \cdot x) \\ &= \sum (S(a_{(1)}) \otimes 1) \cdot (\boldsymbol{\epsilon}(a_{(2)}) x) \quad \text{by invariance} \\ &= (S(a) \otimes 1) \cdot x, \end{aligned}$$

which proves (11).

(b) The proof of (b) is similar and is left for the reader. Statement (b) is mentioned in the textbooks [30, Section 6.20] and [41, Section 6.3.2].

(c) Let $a \in \mathscr{H}$ and $y \in W$. One has

$$\begin{aligned} f_x(a \cdot y) &= f(x \otimes (a \cdot y)) \\ &= f(S^{-1}(a) \cdot x \otimes y) && \text{by part (b)} \\ &= \boldsymbol{\epsilon}(S^{-1}(a)) f(x \otimes y) && \text{by invariance of } x \\ &= \boldsymbol{\epsilon}(a) f_x(y) && \text{since } \boldsymbol{\epsilon}(S^{-1}(a)) = \boldsymbol{\epsilon}(a). \end{aligned}$$

This proves f_x is an $\mathscr{H}$–module homomorphism.

(d) The map $\tilde{g} := \mathrm{id}^{\hat{\otimes}(i-1)} \hat{\otimes} g \hat{\otimes} \mathrm{id}^{\hat{\otimes}(n-i)}$ is also an $\mathscr{H}$–module homomorphism. Hence, for every $a \in \mathscr{H}$ and $x \in (V^{\hat{\otimes} n})^{\mathrm{inv}}$,

$$a \cdot \tilde{g}(x) = \tilde{g}(a \cdot x) = \tilde{g}(\boldsymbol{\varepsilon}(a)x) = \boldsymbol{\varepsilon}(a)\tilde{g}(x).$$

This shows $\tilde{g}(x)$ is invariant. □

2D Left image of an element

Let V and W be topologically free $\mathbb{C}[\![h]\!]$–modules.

Suppose $x \in V \hat{\otimes} W$. Choose a topological basis $\{e(i) \mid i \in I\}$ of W. Then x can be uniquely presented as an h–adically convergent sum

$$x = \sum_{i \in I} x_i \otimes e(i), \tag{12}$$

where $\{x_i \in V \mid i \in I\}$ is 0–convergent. The *left image* V_x of $x \in V \hat{\otimes} W$ is the topological closure (in the h–adic topology of V) of the $\mathbb{C}[\![h]\!]$–span of $\{x_i \mid i \in I\}$. It is easy to show that V_x does not depend on the choice of the topological basis $\{e(i) \mid i \in I\}$ of W.

Proposition 2.5 *Suppose V, W are topologically free $\mathscr{H}$–modules. Let $x \in V \hat{\otimes} W$ and let $V_x \subset V$ be the left image of x.*

(a) *If x is $\mathscr{H}$–invariant, then V_x is $\mathscr{H}$–stable, ie $\mathscr{H} \cdot V_x \subset V_x$.*
(b) *If $(f \hat{\otimes} g)(x) = x$, where $f\colon V \to V$ and $g\colon W \to W$ are $\mathbb{C}[\![h]\!]$–module isomorphisms, then $f(V_x) = V_x$.*

Proof Let $\{e(i) \mid i \in I\}$ be a topological basis of W and let x_i be as in (12).

(a) By Proposition 2.4(a), the $\mathscr{H}$–invariance of x implies that, for every $a \in \mathscr{H}$,

$$\sum_{i \in I} a \cdot x_i \otimes e(i) = \sum_{j \in I} x_j \otimes S^{-1}(a) \cdot e(j). \tag{13}$$

Using the topological basis $\{e(i)\}$, we have the structure constants

$$S^{-1}(a) \cdot e(j) = \sum_{i \in I} a^i_j e(i),$$

where $a^i_j \in \mathbb{C}[\![h]\!]$. Using this expression in (13),

$$\sum_{i \in I} a \cdot x_i \otimes e(i) = \sum_{i \in I} \sum_{j \in I} a^i_j x_j \otimes e(i).$$

The uniqueness of expression of the form (12) shows that

$$a \cdot x_i = \sum_{j \in I} a^i_j x_j \in V_x.$$

Since the $\mathbb{C}[\![h]\!]$–span of x_i is dense in V_x and the action of a is continuous in the h–adic topology of V, we have $a \cdot V_x \subset V_x$.

(b) Using $x = (f \mathbin{\hat{\otimes}} g)(x)$, we have

$$x = \sum_i f(x_i) \otimes g(e_i).$$

Since g is a $\mathbb{C}[\![h]\!]$–module isomorphism, $\{g(e_i)\}$ is a topological basis of W. It follows that V_x is the closure of the $\mathbb{C}[\![h]\!]$–span of $\{f(x_i) \mid i \in I\}$. At the same time, V_x is the closure of the $\mathbb{C}[\![h]\!]$–span of $\{x_i \mid i \in I\}$. Hence, we have $f(V_x) = V_x$. □

2E Adjoint action and ad-invariance

Suppose $\mathscr{H}$ is a topological ribbon Hopf algebra. The (left) adjoint action

$$\mathrm{ad}\colon \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H} \to \mathscr{H}$$

of $\mathscr{H}$ on itself is defined by

$$\mathrm{ad}(x \otimes y) = \sum x_{(1)} y S(x_{(2)}).$$

It is convenient to use an infix notation for ad:

$$x \triangleright y = \mathrm{ad}(x \otimes y).$$

We regard $\mathscr{H}$ as a (topologically free) $\mathscr{H}$–module via the adjoint action, unless otherwise stated. Then $\mathscr{H}^{\hat{\otimes} n}$ becomes a topologically free $\mathscr{H}$–module, for every $n \geq 0$. The action of $x \in \mathscr{H}$ on $y \in \mathscr{H}^{\hat{\otimes} n}$ is denoted by $x \triangleright_n y$.

To emphasize the adjoint action, we say that a $\mathbb{C}[\![h]\!]$–submodule $V \subset \mathscr{H}^{\hat{\otimes} n}$ is *ad-stable* if V is an $\mathscr{H}$–submodule of $\mathscr{H}^{\hat{\otimes} n}$. An element $x \in \mathscr{H}^{\hat{\otimes} n}$ is *ad-invariant* if it is an invariant element of $\mathscr{H}^{\hat{\otimes} n}$ under the adjoint action. For example, an element of $\mathscr{H}$ is ad-invariant if and only if it is central.

For ad-stable submodules $V \subset \mathscr{H}^{\hat{\otimes} n}$ and $W \subset \mathscr{H}^{\hat{\otimes} m}$, a $\mathbb{C}[\![h]\!]$–module homomorphism $f\colon V \to W$ is *ad-invariant* if f is an $\mathscr{H}$–module homomorphism.

In particular, a linear functional $f\colon V \to \mathbb{C}[\![h]\!]$, where $V \subset \mathscr{H}^{\hat{\otimes} n}$, is ad-invariant if V is ad-stable and, for $x \in \mathscr{H}$ and $y \in V$,

$$f(x \triangleright_n y) = \epsilon(x) f(y).$$

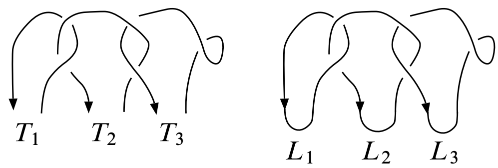


Figure 1: A 3–component bottom tangle $T = T_1 \cup T_2 \cup T_3$ (left) and its closure $\mathrm{cl}(T) = L_1 \cup L_2 \cup L_3$ (right)

The main source of ad-invariant linear functionals comes from quantum traces. Here the quantum trace $\mathrm{tr}_q^V \colon \mathscr{H} \to \mathbb{C}[\![h]\!]$ for a finite-dimensional representation V (ie a topologically free $\mathscr{H}$–module of finite topological rank) is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\boldsymbol{g}x) \quad \text{for } x \in \mathscr{H},$$

where tr^V denotes the trace in V. It is known that $\mathrm{tr}_q^V \colon \mathscr{H} \to \mathbb{C}[\![h]\!]$ is ad-invariant.

2F Bottom tangles

Here we recall the definition of bottom tangles from [23, Section 7.3].

An n–component *bottom tangle* $T = T_1 \cup \cdots \cup T_n$ is a framed tangle in a cube consisting of n arc components $T_1, \dots, T_n$ such that all the endpoints of the T_i are in a bottom line and that, for each i, the component T_i runs from the $2i^{\text{th}}$ endpoint to the $(2i-1)^{\text{st}}$ endpoint, where the endpoints are counted from the left. See Figure 1 (left) for an example. In figures, framings are specified by the blackboard framing convention.

The *closure* $\mathrm{cl}(T)$ of T is the n–component, oriented, ordered framed link in S^3, obtained from T by pasting a "$\cup$–shaped tangle" to each component of L, as depicted in Figure 1 (right). For any oriented, ordered framed link L, there is a bottom tangle whose closure is isotopic to L.

The *linking matrix* of a bottom tangle $T = T_1 \cup \cdots \cup T_n$ is defined as that of the closure T. Thus the linking number of T_i and T_j, $i \neq j$, is defined as the linking number of the corresponding components in $\mathrm{cl}(T)$, and the framing of T_i is defined as the framing of the closure of T_i.

A link or a bottom tangle is called *algebraically split* if the linking matrix is diagonal.

2G Universal invariant and quantum link invariants

Reshetikhin and Turaev [72] constructed quantum invariants of framed links colored by finite-dimensional representations of a ribbon Hopf algebra, eg the quantum group $U_h(\mathfrak{g})$. Lawrence [44], Reshetikhin [70], Ohtsuki [64] and Kauffman [36]

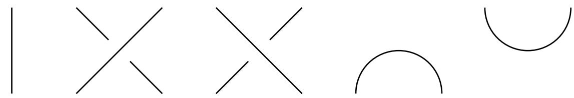


Figure 2: Fundamental tangles: vertical line, positive and negative crossings, local minimum and local maximum. Here the orientations are arbitrary.

constructed "universal quantum invariants" of links and tangles with values in (quotients of) tensor powers of the ribbon Hopf algebra, where the links and tangles are not colored by representations. We recall here construction of link invariants via the universal invariant of bottom tangles. We refer the readers to [23] for details.

Fix a ribbon Hopf algebra $\mathscr{H}$. Let T be a bottom tangle with n components. We choose a diagram for T, which is obtained from copies of *fundamental tangles* — see Figure 2 — by pasting horizontally and vertically. For each copy of fundamental tangle in the diagram of T, we put elements of $\mathscr{H}$ with the rule described in Figure 3.

We set

$$J_T := \sum x_1 \otimes \cdots \otimes x_n \in \mathscr{H}^{\hat{\otimes} n},$$

where each x_i is the product of the elements put on the i^{th} component T_i, with product taken in the order reversing the order of the orientation. The (generally infinite) sum comes from the decompositions of $\mathcal{R}^{\pm 1}$ as (infinite) sums of tensor products. It is known that J_T gives an isotopy invariant of bottom tangles, called the universal invariant of T. Moreover, J_T is ad-invariant [38]; see also [23].

Let $\chi_1, \dots, \chi_n \colon \mathscr{H} \to \mathbb{C}[\![h]\!]$ be ad-invariant. In other words, $\chi_1, \dots, \chi_n$ are $\mathscr{H}$–module homomorphisms. As explained in [23], the quantity

$$(\chi_1 \hat{\otimes} \cdots \hat{\otimes} \chi_n)(J_T) \in \mathbb{C}[\![h]\!]$$

is a *link invariant* of the closure link $\mathrm{cl}(T)$ of T.

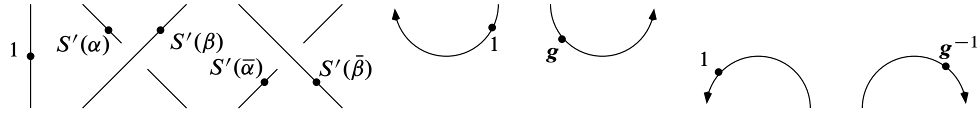


Figure 3: How to put elements of $\mathscr{H}$ on the strings. Here $R = \sum \alpha \otimes \beta$ and $R^{-1} = \sum \bar{\alpha} \otimes \bar{\beta}$. For each string in the positive and the negative crossings, "S'" should be replaced by id if the string is oriented downward and by S otherwise.

In particular, if $\chi_1, \dots, \chi_n$ are the quantum traces $\mathrm{tr}_q^{V_1}, \dots, \mathrm{tr}_q^{V_n}$ in finite-dimensional representations $V_1, \dots, V_n$, respectively, then

$$(\mathrm{tr}_q^{V_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathrm{tr}_q^{V_n})(J_T) \in \mathbb{C}[\![h]\!]$$

is the quantum link invariant for $\mathrm{cl}(T)$ colored by the representations $V_1, \dots, V_n$.

2H Mirror image of bottom tangles

Definition 2.6 A *mirror homomorphism* of a topological ribbon Hopf algebra $\mathscr{H}$ is an h–adically continuous $\mathbb{C}$–algebra homomorphism $\varphi\colon \mathscr{H} \to \mathscr{H}$ satisfying

$$(\varphi \mathbin{\hat{\otimes}} \varphi)\mathcal{R} = \mathcal{R}_{21}^{-1}, \tag{14}$$

$$\varphi(\boldsymbol{g}) = \boldsymbol{g}. \tag{15}$$

In general, such a φ is not a $\mathbb{C}[\![h]\!]$–algebra homomorphism. In fact, what we will have in the future is $\varphi(h) = -h$.

For a bottom tangle T with diagram D let the *mirror image of T* be the bottom tangle whose diagram is obtained from D by switching over/under crossing at every crossing.

Proposition 2.7 *Suppose φ is a mirror homomorphism of a ribbon Hopf algebra $\mathscr{H}$. If T' is the mirror image of an n–component bottom tangle T, then*

$$J_{T'} = \varphi^{\hat{\otimes} n}(J_T).$$

Proof Let D be a diagram of T. By rotations at crossings if necessary, we can assume that the two strands at each crossing of D are oriented downwards. Then at each crossing we assign α and β to the strands if the crossing is positive, and we assign $\bar{\beta}$ and $\bar{\alpha}$ to the strands if it is negative, at the same spots where we would assign α and β if the crossing were positive; see Figure 4. Here $\mathcal{R} = \sum \alpha \otimes \beta$ and $\mathcal{R}^{-1} = \sum \bar{\alpha} \otimes \bar{\beta}$. Condition (14) implies that

$$\sum \bar{\beta} \otimes \bar{\alpha} = \sum \varphi(\alpha) \otimes \varphi(\beta), \quad \sum \alpha \otimes \beta = \sum \varphi(\bar{\beta}) \otimes \varphi(\bar{\alpha}).$$

Together with (15) this shows that the assignments to strands of diagram D' of T' can be obtained by applying φ to the corresponding assignments to strands of D. Since φ is a $\mathbb{C}$–algebra homomorphism, we get $J_{T'} = \varphi^{\hat{\otimes} n}(J_T)$. □

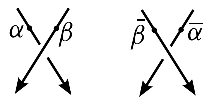


Figure 4: Assignments on positive and negative crossings

2I Braiding and transmutation

Let $\mathcal{R} = \sum \alpha \otimes \beta$ be the R–matrix. The *braiding* for $\mathscr{H}$ and its inverse

$$\boldsymbol{\psi}^{\pm 1}\colon \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H} \to \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H}$$

are given by

$$\text{(16)} \quad \boldsymbol{\psi}(x \otimes y) = \sum (\beta \triangleright y) \otimes (\alpha \triangleright x), \quad \boldsymbol{\psi}^{-1}(x \otimes y) = \sum (S(\alpha) \triangleright y) \otimes (\beta \triangleright x).$$

The maps $\boldsymbol{\mu}$, $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ are $\mathscr{H}$–module homomorphisms. In particular, we have

$$\text{(17)} \qquad x \triangleright yz = \sum (x_{(1)} \triangleright y)(x_{(2)} \triangleright z) \quad \text{for } x, y, z \in \mathscr{H}.$$

In general, Δ and S are not $\mathscr{H}$–module homomorphisms, but the twisted versions of Δ and S, introduced by Majid (see [57; 58]),

$$\underline{\Delta}\colon \mathscr{H} \to \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{H}, \quad \underline{S}\colon \mathscr{H} \to \mathscr{H},$$

defined by

$$\text{(18)} \qquad \underline{\Delta}(x) = \sum x_{(1)} S(\beta) \otimes (\alpha \triangleright x_{(2)}) = \sum (\beta \triangleright x_{(2)}) \otimes \alpha x_{(1)},$$

$$\text{(19)} \qquad \underline{S}(x) = \sum \beta S(\alpha \triangleright x) = \sum S^{-1}(\beta \triangleright x) S(\alpha),$$

for $x \in \mathscr{H}$, are $\mathscr{H}$–module homomorphisms. Geometric interpretations of $\underline{\Delta}$ and $\underline{S}$ are given in [23].

Remark 2.8 $\underline{\mathscr{H}} := (\mathscr{H}, \boldsymbol{\mu}, \boldsymbol{\eta}, \underline{\Delta}, \boldsymbol{\epsilon}, \underline{S})$ forms a braided Hopf algebra in the braided category of topologically free $\mathscr{H}$–modules, called the *transmutation* of $\mathscr{H}$ [57; 58].

2J Clasp bottom tangle

Let C^+ be the *clasp tangle* depicted in Figure 5. We call $\boldsymbol{c} = J_{C^+} \in \mathscr{H}^{\hat{\otimes} 2}$ the *clasp element* for $\mathscr{H}$. With $\mathcal{R} = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta'$, we have

$$\text{(20)} \qquad \boldsymbol{c} = (S \mathbin{\hat{\otimes}} \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) = \sum S(\alpha) S(\beta') \otimes \alpha' \beta.$$

Let C^- be the mirror image of C^+ — see Figure 6 — and $\boldsymbol{c}^- = J_{C^-} \in \mathscr{H}^{\hat{\otimes} 2}$.

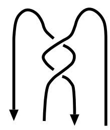
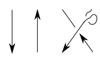


Figure 5: The clasp tangle C^+

Let $(C^+)'$ be the tangle obtained by reversing the orientation of the second component of C^+, and $(C^+)''$ be the result of putting $(C^+)'$ on top of the tangle



(see Figure 6). By the geometric interpretation of $\underline{S}$ — see [23, Formula (8-10)] — we have

$$J_{(C^+)''} = (\mathrm{id} \mathbin{\hat{\otimes}} \underline{S}) J_{C^+}.$$

Since $(C^+)''$ is isotopic to C^-, we have

$$\boldsymbol{c}^- = (\mathrm{id} \mathbin{\hat{\otimes}} \underline{S})(\boldsymbol{c}). \tag{21}$$

2K Hoste moves

It is known that every integral homology 3–sphere can be obtained by surgery on S^3 along an algebraically split link with ± 1 framings.

The following refinement of the Kirby–Fenn–Rourke theorem on framed links was first essentially conjectured by Hoste [28]. (Hoste stated it in a more general form related to Rolfsen's calculus for rationally framed links.)

Theorem 2.9 [24] *Let L and L' be two nonoriented, unordered, algebraically split ± 1–framed links in S^3. Then L and L' give orientation-preserving homeomorphic results of surgery if and only if L and L' are related by a sequence of ambient isotopy and* ***Hoste moves****.* (Here a Hoste move is a Fenn–Rourke (FR) move between two algebraically split, ± 1–framed links; see Figure 7.)

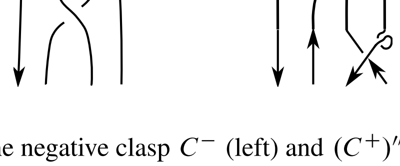


Figure 6: The negative clasp C^- (left) and $(C^+)''$ are isotopic.

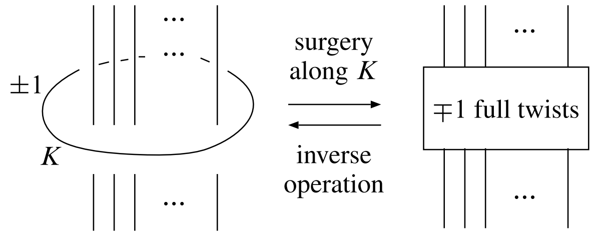


Figure 7: A Hoste move (including the case when there are no vertical strands). Here both these two framed links are algebraically split and ± 1–framed.

Theorem 2.9 implies that, to construct an invariant of integral homology spheres, it suffices to construct an invariant of algebraically split, ± 1–framed links which is invariant under the Hoste moves.

Lemma 2.10 *Suppose f is an invariant of oriented, unordered, algebraically split ± 1–framed links which is invariant under Hoste moves. Then $f(L)$ does not depend on the orientation of the link L. Consequently, f descends to an invariant of integral homology 3–spheres, ie if the results of surgery along two oriented, unordered, algebraically split ± 1–framed links L and L' are homeomorphic integral homology 3–spheres, then $f(L) = f(L')$.*

Proof Suppose K is a component of L so that $L = L_1 \cup K$. We will show that f does not depend on the orientation of K by induction on the unknotting number of K.

First assume that K is an unknot. We first apply the Hoste move to K, then apply the Hoste move in the reverse way, obtaining $L_1 \cup (-K)$, where $-K$ is the orientation-reversal of K. This shows $f(L_1 \cup K) = f(L_1 \cup (-K))$.

Suppose K is an arbitrary knot with positive unknotting number. We can use a Hoste move to realize a self-crossing change of K, reducing the unknotting number. Induction on the unknotting number shows that f does not depend on the orientation of K. □

2L Definition of the invariant J_M for the case when the ground ring is a field

In this subsection, we explain a construction of an invariant of integral homology spheres associated to a ribbon Hopf algebra over a field $\boldsymbol{k}$, equipped with "full twist forms". In this subsection, and only here, we will assume that $\mathscr{H}$ is a ribbon Hopf algebra over a field $\boldsymbol{k}$. This assumption simplifies the definition of the invariant.

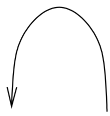


Figure 8: The trivial bottom tangle

2L1 Full twist forms Recall that $\boldsymbol{c} = J_{C^+} \in \mathscr{H} \otimes \mathscr{H}$ is the universal invariant of the clasp bottom tangle and $\boldsymbol{r}$ is the ribbon element.

A pair of ad-invariant linear functionals $\mathcal{T}_+, \mathcal{T}_-\colon \mathscr{H} \to \boldsymbol{k}$ are called *full twist forms* for $\mathscr{H}$ if

(22)
$$(\mathcal{T}_\pm \otimes \mathrm{id})(\boldsymbol{c}) = \boldsymbol{r}^{\pm 1}.$$

The following lemma essentially shows how the universal link invariant behaves under the Hoste move if there are full twist forms:

Lemma 2.11 *Suppose that a ribbon Hopf algebra $\mathscr{H}$ admits full twist forms $(\mathcal{T}_+, \mathcal{T}_-)$. Let $T = T_1 \cup \cdots \cup T_n$ be an n–component bottom tangle $(n \geq 1)$ such that the first component T_1 of T is a 1–component trivial bottom tangle* (see Figure 8). *Let $T' = T_2' \cup \cdots \cup T_n'$ be the $(n-1)$–component bottom tangle obtained from $T \setminus T_1 = T_2 \cup \cdots \cup T_n$ by surgery along the closure of T_1 with framing ± 1* (see Figure 9). *Then we have*

(23)
$$J_{T'} = (\mathcal{T}_\pm \otimes \mathrm{id}^{\otimes (n-1)})(J_T).$$

Proof In this proof we use the universal invariant for tangles that are not bottom tangles. For details, see [23, Section 7.3].

If $T_{p,q}$ is a $(p+q+1)$–component tangle as depicted in Figure 10, left, with $p, q \geq 0$, then we have
$$J_{T_{p,q}} = (\mathrm{id} \otimes \mathrm{id}^{\otimes p} \otimes S^{\otimes q})(\mathrm{id} \otimes \Delta^{[p+q]})(\boldsymbol{c}).$$

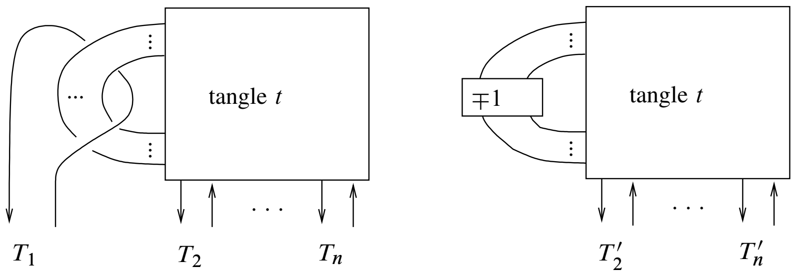


Figure 9: The tangles T (left) and T' (right)

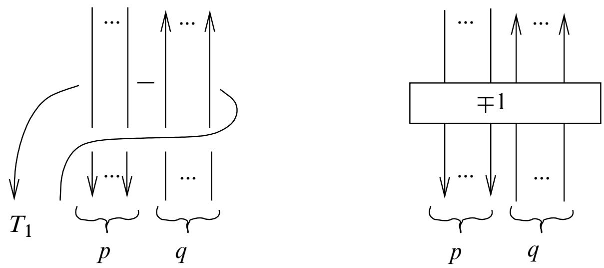


Figure 10: The tangles $T_{p,q}$ (left) and $T'_{p,q;\pm 1}$ (right)

The tangle $T'_{p,q;\pm 1}$ obtained from $T_{p,q} \setminus T_1$ by surgery along the closure of the first component T_1 of $T_{p,q}$ with framing ± 1 (see Figure 10, right) has the universal invariant

$$J_{T'_{p,q;\pm 1}} = (\mathrm{id}^{\otimes p} \otimes S^{\otimes q})\Delta^{[p+q]}(\boldsymbol{r}^{\pm 1}).$$

Since $\mathcal{T}_\pm$ is a full twist form, it follows that

$$(\mathcal{T}_\pm \otimes \mathrm{id}^{\otimes(p+q)})(J_{T_{p,q}}) = J_{T'_{p,q}}.$$

The general case follows from the above case and functoriality of the universal invariant, since T can be obtained from some $T_{p,q}$ by tensoring and composing appropriate tangles. □

2L2 Invariant of integral homology 3–spheres We will show here that a ribbon Hopf algebra $\mathscr{H}$ with full twist forms $\mathcal{T}_\pm$ gives rise to an invariant of integral homology spheres.

Suppose that T is an n–component bottom tangle with zero linking matrix and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$. Let $M = M(T; \varepsilon_1, \dots, \varepsilon_n)$ be the oriented 3–manifold obtained by surgery on S^3 along the framed link $L = L(T; \varepsilon_1, \dots, \varepsilon_n)$, which is the closure link of T with the framing on the i^{th} component switched to ε_i. Since L is an algebraically split link with ± 1 framing on each component, M is an integral homology 3–sphere. Every integral homology 3–sphere can be obtained in this way.

Proposition 2.12 *Suppose $\mathscr{H}$ is a ribbon Hopf algebra with full twist forms $\mathcal{T}_\pm$ and $M = M(T; \varepsilon_1, \dots, \varepsilon_n)$ is an integral homology 3–sphere. Then*

$$J_M := (\mathcal{T}_{\varepsilon_1} \otimes \dots \otimes \mathcal{T}_{\varepsilon_n})(J_T) \in \boldsymbol{k}$$

is an invariant of M. In other words, if $M(T; \varepsilon_1, \dots, \varepsilon_n) \cong M(T'; \varepsilon'_1, \dots, \varepsilon'_{n'})$, then

$$(\mathcal{T}_{\varepsilon_1} \otimes \dots \otimes \mathcal{T}_{\varepsilon_n})(J_T) = (\mathcal{T}_{\varepsilon'_1} \otimes \dots \otimes \mathcal{T}_{\varepsilon'_{n'}})(J_{T'}).$$

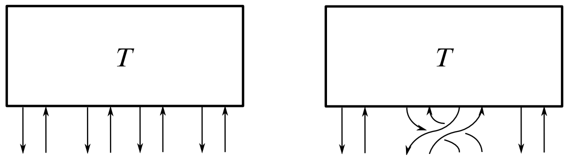


Figure 11: Modification of a bottom tangle with a braid of bands

Proof Since $\mathcal{T}_\pm$ are ad-invariant, $(\mathcal{T}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ depends only on $\varepsilon_1, \ldots, \varepsilon_n$ and the oriented, ordered framed link $\mathrm{cl}(T)$, but not on the choice of T; see eg [23, Section 11.1]. This shows $(\mathcal{T}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ is an invariant of the framed link $L(T; \varepsilon_1, \ldots, \varepsilon_n)$.

We now show that $(\mathcal{T}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ does not depend on the order of the components of L. Suppose $L = L(T; \varepsilon_1, \ldots, \varepsilon_n)$ and L' is the same L, with the orders of the $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ components switched. Then $L' = L(T'; \varepsilon'_1, \ldots, \varepsilon'_n)$, where T' is T on top of a simple braid of bands which switches the $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ components; see Figure 11. Also, $\varepsilon'_j = \varepsilon_j$ for $j \neq i+1,\ i+2$, and $\varepsilon'_{i+1} = \varepsilon_{i+2}$ and $\varepsilon'_{i+2} = \varepsilon_{i+1}$.

According to the geometric interpretation of the braiding [23, Proposition 8.1],

$$J_{T'} = (\mathrm{id}^{\otimes i} \otimes \boldsymbol{\psi} \otimes \mathrm{id}^{\otimes n-i-2})(J_T).$$

By (16),

$$\boldsymbol{\psi}(x \otimes y) = \sum (\beta \triangleright y) \otimes (\alpha \triangleright x), \quad \text{where } \mathcal{R} = \sum \alpha \otimes \beta.$$

Since $(\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon})(\mathcal{R}) = 1$ and $\mathcal{T}_\pm$ are ad-invariant,

$$(\mathcal{T}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon_n})(J_T) = (\mathcal{T}_{\varepsilon'_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon'_n})(J_{T'}).$$

Thus, $(\mathcal{T}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ is an invariant of oriented, unordered framed links.

By Lemma 2.11, $(\mathcal{T}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ is invariant under the Hoste moves. Lemma 2.10 implies that $(\mathcal{T}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{T}_{\varepsilon_n})(J_T)$ descends to an invariant of integral homology 3–spheres. □

2L3 Examples of full twist forms: factorizable case A finite-dimensional, quasi-triangular Hopf algebra over a field $\boldsymbol{k}$ is said to be *factorizable* if the clasp element $\boldsymbol{c} \in \mathscr{H} \otimes_{\boldsymbol{k}} \mathscr{H}$ is nondegenerate in the sense that there exist bases $\{\boldsymbol{c}'(i) \mid i \in I\}$ and $\{\boldsymbol{c}''(i) \mid i \in I\}$ of $\mathscr{H}$ such that

$$\boldsymbol{c} = \sum_{i \in I} \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i).$$

This definition of factorizability is equivalent to the original definition by Reshetikhin and Semenov-Tian-Shansky [71].

Suppose $\mathscr{H}$ is a factorizable ribbon Hopf algebra. The nondegeneracy condition shows that there is a unique bilinear form, called the *clasp form*,

$$\mathscr{L}\colon \mathscr{H}\otimes\mathscr{H}\to \boldsymbol{k},$$

such that, for every $x\in\mathscr{H}$,

$$(\mathscr{L}\otimes \mathrm{id})(x\otimes \boldsymbol{c}) = x, \quad (\mathrm{id}\otimes\mathscr{L})(\boldsymbol{c}\otimes x) = x. \tag{24}$$

Using the ad-invariance of $\boldsymbol{c}$, one can show that $\mathscr{L}\colon \mathscr{H}\otimes\mathscr{H}\to\boldsymbol{k}$ is ad-invariant. Since $\boldsymbol{r}^{\pm 1}$ are ad-invariant, the form $\mathcal{T}_{\pm}\colon \mathscr{H}\to\boldsymbol{k}$ defined by

$$\mathcal{T}_{\pm}(x) := \mathscr{L}(\boldsymbol{r}^{\pm 1}\otimes x) \tag{25}$$

is ad-invariant and satisfies (22), due to (24). Hence, $\mathcal{T}_+$ and $\mathcal{T}_-$ are full twist forms for $\mathscr{H}$, and defines an invariant of integral homology 3–sphere according to Proposition 2.12.

Remark 2.13 Given a finite-dimensional, factorizable, ribbon Hopf algebra $\mathscr{H}$, one can construct the *Hennings invariant* for closed 3–manifolds [27; 37; 65; 38; 55; 77; 83; 23]. The invariant given in Proposition 2.12 constructed from the full twist forms in (25) is equal to the Hennings invariant.

2M Partially defined twist forms and invariant J_M

Let us return to the case when $\mathscr{H}$ is a ribbon Hopf algebra over $\mathbb{C}[\![h]\!]$. Recall that $\mathscr{H}$ is a topologically free $\mathscr{H}$–module with the adjoint action. In general $\mathscr{H}$ does not admit full twist forms.

In the construction of the invariant of integral homology 3–spheres in Proposition 2.12, one first constructs the universal invariant of algebraically split tangles J_T, then feeds the result to the functionals $\mathcal{T}_{\varepsilon_1}\otimes\cdots\otimes\mathcal{T}_{\varepsilon_n}$, which come from the twist forms $\mathcal{T}_{\pm}$. We will show that the conclusion of Proposition 2.12 holds true if the twist forms $\mathcal{T}_{\pm}$ are defined on a submodule large enough so that the domain of $\mathcal{T}_{\varepsilon_1}\otimes\cdots\otimes\mathcal{T}_{\varepsilon_n}$ contains all the values of J_T, with T algebraically split bottom tangles.

2M1 Partially defined twist forms Suppose $\mathscr{X}\subset\mathscr{H}$ is a topologically free $\mathbb{C}[\![h]\!]$–submodule. By Proposition 2.3 all the natural maps $\mathscr{X}^{\otimes n}\to\mathscr{X}^{\hat{\otimes} n}\to\mathscr{H}^{\hat{\otimes} n}$ and $\mathscr{X}\mathbin{\hat{\otimes}}\mathscr{H}^{\hat{\otimes}(n-1)}\to\mathscr{H}^{\hat{\otimes} n}$ are injective. Hence we will consider $\mathscr{X}^{\otimes n}$, $\mathscr{X}^{\hat{\otimes} n}$, and $\mathscr{X}\mathbin{\hat{\otimes}}\mathscr{H}^{\hat{\otimes}(n-1)}$ as submodules of $\mathscr{H}^{\hat{\otimes} n}$. This will explain the meaning of statements like “$\boldsymbol{c}\in\mathscr{X}\mathbin{\hat{\otimes}}\mathscr{H}$”.

Definition 2.14 A twist system $\mathcal{T} = (\mathcal{T}_\pm, \mathscr{X})$ of a topological ribbon Hopf algebra $\mathscr{H}$ consists of a topologically free $\mathbb{C}[\![h]\!]$–submodule $\mathscr{X} \subset \mathscr{H}$ and a pair of $\mathbb{C}[\![h]\!]$–linear functionals $\mathcal{T}_\pm \colon \mathscr{X} \to \mathbb{C}[\![h]\!]$ satisfying the following conditions:

(i) $\mathscr{X}$ is ad-stable (ie $\mathscr{X}$ is stable under the adjoint action of $\mathscr{H}$) and $\mathcal{T}_\pm$ are ad-invariant.

(ii) $\boldsymbol{c} \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$.

(iii) One has
$$(\mathcal{T}_\pm \mathbin{\hat{\otimes}} \mathrm{id})(\boldsymbol{c}) = \boldsymbol{r}^{\pm 1}.$$

Recall that, for an n–component bottom tangle T with zero linking matrix and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$, $M(T; \varepsilon_1, \dots, \varepsilon_n)$ is the integral homology sphere obtained by surgery on S^3 along the framed link $L(T; \varepsilon_1, \dots, \varepsilon_n)$, which is the closure link of T with the framing on the i^{th} component switched to ε_i.

Proposition 2.15 *Suppose $\mathcal{T} = (\mathcal{T}_\pm, \mathscr{X})$ is a twist system of a topological ribbon Hopf algebra $\mathscr{H}$ such that $J_T \in \mathscr{X}^{\hat{\otimes} n}$ for any n–component algebraically split 0–framed bottom tangle T. Let $M = M(T; \varepsilon_1, \dots, \varepsilon_n)$ be an integral homology 3–sphere. Then*
$$J_M := (\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(J_T) \in \mathbb{C}[\![h]\!]$$
is an invariant of M. In other words, if $M(T; \varepsilon_1, \dots, \varepsilon_n) = M(T'; \varepsilon'_1, \dots, \varepsilon'_{n'})$, then
$$(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(J_T) = (\mathcal{T}_{\varepsilon'_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon'_{n'}})(J_{T'}).$$

Proof First we show the following claim, which is a refinement of Lemma 2.11:

Claim *Let T and T' be tangles as in Lemma 2.11. Then $J_T \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}^{\hat{\otimes}(n-1)}$ and*
$$J_{T'} = (\mathcal{T}_\pm \mathbin{\hat{\otimes}} \mathrm{id}^{\hat{\otimes}(n-1)})(J_T) \in \mathscr{H}^{\hat{\otimes}(n-1)}. \tag{26}$$

Proof of claim If $T_{p,q}$ is a $(p+q+1)$–component tangle as depicted in Figure 10 (left) with $p, q \geq 0$, then we have
$$J_{T_{p,q}} = (\mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id}^{\hat{\otimes} p} \mathbin{\hat{\otimes}} S^{\hat{\otimes} q})(\mathrm{id} \mathbin{\hat{\otimes}} \Delta^{[p+q]})(\boldsymbol{c}).$$

Since $\boldsymbol{c} \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$, we have
$$J_{T_{p,q}} \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}^{\hat{\otimes} p+q}.$$

Since T is obtained from $T_{p,q}$ by tensoring and composing appropriate tangles which do not involve the first component, we also have
$$J_T \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}^{\hat{\otimes} n}.$$

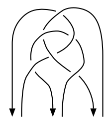


Figure 12: The Borromean tangle

The remaining part of the proof follows exactly the proof of Lemma 2.11. One first verifies the case of $T_{p,q}$ using conditions (ii) and (iii) in the definition of twist system, from which the general case follows. This proves the claim. □

Using the ad-invariance of $\mathcal{T}_\pm$ and (26), one can repeat the proof of Proposition 2.12 verbatim, replacing $\otimes$ by $\hat{\otimes}$ everywhere, to get Proposition 2.15. □

2M2 Values of the universal invariant of algebraically split tangles In Proposition 2.15, we need $J_T \in \mathscr{X}^{\hat{\otimes} n}$ for an n–component bottom tangle T with zero linking matrix. To help prove a statement like that, we use the following result.

Let $\mathcal{K}_n \subset \mathscr{H}^{\hat{\otimes} n}$, $n \geq 0$, be a family of subsets. A $\mathbb{C}[[h]]$–module homomorphism $f\colon \boldsymbol{U}_h^{\hat{\otimes} a} \to \boldsymbol{U}_h^{\hat{\otimes} b}$, $a, b \geq 0$, is said to be $(\mathcal{K}_n)$*–admissible* if we have

$$f_{(i,j)}(\mathcal{K}_{i+j+a}) \subset \mathcal{K}_{i+j+b} \tag{27}$$

for all $i, j \geq 0$. Here $f_{(i,j)} := \mathrm{id}^{\hat{\otimes} i} \mathbin{\hat{\otimes}} f \mathbin{\hat{\otimes}} \mathrm{id}^{\hat{\otimes} j}$.

Proposition 2.16 (see [23, Corollary 9.15]) *Let $\mathcal{K}_n \subset \mathscr{H}^{\hat{\otimes} n}$, $n \geq 0$, be a family of subsets such that*

(i) $1_{\mathbb{C}[[h]]} \in \mathcal{K}_0$, $1_{\mathscr{H}} \in \mathcal{K}_1$, $\boldsymbol{b} \in \mathcal{K}_3$,

(ii) *for $x \in \mathcal{K}_n$ and $y \in \mathcal{K}_m$ one has $x \otimes y \in \mathcal{K}_{n+m}$, and*

(iii) *each of $\boldsymbol{\mu}$, $\boldsymbol{\psi}^{\pm 1}$, $\underline{\Delta}$ and $\underline{S}$ is $(\mathcal{K}_n)$–admissible.*

Then we have $J_T \in \mathcal{K}_n$ for any n–component algebraically split, 0–framed bottom tangle T.

Here $\boldsymbol{b} \in \boldsymbol{U}_h^{\hat{\otimes} 3}$ is the universal invariant of the Borromean bottom tangle depicted in Figure 12.

2N Core subalgebra

We define here a *core subalgebra of a topological ribbon Hopf algebra*, and show that every core subalgebra gives rise to an invariant of integral homology 3–spheres.

In the following we use overline to denote the closure in the h–adic topology of $\mathscr{H}^{\hat{\otimes} n}$.

A *topological Hopf subalgebra* of a topological Hopf algebra $\mathscr{H}$ is a $\mathbb{C}[\![h]\!]$–subalgebra $\mathscr{H}' \subset \mathscr{H}$ such that $\mathscr{H}'$ is topologically free as a $\mathbb{C}[\![h]\!]$–module and

$$\Delta(\mathscr{H}') \subset \mathscr{H}' \mathbin{\hat{\otimes}} \mathscr{H}', \quad S^{\pm 1}(\mathscr{H}') \subset \mathscr{H}'.$$

In general, $\mathscr{H}'$ is not closed in $\mathscr{H}$.

Definition 2.17 A topological Hopf subalgebra $\mathscr{X} \subset \mathscr{H}$ of a topological ribbon Hopf algebra $\mathscr{H}$ is called a *core subalgebra* of $\mathscr{H}$ if:

(i) $\mathscr{X}$ is $\mathscr{H}$–ad-stable, ie it is an $\mathscr{H}$–submodule of $\mathscr{H}$.

(ii) $\mathcal{R} \in \overline{\mathscr{X} \otimes \mathscr{X}}$ and $\boldsymbol{g} \in \overline{\mathscr{X}}$.

(iii) The clasp element $\boldsymbol{c}$, which is contained in $\overline{\mathscr{X} \otimes \mathscr{X}}$ by (ii) (see below), has a presentation

$$\boldsymbol{c} = \sum_{i \in I} \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i), \tag{28}$$

where each of the two sets $\{\boldsymbol{c}'(i) \mid i \in I\}$ and $\{\boldsymbol{c}''(i) \mid i \in I\}$ is

- 0–convergent in $\mathscr{H}$, and
- a topological basis of $\mathscr{X}$.

Some clarifications are in order. As a topological Hopf subalgebra, $\mathscr{X}$ is topologically free as a $\mathbb{C}[\![h]\!]$–module. By Proposition 2.3, all the natural maps $\mathscr{X}^{\otimes n} \to \mathscr{X}^{\hat{\otimes} n} \to \mathscr{H}^{\hat{\otimes} n}$ are injective. We will consider $\mathscr{X}^{\otimes n}$ as a $\mathbb{C}[\![h]\!]$–submodule of $\mathscr{H}^{\hat{\otimes} n}$ in (ii) above when we take its closure in the h–adic topology of $\mathscr{H}^{\hat{\otimes} n}$. Furthermore, since $\mathcal{R}^{-1} = (S \mathbin{\hat{\otimes}} \mathrm{id})(\mathcal{R})$ and $\boldsymbol{g}^{-1} = S(\boldsymbol{g})$, condition (ii) implies that $\mathcal{R}^{\pm 1} \in \overline{\mathscr{X} \otimes \mathscr{X}}$ and $\boldsymbol{g}^{\pm 1} \in \overline{\mathscr{X}}$. Since J_T, the universal invariant of an n–component bottom tangle T, is built from $\mathcal{R}^{\pm 1}$ and $\boldsymbol{g}^{\pm 1}$, condition (ii) implies that $J_T \in \overline{\mathscr{X}^{\otimes n}}$. In particular, $\boldsymbol{c} \in \overline{\mathscr{X} \otimes \mathscr{X}}$.

Remark 2.18 A core subalgebra has properties similar to, but still different from, those of both a minimal Hopf algebra [69] and a factorizable Hopf algebra [71]. Note that the notions of a minimal algebra and a factorizable algebra were introduced only for the case when the ground ring is a field. Over $\mathbb{C}[\![h]\!]$ the picture is much more complicated. For example, in [69] it was shown that a minimal algebra over a field is always finite-dimensional. Here our core algebras are of infinite rank over $\mathbb{C}[\![h]\!]$.

From now on we fix a core subalgebra $\mathscr{X}$ of a topological ribbon Hopf algebra $\mathscr{H}$.

Lemma 2.19 *Suppose $f\colon \mathscr{H} \to \mathscr{H}$ is a $\mathbb{C}[\![h]\!]$–module homomorphism such that $f(\mathscr{X}) \subset \mathscr{X}$. Then $f(\overline{\mathscr{X}}) \subset \overline{\mathscr{X}}$. In particular, $\overline{\mathscr{X}}$ is ad-stable.*

Proof Since f is continuous in the topology of $\mathscr{H}$, we have $f(\overline{\mathscr{X}}) \subset \overline{\mathscr{X}}$. □

2N1 Clasp form associated to a core subalgebra Suppose $\mathscr{X} \subset \mathscr{H}$ is a core subalgebra with the presentation (28) for $\boldsymbol{c}$. Since $\{\boldsymbol{c}'(i)\}$ is a topological basis of $\mathscr{X}$, every $y \in \mathscr{X}$ has its coordinates $y'_i \in \mathbb{C}[\![h]\!]$ such that,

$$y = \sum_{i \in I} y'_i \boldsymbol{c}'(i),$$

where $(y'_i)_{i\in I}$ is 0–convergent, ie $(y'_i)_{i\in I} \in (\mathbb{C}[\![h]\!]^I)_0$. The map $y \mapsto (y'_i)$ is a $\mathbb{C}[\![h]\!]$–module isomorphism from $\mathscr{X}$ to $(\mathbb{C}[\![h]\!]^I)_0$.

The set $\{\boldsymbol{c}''(i)\}$ is a formal basis of $\overline{\mathscr{X}}$, which is a formal series $\mathbb{C}[\![h]\!]$–module. Every $x \in \overline{\mathscr{X}}$ has its coordinates $x''_i \in \mathbb{C}[\![h]\!]$ such that, in the h–adic topology of $\mathscr{H}$,

$$x = \sum_{i \in I} x''_i \boldsymbol{c}''(i), \tag{29}$$

where $(x''_i)_{i\in I} \in \mathbb{C}[\![h]\!]^I$. The map $x \mapsto (x''_i)$ is an $\mathbb{C}[\![h]\!]$–module isomorphism from $\overline{\mathscr{X}}$ to $\mathbb{C}[\![h]\!]^I$.

Define a bilinear form $\mathscr{L} = \langle \cdot, \cdot \rangle\colon \overline{\mathscr{X}} \otimes \mathscr{X} \to \mathbb{C}[\![h]\!]$, called the *clasp form*, by

$$\langle x, y \rangle := \sum_{i \in I} x''_i y'_i.$$

The sum on the right-hand side is convergent since $(y'_i)_{i\in I}$ is 0–convergent. The bilinear form is defined so that $\{\boldsymbol{c}''(i)\}$ and $\{\boldsymbol{c}'(i)\}$ are dual to each other:

$$\langle \boldsymbol{c}''(i), \boldsymbol{c}'(j) \rangle = \delta_{ij}. \tag{30}$$

By continuity (in the h–adic topology), $\mathscr{L}$ extends to a $\mathbb{C}[\![h]\!]$–module map, also denoted by $\mathscr{L}$,

$$\mathscr{L}\colon \overline{\mathscr{X}} \hat{\otimes} \mathscr{X} \to \mathbb{C}[\![h]\!].$$

The following lemma says that the above bilinear form is dual to $\boldsymbol{c}$:

Lemma 2.20 (a) *One has $\boldsymbol{c} \in (\mathscr{X} \hat{\otimes} \mathscr{H}) \cap (\mathscr{H} \hat{\otimes} \mathscr{X})$.*

(b) *For every* $x \in \overline{\mathscr{X}}$ *and* $y \in \mathscr{X}$ *one has*

$$(\mathscr{L} \mathbin{\hat{\otimes}} \mathrm{id})(x \otimes \boldsymbol{c}) = x, \tag{31}$$

$$(\mathrm{id} \mathbin{\hat{\otimes}} \mathscr{L})(\boldsymbol{c} \otimes y) = y. \tag{32}$$

Remark 2.21 By part (a), $\boldsymbol{c} \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$, hence $x \otimes \boldsymbol{c} \in \overline{\mathscr{X}} \mathbin{\hat{\otimes}} \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$. This is the reason why the left-hand side of (31) is well-defined as an element of $\mathscr{H}$. Similarly, the left-hand side of (32) is well-defined. With this well-definedness, all the proofs will be the same as in the case of finite-dimensional vector spaces over a field.

Proof (a) Since $\{\boldsymbol{c}''(i)\}$ is 0–convergent in $\mathscr{H}$, $\boldsymbol{c} = \sum_i \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i) \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$. Similarly, $\boldsymbol{c} \in \mathscr{H} \mathbin{\hat{\otimes}} \mathscr{X}$.

(b) Suppose x has the presentation (29). By (30), we have

$$\langle x, \boldsymbol{c}'(i) \rangle = x''(i). \tag{33}$$

Thus, we have

$$x = \sum_i \langle x, \boldsymbol{c}'(i) \rangle \boldsymbol{c}''(i), \tag{34}$$

which is (31). The identity (32) is proved similarly. □

Because $\boldsymbol{r}^{\pm 1} \in \overline{\mathscr{X}}$, one can define the $\mathbb{C}[\![h]\!]$–module homomorphisms

$$\mathcal{T}_\pm \colon \mathscr{X} \to \mathbb{C}[\![h]\!], \quad \mathcal{T}_\pm(y) = \langle \boldsymbol{r}^{\pm 1}, y \rangle. \tag{35}$$

Since $\boldsymbol{c}$ is ad-invariant, one can expect the following:

Lemma 2.22 (a) *The clasp form* $\mathscr{L} \colon \overline{\mathscr{X}} \mathbin{\hat{\otimes}} \mathscr{X} \to \mathbb{C}[\![h]\!]$ *is ad-invariant, ie it is an* $\mathscr{H}$*–module homomorphism.*

(b) *The maps* $\mathcal{T}_\pm \colon \mathscr{X} \to \mathbb{C}[\![h]\!]$ *are ad-invariant.*

Proof (a) By Proposition 2.4(b), $\mathscr{L}$ is ad-invariant if and only if, for every $a \in \mathscr{H}$, $x \in \overline{\mathscr{X}}$, and $y \in \mathscr{X}$,

$$\langle a \triangleright x, y \rangle = \langle x, S(a) \triangleright y \rangle, \tag{36}$$

which we will prove now.

Since $\boldsymbol{c} = \sum_i \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i)$ is ad-invariant, by Proposition 2.4(a) we have

$$\sum_i S(a) \triangleright \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i) = \sum_i \boldsymbol{c}'(i) \otimes a \triangleright \boldsymbol{c}''(i).$$

Tensoring with x on the left, and applying $\mathscr{L}\otimes \mathrm{id}$,

$$\begin{aligned}\sum_i \langle x, S(a)\triangleright \boldsymbol{c}'(i)\rangle \boldsymbol{c}''(i) &= \sum_i \langle x, \boldsymbol{c}'(i)\rangle a\triangleright \boldsymbol{c}''(i)\\ &= \sum_i x''(i) a\triangleright \boldsymbol{c}''(i)\\ &= a\triangleright\Big(\sum_i x''(i)\boldsymbol{c}''(i)\Big) = a\triangleright x.\end{aligned}$$

Tensoring on the right with $\boldsymbol{c}'(j)$ then applying $\mathscr{L}$, one has

$$\langle x, S(a)\triangleright \boldsymbol{c}'(j)\rangle = \langle a\triangleright x, \boldsymbol{c}'(j)\rangle,$$

which is (36) with $y=\boldsymbol{c}'(j)$. Since $\{\boldsymbol{c}'(j)\}$ is a topological basis of $\mathscr{X}$, (36) holds for any $y\in\mathscr{X}$.

(b) This follows from Proposition 2.4(c). □

Proposition 2.23 *Suppose $f\colon \mathscr{H}\to\mathscr{H}$ and $g\colon \mathscr{H}\to\mathscr{H}$ are $\mathbb{C}[\![h]\!]$–module isomorphisms such that $f(\mathscr{X})=\mathscr{X}$, $g(\mathscr{X})=\mathscr{X}$, and $(f\mathbin{\hat\otimes} g)(\boldsymbol{c})=\boldsymbol{c}$. Then $g(\overline{\mathscr{X}})=\overline{\mathscr{X}}$ and, for every $x\in\overline{\mathscr{X}}$ and $y\in\mathscr{X}$, one has*

$$\langle g(x), f(y)\rangle = \langle x, y\rangle. \tag{37}$$

Proof By Lemma 2.19, $g^{\pm1}(\overline{\mathscr{X}})\subset\overline{\mathscr{X}}$. It follows that $g(\overline{\mathscr{X}})=\overline{\mathscr{X}}$. One has

$$\boldsymbol{c}=\sum_i \boldsymbol{c}'(i)\otimes\boldsymbol{c}''(i)=\sum_i f(\boldsymbol{c}'(i))\otimes g(\boldsymbol{c}''(i)).$$

Since $g(x)\in\overline{\mathscr{X}}$, one can replace x by $g(x)$ in (31),

$$\begin{aligned}g(x) = (\mathscr{L}\mathbin{\hat\otimes}\mathrm{id})(g(x)\otimes\boldsymbol{c}) &= \sum_i (\mathscr{L}\mathbin{\hat\otimes}\mathrm{id})\big(g(x)\otimes f(\boldsymbol{c}'(i))\otimes g(\boldsymbol{c}''(i))\big)\\ &= \sum_i \langle g(x), f(\boldsymbol{c}'(i))\rangle g(\boldsymbol{c}''(i))\\ &= g\Big(\sum_i \langle g(x), f(\boldsymbol{c}'(i))\rangle \boldsymbol{c}''(i)\Big).\end{aligned}$$

Injectivity of g implies

$$x=\sum_i \langle g(x), f(\boldsymbol{c}'(i))\rangle \boldsymbol{c}''(i).$$

Comparing with (34) we have, for every $i\in I$,

$$\langle g(x), f(\boldsymbol{c}'(i))\rangle = \langle x, \boldsymbol{c}'(i)\rangle,$$

which shows that (37) holds for $y = \boldsymbol{c}'(i)$, $i \in I$. Hence, (37) holds for every $y \in \mathscr{X}$, since $\{\boldsymbol{c}'(i)\}$ is a topological basis of $\mathscr{X}$. □

2N2 Twist system from core subalgebra

Proposition 2.24 *The collection $\mathcal{T} = (\mathcal{T}_\pm, \mathscr{X})$ is a twist system for $\mathscr{H}$.*

Proof By definition, $\mathscr{X}$ is ad-stable. By Lemma 2.22, $\mathcal{T}_\pm$ are ad-invariant. By Lemma 2.20(a), $\boldsymbol{c} \in \mathscr{X} \mathbin{\hat{\otimes}} \mathscr{H}$. Finally, (31) with $x = \boldsymbol{r}^{\pm 1}$ gives

$$(\mathcal{T}_\pm \mathbin{\hat{\otimes}} \mathrm{id})\boldsymbol{c} = \boldsymbol{r}^{\pm 1}.$$

This shows $\mathcal{T} = (\mathcal{T}_\pm, \mathscr{X})$ is a twist system. □

2O From core subalgebra to invariant of integral homology 3–spheres

Theorem 2.25 *Let $\mathscr{X}$ be a core subalgebra of a topological ribbon Hopf algebra $\mathscr{H}$, with its associated $\mathscr{H}$–module homomorphisms $\mathcal{T}_\pm : \mathscr{X} \to \mathbb{C}[\![h]\!]$. Assume T is an n–component bottom tangle with 0 linking matrix, $\varepsilon_i \in \{\pm 1\}$ for $i = 1, \dots, n$, and $M = M(T; \varepsilon_1, \dots, \varepsilon_n)$ is the integral homology 3–sphere obtained from S^3 by surgery along $\mathrm{cl}(T)$, with framing of the i^{th} component changed to ε_i.*

Then $J_T \in \mathscr{X}^{\hat{\otimes} n}$, and

$$J_M := (\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(J_T) \in \mathbb{C}[\![h]\!]$$

defines an invariant of integral homology 3–spheres.

By Propositions 2.15 and 2.24, to prove Theorem 2.25, it is sufficient to show the following:

Proposition 2.26 *Suppose $\mathscr{X}$ is a core subalgebra of a topological ribbon Hopf algebra $\mathscr{H}$ and T is an n–component bottom tangle with 0 linking matrix. Then $J_T \in \mathscr{X}^{\hat{\otimes} n}$.*

The rest of this section is devoted to a proof of this proposition based on Proposition 2.16.

2O1 $(\mathscr{X}^{\hat{\otimes} n})$–admissibility The following lemma follows easily from the definition.

Lemma 2.27 *Suppose $f\colon \mathscr{H}^{\hat{\otimes} a} \to \mathscr{H}^{\hat{\otimes} b}$ is a $\mathbb{C}[\![h]\!]$–module homomorphism having a presentation as an h–adically convergent sum $f = \sum_{p\in P} f_p$ such that, for each p, $f_p(\mathscr{X}^{\hat{\otimes} a}) \subset \mathscr{X}^{\hat{\otimes} b}$, where P is a countable set.* (Here, the sum f being h–adically convergent means that, for each $j \geq 0$, we have $f_p(\mathscr{H}^{\hat{\otimes} a}) \subset h^j \mathscr{H}^{\hat{\otimes} b}$ for all but finitely many $p \in P$.) *Then f is $(\mathscr{X}^{\hat{\otimes} n})$–admissible.*

Proposition 2.28 *Each of $\boldsymbol{\mu}$, $\boldsymbol{\psi}^{\pm 1}$, $\underline{\Delta}$ and $\underline{S}$ is $(\mathscr{X}^{\hat{\otimes} n})$–admissible.*

Proof (a) Because $\boldsymbol{\mu}(\mathscr{X} \hat{\otimes} \mathscr{X}) \subset \mathscr{X}$, by Lemma 2.27 $\boldsymbol{\mu}$ is $(\mathscr{X}^{\hat{\otimes} n})$–admissible.

(b) Because $\mathcal{R} \in \overline{\mathscr{X} \otimes \mathscr{X}}$, $\mathcal{R}$ has a presentation

$$\mathcal{R} = \sum_{p\in P} \mathcal{R}_1(p) \otimes \mathcal{R}_2(p), \quad \mathcal{R}_1(p), \mathcal{R}_2(p) \in \mathscr{X},$$

where the sum is convergent in the h–adic topology of $\mathscr{H} \hat{\otimes} \mathscr{H}$. Using the definitions (16)–(19), we have the following presentations as h–adically convergent sums:

$$\boldsymbol{\psi} = \sum_{p\in P} \boldsymbol{\psi}_p^+, \quad \text{where} \quad \boldsymbol{\psi}_p^+(x \otimes y) = \mathcal{R}_2(p) \triangleright y \otimes \mathcal{R}_1(p) \triangleright x,$$

$$\boldsymbol{\psi}^{-1} = \sum_{p\in P} \boldsymbol{\psi}_p^-, \quad \text{where} \quad \boldsymbol{\psi}_p^-(x \otimes y) = S(\mathcal{R}_1(p)) \triangleright y \otimes \mathcal{R}_2(p) \triangleright x,$$

$$\underline{\Delta} = \sum_{p\in P} \underline{\Delta}_p, \quad \text{where} \quad \underline{\Delta}_p(x) = \sum \mathcal{R}_2(p) \triangleright x_{(2)} \otimes \mathcal{R}_1 x_{(1)},$$

$$\underline{S} = \sum_{p\in P} \underline{S}_p, \quad \text{where} \quad \underline{S}_p(x) = \mathcal{R}_2(p) S(\mathcal{R}_1 \triangleright x).$$

Since $\mathcal{R}_1(p), \mathcal{R}_2(p) \in \mathscr{X}$, which is a topological Hopf algebra, $\boldsymbol{\psi}_p^{\pm}(\mathscr{X} \hat{\otimes} \mathscr{X}) \subset \mathscr{X} \hat{\otimes} \mathscr{X}$, $\underline{\Delta}_p(\mathscr{X}) \subset \mathscr{X} \hat{\otimes} \mathscr{X}$ and $\underline{S}_p(\mathscr{X}) \subset \mathscr{X}$. By Lemma 2.27, all of $\boldsymbol{\psi}^{\pm 1}$, $\underline{\Delta}$ and $\underline{S}$ are $(\mathscr{X}^{\hat{\otimes} n})$–admissible. □

2O2 Braided commutator and Borromean tangle We recall from [20; 23] the definitions and properties of the braided commutator for a braided Hopf algebra and a formula for universal invariant of the Borromean tangle.

Define the *braided commutator* $\Upsilon\colon \mathscr{H} \hat{\otimes} \mathscr{H} \to \mathscr{H}$ (for the braided Hopf algebra $\underline{\mathscr{H}}$) by

$$\Upsilon = \boldsymbol{\mu}^{[4]}(\mathrm{id} \hat{\otimes} \boldsymbol{\psi} \hat{\otimes} \mathrm{id})(\mathrm{id} \hat{\otimes} \underline{S} \hat{\otimes} \underline{S} \hat{\otimes} \mathrm{id})(\underline{\Delta} \hat{\otimes} \underline{\Delta}).$$

As noted in [23, Section 9.5], with $\boldsymbol{c} = \sum_i \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i)$, we have

$$\boldsymbol{b} = \sum_{i,j \in I} (\mathrm{id}^{\hat{\otimes} 2} \mathbin{\hat{\otimes}} \Upsilon)\big(\boldsymbol{c}'(i) \otimes \boldsymbol{c}'(j) \otimes \boldsymbol{c}''(j) \otimes \boldsymbol{c}''(i)\big) \tag{38}$$
$$= \sum_{i,j \in I} \boldsymbol{c}'(i) \otimes \boldsymbol{c}'(j) \otimes \Upsilon(\boldsymbol{c}''(j) \otimes \boldsymbol{c}''(i)).$$

Let $\boldsymbol{b}_{i,j}$ be the (i,j)–summand of the right-hand side, so that $\boldsymbol{b} = \sum_{i,j} \boldsymbol{b}_{i,j}$ with $\boldsymbol{b}_{i,j} \in \mathscr{X}^{\hat{\otimes} 3}$ and the sum converging in the h–adic topology of $\mathscr{H}^{\hat{\otimes} 3}$. We want to show that the sum $\sum_{i,j} \boldsymbol{b}_{i,j}$ is convergent in the h–adic topology of $\mathscr{X}^{\hat{\otimes} 3}$.

2O3 Two definitions of the braided commutator From [23, Section 9.3], we have

$$\Upsilon = \boldsymbol{\mu}(\mathrm{ad} \mathbin{\hat{\otimes}} \mathrm{id})(\mathrm{id} \mathbin{\hat{\otimes}} (\underline{S} \mathbin{\hat{\otimes}} \mathrm{id})\underline{\Delta}) \tag{39}$$
$$= \boldsymbol{\mu}(\mathrm{id} \mathbin{\hat{\otimes}} \underline{\mathrm{ad}}^{\mathrm{r}})((\mathrm{id} \mathbin{\hat{\otimes}} \underline{S})\underline{\Delta} \mathbin{\hat{\otimes}} \mathrm{id}), \tag{40}$$

where $\underline{\mathrm{ad}}^{\mathrm{r}}$ is the right-adjoint action (of the braided Hopf algebra $\underline{\mathscr{H}}$) defined by

$$\underline{\mathrm{ad}}^{\mathrm{r}} := \boldsymbol{\mu}^{[3]}(\underline{S} \mathbin{\hat{\otimes}} \mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id})(\boldsymbol{\psi} \mathbin{\hat{\otimes}} \mathrm{id})(\mathrm{id} \mathbin{\hat{\otimes}} \underline{\Delta}).$$

Lemma 2.29 *For $x, y \in \mathscr{H}$, we have*

$$\underline{\mathrm{ad}}^{\mathrm{r}}(x \otimes y) = S^{-1}(y) \triangleright x. \tag{41}$$

Proof In what follows we use $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 = \sum \mathcal{R}'_1 \otimes \mathcal{R}'_2 = \sum \mathcal{R}''_1 \otimes \mathcal{R}''_2 = \sum \mathcal{R}'''_1 \otimes \mathcal{R}'''_2$. One can easily verify

$$\boldsymbol{\psi}(x \otimes y) := \sum (\mathcal{R}_2 \mathcal{R}'_2 \triangleright y) \otimes \mathcal{R}_1 x S(\mathcal{R}'_1). \tag{42}$$

We have

$$\begin{aligned}
\underline{\mathrm{ad}}^{\mathrm{r}}(x \otimes y) &= \boldsymbol{\mu}^{[3]}(\underline{S} \mathbin{\hat{\otimes}} \mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id})(\boldsymbol{\psi} \mathbin{\hat{\otimes}} \mathrm{id})(x \otimes \underline{\Delta}(y)) \\
&= \sum \boldsymbol{\mu}^{[3]}(\underline{S} \mathbin{\hat{\otimes}} \mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id})(\boldsymbol{\psi} \mathbin{\hat{\otimes}} \mathrm{id})(x \otimes y_{\underline{(1)}} \otimes y_{\underline{(2)}}) \\
&= \sum \boldsymbol{\mu}^{[3]}(\underline{S} \mathbin{\hat{\otimes}} \mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id})(\boldsymbol{\psi}(x \otimes y_{\underline{(1)}}) \otimes y_{\underline{(2)}}) \\
&= \sum \boldsymbol{\mu}^{[3]}(\underline{S} \mathbin{\hat{\otimes}} \mathrm{id} \mathbin{\hat{\otimes}} \mathrm{id})(\mathcal{R}_2 \mathcal{R}'_2 \triangleright y_{\underline{(1)}} \otimes \mathcal{R}_1 x S(\mathcal{R}'_1) \otimes y_{\underline{(2)}}) \quad \text{by (42)} \\
&= \sum \boldsymbol{\mu}^{[3]}(\underline{S}(\mathcal{R}_2 \mathcal{R}'_2 \triangleright y_{\underline{(1)}}) \otimes \mathcal{R}_1 x S(\mathcal{R}'_1) \otimes y_{\underline{(2)}}) \\
&= \sum \underline{S}(\mathcal{R}_2 \mathcal{R}'_2 \triangleright y_{\underline{(1)}}) \mathcal{R}_1 x S(\mathcal{R}'_1) y_{\underline{(2)}},
\end{aligned}$$

where $\underline{\Delta}(y) = \sum y_{\underline{(1)}} \otimes y_{\underline{(2)}}$. Using

$$\sum y_{\underline{(1)}} \otimes y_{\underline{(2)}} = \sum (\mathcal{R}''_2 \triangleright y_{(2)}) \otimes \mathcal{R}''_1 y_{(1)}, \quad \underline{S}(w) = \sum S^{-1}(\mathcal{R}'''_2 \triangleright w) S(\mathcal{R}'''_1),$$

we obtain

$$\begin{aligned}\underline{\mathrm{ad}}^{\mathrm{r}}(x\otimes y)&=\sum \underline{S}(\mathcal{R}_2\mathcal{R}'_2\triangleright y_{\underline{(1)}})\mathcal{R}_1 x S(\mathcal{R}'_1)y_{\underline{(2)}}\\&=\sum S^{-1}(\mathcal{R}'''_2\triangleright(\mathcal{R}_2\mathcal{R}'_2\triangleright(\mathcal{R}''_2\triangleright y_{(2)})))S(\mathcal{R}'''_1)\mathcal{R}_1 x S(\mathcal{R}'_1)\mathcal{R}''_1 y_{(1)}\\&=\sum S^{-1}(\mathcal{R}'''_2\mathcal{R}_2\mathcal{R}'_2\mathcal{R}''_2\triangleright y_{(2)})S(\mathcal{R}'''_1)\mathcal{R}_1 x S(\mathcal{R}'_1)\mathcal{R}''_1 y_{(1)}.\end{aligned}$$

Since $\sum\mathcal{R}'''_2\mathcal{R}_2\otimes S(\mathcal{R}'''_1)\mathcal{R}_1=\sum\mathcal{R}'_2\mathcal{R}''_2\otimes S(\mathcal{R}'_1)\mathcal{R}''_1=\mathcal{R}_{21}^{-1}\mathcal{R}_{21}=1\otimes 1$, we obtain

$$\underline{\mathrm{ad}}^{\mathrm{r}}(x\otimes y)=\sum S^{-1}(y_{(2)})x y_{(1)}=S^{-1}(y)\triangleright x.$$

This completes the proof of the lemma. □

By Lemma 2.29 and $\mathrm{ad}(\mathscr{H}\hat{\otimes}\mathscr{X})\subset\mathscr{X}$ we easily obtain

$$\underline{\mathrm{ad}}^{\mathrm{r}}(\mathscr{X}\hat{\otimes}\mathscr{H})\subset\mathscr{X}. \tag{43}$$

Lemma 2.30 *We have*

$$\Upsilon(\mathscr{H}\hat{\otimes}\mathscr{X})\subset\mathscr{X}, \tag{44}$$

$$\Upsilon(\mathscr{X}\hat{\otimes}\mathscr{H})\subset\mathscr{X}. \tag{45}$$

Proof Using (39) and $\mathrm{ad}(\mathscr{H}\hat{\otimes}\mathscr{X})\subset\mathscr{X}$, we have

$$\begin{aligned}\Upsilon(\mathscr{H}\hat{\otimes}\mathscr{X})&=\boldsymbol{\mu}(\mathrm{ad}\hat{\otimes}\mathrm{id})(\mathrm{id}\hat{\otimes}\underline{S}\hat{\otimes}\mathrm{id})(\mathrm{id}\hat{\otimes}\underline{\Delta})(\mathscr{H}\hat{\otimes}\mathscr{X})\\&\subset\boldsymbol{\mu}(\mathrm{ad}\hat{\otimes}\mathrm{id})(\mathrm{id}\hat{\otimes}\underline{S}\hat{\otimes}\mathrm{id})(\mathscr{H}\hat{\otimes}\mathscr{X}\hat{\otimes}\mathscr{X})\\&\subset\boldsymbol{\mu}(\mathrm{ad}\hat{\otimes}\mathrm{id})(\mathscr{H}\hat{\otimes}\mathscr{X}\hat{\otimes}\mathscr{X})\\&\subset\boldsymbol{\mu}(\mathscr{X}\hat{\otimes}\mathscr{X})\\&\subset\mathscr{X}.\end{aligned}$$

Using (40) and (43), we can similarly check that $\Upsilon(\mathscr{X}\hat{\otimes}\mathscr{H})\subset\mathscr{X}$. □

2O4 Borromean tangle

Lemma 2.31 *One has $\boldsymbol{b}\in\mathscr{X}^{\hat{\otimes}3}$.*

Proof Since $\{\boldsymbol{c}''(i)\mid i\in I\}$ is 0–convergent in $\mathscr{H}$, we have $\boldsymbol{c}''(i)=h^{k_i}\tilde{\boldsymbol{c}}''(i)$, where $\tilde{\boldsymbol{c}}''(i)\in\mathscr{H}$ and for any $N\ge 0$ we have $k_i\ge N$ for all but finitely many i.

Recall that, by (38), we have

$$\boldsymbol{b}=\sum_{i,j\in I}\boldsymbol{b}_{i,j},\quad\text{where }\boldsymbol{b}_{i,j}=\boldsymbol{c}'(i)\otimes\boldsymbol{c}'(j)\otimes\Upsilon(\boldsymbol{c}''(j)\otimes\boldsymbol{c}''(i)). \tag{46}$$

By (44) and (45), we have

$$\Upsilon(\boldsymbol{c}''(j)\otimes\boldsymbol{c}''(i)) = h^{k_j}\,\Upsilon(\tilde{\boldsymbol{c}}''(j)\otimes\boldsymbol{c}''(i)) \in h^{k_j}\,\mathscr{X},$$
$$\Upsilon(\boldsymbol{c}''(j)\otimes\boldsymbol{c}''(i)) = h^{k_i}\,\Upsilon(\boldsymbol{c}''(j)\otimes\tilde{\boldsymbol{c}}''(i)) \in h^{k_i}\,\mathscr{X},$$

respectively. Hence,

$$\Upsilon(\boldsymbol{c}''(j)\otimes\boldsymbol{c}''(i)) \in h^{\max(k_i,k_j)}\,\mathscr{X}. \tag{47}$$

Since $\boldsymbol{c}'(i)$, $\boldsymbol{c}'(j)\in\mathscr{X}$, the sum (46) defines an element of $\mathscr{X}^{\hat{\otimes}3}$. □

2O5 Proof of Proposition 2.26 It is clear that $1\in\mathscr{X}^{\hat{\otimes}0}=\mathbb{C}[\![h]\!]$, $1\in\mathscr{X}$ and $\mathscr{X}^{\hat{\otimes}n}\otimes\mathscr{X}^{\hat{\otimes}m}\subset\mathscr{X}^{\hat{\otimes}n+m}$. By Proposition 2.28, each of $\boldsymbol{\mu}$, $\boldsymbol{\psi}^{\pm1}$, $\underline{\Delta}$ and $\underline{S}$ is $(\mathscr{X}^{\hat{\otimes}n})$–admissible. By Lemma 2.31, $\boldsymbol{b}\in\mathscr{X}^{\hat{\otimes}3}$. Hence, by Proposition 2.16, $J_T\in\mathscr{X}^{\hat{\otimes}n}$.

This completes the proof of Proposition 2.26 and also the proof of Theorem 2.25.

2P Integrality of J_M

Theorem 2.32 *Suppose $\mathscr{X}$ is a core subalgebra of a topological ribbon Hopf algebra $\mathscr{H}$ with the associated twist system $\mathcal{T}_\pm:\mathscr{X}\to\mathbb{C}[\![h]\!]$. Assume that there is a family of subsets $\widetilde{\mathcal{K}}_n\subset\mathscr{X}^{\hat{\otimes}n}$, $n\ge 0$, such that:*

(AL1) *$1_{\mathbb{C}[\![h]\!]}\in\widetilde{\mathcal{K}}_0$, $1_{\mathscr{H}}\in\widetilde{\mathcal{K}}_1$, $\boldsymbol{b}\in\widetilde{\mathcal{K}}_3$, each of $\boldsymbol{\psi}^{\pm1}$, $\boldsymbol{\mu}$, $\underline{\Delta}$ and $\underline{S}$ is $(\widetilde{\mathcal{K}}_n)$–admissible, and $x\otimes y\in\widetilde{\mathcal{K}}_{n+m}$ for any $x\in\widetilde{\mathcal{K}}_n$ and $y\in\widetilde{\mathcal{K}}_m$.*

(AL2) *For any $\varepsilon_1,\dots,\varepsilon_n\in\{\pm\}$,*

$$(\mathcal{T}_{\varepsilon_1}\hat{\otimes}\cdots\hat{\otimes}\mathcal{T}_{\varepsilon_n})(\widetilde{\mathcal{K}}_n)\subset\widetilde{\mathcal{K}}_0.$$

Then the invariant J_M of integral homology 3–spheres has values in $\widetilde{\mathcal{K}}_0$.

Proof Suppose T is an n–component bottom tangle T with zero linking matrix. By Proposition 2.16, condition (AL1) implies that $J_T\in\widetilde{\mathcal{K}}_n$. Condition (AL2) implies that

$$J_M=(\mathcal{T}_{\varepsilon_1}\hat{\otimes}\cdots\hat{\otimes}\mathcal{T}_{\varepsilon_n})(J_T)\in\widetilde{\mathcal{K}}_0,$$

where $M=M(T;\varepsilon_1,\dots,\varepsilon_n)$. □

We will construct a core subalgebra $\mathscr{X}$ and a sequence of $\widehat{\mathbb{Z}[q]}$–submodules $\widetilde{\mathcal{K}}_n\subset\mathscr{X}^{\hat{\otimes}n}$ satisfying the assumptions (AL1) and (AL2) of Theorem 2.32 for the quantized universal enveloping algebra (of a simple Lie algebra) with $\widetilde{\mathcal{K}}_0=\widehat{\mathbb{Z}[q]}$. By Theorem 2.32, the corresponding invariant of integral homology 3–spheres takes values in $\widehat{\mathbb{Z}[q]}$. We then show that this invariant specializes to the Witten–Reshetikhin–Turaev invariant at roots of unity. In a sense, the $(\widetilde{\mathcal{K}}_n)$ form an integral version of the $(\mathscr{X}^{\hat{\otimes}n})$. The construction of the integral objects $\widetilde{\mathcal{K}}_n$ is much more complicated than that of $\mathscr{X}$.

	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
d	1	2	2	1	1	1	1	2	3
D	$l+1$	2	2	4	3	2	1	1	1
$h^\vee$	$l+1$	$2l-1$	$l+1$	$2l-2$	12	18	30	9	4

Table 1: Constants d, D and $h^\vee$ of simple Lie algebras

3 Quantized enveloping algebras

In this section we present basic facts about the quantized enveloping algebras associated to a simple Lie algebra $\mathfrak{g}$: the h–adic version $\boldsymbol{U}_h(\mathfrak{g})$, the q–version $\boldsymbol{U}_q(\mathfrak{g})$ and its simply connected version $\breve{\boldsymbol{U}}_q(\mathfrak{g})$. We discuss the well-known braid group actions, various automorphisms of $\boldsymbol{U}_q$, the universal R–matrix and ribbon structure, and Poincaré–Birkhoff–Witt bases. New materials include gradings on the quantized enveloping algebras in Section 3C2, the mirror automorphism φ, and a calculation of the clasp element.

3A Quantized enveloping algebras $\boldsymbol{U}_h$, $\boldsymbol{U}_q$, and $\breve{\boldsymbol{U}}_q$

3A1 Simple Lie algebra Suppose $\mathfrak{g}$ is a finite-dimensional, simple Lie algebra over $\mathbb{C}$ of rank l. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots in the dual space $\mathfrak{h}^*$. Set $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\Pi \subset \mathfrak{h}^*$. Let $Y = \mathbb{Z}\Pi \subset \mathfrak{h}^*_{\mathbb{R}}$ denote the root lattice, $\Phi \subset Y$ the set of all roots, and $\Phi_+ \subset \Phi$ the set of all positive roots. Denote by t the number of positive roots, $t = |\Phi_+|$. Let $(\cdot,\cdot)$ denote the invariant inner product on $\mathfrak{h}^*_{\mathbb{R}}$ such that $(\alpha,\alpha) = 2$ for every short root α. For $\alpha \in \Phi$, set $d_\alpha = \frac{1}{2}(\alpha,\alpha) \in \{1,2,3\}$. Let X be the *weight lattice*, ie $X \subset \mathfrak{h}^*_{\mathbb{R}}$ is the $\mathbb{Z}$–span of the *fundamental weights* $\breve{\alpha}_1, \dots, \breve{\alpha}_l \in \mathfrak{h}^*_{\mathbb{R}}$, which are defined by $(\breve{\alpha}_i, \alpha_j) = \delta_{ij} d_{\alpha_i}$.

For $\gamma = \sum_{i=1}^l k_i \alpha_i \in Y$, let $\mathrm{ht}(\gamma) = \sum_i k_i$. Let ρ be the half-sum of positive roots, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha$. It is known that $\rho = \sum_{i=1}^l \breve{\alpha}_i$.

We list all simple Lie algebras and their constants in Table 1.

3A2 Base rings Let v be an indeterminate and set $\mathcal{A} := \mathbb{Z}[v^{\pm1}] \subset \mathbb{C}(v)$. We regard $\mathcal{A}$ also as a subring of $\mathbb{C}[\![h]\!]$, with $v = \exp\bigl(\frac{1}{2}h\bigr)$. Set $q = v^2$.

Remark 3.1 We will follow mostly Jantzen's book [30]. However, our v, q and h are equal to "q", "q^2" and "$-h$", respectively, of [30]. Since $q = v^2$, one could avoid using either q or v. We will use both q and v because, on the one hand, the use of half-integer powers of q would be cumbersome and, on the other hand, we would like to stress that many constructions in quantized enveloping algebras can be done over $\mathbb{Z}[q^{\pm 1}]$.

For $\alpha \in \Phi$ and integers $n, k \geq 0$, set

$$v_\alpha := v^{d_\alpha}, \quad q_\alpha := q^{d_\alpha} = v_\alpha^2,$$
$$[n]_\alpha := \frac{v_\alpha^n - v_\alpha^{-n}}{v_\alpha - v_\alpha^{-1}}, \quad [n]_\alpha! := \prod_{i=1}^{n} [i]_\alpha, \quad \begin{bmatrix} n \\ k \end{bmatrix}_\alpha := \prod_{i=1}^{k} \frac{[n-i+1]_\alpha}{[i]_\alpha},$$
$$\{n\}_\alpha := v_\alpha^n - v_\alpha^{-n}, \quad \{n\}_\alpha! := \prod_{i=1}^{n} \{i\}_\alpha.$$

When α is a short root, we sometimes suppress the subscript α in these expressions.

Recall that, for $n \geq 0$ and any element x in a $\mathbb{Z}[q]$–algebra,

$$(x;q)_n := \prod_{j=0}^{n-1} (1 - xq^j).$$

3A3 The algebra $\boldsymbol{U_h}$ The quantized enveloping algebra $\boldsymbol{U}_h = \boldsymbol{U}_h(\mathfrak{g})$ is defined as the h–adically complete $\mathbb{C}[\![h]\!]$–algebra, topologically generated by E_α, F_α and H_α for $\alpha \in \Pi$, subject to the relations

$$H_\alpha H_\beta = H_\beta H_\alpha, \tag{48}$$

$$H_\alpha E_\beta - E_\beta H_\alpha = (\alpha, \beta) E_\beta, \quad H_\alpha F_\beta - F_\beta H_\alpha = -(\alpha, \beta) F_\beta, \tag{49}$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{v_\alpha - v_\alpha^{-1}}, \quad \text{where } K_\alpha = \exp\bigl(\tfrac{1}{2} h H_\alpha\bigr), \tag{50}$$

$$\sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_\alpha E_\alpha^{r-s} E_\beta E_\alpha^s = 0, \quad \text{where } r = 1 - (\beta, \alpha)/d_\alpha, \tag{51}$$

$$\sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_\alpha F_\alpha^{r-s} F_\beta F_\alpha^s = 0, \quad \text{where } r = 1 - (\beta, \alpha)/d_\alpha. \tag{52}$$

We also write E_i, F_i and K_i, respectively, for E_{α_i}, F_{α_i} and K_{α_i} for $i = 1, \dots, l$.

For every $\lambda = \sum_{\alpha \in \Pi} k_\alpha \alpha \in \mathfrak{h}_{\mathbb{R}}^*$, define $H_\lambda = \sum_\alpha k_\alpha H_\alpha$ and $K_\lambda = \exp\bigl(\frac{1}{2} h H_\lambda\bigr)$. In particular, one can define $\breve{K}_\alpha := K_{\breve{\alpha}}$ for $\alpha \in \Pi$.

3A4 Hopf algebra structure The algebra $\boldsymbol{U}_h$ has the structure of a complete Hopf algebra over $\mathbb{C}[\![h]\!]$, where the comultiplication, counit and antipode are given by

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \quad \epsilon(E_\alpha) = 0, \quad S(E_\alpha) = -K_\alpha^{-1} E_\alpha,$$
$$\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \quad \epsilon(F_\alpha) = 0, \quad S(F_\alpha) = -F_\alpha K_\alpha,$$
$$\Delta(H_\alpha) = H_\alpha \otimes 1 + 1 \otimes H_\alpha, \quad \epsilon(H_\alpha) = 0, \quad S(H_\alpha) = -H_\alpha.$$

3A5 The algebra $\boldsymbol{U_q}$ and its simply connected version $\breve{\boldsymbol{U}}_{\boldsymbol{q}}$ Let $\boldsymbol{U}_q$ denote the $\mathbb{C}(v)$–subalgebra of $\boldsymbol{U}_h[h^{-1}] = \boldsymbol{U}_h \otimes_{\mathbb{C}[h]} \mathbb{C}[h, h^{-1}]$ generated by E_α, F_α and $K_\alpha^{\pm 1}$ for all $\alpha \in \Pi$. Alternatively, $\boldsymbol{U}_q$ is defined to be the $\mathbb{C}(v)$–subalgebra generated by the elements K_α, K_α^{-1}, E_α, F_α ($\alpha \in \Pi$), with relations (50)–(52) and

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1, \tag{53}$$

$$K_\beta E_\alpha = v^{(\beta,\alpha)} E_\alpha K_\beta, \quad K_\beta F_\alpha = v^{-(\beta,\alpha)} E_\alpha K_\beta, \tag{54}$$

for α, $\beta \in \Pi$.

The algebra $\boldsymbol{U}_q$ inherits a Hopf algebra structure from $\boldsymbol{U}_h[h^{-1}]$, where

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \epsilon(K_\alpha) = 1, \quad S(K_\alpha) = K_\alpha^{-1}.$$

Similarly, the simply connected version $\breve{\boldsymbol{U}}_q$ is the $\mathbb{C}(v)$–subalgebra of $\boldsymbol{U}_h[h^{-1}]$ generated by E_α, F_α and $\breve{K}_\alpha^{\pm 1}$ for all $\alpha \in \Pi$. Again $\breve{\boldsymbol{U}}_q$ is a $\mathbb{C}(v)$–Hopf algebra, which contains $\boldsymbol{U}_q$ as a Hopf subalgebra. Let $\breve{\boldsymbol{U}}_q^0$ be the $\mathbb{C}(v)$–algebra generated by $\breve{K}_\alpha^{\pm 1}$ for $\alpha \in \Pi$. Then

$$\breve{\boldsymbol{U}}_q = \breve{\boldsymbol{U}}_q^0 \boldsymbol{U}_q. \tag{55}$$

The simply connected version $\breve{\boldsymbol{U}}_q$ has been studied in [14; 18; 11] in connection with quantum adjoint action and various duality results. We need the simply connected version $\breve{\boldsymbol{U}}_q(\mathfrak{g})$ for a duality result, and also for the description of the R–matrix.

3B Automorphisms

There are unique h–adically continuous $\mathbb{C}$–algebra automorphisms ι_{bar}, φ, ω of $\boldsymbol{U}_h$ defined by

$$\iota_{\mathrm{bar}}(h) = -h, \quad \iota_{\mathrm{bar}}(H_\alpha) = H_\alpha, \quad \iota_{\mathrm{bar}}(E_\alpha) = E_\alpha, \quad \iota_{\mathrm{bar}}(F_\alpha) = F_\alpha,$$
$$\omega(h) = h, \quad \omega(H_\alpha) = -H_\alpha, \quad \omega(E_\alpha) = F_\alpha, \quad \omega(F_\alpha) = E_\alpha,$$
$$\varphi(h) = -h, \quad \varphi(H_\alpha) = -H_\alpha, \quad \varphi(E_\alpha) = -F_\alpha K_\alpha, \quad \varphi(F_\alpha) = -K_\alpha^{-1} E_\alpha,$$

and a unique h–adically continuous $\mathbb{C}$–algebra antiautomorphism τ defined by

$$\tau(h) = h, \quad \tau(H_\alpha) = -H_\alpha, \quad \tau(E_\alpha) = E_\alpha, \quad \tau(F_\alpha) = F_\alpha.$$

The map ι_{bar} is the bar operator of [54], and τ and ω are the same τ and ω in [30]. All three are involutive, ie $\tau^2 = \iota_{\mathrm{bar}}^2 = \omega^2 = \mathrm{id}$. The restrictions of ι_{bar}, φ, τ and ω to $\boldsymbol{U}_h \cap \boldsymbol{U}_q$ naturally extend to maps from $\boldsymbol{U}_q$ to $\boldsymbol{U}_q$, and we have

$$\begin{aligned} \tau(v) = \omega(v) &= v, & \tau(K_\alpha) = \omega(K_\alpha) &= K_\alpha^{-1}, \\ \iota_{\mathrm{bar}}(v) &= v^{-1}, & \iota_{\mathrm{bar}}(K_\alpha) &= K_\alpha^{-1}, \\ \varphi(v) &= v^{-1}, & \varphi(K_\alpha) &= K_\alpha. \end{aligned}$$

Unlike ι_{bar}, τ and ω, the map φ is a $\mathbb{C}$–Hopf algebra homomorphism:

Proposition 3.2 *The $\mathbb{C}$–algebra automorphism φ commutes with S and Δ, ie*

$$\varphi S = S\varphi, \quad (\varphi \mathbin{\hat{\otimes}} \varphi)\Delta = \Delta\varphi.$$

Furthermore, $\varphi = \iota_{\mathrm{bar}}\tau\omega S = \iota_{\mathrm{bar}}\omega\tau S = S\iota_{\mathrm{bar}}\tau\omega$ and

$$\varphi^2(x) = S^2(x) = K_{-2\rho} x K_{2\rho}. \tag{56}$$

Proof All the statements can easily be checked to hold on the generators h, H_α, E_α and F_α. □

3C Gradings by root lattice

3C1 *Y*–grading There are Y–gradings on $\boldsymbol{U}_h$ and $\boldsymbol{U}_q$ defined by

$$|E_\alpha| = \alpha, \quad |F_\alpha| = -\alpha, \quad |H_\alpha| = |K_\alpha| = 0.$$

For a subset $A \subset \boldsymbol{U}_h$, denote by A_μ, $\mu \in Y$, the set of all elements of Y–grading μ in A.

We frequently use the following simple fact: if x is Y–homogeneous and $\beta \in Y$, then

$$K_\beta x = v^{(\beta,|x|)} x K_\beta. \tag{57}$$

In the language of representation theory, $x \in \boldsymbol{U}_h$ has Y–grading $\beta \in Y$ if and only if it is an element of weight β in the adjoint representation of $\boldsymbol{U}_h$.

3C2 $(Y/2Y)$–grading and the even part of U_q

Proposition 3.3 *There is a unique $(Y/2Y)$–grading on the $\mathbb{C}(v)$–algebra U_q satisfying*

$$\deg(K_\alpha) \equiv \alpha, \quad \deg(E_\alpha) \equiv 0, \quad \deg(F_\alpha) \equiv \alpha \pmod{2Y}$$

for $\alpha \in \Pi$.

Proof Using the defining relations (50)–(54) for U_q, one checks that the $(Y/2Y)$–grading is well defined. □

The degree-0 part of U_q in the $(Y/2Y)$–grading, which is generated by $K_\alpha^{\pm 2}$, E_α and $F_\alpha K_\alpha$ for $\alpha \in \Pi$, is called the *even part* of U_q and denoted by U_q^{ev}. Elements of U_q^{ev} are said to be *even*.

For each $\alpha \in Y$, the degree $(\alpha \bmod 2Y)$ part of U_q is $K_\alpha U_q^{\mathrm{ev}}$.

Lemma 3.4 (a) *Suppose $\mu \in Y$. Let $(U_q^{\mathrm{ev}})_\mu$ be the grading μ part of U_q^{ev}. Then*

$$S((U_q^{\mathrm{ev}})_\mu) \subset K_\mu U_q^{\mathrm{ev}}, \quad \Delta((U_q^{\mathrm{ev}})_\mu) \subset \bigoplus_{\lambda \in Y} K_\lambda (U_q^{\mathrm{ev}})_{\mu-\lambda} \otimes (U_q^{\mathrm{ev}})_\lambda.$$

In particular, $\Delta(U_q^{\mathrm{ev}}) \subset U_q \otimes U_q^{\mathrm{ev}}$.

(b) *The adjoint action preserves the even part, ie $U_q \triangleright U_q^{\mathrm{ev}} \subset U_q^{\mathrm{ev}}$.*

(c) *Each of ι_{bar}, τ and φ leaves U_q^{ev} stable, ie $f(U_q^{\mathrm{ev}}) \subset U_q^{\mathrm{ev}}$ for $f = \iota_{\mathrm{bar}}, \tau, \varphi$.*

Proof (a) Suppose $x \in (U_q^{\mathrm{ev}})_\mu$. We have to show that

$$S(x) \in K_\mu U_q^{\mathrm{ev}} \quad \text{and} \quad \Delta(x) \in \bigoplus_{\lambda \in Y} K_\lambda (U_q^{\mathrm{ev}})_{\mu-\lambda} \otimes (U_q^{\mathrm{ev}})_\lambda.$$

If the statements hold for $x = x_1 \in (U_q^{\mathrm{ev}})_{\mu_1}$ and $x = x_2 \in (U_q^{\mathrm{ev}})_{\mu_2}$, then they hold for $x = x_1 x_2 \in (U_q^{\mathrm{ev}})_{\mu_1+\mu_2}$. Since U_q^{ev} is generated as an algebra by $K_\alpha^{\pm 2} \in (U_q^{\mathrm{ev}})_0$, $E_\alpha \in (U_q^{\mathrm{ev}})_\alpha$ and $F_\alpha K_\alpha \in (U_q^{\mathrm{ev}})_{-\alpha}$, it is enough to prove the statements when x is one of $K_\alpha^{\pm 2}$, E_α or $F_\alpha K_\alpha$. For these special values of x, the explicit formulas of $S(x)$ and $\Delta(x)$ are given in Section 3A4, from which the statements follow immediately.

(b) For $x \in U_q$, we have the following explicit formulas for the adjoint actions:

$$\begin{aligned} K_\alpha \triangleright x &= K_\alpha x K_\alpha^{-1}, \\ E_\alpha \triangleright x &= E_\alpha x - K_\alpha x K_\alpha^{-1} E_\alpha, \\ F_\alpha \triangleright x &= (F_\alpha x - x F_\alpha) K_\alpha. \end{aligned} \tag{58}$$

If x is even, then all the right-hand sides of the above are even. Since $\boldsymbol{U}_q$ is generated by K_α, E_α and F_α, we have $\boldsymbol{U}_q \triangleright \boldsymbol{U}_q^{\mathrm{ev}} \subset \boldsymbol{U}_q^{\mathrm{ev}}$.

(c) One can check directly that each of ι_{bar}, τ and φ maps any of the generators $K_\alpha^{\pm 2}$, E_α and $F_\alpha K_\alpha$ of $\boldsymbol{U}_q^{\mathrm{ev}}$ to an element of $\boldsymbol{U}_q^{\mathrm{ev}}$. □

Remark 3.5 In Section 6, we refine the $Y/2Y$–grading of the $\mathbb{C}(v)$–algebra $\boldsymbol{U}_q$ to a grading of the $\mathbb{C}(v)$–algebra $\boldsymbol{U}_q$ by a noncommutative $\mathbb{Z}/2\mathbb{Z}$–extension of $Y/2Y$.

By (55), $\breve{\boldsymbol{U}}_q = \breve{\boldsymbol{U}}_q^0 \boldsymbol{U}_q$, where $\breve{\boldsymbol{U}}_q^0 = \mathbb{C}(v)[\breve{K}_1^{\pm 1}, \dots, \breve{K}_l^{\pm 1}]$. Here we set $\breve{K}_i = \breve{K}_{\alpha_i}$ for $i = 1, \dots, l$. Let $\breve{\boldsymbol{U}}_q^{\mathrm{ev},0} = \mathbb{C}(v)[\breve{K}_1^{\pm 2}, \dots, \breve{K}_l^{\pm 2}]$ and

$$\breve{\boldsymbol{U}}_q^{\mathrm{ev}} := \breve{\boldsymbol{U}}_q^{\mathrm{ev},0} \boldsymbol{U}_q^{\mathrm{ev}}.$$

Lemma 3.6 *One has $\breve{\boldsymbol{U}}_q \triangleright \breve{\boldsymbol{U}}_q^{\mathrm{ev}} \subset \breve{\boldsymbol{U}}_q^{\mathrm{ev}}$ and $\breve{\boldsymbol{U}}_q \triangleright \boldsymbol{U}_q^{\mathrm{ev}} \subset \boldsymbol{U}_q^{\mathrm{ev}}$.*

Proof The proof is similar to that of Lemma 3.4(b). □

3D Triangular decompositions and their even versions

Let $\boldsymbol{U}_h^+$ (resp. $\boldsymbol{U}_h^-$, $\boldsymbol{U}_h^0$) be the h–adically closed $\mathbb{C}[\![h]\!]$–subalgebra of $\boldsymbol{U}_h$ topologically generated by E_α (resp. F_α, H_α) for $\alpha \in \Pi$.

Let $\boldsymbol{U}_q^+$ (resp. $\boldsymbol{U}_q^-$, $\boldsymbol{U}_q^0$) denote the $\mathbb{C}(v)$–subalgebra of $\boldsymbol{U}_q$ generated by E_α (resp. F_α, $K_\alpha^{\pm 1}$) for $\alpha \in \Pi$.

It is known that the multiplication map

$$\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^0 \otimes \boldsymbol{U}_q^+ \to \boldsymbol{U}_q, \quad x \otimes x' \otimes x'' \mapsto x x' x'',$$

is an isomorphism of $\mathbb{C}(v)$–vector spaces. This fact is called the triangular decomposition of $\boldsymbol{U}_q$. Similarly,

$$\boldsymbol{U}_h^- \mathbin{\hat{\otimes}} \boldsymbol{U}_h^0 \mathbin{\hat{\otimes}} \boldsymbol{U}_h^+ \to \boldsymbol{U}_h, \quad x \otimes x' \otimes x'' \mapsto x x' x'',$$

is an isomorphism of $\mathbb{C}[\![h]\!]$–modules. These triangular decompositions descend to various subalgebras of $\boldsymbol{U}_q$ and $\boldsymbol{U}_h$, which we will introduce later.

We need also an even version of triangular decomposition for $\boldsymbol{U}_q^{\mathrm{ev}}$. Although $\boldsymbol{U}_q^+ \subset \boldsymbol{U}_q^{\mathrm{ev}}$, the negative part $\boldsymbol{U}_q^-$ is not even.

Let $\boldsymbol{U}_q^{\mathrm{ev},-} := \varphi(\boldsymbol{U}_q^+)$, which is the $\mathbb{C}(v)$–subalgebra of $\boldsymbol{U}_q^{\mathrm{ev}}$ generated by $F_\alpha K_\alpha = -\varphi(E_\alpha)$, $\alpha \in \Pi$. Then $\boldsymbol{U}_q^{\mathrm{ev},-} \subset \boldsymbol{U}_q^{\mathrm{ev}}$. Let $\boldsymbol{U}_q^{\mathrm{ev},0}$ be the even part of $\boldsymbol{U}_q^0$, ie

$$\boldsymbol{U}_q^{\mathrm{ev},0} := \boldsymbol{U}_q^{\mathrm{ev}} \cap \boldsymbol{U}_q^0 = \mathbb{C}(v)[K_1^{\pm 2}, \dots, K_l^{\pm 2}].$$

Using (57), we obtain the isomorphisms of $\mathbb{C}(v)$–vector spaces and $\mathbb{C}[\![h]\!]$–modules

$$\boldsymbol{U}_q^{\mathrm{ev},-}\otimes \boldsymbol{U}_q^0\otimes \boldsymbol{U}_q^+ \xrightarrow{\cong} \boldsymbol{U}_q, \quad x\otimes y\otimes z\mapsto xyz, \tag{59}$$

$$\boldsymbol{U}_q^{\mathrm{ev},-}\otimes \boldsymbol{U}_q^{\mathrm{ev},0}\otimes \boldsymbol{U}_q^+ \xrightarrow{\cong} \boldsymbol{U}_q^{\mathrm{ev}}, \quad x\otimes y\otimes z\mapsto xyz, \tag{60}$$

$$\boldsymbol{U}_h^{\mathrm{ev},-}\mathbin{\hat\otimes} \boldsymbol{U}_h^0\mathbin{\hat\otimes} \boldsymbol{U}_h^+ \xrightarrow{\cong} \boldsymbol{U}_h, \quad x\otimes y\otimes z\mapsto xyz, \tag{61}$$

where we set $\boldsymbol{U}_h^{\mathrm{ev},-}=\varphi(\boldsymbol{U}_h^+)$, which is the h–adically closed $\mathbb{C}[\![h]\!]$–subalgebra of $\boldsymbol{U}_h$ topologically generated by $F_\alpha K_\alpha$, $\alpha\in\Pi$. We call (59), (60), and (61) the *even triangular decomposition* of $\boldsymbol{U}_q$, $\boldsymbol{U}_q^{\mathrm{ev}}$ and $\boldsymbol{U}_h$, respectively.

3E Braid group action

3E1 Braid group and Weyl group The *braid group* for the root system Φ has the presentation with generators T_α for $\alpha\in\Pi$ and with relations

$$\begin{aligned}
T_\alpha T_\beta &= T_\beta T_\alpha && \text{for } \alpha,\beta\in\Pi \text{ with } (\alpha,\beta)=0,\\
T_\alpha T_\beta T_\alpha &= T_\beta T_\alpha T_\beta && \text{for } \alpha,\beta\in\Pi \text{ with } (\alpha,\beta)=-1,\\
T_\alpha T_\beta T_\alpha T_\beta &= T_\beta T_\alpha T_\beta T_\alpha && \text{for } \alpha,\beta\in\Pi \text{ with } (\alpha,\beta)=-2,\\
T_\alpha T_\beta T_\alpha T_\beta T_\alpha T_\beta &= T_\beta T_\alpha T_\beta T_\alpha T_\beta T_\alpha && \text{for } \alpha,\beta\in\Pi \text{ with } (\alpha,\beta)=-3.
\end{aligned}$$

The Weyl group $\mathfrak{W}$ of Φ is the quotient of braid group by the relations $T_\alpha^2=1$ for $\alpha\in\Pi$. We denote the generator in $\mathfrak{W}$ corresponding to T_α by s_α. We set $T_i=T_{\alpha_i}$ and $s_i=s_{\alpha_i}$ for $i=1,\dots,l$.

Suppose $\boldsymbol{i}=(i_1,\dots,i_k)$ with $i_j\in\{1,2,\dots,l\}$. Let $w(\boldsymbol{i})=s_{i_1}s_{i_2}\cdots s_{i_k}\in\mathfrak{W}$. If there is no shorter sequence $\boldsymbol{j}$ such that $w(\boldsymbol{i})=w(\boldsymbol{j})$, then we say that the sequence $\boldsymbol{i}$ is *reduced*, and $w(\boldsymbol{i})$ *has length* k. It is known that the length of any reduced sequence is less than or equal to $t:=|\Phi_+|$, the number of positive roots of $\mathfrak{g}$. A sequence $\boldsymbol{i}$ is called *longest reduced* if $\boldsymbol{i}$ is reduced and has length t. There is a unique element $w_0\in\mathfrak{W}$ such that for any longest reduced sequence $\boldsymbol{i}$ one has $w(\boldsymbol{i})=w_0$.

3E2 Braid group action As described in [30, Chapter 8], there is an action of the braid group on the $\mathbb{C}(v)$–algebra $\boldsymbol{U}_q$. For $\alpha\in\Pi$, $T_\alpha\colon \boldsymbol{U}_q\to\boldsymbol{U}_q$ is the $\mathbb{C}(v)$–algebra automorphism defined by

$$T_\alpha(K_\gamma)=K_{s_\alpha(\gamma)},\quad T_\alpha(E_\alpha)=-F_\alpha K_\alpha,\quad T_\alpha(F_\alpha)=-K_\alpha^{-1}E_\alpha,$$

$$T_\alpha(E_\beta)=\sum_{i=0}^{r}(-1)^i v_\alpha^{-i}E_\alpha^{(r-i)}E_\beta E_\alpha^{(i)},\quad \text{with } r=-(\beta,\alpha)/d_\alpha,$$

$$T_\alpha(F_\beta)=\sum_{i=0}^{r}(-1)^i v_\alpha^{i}F_\alpha^{(i)}F_\beta F_\alpha^{(r-i)},\quad \text{with } r=-(\beta,\alpha)/d_\alpha,$$

where $\gamma \in Y$ and $\beta \in \Pi \setminus \{\alpha\}$. The restriction of T_α to $\boldsymbol{U}_q \cap \boldsymbol{U}_h$ extends to a continuous $\mathbb{C}[\![h]\!]$–algebra automorphism T_α of $\boldsymbol{U}_h$ by setting

$$T_\alpha(H_\gamma) = H_{s_\alpha(\gamma)} \quad \text{for } \gamma \in Y.$$

Remark 3.7 Our T_α is the same as T_α in [30]. Our $T_i = T_{\alpha_i}$ is $T''_{i,1}$ in [54], or $\widetilde{T}_i^{-1}$ in [52].

One can easily check that

$$T_\alpha^{\pm 1}(K_\beta \boldsymbol{U}_q^{\mathrm{ev}}) \subset K_{s_\alpha(\beta)} \boldsymbol{U}_q^{\mathrm{ev}}. \tag{62}$$

for $\alpha \in \Pi$ and $\beta \in Y$. In particular, the even part $\boldsymbol{U}_q^{\mathrm{ev}}$ is stable under $T_\alpha^{\pm 1}$. Thus, we have:

Proposition 3.8 *The even part $\boldsymbol{U}_q^{\mathrm{ev}}$ is stable under the action of the braid group.*

3F PBW-type bases

3F1 Root vectors Suppose $\boldsymbol{i} = (i_1, \dots, i_t)$ is a longest reduced sequence. For $j \in \{1, \dots, t\}$, set

$$\gamma_j = \gamma_j(\boldsymbol{i}) := s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j}).$$

It is known that $\gamma_1, \dots, \gamma_t$ are distinct positive roots and $\{\gamma_1, \dots, \gamma_t\} = \Phi_+$. The elements

$$E_{\gamma_j}(\boldsymbol{i}) := T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{j-1}}}(E_{\alpha_{i_j}}) \quad \text{and} \quad F_{\gamma_j}(\boldsymbol{i}) := T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{j-1}}}(F_{\alpha_{i_j}})$$

are called *root vectors corresponding to* $\boldsymbol{i}$. The Y–grading of the root vectors are $|E_{\gamma_j}(\boldsymbol{i})| = \gamma_j = -|F_{\gamma_j}(\boldsymbol{i})|$. It is known that $E_{\gamma_j}(\boldsymbol{i}) \in \boldsymbol{U}_q^+$ and $F_{\gamma_j}(\boldsymbol{i}) \in \boldsymbol{U}_q^-$.

In general, $E_{\gamma_j}(\boldsymbol{i})$ and $F_{\gamma_j}(\boldsymbol{i})$ depend on $\boldsymbol{i}$, but if γ_j is a simple root, ie $\gamma_j = \alpha \in \Pi$, then we have $E_{\gamma_j}(\boldsymbol{i}) = E_\alpha$ and $F_{\gamma_j}(\boldsymbol{i}) = F_\alpha$.

3F2 PBW-type bases Fix a longest reduced sequence $\boldsymbol{i}$. In what follows, we often suppress $\boldsymbol{i}$ and write $E_\gamma = E_\gamma(\boldsymbol{i})$ and $F_\gamma = F_\gamma(\boldsymbol{i})$ for all $\gamma \in \Phi_+$.

The divided powers $E_\gamma^{(n)}$ and $F_\gamma^{(n)}$ for $\gamma \in \Phi_+$ and $n \geq 0$ are defined by

$$E_\gamma^{(n)} := E_\gamma^n/[n]_\gamma! \quad \text{and} \quad F_\gamma^{(n)} = F_\gamma^n/[n]_\gamma!.$$

Following Bourbaki, we denote by $\mathbb{N}$ the set of nonnegative integers. For $\boldsymbol{n} \in \mathbb{N}^t$, define

$$F^{(\boldsymbol{n})} = \overleftarrow{\prod_{\gamma_j \in \Phi_+}} F_{\gamma_j}^{(n_j)}, \quad E^{(\boldsymbol{n})} = \overleftarrow{\prod_{\gamma_j \in \Phi_+}} E_{\gamma_j}^{(n_j)}.$$

Here $\overleftarrow{\prod}_{\gamma_j \in \Phi_+}$ means to take the product in the reverse order of $(\gamma_1, \gamma_2, \dots, \gamma_t)$. For example,

$$F^{(\boldsymbol{n})} = \overleftarrow{\prod_{\gamma_j \in \Phi_+}} F_{\gamma_j}^{(n_j)} = F_{\gamma_t}^{(n_t)} F_{\gamma_{t-1}}^{(n_{t-1})} \cdots F_{\gamma_1}^{(n_1)}.$$

The set $\{E^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ is a basis of the $\mathbb{C}(v)$–vector space $\boldsymbol{U}_q^+$ and a topological basis of $\boldsymbol{U}_h$.

Similarly, the set $\{F^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ is a basis of $\boldsymbol{U}_q^-$ and a topological basis of $\boldsymbol{U}_h^-$.

On the other hand, $\{K_\gamma \mid \gamma \in Y\}$ is a $\mathbb{C}(v)$–basis of $\boldsymbol{U}_q^0$ and $\{H^{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathbb{N}^l\}$, where $H^{\boldsymbol{k}} = \prod_{j=1}^l H_j^{k_j}$ for $\boldsymbol{k} = (k_1, \dots, k_l)$, is a topological basis of $\boldsymbol{U}_h^0$.

Combining these bases and using the even triangular decompositions (59)–(61), we get the following proposition, which describes the Poincaré–Birkhoff–Witt (PBW) bases of $\boldsymbol{U}_q$, $\boldsymbol{U}_q^{\mathrm{ev}}$ and $\boldsymbol{U}_h$:

Proposition 3.9 *For any longest reduced sequence $\boldsymbol{i}$,*

$$\begin{aligned}
&\{F^{(\boldsymbol{m})} K_{\boldsymbol{m}} K_\gamma E^{(\boldsymbol{n})} \mid \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t,\ \gamma \in Y\} && \text{is a } \mathbb{C}(v)\text{–basis for } \boldsymbol{U}_q,\\
&\{F^{(\boldsymbol{m})} K_{\boldsymbol{m}} K_\gamma^2 E^{(\boldsymbol{n})} \mid \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t, \gamma \in Y\} && \text{is a } \mathbb{C}(v)\text{–basis for } \boldsymbol{U}_q^{\mathrm{ev}},\\
&\{F^{(\boldsymbol{m})} K_{\boldsymbol{m}} H^{\boldsymbol{k}} E^{(\boldsymbol{n})} \mid \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t, \boldsymbol{k} \in \mathbb{N}^l\} && \text{is a topological basis for } \boldsymbol{U}_h,
\end{aligned}$$

where

$$K_{\boldsymbol{n}} := \prod_{j=1}^t K_{\gamma_j}^{n_j} = K_{-|F^{(\boldsymbol{n})}|} \quad \text{for } \boldsymbol{n} = (n_1, \dots, n_t) \in \mathbb{N}^t. \tag{63}$$

3G *R*–matrix

3G1 Quasi-*R*–matrix Fix a longest reduced sequence $\boldsymbol{i}$. Recall that $\{k\}_\alpha = v_\alpha^k - v_\alpha^{-k}$.

The *quasi-R–matrix* $\Theta \in \boldsymbol{U}_h^{\hat{\otimes} 2}$ is (see [30; 54]) defined by

$$\Theta = \sum_{\boldsymbol{n} \in \mathbb{N}^t} F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}}, \tag{64}$$

where, for $\boldsymbol{n} = (n_1, \dots, n_t) \in \mathbb{N}^t$,

$$E_{\boldsymbol{n}} := E^{(\boldsymbol{n})} \prod_{j=1}^t \{n_j\}_{\gamma_j}! = \overleftarrow{\prod_{\gamma_j \in \Phi_+}} ((v_{\gamma_j} - v_{\gamma_j}^{-1}) E_{\gamma_j})^{n_j}, \tag{65}$$

$$F_{\boldsymbol{n}} := F^{(\boldsymbol{n})} \prod_{j=1}^t (-1)^{n_j} v_{\gamma_j}^{-n_j(n_j-1)/2} = \overleftarrow{\prod_{\gamma_j \in \Phi_+}} (-1)^{n_j} v_{\gamma_j}^{-n_j(n_j-1)/2} F_{\gamma_j}^{(n_j)}. \tag{66}$$

It is known that Θ does not depend on $\boldsymbol{i}$ and

(67) $$\Theta^{-1} = (\iota_{\mathrm{bar}} \otimes \iota_{\mathrm{bar}})(\Theta) = \sum_{\boldsymbol{n}\in\mathbb{N}^t} F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}},$$

where

$$F'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(F_{\boldsymbol{n}}) \quad \text{and} \quad E'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(E_{\boldsymbol{n}}).$$

3G2 Universal *R*–matrix and ribbon element Define an inner product on $\mathfrak{h}_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{H_\alpha \mid \alpha \in \Pi\}$ by $(H_\alpha, H_\beta) = (\alpha, \beta)$. Recall that the $\breve{\alpha}$ are the fundamental weights. Let $\breve{H}_\alpha = H_{\breve{\alpha}}$. Then the set $\{\breve{H}_\alpha/d_\alpha \mid \alpha \in \Pi\}$ is dual to $\{H_\alpha \mid \alpha \in \Pi\}$ with respect to the inner product, ie $(H_\alpha, \breve{H}_\beta/d_\beta) = \delta_{\alpha,\beta}$ for $\alpha,\ \beta \in \Pi$. Define the *diagonal part*, or the *Cartan part*, of the R–matrix by

(68) $$\mathcal{D} = \exp\biggl(\frac{h}{2} \sum_{\alpha\in\Pi} (H_\alpha \otimes \breve{H}_\alpha/d_\alpha)\biggr) \in (\boldsymbol{U}_h^0)^{\hat{\otimes} 2}.$$

We have $\mathcal{D} = \mathcal{D}_{21}$, where $\mathcal{D}_{21} \in (\boldsymbol{U}_h^0)^{\hat{\otimes} 2}$ is obtained from $\mathcal{D}$ by permuting the first and the second tensorands.

A simple calculation shows that, for Y–homogeneous $x, y \in \boldsymbol{U}_h$, we have

(69) $$\mathcal{D}(x \otimes y)\mathcal{D}^{-1} = xK_{|y|} \otimes K_{|x|}y.$$

The universal R–matrix and its inverse are given by

(70) $$\mathcal{R} = \mathcal{D}\Theta^{-1} \quad \text{and} \quad \mathcal{R}^{-1} = \Theta\mathcal{D}^{-1}.$$

Note that our R–matrix is the inverse of the R–matrix in [30].

The quasitriangular Hopf algebra $(\boldsymbol{U}_h, \mathcal{R})$ has a ribbon element $\boldsymbol{r}$ whose corresponding balanced element (see Section 2B) is given by $\boldsymbol{g} = K_{-2\rho}$. For Y–homogeneous $x \in \boldsymbol{U}_h$ we have

(71) $$S^2(x) = K_{-2\rho} x K_{2\rho} = q^{-(\rho,|x|)} x.$$

With $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$, the ribbon element and its inverse are given by

$$\boldsymbol{r} = \sum S(\mathcal{R}_1) K_{-2\rho} \mathcal{R}_2 \quad \text{and} \quad \boldsymbol{r}^{-1} = \sum \mathcal{R}_1 K_{2\rho} \mathcal{R}_2 = \sum \mathcal{R}_2 K_{-2\rho} \mathcal{R}_1.$$

One has $\boldsymbol{r} = J_T$ and $\boldsymbol{r}^{-1} = J_{T'}$, where T and T' are the bottom tangles in Figure 13.

Using (64) and (70), we obtain

(72) $$\boldsymbol{r} = \sum_{\boldsymbol{n}\in\mathbb{N}^t} F_{\boldsymbol{n}} K_{\boldsymbol{n}} \boldsymbol{r}_0 E_{\boldsymbol{n}}, \quad \boldsymbol{r}^{-1} = \sum_{\boldsymbol{n}\in\mathbb{N}^t} F'_{\boldsymbol{n}} K_{\boldsymbol{n}}^{-1} \boldsymbol{r}_0^{-1} E'_{\boldsymbol{n}},$$

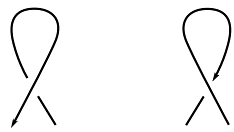


Figure 13: Tangles T (left) and T' (right) determining the ribbon element $\boldsymbol{r}$ and its inverse

where $K_{\boldsymbol{n}}$ is given by (63) and

$$\boldsymbol{r}_0 := K_{-2\rho}\boldsymbol{\mu}(\mathcal{D}^{-1}) = K_{-2\rho}\exp\Big(-\frac{h}{2}\sum_{\alpha\in\Pi} H_\alpha \breve{H}_\alpha/d_\alpha\Big).$$

We also have

$$S(\boldsymbol{r}) = \underline{S}(\boldsymbol{r}) = \boldsymbol{r}. \tag{73}$$

3H Mirror homomorphism $\boldsymbol{\varphi}$

We defined the $\mathbb{C}$–algebra homomorphism φ in Section 3B.

Proposition 3.10 *The $\mathbb{C}$–automorphism φ is a mirror homomorphism for $\boldsymbol{U}_h$, ie*

$$\varphi(K_{2\rho}) = K_{2\rho}, \tag{74}$$

$$(\varphi \mathbin{\hat{\otimes}} \varphi)(\mathcal{R}) = (\mathcal{R}^{-1})_{21}. \tag{75}$$

Consequently, if T' is the mirror image of an n–component bottom tangle T, then $J_{T'} = \varphi^{\hat{\otimes} n}(J_T)$.

Proof Identity (74) is part of the definition of φ. One could prove (75) by direct calculations. Here is an alternative proof using known identities:

By Proposition 3.2, $\varphi = \iota_{\mathrm{bar}}\omega\tau S$. Hence, (75) follows from the following four known identities:

$$\begin{aligned}
(S \mathbin{\hat{\otimes}} S)(\mathcal{R}) &= \mathcal{R} && \text{by (10),}\\
(\tau \mathbin{\hat{\otimes}} \tau)(\mathcal{R}) = (\tau \mathbin{\hat{\otimes}} \tau)(\mathcal{D}\Theta^{-1}) &= \Theta^{-1}\mathcal{D} && \text{by [30, 7.1(2)],}\\
(\omega \mathbin{\hat{\otimes}} \omega)(\Theta^{-1}\mathcal{D}) &= \Theta_{21}^{-1}\mathcal{D} && \text{by [30, 7.1(3)],}\\
(\iota_{\mathrm{bar}} \mathbin{\hat{\otimes}} \iota_{\mathrm{bar}})(\Theta_{21}^{-1}\mathcal{D}) = \Theta_{21}\mathcal{D}^{-1} &= \mathcal{R}_{21}^{-1} && \text{by (67).}
\end{aligned}$$

This shows φ is a mirror homomorphism. By Proposition 2.7, $J_{T'} = \varphi^{\hat{\otimes} n}(J_T)$. □

Because the negative twist is the mirror image of the positive one, we have the following:

Corollary 3.11 *One has $\varphi(\boldsymbol{r}) = \boldsymbol{r}^{-1}$.*

3I Clasp element and quasiclasp element

Here we calculate explicitly the value of the clasp element $\boldsymbol{c} = J_{C^+} \in \boldsymbol{U}_h \mathbin{\hat{\otimes}} \boldsymbol{U}_h$, which is the universal invariant of the clasp tangle C^+ of Figure 5. Recall that we have defined $E_{\boldsymbol{n}}$, $F_{\boldsymbol{n}}$ and $\mathcal{D}$ in Section 3G. We call

$$\Gamma := \boldsymbol{c}\mathcal{D}^2 \tag{76}$$

the *quasiclasp element*. Like the quasi-*R*–matrix, the quasiclasp element enjoys better integrality than the clasp element itself.

Lemma 3.12 *Fix a longest reduced sequence $\boldsymbol{i}$. We have*

$$\boldsymbol{c} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^t} q^{-(\rho,|E_{\boldsymbol{n}}|)}(F_{\boldsymbol{m}}K_{\boldsymbol{m}} \otimes F_{\boldsymbol{n}}K_{\boldsymbol{n}})(\mathcal{D}^{-2})(E_{\boldsymbol{n}} \otimes E_{\boldsymbol{m}}), \tag{77}$$

$$\Gamma = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^t} q^{-(\rho,|E_{\boldsymbol{n}}|)+(|E_{\boldsymbol{m}}|,|E_{\boldsymbol{n}}|)}(F_{\boldsymbol{m}}K_{\boldsymbol{m}}^{-1}E_{\boldsymbol{n}} \otimes F_{\boldsymbol{n}}K_{\boldsymbol{n}}^{-1}E_{\boldsymbol{m}}), \tag{78}$$

$$\boldsymbol{c} = (\varphi \otimes \underline{S}^{-1}\varphi)(\boldsymbol{c}). \tag{79}$$

Proof Let $\mathcal{D}^{-2} = \sum(\mathcal{D}^{-2})_1 \otimes (\mathcal{D}^{-2})_2$ and $\mathcal{R}^{-1} = \sum \bar{\mathcal{R}}_1 \otimes \bar{\mathcal{R}}_2 = \sum \bar{\mathcal{R}}'_1 \otimes \bar{\mathcal{R}}'_2$. By (20), we obtain

$$\boldsymbol{c} = \sum S(\mathcal{R}_1)S(\mathcal{R}'_2) \otimes \mathcal{R}'_1\mathcal{R}_2 = \sum \bar{\mathcal{R}}_1 S^2(\bar{\mathcal{R}}'_2) \otimes \bar{\mathcal{R}}'_1\bar{\mathcal{R}}_2.$$

We have

$$\mathcal{R}^{-1} = \Theta\mathcal{D}^{-1} = \sum_{\boldsymbol{m}\in\mathbb{N}^t} F_{\boldsymbol{m}}(\mathcal{D}^{-1})_1 \otimes E_{\boldsymbol{m}}(\mathcal{D}^{-1})_2 = \sum_{\boldsymbol{m}\in\mathbb{N}^t} F_{\boldsymbol{m}}K_{\boldsymbol{m}}(\mathcal{D}^{-1})_1 \otimes (\mathcal{D}^{-1})_2 E_{\boldsymbol{m}}.$$

Substituting this into the formula for $\boldsymbol{c}$, we obtain

$$\boldsymbol{c} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^t} F_{\boldsymbol{m}}K_{\boldsymbol{m}}(\mathcal{D}^{-2})_1 S^2(E_{\boldsymbol{n}}) \otimes F_{\boldsymbol{n}}K_{\boldsymbol{n}}(\mathcal{D}^{-2})_2 E_{\boldsymbol{m}},$$

which, using (71), is (77). Identity (78) follows from (77), via (69).

Since C^- is the mirror image of C^+, by Proposition 3.10 $\boldsymbol{c}^- = (\varphi \otimes \varphi)(\boldsymbol{c})$, which, together with (21), gives (79). □

From [58, Proposition 2.1.14], one has

$$(\mathrm{id} \otimes S^2)(\boldsymbol{c}) = \boldsymbol{c}_{21}. \tag{80}$$

4 Core subalgebra of $U_{\sqrt{h}}$ and quantum Killing form

In this section we construct a core subalgebra X_h of the ribbon Hopf algebra

$$U_{\sqrt{h}} := U_h \mathbin{\hat{\otimes}}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!],$$

which is the extension of U_h when the ground ring is $\mathbb{C}[\![\sqrt{h}]\!]$. We will use the Drinfel'd dual V_h of U_h to construct X_h. To show that X_h is a Hopf algebra we use a stability principle established in Section 4C, which also finds applications later. We then discuss the clasp form of X_h which turns out to coincide with the well-known quantum Killing form (or Rosso form) when restricted to U_q. Thus, we get a geometric interpretation of the quantum Killing form.

4A A dual of U_h

Fix a longest reduced sequence $\boldsymbol{i}$. For $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ let

$$\|\boldsymbol{n}\| = \sum_{j=1}^{k} n_j.$$

Let us recall the topological basis of U_h described in Proposition 3.9. For $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in \mathbb{N}^t \times \mathbb{N}^l \times \mathbb{N}^t$, let

$$e_h(\boldsymbol{n}) = F^{(\boldsymbol{n}_1)} K_{\boldsymbol{n}_1} H^{\boldsymbol{n}_2} E^{(\boldsymbol{n}_3)},$$

where $F^{(\boldsymbol{n}_1)}$, $K_{\boldsymbol{n}_1}$, $H^{\boldsymbol{n}_2}$ and $E^{(\boldsymbol{n}_3)}$ are as defined in Section 3F2. By Proposition 3.9,

$$\{e_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$$

is a topological basis of U_h.

Let V_h be the closure (in the h–adic topology of U_h) of the $\mathbb{C}[\![h]\!]$–span of the set

$$\{h^{\|\boldsymbol{n}\|} e_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}. \tag{81}$$

Then V_h is a formal series $\mathbb{C}[\![h]\!]$–module, having the above set (81) as a formal basis. (See Example 2.2 of Section 2A.) Every $x \in V_h$ has a unique presentation of the form

$$x = \sum_{\boldsymbol{n} \in \mathbb{N}^{t+l+t}} x_{\boldsymbol{n}} (h^{\|\boldsymbol{n}\|} e_h(\boldsymbol{n})),$$

where $x_{\boldsymbol{n}} \in \mathbb{C}[\![h]\!]$. The map $x \mapsto (x_{\boldsymbol{n}})_{\boldsymbol{n} \in I}$ is a $\mathbb{C}[\![h]\!]$–module isomorphism between V_h and $\mathbb{C}[\![h]\!]^I$, with $I = \mathbb{N}^{t+l+t}$.

In the terminology of Drinfel'd [17], V_h is a "quantized formal series Hopf algebra" (QFSH algebra); see also [12]. As part of his duality principle, Drinfel'd associates a QFSH algebra to every so-called "quantum universal enveloping algebra" (QUE algebra). Gavarini [18] gave a detailed treatment of this duality and showed that the above-defined $\boldsymbol{V}_h$ is the QFSH algebra associated to $\boldsymbol{U}_h$, which is a QUE algebra.

For $n \geq 0$ let $\boldsymbol{V}_h^{\bar{\otimes} n}$ be the topological closure of $\boldsymbol{V}_h^{\otimes n}$ in $\boldsymbol{U}_h^{\hat{\otimes} n}$. Then $\boldsymbol{V}_h^{\bar{\otimes} n}$ is the n^{th} tensor power of $\boldsymbol{V}_h$ in the category of QFSH algebras; see [18, Section 3.5]. The result of Drinfel'd, proved in detail by Gavarini [18], says that $\boldsymbol{V}_h$ is a Hopf algebra in the category of QFSH algebras, where the Hopf algebra structure of $\boldsymbol{V}_h$ is the restriction of the Hopf algebra structure of $\boldsymbol{U}_h$. Thus, we have the following:

Proposition 4.1 *One has*

$$\mu(\boldsymbol{V}_h^{\bar{\otimes} 2}) \subset \boldsymbol{V}_h, \quad \Delta(\boldsymbol{V}_h) \subset \boldsymbol{V}_h^{\bar{\otimes} 2}, \quad S(\boldsymbol{V}_h) \subset \boldsymbol{V}_h.$$

For completeness, we give an independent proof of Proposition 4.1 in Appendix A. Yet another proof can be obtained from Proposition 5.10.

Proposition 4.2 *Fix a longest reduced sequence* $\boldsymbol{i}$. *Then* $\boldsymbol{V}_h$ *is the topological closure (in the* h*–adic topology of* $\boldsymbol{U}_h$*) of the* $\mathbb{C}[\![h]\!]$*–algebra generated by* hH_α, $hF_\gamma(\boldsymbol{i})$ *and* $hE_\gamma(\boldsymbol{i})$ *with* $\alpha \in \Pi$ *and* $\gamma \in \Phi_+$.

Proof Let $\boldsymbol{V}'_h$ be the topological closure (in the h–adic topology of $\boldsymbol{U}_h$) of the $\mathbb{C}[\![h]\!]$–algebra generated by hH_α, hF_γ and hE_γ with $\alpha \in \Pi$ and $\gamma \in \Phi_+$. One can easily check $K_\gamma \in \boldsymbol{V}'_h$ for $\gamma \in Y$. The set $\{h^{\|\boldsymbol{n}\|}\boldsymbol{e}_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ is a formal basis of $\boldsymbol{V}_h$.

When $\boldsymbol{n} = (n_1, \dots, n_{t+l+t}) \in \mathbb{N}^{t+l+t}$ is such that all $n_j = 0$ except for one that is equal to 1, then the basis element $h^{\|\boldsymbol{n}\|}\boldsymbol{e}_h(\boldsymbol{n})$ is one of hH_α, $hF_\gamma K_\gamma$ or hE_γ. It follows that hH_α, hF_γ, $hE_\gamma \in \boldsymbol{V}_h$, and hence $\boldsymbol{V}'_h \subset \boldsymbol{V}_h$.

From the definition of $\boldsymbol{e}_h(\boldsymbol{n})$, for any $\boldsymbol{n} = (\boldsymbol{m}, \boldsymbol{k}, \boldsymbol{u}) \in \mathbb{N}^t \times \mathbb{N}^l \times \mathbb{N}^t$,

$$h^{\|\boldsymbol{n}\|}\boldsymbol{e}_h(\boldsymbol{n}) = a \overleftarrow{\prod_{\gamma_j \in \Phi_+}} (hF_{\gamma_j})^{m_j} \prod_{\gamma_j \in \Phi_+} K_{\gamma_j}^{m_j} \prod_{j=1}^{l} (hH_j)^{k_j} \overleftarrow{\prod_{\gamma_j \in \Phi_+}} (hE_{\gamma_j})^{u_j}, \tag{82}$$

where $\boldsymbol{m} = (m_1, \dots, m_t)$, $\boldsymbol{k} = (k_1, \dots, k_l)$, $\boldsymbol{u} = (u_1, \dots, u_t)$, and

$$a = \frac{1}{\prod_{j=1}^{t} [m_j]_{\gamma_j}! [u_j]_{\gamma_j}!}$$

is a unit in $\mathbb{C}[\![h]\!]$. Since the right-hand side of (82) is in $\boldsymbol{V}'_h$, we have $\boldsymbol{V}_h \subset \boldsymbol{V}'_h$. Thus, $\boldsymbol{V}_h = \boldsymbol{V}'_h$. □

4B Ad-stability and φ–stability of V_h

Recall that we defined the left image of an element $x \in \boldsymbol{U}_h \mathbin{\hat{\otimes}} \boldsymbol{U}_h$ in Section 2D.

Proposition 4.3 *The module $\boldsymbol{V}_h$ is the left image of the clasp element $\boldsymbol{c}$ in $\boldsymbol{U}_h \mathbin{\hat{\otimes}} \boldsymbol{U}_h$. Moreover, $\boldsymbol{V}_h$ is ad-stable, ie $\boldsymbol{U}_h \triangleright \boldsymbol{V}_h \subset \boldsymbol{V}_h$.*

Proof For $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in \mathbb{N}^{t+l+t}$ let

$$e''_h(\boldsymbol{n}) = F^{(\boldsymbol{n}_3)} K_{\boldsymbol{n}_3} \breve{H}^{\boldsymbol{n}_2} E^{(\boldsymbol{n}_1)},$$

where $\breve{H}^{\boldsymbol{k}} = \prod_{j=1}^{l} \breve{H}_{\alpha_j}^{k_j}$ for $\boldsymbol{k} = (k_1, \dots, k_l) \in \mathbb{N}^l$.

Then $\{e''_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ is a topological basis of $\boldsymbol{U}_h$. From (77),

$$\boldsymbol{c} = \sum_{\boldsymbol{n} \in \mathbb{N}^{t+l+t}} u_h(\boldsymbol{n}) h^{\|\boldsymbol{n}\|} e_h(\boldsymbol{n}) \otimes e''_h(\boldsymbol{n}), \tag{83}$$

where $u_h(\boldsymbol{n})$ is a unit in $\mathbb{C}[\![h]\!]$ for each $\boldsymbol{n} \in \mathbb{N}^{t+l+t}$. The exact value of $u_h(\boldsymbol{n})$ is as follows: for $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in \mathbb{N}^{t+l+t}$,

$$u_h(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = q^{-(\rho, |E_{\boldsymbol{n}_3}|)} u''_h(\boldsymbol{n}_1) u'_h(\boldsymbol{n}_2) u''_h(\boldsymbol{n}_3), \tag{84}$$

where, for $\boldsymbol{k} = (k_1, \dots, k_l) \in \mathbb{N}^l$ and $\boldsymbol{m} = (m_1, \dots, m_t) \in \mathbb{N}^t$,

$$u'_h(\boldsymbol{k}) = \prod_{j=1}^{l} \frac{(-1)^{k_j}}{k_j!\, d_{\alpha_j}^{k_j}}, \quad u''_h(\boldsymbol{m}) = \prod_{j=1}^{t} \frac{v_{\gamma_j}^{-m_j^2} (q_{\gamma_j}; q_{\gamma_j})_{m_j}}{h^{m_j}}.$$

By definition, the left image of $\boldsymbol{c}$ is the topological closure of the $\mathbb{C}[\![h]\!]$–span of $\{u_h(\boldsymbol{n}) h^{\|\boldsymbol{n}\|} e_h(\boldsymbol{n})\}$, which is the same as $\boldsymbol{V}_h$, since the $u_h(\boldsymbol{n})$ are units in $\mathbb{C}[\![h]\!]$.

Since $\boldsymbol{c}$ is ad-invariant, by Proposition 2.5 we have $\boldsymbol{U}_h \triangleright \boldsymbol{V}_h \subset \boldsymbol{V}_h$. □

Remark 4.4 Proposition 4.3 shows that $\boldsymbol{V}_h$ does not depend on the choice of the longest reduced sequence $\boldsymbol{i}$.

Proposition 4.5 *One has $\varphi(\boldsymbol{V}_h) \subset \boldsymbol{V}_h$, ie $\boldsymbol{V}_h$ is φ–stable.*

Proof By Lemma 3.12, $\boldsymbol{c} = (\varphi \otimes \underline{S}^{-1}\varphi)(\boldsymbol{c})$. Note that $\underline{S}^{-1}\varphi$ is a $\mathbb{C}[\![h]\!]$–linear automorphism of $\boldsymbol{U}_h$. By Proposition 2.5(b), φ leaves stable the left image of $\boldsymbol{c}$, ie $\varphi(\boldsymbol{V}_h) = \boldsymbol{V}_h$. □

4C Extension of ground ring and stability principle

Let $\sqrt{h}$ be an indeterminate such that $h = (\sqrt{h})^2$. Then $\mathbb{C}[\![h]\!] \subset \mathbb{C}[\![\sqrt{h}]\!]$. For a $\mathbb{C}[\![h]\!]$–module homomorphism $f\colon V \to V'$, we often use the same symbol f to denote $f \mathbin{\widehat{\otimes}} \mathrm{id} : V \mathbin{\widehat{\otimes}}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!] \to V' \mathbin{\widehat{\otimes}}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!]$.

Suppose the following data are given:

(i) a topologically free $\mathbb{C}[\![h]\!]$–module V with a topological basis $\{e(i) \mid i \in I\}$, and

(ii) a function $a\colon I \to \mathbb{C}[\![h]\!]$ such that $a(i) \neq 0$ and $\{a(i) \mid i \in I\}$ is 0–convergent.

Let $V(\sqrt{a})$ be the topologically free $\mathbb{C}[\![\sqrt{h}]\!]$–module with the topological basis $\{\sqrt{a(i)}e(i) \mid i \in I\}$ and let $V(a) \subset V$ be the closure (in the h–adic topology of V) of the $\mathbb{C}[\![h]\!]$–span of $\{a(i)e(i) \mid i \in I\}$. We call $(V, V(\sqrt{a}), V(a))$ a *topological dilatation triple* defined by the data given in (i) and (ii).

Proposition 4.6 (stability principle) *Suppose*

$$(V, V(\sqrt{a}), V(a)) \quad \text{and} \quad (V', V'(\sqrt{a'}), V'(a'))$$

are two topological dilatation triples and $f\colon V \to V'$ is a $\mathbb{C}[\![h]\!]$–module homomorphism such that $f(V(a)) \subset V'(a')$. Then $f(V(\sqrt{a})) \subset V'(\sqrt{a'})$.

Proof We first prove:

Claim *If $x_1, x_2, x_3 \in \mathbb{C}[\![h]\!]$ with $x_3 \neq 0$ and $x_1x_2/x_3 \in \mathbb{C}[\![h]\!]$, then*

$$x_1\sqrt{x_2/x_3} \in \mathbb{C}[\![\sqrt{h}]\!].$$

Proof of claim Let $x_i = h^{k_i} y^i$, where y_i is invertible in $\mathbb{C}[\![h]\!]$. The assumption $x_1x_2/x_3 \in \mathbb{C}[\![h]\!]$ means $k_1 + k_2 \geq k_3$. Then $k_1 + \frac{1}{2}k_2 \geq \frac{1}{2}(k_1 + k_2) \geq \frac{1}{2}k_3$, which implies the claim. □

Let us now prove the proposition. The $\mathbb{C}[\![h]\!]$–module $V(a)$ is a formal series $\mathbb{C}[\![h]\!]$–module with formal basis $\{a(i)e(i) \mid i \in I\}$; see Example 2.2. Every $x \in V(a)$ has a unique presentation as an h–adically convergent sum

$$x = \sum_{i \in I} x_i(a(i)e(i)), \quad \text{where } (x_i)_{i \in I} \in \mathbb{C}[\![h]\!]^I.$$

Using the topological bases $\{e(i) \mid i \in I\}$ of V and $\{e'(i') \mid i' \in I'\}$ of V', we have

$$f(e(i)) = \sum_{j \in I'} f_i^j e'(j),$$

where $f_i^j \in \mathbb{C}[\![h]\!]$ and, for a fixed i, $\{f_i^j \mid j \in I'\}$ is 0–convergent. Multiplying by appropriate powers of $\sqrt{a(i)}$, we get

$$f(a(i)e(i)) = \sum_{j\in I'} \tilde{f}_i^j (a'(j)e'(j)), \qquad \text{where } \tilde{f}_i^j = \frac{a(i)}{a'(j)} f_i^j,$$

$$f(\sqrt{a(i)}e(i)) = \sum_{j\in I'} \tilde{\tilde{f}}_i^j (\sqrt{a'(j)}e'(j)), \quad \text{where } \tilde{\tilde{f}}_i^j = \frac{\sqrt{a(i)}}{\sqrt{a'(j)}} f_i^j. \tag{85}$$

The assumption $f(V(a)) \subset V'(a')$ implies that $\tilde{f}_i^j \in \mathbb{C}[\![h]\!]$, which, together with $f_i^j \in \mathbb{C}[\![h]\!]$ and the claim, shows that $\tilde{\tilde{f}}_i^j \in \mathbb{C}[\![\sqrt{h}]\!]$. Equation (85) shows that $f(V(\sqrt{a})) \subset V'(\sqrt{a'})$. This proves the proposition. □

4D Definition of X_h

Fix a longest reduced sequence $\boldsymbol{i}$. Recall that $\{e_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ is a topological basis of U_h; see Section 4A. Let $a\colon \mathbb{N}^{t+l+t} \to \mathbb{C}[\![h]\!]$ be the function defined by $a(\boldsymbol{n}) = h^{\|\boldsymbol{n}\|}$ and consider the topological dilatation triple $(U_h, U_h(\sqrt{a}), U_h(a))$. Denote the middle one by X_h, ie $U_h(\sqrt{a}) = X_h$. Later we show that X_h does not depend on $\boldsymbol{i}$.

By definition, $U_h(a)$ is the closure (in the h–adic topology of U_h) of the $\mathbb{C}[\![h]\!]$–span of $\{h^{\|\boldsymbol{n}\|} e_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$. Thus, $U_h(a) = V_h$.

Also by definition, X_h is the topologically free $\mathbb{C}[\![\sqrt{h}]\!]$–module with the topological basis

$$\{h^{\|\boldsymbol{n}\|/2} e_h(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}. \tag{86}$$

Note that X_h is a submodule of $U_{\sqrt{h}} = U_h \hat{\otimes}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!]$.

The topological closure $\overline{X}_h$ of X_h in $U_{\sqrt{h}}$ is a formal series $\mathbb{C}[\![\sqrt{h}]\!]$–module with (86) as a formal basis.

Theorem 4.7 *The $\mathbb{C}[\![\sqrt{h}]\!]$–module X_h is a topological Hopf subalgebra of $U_{\sqrt{h}}$. Moreover $U_{\sqrt{h}} \triangleright X_h \subset X_h$, ie X_h is ad-stable, and $\varphi(X_h) \subset X_h$.*

Proof We will show that X_h is closed under all the Hopf algebra operations of $U_{\sqrt{h}}$.

Let us first show that X_h is closed under the coproduct. Both (U_h, X_h, V_h) and $(U_h^{\hat{\otimes} 2}, X_h^{\hat{\otimes} 2}, V_h^{\overline{\otimes} 2})$ are topological dilatation triples, and

$$\Delta(U_h) \subset U_h^{\hat{\otimes} 2} \quad \text{and} \quad \Delta(V_h) \subset V_h^{\overline{\otimes} 2}$$

(see Proposition 4.1). Hence, by the stability principle (Proposition 4.6), $\Delta(X_h) \subset X_h^{\hat{\otimes} 2}$.

Similarly, applying stability principle to all the operations of a Hopf algebra, namely $\boldsymbol{\mu}$, $\boldsymbol{\eta}$, $\boldsymbol{\Delta}$, $\boldsymbol{\epsilon}$ and S (using Proposition 4.1), as well as the adjoint actions (using Proposition 4.3) and the map φ (using Proposition 4.5) we get the results. □

Corollary 4.8 *Fix a longest reduced sequence* $\boldsymbol{i}$ *. The* $\mathbb{C}[\![\sqrt{h}]\!]$*–algebra* $\boldsymbol{X}_h$ *is the topologically complete subalgebra of* $\boldsymbol{U}_{\sqrt{h}}$ *generated by* $\sqrt{h}H_\alpha$, $\sqrt{h}E_\gamma(\boldsymbol{i})$ *and* $\sqrt{h}F_\gamma(\boldsymbol{i})$ *with* $\alpha \in \Pi$ *and* $\gamma \in \Phi_+$*.*

Proof Since $\boldsymbol{X}_h$ is an algebra, the proof is the same as that of Proposition 4.2. □

4E $\boldsymbol{X}_h$ is a core subalgebra of $\boldsymbol{U}_{\sqrt{h}}$

Recall that core subalgebras were introduced in Section 2N.

Theorem 4.9 *The subalgebra* $\boldsymbol{X}_h$ *is a core subalgebra of the topological ribbon Hopf algebra* $\boldsymbol{U}_{\sqrt{h}}$*.*

Proof For the convenience of the reader, we recall the definition of a core subalgebra: $\boldsymbol{X}_h$ is a core subalgebra of $\boldsymbol{U}_{\sqrt{h}}$ if $\boldsymbol{X}_h$ is a topological Hopf subalgebra of $\boldsymbol{U}_{\sqrt{h}}$ and the following holds:

(i) $\boldsymbol{X}_h$ is $\boldsymbol{U}_{\sqrt{h}}$–stable.

(ii) $\mathcal{R} \in \overline{\boldsymbol{X}_h \otimes \boldsymbol{X}_h}$ and $K_{2\rho} \in \overline{\boldsymbol{X}}_h$.

(iii) The clasp element $\boldsymbol{c}$ has a presentation

$$\boldsymbol{c} = \sum_{i \in I} \boldsymbol{c}'(i) \otimes \boldsymbol{c}''(i),$$

where each of $\{\boldsymbol{c}'(i)\}$ and $\{\boldsymbol{c}''(i)\}$ is 0–convergent in $\boldsymbol{U}_{\sqrt{h}}$ and is a topological basis of $\boldsymbol{X}_h$.

Let us look at all three statements.

(i) By Theorem 4.7, $\boldsymbol{X}_h$ is a topological Hopf subalgebra of $\boldsymbol{U}_{\sqrt{h}}$ and (i) holds.

(ii) Since $\sqrt{h}H_\alpha \in \boldsymbol{X}_h$ (see Corollary 4.8), $K_{\pm 2\rho} = \exp\big(\pm \sum_{\alpha \in \Phi_+} hH_\alpha\big) \in \boldsymbol{X}_h \subset \overline{\boldsymbol{X}}_h$.

By (70), $\mathcal{R}^{-1} = \Theta\mathcal{D}^{-1}$, where

$$\Theta = \sum_{\boldsymbol{n} \in \mathbb{N}^t} F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}} \quad \text{and} \quad \mathcal{D}^{-1} = \exp\bigg(-\frac{h}{2} \sum_{\alpha \in \Pi} H_\alpha \otimes \breve{H}_\alpha / d_\alpha\bigg).$$

As $\sqrt{h}H_\alpha, \sqrt{h}\breve{H}_\alpha \in \boldsymbol{X}_h$, one has $\mathcal{D}^{-1} \in \overline{\boldsymbol{X}_h \otimes \boldsymbol{X}_h}$.

Using the definition (65)–(66) of $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$, and Corollary 4.8, we have

$$F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}} \sim \overleftarrow{\prod_{\gamma_j \in \Phi_+}} (hF_{\gamma_j} \otimes E_{\gamma_j})^{n_j} \in \boldsymbol{X}_h \otimes \boldsymbol{X}_h,$$

where $a \sim b$ means $a = ub$ for some unit u in $\mathbb{C}[\![h]\!]$. Hence, $\Theta = \sum F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}} \in \overline{\boldsymbol{X}_h \otimes \boldsymbol{X}_h}$. It follows that $\mathcal{R}^{-1} = \Theta \mathcal{D}^{-1} \in \overline{\boldsymbol{X}_h \otimes \boldsymbol{X}_h}$. Since $\mathcal{R} = (\mathrm{id} \mathbin{\hat{\otimes}} S)(\mathcal{R}^{-1})$, we also have $\mathcal{R} \in \overline{\boldsymbol{X}_h \otimes \boldsymbol{X}_h}$. Thus (ii) holds.

(iii) Let $I = \mathbb{N}^{t+l+t}$ and, for $\boldsymbol{n} \in I$,

$$c'(\boldsymbol{n}) = h^{\|\boldsymbol{n}\|/2} e_h(\boldsymbol{n}), \quad c''(\boldsymbol{n}) = u_h(\boldsymbol{n}) h^{\|\boldsymbol{n}\|/2} e''_h(\boldsymbol{n}), \tag{87}$$

where $u_h(\boldsymbol{n})$ is the unit of $\mathbb{C}[\![h]\!]$ in (84). By (83),

$$\boldsymbol{c} = \sum_{\boldsymbol{n}} c'(\boldsymbol{n}) \otimes c''(\boldsymbol{n}).$$

By definition, $\{c'(\boldsymbol{n})\}$ is a topological basis of $\boldsymbol{X}_h$. Since $\{\breve{H}_\alpha \mid \alpha \in \Pi\}$ is a basis of $\mathfrak{h}^*_{\mathbb{R}}$, $\{c''(\boldsymbol{n})\}$ is also a topological basis of $\boldsymbol{X}_h$. The factors $h^{\|\boldsymbol{n}\|/2}$ in (87) show that each set $\{c'(\boldsymbol{n})\}$ and $\{c''(\boldsymbol{n})\}$ is 0–convergent. Hence (iii) holds. This completes the proof of the theorem. □

By Theorem 2.25, the core subalgebra $\boldsymbol{X}_h$ gives rise to an invariant $J_M \in \mathbb{C}[\![\sqrt{h}]\!]$ of integral homology 3–spheres M, via the twists $\mathcal{T}_\pm$, which we will study in the next subsections.

4F Quantum Killing form

Since $\boldsymbol{X}_h$ is a core subalgebra of $\boldsymbol{U}_{\sqrt{h}}$, according to Section 2N one has a clasp form, which is a $\boldsymbol{U}_{\sqrt{h}}$–module homomorphism

$$\mathscr{L}\colon \overline{\boldsymbol{X}}_h \mathbin{\hat{\otimes}} \boldsymbol{X}_h \to \mathbb{C}[\![\sqrt{h}]\!], \tag{88}$$

defined by

$$\mathscr{L}(c''(\boldsymbol{n}) \otimes c'(\boldsymbol{m})) = \delta_{\boldsymbol{n},\boldsymbol{m}} \quad \text{for } \boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^{t+l+t}, \tag{89}$$

where $c''(\boldsymbol{n})$ and $c'(\boldsymbol{m})$ are given by (87). We also denote $\mathscr{L}(x \otimes y)$ by $\langle x, y \rangle$.

Let us calculate explicitly the form $\mathscr{L}$. Recall that $F_{\boldsymbol{n}}, E_{\boldsymbol{n}} \in \boldsymbol{U}_q$ were defined by (65) and (66), which depend on a longest reduced sequence.

Proposition 4.10 *Fix a longest reduced sequence* $\boldsymbol{i}$. *For* $\boldsymbol{m}$, $\boldsymbol{n}$, $\boldsymbol{n}'$, $\boldsymbol{m}' \in \mathbb{N}^t$, $\boldsymbol{k}$, $\boldsymbol{k}' \in \mathbb{N}^l$, α, $\beta \in Y$ *and* k, $l \in \mathbb{N}$, *one has*

$$\langle F_{\boldsymbol{m}} K_{\boldsymbol{m}} h^{\frac{k}{2}} H_\alpha^k E_{\boldsymbol{n}}, F_{\boldsymbol{n}'} K_{\boldsymbol{n}'} h^{\frac{l}{2}} H_\beta^l E_{\boldsymbol{m}'} \rangle = \delta_{k,l} \delta_{\boldsymbol{m},\boldsymbol{m}'} \delta_{\boldsymbol{n},\boldsymbol{n}'} q^{(\rho,|E_{\boldsymbol{n}}|)} (-1)^k k! (\alpha,\beta)^k, \tag{90}$$

$$\langle F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_\mu E_{\boldsymbol{n}}, F_{\boldsymbol{n}'} K_{\boldsymbol{n}'} K_{\mu'} E_{\boldsymbol{m}'} \rangle = \delta_{\boldsymbol{m},\boldsymbol{m}'} \delta_{\boldsymbol{n},\boldsymbol{n}'} q^{(\rho,|E_{\boldsymbol{n}}|)} v^{-(\mu,\mu')/2}. \tag{91}$$

Proof Formula (90) is obtained from (89) by a simple calculation, using the definition (87) of $\boldsymbol{c}'(\boldsymbol{n})$ and $\boldsymbol{c}''(\boldsymbol{n})$. Formula (91) is obtained from (90) using the expansion $K_\mu = \exp\bigl(\frac{1}{2} h H_\mu\bigr) = \sum_k h^k H_\mu^k / (2^k k!)$. □

Suppose $x, y \in \boldsymbol{U}_q$. There are nonzero a, $b \in \mathbb{C}[v^{\pm 1}]$ such that ax, $by \in \boldsymbol{X}_h$. By (91), $\langle ax, by \rangle \in \mathbb{C}[v^{\pm 1/2}]$. Hence we can define $\langle x, y \rangle = \langle ax, by \rangle / (ab) \in \mathbb{C}(v^{1/2})$. Thus, we have a $\mathbb{C}(v)$–bilinear form

$$\langle \cdot, \cdot \rangle \colon \boldsymbol{U}_q \otimes \boldsymbol{U}_q \to \mathbb{C}(v^{1/2}). \tag{92}$$

Remark 4.11 The form we construct is not new. On $\boldsymbol{U}_q$ the form $\mathscr{L}$ is exactly the quantum Killing form (or the Rosso form) [74; 81] (see [30]), which was constructed via an elaborate process. For example, if one defines the quantum Killing form by (91), then it not easy to check the ad-invariance of the quantum Killing form. Essentially here we give a geometric characterization of the quantum Killing form: it is the dual of the clasp element $\boldsymbol{c}$. The ad-invariance of the quantum Killing form then follows right away from the ad-invariance of $\boldsymbol{c}$. We also determine the space $\boldsymbol{X}_h$, which in a sense is the biggest space for which the quantum Killing form can be defined (with values in $\mathbb{C}[\![h]\!]$).

4G Properties of quantum Killing form

We again emphasize that the form $\mathscr{L}$ is ad-invariant, ie the map $\mathscr{L}$ in (88) is a $\boldsymbol{U}_{\sqrt{h}}$–module homomorphism; see Lemma 2.22. It follows that the form (92) is $\boldsymbol{U}_q$–ad-invariant.

Since each of $\{\boldsymbol{c}'(\boldsymbol{n})\}$ and $\{\boldsymbol{c}''(\boldsymbol{n})\}$ is a topological basis of $\boldsymbol{X}_h$ and they are dual to each other, the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate.

From (90), we see that the quantum Killing form is *triangular* in the following sense. Let $x, x' \in \boldsymbol{X}_h \cap \boldsymbol{U}_h^{\mathrm{ev},-}$, $y, y' \in \boldsymbol{X}_h \cap \boldsymbol{U}_h^0$, and $z, z' \in \boldsymbol{X}_h \cap \boldsymbol{U}_h^+$; then

$$\langle xyz, x'y'z' \rangle = \langle x, z' \rangle \langle y, y' \rangle \langle z, x' \rangle. \tag{93}$$

The quantum Killing form is uniquely determined up to a scalar by the ad-invariant, nondegenerate, and triangular properties; see [33, Theorem 4.8].

The quantum Killing form is not symmetric. In fact, for $x, y \in \boldsymbol{X}_h$, we have

$$\langle y, x\rangle = \langle x, S^2(y)\rangle = \langle S^{-2}(x), y\rangle,$$

which follows from the identity $(\mathrm{id} \otimes S^2)(\boldsymbol{c}) = \boldsymbol{c}_{21}$. If y is central, then $S^2(y) = K_{-2\rho} y K_{2\rho} = y$. Hence,

$$\langle x, y\rangle = \langle y, x\rangle \quad \text{if } y \text{ is central.} \tag{94}$$

The quantum Killing form extends to a multilinear form

$$\langle\cdot,\cdot\rangle\colon \overline{\boldsymbol{X}_h^{\otimes n}} \mathbin{\hat{\otimes}} \boldsymbol{X}_h^{\hat{\otimes} n} \to \mathbb{C}[\![\sqrt{h}]\!],$$

where $\overline{\boldsymbol{X}_h^{\otimes n}}$ is the topological closure of $\boldsymbol{X}_h^{\otimes n}$, given by

$$\langle x_1 \otimes \ldots \otimes x_n, y_1 \otimes \ldots \otimes y_n\rangle = \prod_{j=1}^{n} \langle x_j, y_j\rangle.$$

Lemma 4.12 *Suppose x, y and z are elements of $\boldsymbol{X}_h^0 = \boldsymbol{X}_h \cap \boldsymbol{U}_{\sqrt{h}}^0$. Then*

$$\langle xy, z\rangle = \langle x \otimes y, \Delta(z)\rangle. \tag{95}$$

Proof This follows from (90), with $\boldsymbol{n} = \boldsymbol{m} = 0$. □

Note that (95) does not hold for general $x, y, z \in \boldsymbol{X}_h$.

4H Twist system associated to $\boldsymbol{X}_h$ and an invariant of integral homology 3–spheres

According to the result of Section 2M, the core subalgebra $\boldsymbol{X}_h$ gives rise to a twist system $\mathcal{T}_\pm\colon \boldsymbol{X}_h \to \mathbb{C}[\![\sqrt{h}]\!]$, defined by

$$\mathcal{T}_\pm(x) = \langle \boldsymbol{r}^{\pm 1}, x\rangle$$

and an invariant $J_M \in \mathbb{C}[\![\sqrt{h}]\!]$ of integral homology 3–spheres M. Recall that J_M is defined as follows. Suppose T is an n–component bottom tangle with 0 linking matrix and $\varepsilon_i \in \{-1, 1\}$, and M is obtained from S^3 by surgery along the closure link $\mathrm{cl}(T)$ with the framing of the i^{th} component switched to ε_i. Then

$$J_M = (\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(J_T).$$

In the next few sections we will show that $J_M \in \widehat{\mathbb{Z}[q]}$.

Let us calculate the values of $\mathcal{T}_\pm$ on basis elements. Recall that

$$\boldsymbol{r}_0 = K_{-2\rho} \exp\Big(-\frac{1}{2}\sum_{\alpha\in\Pi} H_\alpha \breve{H}_\alpha / d_\alpha\Big).$$

Proposition 4.13 (a) *Fix a longest reduced sequence* $\boldsymbol{i}$. *For* $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t$, $\gamma \in Y$ *and* $x \in \boldsymbol{X}_h^0$, *one has*

$$\mathcal{T}_+(F_{\boldsymbol{m}} K_{\boldsymbol{m}} x E_{\boldsymbol{n}}) = \delta_{\boldsymbol{m},\boldsymbol{n}} q^{(\rho,|E_{\boldsymbol{n}}|)} \langle \boldsymbol{r}_0, x\rangle, \tag{96}$$

$$\langle \boldsymbol{r}_0, K_\gamma\rangle = v^{(\gamma,\rho)-\frac{1}{4}(\gamma,\gamma)} \in \mathbb{Z}[v^{\pm 1/2}]. \tag{97}$$

(b) *For every* $x \in \boldsymbol{X}_h$, *one has*

$$\mathcal{T}_-(x) = \mathcal{T}_+(\varphi(x)). \tag{98}$$

Proof (a) By (72),

$$\boldsymbol{r} = \sum_{\boldsymbol{n}\in\mathbb{N}^t} F_{\boldsymbol{n}} K_{\boldsymbol{n}} \boldsymbol{r}_0 E_{\boldsymbol{n}}.$$

Identity (96) follows from the triangular property of the quantum Killing form. The identities in (97) follow from a calculation using (90) and the explicit expression of $\boldsymbol{r}_0$.

(b) By (79), $\boldsymbol{c} = (\varphi \mathbin{\hat\otimes} \underline{S}^{-1}\varphi)(\boldsymbol{c})$. By Proposition 2.23, for $y \in \overline{\boldsymbol{X}}_h$ and $x \in \boldsymbol{X}_h$ one has

$$\langle y, x\rangle = \langle \underline{S}^{-1}\varphi(y), \varphi(x)\rangle. \tag{99}$$

By Corollary 3.11 and (73), $\underline{S}^{-1}\varphi(\boldsymbol{r}^{-1}) = \boldsymbol{r}$. Using (99) with $y = \boldsymbol{r}^{-1}$, we get (98). □

4I Twist forms on U_q

By construction we have twist forms $\mathcal{T}_\pm\colon \boldsymbol{X}_h \to \mathbb{C}[\![\sqrt{h}]\!]$, with domain $\boldsymbol{X}_h$ and codomain $\mathbb{C}[\![\sqrt{h}]\!]$. We can change the domain to get a better image space.

By Proposition 4.13, for $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t$ and $\gamma \in Y$,

$$\mathcal{T}_+(F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_{2\gamma} E_{\boldsymbol{n}}) = \delta_{\boldsymbol{m},\boldsymbol{n}} q^{(\rho,|E_{\boldsymbol{n}}|)} v^{2(\gamma,\rho)-(\gamma,\gamma)} \in \mathbb{Z}[q^{\pm 1}] \subset \mathbb{Z}[v^{\pm 1}]. \tag{100}$$

Because $\{F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_{2\gamma} E_{\boldsymbol{n}} \mid \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^t,\ \gamma \in Y\}$ is a $\mathbb{C}(v)$–basis of $\boldsymbol{U}_q^{\mathrm{ev}}$, we have

$$\mathcal{T}_+(\boldsymbol{U}_q^{\mathrm{ev}} \cap \boldsymbol{X}_h) \subset \mathbb{C}(v) \cap \mathbb{C}[\![\sqrt{h}]\!].$$

Using $\mathcal{T}_-(x) = \mathcal{T}_+(\varphi(x))$ (see Proposition 4.13) and the fact that both $\boldsymbol{U}_q^{\mathrm{ev}}$ and $\boldsymbol{X}_h$ are φ–stable, we also have

$$\mathcal{T}_-(\boldsymbol{U}_q^{\mathrm{ev}} \cap \boldsymbol{X}_h) \subset \mathbb{C}(v) \cap \mathbb{C}[\![\sqrt{h}]\!].$$

Because $\boldsymbol{U}_q^{\mathrm{ev}} \cap \boldsymbol{X}_h$ spans $\boldsymbol{U}_q^{\mathrm{ev}}$ over $\mathbb{C}(v)$, we can extend the restriction of $\mathcal{T}_\pm$ on $\boldsymbol{U}_q^{\mathrm{ev}} \cap \boldsymbol{X}_h$ to $\mathbb{C}(v)$–linear maps, also denoted by $\mathcal{T}_\pm$:

$$\mathcal{T}_\pm\colon \boldsymbol{U}_q^{\mathrm{ev}} \to \mathbb{C}(v).$$

The values of $\mathcal{T}_+$ on the basis elements are given by (100). It is clear that

$$\mathcal{T}_\pm(\boldsymbol{U}_\mathbb{Z}^{\mathrm{ev}}) \subset \mathbb{Q}(v). \tag{101}$$

5 Integral core subalgebra

In Section 4 we constructed a core subalgebra $\boldsymbol{X}_h$ of $\boldsymbol{U}_{\sqrt{h}}$ that gives rise to an invariant J_M of integral homology 3–spheres with values in $\mathbb{C}[\![\sqrt{h}]\!]$. In order to show that J_M takes values in $\widehat{\mathbb{Z}[q]}$ we need an integral version of the core algebra. This section is devoted to an integral form $\boldsymbol{X}_\mathbb{Z}$ of the core algebra $\boldsymbol{X}_h$.

In order to construct $\boldsymbol{X}_\mathbb{Z}$ we first introduce Lusztig's integral form $\boldsymbol{U}_\mathbb{Z}$ and De Concini and Procesi's integral form $\boldsymbol{V}_\mathbb{Z}$. Then we construct $\boldsymbol{X}_\mathbb{Z}$ so that $(\boldsymbol{U}_\mathbb{Z}, \boldsymbol{X}_\mathbb{Z}, \boldsymbol{V}_\mathbb{Z})$ forms an *integral dilatation triple* corresponding to the topological dilatation triple $(\boldsymbol{U}_h, \boldsymbol{X}_h, \boldsymbol{V}_h)$.

Lusztig introduced $\boldsymbol{U}_\mathbb{Z}$ in connection with his discovery (independently with Kashiwara) of canonical bases. De Concini and Procesi introduced $\boldsymbol{V}_\mathbb{Z}$ in connection with their study of geometric aspects of quantized enveloping algebras. For the study of the integrality of quantum invariants, Lusztig's integral form $\boldsymbol{U}_\mathbb{Z}$ is too big: it does not have necessary integrality properties. For example, the quantum Killing form $\langle x, y\rangle$ with $x, y \in \boldsymbol{U}_\mathbb{Z}$ belongs to $\mathbb{Q}(v^{1/2})$ but not to $\mathbb{Z}[v^{\pm 1/2}]$ in general. On the other hand, De Concini and Procesi's form $\boldsymbol{V}_\mathbb{Z}$ is too small, in the sense that completed tensor powers of $\boldsymbol{V}_\mathbb{Z}$ do not contain the universal invariant of general bottom tangles. (Recently, however, Suzuki [78; 79] proved that, for $\mathfrak{g} = \mathrm{sl}_2$, the universal invariant of *ribbon* and *boundary* bottom tangles is contained in completed tensor powers of $\boldsymbol{V}_\mathbb{Z}$.) Our integral form $\boldsymbol{X}_\mathbb{Z}$ is the perfect middle ground, since it is big enough to contain quantum link invariants and small enough to have the necessary integrality. We believe that $\boldsymbol{X}_\mathbb{Z}$ is the right integral form for the study of quantum invariants of links and 3–manifolds.

We will show that De Concini and Procesi's $\boldsymbol{V}_\mathbb{Z}$ is "almost" dual to Lusztig's $\boldsymbol{U}_\mathbb{Z}$ under the quantum Killing form; see the precise statement in Proposition 5.15. This fact can be interpreted as an integral version of the duality of Drinfel'd [17] and Gavarini [18]. Using the duality we then show that the even part of $\boldsymbol{V}_\mathbb{Z}$ is invariant under the adjoint action of $\boldsymbol{U}_\mathbb{Z}$, an important result which will be used frequently later. We then show that the twist forms have nice integrality on $\boldsymbol{X}_\mathbb{Z}$.

5A Dilatation of based free modules

Let $\widetilde{\mathcal{A}}$ be the extension ring of $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ obtained by adjoining all $\sqrt{\phi_n(q)}$, $n = 1, 2, \dots$, to $\mathcal{A}$. Here $\phi_n(q)$ is the n^{th} cyclotomic polynomial and $q = v^2$. One reason why working over $\widetilde{\mathcal{A}}$ is not too much a sacrifice is the following:

Lemma 5.1 *One has* $\widetilde{\mathcal{A}} \cap \mathbb{Q}(q) = \mathbb{Z}[q^{\pm 1}]$.

Proof Since $\sqrt{\phi_k(q)}$ is integral over $\mathbb{Z}[q^{\pm 1}]$, $\widetilde{\mathcal{A}}$ is integral over $\mathbb{Z}[q^{\pm 1}]$. Hence $\widetilde{\mathcal{A}} \cap \mathbb{Q}(q) = \mathbb{Z}[q^{\pm 1}]$. □

Suppose V is a *based free $\mathcal{A}$–module*, ie a free $\mathcal{A}$–module equipped with a preferred basis $\{e(i) \mid i \in I\}$. Assume $a\colon I \to \mathcal{A}$ is a function such that $a(i)$ is a product of cyclotomic polynomials in q for every $i \in I$. In particular, $a(i) \neq 0$ and $\sqrt{a(i)} \in \widetilde{\mathcal{A}}$. The based free $\mathcal{A}$–module $V(a) \subset V$, with preferred basis $\{a(i)e(i) \mid i \in I\}$, is called a *dilatation of* V, with dilatation factors $a(i)$. Let $V(\sqrt{a})$ be the based free $\widetilde{\mathcal{A}}$–module with preferred basis $\{\sqrt{a(i)}\,e(i) \mid i \in I\}$. We call $(V, V(\sqrt{a}), V(a))$ a *dilatation triple* determined by the based free $\mathcal{A}$–module V and the function a.

We will introduce the Lusztig integral form $\boldsymbol{U}_{\mathbb{Z}}$, the integral core algebra $\boldsymbol{X}_{\mathbb{Z}}$ and the De Concini–Procesi integral form $\boldsymbol{V}_{\mathbb{Z}}$ so that $(\boldsymbol{U}_{\mathbb{Z}}, \boldsymbol{X}_{\mathbb{Z}}, \boldsymbol{V}_{\mathbb{Z}})$ is a dilatation triple.

5B Lusztig's integral form $\boldsymbol{U}_{\mathbb{Z}}$

Let $\boldsymbol{U}_{\mathbb{Z}}$ be the $\mathcal{A}$–subalgebra of $\boldsymbol{U}_q$ generated by all $E_\alpha^{(n)}$, $F_\alpha^{(n)}$ and $K_\alpha^{\pm 1}$ with $\alpha \in \Pi$ and $n \in \mathbb{N}$. Set $\boldsymbol{U}_{\mathbb{Z}}^* = \boldsymbol{U}_{\mathbb{Z}} \cap \boldsymbol{U}_q^*$ for $* = -,\ 0,\ +$.

Let us collect some well-known facts about $\boldsymbol{U}_{\mathbb{Z}}$. Recall that $E^{(\boldsymbol{n})}$ and $F^{(\boldsymbol{n})}$, defined for $\boldsymbol{n} \in \mathbb{N}^t$ in Section 3F2, depend on the choice of a longest reduced sequence.

Proposition 5.2 *Fix a longest reduced sequence* $\boldsymbol{i}$.

(a) *The $\mathcal{A}$–algebra $\boldsymbol{U}_{\mathbb{Z}}$ is a Hopf subalgebra of $\boldsymbol{U}_q$ and satisfies the triangular decomposition*

$$\boldsymbol{U}_{\mathbb{Z}}^- \otimes \boldsymbol{U}_{\mathbb{Z}}^0 \otimes \boldsymbol{U}_{\mathbb{Z}}^+ \xrightarrow{\cong} \boldsymbol{U}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz.$$

Moreover, $\boldsymbol{U}_{\mathbb{Z}}$ is stable under the action of $T_\alpha^{\pm 1}$, $\alpha \in \Pi$.

(b) *The set $\{F^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ is a free $\mathcal{A}$–basis of the $\mathcal{A}$–module $\boldsymbol{U}_{\mathbb{Z}}^-$. Similarly, $\{E^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ is a free $\mathcal{A}$–basis of $\boldsymbol{U}_{\mathbb{Z}}^+$.*

(c) *The Cartan part* $U^0_{\mathbb{Z}}$ *is the* $\mathcal{A}$*–subalgebra of* U^0_q *generated by*

$$K_\alpha^{\pm 1} \quad \text{and} \quad \frac{(K_\alpha^2; q_\alpha)_n}{(q_\alpha; q_\alpha)_n}$$

for $\alpha \in \Pi$ *and* $n \in \mathbb{N}$.

(d) *The algebra* $U_{\mathbb{Z}}$ *is stable under* ι_{bar}, τ *and* φ. *Moreover* $U^-_{\mathbb{Z}}$ *is stable under* ι_{bar} *and* τ.

Proof Parts (a)–(c) are proved in [53; 54, Proposition 41.1.3]. Part (d) can be proved by noticing that each of ι_{bar}, τ and φ maps each of the generators $E_\alpha^{(n)}$, $F_\alpha^{(n)}$ and $K_\alpha^{\pm 1}$ of $U_{\mathbb{Z}}$ into $U_{\mathbb{Z}}$, and each of ι_{bar} and τ maps each of the generators $F_\alpha^{(n)}$ of $U^-_{\mathbb{Z}}$ into $U^-_{\mathbb{Z}}$. □

We will consider $U^-_{\mathbb{Z}}$ and $U^+_{\mathbb{Z}}$ as based free $\mathcal{A}$–modules with preferred bases described in Proposition 5.2(b). Later we will find a preferred basis for the Cartan part $U^0_{\mathbb{Z}}$.

Let $U^{\mathrm{ev}}_{\mathbb{Z}} = U_{\mathbb{Z}} \cap U^{\mathrm{ev}}_q$ be the even part of $U_{\mathbb{Z}}$. From the triangulation of $U_{\mathbb{Z}}$ we have the following *even triangulation* of $U_{\mathbb{Z}}$ and $U^{\mathrm{ev}}_{\mathbb{Z}}$:

$$U^{\mathrm{ev},-}_{\mathbb{Z}} \otimes U^0_{\mathbb{Z}} \otimes U^+_{\mathbb{Z}} \xrightarrow{\cong} U_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz, \tag{102}$$

$$U^{\mathrm{ev},-}_{\mathbb{Z}} \otimes U^{\mathrm{ev},0}_{\mathbb{Z}} \otimes U^+_{\mathbb{Z}} \xrightarrow{\cong} U^{\mathrm{ev}}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz. \tag{103}$$

Here, $U^{\mathrm{ev},0}_{\mathbb{Z}} = U^{\mathrm{ev}}_{\mathbb{Z}} \cap U^{\mathrm{ev},0}_q$, with $U^0_q = \mathbb{C}(v)[K_\alpha^{\pm 2}, \alpha \in \Pi]$, and $U^{\mathrm{ev},-}_{\mathbb{Z}} = U^{\mathrm{ev}}_{\mathbb{Z}} \cap U^{\mathrm{ev},-}_q = \varphi(U^+_{\mathbb{Z}})$.

From Proposition 5.2(b) and $U^{\mathrm{ev},-}_{\mathbb{Z}} = \varphi(U^+_{\mathbb{Z}})$, we have the following:

Proposition 5.3 *The set* $\{F^{(\boldsymbol{n})} K_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ *is a free* $\mathcal{A}$*–basis of the* $\mathcal{A}$*–module* $U^{\mathrm{ev},-}_{\mathbb{Z}}$.

We will consider $U^{\mathrm{ev},-}_{\mathbb{Z}}$ as a based free $\mathcal{A}$–module with the above preferred basis.

5C De Concini–Procesi integral form $V_{\mathbb{Z}}$

Let $V_{\mathbb{Z}}$ be the smallest $\mathcal{A}$–subalgebra of $U_{\mathbb{Z}}$ which is invariant under the action of the braid group and contains $(1-q_\alpha)E_\alpha$, $(1-q_\alpha)F_\alpha$ and $K_\alpha^{\pm 1}$ for $\alpha \in \Pi$. For $* = 0, +, -$, set $V^*_{\mathbb{Z}} = V_{\mathbb{Z}} \cap U^*_q$.

Remark 5.4 In the original definition, De Concini and Procesi [15, Section 12] used the ground ring $\mathbb{Q}[v^{\pm 1}]$ instead of $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$. Our $V_{\mathbb{Z}}$ is denoted by A in [15].

Fix a longest reduced sequence $\boldsymbol{i}$. For $\boldsymbol{n} = (n_1, \dots, n_t) \in \mathbb{N}^t$, let

$$(q;q)_{\boldsymbol{n}} = \prod_{j=1}^{t} (q_{\gamma_j}; q_{\gamma_j})_{n_j}. \tag{104}$$

Note that $(q;q)_{\boldsymbol{n}}$ depends on $\boldsymbol{i}$, since $\gamma_j = \gamma_j(\boldsymbol{i})$ depends on $\boldsymbol{i}$.

Proposition 5.5 *Fix a longest reduced sequence* $\boldsymbol{i}$.

(a) *The* $\mathcal{A}$*–algebra* $\boldsymbol{V}_{\mathbb{Z}}$ *is a Hopf subalgebra of* $\boldsymbol{U}_{\mathbb{Z}}$.

(b) *We have* $\boldsymbol{V}^0_{\mathbb{Z}} = \mathcal{A}[K_1^{\pm 1}, \dots, K_l^{\pm 1}]$ *and the triangular decomposition*

$$\boldsymbol{V}^-_{\mathbb{Z}} \otimes \boldsymbol{V}^0_{\mathbb{Z}} \otimes \boldsymbol{V}^+_{\mathbb{Z}} \xrightarrow{\cong} \boldsymbol{V}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz.$$

(c) *The set* $\{(q;q)_{\boldsymbol{n}} F^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ *is a free* $\mathcal{A}$*–basis of the* $\mathcal{A}$*–module* $\boldsymbol{V}^-_{\mathbb{Z}}$. *Similarly,* $\{(q;q)_{\boldsymbol{n}} E^{(\boldsymbol{n})} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ *is a free* $\mathcal{A}$*–basis of* $\boldsymbol{V}^+_{\mathbb{Z}}$.

Proof The proofs for the case when $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ is replaced by $\mathbb{Q}[v^{\pm 1}]$ were given in [15, Section 12]. The proofs there remain valid for $\mathcal{A}$. Note that in [15] our $\boldsymbol{V}_{\mathbb{Z}}$ is denoted by A. □

The even part $\boldsymbol{V}^{\mathrm{ev}}_{\mathbb{Z}} := \boldsymbol{V}_{\mathbb{Z}} \cap \boldsymbol{U}^{\mathrm{ev}}_q$ is an $\mathcal{A}$–subalgebra of $\boldsymbol{V}_{\mathbb{Z}}$. From the triangular decomposition of $\boldsymbol{V}_{\mathbb{Z}}$, we have the *even triangular decompositions*

$$\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}} \otimes \boldsymbol{V}^{\mathrm{ev},0}_{\mathbb{Z}} \otimes \boldsymbol{V}^+_{\mathbb{Z}} \xrightarrow{\cong} \boldsymbol{V}^{\mathrm{ev}}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz, \tag{105}$$

$$\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}} \otimes \boldsymbol{V}^0_{\mathbb{Z}} \otimes \boldsymbol{V}^+_{\mathbb{Z}} \xrightarrow{\cong} \boldsymbol{V}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz, \tag{106}$$

where $\boldsymbol{V}^{\mathrm{ev},0}_{\mathbb{Z}} := \boldsymbol{V}_{\mathbb{Z}} \cap \boldsymbol{U}^{\mathrm{ev},0}_q = \mathcal{A}[K_1^{\pm 2}, \dots, K_l^{\pm 2}]$ and $\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}} := \boldsymbol{V}_{\mathbb{Z}} \cap \boldsymbol{U}^{\mathrm{ev},-}_q = \varphi(\boldsymbol{V}^+_{\mathbb{Z}})$.

From Proposition 5.2(b) and $\boldsymbol{U}^{\mathrm{ev},-}_q = \varphi(\boldsymbol{U}^+_q)$, we have the following:

Proposition 5.6 *The set* $\{(q;q)_{\boldsymbol{n}} F^{(\boldsymbol{n})} K_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^t\}$ *is a free* $\mathcal{A}$*–basis of the* $\mathcal{A}$*–module* $\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}}$.

We will consider $\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}}$ as a based free $\mathcal{A}$–module with the above preferred basis. Then $\boldsymbol{V}^{\mathrm{ev},-}_{\mathbb{Z}}$ is a dilatation of $\boldsymbol{U}^{\mathrm{ev},-}_{\mathbb{Z}}$. Similarly, we consider $\boldsymbol{V}^+_{\mathbb{Z}}$ as a based free $\mathcal{A}$–module with preferred basis given in Proposition 5.5. Then $\boldsymbol{V}^+_{\mathbb{Z}}$ is a dilatation of $\boldsymbol{U}^+_{\mathbb{Z}}$.

5D Preferred bases for $U_{\mathbb{Z}}^{0}$ and $V_{\mathbb{Z}}^{0}$

We will equip $U_{\mathbb{Z}}^{0}$ and $V_{\mathbb{Z}}^{0}$ with preferred $\mathcal{A}$–bases such that $V_{\mathbb{Z}}^{0}$ is a dilatation of $U_{\mathbb{Z}}^{0}$. Recall that $K_j = K_{\alpha_j}$ and $q_j = q_{\alpha_j}$.

For $\boldsymbol{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_l) \in \{0,1\}^l$ let

$$Q^{\mathrm{ev}}(\boldsymbol{n}) := \prod_{j=1}^{l} \frac{K_j^{-2\lfloor n_j/2\rfloor}(q_j^{-\lfloor (n_j-1)/2\rfloor} K_j^2; q_j)_{n_j}}{(q_j;q_j)_{n_j}}, \tag{107}$$

$$Q(\boldsymbol{n}, \boldsymbol{\delta}) := Q^{\mathrm{ev}}(\boldsymbol{n}) \prod_{j=1}^{l} K_j^{\delta_j}, \tag{108}$$

$$(q;q)_{\boldsymbol{n}} := \prod_{j=1}^{l} (q_j;q_j)_{n_j}. \tag{109}$$

Proposition 5.7 (a) *The sets* $\{Q^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}$ *and* $\{(q;q)_{\boldsymbol{n}} Q^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}$ *are* $\mathcal{A}$*–bases of* $U_{\mathbb{Z}}^{\mathrm{ev},0}$ *and* $V_{\mathbb{Z}}^{\mathrm{ev},0}$*, respectively.*

(b) *The sets* $\{Q(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^l,\ \boldsymbol{\delta} \in \{0,1\}^l\}$ *and* $\{(q;q)_{\boldsymbol{n}} Q(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^l,\ \boldsymbol{\delta} \in \{0,1\}^l\}$ *are* $\mathcal{A}$*–bases of* $U_{\mathbb{Z}}^{\mathrm{ev}}$ *and* $V_{\mathbb{Z}}^{\mathrm{ev}}$*, respectively.*

The proof is not difficult, since $U_{\mathbb{Z}}^{0}$ and $V_{\mathbb{Z}}^{0}$ are $\mathcal{A}$–subalgebras of the commutative algebra $\mathbb{Q}(v)[K_1^{\pm 1}, \dots, K_l^{\pm 1}]$, though it involves some calculation. We give a proof of Proposition 5.7 in Appendix B.

Remark 5.8 In [52], Lusztig gave a similar, but different, basis of $U_{\mathbb{Z}}^{0}$. Our basis can be obtained from Lusztig's by an upper triangular matrix, and hence a proof of the proposition can be obtained this way. We chose the basis in Proposition 5.7 instead of Lusztig's one for orthogonality reasons.

5E Preferred bases of $U_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$

Recall that we have defined $(q;q)_{\boldsymbol{n}}$ in two cases depending on the length of $\boldsymbol{n}$ — see (104) and (109) — either $\boldsymbol{n} = (n_1, \dots, n_t) \in \mathbb{N}^t$, in which case

$$(q;q)_{\boldsymbol{n}} = \prod_{j=1}^{t} (q_{\gamma_j}; q_{\gamma_j})_{n_j},$$

or $\boldsymbol{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$, then

$$(q;q)_{\boldsymbol{n}} = \prod_{j=1}^{l} (q_{\alpha_j}; q_{\alpha_j})_{n_j}.$$

The first one depends on a longest reduced sequence since γ_j does, while the second one does not.

Introduce another $(q;q)_{\boldsymbol{n}}$, with length $n = 2t + l$. For $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in \mathbb{N}^{t+l+t}$, where $\boldsymbol{n}_1$, $\boldsymbol{n}_3 \in \mathbb{N}^t$ and $\boldsymbol{n}_2 \in \mathbb{N}^l$, define

$$(q;q)_{\boldsymbol{n}} := (q;q)_{\boldsymbol{n}_1}(q;q)_{\boldsymbol{n}_2}(q;q)_{\boldsymbol{n}_3}. \tag{110}$$

Further, if $\boldsymbol{\delta} \in \{0,1\}^l$, let

$$\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}) := F^{(\boldsymbol{n}_1)} K_{\boldsymbol{n}_1} Q^{\mathrm{ev}}(\boldsymbol{n}_2) E^{(\boldsymbol{n}_3)}, \quad \boldsymbol{e}(\boldsymbol{n},\boldsymbol{\delta}) := F^{(\boldsymbol{n}_1)} K_{\boldsymbol{n}_1} Q(\boldsymbol{n}_2,\boldsymbol{\delta}) E^{(\boldsymbol{n}_3)}. \tag{111}$$

Proposition 5.9 (a) *The set*

$$\{\boldsymbol{e}(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t},\ \boldsymbol{\delta} \in \{0,1\}^l\}$$

and its dilated set

$$\{(q;q)_{\boldsymbol{n}} \boldsymbol{e}(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t},\ \boldsymbol{\delta} \in \{0,1\}^l\}$$

are $\mathcal{A}$–bases of $\boldsymbol{U}_{\mathbb{Z}}$ and $\boldsymbol{V}_{\mathbb{Z}}$, respectively.

(b) *The set*

$$\{\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$$

and its dilated set

$$\{(q;q)_{\boldsymbol{n}} \boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$$

are $\mathcal{A}$–bases of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ and $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$, respectively.

Proof The proposition follows from the even triangular decompositions of $\boldsymbol{U}_{\mathbb{Z}}$ and $\boldsymbol{V}_{\mathbb{Z}}$, together with the bases of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},-}$, $\boldsymbol{U}_{\mathbb{Z}}^{0}$ and $\boldsymbol{U}_{\mathbb{Z}}^{+}$, and $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},-}$, $\boldsymbol{V}_{\mathbb{Z}}^{0}$ and $\boldsymbol{V}_{\mathbb{Z}}^{+}$ in Propositions 5.2, 5.5 and 5.7. □

We will consider $\boldsymbol{U}_{\mathbb{Z}}$, $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$, $\boldsymbol{V}_{\mathbb{Z}}$ and $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ as based free $\mathcal{A}$–modules with the preferred bases described in the above proposition. Then $\boldsymbol{V}_{\mathbb{Z}}$ is a dilatation of $\boldsymbol{U}_{\mathbb{Z}}$ and $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is a dilatation of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$.

5F Relation between $\boldsymbol{V}_{\mathbb{Z}}$ and $\boldsymbol{V}_h$

Proposition 5.10 (a) *One has $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}} \subset \boldsymbol{V}_h$.*

(b) *Moreover, $\boldsymbol{V}_h$ is the topological closure (in the h–adic topology of $\boldsymbol{U}_h$) of the $\mathbb{C}[\![h]\!]$–span of $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$. Consequently, $\boldsymbol{V}_h$ is also the topological closure (in the h–adic topology of $\boldsymbol{U}_h$) of $\boldsymbol{V}_{\mathbb{Z}}$.*

Proof (a) It is clear that $V_{\mathbb{Z}}^{\mathrm{ev}} \subset V_{\mathbb{Z}}$. Let us prove $V_{\mathbb{Z}} \subset V_h$.

Fix a longest reduced sequence $\boldsymbol{i}$. By Proposition 4.2, V_h is the topological closure of the $\mathbb{C}[\![h]\!]$–subalgebra generated by hH_α, hF_γ and hE_γ, with $\alpha \in \Pi$ and $\gamma \in \Phi_+$.

For every $\gamma \in \Phi_+$, there is a unit u in $\mathbb{C}[\![h]\!]$ such that $1-q_\gamma = hu$ and

$$(1-q_\gamma)F_\gamma = u(hF_\gamma) \in V_h. \tag{112}$$

Similarly, $(1-q_\gamma)E_\gamma \in V_h$. We already have $K_\alpha^{\pm 1} \in V_h$. Since $(1-q_\gamma)F_\gamma$, $(1-q_\gamma)E_\gamma$ and $K_\alpha^{\pm 1}$ generate $V_{\mathbb{Z}}$ as an $\mathcal{A}$–algebra and V_h is an $\mathcal{A}$–algebra, we have $V_{\mathbb{Z}} \subset V_h$.

(b) Let V_h' be the topological closure of the $\mathbb{C}[\![h]\!]$–span of $V_{\mathbb{Z}}^{\mathrm{ev}}$. We have to show that $V_h' = V_h$. From part (a) we now have that $V_h' \subset V_h$. It remains to show $V_h \subset V_h'$. It is easy to see that V_h' is a $\mathbb{C}[\![h]\!]$–algebra.

Since $K_\alpha^2 \in V_{\mathbb{Z}}^{\mathrm{ev}}$ and

$$hH_\alpha = \log(K_\alpha^2) = -\sum_{n=1}^{\infty} \frac{(1-K_\alpha^2)^n}{n},$$

we have $hH_\alpha \in V_h'$ for any $\alpha \in \Pi$. It follows that $K_\alpha^{\pm 1} = \exp(\pm hH_\alpha/2) \in V_h'$.

From (112),

$$hF_\gamma = u^{-1}(1-q_\gamma)(F_\gamma K_\gamma)K_\gamma^{-1} \in V_h', \quad hE_\gamma = u^{-1}(1-q_\gamma)E_\gamma \in V_h'.$$

Thus, hH_α, hF_γ and hE_γ are in V_h' for any $\alpha \in \Pi$ and $\gamma \in \Phi_+$. Since V_h is the topological closure of the $\mathbb{C}[\![h]\!]$–algebra generated by hH_α, hF_γ and hE_γ, we have $V_h \subset V_h'$. This completes the proof of the proposition. □

Corollary 5.11 *The algebra V_h is stable under the braid group action, ie $T_\alpha^{\pm 1}(V_h) \subset V_h$ for any $\alpha \in \Pi$.*

Proof Since $V_{\mathbb{Z}}$ is invariant under the braid group action and V_h is the topological closure of the $\mathbb{C}[\![h]\!]$–span of $V_{\mathbb{Z}}$, the algebra V_h is also invariant under the braid group action. □

Remark 5.12 Using Corollary 5.11 one can easily prove that V_h is the smallest $\mathbb{C}[\![h]\!]$–subalgebra of U_h which

(i) contains hE_α, hF_α and hH_α for $\alpha \in \Pi$,

(ii) is stable under the action of the braid group,

(iii) is closed in the h–adic topology of U_h.

5G Stability of $V_{\mathbb{Z}}$ under ι_{bar}, τ and φ

By Proposition 5.2, $U_{\mathbb{Z}}$ is stable under ι_{bar}, τ and φ.

Proposition 5.13 *The algebra $V_{\mathbb{Z}}$ is stable under each of τ, φ and ι_{bar}.*

Proof Recall that $V_{\mathbb{Z}}$ is the smallest $\mathcal{A}$–subalgebra of $U_{\mathbb{Z}}$ containing $(1-q_\alpha)E_\alpha$, $(1-q_\alpha)F_\alpha$ and K_α for $\alpha \in \Pi$ and is stable under the action of the braid group. Let f be one of τ, φ or ι_{bar}.

Claim 1 *$f(V_{\mathbb{Z}})$ is stable under the braid group action.*

Proof of Claim 1 (i) $\boldsymbol{f=\tau}$ By [30, Formula 8.14.10], $\tau T_\alpha = T_\alpha^{-1}\tau$ for every $\alpha \in \Pi$. Since the T_α generate the braid group, we conclude that, like $V_{\mathbb{Z}}$, $\tau(V_{\mathbb{Z}})$ is also stable under the braid group.

(ii) $\boldsymbol{f=\varphi}$ Recall that S is the antipode. By Proposition 3.2, $\varphi = S\kappa = \kappa S$, where $\kappa = \iota_{\mathrm{bar}}\tau\omega$ is a $\mathbb{C}$–antiautomorphism of U_h. Our κ is the same κ in [15], where it was observed that κ commutes with the action of the braid group, ie $\kappa T_\alpha = T_\alpha \kappa$ for $\alpha \in \Pi$. It follows that $\kappa(V_{\mathbb{Z}})$ is stable under the braid group. Since $\varphi(V_{\mathbb{Z}}) = \kappa S(V_{\mathbb{Z}}) = \kappa(V_{\mathbb{Z}})$, $\varphi(V_{\mathbb{Z}})$ is stable under the braid group.

(iii) $\boldsymbol{f=\iota_{\mathrm{bar}}}$ Checking on the generators, one has $\iota_{\mathrm{bar}} = \kappa\tau\omega$.

By [30, Formula 8.14.9], if $x \in U_q$ is Y–homogeneous, then $T_\alpha(\omega(x)) \sim \omega T_\alpha(x)$, where $x \sim y$ means $x = uy$ for some unit $u \in \mathcal{A}$. As $V_{\mathbb{Z}}$ has an $\mathcal{A}$–basis consisting of Y–homogeneous elements (see Proposition 5.9), we conclude that $\omega(V_{\mathbb{Z}})$ is stable under the braid group. The results of (i) and (ii) show that $\iota_{\mathrm{bar}}(V_{\mathbb{Z}}) = \kappa\tau\omega(V_{\mathbb{Z}})$ is stable under the braid group.

This completes the proof of Claim 1. □

Claim 2 *One has $V_{\mathbb{Z}} \subset f(V_{\mathbb{Z}})$.*

Proof of Claim 2 Using the explicit formulas of f^{-1} in Section 3B, one sees that each of $f^{-1}((1-q_\alpha)E_\alpha)$, $f^{-1}((1-q_\alpha)F_\alpha)$ and $f^{-1}(K_\alpha)$ is in $V_{\mathbb{Z}}$. It follows that each of $(1-q_\alpha)E_\alpha$, $(1-q_\alpha)F_\alpha$ and K_α is in $f(V_{\mathbb{Z}})$. Together with Claim 1, this implies $f(V_{\mathbb{Z}})$ is an algebra stable under the braid group and contains $f^{-1}((1-q_\alpha)E_\alpha)$, $f^{-1}((1-q_\alpha)F_\alpha)$ and $f^{-1}(K_\alpha)$. Hence $f(V_{\mathbb{Z}}) \supset V_{\mathbb{Z}}$. This completes the proof of Claim 2. □

Since τ and ι_{bar} are involutions and $\varphi^2(x) = K_{-2\rho}xK_{2\rho}$ (by Proposition 3.2), we have $f^2(V_{\mathbb{Z}}) = V_{\mathbb{Z}}$. Applying f to $V_{\mathbb{Z}} \subset f(V_{\mathbb{Z}})$, we get $f(V_{\mathbb{Z}}) \subset f^2(V_{\mathbb{Z}}) = V_{\mathbb{Z}}$. Hence, $V_{\mathbb{Z}} = f(V_{\mathbb{Z}})$. □

5H Simply connected version of $U_{\mathbb{Z}}$

Recall that the simply connected version $\breve{U}_q$ is obtained from U_q by replacing the Cartan part $U_q^0 = \mathbb{C}(v)[K_1^{\pm 1}, \dots, K_l^{\pm 1}]$ with the bigger $\breve{U}_q^0 = \mathbb{C}(v)[\breve{K}_1^{\pm 1}, \dots, \breve{K}_l^{\pm 1}]$. We introduce an analog of Lusztig's integral form for $\breve{U}_q$ here.

The $\mathbb{C}(v)$–algebra homomorphism $\breve{\iota}\colon U_q^0 \to \breve{U}_q^0$, defined by $\breve{\iota}(K_\alpha) = \breve{K}_\alpha$, $\alpha \in \Pi$, is a Hopf algebra homomorphism. Let

$$\breve{U}_{\mathbb{Z}}^0 := \breve{\iota}(U_{\mathbb{Z}}^0), \quad \breve{U}_{\mathbb{Z}}^{\mathrm{ev},0} := \breve{\iota}(U_{\mathbb{Z}}^{\mathrm{ev},0}).$$

Then $\breve{U}_{\mathbb{Z}}^0$ and $\breve{U}_{\mathbb{Z}}^{\mathrm{ev},0}$ are $\mathcal{A}$–Hopf subalgebra of $\breve{U}_q^{\mathrm{ev},0}$. Define

$$\breve{U}_{\mathbb{Z}} := \breve{U}_{\mathbb{Z}}^0 U_{\mathbb{Z}}, \quad \breve{U}_{\mathbb{Z}}^{\mathrm{ev}} := \breve{U}_{\mathbb{Z}}^{\mathrm{ev},0} U_{\mathbb{Z}}^{\mathrm{ev}}.$$

For $\boldsymbol{m} \in \mathbb{N}^l$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_l) \in \{0,1\}^l$, define

$$\breve{Q}^{\mathrm{ev}}(\boldsymbol{m}) := \breve{\iota}(Q^{\mathrm{ev}}(\boldsymbol{m})), \quad \breve{Q}(\boldsymbol{m}, \boldsymbol{\delta}) := \breve{\iota}(Q(\boldsymbol{m}, \boldsymbol{\delta})),$$

and, furthermore, for $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in \mathbb{N}^{t+l+t}$ define

$$\breve{e}^{\mathrm{ev}}(\boldsymbol{n}) := F^{(\boldsymbol{n}_3)} K_{\boldsymbol{n}_3} \breve{Q}^{\mathrm{ev}}(\boldsymbol{n}_2) E^{(\boldsymbol{n}_1)}, \quad \breve{e}(\boldsymbol{n}, \boldsymbol{\delta}) := \breve{e}^{\mathrm{ev}}(\boldsymbol{n}) \prod_{j=1}^{l} \breve{K}_{\alpha_j}^{\delta_j}. \tag{113}$$

Proposition 5.14 (a) *$\breve{U}_{\mathbb{Z}}$ is an $\mathcal{A}$–Hopf subalgebra of $\breve{U}_q$ and $\breve{U}_{\mathbb{Z}}^{\mathrm{ev}}$ is an $\mathcal{A}$–subalgebra of $\breve{U}_{\mathbb{Z}}$. We also have the even triangular decompositions*

$$U_{\mathbb{Z}}^{\mathrm{ev},-} \otimes \breve{U}_{\mathbb{Z}}^{\mathrm{ev},0} \otimes U_{\mathbb{Z}}^{+} \xrightarrow{\cong} \breve{U}_{\mathbb{Z}}^{\mathrm{ev}}, \quad x \otimes y \otimes z \mapsto xyz, \tag{114}$$

$$U_{\mathbb{Z}}^{\mathrm{ev},-} \otimes \breve{U}_{\mathbb{Z}}^{0} \otimes U_{\mathbb{Z}}^{+} \xrightarrow{\cong} \breve{U}_{\mathbb{Z}}, \quad x \otimes y \otimes z \mapsto xyz. \tag{115}$$

(b) *The sets $\{\breve{e}(\boldsymbol{n}, \boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}, \boldsymbol{\delta} \in \{0,1\}^l\}$ and $\{\breve{e}^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ are $\mathcal{A}$–bases of $\breve{U}_{\mathbb{Z}}$ and $\breve{U}_{\mathbb{Z}}^{\mathrm{ev}}$, respectively.*

(c) *One has $\breve{U}_{\mathbb{Z}} \triangleright \breve{U}_{\mathbb{Z}}^{\mathrm{ev}} \subset \breve{U}_{\mathbb{Z}}^{\mathrm{ev}}$. Consequently, $U_{\mathbb{Z}} \triangleright \breve{U}_{\mathbb{Z}}^{\mathrm{ev}} \subset \breve{U}_{\mathbb{Z}}^{\mathrm{ev}}$.*

Proof (a) As an $\mathcal{A}$–module, $U_{\mathbb{Z}}^0$ is spanned by

$$f_{\alpha,m,n,k} := \frac{K_\alpha^m (q_\alpha^n K_\alpha^2; q_\alpha)_k}{(q_\alpha; q_\alpha)_k},$$

with $\alpha \in \Pi$, $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Hence, $\breve{U}_{\mathbb{Z}}^0 = \breve{\iota}(U_{\mathbb{Z}}^0)$ is $\mathcal{A}$–spanned by $\breve{f}_{\alpha,m,n,k} := \breve{\iota}(f_{\alpha,m,n,k})$. If $x \in U_{\mathbb{Z}}$ is Y–homogeneous then, using (57), which describes the commutation between $\breve{K}_\alpha$ and x,

$$\breve{f}_{\alpha,m,n,k}\, x = v^{m(|x|,\breve{\alpha})} x \breve{f}_{\alpha,m,n',k}, \tag{116}$$

where $n' = n + (|x|, \alpha)/d_\alpha \in \mathbb{Z}$. Hence, $\boldsymbol{U}_{\mathbb{Z}}$ commutes with $\breve{\boldsymbol{U}}^0_{\mathbb{Z}}$ in the sense that $\boldsymbol{U}_{\mathbb{Z}}\breve{\boldsymbol{U}}^0_{\mathbb{Z}} = \breve{\boldsymbol{U}}^0_{\mathbb{Z}}\boldsymbol{U}_{\mathbb{Z}}$. Since both $\boldsymbol{U}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}^0_{\mathbb{Z}}$ are $\mathcal{A}$–Hopf subalgebras of $\boldsymbol{U}_h$ and they commute in the above sense, $\breve{\boldsymbol{U}}_{\mathbb{Z}} = \breve{\boldsymbol{U}}^0_{\mathbb{Z}}\boldsymbol{U}_{\mathbb{Z}}$ is an $\mathcal{A}$–Hopf subalgebra of $\breve{\boldsymbol{U}}_q$.

Identity (116) also shows that each of $\breve{\boldsymbol{U}}^{\mathrm{ev},0}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}^0_{\mathbb{Z}}$ commutes with each of $\boldsymbol{U}^-_{\mathbb{Z}}$, $\boldsymbol{U}^+_{\mathbb{Z}}$ and $\boldsymbol{U}^0_{\mathbb{Z}}$. Hence, $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} = \breve{\boldsymbol{U}}^{\mathrm{ev},0}_{\mathbb{Z}}\boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}$ is an $\mathcal{A}$–subalgebra of $\breve{\boldsymbol{U}}_{\mathbb{Z}}$. The triangular decompositions for $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ follows from those of $\boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}$ and $\boldsymbol{U}_{\mathbb{Z}}$.

(b) Combining the bases $\{F^{(\boldsymbol{n}_3)}K_{\boldsymbol{n}_3}\}$ of $\boldsymbol{U}^{\mathrm{ev},-}_{\mathbb{Z}}$ (see Proposition 5.3), $\{\breve{Q}^{\mathrm{ev}}(\boldsymbol{n}_2)\}$ of $\breve{\boldsymbol{U}}^{\mathrm{ev},0}_{\mathbb{Z}}$ (by Proposition 5.7 and isomorphism $\breve{\iota}$), $\{E^{(\boldsymbol{n}_1)}\}$ of $\boldsymbol{U}^+_{\mathbb{Z}}$ (see Proposition 5.2), and the even triangular decompositions of $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$, we get the bases of $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$ as described.

(c) Since $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ contains $E^{(n)}_\alpha$, $F^{(n)}_\alpha$ and $K^{\pm 1}_\alpha$, which generate $\boldsymbol{U}_{\mathbb{Z}}$, we have $\boldsymbol{U}_{\mathbb{Z}} \subset \breve{\boldsymbol{U}}_{\mathbb{Z}}$. Let us prove

$$\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} \subset \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}.$$

From the triangular decompositions of $\breve{\boldsymbol{U}}_{\mathbb{Z}}$, $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}_q$, we see that $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} = \breve{\boldsymbol{U}}_{\mathbb{Z}} \cap \breve{\boldsymbol{U}}^{\mathrm{ev}}_q$.

Since $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ is a Hopf algebra, we have $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} \subset \breve{\boldsymbol{U}}_{\mathbb{Z}}$. By Lemma 3.6,

$$\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} \subset \breve{\boldsymbol{U}}_q \triangleright \breve{\boldsymbol{U}}^{\mathrm{ev}}_q \subset \breve{\boldsymbol{U}}^{\mathrm{ev}}_q.$$

Hence, $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}} \subset \breve{\boldsymbol{U}}_{\mathbb{Z}} \cap \breve{\boldsymbol{U}}^{\mathrm{ev}}_q = \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$. This finishes the proof of the proposition. □

5I Integral duality with respect to quantum Killing form

Recall that $\{e^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ is an $\mathcal{A}$–basis of $\boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}$ (see Proposition 5.9) and $\{\breve{e}^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}$ is an $\mathcal{A}$–basis of $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$ (see Proposition 5.14). We will show that these two bases are orthogonal to each other with respect to the quantum Killing form.

Recall that we defined $(q;q)_{\boldsymbol{n}} = (q;q)_{\boldsymbol{n}_1}(q;q)_{\boldsymbol{n}_2}(q;q)_{\boldsymbol{n}_3}$; see Section 5E.

Proposition 5.15 (a) *For $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^{t+l+t}$, there exists a unit $u(\boldsymbol{n}) \in \mathcal{A}$ such that*

$$\langle e^{\mathrm{ev}}(\boldsymbol{n}), \breve{e}^{\mathrm{ev}}(\boldsymbol{m})\rangle = \delta_{\boldsymbol{n},\boldsymbol{m}} \frac{u(\boldsymbol{n})}{(q;q)_{\boldsymbol{n}}}.$$

(b) *The $\mathcal{A}$–module $\boldsymbol{V}^{\mathrm{ev}}_{\mathbb{Z}}$ is the $\mathcal{A}$–dual of $\breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}$ in $\boldsymbol{U}^{\mathrm{ev}}_q$ with respect to the quantum Killing form, ie*

$$\boldsymbol{V}^{\mathrm{ev}}_{\mathbb{Z}} = \{x \in \boldsymbol{U}^{\mathrm{ev}}_q \mid \langle x, y\rangle \in \mathcal{A} \text{ for all } y \in \breve{\boldsymbol{U}}^{\mathrm{ev}}_{\mathbb{Z}}\}.$$

Proof Define the following units in $\mathcal{A}$. For $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{N}^t$ and $\boldsymbol{k}=(k_1,\dots,k_l)\in\mathbb{N}^l$ let

$$u_1(\boldsymbol{m})=\prod_{j=1}^{t} v_{\gamma_j}^{m_j^2}, \quad u_2(\boldsymbol{k})=\prod_{j=1}^{l} q_{\alpha_j}^{-\lfloor (k_j+1)/2\rfloor^2}.$$

For $\boldsymbol{n}=(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)\in\mathbb{N}^{t+l+t}$, let

$$u(\boldsymbol{n})=q^{(\rho,|E_{\boldsymbol{n}_3}|)}u_1(\boldsymbol{n}_1)u_2(\boldsymbol{n}_2)u_1(\boldsymbol{n}_3).$$

(a) We will use the following lemma, whose proof will be given in Appendix B:

Lemma 5.16 *For $\boldsymbol{k}$, $\boldsymbol{k}'\in\mathbb{N}^l$, one has*

$$\langle Q^{\mathrm{ev}}(\boldsymbol{k}), \breve{Q}^{\mathrm{ev}}(\boldsymbol{k}')\rangle=\delta_{\boldsymbol{k},\boldsymbol{k}'}u_2(\boldsymbol{k})/(q;q)_{\boldsymbol{k}}. \tag{117}$$

For $\boldsymbol{p}\in\mathbb{N}^t$, using the definition of $E_{\boldsymbol{p}}$ and $F_{\boldsymbol{p}}$ in Section 3G1, we have

$$F^{(\boldsymbol{p})}\otimes E^{(\boldsymbol{p})}=\frac{u_1(\boldsymbol{p})}{(q;q)_{\boldsymbol{p}}}(F_{\boldsymbol{p}}\otimes E_{\boldsymbol{p}}).$$

Suppose $\boldsymbol{n}=(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)$ and $\boldsymbol{m}=(\boldsymbol{m}_1,\boldsymbol{m}_2,\boldsymbol{m}_3)$ are in $\mathbb{N}^{t+l+t}$. Using the definition of $\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n})$ and $\breve{\boldsymbol{e}}^{\mathrm{ev}}(\boldsymbol{n})$ from (111) and (113), the triangular property of the quantum Killing form, and formulas (91) and (117),

$$\begin{aligned}\langle \boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}),\breve{\boldsymbol{e}}^{\mathrm{ev}}(\boldsymbol{m})\rangle &= \langle F^{(\boldsymbol{n}_1)}K_{\boldsymbol{n}_1},E^{(\boldsymbol{m}_1)}\rangle\langle Q^{\mathrm{ev}}(\boldsymbol{n}_2),\breve{Q}^{\mathrm{ev}}(\boldsymbol{m}_2)\rangle\langle E^{(\boldsymbol{n}_3)},F^{(\boldsymbol{m}_3)}K_{\boldsymbol{m}_3}\rangle\\ &= \frac{\delta_{\boldsymbol{n}_1,\boldsymbol{m}_1}u_1(\boldsymbol{n}_1)}{(q;q)_{\boldsymbol{n}_1}}\frac{\delta_{\boldsymbol{n}_2,\boldsymbol{m}_2}u_2(\boldsymbol{n}_2)}{(q;q)_{\boldsymbol{n}_2}}\frac{\delta_{\boldsymbol{n}_3,\boldsymbol{m}_3}q^{(\rho,|E_{\boldsymbol{n}_3}|)}u_1(\boldsymbol{n}_3)}{(q;q)_{\boldsymbol{n}_3}}\\ &= \frac{\delta_{\boldsymbol{n},\boldsymbol{m}}u(\boldsymbol{n})}{(q;q)_{\boldsymbol{n}}}.\end{aligned}$$

(b) By Proposition 5.9, $\{(q;q)_{\boldsymbol{n}}\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n})\mid \boldsymbol{n}\in\mathbb{N}^{t+l+t}\}$ is an $\mathcal{A}$–basis of $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ and a $\mathbb{C}(v)$–basis of $\boldsymbol{U}_q^{\mathrm{ev}}$ and, by Proposition 5.14, $\{\breve{\boldsymbol{e}}^{\mathrm{ev}}(\boldsymbol{n})\mid \boldsymbol{n}\in\mathbb{N}^{t+l+t}\}$ is an $\mathcal{A}$–basis of $\breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}}$. Part (b) follows from the orthogonality of part (a). □

Remark 5.17 From the orthogonality of Proposition 5.15, we can show that

$$\boldsymbol{c}=\sum_{\boldsymbol{n}\in\mathbb{N}^{t+l+t}}\frac{(q;q)_{\boldsymbol{n}}}{u(\boldsymbol{n})}\breve{\boldsymbol{e}}^{\mathrm{ev}}(\boldsymbol{n})\otimes\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}). \tag{118}$$

5J Invariance of $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ under adjoint action of $\boldsymbol{U}_{\mathbb{Z}}$

The adjoint action makes $\boldsymbol{U}_{\mathbb{Z}}$ a $\boldsymbol{U}_{\mathbb{Z}}$–module. The following result, showing that $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is a $\boldsymbol{U}_{\mathbb{Z}}$–submodule of $\boldsymbol{U}_{\mathbb{Z}}$, is important for us and will be used frequently.

Theorem 5.18 *We have $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$. In particular, $\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$, ie $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–ad-stable.*

Proof By Proposition 5.14, $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}} \subset \breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}}$ and, by Proposition 5.15, $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is the $\mathcal{A}$–dual of $\breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}}$ with respect to the quantum Killing form. Further, the quantum Killing form is ad-invariant. Hence, one also has $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$, as the following argument shows: Recall that we already have $\breve{\boldsymbol{U}}_{\mathbb{Z}} \triangleright \boldsymbol{U}_q^{\mathrm{ev}} \subset \boldsymbol{U}_q^{\mathrm{ev}}$ (see Lemma 3.6). Suppose $a \in \breve{\boldsymbol{U}}_{\mathbb{Z}}$ and $x \in \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$. We will show $a \triangleright x \in \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$. We have

$$\begin{aligned} a \triangleright x \in \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \quad &\iff \quad \langle a \triangleright x, y\rangle \in \mathcal{A} \quad \text{for all } y \in \breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}} \\ &\iff \quad \langle x, S(a) \triangleright y\rangle \in \mathcal{A} \quad \text{for all } y \in \breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}}, \end{aligned}$$

where the first equivalence is by duality (Proposition 5.15) and the second is by ad-invariance (Proposition 2.4(b)).

Since $S(a) \triangleright y \in \breve{\boldsymbol{U}}_{\mathbb{Z}}^{\mathrm{ev}}$, the last statement $\langle x, S(a) \triangleright y\rangle \in \mathcal{A}$ holds by Proposition 5.15. Thus we have proved that $\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$. □

Remark 5.19 We do *not* have $\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}} \subset \boldsymbol{V}_{\mathbb{Z}}$ in general. For example, when $\mathfrak{g} = A_2$ and $\alpha \neq \beta \in \Pi$,

$$E_\alpha \triangleright K_\beta = (v-1)K_\beta E_\alpha \notin \boldsymbol{V}_{\mathbb{Z}}.$$

However, when $\mathfrak{g} = A_1$, we do have $\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{V}_{\mathbb{Z}} \subset \boldsymbol{V}_{\mathbb{Z}}$, as easily follows from [78, Proposition 3.2], where a more refined statement is given.

5K Extension from $\mathcal{A}$ to $\tilde{\mathcal{A}}$: stability principle

Recall that $\tilde{\mathcal{A}}$ is obtained from $\mathcal{A}$ by adjoining all square roots $\sqrt{\phi_k(q)}$, $k = 1, 2, \dots,$ of cyclotomic polynomials $\phi_k(q)$.

Suppose V is a based free $\mathcal{A}$–module with preferred basis $\{e(i) \mid i \in I\}$ and $a \colon I \to \mathcal{A}$ is a function such that, for every $i \in I$, $a(i)$ is a product of cyclotomic polynomials in q. We already defined the dilatation triple $(V, V(\sqrt{a}), V(a))$ in Section 5A. Recall that $V(a)$ is the free $\mathcal{A}$–module with basis $\{a(i)e(i) \mid i \in I\}$ and $V(\sqrt{a})$ is the free $\tilde{\mathcal{A}}$–module with basis $\{\sqrt{a(i)}e(i) \mid i \in I\}$.

For any $\mathcal{A}$–module homomorphism $f \colon V_1 \to V_2$ we also use the same notation f to denote the linear extension $f \otimes \mathrm{id} \colon V_1 \otimes_{\mathcal{A}} \tilde{\mathcal{A}} \to V_2 \otimes_{\mathcal{A}} \tilde{\mathcal{A}}$, which is an $\tilde{\mathcal{A}}$–module homomorphism.

Proposition 5.20 (stability principle) *Let*

$$(V_1, V_1(\sqrt{a_1}), V_1(a_1)) \quad \text{and} \quad (V_2, V_2(\sqrt{a_2}), V_2(a_2))$$

be two dilatation triples and let $f\colon V_1 \to V_2$ *be an* $\mathcal{A}$*–module homomorphism. If* $f(V_1(a_1)) \subset V_2(a_2)$*, then* $f(V_1(\sqrt{a_1})) \subset V_2(\sqrt{a_2})$.

Proof First we prove the following:

Claim *Suppose* $a, b, c \in \mathcal{A}$ *with* b *and* c *products of cyclotomic polynomials* $\phi_k(q)$*. If* $ab/c \in \mathcal{A}$ *then* $a\sqrt{b/c} \in \widetilde{\mathcal{A}}$.

Proof of claim Since $\mathcal{A}$ is a unique factorization domain, one can assume that b and c are coprime. Then a must be divisible by c, say $a = a'c$ with $a' \in \mathcal{A}$. Then $a\sqrt{b/c} = a'\sqrt{bc'} \in \widetilde{\mathcal{A}}$, which proves the claim. □

The proof of the proposition is now parallel to that in the topological case (Proposition 4.6). Using the bases $\{e_1(i) \mid i \in I_1\}$ and $\{e_2(i) \mid i \in I_2\}$ of V_1 and V_2, we can write

$$f(e_1(i)) = \sum_{k \in I_2} f_i^k e_2(k),$$

where $f_i^k = 0$ except for a finite number of k (when i is fixed) and $f_i^k \in \mathcal{A}$.

Multiplying by $a_1(i)$ and $\sqrt{a_1(i)}$, we get

$$f(a_1(i)e_1(i)) = \sum_{k \in I_2} f_i^k \frac{a_1(i)}{a_2(k)} (a_2(k)e_2(k)), \tag{119}$$

$$f(\sqrt{a_1(i)}e_1(i)) = \sum_{k \in I_2} f_i^k \sqrt{\frac{a_1(i)}{a_2(k)}} (\sqrt{a_2(k)}e_2(k)). \tag{120}$$

Since $f(V_1(a_1)) \subset V_2(a_2)$, (119) implies that $f_i^k a_1(i)/a_2(k) \in \mathcal{A}$, which, together with $f_i^k \in \mathcal{A}$ and the claim, implies that

$$f_i^k \sqrt{\frac{a_1(i)}{a_2(k)}} \in \widetilde{\mathcal{A}}.$$

Now (120) shows that $f(V_1(\sqrt{a_1})) \subset V_2(\sqrt{a_2})$. □

5L The integral core subalgebra $X_{\mathbb{Z}}$

By Proposition 5.9, we can consider $\boldsymbol{U}_{\mathbb{Z}}$ as a based free $\mathcal{A}$–module with the preferred basis $\{\boldsymbol{e}(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t},\ \boldsymbol{\delta} \in \{0,1\}^l\}$.

Let $a\colon \mathbb{N}^{t+l+t} \times \{0,1\}^l \to \mathcal{A}$ be the function defined by $a(\boldsymbol{n},\boldsymbol{\delta}) = (q;q)_{\boldsymbol{n}}$, where $(q;q)_{\boldsymbol{n}}$ is defined by (110). We consider the dilatation triple $(\boldsymbol{U}_{\mathbb{Z}}, \boldsymbol{U}_{\mathbb{Z}}(\sqrt{a}), \boldsymbol{U}_{\mathbb{Z}}(a))$. By Proposition 5.9, $\boldsymbol{U}_{\mathbb{Z}}(a)$ is $\boldsymbol{V}_{\mathbb{Z}}$.

Let $\boldsymbol{X}_{\mathbb{Z}}$ be $\boldsymbol{U}_{\mathbb{Z}}(\sqrt{a})$, which by definition is the free $\widetilde{\mathcal{A}}$–module with basis

(121) $$\{\sqrt{(q;q)_{\boldsymbol{n}}}\,\boldsymbol{e}(\boldsymbol{n},\boldsymbol{\delta}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t},\ \boldsymbol{\delta} \in \{0,1\}^l\}.$$

The even part $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ of $\boldsymbol{X}_{\mathbb{Z}}$ is defined to be the $\widetilde{\mathcal{A}}$–submodule spanned by

(122) $$\{\sqrt{(q;q)_{\boldsymbol{n}}}\,\boldsymbol{e}^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l+t}\}.$$

Then $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} = \boldsymbol{X}_{\mathbb{Z}} \cap (\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}})$, and $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}, \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}, \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}})$ is a dilatation triple.

Theorem 5.21 (a) *The $\widetilde{\mathcal{A}}$–module $\boldsymbol{X}_{\mathbb{Z}}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}$.*

(b) *The $\widetilde{\mathcal{A}}$–module $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is an $\widetilde{\mathcal{A}}$–subalgebra of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}$. Further, $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is*
- (i) *$\boldsymbol{U}_{\mathbb{Z}}$–ad-stable,*
- (ii) *stable under the action of the braid group, and*
- (iii) *stable under ι_{bar} and φ.*

(c) *The core algebra $\boldsymbol{X}_h$ is the $\sqrt{h}$–adic completion of the $\mathbb{C}[\![\sqrt{h}]\!]$–span of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ (or $\boldsymbol{X}_{\mathbb{Z}}$) in $\boldsymbol{U}_{\sqrt{h}}$.*

Proof (a) Let us show that $\Delta(\boldsymbol{X}_{\mathbb{Z}}) \subset \boldsymbol{X}_{\mathbb{Z}} \otimes \boldsymbol{X}_{\mathbb{Z}}$. Since $(\boldsymbol{U}_{\mathbb{Z}}, \boldsymbol{X}_{\mathbb{Z}}, \boldsymbol{V}_{\mathbb{Z}})$ is a dilatation triple, $(\boldsymbol{U}_{\mathbb{Z}} \otimes \boldsymbol{U}_{\mathbb{Z}}, \boldsymbol{X}_{\mathbb{Z}} \otimes \boldsymbol{X}_{\mathbb{Z}}, \boldsymbol{V}_{\mathbb{Z}} \otimes \boldsymbol{V}_{\mathbb{Z}})$ is also a dilatation triple. We have $\Delta(\boldsymbol{U}_{\mathbb{Z}}) \subset \boldsymbol{U}_{\mathbb{Z}} \otimes \boldsymbol{U}_{\mathbb{Z}}$ and $\Delta(\boldsymbol{V}_{\mathbb{Z}}) \subset \boldsymbol{V}_{\mathbb{Z}} \otimes \boldsymbol{V}_{\mathbb{Z}}$. By the stability principle (Proposition 5.20), we have $\Delta(\boldsymbol{X}_{\mathbb{Z}}) \subset \boldsymbol{X}_{\mathbb{Z}} \otimes \boldsymbol{X}_{\mathbb{Z}}$, ie $\boldsymbol{X}_{\mathbb{Z}}$ is an $\widetilde{\mathcal{A}}$–coalgebra.

Similarly, applying the stability principle to all the operations of a Hopf algebra, we conclude that $\boldsymbol{X}_{\mathbb{Z}}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}$.

(b) Because $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is an $\mathcal{A}$–subalgebra of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$, the stability principle for the dilatation triple $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}, \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}, \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}})$ shows that $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is an $\widetilde{\mathcal{A}}$–algebra.

By Theorem 5.18, $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–ad-stable; and, by Proposition 5.13, $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ is stable under ι_{bar} and φ. Since $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–ad-stable is stable under ι_{bar} and φ (by Proposition 5.2), the stability principle proves that $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is (i) $\boldsymbol{U}_{\mathbb{Z}}$–ad-stable, (ii) stable under the action of the braid groups, and (iii) stable under ι_{bar} and φ.

(c) Each element of the basis (121) of $\boldsymbol{X}_{\mathbb{Z}}$ is in $\boldsymbol{X}_h$. Hence, $\boldsymbol{X}_{\mathbb{Z}} \subset \boldsymbol{X}_h$. On the other hand, the $\widetilde{\mathcal{A}}$–basis (122) of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is also a topological basis of $\boldsymbol{X}_h$. Hence, $\boldsymbol{X}_h$ is the $\sqrt{h}$–adic completion of the $\mathbb{C}[\![\sqrt{h}]\!]$–span of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ in $\boldsymbol{U}_{\sqrt{h}}$. □

Corollary 5.22 (a) *The core algebra X_h is stable under the action of the braid group.*

(b) *The core algebra X_h is the smallest $\sqrt{h}$–adically completed topological $\mathbb{C}[\![\sqrt{h}]\!]$–subalgebra of $U_{\sqrt{h}}$ which* (i) *is closed in the $\sqrt{h}$–adic topology,* (ii) *contains $\sqrt{h}E_\alpha$, $\sqrt{h}F_\alpha$ and $\sqrt{h}H_\alpha$ for each $\alpha \in \Pi$, and* (iii) *is invariant under the action of the braid group.*

Proof (a) Since X_h is the $\sqrt{h}$–adic completion of the $\mathbb{C}[\![\sqrt{h}]\!]$–span of $X_{\mathbb{Z}}^{\mathrm{ev}}$, which is stable under the action of the braid group, X_h is also stable under the action of the braid group.

(b) Let X'_h be the smallest completed subalgebra of $U_{\sqrt{h}}$ satisfying (i), (ii), and (iii). Since X_h satisfies (i), (ii), and (iii), we have $X'_h \subset X_h$.

For each $\gamma \in \Phi_+$, E_γ and F_γ are obtained from E_α and F_α for $\alpha \in \Pi$ by actions of the braid group. Thus X'_h contains all $\sqrt{h}E_\gamma$ and $\sqrt{h}F_\gamma$ for $\gamma \in \Phi_+$ and $\sqrt{h}H_\alpha$ for $\alpha \in \Pi$, which generate X_h as an algebra (after h–adic completion). It follows that $X_h \subset X'_h$. Hence, $X_h = X'_h$. □

Remark 5.23 The disadvantage of $X_{\mathbb{Z}}$ is its ground ring is $\widetilde{\mathcal{A}}$, not $\mathcal{A}$. Let us define

$$X_{\mathcal{A}} = X_{\mathbb{Z}} \cap U_{\mathbb{Z}}.$$

Then $X_{\mathcal{A}}$ is an $\mathcal{A}$–algebra. However, $X_{\mathcal{A}}$ is not an $\mathcal{A}$–Hopf algebra in the usual sense, since

$$\Delta(X_{\mathcal{A}}) \not\subset X_{\mathcal{A}} \otimes_{\mathcal{A}} X_{\mathcal{A}}.$$

Let us define a new tensor product

$$(X_{\mathcal{A}})^{\boxtimes n} := X_{\mathbb{Z}}^{\otimes n} \cap U_{\mathbb{Z}}^{\otimes n}, \quad (X_{\mathcal{A}}^{\mathrm{ev}})^{\boxtimes n} := (X_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap (U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}. \tag{123}$$

Then we have

$$\Delta(X_{\mathcal{A}}) = \Delta(X_{\mathbb{Z}} \cap U_{\mathbb{Z}}) \subset (X_{\mathbb{Z}} \otimes X_{\mathbb{Z}}) \cap (U_{\mathbb{Z}} \otimes U_{\mathbb{Z}}) = X_{\mathcal{A}} \boxtimes X_{\mathcal{A}}.$$

Hence, $X_{\mathcal{A}}$, with this new tensor power, is a Hopf algebra, which is a Hopf subalgebra of both $X_{\mathbb{Z}}$ and $U_{\mathbb{Z}}$.

What we will prove later implies that if T is an n–component bottom tangle with 0 linking matrix then

$$J_T \in \varprojlim_k (X_{\mathcal{A}}^{\mathrm{ev}})^{\boxtimes n}/((q;q)_k).$$

However, we will not use $X_{\mathcal{A}}$ in this paper.

5M Integrality of twist forms $\mathcal{T}_\pm$ on $X_{\mathbb{Z}}^{\mathrm{ev}}$

Recall that we have twist forms $\mathcal{T}_\pm\colon X_h \to \mathbb{C}[\![\sqrt{h}]\!]$. By Theorem 5.21, $X_{\mathbb{Z}} \subset X_h$.

The embedding $\mathcal{A} \hookrightarrow \mathbb{C}[\![h]\!]$ by $v = \exp\bigl(\frac{1}{2}h\bigr)$ extends to an embedding $\widetilde{\mathcal{A}} \to \mathbb{C}[\![\sqrt{h}]\!]$. Although there are many extensions, it is easy to see that the image of the extended embedding does not depend on the extension, because the two roots of $\phi_k(q)$ are inverse (with respect to addition) to each other.

Proposition 5.24 *One has $\mathcal{T}_\pm(X_{\mathbb{Z}}^{\mathrm{ev}}) \subset \widetilde{\mathcal{A}}$.*

The proof of this proposition will occupy the rest of this section (Sections 5M1–5M4.)

5M1 Integrality on the Cartan part

Lemma 5.25 (a) *The Cartan part $X_{\mathbb{Z}}^{\mathrm{ev},0}$ of $X_{\mathbb{Z}}^{\mathrm{ev}}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $X_{\mathbb{Z}}$.*

(b) *Suppose $x, y \in X_{\mathbb{Z}}^{\mathrm{ev},0}$ and $\lambda \in X$. Then $\langle x, y\rangle \in \widetilde{\mathcal{A}}$ and $\langle x, K_{2\lambda}\rangle \in \widetilde{\mathcal{A}}$.*

Proof (a) Since $X_{\mathbb{Z}}^{\mathrm{ev},0}$ is an $\widetilde{\mathcal{A}}$–subalgebra of the commutative cocommutative Hopf algebra $X_{\mathbb{Z}}^0$, we need to check that $\Delta(X_{\mathbb{Z}}^{\mathrm{ev},0}) \subset X_{\mathbb{Z}}^{\mathrm{ev},0} \otimes X_{\mathbb{Z}}^{\mathrm{ev},0}$. This follows from the fact that $X_{\mathbb{Z}}^0$ is an $\mathcal{A}$–Hopf algebra, and $\Delta(K_\alpha^2) = K_\alpha^2 \otimes K_\alpha^2$.

(b) Recall that $\breve{\iota}\colon U_q^0 \to \breve{U}_q^0$ is the algebra homomorphism defined by $\breve{\iota}(K_\alpha) = \breve{K}_\alpha$. Recall that $(U_{\mathbb{Z}}^{\mathrm{ev},0}, X_{\mathbb{Z}}^{\mathrm{ev},0}, V_{\mathbb{Z}}^{\mathrm{ev},0})$ is a dilatation triple. We have $\breve{U}_{\mathbb{Z}}^{\mathrm{ev},0} = \breve{\iota}(U_{\mathbb{Z}}^{\mathrm{ev},0})$. Define

$$\breve{X}_{\mathbb{Z}}^{\mathrm{ev},0} = \breve{\iota}(X_{\mathbb{Z}}^{\mathrm{ev},0}) \quad \text{and} \quad \breve{V}_{\mathbb{Z}}^{\mathrm{ev},0} = \breve{\iota}(V_{\mathbb{Z}}^{\mathrm{ev},0}).$$

Then $(\breve{U}_{\mathbb{Z}}^{\mathrm{ev},0}, \breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}, \breve{V}_{\mathbb{Z}}^{\mathrm{ev},0})$ is also a dilatation triple, and $X_{\mathbb{Z}}^{\mathrm{ev},0}$ and $\breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ are free $\widetilde{\mathcal{A}}$–modules with respective bases

$$\{\sqrt{(q;q)_{\boldsymbol{n}}}\, Q(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\} \quad \text{and} \quad \{\sqrt{(q;q)_{\boldsymbol{n}}}\, \breve{Q}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}. \tag{124}$$

Since the inclusion $U_{\mathbb{Z}}^{\mathrm{ev},0} \hookrightarrow \breve{U}_{\mathbb{Z}}^{\mathrm{ev},0}$ maps $V_{\mathbb{Z}}^{\mathrm{ev},0}$ into $\breve{V}_{\mathbb{Z}}^{\mathrm{ev},0}$, Proposition 5.20 shows that $X_{\mathbb{Z}}^{\mathrm{ev},0} \subset \breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}$. In particular, $y \in \breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}$.

The orthogonality (117) and bases (124) show that if $x \in X_{\mathbb{Z}}^{\mathrm{ev},0}$ and $y \in \breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ then $\langle x, y\rangle \in \widetilde{\mathcal{A}}$. Since $K_{2\lambda} \in \breve{X}_{\mathbb{Z}}^{\mathrm{ev},0}$, we also have $\langle x, K_{2\lambda}\rangle \in \widetilde{\mathcal{A}}$. □

5M2 Diagonal part of the ribbon element The diagonal part $\boldsymbol{r}_0$ of the ribbon element (see Section 3G) is given by

$$\boldsymbol{r}_0 = K_{-2\rho} \exp\Bigl(-h \sum_{\alpha\in\Pi} H_\alpha \breve{H}_\alpha / d_\alpha\Bigr).$$

For $\alpha \in \Pi$ let the α–part of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ be $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha} := \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0} \cap \widetilde{\mathcal{A}}[K_\alpha^{\pm 2}]$.

Lemma 5.26 (a) *Each $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ and $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0} = \bigotimes_{\alpha\in\Pi} \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$.*

(b) *$\langle \boldsymbol{r}_0, \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}\rangle \in \widetilde{\mathcal{A}}$ for any $\alpha \in \Pi$.*

Proof (a) By definition, $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ has $\widetilde{\mathcal{A}}$–basis $\{\sqrt{(q;q)_{\boldsymbol{n}}}\, Q^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}$, where $Q^{\mathrm{ev}}(\boldsymbol{n}) = \prod_{j=1}^{l} Q(\alpha_j; n_j)$, with

$$Q(\alpha;n) = K_\alpha^{-2\lfloor n/2\rfloor} \frac{(q_\alpha^{-\lfloor (n-1)/2\rfloor} K_\alpha^2; q_\alpha)_n}{(q_\alpha;q_\alpha)_n}.$$

It follows that $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$ is the $\widetilde{\mathcal{A}}$–module spanned by $\sqrt{(q_\alpha;q_\alpha)_n}\, Q(\alpha;n)$ and $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0} = \bigotimes_{\alpha\in\Pi} \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$. Because $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ is an $\widetilde{\mathcal{A}}$–Hopf algebra (Lemma 5.25), $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$.

(b) We need to show that $\langle \boldsymbol{r}_0, \sqrt{(q_\alpha;q_\alpha)_n}\, Q(\alpha;n)\rangle \in \widetilde{\mathcal{A}}$ for every $n \in \mathbb{N}$. Fix such an n.

Let $\mathcal{I}$ be the ideal of $\mathbb{Z}[q_\alpha^{\pm 1}, K_\alpha^{\pm 2}]$ generated by elements of the form $(q_\alpha^m K_\alpha^2; q_\alpha)_n$, $m \in \mathbb{N}$. Then $(q_\alpha;q_\alpha)_n Q(\alpha;n) \in \mathcal{I}$. By (96),

$$\langle \boldsymbol{r}_0, K_{2\alpha}^k\rangle = q_\alpha^{-k^2+k}.$$

With $z = K_{2\alpha}$, the $\mathbb{Z}[q_\alpha^{\pm 1}]$–linear map $\mathcal{L}_*\colon \mathbb{Z}[q_\alpha^{\pm 1}, K_\alpha^{\pm 2}] \to \mathbb{Z}[q_\alpha^{\pm 1}]$ defined by $\mathcal{L}_*(K_{2\alpha}^k) = \langle \boldsymbol{r}_0, K_{2\alpha}^k\rangle$ equals the map $\mathcal{L}_{-x^2+x}\colon \mathbb{Z}[q_\alpha^{\pm 1}, z^{\pm 1}] \to \mathbb{Z}[q_\alpha^{\pm 1}]$ of [7]. By [7, Theorem 3.2], for any $f \in \mathcal{I}$,

$$\mathcal{L}_*(f) \in \frac{(q_\alpha;q_\alpha)_n}{(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}} \mathbb{Z}[q_\alpha^{\pm 1}].$$

As $(q_\alpha;q_\alpha)_n Q(\alpha;n) \in \mathcal{I}$, one has

$$\langle \boldsymbol{r}_0, (q_\alpha;q_\alpha)_n Q(\alpha;n)\rangle = \mathcal{L}_*((q_\alpha;q_\alpha)_n Q(\alpha;n)) \in \frac{(q_\alpha;q_\alpha)_n}{(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}} \mathbb{Z}[q_\alpha^{\pm 1}]. \tag{125}$$

We have

$$\left(\frac{\sqrt{(q_\alpha;q_\alpha)_n}}{(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}}\right)^2 = \left(\frac{(q_\alpha;q_\alpha)_n}{(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}}\right) \in \mathbb{Z}[q_\alpha^{\pm 1}],$$

where the last inclusion follows from the integrality of the quantum binomial coefficients. Hence, from (125),

$$\langle \boldsymbol{r}_0^\alpha, \sqrt{(q_\alpha;q_\alpha)_n}\, Q(\alpha;n)\rangle \in \frac{\sqrt{(q_\alpha;q_\alpha)_n}}{(q_\alpha;q_\alpha)_{\lfloor n/2\rfloor}}\mathbb{Z}[q_\alpha^{\pm 1}] \in \widetilde{\mathcal{A}}.$$

This completes the proof of the lemma. □

Remark 5.27 Theorem 3.2 of [7], used in the proof of the above lemma, is one of the main technical results of [7] and is difficult to prove. Its proof uses Andrews' generalization of the Rogers–Ramanujan identity. Actually, only a special case of [7, Theorem 3.2] is used here. This special case can be proved using other methods.

5M3 Integrality of $\boldsymbol{r}_0$

Lemma 5.28 *Suppose $x \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$; then $\langle \boldsymbol{r}_0, x\rangle \in \widetilde{\mathcal{A}}$.*

Proof We first prove the following claim:

Claim *If $\langle \boldsymbol{r}_0, x\rangle \in \widetilde{\mathcal{A}}$ for all $x \in \mathscr{H}_1$ and all $x \in \mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are $\widetilde{\mathcal{A}}$–Hopf subalgebras of $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$, then $\langle \boldsymbol{r}_0, x\rangle \in \widetilde{\mathcal{A}}$ for all $x \in \mathscr{H}_1\mathscr{H}_2$.*

Proof of claim Suppose $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$. Using the Hopf dual property of the quantum Killing form on the Cartan part (95), we have

$$\langle \boldsymbol{r}_0, xy\rangle = \langle \Delta(\boldsymbol{r}_0), x \otimes y\rangle.$$

A simple calculation shows that $\Delta(\boldsymbol{r}_0) = (\boldsymbol{r}_0 \otimes \boldsymbol{r}_0)\mathcal{D}^{-2}$, where $\mathcal{D}$ is the diagonal part of the R–matrix; see (68). We have

$$\mathcal{D}^{-2} = \exp\Big(-h\sum_\alpha H_\alpha \otimes \breve{H}_\alpha/d_\alpha\Big).$$

Writing $\mathcal{D}^{-2} = \sum \delta_1 \otimes \delta_2$, we have

$$\begin{aligned}\langle \boldsymbol{r}_0, xy\rangle &= \sum \langle \boldsymbol{r}_0\delta_1, x\rangle\langle \boldsymbol{r}_0\delta_2, y\rangle \\ &= \sum \langle \boldsymbol{r}_0, x_{(1)}\rangle\langle \delta_1, x_{(2)}\rangle\langle \boldsymbol{r}_0, y_{(1)}\rangle\langle \delta_2, y_{(2)}\rangle \\ &= \sum \langle \boldsymbol{r}_0, x_{(1)}\rangle\langle \boldsymbol{r}_0, y_{(1)}\rangle\langle x_{(2)}, y_{(2)}\rangle,\end{aligned} \tag{126}$$

where in the last identity we use the fact that $\sum\langle \delta_1, x\rangle\langle \delta_2, y\rangle = \langle x, y\rangle$, which is easy to prove. (Note that, on the Cartan part $\boldsymbol{X}_h^0$, the quantum Killing form is the dual of $\mathcal{D}^{-2}$, which is the Cartan part of the clasp element $\boldsymbol{c}$.)

Since $x_{(1)} \in \mathscr{H}_1$ and $y_{(1)} \in \mathscr{H}_2$, we have $\langle \boldsymbol{r}_0, x_{(1)}\rangle\langle \boldsymbol{r}_0, y_{(1)}\rangle \in \widetilde{\mathcal{A}}$. By Lemma 5.25(b), $\langle x_{(2)}, y_{(2)}\rangle \in \widetilde{\mathcal{A}}$. Hence, (126) shows that $\langle \boldsymbol{r}_0, xy\rangle \in \widetilde{\mathcal{A}}$. This proves the claim. □

By Lemma 5.26, $X_{\mathbb{Z}}^{\mathrm{ev},0} = \bigotimes_{\alpha\in\Pi} X_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$, each $X_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}$ is a Hopf subalgebra of $X_{\mathbb{Z}}^{\mathrm{ev},0}$, and $\langle \boldsymbol{r}_0, X_{\mathbb{Z}}^{\mathrm{ev},0,\alpha}\rangle \subset \widetilde{\mathcal{A}}$. Hence, from the claim we have $\langle \boldsymbol{r}_0, X_{\mathbb{Z}}^{\mathrm{ev},0}\rangle \subset \widetilde{\mathcal{A}}$. □

5M4 Proof of Proposition 5.24

Proof We have to show that $\mathcal{T}_{\pm}(x) \in \widetilde{\mathcal{A}}$ for every $x \in X_{\mathbb{Z}}^{\mathrm{ev}}$. First we will show $\mathcal{T}_+(x) \in \widetilde{\mathcal{A}}$.

It is enough to consider the case $x = \sqrt{(q;q)_{\boldsymbol{n}}}\, e^{\mathrm{ev}}(\boldsymbol{n})$ with $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{k}, \boldsymbol{n}_3) \in \mathbb{N}^{t+l+t}$, since $X_{\mathbb{Z}}^{\mathrm{ev}}$ is $\widetilde{\mathcal{A}}$–spanned by elements of this form. By the triangular property (93) of the quantum Killing form and (96),

$$\mathcal{T}_+(x) = \delta_{\boldsymbol{n}_1,\boldsymbol{n}_3} q^{(\rho,|E_{\boldsymbol{n}_1}|)} \langle \boldsymbol{r}_0, \sqrt{(q;q)_{\boldsymbol{k}}}\, Q(\boldsymbol{k})\rangle \in \widetilde{\mathcal{A}}.$$

Here the last inclusion follows from Lemma 5.28. This proves the statement for $\mathcal{T}_+$.

By Theorem 5.21, $X_{\mathbb{Z}}^{\mathrm{ev}}$ is φ–stable. By (98), we have $\mathcal{T}_-(x) = \mathcal{T}_+(\varphi(x)) \in \widetilde{\mathcal{A}}$. □

5N More on integrality of $\boldsymbol{r}_0$

Lemma 5.29 *Suppose $y \in X_{\mathbb{Z}}^{\mathrm{ev},0}$. Then $\langle \boldsymbol{r}_0^{\pm 1}, K_{2\rho} y\rangle \in v^{(\rho,\rho)}\widetilde{\mathcal{A}}$.*

Proof Since $X_{\mathbb{Z}}^{\mathrm{ev},0}$ is a Hopf algebra (Lemma 5.25), we have $\Delta(y) = \sum y_{(1)} \otimes y_{(2)}$ with $y_{(1)}, y_{(2)} \in X_{\mathbb{Z}}^{\mathrm{ev},0}$. Using (126), then (97), we have

$$\langle \boldsymbol{r}_0, K_{2\rho} y\rangle = \sum \langle \boldsymbol{r}_0, K_{2\rho}\rangle \langle \boldsymbol{r}_0, y_{(1)}\rangle \langle K_{2\rho}, y_{(2)}\rangle = v^{(\rho,\rho)} \sum \langle \boldsymbol{r}_0, y_{(1)}\rangle \langle K_{2\rho}, y_{(2)}\rangle,$$

where we use $\langle \boldsymbol{r}_0, K_{2\rho}\rangle = v^{(\rho,\rho)}$, which follows from an easy calculation. The second factor $\langle \boldsymbol{r}_0, y_{(1)}\rangle$ is in $\widetilde{\mathcal{A}}$ by Lemma 5.28. The last factor $\langle K_{2\rho}, y_{(2)}\rangle$ is in $\widetilde{\mathcal{A}}$. Thus, we have $\langle \boldsymbol{r}_0, K_{2\rho} y\rangle \in v^{(\rho,\rho)}\widetilde{\mathcal{A}}$.

Using (98), the fact that $X_{\mathbb{Z}}^{\mathrm{ev},0}$ is φ–stable and the above case for $\boldsymbol{r}_0$, we have

$$\langle \boldsymbol{r}_0^{-1}, K_{2\rho} y\rangle = \langle \boldsymbol{r}_0, \varphi(K_{2\rho} y)\rangle = \langle \boldsymbol{r}_0, K_{2\rho}\varphi(y)\rangle \in v^{(\rho,\rho)}\widetilde{\mathcal{A}}.$$

This completes the proof of the lemma. □

6 Gradings

In Section 3C, we defined the Y–grading and the $Y/2Y$–grading on $\boldsymbol{U}_q$. In this section we define a grading of $\boldsymbol{U}_q$ by a group G, which is a (possibly noncommutative) central $\mathbb{Z}/2\mathbb{Z}$–extension of $Y \times (Y/2Y)$, thus refining both the gradings by Y and $Y/2Y$. This grading is extended to the tensor powers of $\boldsymbol{U}_q$.

The reason for the introduction of the G–grading is the following: The integral core subalgebra $X_{\mathbb{Z}}$ will be enough for us to show that the invariant J_M of integral homology 3–spheres, a priori belonging to $\mathbb{C}[\![\sqrt{h}]\!]$, is in

$$\varprojlim_k \mathbb{Z}[v^{\pm 1}]/((q;q)_k).$$

But we want to show that J_M belongs to a smaller ring, $\widehat{\mathbb{Z}[q]} = \varprojlim_k \mathbb{Z}[q^{\pm 1}]/((q;q)_k)$, and the G–grading will be helpful in the proof. In Section 7 we will show that quantum link invariants of algebraically split bottom tangles belong to a certain homogeneous part of this G–grading.

6A The groups G and G^{ev}

Let G denote the group generated by the elements $\dot{v}$, $\dot{K}_\alpha$ and $\dot{e}_\alpha$ ($\alpha \in \Pi$) with the relations

$$\dot{v} \text{ central}, \quad \dot{v}^2 = \dot{K}_\alpha^2 = 1, \quad \dot{K}_\alpha \dot{K}_\beta = \dot{K}_\beta \dot{K}_\alpha,$$
$$\dot{K}_\alpha \dot{e}_\beta = \dot{v}^{(\alpha,\beta)} \dot{e}_\beta \dot{K}_\alpha, \quad \dot{e}_\alpha \dot{e}_\beta = \dot{v}^{(\alpha,\beta)} \dot{e}_\beta \dot{e}_\alpha.$$

Let G^{ev} be the subgroup of G generated by $\dot{v}$ and $\dot{e}_\alpha$ ($\alpha \in \Pi$).

Remark 6.1 The groups G and G^{ev} are abelian if and only if $\mathfrak{g}$ is of type A_1 or B_n ($n \geq 2$).

Define a homomorphism $G \to Y$, $g \mapsto |g|$, by

$$|\dot{v}| = |\dot{K}_\alpha| = 0, \quad |\dot{e}_\alpha| = \alpha \quad (\alpha \in \Pi).$$

For $\gamma = \sum_i m_i \alpha_i \in Y$, set

$$\dot{K}_\gamma = \prod_i \dot{K}_{\alpha_i}^{m_i}, \quad \dot{e}_\gamma = \prod_i \dot{e}_{\alpha_i}^{m_i} = \dot{e}_{\alpha_1}^{m_1} \cdots \dot{e}_{\alpha_l}^{m_l}.$$

Note that $\dot{e}_\gamma$ depends on the order of the simple roots $\alpha_1, \dots, \alpha_l \in \Pi$.

One can easily verify the following commutation rules:

$$g \dot{K}_\lambda = \dot{v}^{(|g|,\lambda)} \dot{K}_\lambda g \quad \text{for } g \in G,\ \lambda \in Y, \tag{127}$$

$$g g' = \dot{v}^{(|g|,|g'|)} g' g \quad \text{for } g, g' \in G^{\mathrm{ev}}. \tag{128}$$

Let N be the subgroup of G generated by $\dot{v}$. Then N has order 2 and is a subgroup of the center of G. Note that $G/N \cong Y \times (Y/2Y)$ and $G^{\mathrm{ev}}/N \cong Y$.

6A1 Tensor products of G and G^{ev} By $G \otimes G = G \otimes_N G$, we mean the "tensor product over N" of two copies of G, ie

$$G \otimes G := (G \times G)/((\dot{v}x, y) \sim (x, \dot{v}y)).$$

Similarly, we can define $G \otimes G^{\mathrm{ev}}$, $G^{\mathrm{ev}} \otimes G^{\mathrm{ev}}$, etc, which are subgroups of $G \otimes G$. Denote by $x \otimes y$ the element in $G \otimes G$ represented by (x, y). Thus we have $\dot{v}x \otimes y = x \otimes \dot{v}y$.

Similarly, we can also define the tensor powers $G^{\otimes n} = G \otimes \cdots \otimes G$ and $(G^{\mathrm{ev}})^{\otimes n} = G^{\mathrm{ev}} \otimes \cdots \otimes G^{\mathrm{ev}} \subset G^{\otimes n}$ (each with n tensorands). Define a homomorphism $\iota_n \colon N \to G^{\otimes n}$ by

$$\iota_n(\dot{v}^k) = \dot{v}^k \otimes 1^{\otimes(n-1)}, \quad k = 0, 1.$$

We have

$$G^{\otimes n}/\iota_n(N) \cong Y^n \times (Y/2Y)^n, \quad (G^{\mathrm{ev}})^{\otimes n}/\iota_n(N) \cong Y^n.$$

For $n = 0$, we set

$$G^{\otimes 0} = (G^{\mathrm{ev}})^{\otimes 0} = N.$$

6B G–grading of $\boldsymbol{U}_q$

By a G–grading of $\boldsymbol{U}_q$ we mean a direct sum decomposition of $\mathbb{C}(q)$–vector spaces

$$\boldsymbol{U}_q = \bigoplus_{g \in G} [\boldsymbol{U}_q]_g$$

such that $1 \in [\boldsymbol{U}_q]_1$ and $[\boldsymbol{U}_q]_g [\boldsymbol{U}_q]_{g'} \subset [\boldsymbol{U}_q]_{gg'}$ for $g, g' \in G$. If $x \in [\boldsymbol{U}_q]_g$, we write $\deg_G(x) = g$.

Proposition 6.2 *There is a unique G–grading on $\boldsymbol{U}_q$ such that*

$$\deg_G(v) = \dot{v}, \quad \deg_G(K_{\pm\alpha}) = \dot{K}_\alpha, \quad \deg_G(E_\alpha) = \dot{v}^{d_\alpha} \dot{e}_\alpha, \quad \deg_G(F_\alpha) = \dot{e}_\alpha^{-1} \dot{K}_\alpha.$$

Proof Since $v^{\pm 1}$, $K_{\pm\alpha}$, E_α and F_α generate the $\mathbb{C}(q)$–algebra $\boldsymbol{U}_q$, the uniqueness is clear. Let us prove the existence of the G–grading.

Let $\widetilde{\boldsymbol{U}}_q$ denote the free $\mathbb{C}(q)$–algebra generated by the elements $\tilde{v}$, $\tilde{v}^{-1}$, $\widetilde{K}_\alpha$, $\widetilde{K}_\alpha^{-1}$, $\widetilde{E}_\alpha$ and $\widetilde{F}_\alpha$. We can define a G–grading of $\widetilde{\boldsymbol{U}}_q$ by

$$\deg_G(\tilde{v}^{\pm 1}) = \dot{v}, \quad \deg_G(\widetilde{K}_\alpha^{\pm 1}) = \dot{K}_\alpha, \quad \deg_G(\widetilde{E}_\alpha) = \dot{v}^{d_\alpha} \dot{e}_\alpha, \quad \deg_G(\widetilde{F}_\alpha) = \dot{e}_\alpha^{-1} \dot{K}_\alpha.$$

The kernel of the obvious homomorphism $\widetilde{\boldsymbol{U}}_q \to \boldsymbol{U}_q$ is the two-sided ideal in $\widetilde{\boldsymbol{U}}_q$ generated by the defining relations of the $\mathbb{C}(q)$–algebra $\boldsymbol{U}_q$:

$$\begin{gathered}
\widetilde{v}\widetilde{v}^{-1} = \widetilde{v}^{-1}\widetilde{v} = 1, \quad \widetilde{v}^2 = q, \quad \widetilde{v} \text{ central}, \\
\widetilde{K}_\alpha \widetilde{K}_\alpha^{-1} = \widetilde{K}_\alpha^{-1}\widetilde{K}_\alpha = 1, \quad \widetilde{K}_\alpha\widetilde{K}_\beta = \widetilde{K}_\beta\widetilde{K}_\alpha, \\
\widetilde{K}_\alpha \widetilde{E}_\beta \widetilde{K}_\alpha^{-1} = \widetilde{v}^{(\alpha,\beta)}\widetilde{E}_\beta, \quad \widetilde{K}_\alpha \widetilde{F}_\beta \widetilde{K}_\alpha^{-1} = \widetilde{v}^{-(\alpha,\beta)}\widetilde{F}_\beta, \\
\widetilde{E}_\alpha\widetilde{F}_\beta - \widetilde{F}_\beta\widetilde{E}_\alpha = \delta_{\alpha,\beta}(q^{d_\alpha}-1)^{-1}\widetilde{v}^{d_\alpha}(\widetilde{K}_\alpha - \widetilde{K}_\alpha^{-1}), \\
\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \widetilde{\begin{bmatrix} 1-a_{\alpha\beta} \\ s \end{bmatrix}}_\alpha \widetilde{E}_\alpha^{1-a_{\alpha\beta}-s}\widetilde{E}_\beta\widetilde{E}_\alpha^s = 0 \quad (\alpha \neq \beta), \\
\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \widetilde{\begin{bmatrix} 1-a_{\alpha\beta} \\ s \end{bmatrix}}_\alpha \widetilde{F}_\alpha^{1-a_{\alpha\beta}-s}\widetilde{F}_\beta\widetilde{F}_\alpha^s = 0 \quad (\alpha \neq \beta).
\end{gathered}$$

Here, for $n,\ s \ge 0$, $\widetilde{\left[\begin{smallmatrix} n \\ s \end{smallmatrix}\right]}_\alpha$ is obtained from $\left[\begin{smallmatrix} n \\ s \end{smallmatrix}\right]_\alpha \in \mathbb{Z}[v_\alpha, v_\alpha^{-1}]$ by replacing $v_\alpha^{\pm 1}$ by $\widetilde{v}^{\pm d_\alpha}$. Since the above relations are homogeneous in the G–grading of $\widetilde{\boldsymbol{U}}_q$, the assertion holds. □

From the definition, we have

$$\boldsymbol{U}_q^{\mathrm{ev}} = \bigoplus_{g \in G^{\mathrm{ev}}} [\boldsymbol{U}_q]_g.$$

We say that $x \in \boldsymbol{U}_q$ is *G–homogeneous*, and write $\dot{x} = g$, if $x \in [\boldsymbol{U}_q]_g$ for some $g \in G$. Similarly, we say $x \in \boldsymbol{U}_q$ is *G^{ev}–homogeneous* if $x \in [\boldsymbol{U}_q]_g$ for some $g \in G^{\mathrm{ev}}$.

6B1 The $G^{\otimes m}$–grading of $U_q^{\otimes m}$ For $m \ge 1$, $\boldsymbol{U}_q^{\otimes m}$ is $G^{\otimes m}$–graded:

$$\boldsymbol{U}_q^{\otimes m} = \bigoplus_{g \in G^{\otimes m}} [\boldsymbol{U}_q^{\otimes m}]_g,$$

where, for $g = g_1 \otimes \cdots \otimes g_m \in G^{\otimes m}$ ($g_i \in G$), we set

$$[\boldsymbol{U}_q^{\otimes m}]_g = j([\boldsymbol{U}_q]_{g_1} \otimes_{\mathbb{C}(q)} \cdots \otimes_{\mathbb{C}(q)} [\boldsymbol{U}_q]_{g_m}) \subset \boldsymbol{U}_q^{\otimes m},$$

where

$$j\colon \boldsymbol{U}_q \otimes_{\mathbb{C}(q)} \cdots \otimes_{\mathbb{C}(q)} \boldsymbol{U}_q \to \boldsymbol{U}_q \otimes_{\mathbb{C}(v)} \cdots \otimes_{\mathbb{C}(v)} \boldsymbol{U}_q = \boldsymbol{U}_q^{\otimes m}$$

is the natural map.

Note that $\mathbb{C}(v) = \boldsymbol{U}_q^{\otimes 0}$ is N–graded (where $N = G^{\otimes 0}$): $[\mathbb{C}(v)]_{\dot{v}^k} = v^k\mathbb{C}(q)$, $k = 0, 1$.

6B2 Total G–grading of $U_q^{\otimes m}$ and G–grading-preserving map For $g \in G$ and $m \geq 0$ set

$$[U_q^{\otimes m}]_g := \sum_{g_1,\dots,g_m \in G;\, g_1\cdots g_m = g} [U_q^{\otimes m}]_{g_1 \otimes \cdots \otimes g_m}.$$

This gives a G–grading of the $\mathbb{C}(q)$–module $U_q^{\otimes m}$ for each $m \geq 0$. (If $m = 0$, we have $[U_q^{\otimes 0}]_{\dot{v}^k} = [\mathbb{C}(v)]_{\dot{v}^k} = v^k \mathbb{C}(q)$ for $k = 0, 1$ and $[U_q^{\otimes 0}]_g = 0$ for $g \in G \setminus \{1, \dot{v}\}$.)

A $\mathbb{C}[\![h]\!]$–module map $f\colon U_h^{\hat{\otimes} n} \to U_h^{\hat{\otimes} m}$ is said *to preserve the G–grading* if, for every $g \in G$, $f([U_{\mathbb{Z}}^{\otimes n}]_g) \subset [U_{\mathbb{Z}}^{\otimes m}]_g$. Here

$$[U_{\mathbb{Z}}^{\otimes n}]_g = [U_q^{\otimes n}]_g \cap U_{\mathbb{Z}}^{\otimes n}.$$

6C Multiplication, unit and counit

From the definition of the G–grading, we have the following:

Proposition 6.3 *Each of μ, η and ϵ preserves the G–grading, ie*

$$\mu([U_q^{\otimes 2}]_g) \subset [U_q]_g, \quad \eta([\mathbb{C}(v)]_g) \subset [U_q]_g, \quad \epsilon([U_q]_g) \subset [\mathbb{C}(v)]_g.$$

6D Bar involution ι_{bar} and mirror automorphism φ

From the definition one immediately has the following:

Lemma 6.4 *The bar involution $\iota_{\mathrm{bar}}\colon U_h \to U_h$ preserves the G–grading.*

Let $\dot{\varphi}\colon G \to G$ be the automorphism defined by $\dot{\varphi}(\dot{v}) = \dot{v}$, $\dot{\varphi}(\dot{K}_\alpha) = \dot{K}_\alpha$ and $\dot{\varphi}(\dot{e}_\alpha) = v^{d_\alpha} \dot{e}_\alpha^{-1}$. From the definition of $\dot{\varphi}$ one has the following:

Lemma 6.5 *For $g \in G$, we have $\varphi([U_q]_g) \subset [U_q]_{\dot{\varphi}(g)}$.*

6E Antipode

Define a function $\dot{S}\colon G \to G$ by

$$\dot{S}(g\dot{K}_\gamma) = \dot{K}_{\gamma+|g|} g = \dot{v}^{(|g|,\gamma)} g \dot{K}_{\gamma+|g|}$$

for $g \in G^{\mathrm{ev}}$ and $\gamma \in Y$. One can easily verify that $\dot{S}$ is an involutive antiautomorphism.

Lemma 6.6 *For $g \in G$, we have $S([U_q]_g) \subset [U_q]_{\dot{S}(g)}$. In particular, if $y = S(x)$ with x G^{ev}–homogeneous, then*

$$\dot{y} = \dot{x}\dot{K}_{|x|} = \dot{K}_{|x|}\dot{x}. \tag{129}$$

Here $\dot{y} = \deg_G(y)$ and $\dot{x} = \deg_G(x)$.

Proof It is easy to check that if $x = v,\ K_\alpha,\ E_\alpha,\ F_\alpha$, then $S(x)$ is homogeneous of degree $\dot{S}(g)$. If $x, y \in \boldsymbol{U}_q$ are homogeneous of degrees $\dot{x}, \dot{y} \in G$, respectively, then $S(xy) = S(y)S(x)$ is homogeneous of degree $\dot{S}(\dot{y})\dot{S}(\dot{x}) = \dot{S}(\dot{x}\dot{y})$. Hence, by induction, we deduce that, for each monomial x in the generators, $S(x)$ is homogeneous of degree $\dot{S}(\dot{x})$. This completes the proof. □

6F Braid group action

Let $\alpha \in \Pi$. Define a function $\dot{T}_\alpha\colon G \to G$ by

$$\dot{T}_\alpha(g\dot{K}_\gamma) = \dot{v}_\alpha^{r(r+1)/2}\dot{e}_\alpha^{r} g\dot{K}_{s_\alpha(\gamma)}, \quad \text{where } r = -(|g|,\alpha)/d_\alpha,$$

for $g \in G^{\mathrm{ev}}$ and $\gamma \in Y$. Note that $\dot{T}_\alpha$ is an involutive automorphism of G, satisfying $\dot{T}_\alpha(G^{\mathrm{ev}}) \subset G^{\mathrm{ev}}$.

Lemma 6.7 *If $g \in G$, then we have*

$$T_\alpha([\boldsymbol{U}_q]_g) \subset [\boldsymbol{U}_q]_{\dot{T}_\alpha(g)}.$$

Proof It suffices to check that $T_\alpha(x) \in [\boldsymbol{U}_q]_{\dot{T}_\alpha(\deg_G(x))}$ for each generator x of $\boldsymbol{U}_q$, which follows from the definitions. □

6G Quasi-*R*–matrix

For $\lambda \in Y$, set

$$\dot{\theta}_\lambda = \dot{e}_\lambda^{-1}\dot{K}_\lambda \otimes \dot{e}_\lambda \in G^{\otimes 2}.$$

We have $\dot{\theta}_0 = 1 \otimes 1$. Note that $\dot{\theta}_\lambda$ does not depend on the order of the simple roots $\alpha_1, \dots, \alpha_l$.

Lemma 6.8 *For $\lambda, \mu \in Y$, we have*

$$\dot{\theta}_\lambda\dot{\theta}_\mu = \dot{\theta}_{\lambda+\mu}.$$

Proof We have

$$\begin{aligned}
\dot{\theta}_\lambda\dot{\theta}_\mu &= (\dot{e}_\lambda^{-1}\dot{K}_\lambda \otimes \dot{e}_\lambda)(\dot{e}_\mu^{-1}\dot{K}_\mu \otimes \dot{e}_\mu)\\
&= \dot{e}_\lambda^{-1}\dot{K}_\lambda\dot{e}_\mu^{-1}\dot{K}_\mu \otimes \dot{e}_\lambda\dot{e}_\mu\\
&= \dot{e}_\mu^{-1}\dot{e}_\lambda^{-1}\dot{K}_\lambda\dot{K}_\mu \otimes \dot{e}_\lambda\dot{e}_\mu\\
&= (\dot{e}_\lambda\dot{e}_\mu)^{-1}\dot{K}_{\lambda+\mu} \otimes \dot{e}_\lambda\dot{e}_\mu\\
&= \dot{e}_{\lambda+\mu}^{-1}\dot{K}_{\lambda+\mu} \otimes \dot{e}_{\lambda+\mu}\\
&= \dot{\theta}_{\lambda+\mu}.
\end{aligned}$$

□

The automorphism $\dot{T}_\alpha\colon G \to G$ induces an automorphism

$$\dot{T}_\alpha^{\otimes 2}\colon G^{\otimes 2} \to G^{\otimes 2}, \quad g_1 \otimes g_2 \mapsto \dot{T}_\alpha(g_1) \otimes \dot{T}_\alpha(g_2).$$

Lemma 6.9 *If $\alpha \in \Pi$ and $\lambda \in Y$, then we have*

$$\dot{T}_\alpha^{\otimes 2}(\dot{\theta}_\lambda) = \dot{\theta}_{s_\alpha(\lambda)}.$$

Proof We have

$$\begin{aligned} \dot{T}_\alpha^{\otimes 2}(\dot{\theta}_\lambda) &= \dot{T}_\alpha^{\otimes 2}(\dot{e}_\lambda^{-1}\dot{K}_\lambda \otimes \dot{e}_\lambda) \\ &= \dot{T}_\alpha(\dot{e}_\lambda^{-1})\dot{T}_\alpha(\dot{K}_\lambda) \otimes \dot{T}_\alpha(\dot{e}_\lambda) \\ &= \dot{T}_\alpha(\dot{e}_\lambda)^{-1}\dot{K}_{s_\alpha(\lambda)} \otimes \dot{T}_\alpha(\dot{e}_\lambda). \end{aligned}$$

Hence it suffices to show that

$$\dot{T}_\alpha(\dot{e}_\lambda)^{-1} \otimes \dot{T}_\alpha(\dot{e}_\lambda) = \dot{e}_{s_\alpha(\lambda)}^{-1} \otimes \dot{e}_{s_\alpha(\lambda)}, \tag{130}$$

which can be verified using the fact that there is $k \in \{0,1\}$ such that $\dot{T}_\alpha(\dot{e}_\lambda) = \dot{v}^k \dot{e}_{s_\alpha(\lambda)}$. □

Recall that Θ is the quasi-R–matrix and its definition is given in Section 3G1. For $\gamma \in Y_+$, let $\Theta_\gamma \in \boldsymbol{U}_q^{\otimes 2}$ denote the weight-$(-\gamma,\gamma)$ part of Θ, so that we have $\Theta = \sum_{\gamma \in Y_+} \Theta_\gamma$. Similarly, let $\bar{\Theta}_\gamma$ denote the weight-$(-\gamma,\gamma)$ part of $\bar{\Theta} = \Theta^{-1}$.

Lemma 6.10 *For $\gamma \in Y_+$, we have $\Theta_\gamma, \bar{\Theta}_\gamma \in [\boldsymbol{U}_q^{\otimes 2}]_{\dot{\theta}_\gamma}$.*

Proof Suppose $\boldsymbol{i} = (i_1,\dots,i_t)$ is a longest reduced sequence. Note that

$$\Theta_\gamma = \sum_{\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t,\ |E_{\boldsymbol{m}}(\boldsymbol{i})|=\gamma} \Theta_{m_t}^{[t]} \cdots \Theta_{m_1}^{[1]},$$

where we set

$$\Theta_n^{[i]} := (T_{\alpha_{j_1}} \cdots T_{\alpha_{j_{i-1}}})^{\otimes 2}((-1)^n v_{\alpha_{j_i}}^{-n(n-1)/2} F_{\alpha_{j_i}}^{(n)} \otimes \bar{E}_{\alpha_{j_i}}^n).$$

For each $\alpha \in \Pi$, we have

$$(-1)^n v_\alpha^{-n(n-1)/2} F_\alpha^{(n)} \otimes \bar{E}_\alpha^n \in [\boldsymbol{U}_q^{\otimes 2}]_{\dot{\theta}_{n\alpha}}.$$

By Lemma 6.9, we deduce that $\Theta_n^{[i]} \in \boldsymbol{U}_q^{\otimes 2}$ is homogeneous of degree

$$(\dot{T}_{\alpha_{j_1}} \cdots \dot{T}_{\alpha_{j_{i-1}}})^{\otimes 2}(\dot{\theta}_{n\alpha_{j_i}}) = \dot{\theta}_{n s_{\alpha_{j_1}} \cdots s_{\alpha_{j_{i-1}}}(\alpha_{j_i})}.$$

Hence, it follows that Θ_γ is homogeneous of degree $\dot{\theta}_\gamma$. The case of $\bar{\Theta}_\gamma$ follows from $\Theta^{-1} = (\iota_{\mathrm{bar}} \otimes \iota_{\mathrm{bar}})(\Theta)$ and Lemma 6.4, which says ι_{bar} preserves the G–grading. □

Corollary 6.11 *Fix a longest reduced sequence* $\boldsymbol{i}$ *. For* $\boldsymbol{m}\in\mathbb{N}^t$ *and* $\gamma\in Y$ *,*

$$E_{\boldsymbol{m}}\otimes K_{\boldsymbol{m}}F_{\boldsymbol{m}},\ E'_{\boldsymbol{m}}\otimes K_{\boldsymbol{m}}F'_{\boldsymbol{m}}\in[\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}]_{\dot{e}_{\lambda_{\boldsymbol{m}}}\otimes\dot{e}_{\lambda_{\boldsymbol{m}}}^{-1}}\subset[\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}]_1, \tag{131}$$

$$F_{\boldsymbol{m}}K_{\boldsymbol{m}}K_{2\gamma}E_{\boldsymbol{m}}\in[U_{\mathbb{Z}}]_1. \tag{132}$$

(Here $\lambda_{\boldsymbol{m}}=|E_{\boldsymbol{m}}|=|E'_{\boldsymbol{m}}|$.)

Proof We have $\Theta=\sum_{\boldsymbol{m}}F_{\boldsymbol{m}}\otimes E_{\boldsymbol{m}}$ and $\Theta^{-1}=\sum_{\boldsymbol{m}}F'_{\boldsymbol{m}}\otimes E'_{\boldsymbol{m}}$. Hence, (131) follows from Lemma 6.10. In turn, (132) follows from (131), because $\dot{K}_{2\gamma}=1$. □

6H Twist forms

Recall that we have defined $\mathcal{T}_\pm\colon \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\to\mathbb{Q}(v)$; see Section 4I.

Proposition 6.12 *Both maps* $\mathcal{T}_\pm\colon \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\to\mathbb{Q}(v)$ *preserve the G–grading, ie*

$$\mathcal{T}_\pm([\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}]_g)\subset[\mathbb{Q}(v)]_g.$$

Proof (a) First we consider the case of $\mathcal{T}_+$. The set

$$\{F_{\boldsymbol{m}}K_{\boldsymbol{m}}K_{2\gamma}E_{\boldsymbol{n}}\mid \boldsymbol{n},\boldsymbol{m}\in\mathbb{N}^t,\ \gamma\in Y\}$$

is a $\mathbb{Q}(v)$–basis of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)$. Hence,

$$\{v^\delta F_{\boldsymbol{m}}K_{\boldsymbol{m}}K_{2\gamma}E_{\boldsymbol{n}}\mid \boldsymbol{n},\boldsymbol{m}\in\mathbb{N}^t,\ \gamma\in Y,\ \delta\in\{0,1\}\}$$

is a $\mathbb{Q}(q)$–basis of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)$. Each element of this basis is G–homogeneous. By (101),

$$\mathcal{T}_+(v^\delta F_{\boldsymbol{m}}K_{\boldsymbol{m}}K_{2\gamma}E_{\boldsymbol{n}})=\delta_{\boldsymbol{n},\boldsymbol{m}}v^\delta q^{(\gamma,\rho)-(\gamma,\gamma)/2}\in v^\delta\mathbb{Z}[q^{\pm1}]=[\mathcal{A}]_{\dot{v}^\delta}.$$

By Corollary 6.11, the G–grading of $v^\delta F_{\boldsymbol{m}}K_{\boldsymbol{m}}K_{2\gamma}E_{\boldsymbol{m}}$ is $\dot{v}^\delta$. Hence, we have

$$\mathcal{T}_+([\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)]_g)\subset[\mathbb{Q}(v)]_g. \tag{133}$$

(b) Now consider $\mathcal{T}_-$. Using (98), Lemma 6.5 and (133), we have

$$\begin{aligned}\mathcal{T}_-([\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)]_g)&=\mathcal{T}_+\big(\varphi([\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)]_g)\big)\\&\subset\mathcal{T}_+\big([\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes_{\mathcal{A}}\mathbb{Q}(v)]_{\dot{\varphi}(g)}\big)\subset[\mathbb{Q}(v)]_{\dot{\varphi}(g)}=[\mathbb{Q}(v)]_g,\end{aligned}$$

where the last identity follows from the fact that, for the involution $\dot{\varphi}$, we have $\dot{\varphi}(1)=1$ and $\dot{\varphi}(v)=v$, and, for any $g\notin\{1,v\}$, we have $[\mathbb{Q}(v)]_g=0$. □

6I Coproduct

Lemma 6.13 *Suppose $x \in \boldsymbol{U}_q$ is G^{ev}–homogeneous. There exists a presentation*

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$$

such that, for each $x_{(1)} \otimes x_{(2)}$,

(i) $x_{(1)}$ *is G–homogeneous,*

(ii) $x_{(2)}$ *and* $x_{(1)} K_{|x_{(2)}|}$ *are* G^{ev}*–homogeneous, and*

$$\dot{x}_{(1)} \dot{K}_{|x_{(2)}|} \dot{x}_{(2)} = \dot{x} = \dot{x}_{(2)} \dot{K}_{|x_{(2)}|} \dot{x}_{(1)}. \tag{134}$$

Remark 6.14 A presentation of $\Delta(x)$ as in Lemma 6.13 is called a *G–good presentation*. When x is G^{ev}–homogeneous, we always use a G–good presentation for $\Delta(x)$.

Proof Suppose x, y are G^{ev}–homogeneous. If $\Delta(x) = \sum_i x'_i \otimes x''_i$ and $\Delta(y) = \sum_j y'_j \otimes y''_j$ are G–good presentations of x and y, respectively, then it is easy to check that $\sum_{i,j} x'_i y'_j \otimes x''_i y''_j$ is a G–good presentation of $\Delta(xy)$. Hence, one needs only to check the statement for x equal to the generators $K_{2\alpha}$, E_α and $F_\alpha K_\alpha$ of $\boldsymbol{U}_q^{\mathrm{ev}}$. For each of these generators, the defining formulas of Δ show that the statement holds. □

6J Adjoint action

Define a map

$$\dot{\mathrm{ad}}\colon G \otimes G^{\mathrm{ev}} \to G^{\mathrm{ev}}$$

by

$$\dot{\mathrm{ad}}(g \dot{K}_\lambda \otimes g') = \dot{v}^{(\lambda, |g'|)} g g'$$

for $\lambda \in Y$ and g, $g' \in G^{\mathrm{ev}}$. Note that for g, $g' \in G^{\mathrm{ev}}$ we have $\dot{\mathrm{ad}}(g \otimes g') = gg'$.

Lemma 6.15 *For g, $g' \in G^{\mathrm{ev}}$ and $\gamma \in Y$, we have*

$$\mathrm{ad}([\boldsymbol{U}_q \otimes \boldsymbol{U}_q^{\mathrm{ev}}]_{g \dot{K}_\gamma \otimes g'}) \subset [\boldsymbol{U}_q^{\mathrm{ev}}]_{\dot{\mathrm{ad}}(g \dot{K}_\gamma \otimes g')}.$$

In particular, if $z = x \triangleright y$ and both x and y are G^{ev}–homogeneous, then z is G^{ev}–homogeneous with

$$\dot{z} = \dot{x} \dot{y}.$$

Proof Suppose x and y are G^{ev}–homogeneous and $\gamma \in Y$. Choose a G–good presentation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ (see Section 6I). By definition,

$$\begin{aligned}(xK_\gamma) \triangleright y &= \sum x_{(1)} K_\gamma y S(x_{(2)} K_\gamma) \\ &= \sum x_{(1)} K_\gamma y K_\gamma^{-1} S(x_{(2)}) = \sum v^{(\gamma,|y|)} x_{(1)} y S(x_{(2)}).\end{aligned}$$

The term of the last sum is in $[\boldsymbol{U}_q]_u$, where

$$\begin{aligned}u &= \dot{v}^{(\gamma,|y|)} \dot{x}_{(1)} \dot{y} \dot{S}(\dot{x}_{(2)}) \\ &= \dot{v}^{(\gamma,|y|)} \dot{x}_{(1)} \dot{y} \dot{K}_{|x_{(2)}|} \dot{x}_{(2)} && \text{by Lemma 6.6} \\ &= \dot{v}^{(\gamma,|y|)} \dot{x}_{(1)} \dot{K}_{|x_{(2)}|} \dot{x}_{(2)} \dot{y} \\ &= \dot{v}^{(\gamma,|y|)} \dot{x} \dot{y} && \text{by (134).}\end{aligned}$$

Hence we have the assertion. □

7 Integral values of J_M

By Theorem 2.25, the core subalgebra $\boldsymbol{X}_h$, constructed in Section 4, gives rise to an invariant J_M of integral homology 3–spheres. A priori, $J_M \in \mathbb{C}[\![\sqrt{h}]\!]$. The main purpose of this section is to show that $J_M \in \widehat{\mathbb{Z}[q]}$ for any integral homology 3–sphere M. To prove this fact we will construct a family of $\mathcal{A}$–submodules $\widetilde{\mathcal{K}}_n \subset \boldsymbol{X}_h^{\hat{\otimes} n}$ satisfying conditions (AL1) and (AL2) of Theorem 2.32, with $\widetilde{\mathcal{K}}_0 = \widehat{\mathbb{Z}[q]}$. Then, by Theorem 2.32, $J_M \in \widetilde{\mathcal{K}}_0 = \widehat{\mathbb{Z}[q]}$.

7A Module $\widetilde{\mathcal{K}}_n$

For $n \geq 0$, let $[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}]_1$ denote the G–grading-1 part of $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}$. Define

$$\mathcal{K}_n := (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}]_1 \subset (\boldsymbol{X}_h^{\hat{\otimes} n} \cap \boldsymbol{U}_h^{\hat{\otimes} n}). \tag{135}$$

In the notation of (123), $\mathcal{K}_n = [\boldsymbol{X}_{\mathcal{A}}^{\boxtimes n}]_1$. For example, $\mathcal{K}_0 = \mathbb{Z}[q^{\pm 1}]$. Define filtrations on $\mathcal{K}_n$ by

$$\mathcal{F}_k(\mathcal{K}_n) := (q;q)_k \mathcal{K}_n \subset (h^k \boldsymbol{X}_h^{\hat{\otimes} n} \cap h^k \boldsymbol{U}_h^{\hat{\otimes} n}).$$

Let $\widetilde{\mathcal{K}}_n$ be the completion of $\mathcal{K}_n$ by the filtrations $\mathcal{F}_k(\mathcal{K}_n)$ in $\boldsymbol{U}_h^{\hat{\otimes} n}$, ie

$$\widetilde{\mathcal{K}}_n := \left\{ x = \sum_{k=0}^{\infty} x_k \;\middle|\; x_k \in \mathcal{F}_k(\mathcal{K}_n) \right\} \subset (\boldsymbol{X}_h^{\hat{\otimes} n} \cap \boldsymbol{U}_h^{\hat{\otimes} n}).$$

Since $\mathcal{K}_0 = \mathbb{Z}[q^{\pm 1}]$, we have $\widetilde{\mathcal{K}}_0 = \widehat{\mathbb{Z}[q]}$. Each $\widetilde{\mathcal{K}}_n$ has the structure of a complete $\widehat{\mathbb{Z}[q]}$–module.

Proposition 7.1 *The family* $(\widetilde{\mathcal{K}}_n)$ *satisfies condition* (AL2) *of Theorem 2.32. Namely, if* $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ *and* $x \in \widetilde{\mathcal{K}}_n$ *then*

$$(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x) \in \widetilde{\mathcal{K}}_0 = \widehat{\mathbb{Z}[q]}.$$

Proof By definition, x has a presentation

$$x = \sum_{k=0}^{\infty} (q;q)_k x_k$$

with $x_k \in \mathcal{K}_n \subset \boldsymbol{X}_h$. Since $\mathcal{T}_\pm$ are continuous on the h–adic topology of $\boldsymbol{X}_h^{\hat{\otimes} n}$, we have

$$(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x) = \sum_{k=0}^{\infty} (q;q)_k (\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x_k) \in \mathbb{C}[\![\sqrt{h}]\!]. \tag{136}$$

Since $x_k \in \mathcal{K}_n \subset (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}$, by Proposition 5.24 $(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x_k) \subset \widetilde{\mathcal{A}}$.

Since $x_k \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}]_1$, by Proposition 6.12 $(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x_k) \subset [\mathbb{Q}(v)]_1 = \mathbb{Q}(q)$. Hence,

$$(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x_k) \in \widetilde{\mathcal{A}} \cap \mathbb{Q}(q) = \mathbb{Z}[q^{\pm 1}],$$

where the last identity is Lemma 5.1. From (136), $(\mathcal{T}_{\varepsilon_1} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathcal{T}_{\varepsilon_n})(x) \in \widehat{\mathbb{Z}[q]}$. □

7B Finer version of $\widetilde{\mathcal{K}}_n$

We will show that, for an n–component bottom tangle T with 0 linking matrix, $J_T \in \widetilde{\mathcal{K}}_n$. Then Proposition 7.1 will show that $J_M \in \widehat{\mathbb{Z}[q]}$ for any integral homology 3–spheres.

For the purpose of proving that J_M recovers the Witten–Reshetikhin–Turaev invariant, we want J_T to belong to smaller subsets of $\widetilde{\mathcal{K}}_n$, which we will describe here.

Suppose $\mathcal{U}$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$. Define (with the convention $\mathcal{U}^{\otimes 0} = \mathcal{A}$)

$$\mathcal{K}_n(\mathcal{U}) := \mathcal{K}_n \cap \mathcal{U}^{\otimes n}, \quad \mathcal{F}_k(\mathcal{K}_n(\mathcal{U})) := \mathcal{F}_k(\mathcal{K}_n) \cap \mathcal{U}^{\otimes n}.$$

Let $\widetilde{\mathcal{K}}_n(\mathcal{U})$ be the completion of $\mathcal{K}_n(\mathcal{U})$ with respect to the filtration $(\mathcal{F}_k(\mathcal{K}_n))$ in $\boldsymbol{U}_h$, ie

$$\widetilde{\mathcal{K}}_n(\mathcal{U}) := \left\{ x = \sum_{k=0}^{\infty} x_k \;\middle|\; x_k \in \mathcal{F}_k(\mathcal{K}_n(\mathcal{U})) \right\}.$$

Since $\mathcal{F}_k(\mathcal{K}_n(\mathcal{U})) \subset \mathcal{F}_k(\mathcal{K}_n)$ we have $\widetilde{\mathcal{K}}_n(\mathcal{U}) \subset \widetilde{\mathcal{K}}_n \subset \boldsymbol{X}_h^{\hat{\otimes} n}$. We always have $\widetilde{\mathcal{K}}_0(\mathcal{U}) = \widetilde{\mathcal{K}}_0 = \widehat{\mathbb{Z}[q]}$.

7C Values of universal invariant of algebraically split bottom tangle

Throughout we fix a longest reduced sequence $\boldsymbol{i}$.

Recall that $\Gamma = \boldsymbol{c}\mathcal{D}^2$ is the quasiclasp element; see Section 3I. By Lemma 3.12,

$$\Gamma = \sum_{\boldsymbol{n}\in\mathbb{N}^{2t}} \Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}),$$

where, for $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2) \in \mathbb{N}^t \times \mathbb{N}^t$,

$$\Gamma_1(\boldsymbol{n}) = q^{-(\rho - |E_{\boldsymbol{n}_1}|, |E_{\boldsymbol{n}_2}|)} F_{\boldsymbol{n}_1} K_{\boldsymbol{n}_1}^{-1} E_{\boldsymbol{n}_2}, \quad \Gamma_2(\boldsymbol{n}) = F_{\boldsymbol{n}_2} K_{\boldsymbol{n}_2}^{-1} E_{\boldsymbol{n}_1}. \tag{137}$$

Proposition 7.2 *Suppose $\mathcal{U}$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$ such that $K_\alpha \in \mathcal{U}$ for all $\alpha \in \Pi$, and $F_{\boldsymbol{m}} \otimes E_{\boldsymbol{m}}$, $F'_{\boldsymbol{m}} \otimes E'_{\boldsymbol{m}} \in \mathcal{U} \otimes \mathcal{U}$ for all $\boldsymbol{m} \in \mathbb{N}^t$.*

Then the family $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$ satisfies condition (AL1) *of Theorem 2.32. Namely, the following statements hold:*

- (i) $1_{\mathbb{C}[\![h]\!]} \in \widetilde{\mathcal{K}}_0(\mathcal{U})$, $1_{\boldsymbol{U}_h} \in \widetilde{\mathcal{K}}_1(\mathcal{U})$, *and* $x \otimes y \in \widetilde{\mathcal{K}}_{n+m}(\mathcal{U})$ *whenever* $x \in \widetilde{\mathcal{K}}_n(\mathcal{U})$ *and* $y \in \widetilde{\mathcal{K}}_m(\mathcal{U})$.
- (ii) *Each of* $\boldsymbol{\mu}$, $\boldsymbol{\psi}^{\pm 1}$, $\underline{\Delta}$ *and* $\underline{S}$ *is* $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$*–admissible.*
- (iii) *The Borromean element* $\boldsymbol{b}$ *belongs to* $\widetilde{\mathcal{K}}_3(\mathcal{U})$.

Note that, under the assumption of Proposition 7.2, we have $\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \in \mathcal{U} \otimes \mathcal{U}$ for all $\boldsymbol{n} \in \mathbb{N}^{2t}$.

Before embarking on the proof of the proposition, let us record some consequences.

Theorem 7.3 *Suppose $\mathcal{U}$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$ satisfying the assumption of Proposition 7.2. Then:*

- (a) *For any n–component bottom tangle T with 0 linking matrix, $J_T \in \widetilde{\mathcal{K}}_n(\mathcal{U})$. In particular, $J_T \in \widetilde{\mathcal{K}}_n$.*
- (b) $J_M \in \widehat{\mathbb{Z}[q]}$ *for any integral homology 3–sphere M.*

Proof (a) By Proposition 2.16, Proposition 7.2(i)–(iii) imply that $J_T \in \widetilde{\mathcal{K}}_n(\mathcal{U}) \subset \widetilde{\mathcal{K}}_n$.

(b) By Propositions 7.2 and 7.1, $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$ satisfies both conditions (AL1) and (AL2) of Theorem 2.32. By Theorem 2.32, $J_M \in \widetilde{\mathcal{K}}_0(\mathcal{U}) = \widehat{\mathbb{Z}[q]}$. □

The remaining part of the section is devoted to the proof of Proposition 7.2. Statement (i) of Proposition 7.2 follows trivially from the definitions. We will prove (ii) and (iii) in this section. We fix $\mathcal{U}$ satisfying the assumptions of Proposition 7.2.

Remarks 7.4 (a) One can relax the assumption of Proposition 7.2, requiring only that $K_\alpha \in \mathcal{U}$ for all $\alpha \in \Pi$ and that both Θ and Θ^{-1} are in the topological closure of $\mathcal{U} \otimes \mathcal{U}$ (in the h–adic topology of $\boldsymbol{U}_h \hat{\otimes} \boldsymbol{U}_h$).

(b) Almost identical proof shows that Theorem 7.3 holds true if $\mathcal{U}$ is an $\widetilde{\mathcal{A}}$–Hopf subalgebra of $\boldsymbol{X}_{\mathbb{Z}}$ instead of $\mathcal{U} \subset \boldsymbol{U}_{\mathbb{Z}}$.

7D Quasi-*R*–matrix

Recall that $\Theta = \sum_{\boldsymbol{n} \in \mathbb{N}^t} F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}}$; see Section 3G. For a multiindex $\boldsymbol{n} = (n_1, \dots, n_k)$ in $\mathbb{N}^k$ let $\max(\boldsymbol{n}) = \max_j(n_j)$ and

$$o(\boldsymbol{n}) := (q;q)_{\lfloor \max(\boldsymbol{n})/2 \rfloor} \in \mathbb{Z}[q^{\pm 1}]. \tag{138}$$

Lemma 7.5 *For each $\boldsymbol{n} \in \mathbb{N}^t$, we have*

$$E_{\boldsymbol{n}},\ E'_{\boldsymbol{n}} \in o(\boldsymbol{n}) \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}, \tag{139}$$

$$K_{\boldsymbol{n}} F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}},\ K_{\boldsymbol{n}} F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}} \in o(\boldsymbol{n})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}). \tag{140}$$

Proof We write $x \sim y$ if $x = uy$ with u a unit in $\mathcal{A}$. From the definition of $E_{\boldsymbol{n}}$ (see Section 3G),

$$E_{\boldsymbol{n}} \sim (q;q)_{\boldsymbol{n}} E^{(\boldsymbol{n})} \in (q;q)_{\boldsymbol{n}} \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \subset o(\boldsymbol{n}) \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}.$$

Recall that $E'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(E_{\boldsymbol{n}})$ and $F'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(F_{\boldsymbol{n}})$. Since ι_{bar} preserves the even part (Lemma 3.4) and ι_{bar} leaves both $\boldsymbol{U}_{\mathbb{Z}}$ and $\boldsymbol{X}_{\mathbb{Z}}$ stable (Proposition 5.2 and Theorem 5.21), ι_{bar} leaves both $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ and $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ stable. Hence, we have

$$E'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(E_{\boldsymbol{n}}) \subset o(\boldsymbol{n}) \iota_{\mathrm{bar}}(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}) = o(\boldsymbol{n}) \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}},$$

which proves (139). Let us now prove (140). We have

$$\begin{aligned} K_{\boldsymbol{n}} F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}} &\sim (q;q)_{\boldsymbol{n}} F^{(\boldsymbol{n})} K_{\boldsymbol{n}} \otimes E^{(\boldsymbol{n})} \\ &\sim \sqrt{(q;q)_{\boldsymbol{n}}} \big(\sqrt{(q;q)_{\boldsymbol{n}}}\, F^{(\boldsymbol{n})} K_{\boldsymbol{n}} \big) \otimes E^{(\boldsymbol{n})} \\ &\in \sqrt{(q;q)_{\boldsymbol{n}}}\, \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \subset o(\boldsymbol{n}) \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}. \end{aligned}$$

Applying ι_{bar}, we get

$$K_{\boldsymbol{n}}^{-1} F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}} \in o(\boldsymbol{n})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}).$$

Since $K_{\boldsymbol{n}}^2 \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$, we also have $K_{\boldsymbol{n}} F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}} \in o(\boldsymbol{n})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})$. □

7E On $\mathcal{F}_k(\mathcal{K}_n)$

Lemma 7.6 *For any* $k, n \in \mathbb{N}$, *one has*

$$\mathcal{F}_k(\mathcal{K}_n) = (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}]_1 = (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U_{\mathbb{Z}})^{\otimes n}]_1, \tag{141}$$

$$(q;q)_k (X_{\mathbb{Z}})^{\otimes n} \cap ((U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}) = (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}. \tag{142}$$

Proof The preferred basis (121) of $X_{\mathbb{Z}}$ is a dilatation of the preferred basis of $U_{\mathbb{Z}}$ (described in Proposition 5.9). The basis of $U_{\mathbb{Z}}$ gives rise to an $\mathcal{A}$–basis $\{e(i) \mid i \in I\}$ of $U_{\mathbb{Z}}^{\otimes n}$ in a natural way. By construction, there is a function $a\colon I \to \widetilde{\mathcal{A}}$ such that $\{a(i)e(i) \mid i \in I\}$ is an $\widetilde{\mathcal{A}}$–basis of $X_{\mathbb{Z}}^{\otimes n}$. Further, there is a subset $I^{\mathrm{ev}} \subset I$ such that $\{e(i) \mid i \in I^{\mathrm{ev}}\}$ is an $\mathcal{A}$–basis of $(U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}$ and $\{a(i)e(i) \mid i \in I\}$ is an $\widetilde{\mathcal{A}}$–basis of $(X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}$.

Using the $\widetilde{\mathcal{A}}$–basis $\{e(i) \mid i \in I\}$, every $x \in (U_{\mathbb{Z}}^{\otimes n} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}})$ has unique presentation

$$x = \sum_{I \in I} x_i e(i), \quad x_i \in \widetilde{\mathcal{A}}.$$

Then

(a) $x \in U_{\mathbb{Z}}^{\otimes n}$ if and only if $x_i \in \mathcal{A}$ for all $i \in I$.

(b) $x \in (U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}$ if and only if $x_i \in \mathcal{A}$ for all $i \in I$ and $x_i = 0$ for $i \notin I^{\mathrm{ev}}$.

(c) $x \in ((U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}})$ if and only if $x_i = 0$ for $i \notin I^{\mathrm{ev}}$.

(d) $x \in (q;q)_k (X_{\mathbb{Z}})^{\otimes n}$ if and only if $x_i \in (q;q)_k a(i)\widetilde{\mathcal{A}}$ for all $i \in I$.

(e) $x \in (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}$ if and only if $x_i \in (q;q)_k a(i)\widetilde{\mathcal{A}}$ for all $i \in I$ and $x_i = 0$ for $i \notin I^{\mathrm{ev}}$.

Proof of (142) By (c) and (d), $x \in (q;q)_k (X_{\mathbb{Z}})^{\otimes n} \cap ((U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}})$ if and only if $x_i \in (q;q)_k a(i)\widetilde{\mathcal{A}}$ for all $i \in I$ and $x_i = 0$ for $i \notin I^{\mathrm{ev}}$, which, by (e), is equivalent to $x \in (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}$. Hence we have (142). □

Proof of (141) Since $\mathcal{F}_k(\mathcal{K}_n) = (q;q)_k \big((X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}]_1\big)$, we have

$$\mathcal{F}_k(\mathcal{K}_n) \subset (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}]_1 \subset (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U_{\mathbb{Z}})^{\otimes n}]_1.$$

It remains to prove the converse inclusion, ie if $y \in (q;q)_k (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U_{\mathbb{Z}})^{\otimes n}]_1$ then $x := y/(q;q)_k$ belongs to $(X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap [(U^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n}]_1$. By definition,

$$x \in (X^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes n} \cap \frac{1}{(q;q)_k} [(U_{\mathbb{Z}})^{\otimes n}]_1,$$

and we need to show $x \in [(U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}]_1$. Since both y and $(q;q)_k$ have G–grading 1, $x = y/(q;q)_k$ is an element of $(U_q)^{\otimes n}$ of G–grading 1. It remains to show that $x \in (U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}$.

Because $x \in (X_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}$, (e) implies that $x_i \in \widetilde{\mathcal{A}}$ and $x_i = 0$ if $i \notin I^{\mathrm{ev}}$. Because $x \in (1/(q;q)_k)(U_{\mathbb{Z}})^{\otimes n}$, (a) implies $x_i \in \mathbb{Q}(v)$. It follows that $x_i \in \widetilde{\mathcal{A}} \cap \mathbb{Q}(v) = \mathcal{A}$ and $x_i = 0$ if $i \notin I^{\mathrm{ev}}$. By (b), $x \in (U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}$. □

7F Admissibility decomposition

Suppose $f\colon (U_h)^{\hat{\otimes} a} \to (U_h)^{\hat{\otimes} b}$ is a $\mathbb{C}[\![h]\!]$–module homomorphism. We also use f to denote its natural extension $f\colon (U_{\sqrt{h}})^{\hat{\otimes} a} \to (U_{\sqrt{h}})^{\hat{\otimes} b}$, where $U_{\sqrt{h}} = U_h \hat{\otimes}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!]$.

Recall that f *preserves the G^{ev}–grading* if, for every $g \in G^{\mathrm{ev}}$,

$$f\big([(U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes a}]_g\big) \subset [(U_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes b}]_g,$$

and f is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible if for every $i, j \in \mathbb{N}$,

$$f_{(i,j)}(\widetilde{\mathcal{K}}_{i+a+j}(\mathcal{U})) \subset \widetilde{\mathcal{K}}_{i+b+j}(\mathcal{U}),$$

where $f_{(i,j)} = \mathrm{id}^{\hat{\otimes} i} \mathbin{\hat{\otimes}} f \mathbin{\hat{\otimes}} \mathrm{id}^{\hat{\otimes} j}$.

The following definition is useful in showing a map is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.

Definition 7.7 Suppose $f\colon (U_h)^{\hat{\otimes} a} \to (U_h)^{\hat{\otimes} b}$ is a $\mathbb{C}[\![h]\!]$–module homomorphism. An *admissibility decomposition* for f is a decomposition $f = \sum_{p \in P_f} f_p$ as an h–adically converging sum of $\mathbb{C}[\![h]\!]$–module homomorphisms

$$f_p\colon (U_h)^{\hat{\otimes} a} \to (U_h)^{\hat{\otimes} b}$$

over a set P_f, satisfying the following conditions:

- (A) f_p preserves the G^{ev}–gradings for $p \in P_f$.
- (B) $f_p(\mathcal{U}^{\otimes a}) \subset \mathcal{U}^{\otimes b}$ for $p \in P_f$.
- (C) There are $m_p \in \mathbb{N}$ for $p \in P_f$ such that $\lim_{p \in P_f} m_p = \infty$ and for each $p \in P_f$ we have
$$f_p((X_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes a}) \subset (q;q)_{m_p} (X_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes b}.$$

Here, $\lim_{p \in P_f} m_p = \infty$ means that if $n \geq 0$ then $m_p \geq n$ for all but a finite number of $p \in P_f$. By definition, if P_f is finite then we always have $\lim_{p \in P_f} k_p = \infty$.

Lemma 7.8 *If $f\colon (U_h)^{\hat{\otimes} a} \to (U_h)^{\hat{\otimes} b}$ has an admissibility decomposition then f is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.*

Proof Recall that $\mathcal{F}_k(\mathcal{K}_n(\mathcal{U})) = \mathcal{F}_k(\mathcal{K}_n) \cap \mathcal{U}^{\otimes n}$. From (141),

$$\mathcal{F}_k(\mathcal{K}_n(\mathcal{U})) = (q;q)_k(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n}]_1 \cap \mathcal{U}^{\otimes n}. \tag{143}$$

Let $f = \sum_{p \in P} f_p$ be an admissibility decomposition of f. Suppose $x \in \widetilde{\mathcal{K}}_{i+a+j}(\mathcal{U})$ with presentation

$$x = \sum x_k, \quad x_k \in \mathcal{F}_k(\mathcal{K}_{i+a+j}(\mathcal{U})).$$

Then, with the h–adic topology, we have

$$f_{(i,j)}(x) = \sum_{k,p} (f_p)_{(i,j)}(x_k).$$

We look at each term of the right-hand side. Since $x_k \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes i+a+j}]_1$, (A) implies that

$$(f_p)_{(i,j)}(x_k) \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes i+b+j}]_1. \tag{144}$$

Since $x_k \in \mathcal{U}^{\otimes i+a+j}$, condition (B) implies that

$$(f_p)_{(i,j)}(x_k) \in \mathcal{U}^{\otimes i+a+j}. \tag{145}$$

We have $x_k = (q;q)_k y_k$ with $y_k \in (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes i+a+j}$. By condition (C),

$$\begin{aligned}(f_p)_{(i,j)}(x_k) = (q;q)_k (f_p)_{(i,j)}(y_k) &\in (q;q)_k (q;q)_{m_p} (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes i+b+j} \\ &\subset (q;q)_{m(k,p)} (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes i+b+j},\end{aligned}$$

where $m(k,p) = \max(k, m_p)$. Together with (144), (145) and (143), this implies

$$(f_p)_{(i,j)}(x_k) \in \mathcal{F}_{m(k,p)}(\mathcal{K}_{i+b+j}).$$

Condition (C) implies that

$$\lim_{(k,p) \in \mathbb{N} \times P_f} m(k,p) = \infty.$$

Hence, $f_{(i,j)}(x) = \sum_{k,p} (f_p)_{(i,j)}(x_k)$ belongs to $\widetilde{\mathcal{K}}_{i+b+j}$. □

Remark 7.9 It is not difficult to show that the set of $(\widetilde{\mathcal{K}}_n)$–admissible maps are closed under composition and tensor product. Thus there is a monoidal category whose objects are nonnegative integers and whose morphisms from m to n are $(\widetilde{\mathcal{K}}_n)$–admissible $\mathbb{C}[\![h]\!]$–module homomorphisms from $\boldsymbol{U}_h^{\hat{\otimes} m}$ to $\boldsymbol{U}_h^{\hat{\otimes} n}$. According to Lemma 7.11, this category is braided with $\psi_{1,1} = \boldsymbol{\psi}$, and contains a braided Hopf algebra structure.

7G Admissibility of $\boldsymbol{\mu}$

Lemma 7.10 *The multiplication $\boldsymbol{\mu} : \boldsymbol{U}_h \hat{\otimes} \boldsymbol{U}_h \to \boldsymbol{U}_h$ is $(\widetilde{\mathcal{K}}_n)$–admissible.*

Proof We show that the trivial decomposition, $P_{\boldsymbol{\mu}} = \{0\}$ and $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$, is an admissibility decomposition for $\boldsymbol{\mu}$:

(A) The fact that $\boldsymbol{\mu}$ preserves the G^{ev}–grading is part of Proposition 6.3.

(B) Since $\mathcal{U}$ is a subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$, we have $\boldsymbol{\mu}(\mathcal{U} \otimes \mathcal{U}) \subset \mathcal{U}$.

(C) Since $\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$ is an $\widetilde{\mathcal{A}}$–algebra, we have $\boldsymbol{\mu}(\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}) \subset \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$, which proves (C).

By Lemma 7.8, $\boldsymbol{\mu}$ is $(\widetilde{\mathcal{K}}_n)$–admissible. □

7H Admissibility of $\boldsymbol{\psi}$

Lemma 7.11 *Each of $\boldsymbol{\psi}^{\pm 1}$ is $(\widetilde{\mathcal{K}}_n)$–admissible.*

Proof First consider $\boldsymbol{\psi}$. Using (70) and (67), we obtain $\boldsymbol{\psi} = \sum_{\boldsymbol{m} \in P_{\boldsymbol{\psi}}} \boldsymbol{\psi}_{\boldsymbol{m}}$, where $P_{\boldsymbol{\psi}} = \mathbb{N}^t$ and

$$\boldsymbol{\psi}_{\boldsymbol{m}}(x \otimes y) = v^{(|y|+\lambda_{\boldsymbol{m}}, |x|-\lambda_{\boldsymbol{m}})}(E'_{\boldsymbol{m}} \triangleright y) \otimes (F'_{\boldsymbol{m}} \triangleright x), \tag{146}$$

with $\lambda_{\boldsymbol{m}} = |E'_{\boldsymbol{m}}|$. We will show this is an admissibility decomposition of $\boldsymbol{\psi}$.

(A) Suppose $x, y \in \boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}$ are G^{ev}–homogeneous. By Lemma 6.10,

$$E'_{\boldsymbol{m}} \otimes F'_{\boldsymbol{m}} \in [\boldsymbol{U}_{\mathbb{Z}} \otimes \boldsymbol{U}_{\mathbb{Z}}]_{\dot{e}_{\lambda_{\boldsymbol{m}}} \otimes \dot{e}^{-1}_{\lambda_{\boldsymbol{m}}} \dot{K}_{\boldsymbol{m}}}.$$

From (146) and Lemma 6.15, we have $\boldsymbol{\psi}_{\boldsymbol{m}}(x \otimes y) \in [(\boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}})^{\otimes 2}]_u$, where

$$\begin{aligned} u &= \dot{v}^{(|y|+\lambda_{\boldsymbol{m}}, |x|-\lambda_{\boldsymbol{m}})} \,\dot{\mathrm{ad}}(\dot{e}_{\lambda_{\boldsymbol{m}}} \otimes \dot{y}) \,\dot{\mathrm{ad}}(\dot{e}^{-1}_{\lambda_{\boldsymbol{m}}} \dot{K}_{\lambda_{\boldsymbol{m}}} \otimes \dot{x}) \\ &= \dot{v}^{(|y|+\lambda_{\boldsymbol{m}}, |x|-\lambda_{\boldsymbol{m}})+(\lambda_{\boldsymbol{m}}, |x|)} \dot{e}_{\lambda_{\boldsymbol{m}}} \dot{y} \dot{e}^{-1}_{\lambda_{\boldsymbol{m}}} \dot{x} \\ &= \dot{v}^{(|y|+\lambda_{\boldsymbol{m}}, |x|-\lambda_{\boldsymbol{m}})+(\lambda_{\boldsymbol{m}}, |x|)+(\lambda_{\boldsymbol{m}}, |y|)+(|x|,|y|)} \dot{x} \dot{y} \\ &= \dot{x} \dot{y}. \end{aligned}$$

This shows that $\boldsymbol{\psi}_{\boldsymbol{m}}$ preserves the G^{ev}–grading.

(B) By the assumptions on $\mathcal{U}$, $E'_{\boldsymbol{m}} \otimes F'_{\boldsymbol{m}} \in \mathcal{U} \otimes \mathcal{U}$ and $\mathcal{U}$ is a Hopf algebra. Now (146) shows that $\boldsymbol{\psi}_{\boldsymbol{m}}(\mathcal{U} \otimes \mathcal{U}) \subset \mathcal{U} \otimes \mathcal{U}$. This proves (B).

(C) By (139), $E'_{\boldsymbol{m}} \otimes F'_{\boldsymbol{m}} \in o(\boldsymbol{m}) \boldsymbol{U}_{\mathbb{Z}} \otimes \boldsymbol{U}_{\mathbb{Z}}$. Hence, from (146),

$$\boldsymbol{\psi}_{\boldsymbol{m}}(\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}) \subset o(\boldsymbol{m})(\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}) \otimes (\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}) \subset o(\boldsymbol{m}) \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}},$$

where for the last inclusion we use Theorem 5.21(a), which in particular says $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–stable. This establishes (C) of Definition 7.7.

By Lemma 7.8, $\boldsymbol{\psi}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.

Now consider the case $\boldsymbol{\psi}^{-1}$. By computation, we obtain $\boldsymbol{\psi}^{-1} = \sum_{\boldsymbol{m}\in\mathbb{N}^t} (\boldsymbol{\psi}^{-1})_{\boldsymbol{m}}$, where

$$(\boldsymbol{\psi}^{-1})_{\boldsymbol{m}}(x \otimes y) = v^{-(|x|,|y|)}(F_{\boldsymbol{m}} \triangleright y) \otimes (E_{\boldsymbol{m}} \triangleright x)$$

for homogeneous $x, y \in \boldsymbol{U}_h$. The proof is similar to the case of $\boldsymbol{\psi}$. □

Remark 7.12 One can check that $\boldsymbol{\psi}^{-1} = (\varphi \mathbin{\hat{\otimes}} \varphi)\boldsymbol{\psi}(\varphi^{-1} \mathbin{\hat{\otimes}} \varphi^{-1})$. Hence, the admissibility of $\boldsymbol{\psi}^{-1}$ can also be derived from that of $\boldsymbol{\psi}$.

7I Admissibility of $\underline{\Delta}$

Lemma 7.13 *The braided coproduct $\underline{\Delta}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.*

Proof Suppose $x \in \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ is G^{ev}–homogeneous. By a simple calculation, we have

$$\underline{\Delta} = \sum_{\boldsymbol{m}\in\mathbb{N}^t} \underline{\Delta}_{\boldsymbol{m}}, \tag{147}$$

where, with $\lambda_{\boldsymbol{m}} := |E'_{\boldsymbol{m}}|$,

$$\underline{\Delta}_{\boldsymbol{m}}(x) = \sum v^{-(|x_{(2)}|,\lambda_{\boldsymbol{m}})}(E'_{\boldsymbol{m}} \triangleright x_{(2)}) \otimes (K_{\boldsymbol{m}} F'_{\boldsymbol{m}})(K_{|x_{(2)}|} x_{(1)}). \tag{148}$$

(A) By Corollary 6.11, $E'_{\boldsymbol{m}} \otimes K_{\boldsymbol{m}} F'_{\boldsymbol{m}} \in [\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}]_{\dot{e}_{\boldsymbol{m}} \otimes \dot{e}_{\boldsymbol{m}}^{-1}}$. We will use a G–good presentation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ (see Section 6I). From Lemma 6.15, each summand of the right-hand side of (148) is in $[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}]_u$, where

$$u = \dot{v}^{-(|x_{(2)}|,\lambda_{\boldsymbol{m}})} \dot{e}_{\lambda_{\boldsymbol{m}}} \dot{x}_{(2)} \dot{e}_{\lambda_{\boldsymbol{m}}}^{-1} \dot{K}_{|x_{(2)}|} \dot{x}_{(1)} = \dot{x}_{(2)} \dot{K}_{|x_{(2)}|} \dot{x}_{(1)} = \dot{x}.$$

Here the last identity is (134). Thus, $\underline{\Delta}_{\boldsymbol{m}}$ preserves the G^{ev}–grading.

(B) Since $K_\alpha \in \mathcal{U}$ and $E'_{\boldsymbol{m}} \otimes K_{\boldsymbol{m}} F'_{\boldsymbol{m}} \in \mathcal{U} \otimes \mathcal{U}$, (148) shows that $\underline{\Delta}_{\boldsymbol{m}}(\mathcal{U}) \subset \mathcal{U} \otimes \mathcal{U}$.

(C) Let $x \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$. By an argument similar to the proof of Lemma 6.13, we see that

$$x_{(2)} \otimes K_{|x_{(2)}|} x_{(1)} \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}.$$

By (140),

$$E'_{\boldsymbol{m}} \otimes K_{\boldsymbol{m}} F'_{\boldsymbol{m}} \in o(\boldsymbol{m}) \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}.$$

Hence, from (148),

$$\underline{\Delta}_{\boldsymbol{m}}(x) \in o(\boldsymbol{m})(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \triangleright \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}) \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \subset o(\boldsymbol{m}) \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}},$$

where for the last inclusion we again use the fact that $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–stable (Theorem 5.21). This shows (C) of Definition 7.7 holds. By Lemma 7.8, $\underline{\Delta}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible. □

7J Admissibility of $\underline{S}$

Lemma 7.14 *The braided antipode $\underline{S}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.*

Proof By computation, we obtain $\underline{S} = \sum_{\boldsymbol{m}\in\mathbb{N}^t} \underline{S}_{\boldsymbol{m}}$, where

$$\underline{S}_{\boldsymbol{m}}(x) = S^{-1}(E_{\boldsymbol{m}} \triangleright x)F_{\boldsymbol{m}}K_{-|x|} \tag{149}$$

for Y–homogeneous $x \in \boldsymbol{U}_h$. We will assume x is G^{ev}–homogeneous.

(A) By Lemma 6.6, we have $\underline{S}_{\boldsymbol{m}}(x) \in [\boldsymbol{U}_{\mathbb{Z}}]_g$, where

$$\begin{aligned} g &= \dot{S}^{-1}(\dot{\mathrm{ad}}(\dot{e}_{\lambda_{\boldsymbol{m}}} \otimes \dot{x}))\dot{e}_{\lambda_{\boldsymbol{m}}}^{-1}\dot{K}_{\lambda_{\boldsymbol{m}}}\dot{K}_{-|\dot{x}|} = \dot{S}^{-1}(\dot{e}_{\lambda_{\boldsymbol{m}}}\dot{x})\dot{e}_{\lambda_{\boldsymbol{m}}}^{-1}\dot{K}_{\lambda_{\boldsymbol{m}}}\dot{K}_{-|\dot{x}|} \\ &= \dot{K}_{|\dot{x}|}\dot{x}\dot{K}_{\lambda_{\boldsymbol{m}}}\dot{e}_{\lambda_{\boldsymbol{m}}}\dot{e}_{\lambda_{\boldsymbol{m}}}^{-1}\dot{K}_{\lambda_{\boldsymbol{m}}}\dot{K}_{-|\dot{x}|} = \dot{x}. \end{aligned}$$

(B) Since $K_\alpha \in \mathcal{U}$ and $E_{\boldsymbol{m}} \otimes F_{\boldsymbol{m}} \in \mathcal{U} \otimes \mathcal{U}$, (149) shows that $\underline{S}_{\boldsymbol{m}}(\mathcal{U}) \subset \mathcal{U}$.

(C) We rewrite (149) as

$$\underline{S}_{\boldsymbol{m}}(x) = v^{-(|x|,|E_{\boldsymbol{m}}|)}S^{-1}(E_{\boldsymbol{m}} \triangleright x)K_{-|E_{\boldsymbol{m}}|-|x|}K_{\boldsymbol{m}}F_{\boldsymbol{m}}. \tag{150}$$

By (140), $E_{\boldsymbol{m}} \otimes K_{\boldsymbol{m}}F_{\boldsymbol{m}} \in o(\boldsymbol{m})(\boldsymbol{U}_{\mathbb{Z}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})$. Since $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ is $\boldsymbol{U}_{\mathbb{Z}}$–stable,

$$E_{\boldsymbol{m}} \triangleright x \otimes K_{\boldsymbol{m}}F_{\boldsymbol{m}} \in o(\boldsymbol{m})(\boldsymbol{U}_{\mathbb{Z}} \triangleright \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}) \subset o(\boldsymbol{m})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}).$$

Hence, from (150) we have

$$\underline{S}_{\boldsymbol{m}}(x) \in o(\boldsymbol{m})\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}},$$

which proves property (C).

By Lemma 7.8, $\underline{S}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible. □

Thus, Proposition 7.2(ii) holds.

7K Borromean tangle

The goal now is to establish Proposition 7.2(iii). Namely, we will show that $\boldsymbol{b} \in \widetilde{\mathcal{K}}_3$, where $\boldsymbol{b}$ is the universal invariant of the Borromean bottom tangle.

First we recall (38), which describes $\boldsymbol{b}$ through the clasp element $\boldsymbol{c}$ using the braided commutator. With $\boldsymbol{c} = \sum_{\boldsymbol{n}\in\mathbb{N}^{2t}}[\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n})]\mathcal{D}^{-2}$, (38) says

$$\boldsymbol{b} = \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{N}^{2t}} \boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}},$$

where, for $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^{2t}$,

$$\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} := (\mathrm{id}^{\hat{\otimes}2} \mathbin{\hat{\otimes}} \Upsilon)\big([\Gamma_1(\boldsymbol{n}) \otimes \Gamma_1(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{n})]\mathcal{D}_{14}^{-2}\mathcal{D}_{23}^{-2}\big). \tag{151}$$

Here, if $x = \sum x' \otimes x''$ then $x_{14} = \sum x' \otimes 1 \otimes 1 \otimes x''$ and $x_{23} = \sum 1 \otimes x' \otimes x'' \otimes 1$.

Lemma 7.15 *For $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^{2t}$ one has $\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} \in o(\boldsymbol{n},\boldsymbol{m})\widetilde{\mathcal{K}}_3(\mathcal{U})$. Thus, $\boldsymbol{b} \in \widetilde{\mathcal{K}}_3(\mathcal{U})$.* (Recall that $o(\boldsymbol{n},\boldsymbol{m}) = (q;q)_{\lfloor \max(\boldsymbol{n},\boldsymbol{m})/2 \rfloor}$.)

The remaining part of this section is devoted to the proof of Lemma 7.15.

7K1 Quasiclasp element Recall that $\Gamma_1(\boldsymbol{n})$ and $\Gamma_2(\boldsymbol{n})$ are given by (137) for $\boldsymbol{n} \in \mathbb{N}^{2t}$.

Lemma 7.16 *Suppose $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2) \in \mathbb{N}^t \times \mathbb{N}^t$. Then*

$$\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \in \mathcal{K}_2 = (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2} \cap [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}]_1, \tag{152}$$

$$\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \in o(\boldsymbol{n})\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}. \tag{153}$$

Proof We write $x \sim y$ if $x = uy$ with u a unit in $\mathcal{A}$. Note that $\sqrt{(q;q)_{\boldsymbol{n}_1}}F^{(\boldsymbol{n}_1)}$ and $\sqrt{(q;q)_{\boldsymbol{n}_2}}E^{(\boldsymbol{n}_2)}$ are in $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$, as they are among the preferred basis elements. Using the definition (137) of $\Gamma_1(\boldsymbol{n})$ and $\Gamma_2(\boldsymbol{n})$, we have

$$\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \sim (q;q)_{\boldsymbol{n}_1}(q;q)_{\boldsymbol{n}_2} F^{(\boldsymbol{n}_1)}K_{\boldsymbol{n}_1}^{-1}E^{(\boldsymbol{n}_2)} \otimes F^{(\boldsymbol{n}_2)}K_{\boldsymbol{n}_2}^{-1}E^{(\boldsymbol{n}_1)} \in (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}. \tag{154}$$

From Corollary 6.11, $\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n})$, which is in $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}$, has G–grading equal to

$$(\dot{e}_{\boldsymbol{n}_1}^{-1}\dot{e}_{\boldsymbol{n}_2})(\dot{e}_{\boldsymbol{n}_2}^{-1}\dot{e}_{\boldsymbol{n}_1}) = 1.$$

This shows $\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \in (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2} \cap [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}]_1 = \mathcal{K}_2$. This proves (152).

Because $\sqrt{(q;q)_{\boldsymbol{n}_1}(q;q)_{\boldsymbol{n}_2}}F^{(\boldsymbol{n}_1)}K_{\boldsymbol{n}_1}^{-1}E^{(\boldsymbol{n}_2)} \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}$ and $F^{(\boldsymbol{n}_2)}K_{\boldsymbol{n}_2}^{-1}E^{(\boldsymbol{n}_1)} \in \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$, from (154) we have

$$\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n}) \in \sqrt{(q;q)_{\boldsymbol{n}_1}(q;q)_{\boldsymbol{n}_2}}(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}) \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \subset o(\boldsymbol{n})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}) \otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}.$$

This proves (153). □

7K2 Decomposition of $\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}}$ Recall that $\mathcal{D} = \exp\big(\frac{1}{2}h\sum_{\alpha\in\Pi} H_\alpha \otimes \breve{H}_\alpha/d_\alpha\big)$ is the diagonal part of the R–matrix. We will freely use the following well-known properties of $\mathcal{D}$:

$$(\Delta \otimes 1)(\mathcal{D}) = \mathcal{D}_{13}\mathcal{D}_{23}, \quad (\boldsymbol{\epsilon} \otimes 1)(\mathcal{D}) = 1, \quad (S \otimes 1)(\mathcal{D}) = \mathcal{D}^{-1},$$

where $\mathcal{D}_{13} = \sum \mathcal{D}_1 \otimes 1 \otimes \mathcal{D}_2$ and $\mathcal{D}_{23} = 1 \otimes \mathcal{D}$ are in $\boldsymbol{U}_h^{\hat{\otimes} 3}$. In the sequel we set

$$\mathcal{D}^{-2} = \sum \delta_1 \otimes \delta_2 = \sum \delta_1' \otimes \delta_2'.$$

Recall (151):

$$\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} = (\mathrm{id}^{\otimes 2} \otimes \Upsilon)\big([\Gamma_1(\boldsymbol{n}) \otimes \Gamma_1(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{n})]\mathcal{D}_{14}^{-2}\mathcal{D}_{23}^{-2}\big).$$

By (39), Υ is the composition of four maps:

$$\Upsilon = \boldsymbol{\mu} \circ (\mathrm{ad} \mathbin{\hat{\otimes}} \mathrm{id}) \circ (\mathrm{id} \mathbin{\hat{\otimes}} \underline{S} \mathbin{\hat{\otimes}} \mathrm{id}) \circ (\mathrm{id} \mathbin{\hat{\otimes}} \underline{\Delta}).$$

Using the above decomposition, one gets

$$\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} = f^{\mu} \circ f^{\mathrm{ad}}_{\boldsymbol{m}} \circ f^{\underline{S}} \circ f^{\underline{\Delta}}(\Gamma_1(\boldsymbol{n}) \otimes \Gamma_2(\boldsymbol{n})), \tag{155}$$

where

$$f^{\underline{\Delta}} \colon \boldsymbol{U}_h^{\hat{\otimes} 2} \to \boldsymbol{U}_h^{\hat{\otimes} 3}, \quad f^{\underline{\Delta}}(x) = [(\mathrm{id} \mathbin{\hat{\otimes}} \underline{\Delta})(x\mathcal{D}^{-2})]\mathcal{D}^2_{12}\mathcal{D}^2_{13}, \tag{156}$$

$$f^{\underline{S}} \colon \boldsymbol{U}_h^{\hat{\otimes} 3} \to \boldsymbol{U}_h^{\hat{\otimes} 3}, \quad f^{\underline{S}}(x) = [(\mathrm{id} \mathbin{\hat{\otimes}} \underline{S} \mathbin{\hat{\otimes}} \mathrm{id})(x\mathcal{D}^{-2}_{12})]\mathcal{D}^{-2}_{12}, \tag{157}$$

$$f^{\mathrm{ad}}_{\boldsymbol{m}} \colon \boldsymbol{U}_h^{\hat{\otimes} 3} \to \boldsymbol{U}_h^{\hat{\otimes} 4}, \tag{158}$$

$$f^{\mathrm{ad}}_{\boldsymbol{m}}(x) = (\mathrm{id}^{\hat{\otimes} 2} \mathbin{\hat{\otimes}} \mathrm{ad} \mathbin{\hat{\otimes}} \mathrm{id})\big([x_1 \otimes \Gamma_1(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{m}) \otimes x_2 \otimes x_3]\mathcal{D}^{-2}_{23}\mathcal{D}^2_{14}\big)\mathcal{D}^{-2}_{13},$$

$$f^{\mu} \colon \boldsymbol{U}_h^{\hat{\otimes} 4} \to \boldsymbol{U}_h^{\hat{\otimes} 3}, \quad f^{\mu}(x) = (\mathrm{id}^{\hat{\otimes} 2} \mathbin{\hat{\otimes}} \boldsymbol{\mu})(x\mathcal{D}^2_{13}\mathcal{D}^{-2}_{14}). \tag{159}$$

Similarly, using (40) instead of (39), we have

$$\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} = \tilde{f}^{\mu} \circ \tilde{f}^{\mathrm{ad}}_{\boldsymbol{n}} \circ f^{\underline{S}} \circ f^{\underline{\Delta}}(\Gamma_1(\boldsymbol{m}) \otimes \Gamma_2(\boldsymbol{m})), \tag{160}$$

where $f^{\underline{\Delta}}$ and $f^{\underline{S}}$ are as above, and

$$\tilde{f}^{\mathrm{ad}}_{\boldsymbol{n}} \colon \boldsymbol{U}_h^{\hat{\otimes} 3} \to \boldsymbol{U}_h^{\hat{\otimes} 4}, \tag{161}$$

$$\tilde{f}^{\mathrm{ad}}_{\boldsymbol{n}}(x) = \big[(\mathrm{id}^{\hat{\otimes} 3} \mathbin{\hat{\otimes}} \underline{\mathrm{ad}}^r)\big([\Gamma_1(\boldsymbol{n}) \otimes x_1 \otimes x_2 \otimes x_3 \otimes \Gamma_2(\boldsymbol{n})]\mathcal{D}^{-2}_{24}\mathcal{D}^{-2}_{15}\big)\big]\mathcal{D}^2_{24},$$

$$\tilde{f}^{\mu} \colon \boldsymbol{U}_h^{\hat{\otimes} 4} \to \boldsymbol{U}_h^{\hat{\otimes} 3}, \quad f^{\mu}(x) = (\mathrm{id}^{\hat{\otimes} 2} \mathbin{\hat{\otimes}} \boldsymbol{\mu})(x\mathcal{D}^2_{23}\mathcal{D}^{-2}_{24}). \tag{162}$$

We will prove that each of $f^{\underline{\Delta}}$, $f^{\underline{S}}$ and f^{μ} is $(\tilde{\mathcal{K}}_n)$–admissible, while each of $f^{\mathrm{ad}}_{\boldsymbol{n}}$ and $\tilde{f}^{\mathrm{ad}}_{\boldsymbol{n}}$ maps $\tilde{\mathcal{K}}_3$ to $o(\boldsymbol{n})\tilde{\mathcal{K}}_4$. From here, Lemma 7.15 will follow easily.

7K3 Extended adjoint action To study the maps $f^{\underline{\Delta}}$, $f^{\underline{S}}$, $f^{\mathrm{ad}}_{\boldsymbol{m}}$ and $\tilde{f}^{\mathrm{ad}}_{\boldsymbol{n}}$, we need the following extended adjoint action: For $a \in \boldsymbol{U}_{\sqrt{h}} = \boldsymbol{U}_h \mathbin{\hat{\otimes}}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!]$ and Y–homogeneous $x, y \in \boldsymbol{U}_{\sqrt{h}}$, define

$$a \blacktriangleright (y \otimes x) := \big[(\mathrm{id} \otimes \mathrm{ad}_a)((y \otimes x)\mathcal{D}^2)\big]\mathcal{D}^{-2} = \sum y K_{2|a_{(2)}|} \otimes a_{(1)} x S(a_{(2)}). \tag{163}$$

It is easy to check that $(a \otimes x \otimes y) \mapsto a \blacktriangleright (x \otimes y)$ gives rise to an action of $\boldsymbol{U}_{\sqrt{h}}$ on $\boldsymbol{U}_{\sqrt{h}} \mathbin{\hat{\otimes}} \boldsymbol{U}_{\sqrt{h}}$.

Lemma 7.17 (a) *If $a, x, y \in \boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}$ are G^{ev}–homogeneous, then*

$$a \blacktriangleright (y \otimes x) \in [\boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{U}^{\mathrm{ev}}_{\mathbb{Z}}]_{\dot{y} \otimes \dot{a}\dot{x}}. \tag{164}$$

(b) *One has $\mathcal{U} \blacktriangleright (\mathcal{U} \otimes \mathcal{U}) \subset \mathcal{U} \otimes \mathcal{U}$.*

(c) *One has*

$$\boldsymbol{U}_{\mathbb{Z}} \blacktriangleright (\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}) \subset \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}.$$

Proof (a) The right-hand side of (163) shows that $a \blacktriangleright (y \otimes x)$ has $G^{\otimes 2}$–grading equal to

$$\dot{y} \otimes \dot{a}_{(1)}\dot{x}\dot{S}(\dot{a}_{(2)}) = \dot{y} \otimes \dot{a}_{(1)}\dot{x}\dot{a}_{(2)}\dot{K}_{|a_{(2)}|} = \dot{y} \otimes \dot{a}_{(1)}\dot{a}_{(2)}\dot{K}_{|a_{(2)}|}\dot{x} = \dot{y} \otimes \dot{a}\dot{x},$$

where we use $\dot{a}_{(1)}\dot{a}_{(2)}\dot{K}_{|a_{(2)}|} = \dot{a}$ from (134). This shows (164).

(b) By the assumptions on $\mathcal{U}$, $K_\alpha^{\pm 1} \in \mathcal{U}$ and $\mathcal{U}$ is a Hopf algebra. Hence, (b) follows from (163).

(c) Suppose $a \in \boldsymbol{U}_{\mathbb{Z}}$ and $x, y \in \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$; we need to show that $a \blacktriangleright (y \otimes x) \in \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$. Because $(ab) \blacktriangleright (y \otimes x) = a \blacktriangleright (b \blacktriangleright (y \otimes x))$, it is sufficient to consider the case when a is one of the generators $E_\alpha^{(n)}$, $F_\alpha^{(n)}$ or $K_\alpha^{\pm 1}$ of $\boldsymbol{U}_{\mathbb{Z}}$, where $\alpha \in \Pi$ and $n \in \mathbb{N}$. The cases $a = K_\alpha^{\pm 1}$ are trivial.

For $a = E_\alpha^{(n)}$, a calculation by induction on n shows that

$$\begin{aligned} E_\alpha^{(n)} \blacktriangleright (y \otimes x) &= \sum_{j=0}^{n} (-1)^n v_\alpha^{2jn + \binom{n+1}{2}} y (K_\alpha^2; q_\alpha)_{n-j} \otimes E_\alpha^{(n-j)} (E_\alpha^{(j)} \triangleright x) \\ &= \sum_{j=0}^{n} (-1)^n v_\alpha^{2jn + \binom{n+1}{2}} \left[y \frac{(K_\alpha^2; q_\alpha)_{n-j}}{\sqrt{(q_\alpha; q_\alpha)_{n-j}}} \right] \\ &\qquad\qquad \otimes \left[\sqrt{(q_\alpha; q_\alpha)_{n-j}}\, E_\alpha^{(n-j)} \right] [E_\alpha^{(j)} \triangleright x]. \end{aligned}$$

The right-hand side belongs to $\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}} \otimes \boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$, since each factor in square brackets is in $\boldsymbol{X}^{\mathrm{ev}}_{\mathbb{Z}}$.

The case $a = F_\alpha^{(n)}$ can be handled by a similar calculation, or can be derived from the already-proved case $a = \varphi(F_\alpha) = K_\alpha^{-1} E_\alpha$, using

$$(\varphi \otimes \varphi)(a \blacktriangleright (y \otimes x)) = \varphi(a) \blacktriangleright (\varphi(y) \otimes \varphi(x)),$$

which follows from the fact that φ commutes with S, Δ and $\varphi(K_\alpha) = K_\alpha$. □

7K4 The map f^{Δ}

Lemma 7.18 *The map $f^{\Delta}\colon \boldsymbol{U}_h^{\hat{\otimes} 2} \to \boldsymbol{U}_h^{\hat{\otimes} 3}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.*

Proof Using the definition (156) and the decomposition (147) of $\underline{\Delta}$ we have $f^{\underline{\Delta}} = \sum_{\boldsymbol{u}\in\mathbb{N}^t} f_{\boldsymbol{u}}^{\underline{\Delta}}$, where

$$f_{\boldsymbol{u}}^{\underline{\Delta}}(y\otimes x) = \Big[\sum y\delta_1 \otimes \underline{\Delta}_{\boldsymbol{u}}(x\delta_2)\Big]\mathcal{D}_{12}^2\mathcal{D}_{13}^2.$$

We will show that $f^{\underline{\Delta}} = \sum_{\boldsymbol{u}\in\mathbb{N}^t} f_{\boldsymbol{u}}^{\underline{\Delta}}$ is an admissible decomposition. Using the definitions, we have

$$\begin{aligned}
&\sum y\delta_1 \otimes \underline{\Delta}_{\boldsymbol{u}}(x\delta_2)\\
&\qquad = \sum y\delta_1 \otimes E'_{\boldsymbol{u}} \triangleright (x_{(2)}(\delta_2)_{(2)}) \otimes K_{\lambda_{\boldsymbol{u}}+|x_{(2)}|}F'_{\boldsymbol{u}}x_{(1)}(\delta_2)_{(1)}\\
&\qquad = \sum y\delta_1\delta'_1 \otimes E'_{\boldsymbol{u}} \triangleright (x_{(2)}\delta_2) \otimes K_{\lambda_{\boldsymbol{u}}+|x_{(2)}|}F'_{\boldsymbol{u}}x_{(1)}\delta'_2\\
&\qquad = \sum y\delta_1\delta'_1 \otimes (E'_{\boldsymbol{u}})_{(1)}x_{(2)}\delta_2 S((E'_{\boldsymbol{u}})_{(2)}) \otimes K_{\lambda_{\boldsymbol{u}}+|x_{(2)}|}F'_{\boldsymbol{u}}x_{(1)}\delta'_2\\
&\qquad = \sum yK_{2|(E'_{\boldsymbol{u}})_{(2)}|}\delta_1\delta'_1 \otimes (E'_{\boldsymbol{u}})_{(1)}x_{(2)} S((E'_{\boldsymbol{u}})_{(2)})\delta_2 \otimes K_{\lambda_{\boldsymbol{u}}+|x_{(2)}|}F'_{\boldsymbol{u}}x_{(1)}\delta'_2\\
&\qquad = \Big(\sum yK_{2|(E'_{\boldsymbol{u}})_{(2)}|} \otimes (E'_{\boldsymbol{u}})_{(1)}x_{(2)} S((E'_{\boldsymbol{u}})_{(2)}) \otimes K_{\lambda_{\boldsymbol{u}}+|x_{(2)}|}F'_{\boldsymbol{u}}x_{(1)}\Big)\mathcal{D}_{12}^{-2}\mathcal{D}_{13}^{-2}\\
&\qquad = \sum\big(v^{(|x_{(2)}|,\lambda_{\boldsymbol{u}})}E'_{\boldsymbol{u}} \blacktriangleright (y\otimes x_{(2)}) \otimes (K_{\boldsymbol{u}}F'_{\boldsymbol{u}})K_{|x_{(2)}|}x_{(1)}\big)\mathcal{D}_{12}^{-2}\mathcal{D}_{13}^{-2}.
\end{aligned}$$

This shows that

$$f_{\boldsymbol{u}}^{\underline{\Delta}}(y\otimes x) = \sum v^{(|x_{(2)}|,\lambda_{\boldsymbol{u}})}(E'_{\boldsymbol{u}} \blacktriangleright (y\otimes x_{(2)})) \otimes (K_{\boldsymbol{u}}F'_{\boldsymbol{u}})(K_{|x_{(2)}|}x_{(1)}). \tag{165}$$

(A) Suppose $x, y \in \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ are G^{ev}–homogeneous. By G–good presentation (see Section 6I) and Lemma 7.17, all the factors in parentheses on the right-hand side of (165) are in $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$.

From (165) and Lemma 7.17, $f_{\boldsymbol{u}}^{\underline{\Delta}}(y\otimes x) \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}]_g$, where

$$g = \dot{v}^{(|x_{(2)}|,\lambda_{\boldsymbol{u}})}\dot{y}\dot{e}_{\lambda_{\boldsymbol{u}}}\dot{x}_{(2)}\dot{e}_{\lambda_{\boldsymbol{u}}}^{-1}\dot{K}_{|x_{(2)}|}\dot{x}_{(1)} = \dot{y}\dot{x}_{(2)}\dot{K}_{|x_{(2)}|}\dot{x}_{(1)} = \dot{y}\dot{x},$$

with the last equality obtained from (134). This shows $f_{\boldsymbol{u}}^{\underline{\Delta}}$ preserves the G^{ev}–grading.

(B) Suppose $x, y \in \mathcal{U}$. By the assumptions on $\mathcal{U}$, we have that $K_\alpha \in \mathcal{U}$ and $E'_{\boldsymbol{u}} \otimes K_{\boldsymbol{u}}F'_{\boldsymbol{u}} \in \mathcal{U}\otimes\mathcal{U}$. Now Lemma 7.17 shows that the right-hand side of (165) is in $\mathcal{U}^{\otimes 3}$. Thus, $f_{\boldsymbol{u}}^{\underline{\Delta}}(\mathcal{U}^{\otimes 2}) \subset \mathcal{U}^{\otimes 3}$.

(C) By (140), $E'_{\boldsymbol{u}} \otimes K_{\boldsymbol{u}}F'_{\boldsymbol{u}} \in o(\boldsymbol{u})(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}} \otimes \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})$. Lemma 7.17 and (165) show that

$$f_{\boldsymbol{u}}^{\underline{\Delta}}((\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}) \in o(\boldsymbol{u})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}.$$

By Lemma 7.8, $f^{\underline{\Delta}}$ is $(\widetilde{\mathcal{K}}_n)$–admissible. □

7K5 The map $f^{\underline{S}}$

Lemma 7.19 *The map* $f^{\underline{S}}\colon \boldsymbol{U}_h^{\hat{\otimes} 3} \to \boldsymbol{U}_h^{\hat{\otimes} 3}$ *is* $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$*–admissible.*

Proof Using the definition (157) and the decomposition (149) of $\underline{S}$, we have $f^{\underline{S}} = \sum_{\boldsymbol{u}\in\mathbb{N}^t} f^{\underline{S}}_{\boldsymbol{u}}$, where

$$f^{\underline{S}}_{\boldsymbol{u}}(y\otimes x\otimes z) = \Big(\sum y\delta_1 \otimes \underline{S}_{\boldsymbol{u}}(x\delta_2)\otimes z\Big)\mathcal{D}_{12}^{-2}$$
$$= (y\otimes 1\otimes z)\Big(\sum \delta_1 \otimes \underline{S}_{\boldsymbol{u}}(x\delta_2)\otimes 1\Big)\mathcal{D}_{12}^{-2}.$$

Using the definitions, we have

$$\begin{aligned}
&\sum \delta_1 \otimes \underline{S}_{\boldsymbol{u}}(x\delta_2)\\
&\quad= \sum \delta_1 \otimes S^{-1}(E_{\boldsymbol{u}} \triangleright (x\delta_2))F_{\boldsymbol{u}}K_{-|x|}\\
&\quad= \sum \delta_1 \otimes S^{-1}((E_{\boldsymbol{u}})_{(1)}(x\delta_2)S((E_{\boldsymbol{u}})_{(2)}))F_{\boldsymbol{u}}K_{-|x|}\\
&\quad= \sum \delta_1 \otimes (E_{\boldsymbol{u}})_{(2)}S^{-1}(\delta_2)S^{-1}(x)S^{-1}((E_{\boldsymbol{u}})_{(1)})F_{\boldsymbol{u}}K_{-|x|}\\
&\quad= \sum K_{2(|x|+|(E_{\boldsymbol{u}})_{(1)}|+|F_{\boldsymbol{u}}|)}\delta_1 \otimes (E_{\boldsymbol{u}})_{(2)}S^{-1}(x)S^{-1}((E_{\boldsymbol{u}})_{(1)})F_{\boldsymbol{u}}K_{-|x|}S^{-1}(\delta_2)\\
&\quad= \Big(\sum K_{2(|x|+|(E_{\boldsymbol{u}})_{(1)}|+|F_{\boldsymbol{u}}|)} \otimes (E_{\boldsymbol{u}})_{(2)}S^{-1}(x)S^{-1}((E_{\boldsymbol{u}})_{(1)})F_{\boldsymbol{u}}K_{-|x|}\Big)\mathcal{D}^2\\
&\quad= \sum\big[(1\otimes K_{-|x|}F_{\boldsymbol{u}})(S^{-1}\otimes S^{-1})(E_{\boldsymbol{u}} \blacktriangleright (K_{-2|x|}\otimes x))\big]\mathcal{D}^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(166)\quad & f^{\underline{S}}_{\boldsymbol{u}}(y\otimes x\otimes z)\\
&= \sum (y\otimes 1\otimes 1)\Big(\big[(1\otimes K_{-|x|}F_{\boldsymbol{u}})(S^{-1}\otimes S^{-1})(E_{\boldsymbol{u}} \blacktriangleright (K_{-2|x|}\otimes x))\big]\otimes z\Big).
\end{aligned}$$

Assume that $x, y, z \in \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ are G^{ev}–homogeneous. By Lemma 6.10,

$$F_{\boldsymbol{u}}\otimes E_{\boldsymbol{u}} \in [\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}\otimes \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}]_{\dot{e}^{-1}_{\lambda_{\boldsymbol{u}}}\dot{K}_{\lambda_{\boldsymbol{u}}}\otimes \dot{e}_{\lambda_{\boldsymbol{u}}}}.$$

Hence, from Lemma 7.17(a), $f^{\underline{S}}_{\boldsymbol{u}}(y\otimes x\otimes z) \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}]_g$, where

$$g = \dot{y}\otimes \dot{K}_{|x|}\dot{e}^{-1}_{\lambda_{\boldsymbol{u}}}\dot{K}_{\lambda_{\boldsymbol{u}}}\dot{S}^{-1}(\dot{e}_{\lambda_{\boldsymbol{u}}}\dot{x})\otimes \dot{z} = \dot{y}\otimes\dot{x}\otimes\dot{z},$$

where the last equality follows from a simple calculation. Thus,

$$(167)\qquad f^{\underline{S}}_{\boldsymbol{u}}(y\otimes x\otimes z) \in [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}]_{\dot{y}\otimes\dot{x}\otimes\dot{z}}$$

(A) From (167), $f^{\underline{S}}_{\boldsymbol{u}}$ preserves the G^{ev}–grading.

(B) Assume that $x, y, z \in \mathcal{U}$. Since $K_\alpha \in \mathcal{U}$ and $F_{\boldsymbol{u}} \otimes E_{\boldsymbol{u}} \in \mathcal{U} \otimes \mathcal{U}$, Lemma 7.17(b) shows that the right-hand side of (166) is in $\mathcal{U}^{\otimes 3}$.

(C) By (140), $F_{\boldsymbol{u}} \otimes E_{\boldsymbol{u}} \in o(\boldsymbol{u}) \boldsymbol{X}_{\mathbb{Z}} \otimes \boldsymbol{U}_{\mathbb{Z}}$. Lemma 7.17 and (166) show that

$$f_{\boldsymbol{u}}^{S}((\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}) \subset o(\boldsymbol{u})(\boldsymbol{X}_{\mathbb{Z}})^{\otimes 3}.$$

On the other hand, (167) shows that

$$f_{\boldsymbol{u}}^{S}((\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}) \subset ((\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}).$$

Because

$$o(\boldsymbol{u})(\boldsymbol{X}_{\mathbb{Z}})^{\otimes 3} \cap ((\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}) = o(\boldsymbol{u})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}$$

by (142), we have $f_{\boldsymbol{u}}^{S}((\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}) \subset o(\boldsymbol{u})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}$. □

7K6 The maps f^{ad} and $\tilde{f}^{\mathrm{ad}}$

Lemma 7.20 *For $f = f_{\boldsymbol{m}}^{\mathrm{ad}}$ or $f = \tilde{f}_{\boldsymbol{m}}^{\mathrm{ad}}$, one has $f(\mathcal{K}_3(\mathcal{U})) \subset \mathcal{F}_{\lfloor \max \boldsymbol{m}/2 \rfloor}(\mathcal{K}_4(\mathcal{U}))$.*

Proof Assume $x \otimes y \otimes z \in \mathcal{K}_3(\mathcal{U}) = (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3} \cap [(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 3}]_1 \cap \mathcal{U}^{\otimes 3}$. First assume $f = f_{\boldsymbol{m}}^{\mathrm{ad}}$. Recall that

$$\begin{aligned} &f_{\boldsymbol{m}}^{\mathrm{ad}}(x_1 \otimes x_2 \otimes x_3) \\ &\quad = \Big[\sum (\mathrm{id}^{\otimes 2} \otimes \mathrm{ad} \otimes \mathrm{id})(x_1 S(\delta_1) \otimes \Gamma_1(\boldsymbol{m})\delta_1' \otimes \Gamma_2(\boldsymbol{m})\delta_2' \otimes x_2\delta_2 \otimes x_3)\Big]\mathcal{D}_{13}^{-2} \\ &\quad = (x_1 \otimes \Gamma_1(\boldsymbol{m}) \otimes 1 \otimes x_3)\Big[\sum (\mathrm{id}^{\otimes 2} \otimes \mathrm{ad})(S(\delta_1) \otimes \delta_1' \otimes \Gamma_2(\boldsymbol{m})\delta_2' \otimes x_2\delta_2) \otimes 1\Big]\mathcal{D}_{13}^{-2}. \end{aligned}$$

We have

$$\begin{aligned} &(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad})\Big(\sum S(\delta_1) \otimes \delta_1' \otimes x\delta_2' \otimes y\delta_2\Big) \\ &\quad = \sum S(\delta_1) \otimes \delta_1' \otimes (x\delta_2') \triangleright (y\delta_2) \\ &\quad = \sum S(\delta_1) \otimes \delta_1' \otimes x_{(1)}(\delta_2')_{(1)} y\delta_2 S((\delta_2')_{(2)}) S(x_{(2)}) \\ &\quad = \sum S(\delta_1) \otimes K_{-2|y|} \otimes x_{(1)} y\delta_2 S(x_{(2)}) \\ &\quad = \sum K_{2|x_{(2)}|} S(\delta_1) \otimes K_{-2|y|} \otimes x_{(1)} y S(x_{(2)})\delta_2 \\ &\quad = \Big[\sum K_{2|x_{(2)}|} \otimes K_{-2|y|} \otimes x_{(1)} y S(x_{(2)})\Big]\mathcal{D}_{13}^{2} \\ &\quad = \big[(x \blacktriangleright (1 \otimes y))_{13}(1 \otimes K_{-2|y|} \otimes 1)\big]\mathcal{D}_{13}^{2}. \end{aligned}$$

It follows that

$$\begin{aligned} (168) \quad &f_{\boldsymbol{m}}^{\mathrm{ad}}(x_1 \otimes x_2 \otimes x_3) \\ &\quad = \big[(x_1 \otimes \Gamma_1(\boldsymbol{m}) \otimes 1)(\Gamma_2(\boldsymbol{m}) \blacktriangleright (1 \otimes x_2))_{13}(1 \otimes K_{-2|x_2|} \otimes 1)\big] \otimes x_3. \end{aligned}$$

Since $\Gamma_1(\boldsymbol{m})\otimes\Gamma_2(\boldsymbol{m})\in[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 2}]_1$, Lemma 7.17(a) shows that

$$f_{\boldsymbol{m}}^{\mathrm{ad}}(x_1\otimes x_2\otimes x_3)\in[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}]_g,$$

where $g=\dot{x}_1\dot{\Gamma}_1(\boldsymbol{m})\dot{\Gamma}_2(\boldsymbol{m})\dot{x}_2\dot{x}_3=\dot{x}_1\dot{x}_2\dot{x}_3=1$. Thus, the right-hand side of (168) is in $[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}]_1$.

Since $x\otimes y\otimes z\in\mathcal{U}^{\otimes 3}$, Lemma 7.17(b) shows that the right-hand side of (168) is in $\mathcal{U}^{\otimes 4}$.

Since $\Gamma_1(\boldsymbol{m})\otimes\Gamma_2(\boldsymbol{m})\in o(\boldsymbol{m})\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}\otimes\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}}$ by (153), Lemma 7.17(c) shows that

$$f_{\boldsymbol{m}}^{\mathrm{ad}}(x_1\otimes x_2\otimes x_3)\in o(\boldsymbol{m})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}.$$

Hence,

$$f_{\boldsymbol{m}}^{\mathrm{ad}}(x_1\otimes x_2\otimes x_3)\in o(\boldsymbol{m})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}\cap[(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}]_1\cap\mathcal{U}^{\otimes 4}=\mathcal{F}_{\lfloor\max\boldsymbol{m}/2\rfloor}(\mathcal{K}_4(\mathcal{U})),$$

which proves the statement for $f=f_{\boldsymbol{m}}^{\mathrm{ad}}$.

The proof for $f=\tilde{f}_{\boldsymbol{m}}^{\mathrm{ad}}$ is similar: Using the definition and formula (41) for $\underline{\mathrm{ad}}^{\mathrm{r}}$, one gets

$$\tilde{f}_{\boldsymbol{m}}^{\mathrm{ad}}(x_1\otimes x_2\otimes x_3) = (\Gamma_1(\boldsymbol{m})K_{2|x_3|+2\lambda_{\boldsymbol{m}}}\otimes x_1\otimes x_2\otimes 1)\big(S^{-1}(\Gamma_2(\boldsymbol{m}))\blacktriangleright(1\otimes x_3)\big)_{24}. \tag{169}$$

By Lemma 7.17(a), the right-hand side of (169) is in $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}$, with G–grading equal to

$$\dot{\Gamma}_1(\boldsymbol{m})\dot{x}_1\dot{x}_2\dot{x}_3\dot{\Gamma}_2(\boldsymbol{m})=\dot{\Gamma}_1(\boldsymbol{m})\dot{\Gamma}_2(\boldsymbol{m})=1.$$

Again, Lemma 7.17(b) shows that the right-hand side of (169) is in $\mathcal{U}^{\otimes 4}$ and Lemma 7.17(c) shows that it is in $o(\boldsymbol{m})(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes 4}$. Hence, $\tilde{f}_{\boldsymbol{m}}^{\mathrm{ad}}(x_1\otimes x_2\otimes x_3)\in\mathcal{F}_{\lfloor\max\boldsymbol{m}/2\rfloor}\mathcal{K}_4(\mathcal{U})$. □

7K7 The maps f^{μ} and $\tilde{f}^{\mu}$

Lemma 7.21 *Both f^{μ} and $\tilde{f}^{\mu}$ are $(\tilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.*

Proof By definition,

$$\begin{aligned} f^{\mu}(x_1\otimes x_2\otimes x_3\otimes x_4) &= (\mathrm{id}^{\otimes 2}\otimes\boldsymbol{\mu})\Big(\sum x_1\delta_1 S(\delta_1')\otimes x_2\otimes x_3\delta_2'\otimes x_4\delta_2\Big)\\ &= (x_1\otimes x_2\otimes 1)\Big[(\mathrm{id}^{\otimes 2}\otimes\boldsymbol{\mu})\Big(\sum\delta_1 S(\delta_1')\otimes 1\otimes x_3\delta_2'\otimes x_4\delta_2\Big)\Big].\end{aligned}$$

We have

$$(\mathrm{id}\otimes\boldsymbol{\mu})\Big(\sum \delta_1 S(\delta_1')\otimes x\delta_2'\otimes y\delta_2\Big) = \sum \delta_1 S(\delta_1')\otimes x\delta_2' y\delta_2$$
$$= \sum \delta_1 S(\delta_1')K_{2|y|}\otimes xy\delta_2'\delta_2$$
$$= K_{2|y|}\otimes xy.$$

It follows that f^{μ} has the very simple expression

$$f^{\mu}(x_1\otimes x_2\otimes x_3\otimes x_4) = (x_1\otimes x_2\otimes 1)(K_{2|x_4|}\otimes 1\otimes x_3x_4) = x_1K_{2|x_4|}\otimes x_2\otimes x_3x_4.$$

The trivial decomposition for f^{μ} is admissible. Hence, f^{μ} is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible.

Similarly, a simple computation shows that

$$\tilde{f}^{\mu}(x_1\otimes x_2\otimes x_3\otimes x_4) = x_1\otimes x_2K_{2|x_4|}\otimes x_3x_4.$$

The trivial decomposition for $\tilde{f}^{\mu}$ is admissible. Hence, $\tilde{f}^{\mu}$ is $(\widetilde{\mathcal{K}}_n(\mathcal{U}))$–admissible. □

7K8 Proof of Lemma 7.15

Proof First suppose $\max(\boldsymbol{m})\geq\max(\boldsymbol{n})$. By (155)

$$\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}} = f^{\mu}\circ f^{\mathrm{ad}}_{\boldsymbol{m}}\circ f^{\underline{S}}\circ f^{\underline{\Delta}}(\Gamma_1(\boldsymbol{n})\otimes\Gamma_2(\boldsymbol{n})),$$

By (152), $\Gamma_1(\boldsymbol{n})\otimes\Gamma_2(\boldsymbol{n})\in\mathcal{K}_2$. Lemmas 7.18, 7.19, 7.20 and 7.21 show $\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}}\in o(\boldsymbol{m})\widetilde{\mathcal{K}}_3$.

Suppose $\max(\boldsymbol{n})>\max(\boldsymbol{m})$. Using (160) instead of (155), we have $\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}}\in o(\boldsymbol{n})\widetilde{\mathcal{K}}_3$.

Hence $\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}}\in o(\boldsymbol{n},\boldsymbol{m})\widetilde{\mathcal{K}}_3$. As a consequence, $\boldsymbol{b}=\sum\boldsymbol{b}_{\boldsymbol{n},\boldsymbol{m}}\in\widetilde{\mathcal{K}}_3$. □

7L Proof of Proposition 7.2

Proof As noted, statement (i) follows trivially from the definition of $\widetilde{\mathcal{K}}(\mathcal{U})$. Statement (ii) follows from Lemmas 7.10, 7.11, 7.13 and 7.14. Finally, statement (iii) is Lemma 7.15. □

7M Integrality of the quantum link invariant

Lê [47] proved that, for a framed link $L=L_1\cup\dots\cup L_n$ in S^3, the quantum $\mathfrak{g}$ link invariant $J_L(V_{\lambda_1},\dots,V_{\lambda_n})$, up to multiplication by a fractional power of q, is contained in $\mathbb{Z}[q,q^{-1}]$. Here we sketch an alternative proof, using Theorem 7.3 of the following special case for algebraically split framed links.

Theorem 7.22 [47] *Let $L = L_1 \cup \cdots \cup L_n$ be an algebraically split 0–framed link in S^3. Let $\lambda_1, \ldots, \lambda_n \in X_+$ be dominant integral weights. Then we have*

$$J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) \in q^p \mathbb{Z}[q, q^{-1}],$$

where $p = (2\rho, \lambda_1 + \cdots + \lambda_n)$.

It is much easier to prove

$$J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) \in q^p \mathbb{Z}[v, v^{-1}], \tag{170}$$

and the difficult part of the proof is to show that the normalized invariant

$$q^{-p} J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) \in \mathbb{Z}[v, v^{-1}]$$

is contained in $\mathbb{Z}[q, q^{-1}]$. In [47], a result of Andersen [1] on quantum groups at roots of unity is involved in the proof. The main idea of the proof below is, implicitly, the use of the G–grading of the quantum group $\boldsymbol{U}_q$ as $\mathbb{C}(q)$–module described in Section 6.

Sketch proof of Theorem 7.22 Let T be an algebraically split 0–framed bottom tangle such that the closure link of T is L. Recall that the quantum invariant $J_L(V_{\lambda_1}, \ldots, V_{\lambda_n})$ can be defined by using quantum traces:

$$J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) = (\mathrm{tr}_q^{V_{\lambda_1}} \otimes \ldots \otimes \mathrm{tr}_q^{V_{\lambda_n}})(J_T). \tag{171}$$

It is not difficult to prove that, for $1 \le i \le n$, $\lambda \in X_+$, we have

$$(\mathrm{id}^{\otimes i-1} \otimes \mathrm{tr}_q^{V_\lambda} \otimes \mathrm{id}^{\otimes n-i})(\widetilde{\mathcal{K}}_n) \subset q^{(2\rho,\lambda)} \widetilde{\mathcal{K}}_{n-1}. \tag{172}$$

Using (172), one can prove that

$$(\mathrm{tr}_q^{V_{\lambda_1}} \otimes \ldots \otimes \mathrm{tr}_q^{V_{\lambda_n}})(\widetilde{\mathcal{K}}_n) \subset q^p \widetilde{\mathcal{K}}_0 = q^p \widehat{\mathbb{Z}[q]}. \tag{173}$$

Hence, using (171), (173) and Theorem 7.3(a), we have

$$J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) \in (\mathrm{tr}_q^{V_{\lambda_1}} \otimes \ldots \otimes \mathrm{tr}_q^{V_{\lambda_n}})(\widetilde{\mathcal{K}}_n) \subset q^p \widehat{\mathbb{Z}[q]},$$

which, combined with (170), yields $J_L(V_{\lambda_1}, \ldots, V_{\lambda_n}) \in q^p \mathbb{Z}[q, q^{-1}]$, since we have $\mathbb{Z}[v, v^{-1}] \cap \widehat{\mathbb{Z}[q]} = \mathbb{Z}[q, q^{-1}]$. □

8 Recovering the Witten–Reshetikhin–Turaev invariant

In Section 7 we showed that $J_M \in \widehat{\mathbb{Z}[q]}$, where J_M is the invariant (associated to a simple Lie algebra $\mathfrak{g}$) of an integral homology 3–sphere M. Hence we can evaluate J_M at any root of unity. Here we show that by evaluating J_M at a root of unity we recover the Witten–Reshetikhin–Turaev invariant. We also prove Theorem 1.1 and Proposition 1.6 of the introduction.

8A Introduction

Recall that $\mathfrak{g}$ is a simple Lie algebra and $\mathcal{Z}$ is the set of all roots of unity. Suppose $\zeta \in \mathcal{Z}$ and M is closed oriented 3–manifold. Traditionally the Witten–Reshetikhin–Turaev (WRT) invariant (see [73; 3]) $\tau^{\mathfrak{g}}_M(\xi;\zeta) \in \mathbb{C}$ is defined when ζ is a root of unity of order $2Ddk$ with $k > h^\vee$, where $h^\vee$ is the dual Coxeter number, $d \in \{1,2,3\}$ is defined as in Section 3A, and $D = |X/Y|$. Here $\xi = \zeta^{2D}$. In this case, $k - h^\vee$ is called the level of the theory. The definition of $\tau^{\mathfrak{g}}_M(\xi;\zeta)$ can be extended to a bigger set $\mathcal{Z}'_{\mathfrak{g}}$ that contains all roots of unity of order divisible by $2dD$; see Section 8D. For values of d, D and $h^\vee$ of simple Lie algebras, see Table 1 in Section 3A.

This section is devoted to the proofs of the following theorem and its generalizations:

Theorem 8.1 *Suppose M is an integral homology 3–sphere and $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$. Then*

$$\tau^{\mathfrak{g}}_M(\xi;\zeta) = J_M\big|_{q=\xi}.$$

Remarks 8.2 (a) Although ξ is determined by ζ, we use the notation $\tau_M(\xi;\zeta)$ since, in many cases, $\tau^{\mathfrak{g}}_M(\xi;\zeta)$ depends only on ξ, but not a $2D^{\text{th}}$ root ζ of ξ. In that case, we write $\tau_M(\xi)$ instead of $\tau_M(\xi;\zeta)$. The set $\mathcal{Z}_{\mathfrak{g}}$ in Section 1 is defined by $\mathcal{Z}_{\mathfrak{g}} = \{\zeta^{2D} \mid \zeta \in \mathcal{Z}'_{\mathfrak{g}}\}$.

(b) The theorem implies that for an *integral homology 3–sphere*, $\tau^{\mathfrak{g}}_M(\xi;\zeta)$ depends only on ξ, but not a $2D^{\text{th}}$ root ζ of ξ. This does not hold true for general 3–manifolds.

In Sections 8C and 8D we recall the definition of the WRT invariant and define the set $\mathcal{Z}'_{\mathfrak{g}}$. Section 8F contains the proof of a stronger version of Theorem 8.1, based on results proved in later subsections. To prove the main results we introduce an integral form $\mathcal{U}$ of $\boldsymbol{U}_q$ which is sandwiched between Lusztig's integral form $\boldsymbol{U}_{\mathbb{Z}}$ and De Concini and Procesi's integral form $\boldsymbol{V}_{\mathbb{Z}}$. For $\mathfrak{g} = \mathrm{sl}_2$, the algebra $\mathcal{U}$ was considered by Habiro [25; 26]. A large part of the proof is devoted to the determination of the center of a certain completion of $\mathcal{U}$. For this part we use, among other things, integral bases of $\boldsymbol{U}_{\mathbb{Z}}$–modules, the quantum Harish-Chandra isomorphism, and Chevalley's theorem in invariant theory. In Section 8M, we give a geometric interpretation of Drinfel'd's construction of central elements.

8B Finite-rank U_h–modules

Suppose V is a topologically free U_h–module. For $\mu \in X$ the *weight-μ subspace* of V is defined by

$$V_{[\mu]} = \{e \in V \mid H_\alpha(e) = (\alpha, \mu)e \text{ for all } \alpha \in \Pi\},$$

and $\mu \in X$ is called a *weight of* V if $V_{[\mu]} \neq 0$. We call V a *highest weight module* if V is generated by a nonzero element $1_\mu \in V_{[\mu]}$ for some $\mu \in X$ such that $E_\alpha 1_\mu = 0$ for $\alpha \in \Pi$. Then 1_μ is called a *highest weight vector* of V, and μ the *highest weight*.

By *a finite-rank U_h–module*, we mean a U_h–module which is (topologically) free of finite rank as a $\mathbb{C}[\![h]\!]$–module. The theory of finite-rank U_h–modules is well known and is parallel to that of finite-dimensional $\mathfrak{g}$–modules; see eg [12; 30; 54]: every finite-rank U_h–module is the direct sum of irreducible finite-rank U_h–modules. For every dominant integral weight $\lambda \in X_+ := \left\{\sum_{\alpha\in\Pi} k_\alpha \breve{\alpha} \mid k_\alpha \in \mathbb{N}\right\}$, there exists a unique finite-rank irreducible U_h–module with highest weight λ, and every finite-rank irreducible U_h–module is one of V_λ. The Grothendieck ring of finite-rank U_h–modules is naturally isomorphic to that of finite-dimensional $\mathfrak{g}$–modules.

8C Link invariants and symmetries at roots of unity

8C1 Invariants of colored links Suppose L is the closure link of a framed bottom tangle T, with m components. Let $V_1, \dots, V_m$ be finite-rank U_h–modules. Recall that the quantum link invariant [72] can be defined by

$$J_L(V_1, \dots, V_m) = (\mathrm{tr}_q^{V_1} \otimes \cdots \otimes \mathrm{tr}_q^{V_m})(J_T) \in \mathbb{C}[\![h]\!].$$

Actually, $J_L(V_1, \dots, V_m)$ belongs to a subring $\mathbb{Z}[v^{\pm 1/D}]$ of $\mathbb{C}[\![h]\!]$, where $D = |X/Y|$; see [47]. (D is also equal to the determinant of the Cartan matrix.) We say that V_j is the color of the j^{th} component, and consider $J_L(V_1, \dots, V_m)$ as an invariant of colored links, which is a generalization of the famous Jones polynomial [31].

Let U be the trivial knot with 0 framing. For a finite-rank U_h–modules V, $\dim_q(V) := J_U(V)$ is called as the *quantum dimension* of V. It is known that, for $\lambda \in X_+$,

$$\dim_q(V_\lambda) = \frac{\sum_{w\in\mathfrak{W}} \mathrm{sgn}(w) v^{-(2(\lambda+\rho), w(\rho))}}{\sum_{w\in\mathfrak{W}} \mathrm{sgn}(w) v^{-(2(\rho), w(\rho))}} = q^{-(\lambda,\rho)} \prod_{\alpha\in\Phi_+} \frac{q^{(\lambda+\rho,\alpha)} - 1}{q^{(\rho,\alpha)} - 1}. \tag{174}$$

Here $\mathfrak{W}$ is the Weyl group and $\mathrm{sgn}(w)$ is the sign of w as a linear transformation.

One has $\max_{\alpha\in\Phi_+}(\rho, \alpha) = d(h^\vee - 1)$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Hence, if ξ is a root of unity with

$$\mathrm{ord}(\xi) > d(h^\vee - 1), \tag{175}$$

then the denominator of the right-hand side of (174) is not 0 under the evaluation $q=\xi$. For this reason we often make the assumption (175).

8C2 Evaluation at a root of unity Throughout we fix a root of unity $\zeta \in \mathbb{C}$. Let $\xi = \zeta^{2D}$ and $r = \mathrm{ord}(\zeta^{2D})$.

For $f \in \mathbb{C}[v^{\pm 1/D}]$ let $\mathrm{ev}_{v^{1/D}=\zeta}(f)$ be the value of f at $v^{1/D}=\zeta$. Note that if $v^{1/2D}=\zeta$ then $q=\xi$. If $f \in \mathbb{C}[q^{\pm 1}]$ then $\mathrm{ev}_{v^{1/D}=\zeta}(f)$ is the value of f at $q=\xi$.

Suppose $f, g \in \mathbb{C}[v^{\pm 1/D}]$. If $\mathrm{ev}_{v^{1/D}=\zeta}(f) = \mathrm{ev}_{v^{1/D}=\zeta}(g)$, then we say $f=g$ at ζ and write

$$f =_{(\zeta)} g.$$

We say that $\mu \in X$ is a *ζ–period* if for every link L, $\mathrm{ev}_{v^{1/D}=\zeta}(J_L)$ does not change when the color of a component changes from V_λ to $V_{\lambda+\mu}$ for arbitrary $\lambda \in X_+$ such that $\lambda+\mu \in X_+$ (the colors of other components remain unchanged).

The set of all ζ–periods is a subgroup of X. It turns out that if $\mathrm{ord}(\xi) > d(h^\vee - 1)$, then the group of ζ–periods has finite index in X; in [47] it was proved that the group of ζ–periods contains $2rY$, which, in turn, contains $(2rD)X$ (because $DX \subset Y$).

When $\mathrm{ord}(\xi) \le d(h^\vee - 1)$, the behavior of $\mathrm{ev}_{v^{1/D}=\zeta}(J_L)$ is quite different. For example, when $\zeta = 1$, from (174) and the Weyl dimension formula, one can see that $\dim_q(V_\lambda)$ is the dimension of the classical $\mathfrak{g}$–module of highest weight λ. When $\zeta=1$, the action of the ribbon element on any V_λ is the identity, and the braiding action $\boldsymbol{\psi}$ is trivial on any pair of $\boldsymbol{U}_q$–modules. Hence, we have the following:

Proposition 8.3 *For any framed oriented link L with m ordered components and $\mu_1, \dots, \mu_m \in X_+$,*

$$\mathrm{ev}_{v^{1/D}=1}(J_L(V_{\mu_1}, \dots, V_{\mu_m})) = \prod_{j=1}^m \dim(V_{\mu_j}).$$

(Here $\dim(V_{\mu_j})$ is the dimension of the irreducible $\mathfrak{g}$–module with highest weight μ_j.)

8D The WRT invariant of 3–manifolds

Here we recall the definition of the WRT invariant.

8D1 3–manifolds and Kirby moves Suppose L is a framed link in the standard 3–sphere S^3. Surgery along L yields an oriented 3–manifold $M = M(L)$. Surgeries along two framed links L and L' give the same 3–manifold if and only L and L' are related by a finite sequence of Kirby moves, ie handle slide moves and stabilization moves; see eg [39; 40]. If one can find an invariant of unoriented framed links which is invariant under the two Kirby moves, then the link invariant descends to an invariant of 3–manifolds.

8D2 Kirby color Let $\mathcal{B} := \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[v^{\pm 1}]$. We call any $\mathcal{B}$–linear combination of V_λ, $\lambda \in X_+$, *a color*. By linear extension we can define $J_L(V_1, \dots, V_m) \in \mathbb{C}[v^{\pm 1/D}]$ when each V_j is a color.

A color Ω is called a *handle-slide color at level* $v^{1/D} = \zeta$ if

(i) $\mathrm{ev}_{v^{1/D}=\zeta}(J_L(\Omega, \dots, \Omega))$ is an invariant of nonoriented links, and

(ii) $\mathrm{ev}_{v^{1/D}=\zeta}(J_L(\Omega, \dots, \Omega))$ is invariant under the handle slide move.

Let $U_\pm$ be the unknot with framing ± 1. A handle-slide color is called *a Kirby color* (at level $v^{1/D} = \zeta$) if it satisfies the *nondegeneracy condition*

$$J_{U_\pm}(\Omega) \neq_{(\zeta)} 0. \tag{176}$$

Suppose Ω is a Kirby color at level $v^{1/D} = \zeta$, and $M = M(L)$ is the 3–manifold obtained by surgery on S^3 along a framed link L. Then

$$\tau_M(\Omega) := \mathrm{ev}_\zeta\left(\frac{J_L(\Omega, \dots, \Omega)}{(J_{U_\pm}(\Omega))^{\sigma_+}(J_{U_\pm}(\Omega))^{\sigma_-}}\right) \tag{177}$$

is invariant under both Kirby moves, and hence defines an invariant of M. Here σ_+ (resp. σ_-) is the number of positive (resp. negative) eigenvalues of the linking matrix of L.

8D3 Strong Kirby color All the known Kirby colors satisfy a stronger condition on the invariance under the handle slide move, as described below.

A *root color* is any $\mathcal{B}$–linear combinations of V_λ with $\lambda \in Y \cap X_+$. A handle-slide color Ω at level $v^{1/D} = \zeta$ is a *strong handle-slide color* if it satisfies the following: Suppose the first component of L_1 is colored by Ω and other components are colored by *arbitrary root colors* $V_1, \dots, V_m$. Then a handle slide of any other component over the first component does not change the value of the quantum link invariant, evaluated at $v^{1/D} = \zeta$, ie if L_2 is the resulting link after the handle slide then

$$J_{L_1}(\Omega, V_1, \dots, V_m) =_{(\zeta)} J_{L_2}(\Omega, V_1, \dots, V_m). \tag{178}$$

A nondegenerate strong handle-slide color is called a *strong Kirby color*.

8D4 Strong Kirby color exists Let P_ζ be the following half-open parallelepiped, which is a domain of translations of X by elements of the lattice $(2rD)X$,

$$P_\zeta := \left\{ \lambda = \sum_{i=1}^{l} k_i \breve{\alpha}_i \in X_+ \;\middle|\; 0 \le k_i < 2rD \right\}.$$

Let

$$\Omega^{\mathfrak{g}}(\zeta) := \sum_{\lambda \in P_\zeta} \dim_q(V_\lambda) V_\lambda, \quad \Omega^{P\mathfrak{g}}(\zeta) = \sum_{\lambda \in P_\zeta \cap Y} \dim_q(V_\lambda) V_\lambda.$$

In [49], it was proved that both $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are handle-slide colors at level $v^{1/D} = \zeta$ if $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$. Actually, the proof there shows that $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are strong handle-slide colors at level $v^{1/D} = \zeta$. Hence, assuming $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$, $\Omega^{\mathfrak{g}}(\zeta)$ (resp. $\Omega^{P\mathfrak{g}}(\zeta)$) is a strong Kirby color at $v^{1/D} = \zeta$ if and only $\Omega^{\mathfrak{g}}(\zeta)$ (resp. $\Omega^{P\mathfrak{g}}(\zeta)$) is nondegenerate at $v^{1/D} = \zeta$. There are many cases of $v^{1/D} = \zeta$ when both $\operatorname{ord}(\zeta^{2D})$ and $\Omega^{\mathfrak{g}}(\zeta)$ are strong Kirby colors, and there are many cases when one of them is not. Let $\mathcal{Z}'_{\mathfrak{g}}$ (resp. $\mathcal{Z}'_{P\mathfrak{g}}$) be the set of all roots of unity ζ such that $\Omega^{\mathfrak{g}}(\zeta)$ (resp. $\Omega^{P\mathfrak{g}}(\zeta)$) is a strong Kirby color.

For $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$ the $\mathfrak{g}$ WRT invariant of an oriented closed 3–manifold M is defined by

$$\tau^{\mathfrak{g}}_M(\xi;\zeta) = \tau_M(\Omega^{\mathfrak{g}}(\zeta)).$$

Similarly, for $\zeta \in \mathcal{Z}'_{P\mathfrak{g}}$ the $P\mathfrak{g}$ WRT invariant of an oriented closed 3–manifold M is defined by

$$\tau^{P\mathfrak{g}}_M(\xi;\zeta) = \tau_M(\Omega^{P\mathfrak{g}}(\zeta)).$$

Proposition 8.4 *Suppose ζ is a root of unity with $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$. Then $\zeta \in \mathcal{Z}'_{\mathfrak{g}} \cup \mathcal{Z}'_{P\mathfrak{g}}$. More specifically, if $\operatorname{ord}(\zeta^{2D})$ is odd then $\zeta \in \mathcal{Z}'_{P\mathfrak{g}}$ and if $\operatorname{ord}(\zeta^{2D})$ is even then $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$.*

We will give a proof of the proposition in Appendix C.3. Actually, in the appendix we will describe precisely the sets $\mathcal{Z}'_{\mathfrak{g}}$ and $\mathcal{Z}'_{P\mathfrak{g}}$ (for $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$).

The proposition shows that $\mathcal{Z}'_{\mathfrak{g}} \cup \mathcal{Z}'_{P\mathfrak{g}}$ is all $\mathcal{Z}$ except for a finite number of elements. This means that $\tau^{\mathfrak{g}}_M(\xi;\zeta)$ or $\tau^{P\mathfrak{g}}_M(\xi;\zeta)$ can always be defined at all but a finite number of ζ.

Remarks 8.5 (1) If $\operatorname{ord}(\zeta)$ is divisible by $2dD$, the proposition had been well known, since in this case a modular category, and hence a topological quantum field theory (TQFT), can be constructed; see eg [3]. The rigorous construction of the WRT invariant and the corresponding TQFT was first given by Reshetikhin and Turaev [73] for $\mathfrak{g} = \mathrm{sl}_2$.

The construction of TQFT for higher rank Lie algebras (see eg [3; 82]) uses Andersen's theory of tilting modules [2]. In [49], the WRT invariant was constructed without TQFT (and no tilting modules theory). Here we are interested only in the invariants of 3–manifolds, but not the stronger structure TQFT. We don't know if a modular category — the basis ground of a TQFT — can be constructed for every root ζ of unity with $\mathrm{ord}(\zeta^{2D}) > d(h^\vee - 1)$. At least for $\mathfrak{g} = \mathrm{sl}_n$, if the order of ζ is $2 \pmod 4$ and n is even, then according to [10], the corresponding premodular category is not modularizable.

(2) In general, different strong Kirby colors give different 3–manifold invariants. The invariant corresponding to $\Omega^{P\mathfrak{g}}$, called the projective version of the WRT invariant, was first defined in [40] for $\mathfrak{g} = \mathrm{sl}_2$, then in [42] for sl_n, and then in [49] for general Lie algebras. When both $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are nondegenerate, the relation between the two invariants $\tau_M(\Omega^{\mathfrak{g}})$ and $\tau_M(\Omega^{P\mathfrak{g}})$ is simple if $\mathrm{ord}(\zeta^{2D})$ is coprime with dD, but in general the relation is more complicated; see [49].

(3) It is clear that in the definition of $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$, instead of P_ζ one can take any fundamental domain of any group of ζ–periods which has finite index in Y.

8D5 Dependence on $\xi = \zeta^{2D}$ When components of a framed link L are colored by $\Omega^{P\mathfrak{g}}(\zeta)$, J_L takes values in $\mathbb{C}[q^{\pm 1}] \subset \mathbb{C}[q^{\pm 1/2D}]$; see [49]. Hence, the $P\mathfrak{g}$–WRT invariants $\tau_M^{P\mathfrak{g}}(\xi;\zeta)$, if defined, depend only on $\xi = \zeta^{2D}$, but not on any choice of a $2D^{\text{th}}$ root ζ of ξ.

The $\mathfrak{g}$–WRT invariant $\tau_M^{\mathfrak{g}}(\xi;\zeta)$ does depend on a choice of a $2D^{\text{th}}$ root ζ of ξ, even in the case $\mathfrak{g} = \mathrm{sl}_2$. We will see that when M is an integral homology 3–sphere, the $\mathfrak{g}$–WRT invariant of M depends only on $\xi = \zeta^{2D}$, but not on any choice of a $2D^{\text{th}}$ root ζ of ξ. However, there are cases when $\zeta^{2D} = \xi = (\zeta')^{2D}$, but $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$ and $\zeta' \notin \mathcal{Z}'_{\mathfrak{g}}$. For example, suppose $\mathfrak{g} = \mathrm{sl}_2$ and $\xi = \exp(2\pi/(2k+1))$, a root of unity of odd order. Then $\zeta = \exp(2\pi/(8k+4))$ and $\zeta' = i\zeta$ are both 4^{th} roots of ξ (in this case $2D = 4$). But $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$ and $\zeta' \notin \mathcal{Z}'_{\mathfrak{g}}$.

8D6 Trivial color at $\zeta = 1$ and the case when $\mathrm{ord}(\zeta) \le d(h^\vee - 1)$

Proposition 8.6 *Let $\Omega = \mathbb{C}[\![h]\!]$ be the trivial U_h–module. Then Ω is a strong Kirby color at level $\zeta = 1$ and $\tau_M(\Omega) = 1$.*

This follows immediately from Proposition 8.3 and the defining formula (177) of $\tau_M(\Omega)$.

It is not true that the trivial color is a strong Kirby color for all ζ with $\mathrm{ord}(\zeta^{2D}) \le d(h^\vee - 1)$. For example, if $\mathfrak{g} = \mathrm{sl}_6$ and $\mathrm{ord}(\zeta^{2D}) = 4$, then the trivial color is not a strong Kirby color. One can prove that if $n \equiv 0, \pm 1 \pmod r$, then the trivial color is a strong Kirby color for sl_n at level ζ with $r = \mathrm{ord}(\zeta^{2D})$.

Remark 8.7 If $\mathrm{ord}(\zeta) = 2dDk$, then the level of the corresponding TQFT is $k - h^\vee$. Hence, if the level is nonnegative, as assumed by physics, we automatically have $\mathrm{ord}(\zeta^{2D}) > d(h^\vee - 1)$.

8E Stronger version of Theorem 8.1

Proposition 8.4 shows that strong Kirby colors exist at every level ζ if the order of ζ is big enough. Although different Kirby colors at level ζ might define different 3–manifold invariants, we have the following result for *integral homology* 3*–spheres*, which is more general than Theorem 8.1.

Theorem 8.8 *Suppose Ω is a strong Kirby color at level $v^{1/D} = \zeta$ and M is an integral homology 3–sphere. Then*

$$\tau_M(\Omega) = \mathrm{ev}_{v^{1/D}=\zeta}(J_M) = \mathrm{ev}_{q=\xi}(J_M).$$

Remark 8.9 There is no restriction on the order of ζ in the right-hand side of this equation. We do not know how to directly define the WRT invariant with $\mathrm{ord}(\zeta^{2D}) \le d(h^\vee - 1)$.

The remaining part of this section is devoted to a proof of this theorem. Throughout we fix a root of unity ζ and a strong Kirby color Ω at level ζ. Let $\xi = \zeta^{2D}$ and $r = \mathrm{ord}(\xi)$.

8F Reduction of Theorem 8.8 to Proposition 8.10

Here we reduce Theorem 8.8 to Proposition 8.10, which will be proved later.

8F1 Twisted colors $\Omega_\pm$ Suppose the j^{th} component of a link L is colored by $V = V_\lambda$, and L' is obtained from L by increasing the framing of the j^{th} component by 1; then it is known that

$$(179)\quad J_{L'}(\dots, V, \dots) = \mathfrak{f}_\lambda J_L(\dots, V, \dots), \quad \text{where } \mathfrak{f}_\lambda = q^{(\lambda,\lambda+2\rho)/2} = \frac{\mathrm{tr}_q^V(\boldsymbol{r}^{-1})}{\dim_q V}.$$

For example, if $U_\pm$ is the unknot with framing ± 1, then

$$J_{U_\pm}(V_\lambda) = \mathfrak{f}_\lambda^{\pm 1} \dim_q(V_\lambda) = J_U(\mathfrak{f}_\lambda^{\pm 1} V_\lambda).$$

By definition Ω is a finite sum $\Omega = \sum c_\lambda V_\lambda$, where $c_\lambda \in \mathcal{B} = \mathbb{C}[v^{\pm 1}]$. Define the pair $\Omega_\pm$ of $\mathbb{C}$–linear combinations of finite-rank irreducible $\boldsymbol{U}_h$–modules by

$$\Omega_\pm = \sum \frac{\mathrm{ev}_{v^{1/D}=\zeta}(c_\lambda \mathfrak{f}_\lambda^{\pm 1})}{\mathrm{ev}_{v^{1/D}=\zeta}(J_{U_\pm}(\Omega))} V_\lambda.$$

Suppose a distinguished component of L has framing $\varepsilon = \pm 1$ and color Ω, and L' is the same link with the distinguished component having framing 0 and color Ω_ε. Then from (179) and the definition of Ω_ε one has

$$J_L(\dots,\Omega,\dots) =_{(\zeta)} J_{U_\varepsilon}(\Omega) J_{L'}(\dots,\Omega_\varepsilon,\dots). \tag{180}$$

8F2 Reduction of Theorem 8.8 Here we reduce Theorem 8.8 to the following:

Proposition 8.10 *Let Ω be a strong Kirby color. Suppose T is an algebraically split 0–framed bottom tangle T with m ordered components and $(\varepsilon_1,\dots,\varepsilon_m) \in \{\pm 1\}^m$. Then*

$$(\mathrm{tr}_q^{\Omega_{\varepsilon_1}} \otimes \dots \otimes \mathrm{tr}_q^{\Omega_{\varepsilon_m}})(J_T) =_{(\zeta)} (\mathcal{T}_{\varepsilon_1} \otimes \dots \otimes \mathcal{T}_{\varepsilon_m})(J_T).$$

Proof of Theorem 8.8 assuming Proposition 8.10 Suppose T is an m–component bottom tangle, $\varepsilon_1,\dots,\varepsilon_m \in \{\pm 1\}$ and $M = M(T,\varepsilon_1,\dots,\varepsilon_m)$. This means that, if L is the closure link of T and L' is the same L with framing of the i^{th} component switched to ε_i, then M is obtained from S^3 by surgery along L'. Every integral homology 3–sphere can be obtained in this way. By construction,

$$J_M = (\mathcal{T}_{\varepsilon_1} \otimes \dots \otimes \mathcal{T}_{\varepsilon_m})(J_T).$$

From (180) and the definition (177) of $\tau_M(\Omega)$, we have

$$\tau_M(\Omega) = \mathrm{ev}_\zeta((\mathrm{tr}_q^{\Omega_{\varepsilon_1}} \otimes \dots \otimes \mathrm{tr}_q^{\Omega_{\varepsilon_m}})(J_T)).$$

By Proposition 8.10, we have $\tau_M(\Omega) = \mathrm{ev}_{v^{1/D}=\zeta}(J_M)$. This proves Theorem 8.8. □

The rest of this section is devoted to the proof of Proposition 8.10.

8G Integral form $\mathcal{U}$ of U_q

Besides the integral form $\boldsymbol{U}_\mathbb{Z}$ (of Lusztig) and $\boldsymbol{V}_\mathbb{Z}$ (of De Concini and Procesi), we need another integral form $\mathcal{U}$ of $\boldsymbol{U}_q$, with $\boldsymbol{V}_\mathbb{Z} \subset \mathcal{U} \subset \boldsymbol{U}_\mathbb{Z}$. Let

$$\mathcal{U} := \boldsymbol{U}_\mathbb{Z}^- \boldsymbol{V}_\mathbb{Z} = \boldsymbol{U}_\mathbb{Z}^- \boldsymbol{V}_\mathbb{Z}^0 \boldsymbol{V}_\mathbb{Z}^+ = \boldsymbol{U}_\mathbb{Z}^{\mathrm{ev},-} \boldsymbol{V}_\mathbb{Z}^0 \boldsymbol{V}_\mathbb{Z}^+$$

and

$$\mathcal{U}^{\mathrm{ev}} := \mathcal{U} \cap \mathcal{U}_\mathbb{Z}^{\mathrm{ev}} = \boldsymbol{U}_\mathbb{Z}^{\mathrm{ev},-} \boldsymbol{V}_\mathbb{Z}^{\mathrm{ev},0} \boldsymbol{V}_\mathbb{Z}^+.$$

Theorem 8.11 (a) *The $\mathcal{A}$–module $\mathcal{U}$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_\mathbb{Z}$.*

(b) *Each of $\mathcal{U}$ and $\mathcal{U}^{\mathrm{ev}}$ is stable under ι_{bar} and τ.*

(c) *There are even triangular decompositions*

$$\boldsymbol{U}_{\mathbb{Z}}^{ev,-} \otimes \boldsymbol{V}_{\mathbb{Z}}^{0} \otimes \boldsymbol{V}_{\mathbb{Z}}^{+} \xrightarrow{\cong} \mathcal{U}, \qquad x \otimes y \otimes z \mapsto xyz$$
$$\boldsymbol{U}_{\mathbb{Z}}^{ev,-} \otimes \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} \otimes \boldsymbol{V}_{\mathbb{Z}}^{+} \xrightarrow{\cong} \mathcal{U}^{\mathrm{ev}}, \quad x \otimes y \otimes z \mapsto xyz.$$

(d) *For any longest reduced sequence, the sets*

$$\{F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_\gamma E_{\boldsymbol{n}} \mid \boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^t,\ \gamma \in Y\}, \quad \{F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_{2\gamma} E_{\boldsymbol{n}} \mid \boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^t,\ \gamma \in Y\}$$

are $\mathcal{A}$–bases of $\mathcal{U}$ and $\mathcal{U}^{\mathrm{ev}}$, respectively.

(e) *The Hopf algebra $\mathcal{U}$ satisfies the assumptions of Theorem 7.3, ie $K_\alpha^{\pm 1} \in \mathcal{U}$ for $\alpha \in \Pi$ and $F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}}$, $F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}} \in \mathcal{U} \otimes \mathcal{U}$ for $\boldsymbol{n} \in \mathbb{N}^t$.*

(f) *One has $\mathcal{T}_\pm(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A} = \mathbb{Z}[v, v^{-1}]$.*

(g) *For any $n \geq 0$, one has $(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap \mathcal{U}^{\otimes n} = (\mathcal{U}^{\mathrm{ev}})^{\otimes n}$.*

Proof (a) We have the following statement, whose easy proof is dropped:

Claim *If $\mathscr{H}_1$ and $\mathscr{H}_2$ are $\mathcal{A}$–Hopf subalgebras of a Hopf algebra $\mathscr{H}$ such that $\mathscr{H}_2\mathscr{H}_1 \subset \mathscr{H}_1\mathscr{H}_2$, then $\mathscr{H}_1\mathscr{H}_2$ is an $\mathcal{A}$–Hopf subalgebra of $\mathscr{H}$.*

We will apply the claim to $\mathscr{H}_1 = \boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}}^{0}$ and $\mathscr{H}_2 = \boldsymbol{V}_{\mathbb{Z}}$. By checking the explicit formulas of the coproducts and the antipodes of $F_\alpha^{(n)}$ and K_α for $\alpha \in \Pi$, $n \in \mathbb{N}$, which generate the $\mathcal{A}$–algebra $\mathscr{H}_1 = \boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}}^{0}$, we see that $\mathscr{H}_1$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$. Since $\mathscr{H}_2$ is also an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$, it remains to show $\mathscr{H}_2\mathscr{H}_1 \subset \mathscr{H}_1\mathscr{H}_2$.

Given x and y in any Hopf algebra, we have $xy = \sum y_{(2)}(S^{-1}(y_{(1)}) \triangleright x)$. Hence, since $\mathscr{H}_1$ is a Hopf algebra, and $\mathscr{H}_1 \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}$ (Theorem 5.18),

$$\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}\mathscr{H}_1 \subset \mathscr{H}_1(\mathscr{H}_1 \triangleright \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}) \subset \mathscr{H}_1\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}. \tag{181}$$

Because $\boldsymbol{V}_{\mathbb{Z}} = \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}\boldsymbol{V}_{\mathbb{Z}}^{0}$ and $\boldsymbol{V}_{\mathbb{Z}}^{0}\mathscr{H}_1 = \boldsymbol{V}_{\mathbb{Z}}^{0}\boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}}^{0} = \boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}}^{0} = \mathscr{H}_1$, we have

$$\mathscr{H}_2\mathscr{H}_1 = \boldsymbol{V}_{\mathbb{Z}}\mathscr{H}_1 = \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}\boldsymbol{V}_{\mathbb{Z}}^{0}\mathscr{H}_1 = \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}\mathscr{H}_1 \subset \mathscr{H}_1\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \mathscr{H}_1\mathscr{H}_2,$$

where we used (181). By the above claim, $\mathscr{H}_1\mathscr{H}_2$ is an $\mathcal{A}$–Hopf subalgebra of $\boldsymbol{U}_{\mathbb{Z}}$.

(b) Let $f = \iota_{\mathrm{bar}}$ or $f = \tau$. By Propositions 5.2 and 5.13, $f(\boldsymbol{U}_{\mathbb{Z}}^{-}) = \boldsymbol{U}_{\mathbb{Z}}^{-} \subset \boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}} = \mathcal{U}$ and $f(\boldsymbol{V}_{\mathbb{Z}}) = \boldsymbol{V}_{\mathbb{Z}} \subset \boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}} = \mathcal{U}$. Hence $f(\mathcal{U}) = f(\boldsymbol{U}_{\mathbb{Z}}^{-}\boldsymbol{V}_{\mathbb{Z}}) \subset \mathcal{U}$.

By Lemma 3.4, $f(\boldsymbol{U}_q^{\mathrm{ev}}) \subset \boldsymbol{U}_q^{\mathrm{ev}}$. Hence

$$f(\mathcal{U}^{\mathrm{ev}}) = f(\mathcal{U} \cap \boldsymbol{U}_q^{\mathrm{ev}}) \subset f(\mathcal{U}) \cap f(\boldsymbol{U}_q^{\mathrm{ev}}) \subset \mathcal{U} \cap \boldsymbol{U}_q^{\mathrm{ev}} = \mathcal{U}^{\mathrm{ev}}.$$

(c) The even triangular decompositions of $\boldsymbol{U}_{\mathbb{Z}}$ (see Section 5B) imply the even triangular decompositions of $\mathcal{U}$.

(d) Since $F_{\boldsymbol{m}} \sim F^{(\boldsymbol{m})}$ and $E_{\boldsymbol{n}} \sim (q;q)_{\boldsymbol{n}} E^{(\boldsymbol{n})}$, where $a \sim b$ means $a = ub$ with u a unit in $\mathcal{A}$, Propositions 5.3 and 5.5 show that $\{F_{\boldsymbol{m}} K_{\boldsymbol{m}}\}$ and $\{E_{\boldsymbol{n}}\}$ are $\mathcal{A}$–bases of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},-}$ and $\boldsymbol{V}_{\mathbb{Z}}^{+}$, respectively. It is clear that $\{K_\gamma \mid \gamma \in Y\}$ and $\{K_{2\gamma} \mid \gamma \in Y\}$ are $\mathcal{A}$–bases of $\boldsymbol{V}_{\mathbb{Z}}^{0}$ and $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$, respectively. Combining these bases using the even triangular decompositions, we get the desired bases of $\mathcal{U}$ and $\mathcal{U}^{\mathrm{ev}}$.

(e) Since $K_\alpha^{\pm 1}$, $F_{\boldsymbol{n}}$ and $E_{\boldsymbol{m}}$ are among the basis elements described in (d), we have $K_\alpha^{\pm 1} \in \mathcal{U}$ and $F_{\boldsymbol{n}} \otimes E_{\boldsymbol{n}} \in \mathcal{U} \otimes \mathcal{U}$. Since $\mathcal{U}$ is stable under ι_{bar}, and $F'_{\boldsymbol{n}} = \iota_{\mathrm{bar}}(F_{\boldsymbol{n}})$ and $E'_{\boldsymbol{m}} = \iota_{\mathrm{bar}}(E_{\boldsymbol{m}})$, we also have $F'_{\boldsymbol{n}} \otimes E'_{\boldsymbol{n}} \in \mathcal{U} \otimes \mathcal{U}$.

(f) Applying $\mathcal{T}_+$ to a basis element of $\mathcal{U}^{\mathrm{ev}}$ in (d), using (96) and (97),

$$\mathcal{T}_+(F_{\boldsymbol{m}} K_{\boldsymbol{m}} K_{2\gamma} E_{\boldsymbol{n}}) = \delta_{\boldsymbol{n},\boldsymbol{m}} q^{(\rho,|E_{\boldsymbol{n}}|)} q^{(\gamma,\rho)-(\gamma,\gamma)/2} \in \mathbb{Z}[q^{\pm 1}] \subset \mathcal{A}. \tag{182}$$

It follows that $\mathcal{T}_+(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A}$.

Let us now show $\mathcal{T}_-(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A}$. By [30, Section 6.20], for any $x, y \in \boldsymbol{U}_q$, one has

$$\langle \omega S(x), \omega S(y) \rangle = \langle y, x \rangle.$$

Because $\omega S(\boldsymbol{r}^{-1}) = \boldsymbol{r}^{-1}$, and $\langle x, \boldsymbol{r}^{-1} \rangle = \langle \boldsymbol{r}^{-1}, x \rangle = \mathcal{T}_-(x)$ by (94), we have

$$\mathcal{T}_-(x) = \mathcal{T}_-(\omega S(x)),$$

which is the same as $\mathcal{T}_-(x) = \mathcal{T}_-((\omega S)^{-1}(x))$. Hence,

$$\begin{aligned} \mathcal{T}_-(\mathcal{U}^{\mathrm{ev}}) &= \mathcal{T}_-((\omega S)^{-1}(\mathcal{U}^{\mathrm{ev}})) \\ &= \mathcal{T}_+(\varphi \circ (\omega S)^{-1}(\mathcal{U}^{\mathrm{ev}})) \quad \text{by (98)} \\ &= \mathcal{T}_+(\iota_{\mathrm{bar}} \tau(\mathcal{U}^{\mathrm{ev}})) \qquad \text{because } \varphi = \iota_{\mathrm{bar}} \tau \omega S \text{ by Proposition 3.2} \\ &\subset \mathcal{T}_+(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A}, \end{aligned}$$

where we have used part (b), which says $\iota_{\mathrm{bar}} \tau(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{U}^{\mathrm{ev}}$.

(g) It is clear that $(\mathcal{U}^{\mathrm{ev}})^{\otimes n} \subset (\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap \mathcal{U}^{\otimes n}$. Let us prove the converse inclusion.

The $\mathcal{A}$–basis of $\mathcal{U}$ described in (d) is also a $\mathbb{C}(v)$–basis of $\boldsymbol{U}_q$. This basis generates in a natural way an $\mathcal{A}$–basis $\{e(i) \mid i \in I\}$ of $\mathcal{U}^{\otimes n}$, which is also a $\mathbb{C}(v)$–basis of $\boldsymbol{U}_q^{\otimes n}$. There is a subset $I^{\mathrm{ev}} \subset I$ such that $\{e(i) \mid i \in I^{\mathrm{ev}}\}$ is an $\mathcal{A}$–basis of $(\mathcal{U}^{\mathrm{ev}})^{\otimes n}$ and at the same time a $\mathbb{C}(v)$–basis of $(\boldsymbol{U}_q^{\mathrm{ev}})^{\otimes n}$. Using these bases, one can easily show that $(\mathcal{U}^{\mathrm{ev}})^{\otimes n} = (\boldsymbol{U}_q^{\mathrm{ev}})^{\otimes n} \cap \mathcal{U}^{\otimes n}$. Hence,

$$(\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes n} \cap \mathcal{U}^{\otimes n} \subset (\boldsymbol{U}_q^{\mathrm{ev}})^{\otimes n} \cap \mathcal{U}^{\otimes n} = (\mathcal{U}^{\mathrm{ev}})^{\otimes n},$$

which is the converse inclusion. The proof is complete. □

Theorems 8.11(d) and 7.3 give the following:

Corollary 8.12 *If T is an n–component bottom tangle with 0 linking matrix, then $J_T \in \widetilde{\mathcal{K}}_n(\mathcal{U})$.*

Remarks 8.13 (a) In the case $\mathfrak{g} = \mathrm{sl}_2$ the algebra $\mathcal{U}$ was studied by Habiro [25; 26].

(b) The algebra $\mathcal{U}$ is not balanced between E_α and F_α, and $\varphi(\mathcal{U}) \neq \mathcal{U}$.

8H Complexification of $\widetilde{\mathcal{K}}_m(\mathcal{U})$

To accommodate the complex coefficients appearing in the definition of $\Omega_\pm$, we often extend the ground ring from $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ to $\mathcal{B} = \mathbb{C}[v^{\pm 1}]$. Let

$$\widehat{\mathbb{C}[v]} := \varprojlim_k \mathbb{C}[v^{\pm 1}]/(q;q)_k = \varprojlim_k \mathbb{C}[v]/(q;q)_k.$$

By (143),

$$\begin{aligned}\mathcal{F}_k(\mathcal{K}_m(\mathcal{U})) &= (q;q)_k (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes m} \cap [\boldsymbol{U}_{\mathbb{Z}}^{\otimes m}]_1 \cap \mathcal{U}^{\otimes m} \\ &\subset (q;q)_k (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes m} \cap (\mathcal{U}^{\mathrm{ev}})^{\otimes m} \quad \text{by Theorem 8.11(g).}\end{aligned}$$

Let

$$\mathcal{F}_k(\mathcal{K}'_m) := ((q;q)_k (\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}})^{\otimes m} \cap (\mathcal{U}^{\mathrm{ev}})^{\otimes m}) \otimes_{\mathcal{A}} \mathcal{B} \subset h^k (\boldsymbol{X}_h)^{\hat{\otimes} m} \cap h^k \boldsymbol{U}_h^{\hat{\otimes} m}.$$

Define the completion

$$\widetilde{\mathcal{K}}'_m = \left\{ x = \sum_{k=0}^{\infty} x_k \;\middle|\; x_k \in \mathcal{F}_k(\mathcal{K}'_m) \right\} \subset (\boldsymbol{X}_h)^{\hat{\otimes} m} \cap (\boldsymbol{U}_h)^{\hat{\otimes} m}. \tag{183}$$

Then $\widetilde{\mathcal{K}}_m(\mathcal{U}) \subset \widetilde{\mathcal{K}}'_m$, and $\widetilde{\mathcal{K}}'_0 = \widehat{\mathbb{C}[v]}$. We will work with $\widetilde{\mathcal{K}}'_n$ instead of $\widetilde{\mathcal{K}}_n(\mathcal{U})$.

8I Integral basis of V_λ

For $\lambda \in X_+$ recall that V_λ is the finite-rank $\boldsymbol{U}_h$–module of highest weight λ. Let $1_\lambda \in V_\lambda$ be a highest-weight element. It is known that the $\boldsymbol{U}_{\mathbb{Z}}$–module $\boldsymbol{U}_{\mathbb{Z}} \cdot 1_\lambda$ is a free $\mathcal{A}$–module of rank equal to the rank of V_λ over $\mathbb{C}[\![h]\!]$. Further, there is an $\mathcal{A}$–basis of $\boldsymbol{U}_{\mathbb{Z}} \cdot 1_\lambda$ consisting of weight elements; see eg [12]. We call such a basis an *integral basis* of V_λ. For example, the *canonical basis* of Kashiwara [34] and Lusztig [54] is such an integral basis. An integral basis of V_λ is also a topological basis of V_λ.

Recall that $\breve{\boldsymbol{U}}_{\mathbb{Z}} = \breve{\boldsymbol{U}}_{\mathbb{Z}}^0 \boldsymbol{U}_{\mathbb{Z}}$ and $\breve{\boldsymbol{U}}_q = \breve{\boldsymbol{U}}_q^0 \boldsymbol{U}_q$ are the simply connected versions of $\boldsymbol{U}_{\mathbb{Z}}$ and $\boldsymbol{U}_q$, respectively; see Section 5H. For $\lambda \in X_+$ we have the quantum trace map $\mathrm{tr}_q^{V_\lambda}\colon \boldsymbol{U}_h \to \mathbb{C}[\![h]\!]$. This maps extends to $\mathrm{tr}_q^{V_\lambda}\colon \boldsymbol{U}_h[h^{-1}] \to \mathbb{C}[\![h]\!][h^{-1}]$. In particular, if $x \in \breve{\boldsymbol{U}}_q$ then one can define $\mathrm{tr}_q^{V_\lambda}(x) \in \mathbb{C}[\![h]\!][h^{-1}]$.

Lemma 8.14 *Suppose λ is a dominant weight, $\lambda \in X_+$.*

(a) *If* $x \in \boldsymbol{U}_{\mathbb{Z}}$ *then* $\operatorname{tr}_q^{V_\lambda}(x) \in \mathcal{A}$.

(b) *If* $x \in \breve{\boldsymbol{U}}_q$ *then* $\operatorname{tr}_q^{V_\lambda}(x) \in \mathbb{Q}(v^{\pm 1/D})$.

(c) *If* $x \in \breve{\boldsymbol{U}}_{\mathbb{Z}}$ *and* $\lambda \in Y$ *then* $\operatorname{tr}_q^{V_\lambda}(x) \in \mathcal{A}$.

(d) *If* $x \in \boldsymbol{X}_{\mathbb{Z}}$ *then* $\operatorname{tr}_q^{V_\lambda}(x) \in \widetilde{\mathcal{A}}$.

Proof Fix an integral basis of V_λ. Using the basis, each $x \in \boldsymbol{U}_h$ acts on V_λ by a matrix with entries in $\mathbb{C}[\![h]\!]$, called the matrix of x.

(a) If $x \in \boldsymbol{U}_{\mathbb{Z}}$ then its matrix has entries in $\mathcal{A}$. Thus, $\operatorname{tr}_q^{V_\lambda}(x) = \operatorname{tr}^{V_\lambda}(xK_{-2\rho}) \in \mathcal{A}$.

(b) As a $\mathbb{Q}(v)$–algebra, $\breve{\boldsymbol{U}}_q$ is generated by $\boldsymbol{U}_q$ and $\breve{K}_\alpha$, $\alpha \in \Pi$. Since $\boldsymbol{U}_q = \boldsymbol{U}_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C}(v)$, the matrix of $x \in \boldsymbol{U}_q$ has entries in $\mathbb{C}(v)$. For an element e of weight μ, we have $\breve{K}_\alpha(e) = v^{(\breve{\alpha},\mu)}e$. Note that $(\breve{\alpha},\mu) \in (1/D)\mathbb{Z}$. It follows that the matrix of $\breve{K}_\alpha(e)$ has entries in $\mathbb{Q}(v^{\pm 1/D})$. Hence the matrix of every $x \in \breve{\boldsymbol{U}}_q$ has entries in $\mathbb{C}(v^{\pm 1/D})$, and $\operatorname{tr}_q^{V_\lambda}(x) \in \mathbb{C}(v^{\pm 1/D})$.

(c) As an $\mathcal{A}$–algebra, $\breve{\boldsymbol{U}}_{\mathbb{Z}}$ is generated by $\boldsymbol{U}_{\mathbb{Z}}$ and

$$f(\breve{K}_\alpha; n, k) := \breve{K}_\alpha^n \frac{(\breve{K}_\alpha^2; q_\alpha)_k}{(q_\alpha; q_\alpha)_k}, \quad n \in \mathbb{Z},\ k \in \mathbb{N},\ \alpha \in \Pi.$$

When $\lambda \in Y$, all the weights of V_λ are in Y. From the orthogonality between simple roots and fundamental weights we have $(\breve{\alpha},\mu) \in d_\alpha\mathbb{Z}$ for every $\alpha \in \Pi$ and $y \in Y$. Hence,

$$f(v^{(\breve{\alpha},\mu)}; n, k) = v^{n(\breve{\alpha},\mu)} \frac{(q_\alpha^{(\breve{\alpha},\mu)/d_\alpha}; q_\alpha)_k}{(q_\alpha; q_\alpha)_k} \in \mathcal{A}.$$

Suppose $e \in V_\lambda$ has weight $\mu \in Y$. Then

$$f(\breve{K}_\alpha; n, k)(e) = f(v^{(\breve{\alpha},\mu)}; n, k)e \in \mathcal{A}e.$$

Thus, the matrix of $f(\breve{K}_\alpha; n, k)$ on V_λ has entries in $\mathcal{A}$. We conclude that the matrix of every $x \in \breve{\boldsymbol{U}}_{\mathbb{Z}}$ has entries in $\mathcal{A}$, and $\operatorname{tr}_q^{V_\lambda}(x) \in \mathcal{A}$.

(d) Because $\boldsymbol{X}_{\mathbb{Z}} \subset \boldsymbol{U}_{\mathbb{Z}} \otimes_{\mathcal{A}} \widetilde{\mathcal{A}}$, by part (a) we have $\operatorname{tr}_q^{V_\lambda}(x) \in \widetilde{\mathcal{A}}$. □

8J Quantum traces associated to $\boldsymbol{\Omega}_\pm$

Define

$$\widetilde{\mathcal{T}}_\pm\colon \boldsymbol{U}_h \to \mathbb{C}[\![h]\!], \quad \widetilde{\mathcal{T}}_\pm(x) = \operatorname{tr}_q^{\Omega_\pm}(x).$$

Note that $\widetilde{\mathcal{T}}_\pm$, being quantum traces, are ad-invariant. Since $\Omega_\pm$ are $\mathbb{C}$–linear combinations of V_λ, Lemma 8.14 shows that $\widetilde{\mathcal{T}}_\pm$ restricts to a $\mathcal{B}$–linear map from $\boldsymbol{U}_\mathbb{Z} \otimes_\mathcal{A} \mathcal{B}$ to $\mathcal{B} = \mathbb{C}[v^{\pm 1}]$.

Recall that $(\widetilde{\mathcal{K}}'_n)^{\mathrm{inv}}$ denotes the set of elements in $\widetilde{\mathcal{K}}'_n$ which are $\boldsymbol{U}_\mathbb{Z}$–ad-invariants.

Proposition 8.15 *Suppose f is one of $\mathcal{T}_\pm$ or $\widetilde{\mathcal{T}}_\pm$. Then f is $(\mathcal{K}'_m)^{\mathrm{inv}}$–admissible in the sense that, for $m \ge j \ge 1$,*

$$(\mathrm{id}^{\otimes j-1} \otimes f \otimes \mathrm{id}^{\otimes m-j})((\widetilde{\mathcal{K}}'_m)^{\mathrm{inv}}) \subset (\widetilde{\mathcal{K}}'_{m-1})^{\mathrm{inv}}.$$

Proof Recall that $\mathcal{T}_\pm$ and $\widetilde{\mathcal{T}}_\pm$ are ad-invariant. By Proposition 2.4(d) it is enough to prove

$$(\mathrm{id}^{\otimes j-1} \otimes f \otimes \mathrm{id}^{\otimes m-j})(\widetilde{\mathcal{K}}'_m) \subset \widetilde{\mathcal{K}}'_{m-1},$$

which, in turn, will follow from

(184) $$(\mathrm{id}^{\otimes j-1} \otimes f \otimes \mathrm{id}^{\otimes m-j})(\mathcal{F}_k(\mathcal{K}'_m)) \subset \mathcal{F}_k(\mathcal{K}'_{m-1}).$$

Let us prove (184) for $f = \mathcal{T}_\pm$. By Proposition 5.24,

$$(\mathrm{id}^{\otimes j-1} \otimes \mathcal{T}_\pm \otimes \mathrm{id}^{\otimes m-j})((q;q)_k (\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}})^{\otimes m}) \subset (q;q)_k (\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}})^{\otimes m-1}.$$

By Theorem 8.11(f),

$$(\mathrm{id}^{\otimes j-1} \otimes \mathcal{T}_\pm \otimes \mathrm{id}^{\otimes m-j})((\mathcal{U}^{\mathrm{ev}})^{\otimes m}) \subset (\mathcal{U}^{\mathrm{ev}})^{\otimes m-1}.$$

Because $\mathcal{F}_k(\mathcal{K}'_m) = ((q;q)_k (\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}})^{\otimes m} \cap (\mathcal{U}^{\mathrm{ev}})^{\otimes m}) \otimes_\mathcal{A} \mathcal{B}$, we have

$$(\mathrm{id}^{\otimes j-1} \otimes \mathcal{T}_\pm \otimes \mathrm{id}^{\otimes m-j}) \mathcal{F}_k(\mathcal{K}'_m) \subset \mathcal{F}_k(\mathcal{K}'_{m-1}).$$

Let us now prove (184) for $f = \widetilde{\mathcal{T}}_\pm$. Because $\Omega_\pm$ is a $\mathbb{C}$–linear combination of V_λ, by Lemma 8.14(d), $\widetilde{\mathcal{T}}_\pm(\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}}) \subset \widetilde{\mathcal{A}} \otimes_\mathcal{A} \mathcal{B}$. Hence,

(185) $$(\mathrm{id}^{\otimes j-1} \otimes \widetilde{\mathcal{T}}_\pm \otimes \mathrm{id}^{\otimes m-j})((q;q)_k (\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}})^{\otimes m}) \subset (q;q)_k ((\boldsymbol{X}_\mathbb{Z}^{\mathrm{ev}})^{\otimes m-1} \otimes_\mathcal{A} \mathcal{B}).$$

From Lemma 8.14(a), $\widetilde{\mathcal{T}}_\pm(\mathcal{U}) \subset \mathcal{B}$, and hence

$$(\mathrm{id}^{\otimes j-1} \otimes \widetilde{\mathcal{T}}_\pm \otimes \mathrm{id}^{\otimes m-j})((\mathcal{U}^{\mathrm{ev}})^{\otimes m}) \subset (\mathcal{U}^{\mathrm{ev}})^{\otimes m-1} \otimes_\mathcal{A} \mathcal{B},$$

which, together with (185), proves (184). □

8K Actions of Weyl group on U_h^0 and Chevalley's theorem

The Weyl group acts on the Cartan part U_h^0 by *algebra automorphisms* given by $w(H_\lambda) = H_{w(\lambda)}$. Then $w(K_\alpha) = K_{w(\alpha)}$, and $\mathfrak{W}$ restricts and extends to actions on the Cartan parts $U_{\mathbb{Z}}^0$, $V_{\mathbb{Z}}^0$, and X_h^0.

We say an element $x \in U_h^0$ is $\mathfrak{W}$*–invariant* if $w(x) = x$ for every $w \in \mathfrak{W}$, and x is $\mathfrak{W}$*–skew-invariant* if $w(x) = \operatorname{sgn}(w)x$ for every $w \in \mathfrak{W}$. As usual, if $\mathfrak{W}$ acts on V we denote by $V^{\mathfrak{W}}$ the subset of $\mathfrak{W}$–invariant elements.

By Chevalley's theorem [13], there are l homogeneous polynomials $e_1, \dots, e_l$ in $\mathbb{Z}[H_1, \dots, H_l]$ such that $(\mathbb{C}[H_1, \dots, H_l])^{\mathfrak{W}} = \mathbb{C}[e_1, \dots, e_l]$, the polynomial ring freely generated by l elements $e_1, \dots, e_l$.

Suppose the degree of e_i is k_i. Since $\exp(hH_\alpha) = K_\alpha^2$, we have

$$\tilde{e}_i := \exp(h^{k_i} e_i) \in \mathbb{Z}[K_1^{\pm 2}, \dots, K_l^{\pm 2}]^{\mathfrak{W}} \subset (V_{\mathbb{Z}}^{\mathrm{ev},0})^{\mathfrak{W}}. \tag{186}$$

Proposition 8.16 *One has*

$$(U_h^0)^{\mathfrak{W}} = \mathbb{C}[e_1, \dots, e_l][\![h]\!], \tag{187}$$

$$(X_h^0)^{\mathfrak{W}} = \mathbb{C}[h^{k_1/2} e_1, \dots, h^{k_l/2} e_l][\![\sqrt{h}]\!], \tag{188}$$

$$(V_h^0)^{\mathfrak{W}} = \overline{\mathbb{C}[h^{k_1} e_1, \dots, h^{k_l} e_l][\![h]\!]} \tag{189}$$

$$= \overline{\mathbb{C}[\tilde{e}_1, \dots, \tilde{e}_l][\![h]\!]}. \tag{190}$$

(Here the overline in (189) and (190) denotes the topological closure in the h–adic topology of U_h.)

Proof We have

$$(U_h^0)^{\mathfrak{W}} = (\mathbb{C}[H_1, \dots, H_l][\![h]\!])^{\mathfrak{W}} = (\mathbb{C}[H_1, \dots, H_l])^{\mathfrak{W}}[\![h]\!] = \mathbb{C}[e_1, \dots, e_l][\![h]\!],$$

which proves (187). Similarly, using

$$(X_h^0)^{\mathfrak{W}} = (\mathbb{C}[h^{1/2} H_1, \dots, h^{1/2} H_l][\![h]\!])^{\mathfrak{W}},$$
$$(V_h^0)^{\mathfrak{W}} = (\mathbb{C}[hH_1, \dots, hH_l][\![h]\!])^{\mathfrak{W}},$$

we get (188) and (189). We have

$$\tilde{e}_i - 1 = h^{k_i} e_i + h(V_h^0)^{\mathfrak{W}}.$$

It follows that

$$\mathbb{C}[\tilde{e}_1, \dots, \tilde{e}_l][\![h]\!] = \mathbb{C}[h^{k_1} e_1, \dots, h^{k_l} e_l][\![h]\!],$$

from which one has (190). □

8L The Harish-Chandra isomorphism, center of U_h

Let $\mathfrak{Z}(U_h)$ be the center of U_h, which is known to be U_h^{inv}, the ad-invariant subset of U_h. For any subset $V \subset U_h$ write $\mathfrak{Z}(V) = V \cap \mathfrak{Z}(U_h)$, the set of central elements in V.

Let $p_0\colon U_h \to U_h^0$ be the projection corresponding to the triangular decomposition. This means, if $x = x_- x_0 x_+$, where $x_- \in U_h^-$, $x_+ \in U_h^+$ and $x_0 \in U_h^0$, then $p_0(x) = \epsilon(x_-)\epsilon(x_+)x_0$. Here ϵ is the counit.

For $\mu \in X$, define the algebra homomorphism $\mathrm{sh}_\mu\colon U_h^0 \to U_h^0$ by $\mathrm{sh}_\mu(H_\alpha) = H_\alpha + (\alpha, \mu)$. Then $\mathrm{sh}_\mu(K_\alpha) = v^{(\mu,\alpha)} K_\alpha$. Since $v^{(\mu,\alpha)} = \langle K_{-2\mu}, K_\alpha \rangle$, we have

$$\mathrm{sh}_\mu(K_\alpha) = \langle K_{-2\mu}, K_\alpha \rangle K_\alpha. \tag{191}$$

The *Harish-Chandra map* is the $\mathbb{C}[\![h]\!]$–module homomorphism

$$\chi = \mathrm{sh}_{-\rho} \circ p_0\colon U_h \to U_h^0 = \mathbb{C}[H_1, \dots, H_l][\![h]\!].$$

The restriction of χ to the Y–degree 0 part of U_h, denoted χ by abuse of notation, is a $\mathbb{C}[\![h]\!]$–algebra homomorphism, called the *Harish-Chandra homomorphism.*

One has the following description of the center (see eg [12; 74]):

Proposition 8.17 *The restriction of χ on the center $\mathfrak{Z}(U_h)$ is an algebra isomorphism from $\mathfrak{Z}(U_h)$ to $(U_h^0)^{\mathfrak{W}} = \mathbb{C}[H_1, \dots, H_l]^{\mathfrak{W}}[\![h]\!]$.*

Remark 8.18 Suppose $\mathscr{H} \subset U_h$ is any subring satisfying the triangular decomposition (like $U_{\mathbb{Z}}$ or V_h). By definition,

$$\chi(\mathfrak{Z}(\mathscr{H})) \subset (\mathscr{H}^0)^{\mathfrak{W}}. \tag{192}$$

For $\mathscr{H} = U_h$, we have equality in (192) by Proposition 8.17. But in general, the left-hand side is strictly smaller than the right-hand side. For example, one can show that

$$\chi(\mathfrak{Z}(U_{\mathbb{Z}})) \neq (U_{\mathbb{Z}}^0)^{\mathfrak{W}}.$$

Over the ground ring $\mathcal{A}$, the determination of the image of the Harish-Chandra map is difficult. Later we will determine $\chi(\mathfrak{Z}(\mathscr{H}))$ for two cases, $\mathscr{H} = V_{\mathbb{Z}}^{\mathrm{ev}}$, which is defined over $\mathcal{A}$, and $\mathscr{H} = X_h$, which is defined over $\mathbb{C}[\![\sqrt{h}]\!]$. In both cases, the duality with respect to the quantum Killing form will play an important role.

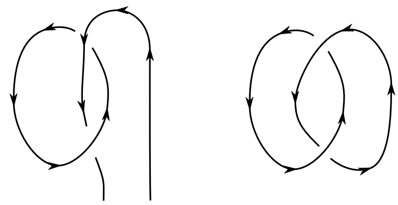


Figure 14: The open Hopf link (left) and the Hopf link

8M From U_h–modules to central elements

In the classical case, the center of the enveloping algebra of $\mathfrak{g}$ is isomorphic to the ring of $\mathfrak{g}$–modules via the character map. We will recall (and modify) here the corresponding fact in the quantized case.

For a dominant weight $\lambda \in X_+$, recall that V_λ is the irreducible $\boldsymbol{U}_h$–module of highest weight λ. Since the map $\mathrm{tr}_q^{V_\lambda}\colon \boldsymbol{U}_h \to \mathbb{C}[\![h]\!]$ is ad-invariant and the clasp element $\boldsymbol{c}$ is ad-invariant, by Proposition 2.4(d) the element

$$z_\lambda := (\mathrm{tr}_q^{V_\lambda} \hat{\otimes} \mathrm{id})(\boldsymbol{c})$$

is in $(\boldsymbol{U}_h)^{\mathrm{inv}} = \mathfrak{Z}(\boldsymbol{U}_h)$. This construction of central elements was sketched in [17], and studied in detail in [33; 4]. Our approach gives a geometric meaning of z_λ as it shows that $z_\lambda = J_T$, where T is the open Hopf link bottom tangle depicted in Figure 14, with the closed component colored by V_λ. Let us summarize some more or less well-known properties of z_λ; see [4; 11; 33].

Proposition 8.19 *Suppose* $\lambda, \lambda' \in X_+$.

(a) *For every* $x \in \breve{\boldsymbol{U}}_q$,

$$\mathrm{tr}_q^{V_\lambda}(x) = \langle z_\lambda, x\rangle. \tag{193}$$

(b) *One has*

$$\chi(z_\lambda) = \sum_{\mu \in X} \dim(V_\lambda)_{[\mu]} K_{-2\mu} = \frac{\sum_{w\in\mathfrak{W}} \mathrm{sgn}(w) K_{-2w(\lambda+\rho)}}{\sum_{w\in\mathfrak{W}} \mathrm{sgn}(w) K_{-2w(\rho)}}. \tag{194}$$

(c) *If L is the Hopf link* (see Figure 14) *then*

$$J_L(V_\lambda, V_{\lambda'}) = \langle z_\lambda, z_{\lambda'}\rangle = \mathrm{tr}_q^{V_\lambda}(z_{\lambda'}), \tag{195}$$

(d) *One has $z_\lambda \in \breve{\boldsymbol{U}}_q^{\mathrm{ev}}$, and if $\lambda \in Y$ then $z_\lambda \in \boldsymbol{U}_q^{\mathrm{ev}}$.*

Proof (a) Recall that the quantum Killing form is the dual to $\boldsymbol{c} = \sum \boldsymbol{c}_1 \otimes \boldsymbol{c}_2$, and $x = \sum \langle \boldsymbol{c}_2, x\rangle \boldsymbol{c}_1$. We have

$$\langle z_\lambda, x\rangle = \Big\langle \sum \operatorname{tr}_q^{V_\lambda}(\boldsymbol{c}_1)\boldsymbol{c}_2, x\Big\rangle = \sum \operatorname{tr}_q^{V_\lambda}(\boldsymbol{c}_1)\langle \boldsymbol{c}_2, x\rangle = \sum \operatorname{tr}_q^{V_\lambda}(\langle \boldsymbol{c}_2, x\rangle \boldsymbol{c}_1) = \operatorname{tr}_q^{V_\lambda}(x).$$

(b) In [30, Chapter 6], it is proved that if $\lambda \in X_+ \cap \frac{1}{2}Y$, then

$$\chi(z_\lambda) = \sum_{\mu \in X} \dim((V_\lambda)_{[\mu]}) K_{-2\mu}, \tag{196}$$

where $\dim((V_\lambda)_{[\mu]})$ is the rank of the weight-μ submodule. Actually, the simple proof in [30, Chapter 6] works for all $\lambda \in X_+$. The second equality of (194) is the famous Weyl character formula; see eg [29].

(c) Let T be the open Hopf link bottom tangle depicted in Figure 14, with the closed component colored by V_λ. Then $J_T = z_\lambda$. We have

$$J_L(V_\lambda, V_{\lambda'}) = \operatorname{tr}_q^{V_{\lambda'}}(J_T) = \langle z_{\lambda'}, J_T\rangle = \langle z_{\lambda'}, z_\lambda\rangle = \langle z_\lambda, z_{\lambda'}\rangle.$$

(d) Joseph and Letzter [32, Section 6.10] (see [4, Proposition 5] for another proof) showed that $z_\lambda \in \breve{\boldsymbol{U}}_q \triangleright K_{-2\lambda}$. Since $K_{-2\lambda} \in \breve{\boldsymbol{U}}_q^{\mathrm{ev}}$, we have $z_\lambda \in \breve{\boldsymbol{U}}_q \triangleright \breve{\boldsymbol{U}}_q^{\mathrm{ev}} \subset \breve{\boldsymbol{U}}_q^{\mathrm{ev}}$, by Lemma 3.6. If $\lambda \in Y$, then $K_{-2\lambda} \in \boldsymbol{U}_q^{\mathrm{ev}}$, hence $z_\lambda \in \boldsymbol{U}_q^{\mathrm{ev}}$ again by Lemma 3.6. □

Note that the right-hand side of (194) makes sense, and is in $(\boldsymbol{U}_q^0)^{\mathfrak{W}}$, for *any* $\lambda \in X$ not necessarily in $X_+ \cap \frac{1}{2}Y$. For *any* $\lambda \in X$, define $z_\lambda \in \mathfrak{Z}(\breve{\boldsymbol{U}}_q)$ by

$$z_\lambda = \chi^{-1}\bigg(\sum_{\mu \in X} \dim(V_\lambda)_{[\mu]} K_{-2\mu}\bigg).$$

If $\lambda + \rho$ and $\lambda' + \rho$ are in the same $\mathfrak{W}$–orbit then, by (194), $z_\lambda = z_{\lambda'}$. On the other hand, if $\lambda + \rho$ is fixed by a nontrivial element of the Weyl group, then $z_\lambda = 0$.

When λ is in the root lattice Y, the right-hand side of (194) is in $\mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]^{\mathfrak{W}}$. Actually, the theory of invariant polynomials says that the right-hand side of (194), with $\lambda \in Y$, gives all $\mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]^W$; see eg [56, Section 2.3]. Hence, we have the following statement:

Proposition 8.20 *The Harish-Chandra homomorphism is an isomorphism of the $\mathcal{A}$–span of $\{z_\alpha \mid \alpha \in Y\}$ onto $\mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]^{\mathfrak{W}}$.*

8N Center of $V_{\mathbb{Z}}^{\mathrm{ev}}$

Lemma 8.21 *Suppose $\beta \in Y$. Then $z_\beta \in V_{\mathbb{Z}}^{\mathrm{ev}}$.*

Proof By Proposition 5.15, $V_{\mathbb{Z}}^{\mathrm{ev}}$ is the $\mathcal{A}$–dual of $\check{U}_{\mathbb{Z}}^{\mathrm{ev}}$ with respect to the quantum Killing form, ie

$$V_{\mathbb{Z}}^{\mathrm{ev}} = \{x \in U_q^{\mathrm{ev}} \mid \langle x, y\rangle \in \mathcal{A} \text{ for all } y \in \check{U}_{\mathbb{Z}}^{\mathrm{ev}}\}.$$

Since $z_\beta \in U_q^{\mathrm{ev}}$ by Proposition 8.19, it is sufficient to show that $\langle z_\beta, y\rangle \in \mathcal{A}$ for any $y \in \check{U}_{\mathbb{Z}}^{\mathrm{ev}}$.

We can assume that β is a dominant weight, $\beta \in X_+ \cap Y$. By Proposition 8.19,

$$\langle z_\beta, y\rangle = \mathrm{tr}_q^{V_\beta}(y) \in \mathcal{A},$$

where the inclusion follows from Lemma 8.14. This shows $z_\beta \in V_{\mathbb{Z}}^{\mathrm{ev}}$. □

Proposition 8.22 (a) *One has $\mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}}) = \mathfrak{Z}(\mathcal{U}^{\mathrm{ev}}) = \mathcal{A}\text{-span}(\{z_\alpha \mid \alpha \in Y\})$.*

(b) *The Harish-Chandra homomorphism maps $\mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}})$ isomorphically onto $(V_{\mathbb{Z}}^{\mathrm{ev},0})^{\mathfrak{W}}$, ie*

$$\chi(\mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}})) = (V_{\mathbb{Z}}^{\mathrm{ev},0})^{\mathfrak{W}} = \mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]^{\mathfrak{W}}. \tag{197}$$

Proof (a) Let us prove the inclusions

$$\mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}}) \subset \mathfrak{Z}(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A}\text{-span}(\{z_\alpha \mid \alpha \in Y\}) \subset \mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}}), \tag{198}$$

which imply that all the terms are the same and prove part (a).

The first inclusion is obvious, since $V_{\mathbb{Z}}^{\mathrm{ev}} \subset \mathcal{U}^{\mathrm{ev}}$, while the third is Lemma 8.21.

Because $\mathcal{U}^{\mathrm{ev},0} = \mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]$, one has $\chi(\mathfrak{Z}(\mathcal{U}^{\mathrm{ev}})) \subset \mathcal{A}[K_{\alpha_1}^{\pm 2}, \dots, K_{\alpha_l}^{\pm 2}]^{\mathfrak{W}}$. Hence, by Proposition 8.20 we have $\mathfrak{Z}(\mathcal{U}^{\mathrm{ev}}) \subset \mathcal{A}\text{-span}(\{z_\alpha \mid \alpha \in Y\})$, which is the second inclusion in (198). This proves (a).

(b) This follows from (a) and Proposition 8.20. □

Proposition 8.23 *The Harish-Chandra map χ is an isomorphism between $\mathfrak{Z}(V_h)$ and $(V_h^0)^{\mathfrak{W}}$.*

Proof Since $\chi(\mathfrak{Z}(V_h)) \subset (V_h^0)^{\mathfrak{W}}$, it remains to show $(V_h^0)^{\mathfrak{W}} \subset \chi(\mathfrak{Z}(V_h))$. By (189),

$$(V_h^0)^{\mathfrak{W}} = \overline{\mathbb{C}[\widetilde{e}_1, \dots, \widetilde{e}_l][\![h]\!]}.$$

By (186) and (197),

$$\widetilde{e}_i \in (V_{\mathbb{Z}}^{\mathrm{ev},0})^{\mathfrak{W}} = \chi(\mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}})) \subset \chi(\mathfrak{Z}(V_h)).$$

Hence, $(V_h^0)^{\mathfrak{W}} \subset \chi(\mathfrak{Z}(V_h))$. This completes the proof of the proposition. □

8O Center of X_h

Proposition 8.24 *The Harish-Chandra map χ is an isomorphism between $\mathfrak{Z}(X_h)$ and $(X_h^0)^{\mathfrak{W}}$.*

Proof By definition, $\chi(\mathfrak{Z}(X_h)) \subset (X_h^0)^{\mathfrak{W}}$. We need to show $\chi^{-1}((X_h^0)^{\mathfrak{W}}) \subset \mathfrak{Z}(X_h)$. Because $\chi^{-1}((X_h^0)^{\mathfrak{W}})$ consists of central elements, one needs only to show that $\chi^{-1}((X_h^0)^{\mathfrak{W}}) \subset X_h$. We will use the stability principle of dilatation triples.

From (187), (188), and (189), the triple $(U_h^0)^{\mathfrak{W}}$, $(X_h^0)^{\mathfrak{W}}$, $(V_h^0)^{\mathfrak{W}}$ forms a topological dilatation triple (see Section 4C).

The triple U_h, X_h, V_h also forms a topological dilatation triple (see Section 4D). Since $\chi^{-1}((U_h^0)^{\mathfrak{W}}) \subset U_h$ and $\chi^{-1}((V_h^0)^{\mathfrak{W}}) \subset V_h$ by Proposition 8.23, one also has $\chi^{-1}((X_h^0)^{\mathfrak{W}}) \subset X_h$, by the stability principle (Proposition 4.6). □

8P Quantum Killing form and Harish-Chandra homomorphism

Since $\chi(x)$ and $\chi(y)$ determine x and y for central $x, y \in U_{\mathbb{Z}}$, one should be able to calculate $\langle x, y\rangle$ in terms of $\chi(x)$ and $\chi(y)$.

Let $\mathbb{D}$ be the denominator of the right-hand side of (194), ie

$$\mathbb{D} := \sum_{w\in\mathfrak{W}} \operatorname{sgn}(w) K_{-w(2\rho)}.$$

By the Weyl denominator formula,

$$\mathbb{D} = \prod_{\alpha\in\Phi_+} (K_\alpha^{-1} - K_\alpha) = K_{2\rho} \prod_{\alpha\in\Phi_+} (K_\alpha^{-2} - 1) \in K_{2\rho} V_{\mathbb{Z}}^{\mathrm{ev}}. \tag{199}$$

Let us define

$$\mathbb{d} := \langle K_{-2\rho}, \mathbb{D}\rangle = \prod_{\alpha\in\Phi_+} (v_\alpha^{-1} - v_\alpha).$$

From the formula for the quantum dimension (174), we have

$$\mathbb{d} \dim_q(V_\lambda) = \langle K_{-2\rho-2\lambda}, \mathbb{D}\rangle. \tag{200}$$

Here is a formula expressing $\langle x, y\rangle$ in terms of $\chi(x), \chi(y)$:

Proposition 8.25 *Suppose $x \in \mathfrak{Z}(\overline{X}_h)$, and $y = z_\lambda$, $\lambda \in Y$. Then*

$$|\mathfrak{W}| \mathbb{d} \langle x, y\rangle = \langle \mathbb{D}\chi(x), \mathbb{D}\chi(y)\rangle \tag{201}$$

Proof As x is central, it acts on V_λ by $c(\lambda,x)\,\mathrm{id}$, where $c(\lambda,x)\in\mathbb{C}[\![h]\!]$. Recall that 1_λ is the highest-weight vector of V_λ. We have $K_\alpha\cdot 1_\lambda = v^{(\alpha,\lambda)}1_\lambda = \langle K_\alpha, K_{-2\lambda}\rangle 1_\lambda$. Hence, for every $z\in \boldsymbol{U}_h^0$,

$$z\cdot 1_\lambda = \langle x, K_{-2\lambda}\rangle 1_\lambda. \tag{202}$$

Since the highest-weight vector 1_λ is killed by all E_α, $\alpha\in\Pi$, we have

$$x\cdot 1_\lambda = p_0(x)\cdot 1_\lambda = \mathrm{sh}_\rho\,\chi(x)\cdot 1_\lambda = \langle \mathrm{sh}_\rho\,\chi(x), K_{-2\lambda}\rangle 1_\lambda \quad \text{by (202).}$$

Thus, $c(\lambda,x) = \langle \mathrm{sh}_\rho\,\chi(x), K_{-2\lambda}\rangle$. Further, by (191),

$$\begin{aligned}
c(\lambda,x) &= \langle \mathrm{sh}_\rho\,\chi(x), K_{-2\lambda}\rangle = \big\langle \langle K_{-2\rho},\chi(x)\rangle \chi(x), K_{-2\lambda}\big\rangle \\
&= \langle K_{-2\rho},\chi(x)\rangle\langle \chi(x), K_{-2\lambda}\rangle \\
&= \langle K_{-2\rho},\chi(x)\rangle\langle K_{-2\lambda},\chi(x)\rangle = \langle K_{-2\rho-2\lambda},\chi(x)\rangle \\
&= \left\langle K_{-2\rho-2\lambda}, \frac{\mathbb{D}\chi(x)}{\mathbb{D}}\right\rangle = \frac{\langle K_{-2\rho-2\lambda},\mathbb{D}\chi(x)\rangle}{\langle K_{-2\rho-2\lambda},\mathbb{D}\rangle} = \frac{\langle K_{-2\rho-2\lambda},\mathbb{D}\chi(x)\rangle}{\mathbb{d}\,\dim_q(V_\lambda)} \\
&= \frac{1}{\mathbb{d}\,\dim_q(V_\lambda)}\left\langle \frac{1}{|\mathfrak{W}|}\sum_{w\in\mathfrak{W}} \mathrm{sgn}(w) K_{-2w(\lambda+\rho)}, \mathbb{D}\chi(x)\right\rangle \\
&= \frac{1}{|\mathfrak{W}|\mathbb{d}\,\dim_q(V_\lambda)}\langle \mathbb{D}\chi(z_\lambda), \mathbb{D}\chi(x)\rangle.
\end{aligned}$$

Here the last equality on line four follows from (200) and the equality on the line five follows from the fact that $\mathbb{D}\chi(x)$ is $\mathfrak{W}$–skew-invariant and the quantum Killing form is $\mathfrak{W}$–invariant on $\boldsymbol{X}_h^0$.

Using (193) and the fact that $x = c(\lambda,x)\,\mathrm{id}$ on V_λ,

$$\langle x, z_\lambda\rangle = \mathrm{tr}_q^{V_\lambda}(x) = c(\lambda,x)\dim_q(V_\lambda) = \frac{1}{|\mathfrak{W}|\mathbb{d}}\langle \mathbb{D}\chi(z_\lambda), \mathbb{D}\chi(x)\rangle,$$

where for the last equality we used the value of $c(\lambda,x)$ calculated above. □

Remark 8.26 It is not difficult to show that Proposition 8.25 holds for every $y\in\mathfrak{Z}(\boldsymbol{X}_h)$.

8Q Center of $\widetilde{\mathcal{K}}_1'$

Recall that $\widetilde{\mathcal{K}}_1'$ is the set of all elements of the form

$$x = \sum x_k, \quad x_k\in\mathcal{F}_k(\mathcal{K}_1').$$

One might expect that every central element of $\widetilde{\mathcal{K}}_1'$ has the same form with x_k central. We don't know if this is true. We have here a weaker statement, which is enough for our purpose. In our presentation, x_k is central, but might not be in $\mathcal{F}_k(\mathcal{K}'_1)$. However, x_k still has enough integrality.

Lemma 8.27 *Suppose $x \in \mathfrak{Z}(\widetilde{\mathcal{K}}'_1)$. There are central elements $x_k \in \mathfrak{Z}(\boldsymbol{X}_h)$ such that*

(a) $|\mathfrak{W}|x = \sum_{k=0}^{\infty}(q;q)_k x_k$,

(b) $(q;q)_k x_k \in \mathfrak{Z}(\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B})$ *for every* $k \geq 0$,

(c) $\mathcal{T}_{\pm}(x_k) \in (1/\mathbb{d})\mathbb{C}[v^{\pm 1}]$ *for every* $k \geq 0$, *and*

(d) $\widetilde{\mathcal{T}}_{\pm}(x_k) \in (1/\mathbb{d})\mathbb{C}[v^{\pm 1}]$ *for every* $k \geq 0$.

Proof (a) Recall that $\mathcal{F}_k(\mathcal{K}'_1) = ((q;q)_k(\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}}) \cap (\mathcal{U}^{\mathrm{ev}})) \otimes_{\mathcal{A}} \mathcal{B}$. Hence, x has a presentation

$$x = \sum_{k=0}^{\infty}(q;q)_k x'_k, \tag{203}$$

where $x'_k \in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B}$ and $(q;q)_k x'_k \in \mathcal{U}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B}$.

Let $y_k = \sum_{w \in \mathfrak{W}} w(\chi(x'_k))$, which is $\mathfrak{W}$–invariant. Then $y_k \in (\boldsymbol{X}_h^0)^{\mathfrak{W}}$. By Proposition 8.24, $x_k := \chi^{-1}(y_k)$ is central and belongs to $\mathfrak{Z}(\boldsymbol{X}_h)$.

Using the $\mathfrak{W}$–invariance of $\chi(x)$ and (203), and the $\mathfrak{W}$–invariance of $\chi(x)$,

$$|\mathfrak{W}|\chi(x) = \sum_{w \in \mathfrak{W}} w(\chi(x)) = \sum_{k=0}^{\infty}(q;q)_k \sum_{w \in \mathfrak{W}} w(\chi(x'_k)) = \sum_{k=0}^{\infty}(q;q)_k y_k.$$

Applying χ^{-1} to the above, we get the form required in (a), $|\mathfrak{W}|x = \sum_{k=0}^{\infty}(q;q)_k x_k$.

(b) Since $(q;q)_k x'_k \in \mathcal{U}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B}$ and $\mathcal{U}^{\mathrm{ev},0} = \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$, one has

$$(q;q)_k y_k = (q;q)_k \sum_{w \in \mathfrak{W}} w(\chi(x'_k)) \in \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} \otimes_{\mathcal{A}} \mathcal{B}.$$

(c) Because $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \subset \mathcal{U}^{\mathrm{ev}}$, we have $\mathcal{T}_{\pm}(\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}}) \subset \mathcal{A}$, by Theorem 8.11(f). From (b), we have

$$(q;q)_k \mathcal{T}_{\pm}(x_k) \in \mathcal{A} \otimes_{\mathcal{A}} \mathcal{B} = \mathcal{B},$$

or

$$\mathcal{T}_{\pm}(x_k) \in \frac{1}{(q;q)_k}\mathcal{B}. \tag{204}$$

A simple calculation shows that $\chi(\boldsymbol{r}) = v^{(\rho,\rho)} K_{2\rho} \boldsymbol{r}_0$. Since $\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$ is an $\widetilde{\mathcal{A}}$–Hopf algebra (Lemma 5.25), we have

$$\Delta(K_{2\rho}\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}) \subset K_{2\rho}\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0} \otimes K_{2\rho}\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}.$$

Since $\mathbb{D}\in K_{2\rho}\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$, we have $\mathbb{D}y_k\in K_{2\rho}\boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$. Hence, $\Delta(\mathbb{D}y_k)=\sum K_{2\rho}y_k'\otimes K_{2\rho}y_k''$, where $y_k', y_k''\in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}\otimes_{\mathcal{A}}\mathcal{B}$. Since $\mathbb{D}K_{\pm 2\rho}\in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$, we have $\Delta(\mathbb{D}K_{\pm 2\rho})=\sum a_1\otimes a_2$ with $a_1, a_2\in \boldsymbol{X}_{\mathbb{Z}}^{\mathrm{ev},0}$. Using (201), we have

$$\begin{aligned}\mathbb{d}\mathcal{T}_{\pm}(x_k) &= \mathbb{d}\langle \boldsymbol{r}^{\pm 1}, x_k\rangle = \langle \mathbb{D}\chi(\boldsymbol{r}^{\pm 1}), \mathbb{D}\chi(x_k)\rangle\\ &= v^{(\rho,\rho)}\langle \mathbb{D}K_{\pm 2\rho}\boldsymbol{r}_0^{\pm 1}, \mathbb{D}y_k\rangle\\ &= v^{(\rho,\rho)}\sum \langle \mathbb{D}K_{\pm 2\rho}, K_{2\rho}y_k'\rangle\langle \boldsymbol{r}_0^{\pm 1}, K_{2\rho}y_k''\rangle \qquad \text{by (95)}\\ &= v^{(\rho,\rho)}\sum \langle a_1, K_{\pm 2\rho}\rangle\langle a_2, y_k'\rangle\langle \boldsymbol{r}_0^{\pm 1}, K_{2\rho}y_k''\rangle \quad \text{again by (95).}\end{aligned}$$

The first two factors $\langle a_1, K_{\pm 2\rho}\rangle$ and $\langle a_2, y_k'\rangle$ are in $\widetilde{\mathcal{B}}$ by Lemma 5.25, where $\widetilde{\mathcal{B}}=\widetilde{\mathcal{A}}\otimes_{\mathcal{A}}\mathcal{B}$. The third factor $\langle \boldsymbol{r}_0^{\pm 1}, K_{2\rho}y_k''\rangle$ is in $v^{(\rho,\rho)}\widetilde{\mathcal{B}}$ by Lemma 5.29. Hence $\mathbb{d}\mathcal{T}_{\pm}(x_k)\in v^{2(\rho,\rho)}\widetilde{\mathcal{B}}=\widetilde{\mathcal{B}}$. Together with (204),

$$\mathbb{d}\,\mathcal{T}_{\pm}(x_k)\in \mathbb{C}(v)\cap\widetilde{\mathcal{B}}=\mathcal{B}.$$

(d) By definition, $\Omega_{\pm}=\sum c_\lambda^{\pm}V_\lambda$, where the sum is finite and $c_\lambda^{\pm}\in\mathbb{C}$. We have

$$\mathbb{d}\,\widetilde{\mathcal{T}}_{\pm}(x_k)=\sum c_\lambda^{\pm}\,\mathbb{d}\,\mathrm{tr}_q^{V_\lambda}(x_k).$$

Hence, to show that $\mathbb{d}\widetilde{\mathcal{T}}_{\pm}(x_k)\in\mathcal{B}$, it is enough to show that $\mathbb{d}\,\mathrm{tr}^{V_\lambda}(x_k)\in\mathcal{B}$ for any $\lambda\in X_+$. Using (193) and (201), we have

$$\begin{aligned}|\mathfrak{W}|\mathbb{d}\,\mathrm{tr}_q^{V_\lambda}(x_k) &= |\mathfrak{W}|\mathbb{d}\langle z_\lambda, x_k\rangle = \langle \mathbb{D}\chi(z_\lambda), \mathbb{D}\chi(x_k)\rangle\\ &= \Big\langle \sum_{w\in\mathfrak{W}} \mathrm{sgn}(w)K_{-2w(\lambda+\rho)}, \mathbb{D}y_k\Big\rangle \qquad \text{by (194)}\\ &= \sum_{w\in\mathfrak{W}}\mathrm{sgn}(w)\langle K_{-2w(\lambda+\rho)}, \mathbb{D}y_k\rangle\\ &= \sum_{w\in\mathfrak{W}}\mathrm{sgn}(w)\langle K_{-2w(\lambda+\rho)}, \mathbb{D}\rangle\langle K_{-2w(\lambda+\rho)}, y_k\rangle.\end{aligned}$$

The second factor $\langle K_{-2w(\lambda+\rho)}, y_k\rangle$ is in $\widetilde{\mathcal{B}}$ by Lemma 5.25. As for the first factor, for any $\mu\in X$,

$$\begin{aligned}\langle K_{2\mu}, \mathbb{D}\rangle &= \Big\langle K_{2\mu}, \prod_{\alpha\in\Phi_+}(K_\alpha-K_\alpha^{-1})\Big\rangle\\ &= \prod_{\alpha\in\Phi_+}(\langle K_{2\mu}, K_\alpha\rangle-\langle K_{2\mu}, K_{-\alpha}\rangle)=\prod_{\alpha\in\Phi_+}(v^{-(\mu,\alpha)}-v^{(\mu,\alpha)})\in\mathbb{C}[v^{\pm 1}].\end{aligned}$$

Hence, $\mathbb{d}\,\mathrm{tr}_q^{V_\lambda}(x_k)\in\widetilde{\mathcal{B}}$.

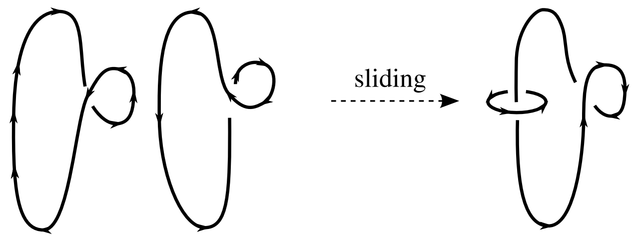


Figure 15: Links L_1 (left) and L_2, which is obtained from L_1 by sliding. Here $\varepsilon = -1$

On the other hand, since $(q;q)x_k \in V_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B}$, we have $\langle z_\lambda, (q;q)x_k\rangle \in \mathcal{B}$. Hence,

$$\mathbb{d}\,\operatorname{tr}_q^{V_\lambda}(x_k) \in \widetilde{\mathcal{B}} \cap \mathbb{C}(v) = \mathbb{C}[v^{\pm 1}].$$

This completes the proof of the lemma. □

8R Comparing $\mathcal{T}$ and $\widetilde{\mathcal{T}}$

Proposition 8.28 *Suppose Ω is a strong Kirby color at level ζ, $x \in (\widetilde{\mathcal{K}}'_m)^{\mathrm{inv}}$, and $\varepsilon_j = \pm 1$ for $j = 1, \dots, m$. Then*

$$\Big(\bigotimes_{j=1}^{m} \widetilde{\mathcal{T}}_{\varepsilon_j}\Big)(x) =_{(\zeta)} \Big(\bigotimes_{j=1}^{m} \mathcal{T}_{\varepsilon_j}\Big)(x).$$

Proof We proceed in three steps:

Step 1 ($m = 1$ and $x \in (V_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B})^{\mathrm{inv}} = \mathfrak{Z}(V_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B})$) By Proposition 8.22, x is a $\mathcal{B}$–linear combination of z_λ, $\lambda \in Y$. We can assume that $x = z_\lambda$ for some $\lambda \in X_+ \cap Y$.

Let L_1 be the disjoint union of $U_{-\varepsilon}$ and U_ε, where the first is colored by V_λ and the second by Ω. Sliding the first component over the second, from L_1 we get a link L_2, which is the Hopf link where the first component has framing 0 and the second has framing ε; see Figure 15. From the strong handle slide invariance (178) we get

$$J_{L_1}(V_\lambda, \Omega) =_{(\zeta)} J_{L_2}(V_\lambda, \Omega). \tag{205}$$

Let us rewrite the left-hand side of (205):

$$J_{L_1}(V_\lambda, \Omega) = J_{U_{-\varepsilon}}(V_\lambda) J_{U_\varepsilon}(\Omega) = \operatorname{tr}_q^{V_\lambda}(\boldsymbol{r}^\varepsilon) J_{U_\varepsilon}(\Omega) = \langle z_\lambda, \boldsymbol{r}^\varepsilon\rangle J_{U_\varepsilon}(\Omega) = \mathcal{T}_\varepsilon(z_\lambda) J_{U_\varepsilon}(\Omega).$$

Let L_0 be the Hopf link with 0 framing on both components. Then the right-hand side of (205) is

$$\begin{aligned} J_{L_2}(V_\lambda,\Omega) &= J_{U_\varepsilon}(\Omega) J_{L_0}(V_\lambda,\Omega_\varepsilon) && \text{by (180)} \\ &=_{(\zeta)} J_{U_\varepsilon}(\Omega)\,\mathrm{tr}_q^{\Omega_\varepsilon}(z_\lambda) && \text{by (195)} \\ &= J_{U_\varepsilon}(\Omega)\widetilde{\mathcal{T}}_\varepsilon(z_\lambda). \end{aligned}$$

Comparing the left-hand side and the right-hand side of (205) we get $\mathcal{T}_\varepsilon(z_\lambda) =_{(\zeta)} \widetilde{\mathcal{T}}_\varepsilon(z_\lambda)$.

Step 2 (**$m = 1$ and x is an arbitrary element of $(\widetilde{\mathcal{K}}'_1)^{\mathrm{inv}} = Z(\widetilde{\mathcal{K}}'_1)$**) Let $x = \sum_{k=0}^\infty (q;q)_k x_k$ be the presentation of x described in Lemma 8.27. Since $x_k \in \mathfrak{Z}(\boldsymbol{X}_h)$ and all $\mathcal{T}_\pm$ and $\widetilde{\mathcal{T}}_\pm$ are continuous in the h–adic topology of $\boldsymbol{X}_h$,

$$\mathcal{T}_\pm(x) = \sum_{k=0}^\infty (q;q)_k \mathcal{T}_\pm(x_k), \quad \widetilde{\mathcal{T}}_\pm(x) = \sum_{k=0}^\infty (q;q)_k \widetilde{\mathcal{T}}_\pm(x_k).$$

Both right-hand sides are in $(1/\mathbb{d})\widehat{\mathbb{C}[v]}$ because $\mathcal{T}_\pm(x_k), \widetilde{\mathcal{T}}_\pm(x_k) \in (1/\mathbb{d})\mathbb{C}[v^{\pm 1}]$ by Lemma 8.27. Since $(q;q)_k =_{(\zeta)} 0$ if $k \ge r$ and $\mathbb{d} \neq_{(\zeta)} 0$, we have

$$\begin{aligned} \mathcal{T}_\pm(x) &=_{(\zeta)} \sum_{k=0}^{r-1} (q;q)_k \mathcal{T}_\pm(x_k) =_{(\zeta)} \mathcal{T}_\pm\Big(\sum_{k=0}^{r-1} (q;q)_k x_k\Big), \\ \widetilde{\mathcal{T}}_\pm(x) &=_{(\zeta)} \sum_{k=0}^{r-1} (q;q)_k \widetilde{\mathcal{T}}_\pm(x_k) =_{(\zeta)} \widetilde{\mathcal{T}}_\pm\Big(\sum_{k=0}^{r-1} (q;q)_k x_k\Big). \end{aligned}$$

By Lemma 8.27(b), the elements in the big parentheses are in $\mathfrak{Z}(\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev}} \otimes_{\mathcal{A}} \mathcal{B})$. Hence, by the result of Step 1, we have $\mathcal{T}_\pm(x) =_{(\zeta)} \widetilde{\mathcal{T}}_\pm(x)$.

Step 3 (**general case**) Define a_k (for $k = 0, 1, \dots, m$) and b_k (for $k = 1, \dots, m$) as follows:

$$a_k = \Big(\bigotimes_{j=1}^{k} \widetilde{\mathcal{T}}_{\varepsilon_j} \otimes \bigotimes_{j=k+1}^{m} \mathcal{T}_{\varepsilon_j}\Big)(x), \quad b_k = \Big(\bigotimes_{j=1}^{k-1} \widetilde{\mathcal{T}}_{\varepsilon_j} \otimes \mathrm{id} \otimes \bigotimes_{j=k+1}^{m} \mathcal{T}_{\varepsilon_j}\Big)(x).$$

Then

$$a_{k-1} = \mathcal{T}_{\varepsilon_k}(b_k) \quad \text{and} \quad a_k = \widetilde{\mathcal{T}}_{\varepsilon_k}(b_k). \tag{206}$$

By Proposition 8.15, $b_k \in (\widetilde{\mathcal{K}}'_1)^{\mathrm{inv}}$. By Step 2,

$$\widetilde{\mathcal{T}}_{\varepsilon_k}(b_k) =_{(\zeta)} \mathcal{T}_{\varepsilon_k}(b_k).$$

Using (206), the above identity becomes $a_{k-1} =_{(\zeta)} a_k$. Since this holds true for $k = 1, 2, \dots, m$, we have $a_0 =_{(\zeta)} a_m$, which is the statement of the proposition. □

8S Proof of Proposition 8.10

By Theorem 7.3, if T is an algebraically split m–component bottom tangle, then $J_T \in \widetilde{\mathcal{K}}_m(\mathcal{U}) \subset \widetilde{\mathcal{K}}'_m$. Hence Proposition 8.10 follows from Proposition 8.28. This also completes the proof of Theorems 8.8 and 8.1.

8T Proof of Theorem 1.1

The existence of the invariant $J_M = J_M^{\mathfrak{g}} \in \widehat{\mathbb{Z}[q]}$ is established by Theorem 7.3. Theorem 8.8 shows that $\mathrm{ev}_\xi(J_M^{\mathfrak{g}}) = \tau_M^{\mathfrak{g}}(\xi)$. The uniqueness of J_M follows from noting

(i) every element of $\widehat{\mathbb{Z}[q]}$ is determined by its values at infinitely many roots of 1 of prime power orders (see Section 1B), and

(ii) $\mathcal{Z}'_{P_{\mathfrak{g}}}$ contains infinitely many such roots of unity (by Proposition 8.4).

This completes the proof of Theorem 1.1.

8U The case $\zeta = 1$, proof of Proposition 1.6

Let Ω be the trivial $\boldsymbol{U}_h$–module $\mathbb{C}[\![h]\!]$. By Proposition 8.6, Ω is a strong Kirby color, and $\tau_M(\Omega) = 1$. By Theorem 8.8, we have $\mathrm{ev}_1(J_M) = 1$. This completes the proof of Proposition 1.6.

Proposition 1.6 can also be proved using the theory of finite-type invariants of integral homology 3–spheres as follows. Note that $\mathrm{ev}_1(J_M)$ is the constant coefficient of the Taylor expansion of J_M at $q = 1$, which is a finite-type invariant of order 0 (see [43], for example). Hence, $\mathrm{ev}_1(J_M)$ is constant on the set of integral homology 3–spheres. For $M = S^3$, we have $\mathrm{ev}_1(J_M) = 1$. Hence, $\mathrm{ev}_1(J_M) = 1$ for any integral homology 3–sphere M.

Appendix A: Another proof of Proposition 4.1

In the main text we take Proposition 4.1 from work of Drinfel'd [17] and Gavarini [18]. Here we give an independent proof.

Each of $\boldsymbol{U}_h^{\le 0} := (\boldsymbol{U}_h^0 \boldsymbol{U}_h^-)^\wedge$ and $\boldsymbol{U}_h^{\ge 0} := (\boldsymbol{U}_h^0 \boldsymbol{U}_h^+)^\wedge$, where $(\cdot)^\wedge$ denotes the h–adic completion, is a Hopf subalgebra of $\boldsymbol{U}_h$, and $\mathcal{R} \in \boldsymbol{U}_h^{\le 0} \mathbin{\hat\otimes} \boldsymbol{U}_h^{\ge 0}$. Let $A_L \subset \boldsymbol{U}_h^{\le 0}$ and $A_R \subset \boldsymbol{U}_h^{\ge 0}$ be the left image (see Section 2D) and right image of $\mathcal{R} \in \boldsymbol{U}_h^{\le 0} \mathbin{\hat\otimes} \boldsymbol{U}_h^{\ge 0}$, respectively. Here the right image is the obvious counterpart of the left image and can be formally defined so that A_R is the left image of $\sigma_{21}(\mathcal{R})$, where

$$\sigma_{21}\colon \boldsymbol{U}_h^{\le 0} \mathbin{\hat\otimes} \boldsymbol{U}_h^{\ge 0} \to \boldsymbol{U}_h^{\ge 0} \mathbin{\hat\otimes} \boldsymbol{U}_h^{\le 0}$$

is the isomorphism given by $\sigma_{21}(x \otimes y) = y \otimes x$.

Explicitly, A_L and A_R are defined as follows: For $\boldsymbol{n} = (\boldsymbol{n}_1, \boldsymbol{n}_2) \in \mathbb{N}^t \times \mathbb{N}^l$ let

$$\mathcal{R}'(\boldsymbol{n}) = F^{(\boldsymbol{n}_1)} H^{\boldsymbol{n}_2}, \quad \mathcal{R}''(\boldsymbol{n}) = E^{(\boldsymbol{n}_1)} \breve{H}^{\boldsymbol{n}_2}.$$

Then $\{\mathcal{R}'(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l}\}$ is a topological basis of $\boldsymbol{U}_h^{\le 0}$ and $\{\mathcal{R}''(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l}\}$ is a topological basis of $\boldsymbol{U}_h^{\ge 0}$. From (70), there are units $f(\boldsymbol{n})$ in $\mathbb{C}[\![h]\!]$ such that

$$\mathcal{R} = \sum_{\boldsymbol{n} \in \mathbb{N}^{t+l}} f(\boldsymbol{n}) h^{\|\boldsymbol{n}\|} \mathcal{R}'(\boldsymbol{n}) \otimes \mathcal{R}''(\boldsymbol{n}).$$

Then A_L and A_R are the topological closures (in $\boldsymbol{U}_h$) of the $\mathbb{C}[\![h]\!]$–span of

$$\{h^{\|\boldsymbol{n}\|} \mathcal{R}'(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l}\} \quad \text{and} \quad \{h^{\|\boldsymbol{n}\|} \mathcal{R}''(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^{t+l}\}, \tag{207}$$

respectively.

For $\mathbb{C}[\![h]\!]$–submodules $\mathscr{H}_1, \mathscr{H}_2 \subset \boldsymbol{U}_h$, let $\overline{\mathscr{H}_1 \otimes \mathscr{H}_2}$, called the *closed tensor product*, be the topological closure of $\mathscr{H}_1 \otimes \mathscr{H}_2$ in the h–adic topology of $\boldsymbol{U}_h \mathbin{\hat\otimes} \boldsymbol{U}_h$.

Proposition A.1 *For each of $A = A_L, A_R$, one has*

$$\boldsymbol{\mu}(\overline{A \otimes A}) \subset A, \quad \Delta(A) \subset \overline{A \otimes A}, \quad S(A) \subset A.$$

This means, each of A_L and A_R is a Hopf algebra in the category where the completed tensor product is replaced by the closed tensor product.

Remark A.2 When the ground ring is a field, the fact that both A_L and A_R are Hopf subalgebras is proved in [69]. Here we modify the proof in [69] for the case when the ground ring is $\mathbb{C}[\![h]\!]$.

Proof We prove the proposition for $A = A_L$ since the case $A = A_R$ is quite analogous.

Let $\tilde{\mathcal{R}}'(\boldsymbol{n}) = f(\boldsymbol{n}) h^{\|\boldsymbol{n}\|} \mathcal{R}'(\boldsymbol{n})$. Then $\mathcal{R} = \sum_{\boldsymbol{n}} \tilde{\mathcal{R}}'(\boldsymbol{n}) \otimes \mathcal{R}''(\boldsymbol{n})$. Using the defining relation $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, we have

$$\sum_{\boldsymbol{n}} \Delta(\tilde{\mathcal{R}}'(\boldsymbol{n})) \otimes \mathcal{R}''(\boldsymbol{n}) = \sum_{\boldsymbol{k},\boldsymbol{m}} \tilde{\mathcal{R}}'(\boldsymbol{m}) \otimes \tilde{\mathcal{R}}'(\boldsymbol{k}) \otimes \mathcal{R}''(\boldsymbol{m}) \mathcal{R}''(\boldsymbol{k}). \tag{208}$$

Since $\{\mathcal{R}''(\boldsymbol{n})\}$ is a topological basis of $\boldsymbol{U}_h^{\ge 0}$, there are structure constants $f^{\boldsymbol{n}}_{\boldsymbol{m},\boldsymbol{k}} \in \mathbb{C}[\![h]\!]$ such that

$$\mathcal{R}''(\boldsymbol{m})\mathcal{R}''(\boldsymbol{k}) = \sum_{\boldsymbol{n}} f^{\boldsymbol{n}}_{\boldsymbol{m},\boldsymbol{k}} \mathcal{R}''(\boldsymbol{n}),$$

and the right-hand side converges. Using the above in (208), we have

$$\Delta(\widetilde{\mathcal{R}}'(\boldsymbol{n})) = \sum_{\boldsymbol{m},\boldsymbol{k}} f^{\boldsymbol{n}}_{\boldsymbol{m},\boldsymbol{k}} \widetilde{\mathcal{R}}'(\boldsymbol{m}) \otimes \widetilde{\mathcal{R}}'(\boldsymbol{k}),$$

with the right-hand side convergent in the h–adic topology of $\boldsymbol{U}_h \mathbin{\hat{\otimes}} \boldsymbol{U}_h$. This proves $\Delta(A_L) \subset \overline{A_L \otimes A_L}$. Actually, we just proved that the coproduct in A_L is dual to the product in $\boldsymbol{U}_h^{\ge 0}$.

Similarly, using $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$, one can easily prove that the product in A_L is dual to the coproduct in $\boldsymbol{U}_h^{\ge 0}$, ie

$$\widetilde{\mathcal{R}}'(\boldsymbol{m})\widetilde{\mathcal{R}}'(\boldsymbol{k}) = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}}^{\boldsymbol{m},\boldsymbol{k}} \widetilde{\mathcal{R}}'(\boldsymbol{n}), \quad \text{where } \Delta(\mathcal{R}''(\boldsymbol{n})) = \sum_{\boldsymbol{m},\boldsymbol{k}} f_{\boldsymbol{n}}^{\boldsymbol{m},\boldsymbol{k}} \mathcal{R}''(\boldsymbol{m}) \otimes \mathcal{R}''(\boldsymbol{k}).$$

This proves that $\boldsymbol{\mu}(\overline{A_L \otimes A_L}) \subset A_L$.

Next we consider the antipode. We have $(S \mathbin{\hat{\otimes}} \mathrm{id})(\mathcal{R}) = (\mathrm{id} \mathbin{\hat{\otimes}} S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}$. Let A'_L be the left image of $\mathcal{R}^{-1}$.

Since S^{-1} is a $\mathbb{C}[\![h]\!]$–module automorphism of $\boldsymbol{U}_h^{\ge 0}$, the identity $(\mathrm{id} \mathbin{\hat{\otimes}} S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}$ shows that $A'_L = A_L$.

Identity $(S \mathbin{\hat{\otimes}} \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ shows that $A'_L = S(A_L)$. Thus, we have $A_L = S(A_L)$. □

Proposition 4.1 follows immediately from:

Proposition A.3 (a) *One has $\overline{A_R A_L} = \overline{A_L A_R}$. It follows that $\overline{A_L A_R}$ is a Hopf algebra with closed tensor products.*

(b) *One has $\overline{A_R A_L} = \boldsymbol{V}_h$.*

Proof (a) We use the following identities in a ribbon Hopf algebra: for every $y \in \boldsymbol{U}_h$ one has

$$\mathcal{R}(y \otimes 1) = \sum_{(y)} (y_{(2)} \otimes y_{(1)}) \mathcal{R}(1 \otimes S(y_{(3)})), \tag{209}$$

$$(y \otimes 1)\mathcal{R} = \sum_{(y)} (1 \otimes S(y_{(1)})) \mathcal{R}(y_{(2)} \otimes y_{(3)}), \tag{210}$$

(211) $$\mathcal{R}(1 \otimes y) = \sum_{(y)} (y_{(3)} \otimes y_{(2)}) \mathcal{R}(S^{-1}(y_{(1)} \otimes 1)),$$

(212) $$(1 \otimes y)\mathcal{R} = \sum_{(y)} (S^{-1}(y_{(1)} \otimes 1)) \mathcal{R}(y_{(1)} \otimes y_{(2)}),$$

which are [69, (6)–(9)]. Suppose $x \in A_L$ and $y \in A_R$. We will show that $xy \in \overline{A_R A_L}$. This will imply that $\overline{A_L A_R} \subset \overline{A_R A_L}$. We only need the fact that A_R is a coalgebra in the closed category: $\Delta(A_R) \subset \overline{A_R \otimes A_R} \subset \boldsymbol{U}_h^{\geq 0} \mathbin{\hat{\otimes}} \boldsymbol{U}_h^{\geq 0}$.

Since $x \in A_L$, we have a presentation

$$x = \sum_{\boldsymbol{n} \in \mathbb{N}^{t+l}} x_{\boldsymbol{n}} \widetilde{\mathcal{R}}'(\boldsymbol{n}), \quad x_{\boldsymbol{n}} \in \mathbb{C}[\![h]\!] \text{ for all } \boldsymbol{n} \in \mathbb{N}^{t+l}.$$

Let $p\colon \boldsymbol{U}_h^{\geq 0} \to \mathbb{C}[\![h]\!]$ be the unique $\mathbb{C}[\![h]\!]$–module homomorphism with $p(\mathcal{R}''(\boldsymbol{n})) = x_{\boldsymbol{n}}$. Then $x = \sum_{\boldsymbol{n}} \widetilde{\mathcal{R}}'(\boldsymbol{n}) p(\mathcal{R}''(n))$. Hence,

$$xy = \sum_{\boldsymbol{n}} \widetilde{\mathcal{R}}'(\boldsymbol{n}) y p(\mathcal{R}''(n)) = \sum_{\boldsymbol{n}} y_{(2)} \widetilde{\mathcal{R}}'(\boldsymbol{n}) p(y_{(1)} \mathcal{R}''(n) S(y_{(3)})) \in \overline{A_R A_L}.$$

Similarly, one can prove $\overline{A_R A_L} \subset \overline{A_L A_R}$, and conclude that $\overline{A_L A_R} = \overline{A_R A_L}$.

(b) The two sets $\{h^{\|\boldsymbol{n}\|} H^{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^l\}$ and $\{h^{\|\boldsymbol{n}\|} \breve{H}^{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{N}^l\}$ span the same $\mathbb{C}[\![h]\!]$–subspace of $\boldsymbol{U}_h^0$. Using spanning sets (207), we see that $\overline{A_L A_R}$ is the topological closure of the $\mathbb{C}[\![h]\!]$–span of

$$\{h^{\|\boldsymbol{n}_1\| + \|\boldsymbol{n}_2\| + \|\boldsymbol{n}_3\|} F^{(\boldsymbol{n}_1)} H^{\boldsymbol{n}_2} E^{(\boldsymbol{n}_3)} \mid \boldsymbol{n}_1, \boldsymbol{n}_3 \in \mathbb{N}^t,\ \boldsymbol{n}_2 \in \mathbb{N}^l\}.$$

Comparing this set with the formal basis (81) of $\boldsymbol{V}_h$, one can easily show that $\boldsymbol{V}_h = \overline{A_L A_R}$. □

Appendix B: Integral duality

B.1 Decomposition of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0}$

Recall that

$$\boldsymbol{U}_q^{\mathrm{ev},0} = \mathbb{C}(v)[K_\alpha^{\pm 2}, \alpha \in \Pi], \quad \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} = \mathcal{A}[K_\alpha^{\pm 2}, \alpha \in \Pi].$$

For a simple root $\alpha \in \Pi$, *the even α–part of $\boldsymbol{U}_{\mathbb{Z}}^0$* is defined to be $\mathcal{I}_\alpha := \mathbb{C}(v)[K_\alpha^{\pm 2}] \cap \boldsymbol{U}_{\mathbb{Z}}^0$. Note that $\mathcal{I}_\alpha$ is an $\mathcal{A}$–Hopf subalgebra of $\mathbb{Q}(v)[K_\alpha^{\pm 2}]$. From Proposition 5.2, $\mathcal{I}_\alpha$ is $\mathcal{A}$–spanned by

(213) $$\left\{ \frac{K_\alpha^{2m} (q_\alpha^n K_\alpha^2; q_\alpha)_k}{(q_\alpha; q_\alpha)_k} \;\middle|\; m, n \in \mathbb{Z},\ k \in \mathbb{N} \right\},$$

and there is an isomorphism

$$\bigotimes_{\alpha\in\Pi} \mathcal{I}_\alpha \xrightarrow{\cong} \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0}, \quad \bigotimes_{\alpha\in\Pi} a_\alpha \mapsto \prod_\alpha a_\alpha. \tag{214}$$

Hence, if one can find $\mathcal{A}$–bases for $\mathcal{I}_\alpha$, then one can combine them using (214) to get an $\mathcal{A}$–basis for $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0}$.

Similarly, let $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} \cap \mathbb{C}(v)[K_\alpha^{\pm 2}] = \mathcal{A}[K_\alpha^{\pm 2}]$ be the *even α–part* of

$$\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} = \mathcal{A}[K_\alpha^{\pm 2}, \alpha \in \Pi].$$

The analog of (214) is much easier for $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$, since in this case it is

$$\bigotimes_{\alpha\in\Pi} \mathcal{A}[K_\alpha^{\pm 2}] \xrightarrow{\cong} \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0} = \mathcal{A}[K_\alpha^{\pm 2}, \alpha \in \Pi], \quad \bigotimes_{\alpha\in\Pi} a_\alpha \mapsto \prod_\alpha a_\alpha. \tag{215}$$

B.2 Bases for $\mathcal{I}_\alpha$ and $\mathcal{A}[x^{\pm 1}]$

Fix $\alpha \in \Pi$, and let $x = K_\alpha^2$ and $y = \breve{K}_\alpha^2$. The even α–part of $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$ is $\mathcal{A}[x^{\pm 1}]$, and $\mathcal{I}_\alpha$, the even α–part of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0}$, is now an $\mathcal{A}$–submodule of $\mathbb{Q}(v)[x^{\pm 1}]$. The quantum Killing form restricts to the $\mathbb{Q}(v)$–bilinear form

$$\langle \cdot,\cdot \rangle\colon \mathbb{Q}(v)[x^{\pm 1}] \otimes \mathbb{Q}(v)[y^{\pm 1}] \to \mathbb{Q}(v), \quad \langle x^m, y^n \rangle = q_\alpha^{-mn}. \tag{216}$$

Let $\breve{\iota}\colon \mathbb{Q}(v)[x^{\pm 1}] \to \mathbb{Q}(v)[y^{\pm 1}]$ be the $\mathbb{Q}(v)$–algebra map defined by $\breve{\iota}(x) = y$. For $n \in \mathbb{N}$, let

$$\begin{aligned} Q'(\alpha;n) &:= x^{-\lfloor n/2\rfloor}(q_\alpha^{-\lfloor (n-1)/2\rfloor}x;q_\alpha)_n, & \breve{Q}'(\alpha,n) &:= \breve{\iota}(Q'(\alpha;n)),\\ Q(\alpha;n) &:= \frac{Q'(\alpha;n)}{(q_\alpha;q_\alpha)_n}, & \breve{Q}(\alpha,n) &:= \breve{\iota}(Q(\alpha;n)). \end{aligned} \tag{217}$$

We will consider $\mathcal{A}[x^{\pm 1}] \subset \mathbb{C}[H_\alpha][\![h]\!]$ by setting $x = \exp(hH_\alpha)$.

Proposition B.1 (a) *The $\mathcal{A}$–module $\mathcal{I}_\alpha$ is the $\mathcal{A}$–dual of $\mathcal{A}[y^{\pm 1}]$ with respect to the form* (216) *in the sense that*

$$\mathcal{I}_\alpha = \big\{ f(x) \in \mathbb{Q}(v)[x^{\pm 1}] \mid \langle f(x), g(y)\rangle \in \mathcal{A} \text{ for all } g(y) \in \mathbb{Q}(v)[y^{\pm 1}] \big\}.$$

(b) *The set $\{Q'(\alpha;n) \mid n \in \mathbb{N}\}$ is an $\mathcal{A}$–basis of $\mathcal{A}[x^{\pm 1}]$.*

(c) *One has the orthogonality*

$$\langle Q(\alpha;n), Q'(\alpha;m)\rangle = \delta_{m,n} q_\alpha^{-\lfloor (n+1)/2\rfloor^2}. \tag{218}$$

(d) *The set $\{Q(\alpha;n) \mid n \in \mathbb{N}\}$ is an $\mathcal{A}$–basis of $\mathcal{I}_\alpha$.*

Proof (a) In Section B.1, $\mathcal{I}_\alpha$ is the $\mathcal{A}$–submodule of $\mathbb{Q}(v)[x^{\pm 1}]$ spanned by the set (213) with K_α^2 replaced by x. This set spans the module of polynomial with q–integral values: By [7, Proposition 2.6], $\mathcal{I}_\alpha$ is exactly the set of all Laurent polynomials $f(x) \in \mathbb{Q}(v)[x^{\pm 1}]$ such that $f(q_\alpha^k) \in \mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ for every $k \in \mathbb{Z}$.

For $f(x) \in \mathbb{Q}(v)[x^{\pm 1}]$, $g(y) \in \mathbb{Q}(v)[y^{\pm 1}]$ and $k \in \mathbb{Z}$, from (216),

$$\langle f(x), y^k \rangle = f(q_\alpha^{-k}), \quad \langle x^k, g(y) \rangle = g(q_\alpha^{-k}). \tag{219}$$

Suppose now $f(x) \in \mathbb{Q}(v)[x^{\pm 1}]$. Since $\{y^k \mid k \in \mathbb{Z}\}$ is an $\mathcal{A}$–basis of $\mathcal{A}[y^{\pm 1}]$,

$$\begin{aligned} f(x) \text{ is in the } \mathcal{A}\text{–dual of } \mathcal{A}[y^{\pm 1}] \quad &\Longleftrightarrow \quad \langle f(x), y^k \rangle \in \mathcal{A} \quad \text{for all } k \in \mathbb{Z} \\ &\Longleftrightarrow \quad f(q_\alpha^{-k}) \in \mathcal{A} \qquad \text{for all } k \in \mathbb{Z} \\ &\Longleftrightarrow \quad f(x) \in \mathcal{I}_\alpha. \end{aligned}$$

This proves part (a).

(b) The bijective map $j\colon \mathbb{N} \to \mathbb{Z}$ given by $j(n) = (-1)^{n+1} \left\lfloor \frac{1}{2}(n+1) \right\rfloor$ defines an order on $\mathbb{Z}$, by $j(0) \prec j(1) \prec j(2) \prec \cdots$. This order looks as follows:

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec -3 \prec \cdots$$

We define an order on the set of monomials $\{x^n \mid n \in \mathbb{Z}\}$ by $x^n \prec x^m$ if $n \prec m$. Using this order, one can define the leading term of a nonzero Laurent polynomial $f(x) \in \mathbb{Q}(v)[x^{\pm 1}]$. One can easily calculate the leading term of $Q'(\alpha; n)$,

$$Q'(\alpha; n) = (-1)^n x^{j(n)} + \text{ lower order terms.} \tag{220}$$

It follows that $\{Q'(\alpha; n) \mid n \in \mathbb{N}\}$ is an $\mathcal{A}$–basis of $\mathcal{A}[x^{\pm 1}]$.

(c) Suppose $m < n$. By (219),

$$\langle Q'(\alpha; n), y^{j(m)} \rangle = Q'(\alpha; n)\big|_{x = q_\alpha^{-j(m)}} = 0,$$

since $x = q_\alpha^{-j(m)}$ annihilates one of the factors of $Q'(\alpha; n)$ when $m < n$. By expanding $\breve{Q}'(\alpha; m)$ using (220), we have

$$\langle Q'(\alpha; n), \breve{Q}'(\alpha; m) \rangle = 0 \quad \text{if } m < n.$$

Similarly, one also has $\langle Q'(\alpha; n), \breve{Q}'(\alpha; m) \rangle = 0$ if $m > n$. It remains to consider the case $m = n$. Using (220), we have

$$\begin{aligned} \langle Q'(\alpha; n), \breve{Q}'(\alpha; n) \rangle &= \langle Q'(\alpha; n), (-1)^n y^{j(n)} \rangle \\ &= (-1)^n Q'(\alpha; n)\big|_{x = q_\alpha^{-j(n)}} = q_\alpha^{-\lfloor (n+1)/2 \rfloor^2} (q_\alpha; q_\alpha)_n, \end{aligned}$$

where the last identity follows from an easy calculation. This proves part (c).

(d) By part (b), $\{\breve{Q}'(\alpha;n) \mid n \in \mathbb{N}\}$ is an $\mathcal{A}$–basis of $\mathcal{A}[y^{\pm 1}]$. Because $\mathcal{I}_\alpha$ is the $\mathcal{A}$–dual of $\mathcal{A}[y^{\pm 1}]$ with respect to the form (216), the orthogonality (218) shows that $\{Q(\alpha;n) \mid n \in \mathbb{N}\}$ is an $\mathcal{A}$–basis of $\mathcal{I}_\alpha$. This proves part (d). □

B.3 Proof of Proposition 5.7

Proof (a) The definition (107) means that, for $\boldsymbol{n} = (n_1, \dots, n_l) \in \mathbb{N}^t$,

$$Q^{\mathrm{ev}}(\boldsymbol{n}) := \prod_{j=1}^{l} Q(\alpha_j; n_j)|_{x=K_j^2}, \quad (q;q)_{\boldsymbol{n}} Q^{\mathrm{ev}}(\boldsymbol{n}) = \prod_{j=1}^{l} Q'(\alpha_j; n_j)|_{x=K_j^2}. \tag{221}$$

By Proposition B.1(d),

$$\{Q(\alpha_j; n)|_{x=K_j^2} \mid n \in \mathbb{N}\}$$

is an $\mathcal{A}$–basis of $\mathcal{I}_{\alpha_j}$. Hence the isomorphism (214) shows that $\{Q^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}$ is an $\mathcal{A}$–basis of $\boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0}$.

Similarly, Proposition B.1(b) and (215) show that $\{(q;q)_{\boldsymbol{n}} Q^{\mathrm{ev}}(\boldsymbol{n}) \mid \boldsymbol{n} \in \mathbb{N}^l\}$ is an $\mathcal{A}$–basis of $\boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}$.

(b) Let $K^{\boldsymbol{\delta}} = \prod_j K_{\alpha_j}^{\delta_j}$ for $\boldsymbol{\delta} = (\delta_1, \dots, \delta_l)$. We have

$$\boldsymbol{V}_{\mathbb{Z}}^0 = \bigoplus_{\boldsymbol{\delta} \in \{0,1\}^l} K^{\boldsymbol{\delta}} \boldsymbol{V}_{\mathbb{Z}}^{\mathrm{ev},0}, \quad \boldsymbol{U}_{\mathbb{Z}}^0 = \bigoplus_{\boldsymbol{\delta} \in \{0,1\}^l} K^{\boldsymbol{\delta}} \boldsymbol{U}_{\mathbb{Z}}^{\mathrm{ev},0},$$

where the first identity is obvious and the second follows from Proposition 5.2. Hence, (b) follows from (a). This completes the proof of Proposition 5.7. □

B.4 Proof of Lemma 5.16

Proof For α, $\beta \in \Pi$, we have $\langle K_\alpha^2, \breve{K}_\beta^2 \rangle = \delta_{\alpha,\beta} q_\alpha$. Hence, with $Q^{\mathrm{ev}}(\boldsymbol{n})$, $\breve{Q}^{\mathrm{ev}}(\boldsymbol{m})$ as in (221),

$$\langle Q^{\mathrm{ev}}(\boldsymbol{n}), \breve{Q}^{\mathrm{ev}}(\boldsymbol{m}) \rangle = \prod_{j=1}^{l} \langle Q(\alpha_j; n_j), \breve{Q}(\alpha_j; m_j) \rangle = \delta_{\boldsymbol{n},\boldsymbol{m}} \prod_{j=1}^{l} q_j^{-\lfloor (n_j+1)/2 \rfloor^2},$$

where the last identity follows from Proposition B.1(c). This proves Lemma 5.16. □

Appendix C: On the existence of the WRT invariant

Here we prove Proposition 8.4 on the existence of strong Kirby colors at every level ζ such that $\mathrm{ord}(\zeta^{2D}) > d(h^\vee - 1)$. We also determine when $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$ and when $\zeta \in \mathcal{Z}'_{P\mathfrak{g}}$, if $\mathrm{ord}(\zeta^{2D}) > d(h^\vee - 1)$.

C.1 Criterion for nonvanishing of Gauss sums

Suppose $\mathfrak{A}$ is a free abelian group of rank l and $\phi\colon \mathfrak{A}\times\mathfrak{A}\to\mathbb{Z}$ is a symmetric $\mathbb{Z}$–bilinear form. Assume further ϕ is even, in the sense that $\phi(x,x)\in 2\mathbb{Z}$ for every $x\in\mathfrak{A}$.

The *quadratic Gauss sum associated to* ϕ *at level* $m\in\mathbb{N}$ is defined by

$$\mathfrak{G}_\phi(m) := \sum_{x\in\mathfrak{A}/m\mathfrak{A}} \exp\left(\pi i \frac{\phi(x,x)}{m}\right).$$

Let $\mathfrak{A}^*_\phi$ be the $\mathbb{Z}$–dual of $\mathfrak{A}$ with respect to ϕ and

$$\ker_\phi(m) := \{x\in\mathfrak{A} \mid \phi(x,y)\in m\mathbb{Z} \text{ for all } y\in\mathfrak{A}\} = m\mathfrak{A}^*_\phi\cap\mathfrak{A}.$$

We have the following well-known criterion for the vanishing of $\mathfrak{G}_\phi(m)$; see [16, Lemma 1].

Lemma C.1 (a) *If m is odd, then $\mathfrak{G}_\phi(m)\neq 0$.*

(b) *$\mathfrak{G}_\phi(m)\neq 0$ if and only if, for every $x\in\ker_\phi(m)$, one has $\frac{1}{2m}\phi(x,x)\in\mathbb{Z}$.*

Lemma C.2 *For every $x\in\ker_\phi(m)$, we have $\frac{1}{2m}\phi(x,x)\in\frac{1}{2}\mathbb{Z}$.*

Proof Because $x\in m\mathfrak{A}^*_\phi$, one has $\phi(x,x)\in m\mathbb{Z}$. Hence, $\frac{1}{2m}\phi(x,x)\in\frac{1}{2}\mathbb{Z}$. □

C.2 Gauss sums on weight lattice

Recall that X and Y are the weight lattice and the root lattice, respectively, in $\mathfrak{h}^*_\mathbb{R}$, which is equipped with the invariant inner product. The $\mathbb{Z}$–dual X^* of X is $\mathbb{Z}$–spanned by α/d_α, $\alpha\in\Pi$.

Lemma C.3 *For $y\in X^*$, we have $(y,y)\in\mathbb{Z}_{(2)} := \{a/b \mid a,b\in\mathbb{Z},\ b$ odd$\}$.*

Proof Suppose $y=\sum k_i\alpha_i/d_i$. Then

$$(y,y) = \sum_i k_i^2\frac{(\alpha_i,\alpha_i)}{d_i^2} + 2\sum_{i<j}\frac{(\alpha_i,\alpha_j)}{d_id_j} = \sum_i k_i^2\frac{2}{d_i} + \sum_{i<j}\frac{2(\alpha_i,\alpha_j)/d_j}{d_i} \in \frac{2}{d}\mathbb{Z}.$$

Since d is one of 1, 2 or 3, we see that $(y,y)\in\mathbb{Z}_{(2)}$. □

Lemma C.4 *Suppose ζ is a root of 1 of order s. Let $r=s/\gcd(s,2D)$ be the order of $\xi=\zeta^{2D}$.*

(a) *Suppose r is odd. Then $\mathfrak{G}^{\mathcal{P}\mathfrak{g}}(\zeta) \neq 0$, where*

$$\mathfrak{G}^{\mathcal{P}\mathfrak{g}}(\zeta) := \sum_{\lambda \in P_\zeta \cap Y} \zeta^{D(\lambda,\lambda+2\rho)} = \sum_{\lambda \in P_\zeta \cap Y} \xi^{(\lambda,\lambda+2\rho)/2}.$$

(b) *Suppose r is even. Then $\mathfrak{G}^{\mathfrak{g}}(\zeta) \neq 0$, where*

$$\mathfrak{G}^{\mathfrak{g}}(\zeta) := \sum_{\lambda \in P_\zeta} \zeta^{D(\lambda,\lambda+2\rho)}.$$

Proof After a Galois transformation of the form $\zeta \to \zeta^k$ with $\gcd(k,s) = 1$ we can assume that $\zeta = \exp(2\pi i/s)$.

(a) The following is the well-known completing the square trick:

$$\begin{aligned}\mathfrak{G}^{\mathcal{P}\mathfrak{g}}(\zeta) &= \sum_{\lambda \in P_\zeta \cap Y} \xi^{(\lambda,\lambda+2\rho(r+1))/2} \quad \text{since } \operatorname{ord}(\xi) = r \\ &= \xi^{-(r+1)^2(\rho,\rho)/2} \sum_{\lambda \in P_\zeta \cap Y} \xi^{(\lambda+(r+1)\rho,\lambda+(r+1)\rho)/2} \\ &= \xi^{-(r+1)^2(\rho,\rho)/2} \sum_{\lambda \in P_\zeta \cap Y} \xi^{(\lambda,\lambda)/2}.\end{aligned}$$

Here the last identity follows because $2\rho \in Y$ and hence $(r+1)\rho \in Y$, since $r+1$ is even and because the shift $\lambda \to \lambda + \beta$ does not change the Gauss sum for any $\beta \in Y$.

The expression $\xi^{(\lambda,\lambda)/2}$, $\lambda \in Y$, is invariant under the translations by vectors in both rY and $2rX$. Hence,

$$\mathfrak{G}^{\mathcal{P}\mathfrak{g}}(\zeta) = \xi^{-(r+1)^2(\rho,\rho)/2} \sum_{\lambda \in P_\zeta \cap Y} \xi^{(\lambda,\lambda)/2} = \xi^{-(r+1)^2(\rho,\rho)/2} \frac{\operatorname{vol}(2rX)}{\operatorname{vol}(rY)} \sum_{\lambda \in Y/rY} \xi^{(\lambda,\lambda)/2}.$$

By Lemma C.1(a) with $\mathfrak{A} = Y$, $\phi(x,y) = (x,y)$ and $m = r$, the right-hand side is nonzero.

(b) Again using the completing the square trick, we get

$$\begin{aligned}\mathfrak{G}^{\mathfrak{g}}(\zeta) &= \zeta^{-D(\rho,\rho)} \sum_{\lambda \in P_\zeta} \zeta^{D(\lambda,\lambda)} \\ &= \zeta^{-D(\rho,\rho)} \sum_{\lambda \in X/2rDX} \exp\Big(\frac{\pi i}{s} 2D(\lambda,\lambda)\Big) \\ &= \zeta^{-D(\rho,\rho)} \Big(\frac{2Dr}{s}\Big)^l \sum_{\lambda \in X/sX} \exp\Big(\frac{\pi i}{s} 2D(\lambda,\lambda)\Big).\end{aligned} \tag{222}$$

Note that $s/\gcd(s,2D)$ is even if and only if

$$\frac{s}{4D} \in \mathbb{Z}_{(2)}. \tag{223}$$

Apply Lemma C.1(b) with $\mathfrak{A} = X$, $\phi(x,y) = 2D(x,y)$ and $m = s$. Then $s\mathfrak{A}^*_\phi = (s/2D)X^*$. Suppose $x \in \ker_\phi(s) = s\mathfrak{A}^*_b \cap \mathfrak{A} \subset s\mathfrak{A}^*_b$. Then $x = (s/2D)y$ with $y \in X^*$. We have

$$\frac{1}{2s}\phi(x,x) = \frac{s}{4D}(y,y) \in \mathbb{Z}_{(2)},$$

where the last inclusion follows from (223) and Lemma C.3. From Lemma C.2 we have

$$\frac{1}{2s}\phi(x,x) \in \frac{1}{2}\mathbb{Z} \cap \mathbb{Z}_{(2)} = \mathbb{Z}.$$

By Lemma C.1(b), the right-hand side of (222) is nonzero. □

C.3 Proof of Proposition 8.4

Proof of Proposition 8.4 By [49, Proposition 2.3 and Theorem 3.3], $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are strong handle-slide colors. Although the formulation in [49] only says that $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are handle-slide colors, the proofs there actually show that $\Omega^{\mathfrak{g}}(\zeta)$ and $\Omega^{P\mathfrak{g}}(\zeta)$ are strong handle-slide colors.

It remains to show that $J_{U_\pm}(\Omega^{\mathfrak{g}}(\zeta)) \neq 0$ if r is even, and $J_{U_\pm}(\Omega^{P\mathfrak{g}}(\zeta)) \neq 0$ if r is odd.

From [49, Section 2.3], with the assumption $\mathrm{ord}(\zeta^{2D}) > d(h^\vee - 1)$, we have

$$\begin{aligned} J_{U_+}(\Omega^{\mathfrak{g}}(\zeta)) &=_{(\zeta)} \frac{\mathfrak{G}^{\mathfrak{g}}(\zeta)}{\prod_{\alpha\in\Phi_+}(1-\xi^{(\alpha,\rho)})}, \\ J_{U_+}(\Omega^{P\mathfrak{g}}(\zeta)) &=_{(\zeta)} \frac{\mathfrak{G}^{P\mathfrak{g}}(\zeta)}{\prod_{\alpha\in\Phi_+}(1-\xi^{(\alpha,\rho)})}. \end{aligned} \tag{224}$$

Further, $J_{U_-}(\Omega^{\mathfrak{g}}(\zeta))$ and $J_{U_-}(\Omega^{P\mathfrak{g}}(\zeta))$ are the complex conjugates of $J_{U_+}(\Omega^{\mathfrak{g}}(\zeta))$ and $J_{U_+}(\Omega^{P\mathfrak{g}}(\zeta))$, respectively.

By Lemma C.4, if $\mathrm{ord}(\zeta^{2D})$ is even then $J_{U_+}(\Omega^{\mathfrak{g}}(\zeta)) \neq 0$, and if $\mathrm{ord}(\zeta^{2D})$ is odd then $J_{U_+}(\Omega^{P\mathfrak{g}}(\zeta)) \neq 0$. This completes the proof of Proposition 8.4. □

C.4 The sets $\mathcal{Z}'_{\mathfrak{g}}$ and $\mathcal{Z}'_{P\mathfrak{g}}$ for each simple Lie algebra

Proposition C.5 (a) *One has $\mathfrak{G}^{\mathfrak{g}}(\zeta) = 0$ in and only in the following cases:*

- $\mathfrak{g} = A_l$ *with l odd and* $\mathrm{ord}(\zeta) \equiv 2 \pmod 4$.
- $\mathfrak{g} = B_l$ *with l odd and* $\mathrm{ord}(\zeta) \equiv 2 \pmod 4$.
- $\mathfrak{g} = B_l$ *with* $l \equiv 2 \pmod 4$ *and* $\mathrm{ord}(\zeta) \equiv 4 \pmod 8$.
- $\mathfrak{g} = C_l$ *and* $\mathrm{ord}(\zeta) \equiv 4 \pmod 8$.

- $\mathfrak{g} = D_l$ *with* l *odd and* $\operatorname{ord}(\zeta) \equiv 2 \pmod 4$.
- $\mathfrak{g} = D_l$ *with* $l \equiv 2 \pmod 4$ *and* $\operatorname{ord}(\zeta) \equiv 4 \pmod 8$.
- $\mathfrak{g} = E_7$ *and* $\operatorname{ord}(\zeta) \equiv 2 \pmod 4$.

(b) *In particular, if* $\operatorname{ord}(\zeta)$ *is odd or* $\operatorname{ord}(\zeta)$ *is divisible by* $2dD$, *then* $\mathfrak{G}^{\mathfrak{g}}(\zeta) \neq 0$.

The proof is a careful, tedious, but not difficult check of the vanishing of the Gaussian sum using Lemma C.1 and the explicit description of the weight lattice for each simple Lie algebra, and we drop the details.

Corollary C.6 *Suppose* $\zeta \in \mathcal{Z}$ *with* $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$. *Then* $\zeta \in \mathcal{Z}'_{\mathfrak{g}}$ *if and only if* ζ *satisfies the condition of Proposition C.5(a).*

Similarly, using Lemma C.1, one can prove the following:

Proposition C.7 *Let* $r = \operatorname{ord}(\xi) = \operatorname{ord}(\zeta^{2D})$.

(a) *One has* $\mathfrak{G}^{P\mathfrak{g}}(\zeta) = 0$ *in and only in the following cases:*
- $\mathfrak{g} = A_l$ *and* $\operatorname{ord}_2(r) = \operatorname{ord}_2(l+1) \geq 1$.
- $\mathfrak{g} = B_l$ *and* $r \equiv 2 \pmod 4$.
- $\mathfrak{g} = C_l$, r *even and* $rl \equiv 4 \pmod 8$.
- $\mathfrak{g} = D_l$, r *even and* $rl \equiv 4 \pmod 8$.
- $\mathfrak{g} = E_7$ *and* $r \equiv 2 \pmod 4$.

Here, $\operatorname{ord}_2(n)$ *is the order of* 2 *in the prime decomposition of the integer* n.

(b) *In particular, if* r *is coprime with* $2^{\operatorname{ord}_2(D)}$, *then* $\mathfrak{G}^{P\mathfrak{g}}(\zeta) \neq 0$.

Corollary C.8 *Suppose* $\operatorname{ord}(\zeta^{2D}) > d(h^\vee - 1)$. *Then* $\zeta \in \mathcal{Z}'_{P\mathfrak{g}}$ *if and only if* ζ *satisfies the condition of Proposition C.7(a).*

List of symbols

Notation	Defined in	Remarks
$\widehat{\mathbb{Z}[q]}$, $(x;q)_n$	1B	
$(\mathbb{C}[\![h]\!]^I)_0$	2A2	
$\mathscr{H}, \boldsymbol{\mu}, \boldsymbol{\eta}, \Delta, \boldsymbol{\epsilon}, S$	2B	Hopf algebra
$\mathcal{R}$	2B, 3G2	R–matrix
$\boldsymbol{r}$	2B, 3G2	ribbon element

$\boldsymbol{g}$	2B	balanced element
$\mathrm{ad}(x \otimes y), x \triangleright y$	2E	adjoint action
tr_q^V	2E	quantum trace
J_T	2G	universal invariant of bottom tangle
$\boldsymbol{\psi}, \underline{\Delta}, \underline{S}$	2I	braiding, transmutation
$\boldsymbol{c}, \boldsymbol{c}^-, C^+, C^-$	2J	clasp, $\boldsymbol{c} = J_{C^+}$
$\mathcal{T}_\pm$	2L, 4H	full twist forms
J_M	2M, 2O	invariant of 3–manifold
Υ	2O2	braided commutator
$\boldsymbol{b}$	2M	universal invariant of Borromean tangle
$\mathscr{L}(x \otimes y), \langle x, y \rangle$	2N, 4F	clasp form
$\mathfrak{g}, l, \mathfrak{h}$	3A1	Lie algebra, its rank, Cartan subalgebra
$d, d_\alpha, t, \mathrm{ht}(\gamma)$	3A1	
X, Y	3A1	weight lattice, root lattice
Π, Φ, Φ_+	3A1	simple roots, all roots, positive roots
$\rho, \alpha_i, \breve{\alpha}_i$	3A1	
$h, v, q, v_\alpha, q_\alpha, \mathcal{A}$	3A2	$q = v^2 = \exp(h)$, $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$
$[n]_\alpha, \{n\}_\alpha, [n]_\alpha!, \{n\}_\alpha!, \begin{bmatrix} n \\ k \end{bmatrix}_\alpha$	3A2	
$\boldsymbol{U}_h, F_\alpha, E_\alpha, H_\lambda, F_i, E_i$	3A3	
$K_\lambda, \breve{K}_\alpha, K_i$	3A3	
$\boldsymbol{U}_q, \breve{\boldsymbol{U}}_q, \breve{\boldsymbol{U}}_q^0$	3A5	
$\iota_{\mathrm{bar}}, \varphi, \omega, \tau$	3B	(anti) automorphisms of $\boldsymbol{U}_h$
$\lvert x \rvert$	3C1	Y–grading
$\boldsymbol{U}_q^{\mathrm{ev}}, \breve{\boldsymbol{U}}_q^{\mathrm{ev}}$	3C2	even grading
$\boldsymbol{U}_h^\pm, \boldsymbol{U}_h^0, \boldsymbol{U}_q^\pm, \boldsymbol{U}_q^0, \boldsymbol{U}_q^{\mathrm{ev},-}, \boldsymbol{U}_q^{\mathrm{ev},0}$	3D	
$\mathfrak{W}, s_\alpha, s_i$	3E	Weyl group, reflection
T_α	3E	braid group action
$E_\gamma, F_\gamma, E^{(\boldsymbol{n})}, F^{(\boldsymbol{n})}, K_{\boldsymbol{n}}$	3F	
$\Theta, E_{\boldsymbol{n}}, F_{\boldsymbol{n}}, E'_{\boldsymbol{n}}, F'_{\boldsymbol{n}}$	3G1	
$D, \breve{H}_\alpha, \boldsymbol{r}_0$	3G2	
Γ	3I	quasiclasp element
$\boldsymbol{U}_{\sqrt{h}}$	4	$\boldsymbol{U}_{\sqrt{h}} := \boldsymbol{U}_h \mathbin{\hat{\otimes}}_{\mathbb{C}[\![h]\!]} \mathbb{C}[\![\sqrt{h}]\!]$
$\lVert \boldsymbol{n} \rVert, \boldsymbol{e}_h(\boldsymbol{n}), \boldsymbol{V}_h, \boldsymbol{V}_h^{\bar{\otimes} n}$	4A	
$\boldsymbol{X}_h$	4D	core subalgebra of $\boldsymbol{U}_{\sqrt{h}}$
$\widetilde{\mathcal{A}}$	5A	
$\boldsymbol{U}_\mathbb{Z}, \boldsymbol{U}_\mathbb{Z}^\pm, \boldsymbol{U}_\mathbb{Z}^0, \boldsymbol{U}_\mathbb{Z}^{\mathrm{ev}}, \boldsymbol{U}_\mathbb{Z}^{\mathrm{ev},-}, \boldsymbol{U}_\mathbb{Z}^{\mathrm{ev},0}$	5B	
$\boldsymbol{V}_\mathbb{Z}, \boldsymbol{V}_\mathbb{Z}^\pm, \boldsymbol{V}_\mathbb{Z}^0, \boldsymbol{V}_\mathbb{Z}^{\mathrm{ev}}, \boldsymbol{V}_\mathbb{Z}^{\mathrm{ev},-}, \boldsymbol{V}_\mathbb{Z}^{\mathrm{ev},0}$	5C	
$(q;q)_{\boldsymbol{n}}$	5C, 5D, 5E	

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