msp

Geometry & Topology 21 (2017) 385–440

The character of the total power operation

TOBIAS BARTHEL
NATHANIEL STAPLETON

We compute the total power operation for the Morava E–theory of any finite group up to torsion. Our formula is stated in terms of the $\mathrm{GL}_n(\mathbb{Q}_p)$–action on the Drinfel'd ring of full level structures on the formal group associated to E–theory. It can be specialized to give explicit descriptions of many classical operations. Moreover, we show that the character map of Hopkins, Kuhn and Ravenel from E–theory to $\mathrm{GL}_n(\mathbb{Z}_p)$–invariant generalized class functions is a natural transformation of global power functors on finite groups.

55N22, 55S25; 55P42

1 Introduction

Power operations and their variants are ubiquitous throughout homotopy theory. The Steenrod operations on mod p cohomology and the Adams operations on topological K–theory are familiar examples. These operations have proven extremely useful; for instance, the Adams operations were used to give a short and elegant proof of the Hopf invariant one problem; see Adams and Atiyah [2]. More generally, many cohomology theories are equipped with this extra structure, which is a consequence of an E_∞–ring structure on the representing spectrum.

It is a theorem of Goerss, Hopkins and Miller [11] that Morava E–theory admits a unique E_∞–ring structure. In the homotopy category, the E_∞–ring structure manifests itself as an H_∞–ring structure which is equivalent to the data of a collection of multiplicative cohomology operations known as total power operations. The study of these operations began in earnest in Ando [3], in which a connection is established between the total power operations and isogenies of the formal group associated to E_n. In this paper, we give a formula for the total power operations applied to a finite group in terms of the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on the Drinfel'd ring of full level structures on the formal group associated to E_n. This is the same action that appears in the local Langlands correspondence; see Carayol [7].

Published: 10 February 2017 DOI: 10.2140/gt.2017.21.385

1.1 Motivation and background

Fix a prime p, a natural number n and a height n formal group law Γ over κ, a perfect field of characteristic p. This data determines the E_∞–ring spectrum E_n known as Morava E–theory. It has been studied extensively because of its close relationship to several areas of mathematics: Work of Devinatz and Hopkins [8] as well as Rognes [19] demonstrates that E_n is a Galois extension of the $K(n)$–local sphere, a fundamental object of the stable homotopy category. It is closely connected to algebraic geometry, as the coefficient ring E_n^0 carries the universal deformation $\mathbb{G}$ of Γ. Morava E–theory is also related to representation theory, being a generalization of p–adic K–theory that admits a well-behaved character theory. We will take advantage of the last two relationships to study the total power operations determined by the unique E_∞–ring structure on E_n.

Let X be a topological space and let $E\Sigma_m \times_{\Sigma_m} X^m$ be the Borel construction for the canonical Σ_m–action on X^m. The total power operations are natural multiplicative nonadditive maps

$$\mathbb{P}_m\colon E_n^0(X) \to E_n^0(E\Sigma_m \times_{\Sigma_m} X^m),$$

defined for all $m > 0$. These maps are quite mysterious and notoriously difficult to compute; see Rezk [18]. At height 2, $\mathbb{P}_m$ has been explicitly determined for X a point when $p = 2$ and $m = 2$ and when $p = 3$ and $m = 3$; see Rezk [17] and Zhu [27]. Above height 2 there have been no explicit computations. Many of the most important operations on E_n, including the Adams operations, Hecke operations and the logarithm — see Rezk [16] — can be built out of the total power operations using various simplifications of $\mathbb{P}_m$.

A useful simplification of $\mathbb{P}_m$ is obtained as the restriction of the total power operation along the diagonal $X \to X^m$. This produces a map

$$P_m\colon E_n^0(X) \to E_n^0(B\Sigma_m \times X) \cong E_n^0(B\Sigma_m) \otimes_{E_n^0} E_n^0(X),$$

the isomorphism being a consequence of the freeness of $E_n^0(B\Sigma_m)$ as a module over E_n^0; see Strickland [25, Proposition 3.6]. Let $I \subset E_n^0(B\Sigma_{p^k})$ be the image of the transfer along the inclusion $\Sigma_{p^{k-1}}^p \subset \Sigma_{p^k}$; then the quotient

$$P_{p^k}/I\colon E_n^0(X) \to E_n^0(B\Sigma_{p^k})/I \otimes_{E_n^0} E_n^0(X)$$

is a ring map; see Bruner, May, McClure and Steinberger [6, Chapter VIII, Proposition 1.4(iv)] and Strickland [25, Lemma 8.11]. Thus it is reasonable to hope that P_{p^k}/I may be attacked using algebraic geometry associated to $\mathbb{G}$. For $X = BA$, where A is a finite abelian group, this was accomplished by Ando [3] and Ando, Hopkins and

Strickland [4]. In contrast to these approaches, we apply a form of character theory available for E_n in order to simplify $\mathbb{P}_m$.

Generalized character theory for Morava E–theory was constructed by Hopkins, Kuhn and Ravenel [12], building on work of Adams [1] on p–adic K–theory and extended by the second author [23]. An introduction to the subject is available in Stapleton [22]. As E–theory is constructed homotopy-theoretically, it is surprising that $E_n^0(BG)$ behaves so much like a (completed) "higher" representation ring.

Let $\mathbb{L} = \mathbb{Z}_p^n$ and $\mathbb{T} = \mathbb{L}^*$, the p–typical Pontryagin dual, so that there is a noncanonical isomorphism $\mathbb{T} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$. Hopkins, Kuhn and Ravenel construct a $p^{-1}E_n^0$–algebra C_0 that corepresents isomorphisms of p–divisible groups between $\mathbb{T}$ and $\mathbb{G}$:

$$\hom(C_0, R) \cong \mathrm{Iso}(R \otimes \mathbb{T}, R \otimes \mathbb{G}).$$

Thus there is an obvious action of $\mathrm{Aut}(\mathbb{T})$, the automorphisms of $\mathbb{T}$, on C_0.

Let $\mathrm{Cl}_n(G, C_0)$ be the ring of generalized class functions on G taking values in C_0. Concretely, $\mathrm{Cl}_n(G, C_0)$ is the set of C_0–valued functions on $\hom(\mathbb{L}, G)_{/\sim}$, the quotient of the set $\hom(\mathbb{L}, G)$ by the conjugation action of G. Note that $\mathrm{Aut}(\mathbb{T})$ is contravariantly isomorphic to $\mathrm{GL}_n(\mathbb{Z}_p)$ by taking the Pontryagin dual. There is an $\mathrm{Aut}(\mathbb{T})$–action on $\mathrm{Cl}_n(G, C_0)$ given by combining the action of $\mathrm{Aut}(\mathbb{T})$ on $\hom(\mathbb{L}, G)_{/\sim}$ by precomposition with the Pontryagin dual and the action of $\mathrm{Aut}(\mathbb{T})$ on C_0 [12, Section 6.3].

In [12, Section 6], Hopkins, Kuhn and Ravenel construct a map of E_n^0–algebras called the generalized character map

$$\chi\colon E_n^0(BG) \to \mathrm{Cl}_n(G, C_0).$$

Theorem C in [12] proves that the induced map

$$C_0 \otimes \chi\colon C_0 \otimes_{E_n^0} E_n^0(BG) \xrightarrow{\cong} \mathrm{Cl}_n(G, C_0)$$

is an isomorphism for any finite group. Theorem C further states that the isomorphism $C_0 \otimes \chi$ is $\mathrm{Aut}(\mathbb{T})$–equivariant and restricts to an isomorphism on fixed points

$$p^{-1}E_n^0(BG) \xrightarrow{\cong} \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}.$$

This brings us to the motivating question for this paper: Does there exist a multiplicative natural transformation on generalized class functions that is compatible with the total

power operation for E_n through the generalized character maps of [12]:

$$\begin{tikzcd} E_n^0(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E_n^0(BG \wr \Sigma_m) \arrow[d, "\chi"] \\ \mathrm{Cl}_n(G, C_0) \arrow[r, dashed, "\exists ?"] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \end{tikzcd}$$

There is no formal reason why this might be possible,[1] and yet the construction of such a natural transformation is one of the goals of this paper.

1.2 Main results

In fact, we construct an infinite family of multiplicative natural transformations that answer the question. In order to state the results precisely we need to establish some notation.

Let $\mathrm{Isog}(\mathbb{T})$ be the monoid of endoisogenies of $\mathbb{T}$, ie the monoid of endomorphisms with finite kernel, and let $\mathrm{Sub}(\mathbb{T})$ be the set of finite subgroups in $\mathbb{T}$. There is an $\mathrm{Aut}(\mathbb{T})$–principal bundle

$$\mathrm{Isog}(\mathbb{T}) \twoheadrightarrow \mathrm{Sub}(\mathbb{T})$$

given by taking an isogeny to its kernel. For each section ϕ of this principal bundle we construct a multiplicative natural transformation

$$\mathbb{P}_m^{\phi}\colon \mathrm{Cl}_n(-, C_0) \Rightarrow \mathrm{Cl}_n(- \wr \Sigma_m, C_0)$$

that is compatible with the total power operation $\mathbb{P}_m$ through the character map χ. We refer the reader to the paragraphs leading up to Definition 5.6 for the explicit formula for $\mathbb{P}_m^{\phi}$.

Theorem A (Theorem 9.1) *For all n, $m \geq 0$, let $\mathbb{P}_m$ be the total power operation for Morava E_n, let ϕ be a section of the principal bundle above and let χ be the generalized character map. There is a commutative diagram*

$$\begin{tikzcd} E_n^0(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E_n^0(BG \wr \Sigma_m) \arrow[d, "\chi"] \\ \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^{\phi}"] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \end{tikzcd}$$

natural in G.

[1] It is not possible to base change $\mathbb{P}_m$ to C_0 because $\mathbb{P}_m$ is not a ring map. Further, simplifications of $\mathbb{P}_m$ such as P_{p^k}/I, which are ring maps, are not E_n^0–algebra maps.

As part of Theorem 10.1, we further prove that each of these "total power operations" $\mathbb{P}_m^{\phi}$ may be restricted to the $\mathrm{Aut}(\mathbb{T})$–fixed points. In Corollary 10.2, we show that the result is independent of ϕ and we call it the "rational total power operation"

$$\mathbb{P}_m^{\mathbb{Q}}\colon p^{-1}E_n^0(BG) \to p^{-1}E_n^0(BG \wr \Sigma_m).$$

Theorem B (Theorem 10.1) *With the notation of Theorem A, there is a commutative diagram*

$$\begin{tikzcd} E_n^0(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E_n^0(BG \wr \Sigma_m) \arrow[d, "\chi"] \\ p^{-1}E_n^0(BG) \arrow[r, "\mathbb{P}_m^{\mathbb{Q}}"'] & p^{-1}E_n^0(BG \wr \Sigma_m). \end{tikzcd}$$

We note that the rationalization $p^{-1}E_n^0(BG)$ retains much of the information contained in $E_n^0(BG)$, so our result approximates the total power operation closely. In fact, $E_n^0(BG)$ is finitely generated and free for many finite groups G. These are the so-called good groups; see Barthel and Stapleton [5], Hopkins, Kuhn and Ravenel [12] and Schuster [21]. Finally, we show that $\mathbb{P}_m^{\mathbb{Q}}$ deserves to be called the rational total power operation by proving it is a global power functor in the sense of Ganter [10, Section 4].

Theorem C (Theorem 10.4, Corollary 10.5) *The rational total power operation $\mathbb{P}_m^{\mathbb{Q}}$ is a global power functor in such a way that the character map*

$$\chi\colon E_n^0(B-) \to p^{-1}E_n^0(B-)$$

is a map of global power functors.

Note that Theorem C is not a direct consequence of Theorem B because Theorem B does not say anything about composites of rational total power operations.

1.3 Outline

In Section 2, we establish notation and terminology that will be used in the rest of the paper.

The first part of this paper, comprised of Section 3 through Section 6, contains results that do not rely on Morava E–theory. We hope that some of the ideas in these sections might be of interest to readers outside of stable homotopy theory. The goal of Section 3 is a thorough study of conjugacy classes of commuting elements in wreath products

$G \wr \Sigma_m$ and their algebrogeometric interpretation. We suspect that this material is well-known, but were unable to locate a reference in the literature. Section 4 deals with the action of the isogenies of $\mathbb{T}$ on the Drinfel'd ring of full level-structures on $\mathbb{G}$. This is fundamental for the rest of the paper and extends work of Hopkins, Kuhn and Ravenel [12]. For all $m \geq 0$ we construct multiplicative natural transformations $\mathbb{P}_m^\phi$ on class functions in Section 5 and study their basic properties. The question of whether one can choose a section ϕ such that the associated transformation $\mathbb{P}_m^\phi$ inherits the structure of a global power functor is considered in Section 6. We give an affirmative answer for heights 1 and 2 by stipulating an explicit solution.

The second part begins in Section 7 with some recollections on Morava E–theory and power operations. As a first step towards the proof of our main theorem, we consider the case of abelian groups in Section 8. Using work of Schlank and Stapleton [20], we extend the algebrogeometric description of the additive total power operations for abelian groups due to Ando, Hopkins and Strickland [4]. Inspired by Artin induction, Section 9 then proves Theorem A by reducing it to the case of abelian groups. Our work culminates in Section 10, where the previous results are combined to descend the multiplicative natural transformations to rationalized E–theory. We prove uniqueness and establish the global power structure on the resulting multiplicative natural transformation.

Acknowledgements It is a pleasure to thank Charles Rezk and Tomer Schlank for their valuable input. Without some of their ideas this project would not have gotten off the ground. We would like to thank Nora Ganter, Justin Noel and Chris Schommer-Pries for helpful conversations, and Drew Heard for useful comments on an earlier draft. The referee and Haynes Miller made many comments leading to an improvement of the exposition of the paper. We also thank the Max Planck Institute for Mathematics for its hospitality.

2 Notation and conventions

Let G be a finite group and fix a prime p and a natural number $n \geq 0$. We set $\mathbb{L} = \mathbb{Z}_p^n$ and let $\hom(\mathbb{L}, G)$ be the set of continuous homomorphisms of groups from $\mathbb{L}$ to G. Fixing a basis of $\mathbb{L}$ gives a bijection

$$\hom(\mathbb{L}, G) \cong \{(g_1, \dots, g_n) \mid [g_i, g_j] = e \text{ and } g_i^{p^k} = e \text{ for some } k > 0\},$$

where the target is the set of n–tuples of pairwise commuting prime-power order elements in G. The group G acts on this set by conjugation; we write $\hom(\mathbb{L}, G)_{/\sim}$

for the set of conjugacy classes. Moreover, we will write $[\alpha]$ for the conjugacy class of a map $\alpha\colon \mathbb{L} \to G$.

Let $G \wr \Sigma_m$ be the wreath product of G with Σ_m, which is constructed as the semidirect product $G^m \rtimes \Sigma_m$ with respect to the permutation action of Σ_m on G^m. The wreath product fits into a short exact sequence

$$1 \to G^m \to G \wr \Sigma_m \xrightarrow{\pi} \Sigma_m \to 1.$$

We will always write π for the projection onto the symmetric group, and represent elements in $G \wr \Sigma_m$ as $(g_1, \dots, g_m; \sigma)$ with $g_i \in G$ and $\sigma \in \Sigma_m$. We may abbreviate this to $(\overline{g}; \sigma)$, where $\overline{g} = (g_1, \dots, g_m)$ and $\overline{g}_i = g_i$. Multiplication in the wreath product is given by

$$(\overline{g}; \sigma)(\overline{h}; \tau) = (\overline{g}(\sigma\overline{h}); \sigma\tau), \tag{1}$$

where $(\sigma\overline{h})_i = h_{\sigma^{-1}i}$.

We call a map $\alpha\colon \mathbb{L} \to \Sigma_m$ transitive if the map represents a transitive $\mathbb{L}$–set of order m. Equivalently, the image $\operatorname{im}\alpha$ of the map α is a transitive abelian subgroup of Σ_m. This can only occur when $m = p^k$. We say that a map

$$\alpha\colon \mathbb{L} \to G \wr \Sigma_{p^k}$$

is transitive if $\pi\alpha$ is transitive. Let $\hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}$ be the set of transitive maps and let $\hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$ be the set of conjugacy classes of transitive maps.

Pontryagin duality is pervasive throughout this paper. We will use the notation $(-)^*$ for the (p–typical) Pontryagin duality endofunctor on abelian groups. Thus for an abelian group A, we write

$$A^* = \hom(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

for the dual abelian group. For a map of abelian groups $A \xrightarrow{f} B$, we write f^* for the dual map. Let $\mathbb{T} = \mathbb{L}^*$; then we have a noncanonical isomorphism

$$\mathbb{T} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n.$$

Let $[p^k]\colon \mathbb{T} \to \mathbb{T}$ the multiplication by p^k map and let $\mathbb{T}[p^k]$ be the p^k–torsion in $\mathbb{T}$.

Let $\mathrm{Sub}_{p^k}(\mathbb{T})$ be the set of subgroups of order p^k in $\mathbb{T}$. Let $\mathrm{Sum}_m(\mathbb{T})$ be the set of formal sums of subgroups $\bigoplus_i H_i$ with $H_i \subset \mathbb{T}$ and $\sum_i |H_i| = m$. It is clear that $\mathrm{Sub}_{p^k}(\mathbb{T})$ is a subset of $\mathrm{Sum}_{p^k}(\mathbb{T})$. Given a subgroup $H \subset \mathbb{T}$ of order p^k, let $f_H\colon H \hookrightarrow \mathbb{T}$ be the inclusion. The Pontryagin dual of f_H,

$$f_H^*\colon \mathbb{L} \twoheadrightarrow H^*,$$

has the associated short exact sequence

$$0 \to \ker(f_H^*) \hookrightarrow \mathbb{L} \twoheadrightarrow H^* \to 0.$$

The kernel of f_H^* is canonically isomorphic to $(\mathbb{T}/H)^*$ and noncanonically isomorphic to $\mathbb{L}$. We set $\mathbb{L}_H = \ker(f_H^*) \subset \mathbb{L}$.

More generally, we define $\mathrm{Sub}_{p^k}(\mathbb{T}, G)$ to be the set of pairs consisting of a subgroup of order p^k, $H \subset \mathbb{T}$, and a conjugacy class in $\hom(\mathbb{L}_H, G)$. We will write such a pair as $(H, [\alpha])$. We define $\mathrm{Sum}_m(\mathbb{T}, G)$ to be the collection of formal sums of pairs $\bigoplus_i (H_i, [\alpha_i])$, where the subgroups $H_i \subset \mathbb{T}$ are such that $\sum_i |H_i| = m$ and

$$[\alpha_i] \in \hom(\mathbb{L}_{H_i}, G)_{/\sim}.$$

If e is the trivial group, then we have

$$\mathrm{Sub}_{p^k}(\mathbb{T}, e) = \mathrm{Sub}_{p^k}(\mathbb{T})$$

and

$$\mathrm{Sum}_m(\mathbb{T}, e) = \mathrm{Sum}_m(\mathbb{T}).$$

Finally, given an isogeny $\phi_H \colon \mathbb{T} \to \mathbb{T}$ with kernel H, it is necessary to have notation for the induced triangle:

$$\begin{tikzcd} & \mathbb{T} \arrow[dl, two heads, "q_H"'] \arrow[dr, two heads, "\phi_H"] & \\ \mathbb{T}/H \arrow[rr, "\cong", "\psi_H"'] & & \mathbb{T} \end{tikzcd}$$

The Pontryagin dual triangle then takes the form:

$$\begin{tikzcd} & \mathbb{L} & \\ \mathbb{L}_H \arrow[ur, hook, "q_H^*"] & & \mathbb{L} \arrow[ul, hook', "\phi_H^*"'] \arrow[ll, "\cong", "\psi_H^*"'] \end{tikzcd}$$

3 Conjugacy classes in wreath products

The purpose of this section is to establish a canonical bijection between conjugacy classes in wreath products and formal sums of subgroups in $\mathbb{T}$:

$$F \colon \hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim} \xrightarrow{\cong} \mathrm{Sum}_m(\mathbb{T}, G).$$

In fact, we have canonical bijections, which are natural in G, as indicated in the table below:

conjugacy classes	subgroups
$\hom(\mathbb{L}, \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$	$\mathrm{Sub}_{p^k}(\mathbb{T})$
$\hom(\mathbb{L}, \Sigma_m)_{/\sim}$	$\mathrm{Sum}_m(\mathbb{T})$
$\hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$	$\mathrm{Sub}_{p^k}(\mathbb{T}, G)$
$\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$	$\mathrm{Sum}_m(\mathbb{T}, G)$

In the case that $\mathbb{L} = \mathbb{Z}_p$, these isomorphisms can be obtained from [26, Section 7.7].

We will use F for any of the bijections in the table (they are all special cases of the bottom entry of the table). The top two bijections are well-known; we will begin by describing F in these cases.

The canonical bijection between $\hom(\mathbb{L}, \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$ and $\mathrm{Sub}_{p^k}(\mathbb{T})$ is constructed as follows: Let $\mathbb{L} \xrightarrow{\alpha} \Sigma_{p^k}$ be a transitive map and let $\mathbb{L}' = \ker \alpha$. The map sends $[\alpha]$ to the kernel of the Pontryagin dual of the map $\mathbb{L}' \hookrightarrow \mathbb{L}$,

$$H = \ker(\mathbb{L}^* \twoheadrightarrow (\mathbb{L}')^*).$$

Since a conjugate map has the same kernel, this assignment does not depend on the chosen representative α for the conjugacy class $[\alpha]$. The resulting subgroup H is an invariant of the conjugacy class $[\alpha]$. There is another description of this subgroup. The map α determines a transitive $\mathbb{L}$–set of order p^k. The lattice $\mathbb{L}'$ is the stabilizer of any element in the set and then H may be formed as the kernel above.

The second bijection is constructed similarly. A map $\alpha\colon \mathbb{L} \to \Sigma_m$ is an $\mathbb{L}$–set X of order m. Let $X = \coprod_i X_i$ be the decomposition of X into transitive $\mathbb{L}$–sets. By following the recipe above, each component gives a subgroup $H_i \subset \mathbb{T}$. Since a conjugate map corresponds to an isomorphic $\mathbb{L}$–set, this collection of subgroups of $\mathbb{T}$ is an invariant of the conjugacy class:

$$[\alpha] \mapsto F([\alpha]) = \bigoplus_i H_i.$$

Now we establish the third and fourth bijections in the table above. Let $\underline{G}$ be a left G–torsor (a free transitive left G–set) and let $\underline{m}$ be a set with $m \in \mathbb{N}$ elements and a trivial G–action. We will make use of three basic lemmas regarding G–sets of the form $\underline{G} \times \underline{m}$.

Let $\overline{x} = (x_1, \dots, x_m)$ be a set of generators for $\underline{G} \times \underline{m}$ as a G–set. This choice of generators induces an isomorphism

$$\overline{x}^*\colon \mathrm{Aut}_G(\underline{G} \times \underline{m}) \cong G \wr \Sigma_m. \tag{2}$$

We may be more explicit about this isomorphism. An automorphism $f \in \mathrm{Aut}_G(\underline{G} \times \underline{m})$ permutes the m G–torsors; call this permutation σ_f. Thus $f(x_i) = g_{\sigma_f(i)} x_{\sigma_f(i)}$ for some $g_{\sigma_f(i)} \in G$. The isomorphism is given by

$$f \mapsto \overline{x}^*(f) = (g_1^{-1}, \dots, g_m^{-1}; \sigma_f) = (\overline{g}^{-1}; \sigma_f).$$

This defines a homomorphism: If $h \in \mathrm{Aut}_G(\underline{G} \times \underline{m})$ maps to

$$h \mapsto (\overline{k}^{-1}; \sigma_h) \in G \wr \Sigma_m,$$

then

$$(h \circ f)(x_i) = h(g_{\sigma_f(i)} x_{\sigma_f(i)}) = g_{\sigma_f(i)} k_{\sigma_h \sigma_f(i)} x_{\sigma_h \sigma_f(i)}$$

and, by (1),

$$\begin{aligned}
(\overline{k}^{-1}(\sigma_h \overline{g}^{-1}); \sigma_h \sigma_f)(x_i) &= ((\overline{k}^{-1}(\sigma_h \overline{g}^{-1}))_{\sigma_h \sigma_f(i)})^{-1} x_{\sigma_h \sigma_f(i)} \\
&= (k^{-1}_{\sigma_h \sigma_f(i)} (\sigma_h \overline{g}^{-1})_{\sigma_h \sigma_f(i)})^{-1} x_{\sigma_h \sigma_f(i)} \\
&= (\sigma_h \overline{g})_{\sigma_h \sigma_f(i)} k_{\sigma_h \sigma_f(i)} x_{\sigma_h \sigma_f(i)} \\
&= g_{\sigma_h^{-1} \sigma_h \sigma_f(i)} k_{\sigma_h \sigma_f(i)} x_{\sigma_h \sigma_f(i)} \\
&= g_{\sigma_f(i)} k_{\sigma_h \sigma_f(i)} x_{\sigma_h \sigma_f(i)}.
\end{aligned}$$

Given a group homomorphism $s\colon G \to K$ and a G–set X, we may functorially form the left K–set

$$K \times_G X,$$

which is the quotient of $K \times X$ by the relation $(ks(g), x) \sim (k, gx)$. This is a K–set through the left action of K on itself. The operation $K \times_G -$ is left adjoint to the functor from K–sets to G–sets given by viewing a K–set as a G–set through s. The K–set

$$K \times_G (\underline{G} \times \underline{m})$$

is a disjoint union of m K–torsors and the unit of the adjunction is a canonical map of G–sets

$$\eta\colon \underline{G} \times \underline{m} \to K \times_G (\underline{G} \times \underline{m}).$$

Given a set of generators $\overline{x}$ of $\underline{G} \times \underline{m}$, $\eta(\overline{x})$ is a set of generators of $K \times_G (\underline{G} \times \underline{m})$ and this induces the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[r, "\overline{x}^*"] \arrow[d] & G \wr \Sigma_m \arrow[d] \\
\mathrm{Aut}_K(K \times_G (\underline{G} \times \underline{m})) \arrow[r, "\eta(\overline{x})^*"] & K \wr \Sigma_m
\end{tikzcd} \tag{3}$$

Although the isomorphism of (2) depends on a choice of generators of $\underline{G} \times \underline{m}$, the next lemma shows that this dependence is removed by passing to conjugacy classes.

Lemma 3.1 *There is a canonical isomorphism, natural in the group G,*

$$\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim} \cong \hom(\mathbb{L}, \mathrm{Aut}_G(\underline{G} \times \underline{m}))_{/\sim}.$$

Proof Let $\overline{x}$ and $\overline{y}$ be sets of generators of $\underline{G} \times \underline{m}$ as a G–set. Then there exists an automorphism of f of $\underline{G} \times \underline{m}$ such that $f(x_i) = y_i$. This automorphism induces an automorphism

$$\mathrm{Aut}_G(\underline{G} \times \underline{m}) \xrightarrow{c_f} \mathrm{Aut}_G(\underline{G} \times \underline{m})$$

given by conjugation with f and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[r, "\overline{x}"] \arrow[d, "c_f"'] & G \wr \Sigma_m \\
\mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[ur, "\overline{y}"'] &
\end{tikzcd}$$

For α and α' conjugate elements in $\hom(\mathbb{L}, \mathrm{Aut}_G(\underline{G} \times \underline{m}))$, it suffices to prove that $\overline{x}^*(\alpha)$ is conjugate to $\overline{y}^*(\alpha')$. Assume that $\alpha' = g\alpha g^{-1}$; then the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[r, "\overline{x}"] \arrow[d, "c_g"] & G \wr \Sigma_m \arrow[dd, dashed] \\
\mathbb{L} \arrow[ur, "\alpha"] \arrow[r, "\alpha'"'] & \mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[d, "c_f"'] \arrow[dr, "\overline{y}"] & \\
 & \mathrm{Aut}_G(\underline{G} \times \underline{m}) \arrow[r, "\overline{x}"'] & G \wr \Sigma_m
\end{tikzcd}$$

The dashed arrow is built by passing through the isomorphisms and is thus given by conjugation. Thus $\overline{x}^*(\alpha)$ is conjugate to $\overline{y}^*(\alpha')$.

The isomorphism is natural by diagram (3). □

A map $\mathbb{L} \to \mathrm{Aut}_G(\underline{G} \times \underline{m})$ corresponds to an $\mathbb{L}$–set structure on $\underline{G} \times \underline{m}$ in the category of G–sets. By adjunction, this corresponds to an $\mathbb{L} \times G$–set of order $|G| \times m$ with a free G–action.

Lemma 3.2 *By adjunction, there is a canonical isomorphism that is natural in the group G between*

$$\hom(\mathbb{L}, \mathrm{Aut}_G(\underline{G} \times \underline{m}))_{/\sim}$$

and the set of isomorphism classes of $\mathbb{L} \times G$–sets of order $|G| \times m$ with a free G–action.

The projection map $\underline{G} \times \underline{m} \to \underline{m}$ induces

$$\mathrm{Aut}_G(\underline{G} \times \underline{m}) \to \mathrm{Aut}(\underline{m}).$$

Thus it makes sense to define

$$\hom(\mathbb{L}, \mathrm{Aut}_G(\underline{G} \times \underline{m}))^{\mathrm{trans}}_{/\sim}$$

to consist of the conjugacy classes that induce a transitive $\mathbb{L}$–action on $\underline{m}$ under the projection. These, in turn, give rise to isomorphism classes of transitive $\mathbb{L} \times G$–sets with a free G–action under the isomorphism of the lemma above. Transitive $\mathbb{L}$–actions on $\underline{m}$ only exist when $\underline{m}$ has cardinality p^k for some k. In the transitive case we will write $\underline{p}^k$ instead of $\underline{m}$.

As usual, isomorphism classes of transitive $\mathbb{L} \times G$–sets of order $|G| \times p^k$ correspond (canonically) to conjugacy classes of index $|G| \times p^k$ subgroups of $\mathbb{L} \times G$. The conjugacy classes of subgroups giving rise to a transitive $\mathbb{L} \times G$–set with a free G–action have a nice description.

Lemma 3.3 *There is a canonical isomorphism, natural in the group G, between conjugacy classes of index $|G| \times p^k$ subgroups M of $\mathbb{L} \times G$ such that $(\mathbb{L} \times G)/M$ has a free G–action and pairs*

$$(\mathbb{L}_H \subseteq \mathbb{L},\ [\alpha_H \colon \mathbb{L}_H \to G])$$

consisting of an index p^k sublattice $\mathbb{L}_H \subset \mathbb{L}$ and a conjugacy class of maps

$$[\alpha_H \colon \mathbb{L}_H \to G].$$

Proof Subgroups $M \subset \mathbb{L} \times G$ with the property that $(\mathbb{L} \times G)/M$ has a free G–action are in correspondence with subgroups of $\mathbb{L} \times G$ that intersect $0 \times G \subset \mathbb{L} \times G$ trivially.

The data of a subgroup with this property has a very simple description. If $M \subset \mathbb{L} \times G$ intersects $0 \times G$ trivially, then the composite

$$M \hookrightarrow \mathbb{L} \times G \to \mathbb{L}$$

is injective, where the last map is the projection: If $m \neq m' \in M$ map to $(l, g) \neq (l, g')$, respectively, then $m(m')^{-1}$ maps to $(0, g(g')^{-1})$, which is not the identity. This contradicts the assumption that the intersection with $0 \times G$ is trivial.

Thus M may be identified with $\mathbb{L}_H$ for some $\mathbb{L}_H \subset \mathbb{L}$ with $H \subset \mathbb{T}$ of order m. Since $\mathbb{L}$ is abelian, a conjugacy class of subgroups is a subgroup. Thus a conjugacy class of subgroups of $\mathbb{L} \times G$ of order $|G| \times p^k$ that intersect $0 \times G$ trivially corresponds to a subgroup $\mathbb{L}_H \subset \mathbb{L}$ of index p^k and a conjugacy class of maps $[\alpha_H \colon \mathbb{L}_H \to G]$.

Next we prove that the isomorphism is natural. Let $s\colon G \to K$ be a group homomorphism. The induced map

$$\mathrm{id} \times s\colon \mathbb{L} \times G \to \mathbb{L} \times K$$

sends conjugacy classes of subgroups to conjugacy classes of subgroups. If $M \subset \mathbb{L} \times G$ intersects $0 \times G$ trivially, then so does $(\mathrm{id} \times s)(M)$.

Also, s induces the map on pairs $(\mathbb{L}_H, [\alpha])$ sending

$$(\mathbb{L}_H, [\alpha]) \mapsto (\mathbb{L}_H, [s\alpha]).$$

Now, by the construction above of $(\mathbb{L}_H, \alpha)$ from a subgroup M that intersects $0 \times G$ trivially, we see that $s(M)$ is sent to $(\mathbb{L}_H, s\alpha)$. Conjugating the subgroup M just conjugates $s\alpha$. □

By construction, the isomorphism of Lemma 3.1 restricts to a canonical isomorphism

$$\hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim} \cong \hom(\mathbb{L}, \mathrm{Aut}_G(\underline{G} \times \underline{p}^k))^{\mathrm{trans}}_{/\sim}.$$

For $[\alpha] \in \hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$, we may use the previous three lemmas to define

$$F([\alpha]) = (H, [\alpha_H]).$$

Proposition 3.4 *The map*

$$F\colon \hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim} \xrightarrow{\cong} \mathrm{Sub}_{p^k}(\mathbb{T}, G),$$

given by $F([\alpha]) = (H, [\alpha_H])$, is a canonical isomorphism that is natural in G.

Proof Combine the canonical isomorphisms of the previous three lemmas. □

It is worth noting an equivalent way to obtain $(H, [\alpha_H])$ from $[\alpha] \in \hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{L}_H \arrow[r, "(\beta_i)"] \arrow[d, "q_H^*"'] & G^{p^k} \arrow[d] \\
\mathbb{L} \arrow[r, "\alpha"] \arrow[dr, "\pi\alpha"'] & G \wr \Sigma_{p^k} \arrow[d, "\pi"] \\
& \Sigma_{p^k}
\end{tikzcd}$$

defining p^k maps $\beta_i\colon \mathbb{L}_H \to G$. A set of generators $\overline{x}$ of $\underline{G} \times \underline{p}^k$ produces an isomorphism

$$\overline{x}^*\colon \mathrm{Aut}_G(\underline{G} \times \underline{p}^k) \to G \wr \Sigma_{p^k}$$

and the image of $(q_H^*, \beta_i)\colon \mathbb{L}_H \to \mathbb{L} \times G$ is the stabilizer of x_i in the transitive $\mathbb{L} \times G$–set determined by the composite $(\overline{x}^*)^{-1}\alpha$. By Lemma 3.3, all of the maps β_i are conjugate and we may take α_H to be any of the β_i.

Corollary 3.5 *We may choose $\alpha \in [\alpha] \in \hom(\mathbb{L}, G \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$ with the property that it factors through the inclusion:*

$$\begin{tikzcd} & (\operatorname{im}\alpha_H) \wr \Sigma_{p^k} \arrow[d, hook] \\ \mathbb{L} \arrow[ur, dashed] \arrow[r, "\alpha"'] & G \wr \Sigma_{p^k} \end{tikzcd}$$

Proof For any $\alpha \in [\alpha]$, the p^k maps

$$\mathbb{L}_H \to G$$

are all conjugate. Thus we may choose elements $g_2, \dots, g_{p^k} \in G$ conjugating them to the first one. The element $(1, g_2, \dots, g_{p^k}; e) \in G \wr \Sigma_{p^k}$ conjugates α to a map satisfying the required condition. □

Since the isomorphism class of a finite $\mathbb{L} \times G$–set is determined by the isomorphism classes of the transitive components of a representative, for $[\alpha] \in \hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$ we may define

$$F([\alpha]) = \bigoplus_i (H_i, [\alpha_{H_i}]),$$

where the pairs $(H_i, [\alpha_{H_i}])$ are determined by the transitive components of the $\mathbb{L} \times G$–set associated to $[\alpha]$.

Proposition 3.6 *There is a canonical isomorphism, which is natural in G,*

$$F\colon \hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim} \xrightarrow{\cong} \mathrm{Sum}_m(\mathbb{T}, G),$$

given by $F([\alpha]) = \bigoplus_i (H_i, [\alpha_{H_i}])$.

Remark 3.7 There is a more geometric interpretation of the above results; see [10, Section 3.3.4, Corollary 3.7]. Conjugacy classes of tuples of commuting elements are the homotopy classes

$$[B\,\mathbb{L}, BG \wr \Sigma_m].$$

The space $B\,\mathbb{L}$ is the p–complete torus. The homotopy classes $[B\,\mathbb{L}, B\Sigma_m]$ are the (isomorphism classes of) m–fold covers of the p–complete torus and the homotopy classes $[B\,\mathbb{L}, BG \wr \Sigma_m]$ are the (isomorphism classes of) principal G–bundles on m–fold covers of the p–complete torus.

Now that we have proved the main result of this section, we draw several consequences and prove related results that will be used in the remaining part of the paper.

Lemma 3.8 *If $\sum_j a_j p^j$ is the p–adic expansion of m, then the inclusion $\prod_j \Sigma_{p^j}^{a_j} \subseteq \Sigma_m$ induces a surjection*

$$\prod_j \mathrm{Sum}_{p^j}(\mathbb{T}, G)^{\times a_j} \twoheadrightarrow \mathrm{Sum}_m(\mathbb{T}, G).$$

Proof This is immediate from the definitions. □

Proposition 3.9 *The map $\Delta\colon G \times \Sigma_{p^k} \to G \wr \Sigma_{p^k}$ induces the map*

$$\mathrm{Sub}_{p^k}(\mathbb{T}) \times \hom(\mathbb{L}, G)_{/\sim} \to \mathrm{Sub}_{p^k}(\mathbb{T}, G)$$

defined by $(H, [\alpha\colon \mathbb{L} \to G]) \mapsto (H, [\alpha q_H^*])$.

Proof Let $\beta\colon \mathbb{L} \to \Sigma_{p^k}$ be transitive with the property that $F([\beta]) = H$; we are interested in the composite

$$\mathbb{L} \xrightarrow{\alpha\times\beta} G \times \Sigma_{p^k} \to G \wr \Sigma_{p^k}.$$

This fits in the commutative diagram

$$\begin{tikzcd}
\mathbb{L}_H \arrow[r, "\alpha q_H^*"] \arrow[d, "q_H^*"'] & G \arrow[r, "\Delta"] \arrow[d] & G^{p^k} \arrow[d] \\
\mathbb{L} \arrow[r, "\alpha\times\beta"] & G \times \Sigma_{p^k} \arrow[r, "\Delta"] \arrow[dr] & G \wr \Sigma_{p^k} \arrow[d] \\
 & & \Sigma_{p^k}
\end{tikzcd}$$

where the top row consists of the kernels of the maps to Σ_{p^k}. It follows that

$$[(\Delta(\alpha \times \beta))_H] = [\alpha(q_H^*)].$$ □

Note that we now have a way of describing conjugacy classes in iterated wreath products of symmetric groups. By Proposition 3.6, there is a bijection

$$\hom(\mathbb{L}, G \wr \Sigma_s \wr \Sigma_t)_{/\sim} \cong \mathrm{Sum}_t(\mathbb{T}, G \wr \Sigma_s).$$

Conjugacy classes from $\mathbb{L}_{H_i}$ to $G \wr \Sigma_s$ satisfy the same kind of formula; thus we may write

$$\bigoplus_i (H_i, [\alpha_{H_i}]) = \bigoplus_i \Big(H_i, \bigoplus_j \big(K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}]\big)\Big),$$

where $K_{i,j} \subset \mathbb{T}/H_i$.

Proposition 3.10 *Let $st = m$; then the natural inclusion*

$$\nabla\colon G \wr \Sigma_s \wr \Sigma_t \to G \wr \Sigma_m$$

induces the map

$$\bigoplus_i \Big(H_i, \bigoplus_j \big(K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}]\big)\Big) \mapsto \bigoplus_{i,j} \big(T_{i,j}, [(\alpha_{H_i})_{K_{i,j}}]\big) = \bigoplus_{i,j} (T_{i,j}, [\alpha_{T_{i,j}}]),$$

where $T_{i,j}$ is defined as the pullback in the following diagram:

$$\begin{tikzcd} T_{i,j} \arrow[r] \arrow[d] & \mathbb{T} \arrow[d] \\ K_{i,j} \arrow[r] & \mathbb{T}/H_i \end{tikzcd}$$

Proof It suffices to show this on $\alpha\colon \mathbb{L} \to G \wr \Sigma_{p^j} \wr \Sigma_{p^t}$ with α transitive and $\alpha_H\colon \mathbb{L}_H \to G \wr \Sigma_{p^j}$ transitive. When α and α_H are transitive, $\nabla\alpha$ is transitive. In this case, we have $K \subset \mathbb{T}/H$ and the pullback T is the kernel of the composite

$$\mathbb{T} \xrightarrow{q_H} \mathbb{T}/H \to (\mathbb{T}/H)/K.$$

Now we check that $[(\alpha_H)_K] = [(\nabla\alpha)_T]$. By Corollary 3.5, $\alpha \in [\alpha]$ can be chosen so that

$$\begin{tikzcd} (\mathbb{L}_H)_K \arrow[r] \arrow[d] & \mathbb{L} \arrow[d, "\alpha"'] & \\ G \arrow[r, "\Delta"] \arrow[d, "="'] & G \wr \Sigma_s \wr \Sigma_t \arrow[r] \arrow[d, "\nabla"] & \Sigma_s \wr \Sigma_t \arrow[d] \\ G \arrow[r, "\Delta"] & G \wr \Sigma_m \arrow[r] & \Sigma_m \end{tikzcd}$$

commutes, where $(\mathbb{L}_H)_K$ is the kernel of the composite $\mathbb{L} \to G \wr \Sigma_s \wr \Sigma_t \to \Sigma_s \wr \Sigma_t$. Since $\Sigma_s \wr \Sigma_t \to \Sigma_m$ is injective, $(\mathbb{L}_H)_K = \mathbb{L}_T$ and $(\alpha_H)_K = \alpha_T$. □

Let $\mathbb{L}' \cong \mathbb{Z}_p^n$ be another rank n lattice and let $\mathbb{T}' = (\mathbb{L}')^*$. Let

$$\sigma\colon \mathbb{T} \xrightarrow{\cong} \mathbb{T}'$$

be an isomorphism. Precomposition with σ^* induces a bijection

$$\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim} \xrightarrow{\cong} \hom(\mathbb{L}', G \wr \Sigma_m)_{/\sim}.$$

We explain the effect of this map on $\mathrm{Sum}_m(\mathbb{T}, G)$.

Proposition 3.11 *Precomposition with σ^* on $\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$ induces the map*

$$\mathrm{Sum}_m(\mathbb{T}, G) \to \mathrm{Sum}_m(\mathbb{T}', G)$$

given by

$$\bigoplus_i (H_i, [\alpha_{H_i}]) \mapsto \bigoplus_i \big(\sigma(H_i), [\alpha_{H_i} \sigma^*_{|\mathbb{L}'_{\sigma(H_i)}}]\big).$$

Proof First we will verify the claim on the subgroups $H_i \subset \mathbb{T}$. If $\mathbb{L}_{H_i}$ stabilizes $x \in \underline{m}_{\pi\alpha}$, then H_i is the kernel of $\mathbb{T} = \mathbb{L}^* \to \mathbb{L}^*_{H_i}$. The stabilizer of $x \in \underline{m}_{\pi\alpha\sigma^*}$ is the preimage of $\mathbb{L}^*_{H_i}$ under σ^*. The kernel of the map $(\mathbb{L}')^* \to ((\sigma^*)^{-1}\mathbb{L}_{H_i})^*$ is σH_i.

Now we may assume that we have a transitive map $\mathbb{L} \xrightarrow{\alpha} G \wr \Sigma_{p^k}$ such that $[\pi\alpha]$ corresponds to H. By Corollary 3.5, we may assume that $\alpha_{|\mathbb{L}_H} = \Delta\alpha_H$. Since $(\sigma^*)^{-1}\mathbb{L}_H = \mathbb{L}_{\sigma H}$, the diagram

$$\begin{tikzcd}
\mathbb{L}'_{\sigma H} \arrow[r] \arrow[d, "\sigma^*_{|\mathbb{L}'_{\sigma(H)}}"'] & \mathbb{L}' \arrow[d, "\sigma^*"] \\
\mathbb{L}_H \arrow[r] \arrow[d, "\alpha_H"'] & \mathbb{L} \arrow[d, "\alpha"] \\
G \arrow[r, "\Delta"] & G \wr \Sigma_{p^k}
\end{tikzcd}$$

shows that $[\alpha_H]$ is sent to $[\alpha_H \sigma^*_{|\mathbb{L}'_{\sigma(H)}}]$. □

There is an obvious left action of $\mathrm{Aut}(\mathbb{T})$ on $\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$ given by precomposition with the Pontryagin dual. The previous proposition gives a formula for this action on $\mathrm{Sum}_m(\mathbb{T}, G)$.

Corollary 3.12 *The left action of $\sigma \in \mathrm{Aut}(\mathbb{T})$ on $\mathrm{Sum}_m(\mathbb{T}, G)$ induced by the action of $\mathrm{Aut}(\mathbb{T})$ on $\hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$ is given by*

$$\bigoplus_i (H_i, [\alpha_{H_i}]) \overset{\sigma}{\mapsto} \bigoplus_i \big(\sigma(H_i), [\alpha_{H_i} \sigma^*_{|\mathbb{L}_{\sigma(H_i)}}]\big).$$

4 The action of isogenies on C_0

Let Γ be a height n formal group over a perfect field κ of characteristic p. In this section we introduce three rings depending on this data: the Lubin–Tate ring E_n, the Drinfel'd ring D_∞, and the rationalized Drinfel'd ring C_0.

The Lubin–Tate ring E_n, which is noncanonically isomorphic to $W(\kappa)[\![u_1, \dots, u_{n-1}]\!]$, first appears in the study of formal groups in [13]. The Lubin–Tate moduli problem associates to a complete local ring R with residue field κ the groupoid $\mathrm{Def}_\Gamma(R)$ with objects deformations of Γ to R and morphisms $\star$–isomorphisms; see [15, Section 4]. The main theorem of Lubin and Tate produces an equivalence of groupoids

$$\mathrm{Spf}(E_n)(R) \simeq \mathrm{Def}_\Gamma(R).$$

The ring E_n carries the universal deformation $\mathbb{G}$ of Γ.

Let D_k be the Drinfel'd ring of full level-k structures on $\mathbb{G}$, which was introduced in [9, Section 4B]. Note that Drinfel'd omits the word "full" as all of his level structures are full. The ring D_k is a complete local E_n–algebra that is finitely generated and free as an E_n–module, with the property that

$$\mathrm{Spf}(D_k) \cong \mathrm{Level}(\mathbb{T}[p^k], \mathbb{G}),$$

where $\mathrm{Level}(\mathbb{T}[p^k], \mathbb{G})$ is the functor sending a complete local E_n–algebra R to the set of level structures $\mathrm{Level}(\mathbb{T}[p^k], R \otimes \mathbb{G})$. In particular, D_k has been studied by homotopy theorists in [12, Sections 6.1 and 6.2; 3, Section 2.4; 4, Part 3; 24, Section 7].

For $H \subset \mathbb{T}[p^k] \subset \mathbb{T}$ a finite subgroup, the formal group $\mathbb{G}/H$ may be defined as a deformation over D_k and thus is classified by a map $Q_H\colon E_n \to D_k$. It follows that there is a (necessarily unique) $\star$–isomorphism

$$D_k \,{}^{Q_H}\!\otimes_{E_n} \mathbb{G} \cong (D_k \otimes_{E_n} \mathbb{G})/H.$$

Let

$$D_\infty = \mathrm{colim}_k\, D_k = \mathrm{colim}_k\, \Gamma\, \mathrm{Level}(\mathbb{T}[p^k], \mathbb{G}),$$

where the maps in the indexing diagram are induced by the precomposition

$$(\mathbb{T}[p^k] \to \mathbb{G}) \mapsto (\mathbb{T}[p^{k-1}] \hookrightarrow \mathbb{T}[p^k] \to \mathbb{G}).$$

We define[2] $\hom(D_\infty, R) = \lim_k \hom(D_k, R)$, an element in this set being an infinite compatible family of level structures.

There are actions of several large groups on D_∞; see [7, Section 1.4]. There is an action of $\mathrm{Aut}(\mathbb{T})$ on D_∞ that plays a prominent role in the theorems in [12]. This action is given by precomposing a level structure $\mathbb{T} \hookrightarrow \mathbb{G}$ with the given automorphism of $\mathbb{T}$. Now let $\mathrm{Isog}(\mathbb{T})$ be the monoid of endoisogenies (endomorphisms with finite kernel) of $\mathbb{T}$. Note that $\mathrm{Isog}(\mathbb{T})$ is contravariantly isomorphic to $M_n^{\det\neq 0}(\mathbb{Z}_p)$, the monoid of $n \times n$ matrices with coefficients in $\mathbb{Z}_p$ that have nonzero determinant. We will extend the opposite of the action of $\mathrm{Aut}(\mathbb{T})$ on D_∞ to an action of $\mathrm{Isog}(\mathbb{T})$ on D_∞. After proving this result, the authors discovered that it seems to be well-known. Because we were unable to locate a proof in the literature, we include the argument.

Proposition 4.1 *The action of the group* $\mathrm{Aut}(\mathbb{T})$ *by* E_n*–algebra maps on* D_∞ *given by precomposing a level structure with the inverse automorphism extends to an action of* $\mathrm{Isog}(\mathbb{T})$ *by ring maps.*

Proof Fix an isogeny $\phi_H\colon \mathbb{T} \to \mathbb{T}$ with kernel H. We want to construct a map

$$D_\infty \xrightarrow{\phi_H} D_\infty$$

by mapping to a cofinal subcategory of the diagram category. For $H \subseteq \mathbb{T}[p^k] \subset \mathbb{T}$, fix $k' \geq k$ such that $\mathbb{T}[p^k] \subseteq \phi_H(\mathbb{T}[p^{k'}])$. Consider the diagram

$$\begin{tikzcd} E_n \arrow[r, "Q_H"] \arrow[d] & D_k \arrow[d] & \\ D_k \arrow[r, dashed] & D_{k'} \arrow[r] & R \end{tikzcd}$$

where Q_H is the map classifying the deformation $\mathbb{G}/H$. We would like to construct the dashed map.

The map $D_{k'} \to R$ is a map of E_n–algebras for the standard E_n–algebra structure on $D_{k'}$. Thus we have a level structure $\mathbb{T}[p^{k'}] \hookrightarrow R \otimes \mathbb{G}$. Taking the quotient by H gives the level structure

$$\mathbb{T}[p^{k'}]/H \cong \phi_H(\mathbb{T}[p^{k'}]) \hookrightarrow (R \otimes \mathbb{G})/H.$$

We put a different E_n–algebra structure on R by using Q_H. We abuse notation and denote the composite $E_n \xrightarrow{Q_H} D_{k'} \to R$ by Q_H. Recall that we have the canonical

[2] We are treating D_∞ as an ind-object since the colimit is not a complete local ring.

isomorphism $(R \otimes_{E_n} \mathbb{G})/H \cong R\ {}^{Q_H}\!\otimes_{E_n} \mathbb{G}$. Precomposition with the inclusion $\mathbb{T}[p^k] \subseteq \phi_H(\mathbb{T}[p^{k'}])$ gives

$$\mathbb{T}[p^k] \hookrightarrow R\ {}^{Q_H}\!\otimes_{E_n} \mathbb{G},$$

which is classified by a map $D_k \to R$ making the whole diagram commute. Now take $R = D_{k'}$.

To check compatibility of the maps, it suffices to consider the following situation. Let $l > k$ and $l' > k'$ be such that $\mathbb{T}[p^l] \subset \phi_H(\mathbb{T}[p^{l'}])$; then the commutative diagram

$$\begin{tikzcd}
\mathbb{T}[p^{k'}] \arrow[r, hook] \arrow[d, hook] & D_{l'} \otimes_{E_n} \mathbb{G} \arrow[d, equal] \\
\mathbb{T}[p^{l'}] \arrow[r, hook] & D_{l'} \otimes_{E_n} \mathbb{G}
\end{tikzcd}$$

gives rise to a commutative diagram:

$$\begin{tikzcd}
\mathbb{T}[p^k] \arrow[r, hook] \arrow[d, hook] & \phi_H(\mathbb{T}[p^{k'}]) \arrow[r, hook] \arrow[d, hook] & (D_{l'} \otimes_{E_n} \mathbb{G})/H \arrow[r, "\cong"] \arrow[d, equal] & D_{l'}\ {}^{Q_H}\!\otimes_{E_n} \mathbb{G} \arrow[d, equal] \\
\mathbb{T}[p^l] \arrow[r, hook] & \phi_H(\mathbb{T}[p^{l'}]) \arrow[r, hook] & (D_{l'} \otimes_{E_n} \mathbb{G})/H \arrow[r, "\cong"] & D_{l'}\ {}^{Q_H}\!\otimes_{E_n} \mathbb{G}
\end{tikzcd}$$

This implies that the diagram

$$\begin{tikzcd}
E_n \arrow[r, "Q_H"] \arrow[d] & D_k \arrow[d] \\
D_k \arrow[r, dashed] \arrow[d] & D_{k'} \arrow[d] \\
D_l \arrow[r, dashed] & D_{l'}
\end{tikzcd}$$

commutes. We can now make precise the (more informal sounding) statement that ϕ_H sends the universal level structure $\mathbb{T} \hookrightarrow D_\infty \otimes_{E_n} \mathbb{G}$ to the composite

$$\mathbb{T} \xrightarrow{\psi_H^{-1}} \mathbb{T}/H \to (D_\infty \otimes_{E_n} \mathbb{G})/H \xrightarrow{\cong} D_\infty\ {}^{Q_H}\!\otimes_{E_n} \mathbb{G}.$$

Finally we show that this action extends the action of $\mathrm{Aut}(\mathbb{T})$ on D_∞ by precomposition with the inverse automorphism. Indeed, by the above, the automorphism $\phi_e \in \mathrm{Aut}(\mathbb{T})$ sends the universal level structure $\mathbb{T} \hookrightarrow D_\infty \otimes_{E_n} \mathbb{G}$ to the composite

$$\mathbb{T} \xrightarrow{\phi_e^{-1}} \mathbb{T} \to D_\infty \otimes_{E_n} \mathbb{G},$$

because in this case $\psi_e = \phi_e$, due to the following diagram:

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi_e"] \arrow[dr, "q_e = \mathrm{id}"'] & \mathbb{T} \\ & \mathbb{T}/e = \mathbb{T} \arrow[u, "\psi_e"'] \end{tikzcd}$$

This concludes the proof. □

We note a corollary of the proof.

Corollary 4.2 *Given an endoisogeny ϕ_H of $\mathbb{T}$ with kernel $H \subset \mathbb{T}$, the following diagram commutes:*

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi_H"] \arrow[dr, "q_H"'] & \mathbb{T} \arrow[r, hook] & D_\infty \; {}^{Q_H}\!\otimes_{E_n} \mathbb{G} \\ & \mathbb{T}/H \arrow[u, "\psi_H"'] \arrow[r, hook] & (D_\infty \otimes_{E_n} \mathbb{G})/H \arrow[u, "\cong"'] \end{tikzcd}$$

The ring

$$C_0 = p^{-1} D_\infty = \mathbb{Q} \otimes D_\infty$$

was introduced in [12, Section 6.2] (where it is called $L(E^0)$) to serve as suitable coefficients for the codomain of the generalized character map. By the previous proposition, C_0 is acted on by $\mathrm{Isog}(\mathbb{T})$. Hopkins, Kuhn and Ravenel [12, Proposition 6.6] showed that $C_0^{\mathrm{Aut}(\mathbb{T})} \cong p^{-1} E_n$. It seems reasonable to conjecture that $C_0^{\mathrm{Isog}(\mathbb{T})} \cong \mathbb{Q}_p$. The ring C_0 also has an algebrogeometric description as it carries the universal isomorphism (of p–divisible groups) $\mathbb{T} \xrightarrow{\cong} \mathbb{G}$.

Remark 4.3 The action defined above in Proposition 4.1 is a right action of $\mathrm{Isog}(\mathbb{T})$ on D_∞. The monoid $\mathrm{Isog}(\mathbb{T})$ is dual to $M_n^{\det \neq 0}(\mathbb{Z}_p)$, which acts on D_∞ on the left. Given $x \in C_0$ and an isogeny ϕ_H, we will write $\phi_H^*(x)$ for the action by the Pontryagin dual. One further remark seems necessary. In [12] the inverse action is defined and they write it as a left action even though it is a right action. This could cause confusion.

Example 4.4 Take $n = 1$ and the height 1 formal group $\widehat{\mathbb{G}}_m$. In this case the Drinfel'd ring D_k is just $\mathbb{Z}_p$ adjoin all primitive $(p^k)^{\text{th}}$ roots of unity. The isogeny $[p^k]$ acts trivially on D_∞. This can be seen in many ways. To see it using the proof above note that Q_H must be the standard algebra structure since $E_n \cong \mathbb{Z}_p$. Thus

$(R \otimes \widehat{\mathbb{G}}_m)/H \cong R \otimes \widehat{\mathbb{G}}_m$ canonically. Now the level structure $\Lambda_k \to R \otimes \widehat{\mathbb{G}}_m$ is precisely the restriction to Λ_k of the one we began with. Since every isogeny has the form up^k for some unit (automorphism) u, we see that

$$D_\infty^{\mathrm{Isog}(\mathbb{Q}_p/\mathbb{Z}_p)} = D_\infty^{\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Z}_p)} = \mathbb{Z}_p.$$

Example 4.5 In fact, the isogeny $[p^k]\colon \mathbb{T} \to \mathbb{T}$ always acts by isomorphisms on D_∞. When $\kappa \subset \mathbb{F}_{p^n}$ and $\mathbb{G}$ is the universal deformation of the Honda formal group, $[p^k]$ acts by the identity. This is due to the fact that, in this case, there is a $\star$–isomorphism

$$(D_k \otimes_{E_n} \mathbb{G})/\,\mathbb{T}[p^k] \cong D_k \otimes_{E_n} \mathbb{G}$$

covering the nk–fold Frobenius (which is the identity) on κ.

5 Multiplicative natural transformations on class functions

The goal of this section is to construct a collection of operations on class functions,

$$\mathbb{P}_m^\phi\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G \wr \Sigma_m, C_0),$$

that are multiplicative, nonadditive, natural in the finite group G. These operations combine the action of isogenies on C_0 with the description of conjugacy classes in wreath products given in Section 3. In Section 9, we will show that the operations constructed in this section are compatible with the total power operation in E–theory.

Recall that $\mathbb{L} = \mathbb{Z}_p^n$ and that $\hom(\mathbb{L}, G)_{/\sim}$ is the set of conjugacy classes of maps from $\mathbb{L}$ to G (see Section 2). Given an isogeny $\phi_H\colon \mathbb{T} \to \mathbb{T}$ with kernel $H \subset \mathbb{T}$, there is an action map $\phi_H^*\colon C_0 \to C_0$. Also recall that $\psi_H\colon \mathbb{T}/H \xrightarrow{\cong} \mathbb{T}$ is the isomorphism induced by ϕ_H and that $\psi_H^*\colon \mathbb{L} \to \mathbb{L}_H$ is the Pontryagin dual.

Definition 5.1 For a finite group G and a subgroup $H \subset \mathbb{T}$, define $\mathrm{Cl}_n^H(G, C_0)$ to be the set of C_0–valued functions on $\hom(\mathbb{L}_H, G)_{/\sim}$. When H is the trivial group we will abbreviate this to $\mathrm{Cl}_n(G, C_0)$. Recall that C_0 depends on n and that $\mathrm{Cl}_n^H(G, C_0)$ depends on the prime p even though it is not part of the notation.

Proposition 5.2 *Given an isogeny $\phi_H\colon \mathbb{T} \to \mathbb{T}$ with kernel H we may produce a natural map of rings*

$$(-)^{\phi_H}\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n^H(G, C_0),$$

defined by $f^{\phi_H}([\alpha]) = \phi_H^* f([\alpha\psi_H^*])$.

Proof Note that the map is well defined.

To prove naturality, let $K \xrightarrow{i} G$ be a map of finite groups. This induces restriction maps $\mathrm{Res}\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(K, C_0)$ and $\mathrm{Res}^H\colon \mathrm{Cl}_n^H(G, C_0) \to \mathrm{Cl}_n^H(K, C_0)$. Now we show that $\mathrm{Res}^H \phi_H = \phi_H \mathrm{Res}$. Let $\alpha\colon \mathbb{L}_H \to K$; then we have

$$\begin{aligned}\mathrm{Res}^H(f^{\phi_H})([\alpha]) &= f^{\phi_H}([i\alpha])\\ &= \phi_H^* f([i\alpha\psi_H^*])\\ &= \phi_H^* \mathrm{Res}(f)([\alpha\psi_H^*])\\ &= \mathrm{Res}(f)^{\phi_H}([\alpha]).\end{aligned}$$

For completeness, we show the map is a ring map. Let $f, g \in \mathrm{Cl}_n(G, C_0)$; then

$$\begin{aligned}(f+g)^{\phi_H}([\alpha]) &= \phi_H^*(f+g)([\alpha\psi_H^*])\\ &= \phi_H^*\big(f([\alpha\psi_H^*]) + g([\alpha\psi_H^*])\big)\\ &= \phi_H^* f([\alpha\psi_H^*]) + \phi_H^* g([\alpha\psi_H^*])\\ &= f^{\phi_H}([\alpha]) + g^{\phi_H}([\alpha])\end{aligned}$$

and

$$\begin{aligned}(fg)^{\phi_H}([\alpha]) &= \phi_H^*(fg)([\alpha\psi_H^*])\\ &= \phi_H^*\big(f([\alpha\psi_H^*])g([\alpha\psi_H^*])\big)\\ &= \phi_H^* f([\alpha\psi_H^*])\phi_H^* g([\alpha\psi_H^*])\\ &= f^{\phi_H}([\alpha])g^{\phi_H}([\alpha]).\end{aligned}$$

□

Corollary 5.3 *When restricted to* $\mathrm{Aut}(\mathbb{T}) \subset \mathrm{Isog}(\mathbb{T})$ *the maps defined above are endomorphisms of* $\mathrm{Cl}_n(G, C_0)$*. This is the inverse of the action of* $\mathrm{GL}_n(\mathbb{Z}_p)$ *on class function described by* [12].

Proof The important thing to note is that $q_e = \mathrm{id}\colon \mathbb{T} \to \mathbb{T}$, so we have a triangle

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi_e"] \arrow[dr, "\mathrm{id}"'] & \mathbb{T} \\ & \mathbb{T}/e = \mathbb{T} \arrow[u, "\psi_e"'] \end{tikzcd}$$

factoring the identity map. Thus $\psi_e = \phi_e$ and Proposition 5.2 produces a map

$$\mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G, C_0)$$

and

$$f^{\phi_e}([\alpha]) = \phi_e^* f([\alpha\psi_e^*]) = \phi_e^* f([\alpha\phi_e^*]),$$

which is the action described in [12] after taking into account Remark 4.3. □

Remark 5.4 We will often use the canonical isomorphism

$$\mathrm{Cl}_n(G, C_0) \otimes_{C_0} \mathrm{Cl}_n(K, C_0) \xrightarrow{\cong} \mathrm{Cl}_n(G \times K, C_0),$$

which follows immediately from the definition of $\mathrm{Cl}_n(-, C_0)$.

Our goal is to construct a family of multiplicative natural transformations

$$\mathrm{Cl}_n(-, C_0) \Longrightarrow \mathrm{Cl}_n(- \wr \Sigma_m, C_0)$$

for all $m \geq 0$.

Let $\mathrm{Sub}(\mathbb{T})$ be the set of finite subgroups of $\mathbb{T}$. We may produce a finite subgroup of $\mathbb{T}$ from an isogeny $\alpha\colon \mathbb{T} \to \mathbb{T}$ by taking the kernel of α. The induced surjection

$$\mathrm{Isog}(\mathbb{T}) \to \mathrm{Sub}(\mathbb{T})$$

makes $\mathrm{Isog}(\mathbb{T})$ into a principal $\mathrm{Aut}(\mathbb{T})$–bundle over $\mathrm{Sub}(\mathbb{T})$, and we denote its set of sections by $\Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$.

Fix a section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ and let $\phi_H = \phi(H)$. Proposition 3.6 and Proposition 4.1 may be combined to construct a map

$$\mathbb{P}_m^{\phi}\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G \wr \Sigma_m, C_0).$$

For $f \in \mathrm{Cl}_n(G, C_0)$, we define

$$\mathbb{P}_m^{\phi}(f)\Big(\bigoplus_i (H_i, [\alpha_i])\Big) = \prod_i f^{\phi_{H_i}}([\alpha_i]) = \prod_i \phi_{H_i}^* f([\alpha_i \psi_{H_i}^*]).$$

Let $F\colon \hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim} \xrightarrow{\cong} \mathrm{Sum}_m(\mathbb{T}, G)$ be the isomorphism of Proposition 3.6. For a conjugacy class $[\alpha] \in \hom(\mathbb{L}, G \wr \Sigma_m)_{/\sim}$, we set

$$\mathbb{P}_m^{\phi}(f)([\alpha]) = \mathbb{P}_m^{\phi}(f)(F([\alpha])) = \mathbb{P}_m^{\phi}(f)\Big(\bigoplus_i (H_i, [\overline{\alpha}_i])\Big).$$

Proposition 5.5 *Each section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ gives a natural multiplicative transformation*

$$\mathbb{P}_m^{\phi}\colon \mathrm{Cl}_n(-, C_0) \Longrightarrow \mathrm{Cl}_n(- \wr \Sigma_m, C_0).$$

Proof Multiplicativity follows from Proposition 5.2. Indeed, for f, $g \in \mathrm{Cl}_n(G, C_0)$ we have

$$\prod_i f^{\phi_H}([\alpha_i]) \prod_i g^{\phi_H}([\alpha_i]) = \prod_i f^{\phi_H}([\alpha_i]) g^{\phi_H}([\alpha_i]) = \prod_i (fg)^{\phi_H}([\alpha_i]).$$

A map of groups $K \xrightarrow{j} G$ induces a map on conjugacy classes $j_*\colon \hom(\mathbb{L}, K)_{/\sim} \to \hom(\mathbb{L}, G)_{/\sim}$ and a restriction map $\mathrm{Cl}_n(G, C_0) \xrightarrow{\mathrm{Res}} \mathrm{Cl}_n(K, C_0)$. We show that the diagram

$$\begin{tikzcd} \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^{\phi}(G)"] \arrow[d, "\mathrm{Res}"] & \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0) \arrow[d, "\mathrm{Res}"] \\ \mathrm{Cl}_n(K, C_0) \arrow[r, "\mathbb{P}_m^{\phi}(K)"] & \mathrm{Cl}_n(K \wr \Sigma_{p^k}, C_0) \end{tikzcd}$$

commutes. We have a commutative diagram:

$$\begin{tikzcd} K \wr \Sigma_{p^k} \arrow[r] \arrow[d] & G \wr \Sigma_{p^k} \arrow[d] \\ \Sigma_{p^k} \arrow[r, "="] & \Sigma_{p^k} \end{tikzcd}$$

Thus

$$j_*\Big(\bigoplus_i (H_i, [\alpha_i])\Big) = \bigoplus_i (H_i, j_*[\alpha_i]),$$

and this implies the result. □

Definition 5.6 We call the map

$$\mathbb{P}_m^{\phi}\colon \mathrm{Cl}_n(-, C_0) \Longrightarrow \mathrm{Cl}_n(- \wr \Sigma_m, C_0)$$

the pseudopower operation associated to $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$.

Remark 5.7 It is important to note that $\mathbb{P}_1^{\phi}$ is not necessarily the identity. It depends on the choice of automorphism $\phi_e \in \mathrm{Isog}(\mathbb{T})$.

Note that there is an action of $\mathrm{Aut}(\mathbb{T})$ on $\Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ given by multiplication on the left. For $\gamma \in \mathrm{Aut}(\mathbb{T})$, $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ and $H \in \mathrm{Sub}(\mathbb{T})$, we have

$$(\gamma\phi)_H = \gamma(\phi_H)\colon \mathbb{T} \to \mathbb{T}\,.$$

This action is compatible with the action of $\mathrm{Aut}(\mathbb{T})$ on class functions.

Proposition 5.8 *For $\gamma \in \mathrm{Aut}(\mathbb{T})$ and $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$, we have*

$$\mathbb{P}_m^{\phi}(f^{\gamma}) = \mathbb{P}_m^{\gamma\phi}(f).$$

Proof It suffices to evaluate on conjugacy classes. By multiplicativity it suffices to check the claim on elements of $\mathrm{Sub}_m(\mathbb{T}, G)$. Let $(H, [\alpha]) \in \mathrm{Sub}_m(\mathbb{T}, G)$; then

$$\mathbb{P}_m^{\phi}(f^{\gamma})(H, [\alpha]) = \phi_H^* f^{\gamma}([\alpha\psi_H^*]) = \phi_H^* \gamma^* f([\alpha\psi_H^*\gamma^*]) = \mathbb{P}_m^{\gamma\phi}(f)(H, [\alpha]). \quad \square$$

6 A total power operation on class functions

In this section we study the basic properties of the pseudopower operations from the previous section. Most importantly, we find that if ϕ satisfies a certain combinatorial identity, then $\mathbb{P}_m^\phi$ equips $\mathrm{Cl}_n(-, C_0)$ with the structure of a global power functor in the sense of [10, Section 4]. We prove that there are sections ϕ satisfying this identity when $n = 1, 2$.

The functor $\mathrm{Cl}_n(-, C_0)$ is a global Green functor in the sense of [10, Section 3]. Transfers, restrictions, and their properties are discussed in [10, Section 3] and in [12, Theorem D]. If $K \subset G$, $f \in \mathrm{Cl}_n(K, C_0)$ and $\alpha\colon \mathbb{L} \to G$, then the formula for the transfer is

$$\mathrm{Tr}(f)([\alpha]) = \sum_{gK \in (G/K)^{\mathrm{im}\,\alpha}} f([g\alpha g^{-1}]).$$

We begin by proving an analogue of [6, Proposition VIII.1.1].

Let $i, j \geq 0$ and consider the maps

$$\begin{aligned} \nabla\colon\ & G \wr \Sigma_i \wr \Sigma_j \to G \wr \Sigma_{ij}, \\ \Delta\colon\ & G \to G \wr \Sigma_m, \\ \Delta_{i,j}\colon\ & G \wr (\Sigma_i \times \Sigma_j) \to G \wr \Sigma_{i+j}, \\ \delta\colon\ & (G \times K) \wr \Sigma_m \to (G \wr \Sigma_m) \times (K \wr \Sigma_m). \end{aligned} \tag{4}$$

Let

$$\mathrm{Tr}_{i,j}\colon \mathrm{Cl}_n(\Sigma_i \times \Sigma_j, C_0) \to \mathrm{Cl}_n(\Sigma_m, C_0)$$

be the transfer along $\Sigma_i \times \Sigma_j \subset \Sigma_m$, where $i + j = m$.

Proposition 6.1 *Let $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ and $m, l \geq 0$; then the pseudopower operations associated to ϕ makes the following diagrams commute:*

$$\begin{tikzcd} \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^\phi \times \mathbb{P}_l^\phi"] \arrow[d, "\mathbb{P}_{m+l}^\phi"'] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \times \mathrm{Cl}_n(G \wr \Sigma_l, C_0) \arrow[d, "\times"] \\ \mathrm{Cl}_n(G \wr \Sigma_{m+l}, C_0) \arrow[r, "\Delta^*_{m,l}"] & \mathrm{Cl}_n(G \wr (\Sigma_m \times \Sigma_l), C_0) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Cl}_n(G, C_0) \times \mathrm{Cl}_n(K, C_0) \arrow[r, "\mathbb{P}_m^\phi \times \mathbb{P}_m^\phi"] \arrow[dr, "\mathbb{P}_m^\phi"'] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \times \mathrm{Cl}_n(K \wr \Sigma_m, C_0) \arrow[d, "\delta^*"] \\ & \mathrm{Cl}_n((G \times K) \wr \Sigma_m) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^\phi"] \arrow[dr, "(\mathbb{P}_1^\phi)^{\times m}"'] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \arrow[d, "\Delta^*"] \\ & \mathrm{Cl}_n(G^m, C_0) \end{tikzcd}$$

$$\mathbb{P}_m^\phi(f_1 + f_2) = \mathbb{P}_m^\phi(f_1) + \mathbb{P}_m^\phi(f_2) + \sum_{j=1}^{m-1} \mathrm{Tr}_{j,m-j}(\mathbb{P}_j^\phi(f_1) \times \mathbb{P}_{m-j}^\phi(f_2))$$

Proof These are all standard. We elaborate on the last formula.

Let $\bigoplus_i (H_i, [\alpha_i]) \in \mathrm{Sum}_m(\mathbb{T}, G)$. Note that

$$\mathbb{P}_m^\phi(f_1 + f_2)\Big(\bigoplus_i (H_i, [\alpha_i])\Big) = \prod_i \big(\phi_{H_i}^* f_1([\alpha\psi_{H_i}^*]) + \phi_{H_i}^* f_2([\alpha\psi_{H_i}^*])\big)$$

and expand this into a sum (without simplifying). Assume that the sum of subgroups is indexed by S, so that

$$\bigoplus_i H_i = \bigoplus_{i \in S} H_i.$$

The formula is completely combinatorial. Let

$$Z_j = \big\{T \subset S \mid \sum_{i \in T} |H_i| = j\big\}.$$

By a standard argument, the transfer equals

$$\begin{aligned} &\mathrm{Tr}_{j,m-j}(\mathbb{P}_j^\phi(f_1) \times \mathbb{P}_{m-j}^\phi(f_2))\Big(\bigoplus_{i \in S} (H_i, [\alpha_i])\Big) \\ &\qquad = \sum_{T \in Z_j} \mathbb{P}_j^\phi(f_1)\Big(\bigoplus_{i \in T} (H_i, [\alpha_i])\Big) \mathbb{P}_{m-j}^\phi(f_2)\Big(\bigoplus_{i \in S \setminus T} (H_i, [\alpha_i])\Big). \end{aligned}$$

As j varies this hits exactly the summands from the expanded sum. □

Let $J \subset \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0)$ be the ideal generated by the image of the transfer along the maps

$$G \wr (\Sigma_i \times \Sigma_j) \to G \wr \Sigma_{p^k}$$

for i, $j > 0$ and $i + j = p^k$. In the special case that $G = e$, we will let $I \subset \mathrm{Cl}_n(\Sigma_{p^k}, C_0)$ be the ideal generated by the transfer along the maps $\Sigma_i \times \Sigma_j \subset \Sigma_{p^k}$.

We now give a description of J in the spirit of Section 3.

Proposition 6.2 *The ideal J consists of the factors of*

$$\mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0) = \prod_{\mathrm{Sum}_{p^k}(\mathbb{T},G)} C_0$$

corresponding to the sums with more than one summand.

Proof This follows immediately from the formula for the transfer. If there is a lift

$$\begin{tikzcd} & G \wr (\Sigma_i \times \Sigma_j) \arrow[d] \\ \mathbb{L} \arrow[ur, dashed] \arrow[r, "\alpha"] & G \wr \Sigma_{p^k} \end{tikzcd}$$

up to conjugacy, then $F([\pi\alpha])$ must have more than one subgroup. Now the transfer of the characteristic class function concentrated on the lift is a generator of the factor corresponding to $[\alpha]$. □

Remark 6.3 We could define J for $m \neq p^k$, but the previous proposition shows that, in this case, $J = \mathrm{Cl}_n(G \wr \Sigma_m, C_0)$.

We define some simplifications of $\mathbb{P}^{\phi}_m$. Consider

$$\mathbb{P}^{\phi}_{p^k}/J\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0)/J,$$
$$P^{\phi}_m = \Delta^* \mathbb{P}^{\phi}_m\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G, C_0) \otimes_{C_0} \mathrm{Cl}_n(\Sigma_m, C_0),$$

where $\Delta\colon G \times \Sigma_{p^k} \to G \wr \Sigma_{p^k}$, and

$$P^{\phi}_{p^k}/I\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G, C_0) \otimes_{C_0} \mathrm{Cl}_n(\Sigma_{p^k}, C_0)/I.$$

Corollary 6.4 *The maps $\mathbb{P}^{\phi}_{p^k}/J$ and $P^{\phi}_{p^k}/I$ are both ring maps.*

Proof This follows immediately from the definition of $\mathbb{P}^{\phi}_m$ and Proposition 6.2. □

Remark 6.5 By construction, the pseudopower operation $\mathbb{P}^{\phi}_m$ restricts to a map

$$\mathrm{Cl}_n(G, D_\infty) \to \mathrm{Cl}_n(G \wr \Sigma_m, D_\infty).$$

Instead of defining $\mathbb{P}^{\phi}_{p^k}/J$ and $P^{\phi}_{p^k}/I$ as above, we could have defined them as the rationalization of a map with domain $\mathrm{Cl}_n(G, D_\infty)$. This is not true of $\mathbb{P}^{\phi}_m$ as it is not additive.

Example 6.6 Let $(H,[\alpha]) \in \mathrm{Sub}_{p^k}(\mathbb{T},G)$. We compute the effect of $P^{\phi}_{p^k}/I$ on $\mathrm{Cl}_n(G,C_0)$. Using Proposition 3.9 we see that

$$\begin{aligned}(P^{\phi}_{p^k}/I)(f)(H,[\alpha]) &= \mathbb{P}^{\phi}_{p^k}(f)(H,[\alpha q^*_H])\\ &= f^{\phi_H}([\alpha q^*_H])\\ &= \phi^*_H f([\alpha q^*_H \psi^*_H])\\ &= \phi^*_H f([\alpha \phi^*_H]).\end{aligned}$$

Example 6.7 When $m = p^{kn}$, there is a further natural simplification of $\mathbb{P}^{\phi}_m$. If we choose ϕ to take $\mathbb{T}[p^k]$ to $[p^k]\colon \mathbb{T} \to \mathbb{T}$, then we may compose $P^{\phi}_{p^{kn}}/I$ further with the map

$$\mathrm{Cl}_n(\Sigma_{p^{kn}},C_0)/I \to C_0$$

that projects onto the factor corresponding to $\mathbb{T}[p^k]$. This gives a map

$$\psi^{p^k}\colon \mathrm{Cl}_n(G,C_0) \to \mathrm{Cl}_n(G,C_0),$$

known as the Adams operation corresponding to $\mathbb{T}[p^k]$. The formula for it can be derived from the previous example:

$$(P^{\phi}_{p^{kn}}/I)(f)(\mathbb{T}[p^k],[\alpha]) = \phi^*_{\mathbb{T}[p^k]} f([\alpha \phi^*_{\mathbb{T}[p^k]}]).$$

When $\kappa \subset \mathbb{F}_{p^n}$ and $\mathbb{G}$ is the universal deformation of the Honda formal group (following Example 4.5), this simplifies to give

$$f([\alpha \phi^*_{\mathbb{T}[p^k]}]).$$

This map is induced by the map on conjugacy classes $\hom(\mathbb{L},G)_{/\sim} \to \hom(\mathbb{L},G)_{/\sim}$ sending

$$[\alpha\colon \mathbb{L} \to G] \mapsto [\mathbb{L} \xrightarrow{p^k} \mathbb{L} \xrightarrow{\alpha} G],$$

recovering the usual formula for the Adams operations on class functions when $n = 1$.

We see that we have analogues of all of the parts of [6, Proposition VIII.1.1] except part (ii), which is the fundamental relation that a global power functor satisfies (see [10, Section 4] as well). Now we classify the sections ϕ that give rise to a pseudopower operation that does satisfy this relation.

Definition 6.8 A section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}),\mathrm{Isog}(\mathbb{T}))$ is a *power section* if, for all $H \subset T \subset \mathbb{T}$,

$$\phi_T = \phi_{T/H}\phi_H,$$

where $T/H = \phi_H(T)$.

Proposition 6.9 *The pseudopower operation $\mathbb{P}_m^\phi$ is a global power functor if and only if ϕ is a power section.*

Proof Assume ϕ is a power section. Let $st = m$; we must show that

$$\begin{tikzcd} \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^\phi"] \arrow[d, "\mathbb{P}_s^\phi"] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \arrow[d, "\nabla^*"] \\ \mathrm{Cl}_n(G \wr \Sigma_s, C_0) \arrow[r, "\mathbb{P}_t^\phi"] & \mathrm{Cl}_n(G \wr \Sigma_s \wr \Sigma_t, C_0) \end{tikzcd}$$

commutes.

Recall that $T_{i,j}$ fits into the following commutative diagram:

$$\begin{tikzcd} T_{i,j} \arrow[r] \arrow[d] & \mathbb{T} \arrow[r, "\phi_{H_i}"] \arrow[d] & \mathbb{T} \\ K_{i,j} \arrow[r] & \mathbb{T}/H_i \arrow[ur, "\psi_{H_i}"'] & \end{tikzcd}$$

Thus we have an induced map

$$\psi_{H_i}/K_{i,j} \colon \mathbb{T}/T_{i,j} \to \mathbb{T}/\psi_{H_i}(K_{i,j}).$$

This fits into the following commutative diagram:

$$\begin{tikzcd} \mathbb{T} \arrow[d, "q_{H_i}"] \arrow[dr, "\phi_{H_i}"] & & \\ \mathbb{T}/H_i \arrow[r, "\psi_{H_i}"] \arrow[d] & \mathbb{T} \arrow[d, "q_{\psi_{H_i}(K_{i,j})}"'] \arrow[dr, "\phi_{T_{i,j}/H_i} = \phi_{\psi_{H_i}(K_{i,j})}"] & \\ \mathbb{T}/T_{i,j} \arrow[r, "\psi_{H_i}/K_{i,j}"'] & \mathbb{T}/\psi_{H_i}(K_{i,j}) \arrow[r, "\psi_{T_{i,j}/H_i}"'] & \mathbb{T} \end{tikzcd} \tag{5}$$

Since ϕ is a power section, we have the relation

$$\phi_{T_{i,j}/H_i}\phi_{H_i} = \phi_{T_{i,j}}.$$

The composite of the left-hand vertical maps is $q_{T_{i,j}}$. Thus we have an equality

$$\psi_{T_{i,j}/H_i}\psi_{H_i}/K_{i,j} = \psi_{T_{i,j}}.$$

Applying Pontryagin duality to this equality gives

$$\psi^*_{T_{i,j}} = \psi^*_{H_i}|_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}\psi^*_{T_{i,j}/H_i}.$$

This relation is used in going from the seventh to the eighth line in the display below. Proposition 3.10 shows that a generalized conjugacy class in $G \wr \Sigma_s \wr \Sigma_t$ has the form

$$\bigoplus_i \Big(H_i, \bigoplus_j (K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}]) \Big).$$

Now we may use the definition of the pseudopower operation to compute the effect of $\mathbb{P}_t^\phi \mathbb{P}_s^\phi$ and $\mathbb{P}_m^\phi$ on a class function $f \in \mathrm{Cl}_n(G, C_0)$. Let $f \in \mathrm{Cl}_n(G, C_0)$; then

$$\begin{aligned}
&\mathbb{P}_t^\phi(\mathbb{P}_s^\phi(f))\Big(\bigoplus_i \Big(H_i, \bigoplus_j (K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big)\Big) \\
&\qquad = \prod_i (\mathbb{P}_s^\phi(f))^{\phi_{H_i}} \Big(\bigoplus_j (K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big) \\
&\qquad = \prod_i \phi_{H_i}^* \mathbb{P}_s^\phi(f)\Big(\Big(\bigoplus_j (K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big)\psi_{H_i}^*\Big) \\
&\qquad = \prod_i \phi_{H_i}^* \mathbb{P}_s^\phi(f)\Big(\bigoplus_j (\psi_{H_i}(K_{i,j}), [(\alpha_{H_i})_{K_{i,j}} \psi_{H_i}^* |_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}])\Big) \\
&\qquad = \prod_i \phi_{H_i}^* \mathbb{P}_s^\phi(f)\Big(\bigoplus_j (\phi_{H_i}(T_{i,j}), [(\alpha_{H_i})_{K_{i,j}} \psi_{H_i}^* |_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}])\Big) \\
&\qquad = \prod_i \prod_j \phi_{H_i}^* (f)^{\phi_{T_{i,j}/H_i}} \big([(\alpha_{H_i})_{K_{i,j}} \psi_{H_i}^* |_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}]\big) \\
&\qquad = \prod_i \prod_j \phi_{H_i}^* \phi_{T_{i,j}/H_i}^* f \big([(\alpha_{H_i})_{K_{i,j}} \psi_{H_i}^* |_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}} \psi_{T_{i,j}/H_i}^*]\big) \\
&\qquad = \prod_i \prod_j \phi_{T_{i,j}}^* f \big([(\alpha_{H_i})_{K_{i,j}} \psi_{T_{i,j}}^*]\big) \\
&\qquad = \mathbb{P}_m^\phi(f)\Big(\bigoplus_{i,j} (T_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big).
\end{aligned}$$

Going from the third to the fourth line of the equalities we have used Proposition 3.11 applied to ψ_{H_i}.

The reverse implication follows immediately. □

Example 6.10 When $n = 1$, there is an obvious power section: for $\mathbb{Z}/p^k \subset \mathbb{Q}_p/\mathbb{Z}_p$ we let $\phi_{\mathbb{Z}/p^k}$ be the multiplication by p^k map on $\mathbb{Q}_p/\mathbb{Z}_p$.

The authors have been unable to find a general method for producing power sections for all n. We now turn to the case of $n = 2$, which is important for applications to

elliptic cohomology. We write elements of $\mathrm{Isog}((\mathbb{Q}_p/\mathbb{Z}_p)^2)$ as matrices with entries in $\mathbb{Z}_p$, but view these as acting on $(\mathbb{Q}_p/\mathbb{Z}_p)^2$.

Lemma 6.11 *For each $H \in \mathrm{Sub}_p((\mathbb{Q}_p/\mathbb{Z}_p)^2)$, there exists a matrix ϕ_H such that*

$$\phi_H^2 = [p] = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad \text{and} \quad \phi_H((\mathbb{Q}_p/\mathbb{Z}_p)^2[p]) = H.$$

Proof Associate to the subgroup generated by $(1/p, i/p)$ the matrix

$$\begin{bmatrix} -i & 1 \\ p-i^2 & i \end{bmatrix}$$

and to the subgroup generated by $(0, 1/p)$ the matrix

$$\begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}. \qquad \square$$

Let us fix endomorphisms for each subgroup of order p as in the above lemma and let N be the submonoid of $\mathrm{End}(\mathbb{Z}_p^2)$ generated by the associated matrices ϕ_H. Note that, if H denotes a finite cyclic subgroup of $(\mathbb{Q}_p/\mathbb{Z}_p)^2$, then there exists a unique ordered tuple of matrices $(\phi_{K_1}, \dots, \phi_{K_i})$ associated to subgroups K_j of order p such that $H = \ker(\phi_{K_i} \cdots \phi_{K_1})$. The next lemma provides a generalization of this observation to arbitrary finite subgroups $H \subset (\mathbb{Q}_p/\mathbb{Z}_p)^2$, thereby producing a normal form for ϕ_H.

Lemma 6.12 *Suppose $H \subset (\mathbb{Q}_p/\mathbb{Z}_p)^2$ and let k be the unique natural number with $(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^k] \subseteq H$ and $(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^{k+1}] \nsubseteq H$. Choose subgroups K_α of order p such that*

$$H = \ker(\phi_{K_i} \cdots \phi_{K_1}), \quad \text{ie } |H| = p^i;$$

then the matrix $[p^k]$ divides $\phi_{K_i} \cdots \phi_{K_1}$ in N. Moreover, writing $\phi_{K_i} \cdots \phi_{K_1} = [p^k]\phi_{L_s} \cdots \phi_{L_1}$, the ϕ_{L_β} are uniquely determined by H.

Proof We prove this by induction on the order of H. We may assume that

$$(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^k] \nsubseteq H_1 := \ker(\phi_{K_{i-1}} \cdots \phi_{K_1}).$$

If not, then we are done by induction on the order.

Thus we are in the situation where $H \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^j$ with $k \leq j$ and $H_1 \cong \mathbb{Z}/p^{k-1} \times \mathbb{Z}/p^j$. By induction, there exist subgroups $L_1, \dots, L_s$ of order p such that

$$\phi_{K_{i-1}} \cdots \phi_{K_1} = [p^{k-1}]\phi_{L_s} \cdots \phi_{L_1}.$$

By definition,

$$\phi_{L_s}[p^{k-1}]\phi_{L_{s-1}}\cdots\phi_{L_1}(H) = [p^{k-1}]\phi_{L_s}\cdots\phi_{L_1}(H) = \phi_{K_{i-1}}\cdots\phi_{K_1}(H) = K_i.$$

Now, since $H \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^j$, where $k \le j$ and $H_1 \cong \mathbb{Z}/p^{k-1} \times \mathbb{Z}/p^j$, we must have that $\phi_{L_s}^{-1}(K_i) = (\mathbb{Q}_p/\mathbb{Z}_p)^2[p]$. To see this, note that $[p^{k-1}]\phi_{L_{s-1}}\cdots\phi_{L_1}(H) \cong \mathbb{Z}/p^2$ would imply that $(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^k] \not\subseteq H$, contradicting our assumption. It follows that

$$K_i = \phi_{L_s}((\mathbb{Q}_p/\mathbb{Z}_p)^2[p]) = L_s.$$

So $\phi_{K_i} = \phi_{L_s}$ and

$$\phi_{K_i}\cdots\phi_{K_1} = \phi_{K_i}[p^{k-1}]\phi_{L_s}\cdots\phi_{L_1} = [p^{k-1}]\phi_{L_s}^2\phi_{L_{s-1}}\cdots\phi_{L_1} = [p^k]\phi_{L_{s-1}}\cdots\phi_{L_1}.$$

This finishes the inductive construction. In order to see uniqueness, let $\phi_{K_i}\cdots\phi_{K_1}$ and $\phi_{K_i'}\cdots\phi_{K_1'}$ be two decompositions corresponding to two composition series for H as above. Assume $(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^k] \subseteq H$ and $(\mathbb{Q}_p/\mathbb{Z}_p)^2[p^{k+1}] \not\subseteq H$; then

$$\phi_{K_i}\cdots\phi_{K_1} = [p^k]\phi_{L_s}\cdots\phi_{L_1}$$

and

$$\phi_{K_i'}\cdots\phi_{K_1'} = [p^k]\phi_{L_s'}\cdots\phi_{L_1'}.$$

We have that $[p^k]H \cong H/[p^k]H$ is cyclic and thus admits a unique decomposition. Therefore, $L_\beta = L_\beta'$ for $1 \le \beta \le s$. □

With this in hand, we can produce power sections for $n = 2$.

Proposition 6.13 *For each choice of endomorphisms for subgroups of order p in $(\mathbb{Q}_p/\mathbb{Z}_p)^2$ satisfying the conclusion of Lemma 6.11, there is a power section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$, ie a section ϕ such that, for $K \subset H$,*

$$\phi_H = \phi_{H/K}\phi_K,$$

where $H/K = \phi_K(H)$.

Proof With notation as in the previous lemma, define ϕ_H to be the composite $\phi_{K_i}\cdots\phi_{K_1}$. The power section condition for ϕ follows from the uniqueness of its normal form, given by Lemma 6.12. □

7 Recollections on Morava E–theory

In this section we recall the facts that we need about Morava E–theory before we can proceed. For any space X we set

$$E_n(X) = E_n^0(X).$$

We will also set $E_n = E_n^0$. By Goerss, Hopkins and Miller [11] there is a unique E_∞–ring structure on the spectrum E_n. In the homotopy category, this gives rise to total power operations [6, Section VIII]. The total power operations are natural multiplicative nonadditive maps

$$\mathbb{P}_m\colon E_n(X) \to E_n(E\Sigma_m \times_{\Sigma_m} X^m)$$

for all $m > 0$ that satisfy the relations of Proposition 7.2.

A common simplification of $\mathbb{P}_m$ is given by restriction along the diagonal $X \xrightarrow{\Delta} X^m$. This gives a map

$$P_m\colon E_n(X) \to E_n(B\Sigma_m \times X) \cong E_n(B\Sigma_m) \otimes_{E_n} E_n(X),$$

the isomorphism being a consequence of the freeness of $E_n(B\Sigma_m)$ over E_n [25, Proposition 3.6]. Let $J \subset E_n(E\Sigma_{p^k} \times_{\Sigma_{p^k}} X^{p^k})$ be the ideal generated by the image of the transfers along the maps

$$E(\Sigma_i \times \Sigma_j) \times_{(\Sigma_i \times \Sigma_j)} X^{p^k} \to E\Sigma_{p^k} \times_{\Sigma_{p^k}} X^{p^k}$$

for i, $j > 0$ and $i + j = p^k$, and let $I \subset E_n(B\Sigma_{p^k})$ be the ideal generated by the image of the same transfer for $X = *$. Neither $\mathbb{P}_m$ nor P_m are additive, but they can both be made so by taking the quotient by J and I, respectively. Thus we have ring maps

$$\mathbb{P}_{p^k}/J\colon E_n(X) \to E_n(E\Sigma_{p^k} \times_{\Sigma_{p^k}} X^{p^k})/J$$

and

$$P_{p^k}/I\colon E_n(X) \to E_n(B\Sigma_{p^k} \times X) \cong E_n(B\Sigma_{p^k})/I \otimes_{E_n} E_n(X).$$

Let us call $\mathbb{P}_{p^k}/J$ the additive total power operation. For $m \neq p^k$ it is still possible to define $\mathbb{P}_m/J$, but these ring maps are uninteresting.

Remark 7.1 We have abused notation by calling these ideals I and J in conflict with the I and J defined prior to Proposition 6.2.

We will apply these power operations to $X = BG$; in this case,

$$E\Sigma_m \times_{\Sigma_m} X^m \simeq BG \wr \Sigma_m.$$

Recall the maps of (4) from Section 6. The relations that we will need are the following:

Proposition 7.2 [6] *For any $x \in E_n(X)$ and i, j, $m \geq 0$, we have*

(1) $\nabla^* \mathbb{P}_{ij}(x) = \mathbb{P}_i \mathbb{P}_j(x)$,

(2) $\Delta^* \mathbb{P}_m(x) = x^m$,

(3) $\Delta_{i,j}^* \mathbb{P}_{i+j}(x) = \delta^*(\mathbb{P}_i(x)\mathbb{P}_j(x))$.

Proof The proof can be found in [6, Chapter VIII]. □

Corollary 7.3 [6, Proposition VIII.1.1] *There is a commutative diagram*

$$\begin{tikzcd} E_n(BG) \arrow[r, "\mathbb{P}_{p^k}"] \arrow[d, "\mathbb{P}_{p^{k-1}}"] & E_n(BG \wr \Sigma_{p^k}) \arrow[d, "\nabla^*"] & \\ E_n(BG \wr \Sigma_{p^{k-1}}) \arrow[r, "\mathbb{P}_p"] & E_n(BG \wr \Sigma_{p^{k-1}} \wr \Sigma_p) \arrow[r] & E_n(BG \wr \Sigma^p_{p^{k-1}}) \end{tikzcd}$$

and the formula for the whole composite is just $x \mapsto \mathbb{P}_{p^{k-1}}(x) \otimes \cdots \otimes \mathbb{P}_{p^{k-1}}(x)$.

Proof The result follows immediately from Proposition 7.2(1) and (3). □

Character theory for Morava E–theory is constructed in [12]. Hopkins, Kuhn and Ravenel construct the ring C_0, which is introduced in Section 4, and produce a natural character map

$$\chi\colon E_n(BG) \to \mathrm{Cl}_n(G, C_0)$$

with the property that the induced map

$$C_0 \otimes_{E_n} E_n(BG) \xrightarrow{\cong} \mathrm{Cl}_n(G, C_0)$$

is an isomorphism. They produce the action of $\mathrm{Aut}(\mathbb{T})$ on $\mathrm{Cl}_n(G, C_0)$, described in Proposition 4.1, and prove further that the above isomorphism induces an isomorphism

$$p^{-1}E_n(BG) \xrightarrow{\cong} \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}.$$

Note that $E_n(BG)$ is a $\mathbb{Z}_p$–algebra, so $p^{-1}E_n(BG) = \mathbb{Q} \otimes E_n(BG)$ is the rationalization of the ring.

Theorem D in [12] discusses the relationship between the character map and transfer maps for E_n and class functions. For $H \subset G$ there is a commutative diagram

$$\begin{tikzcd} E_n(BH) \arrow[r, "\mathrm{Tr}"] \arrow[d, "\chi"'] & E_n(BG) \arrow[d, "\chi"] \\ \mathrm{Cl}_n(H, C_0) \arrow[r, "\mathrm{Tr}"] & \mathrm{Cl}_n(G, C_0) \end{tikzcd}$$

where

$$\mathrm{Tr}\colon \mathrm{Cl}_n(H, C_0) \to \mathrm{Cl}_n(G, C_0)$$

is the transfer introduced in Section 6.

There is a close relationship between the E–cohomology of finite groups and algebraic geometry related to $\mathbb{G}$. We will recall some of these results, and refer the interested reader to [12; 25; 20] for the details.

Let $\mathrm{Hom}(A^*, \mathbb{G})$ be the scheme of maps from A^* to $\mathbb{G}$ and define $\mathrm{Sub}_{p^k}(\mathbb{G})$ to be the scheme of subgroups of order p^k of $\mathbb{G}$, sending an E_n–algebra R to the collection of subgroups $H \subseteq R \otimes \mathbb{G}$ of order p^k. Moreover, for $A^* \subseteq \mathbb{T}'$ a finite abelian group, $\mathrm{Sub}^{A^*}_{p^k|A|}(\mathbb{G} \oplus \mathbb{T}')$ denotes the scheme with underlying functor

$$R \mapsto \{H \subseteq R \otimes \mathbb{G} \oplus \mathbb{T}' : |H| = p^k|A|,\ \mathrm{pr}(H) = A^*\}$$

for an E_n–algebra R, where pr is induced by the natural projection $\mathbb{G} \oplus \mathbb{T}' \to \mathbb{T}'$. We often write $p^{k'} = p^k|A|$. The following table provides the dictionary we need:

topology	algebraic geometry
$E_n(BA)$	$\mathrm{Hom}(A^*, \mathbb{G})$
$E_n(B\Sigma_{p^k})/I$	$\mathrm{Sub}_{p^k}(\mathbb{G})$
$E_n(BA \wr \Sigma_{p^k})/J$	$\mathrm{Sub}^{A^*}_{p^{k'}}(\mathbb{G} \oplus \mathbb{T}')$

These results can be found in [12, Proposition 5.12; 25, Theorem 9.2; 20, Theorem 7.11].

8 The additive total power operation applied to abelian groups

Work of Ando [3] as well as Ando, Hopkins and Strickland [4] gives an algebrogeometric description of the additive power operation P_{p^k}/I applied to finite abelian groups. In this section we recall their result and prove an extension, which is a key step in the proof of the main theorem.

Lemma 8.1 *For any finite group G, there is an induced map*

$$E_n(BG \wr \Sigma_{p^k})/J \to E_n(BG) \otimes_{E_n} E_n(B\Sigma_{p^k})/I.$$

Proof The claim reduces to checking that the following commutative diagram is a homotopy pullback for any i and j with $i + j = p^k$:

$$\begin{tikzcd} B(G \times \Sigma_i \times \Sigma_j) \arrow[r] \arrow[d] & BG \wr (\Sigma_i \times \Sigma_j) \arrow[d] \\ B(G \times \Sigma_{p^k}) \arrow[r] & BG \wr \Sigma_{p^k} \end{tikzcd}$$

Since the square is commutative, it suffices to show that $BG \times \Sigma_i \times \Sigma_j$ has the correct homotopy type. This is obvious for π_k for $k > 1$. To see that the pullback is connected, consider the double coset formula. Every element $g = (g_1, \dots, g_{p^k}; \sigma) \in G \wr \Sigma_{p^k}$ can be factored as $(g_1, \dots, g_{p^k}; e) \circ (1, \dots, 1; \sigma)$, where $(1, \dots, 1; \sigma) \in G \times \Sigma_{p^k}$ and $(g_1, \dots, g_{p^k}; e) \in G \wr (\Sigma_i \times \Sigma_j)$. Finally, π_1 of the pullback is given by the intersection of $G \times \Sigma_{p^k}$ with $G \wr (\Sigma_i \times \Sigma_j)$ inside $G \wr \Sigma_{p^k}$, which clearly is $G \times \Sigma_i \times \Sigma_j$. □

There are two E_n–algebra structures on $E_n(B\Sigma_{p^k})/I$ of interest to us, given by the following two maps: the standard inclusion i induced by $B\Sigma_{p^k} \to *$ and the power operation $P_{p^k}/I \colon E_n \to E_n(B\Sigma_{p^k})/I$. Using Lemma 8.1 and the commutative diagram

$$\begin{tikzcd} E_n \arrow[r, "P_{p^k}/I"] \arrow[d] & E_n(B\Sigma_{p^k})/I \arrow[d] \\ E_n(BA) \arrow[r, "\mathbb{P}_{p^k}/J"] & E_n(BA \wr \Sigma_{p^k})/J \end{tikzcd}$$

we thus obtain a commutative diagram of rings

$$\begin{tikzcd} E_n(BA) \arrow[r, "\mathbb{P}_{p^k}/J"] \arrow[rr, bend left, "P_{p^k}/I"] \arrow[dr] & E_n(BA \wr \Sigma_{p^k})/J \arrow[r] & E_n(BA) \otimes^i_{E_n} E_n(B\Sigma_{p^k})/I \\ & E_n(BA) \otimes^{P_{p^k}/I}_{E_n} E_n(B\Sigma_{p^k})/I \arrow[u] \arrow[ur] & \end{tikzcd} \tag{6}$$

where the superscripts on the tensor product indicate the relevant E_n–algebra structure on $E_n(B\Sigma_{p^k})/I$.

Remark 8.2 In our notation, A plays a very different role than in the setting of [4]. Indeed, they consider a level structure in place of our Σ_{p^k} and S^1 instead of our A; the translation is readily made by passing to torsion subgroups of S^1 and using arbitrary subgroups; see [4, Remark 3.12].

The following result is proven in [3, Lemma 4.2.5] and [4, Proposition 3.21].

Proposition 8.3 (Ando–Hopkins–Strickland) *The additive power operation*

$$E_n(BA) \xrightarrow{P_{p^k}/I} E_n(BA) \otimes^i_{E_n} E_n(B\Sigma_{p^k})/I$$

is the ring of functions on the map

$$\hom(A^*, \mathbb{G}) \times \mathrm{Sub}_{p^k}(\mathbb{G}) \to \hom(A^*, \mathbb{G})$$

induced by

$$(f\colon A^* \to R\otimes\mathbb{G},\ H\subset R\otimes\mathbb{G}) \mapsto (A^* \xrightarrow{f} R\otimes\mathbb{G} \to (R\otimes\mathbb{G})/H \cong R^{P_{p^k}/I}\otimes_{E_n}\mathbb{G}).$$

Fix a map $\mathbb{Z}_p^t = \mathbb{L}' \to A$ with A finite abelian and let $\mathbb{T}' = (\mathbb{L}')^*$. Let $p^{k'} = p^k|A|$. Recall from [20] that there is a formal scheme $\mathrm{Sub}^{A^*}_{p^{k'}}(\mathbb{G}\oplus\mathbb{T}')$ that associates to any E_n–algebra R the collection of subgroup schemes $H\subset R\otimes(\mathbb{G}\oplus\mathbb{T}')$ of order $p^{k'}$ which project onto A^* via the natural map $\mathbb{G}\oplus\mathbb{T}'\to\mathbb{T}'$. The main result of [20] implies that the ring of functions on $\mathrm{Sub}^{A^*}_{p^{k'}}(\mathbb{G}\oplus\mathbb{T}')$ is isomorphic to $E_n(BA\wr\Sigma_{p^k})/J$. Our extension of Proposition 8.3 can now be stated as follows:

Theorem 8.4 *The additive total power operation modulo the transfer*

$$E_n(BA) \xrightarrow{\mathbb{P}_{p^k}/J} E_n(BA\wr\Sigma_{p^k})/J$$

is the ring of functions on the map

$$\mathbb{Q}^*_{p^k}\colon \mathrm{Sub}^{A^*}_{p^{k'}}(\mathbb{G}\oplus\mathbb{T}') \to \hom(A^*,\mathbb{G})$$

given by

$$(H\subset R\otimes(\mathbb{G}\oplus\mathbb{T}')) \mapsto (A^* \to R\otimes\mathbb{G}/K \cong R^{P_{p^k}/I}\otimes_{E_n}\mathbb{G}),$$

where K is the kernel in the map of short exact sequences

$$\begin{tikzcd} K \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & A^* \arrow[d] \\ K \arrow[r] & \mathbb{G} \arrow[r] & \mathbb{G}/K \end{tikzcd}$$

and H maps to $\mathbb{G}$ through the projection $\mathbb{G}\oplus\mathbb{T}' \twoheadrightarrow \mathbb{G}$.

The content of the theorem is that two maps between $E_n(BA)$ and $E_n(BA\wr\Sigma_{p^k})/J$ are in fact the same map. The map $\mathbb{Q}_{p^k}$ is defined algebrogeometrically and the map $\mathbb{P}_{p^k}/J$ is the additive total power operation. By Proposition 8.3, we know that the maps are equal after mapping further to $E_n(BA)\otimes_{E_n}E_n(B\Sigma_{p^k})/I$. But the map

$$E_n(BA\wr\Sigma_{p^k})/J \to E_n(BA)\otimes_{E_n}E_n(B\Sigma_{p^k})/I$$

is not injective in general. We will proceed by building a ring that $E_n(BA\wr\Sigma_{p^k})/J$ injects into, and that can be attacked using Proposition 8.3.

Before giving the proof, we draw a consequence that is of interest in its own right:

Corollary 8.5 *The map*

$$\mathbb{P}_{p^k}/J \otimes_{E_n}^{P_{p^k}/I} E_n(B\Sigma_{p^k})/I \colon E_n(BA) \otimes_{E_n}^{P_{p^k}/I} E_n(B\Sigma_{p^k})/I \to E_n(BA \wr \Sigma_{p^k})/J$$

given in diagram (6) *is an isomorphism.*

Proof This is the map that occurs in Theorem 8.4 base changed to $E_n(B\Sigma_{p^k})/I$. By Theorem 8.4, there is a commutative diagram

$$\begin{tikzcd} E_n(BA) \arrow[r, "\mathbb{P}_{p^k}/J"] \arrow[d, "\wr"] & E_n(BA \wr \Sigma_{p^k})/J \arrow[d, "\wr"] \\ \Gamma\,\hom(A^*, \mathbb{G}) \arrow[r, "\mathbb{Q}_{p^k}"] & \Gamma\,\mathrm{Sub}_{p^{k\prime}}^{A^*}(\mathbb{G} \oplus \mathbb{T}'), \end{tikzcd}$$

where the vertical isomorphisms are canonical. Thus it is enough to show that

$$\mathbb{Q}_{p^k} \otimes_{E_n}^{P_{p^k}/I} E_n(B\Sigma_{p^k})/I$$

is an isomorphism. Proposition 6.5 of [20] implies that it is an isomorphism. □

The idea of the proof of Theorem 8.4 is to reduce the claim to Proposition 8.3 by probing $E_n(BA \wr \Sigma_{p^k})/J$ by an appropriate family $\mathcal{F}(A \wr \Sigma_{p^k})$ of abelian subgroups of $A \wr \Sigma_{p^k}$ that captures all of the transitive conjugacy classes. For each of the abelian subgroups $M \in \mathcal{F}(A \wr \Sigma_{p^k})$, we show how the composite

$$E_n(BA) \xrightarrow{\mathbb{P}_{p^k}/J} E_n(BA \wr \Sigma_{p^k})/J \to E_n(BM)/I_M$$

can be attacked using Proposition 8.3, where I_M is the ideal in $E_n(BM)$ generated by the image of the transfer along all proper subgroups of M. This ring has been studied in [4] and is closely related to M–level structures on $\mathbb{G}$.

Definition 8.6 For each $[\alpha] \in \hom(\mathbb{L}, A \wr \Sigma_{p^k})^{\mathrm{trans}}_{/\sim}$, choose a representative that satisfies Corollary 3.5. Let $\mathcal{F}(A \wr \Sigma_{p^k})$ be the set of images of these representatives.

By definition these subgroups fit into the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[d, "\|"'] & M \arrow[d] \arrow[dr, "\pi\alpha"] & \\ K \arrow[r] \arrow[d] & K \wr \Sigma_{p^k} \arrow[r] \arrow[d] & \Sigma_{p^k} \arrow[d, "\|"] \\ A \arrow[r] & A \wr \Sigma_{p^k} \arrow[r] & \Sigma_{p^k} \end{tikzcd} \tag{7}$$

where K denotes the pullback or, equivalently, the kernel of $\pi\alpha$ and $M \in \mathcal{F}(A \wr \Sigma_{p^k})$.

Lemma 8.7 *For any abelian group A, the map*

$$E_n(BA \wr \Sigma_{p^k})/J \to \prod_{M \in \mathcal{F}(A \wr \Sigma_{p^k})} E_n(BM)/I_M$$

induced by restriction is injective.

Proof We first need to show that the map exists. The map exists if the homotopy pullback of the diagram

$$\begin{array}{ccc} & & B(A \wr (\Sigma_i \times \Sigma_j)) \\ & & \downarrow \\ BM & \longrightarrow & B(A \wr \Sigma_{p^k}) \end{array}$$

is a disjoint union of classifying spaces of proper subgroups of M when $i, j > 1$ and $i + j = p^k$. The classifying spaces in the homotopy pullback are of the form

$$B(gMg^{-1} \cap (A \wr (\Sigma_i \times \Sigma_j)),$$

where g is a representative of a double coset in

$$M \backslash (A \wr \Sigma_{p^k})/(A \wr (\Sigma_i \times \Sigma_j)).$$

The conjugate of a transitive subgroup is transitive and $A \wr (\Sigma_i \times \Sigma_j)$ is not transitive, thus the intersection cannot be all of M.

Since $E_n(BA \wr \Sigma_{p^k})/J$ is a finitely generated free E_n–module [20, Proposition 5.3], we may check injectivity after applying the character map. This allows us to check the claim on class functions, where it follows immediately from the construction of $\mathcal{F}(A \wr \Sigma_{p^k})$. □

We now consider two maps between M and $M \wr \Sigma_{p^k}$. The first is the composite

$$M \hookrightarrow K \wr \Sigma_{p^k} \hookrightarrow M \wr \Sigma_{p^k}$$

and the second is the composite

$$M \xrightarrow{\mathrm{id} \times \pi\alpha} M \times \Sigma_{p^k} \xrightarrow{\Delta} M \wr \Sigma_{p^k}.$$

Applying E–cohomology to these maps gives the same map because they are conjugate:

Lemma 8.8 *The square*

$$\begin{tikzcd} E_n(BM \wr \Sigma_{p^k})/J \arrow[r] \arrow[d] & E_n(BM) \otimes^i_{E_n} E_n(B\Sigma_{p^k})/I \arrow[d] \\ E_n(BK \wr \Sigma_{p^k})/J \arrow[r] & E_n(BM)/I_M \end{tikzcd}$$

commutes, where the maps are induced by the maps described just above.

Proof We show that the two maps $M \rightrightarrows M \wr \Sigma_{p^k}$ are conjugate. The composites

$$M \rightrightarrows M \wr \Sigma_{p^k} \to \Sigma_{p^k}$$

coincide by construction. Thus, by Proposition 3.4, it suffices to show that the kernel K factors through $K \xrightarrow{\Delta} K \wr \Sigma_{p^k}$ by the identity map in each case. But this follows from the commutativity of the following two diagrams:

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[dd, "\iota"'] & M \arrow[d] \arrow[ddr] & \\ & M \times \Sigma_{p^k} \arrow[d] & \\ M \arrow[r, "\Delta"] & M \wr \Sigma_{p^k} \arrow[r] & \Sigma_{p^k} \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r] \arrow[d, "\|"] \arrow[dd, bend right, "\iota"'] & M \arrow[d] \arrow[ddr] & \\ K \arrow[r] \arrow[d] & K \wr \Sigma_{p^k} \arrow[d] \arrow[dr] & \\ M \arrow[r, "\Delta"] & M \wr \Sigma_{p^k} \arrow[r] & \Sigma_{p^k} \end{tikzcd}$$

Thus we have a commutative diagram in E–cohomology without taking any quotients. For each ring we may take the quotient by the appropriate transfer ideal to get the square in the statement of the lemma. □

We are now ready to put the pieces together to prove the main theorem of this section.

Proof of Theorem 8.4 By Lemma 8.7, we may reduce to showing that the following two composites coincide:

$$E_n(BA) \underset{\mathbb{Q}_{p^k}}{\overset{\mathbb{P}_{p^k}/J}{\rightrightarrows}} E_n(BA \wr \Sigma_{p^k})/J \hookrightarrow \prod_{M \in \mathcal{F}(A \wr \Sigma_{p^k})} E_n(BM)/I_M$$

We will prove this one factor at a time. Let $M \in \mathcal{F}(A \wr \Sigma_{p^k})$ and consider the diagram:

$$\begin{tikzcd}
E_n(BM) \arrow[r, shift left, "\mathbb{P}_{p^k}/J"] \arrow[r, shift right, "\mathbb{Q}_{p^k}"'] \arrow[d, two heads, "f"'] & E_n(BM \wr \Sigma_{p^k})/J \arrow[r] \arrow[d] & E_n(BM) \otimes^i_{E_n} E_n(B\Sigma_{p^k})/I \arrow[d] \\
E_n(BK) \arrow[r, shift left, "\mathbb{P}_{p^k}/J"] \arrow[r, shift right, "\mathbb{Q}_{p^k}"'] & E_n(BK \wr \Sigma_{p^k})/J \arrow[r] & E_n(BM)/I_M \\
E_n(BA) \arrow[r, shift left, "\mathbb{P}_{p^k}/J"] \arrow[r, shift right, "\mathbb{Q}_{p^k}"'] \arrow[u] & E_n(BA \wr \Sigma_{p^k})/J \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The diagram commutes, where the $\mathbb{Q}_{p^k}$'s commute with $\mathbb{Q}_{p^k}$'s and the $\mathbb{P}_{p^k}/J$'s commute with $\mathbb{P}_{p^k}/J$'s; for the top right square this follows from Lemma 8.8. The composites on the top row are equal by Proposition 8.3. The surjectivity of f implies that the middle two composites are equal. But now that these are equal, the composites from $E_n(BA)$ to $E_n(BM)/I_M$ must be equal. This proves the claim. □

9 The character of the total power operation

In this section we establish the relevance of the multiplicative natural transformations constructed in Section 5. The goal of this section is to prove the following theorem.

Theorem 9.1 *Let $\mathbb{P}_m\colon E_n(BG) \to E_n(BG \wr \Sigma_m)$ be the total power operation for Morava E–cohomology applied to BG, and let $\chi\colon E_n(BG) \to \mathrm{Cl}_n(G, C_0)$ be the character map. For all $n \geq 0$, all $m \geq 0$ and any section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$, there is a commutative diagram*

$$\begin{tikzcd}
E_n(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & E_n(BG \wr \Sigma_m) \arrow[d, "\chi"] \\
\mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_m^{\phi}"] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0)
\end{tikzcd}$$

which is natural in G.

We will prove the theorem in three steps. First, our extension of the work of Ando, Hopkins and Strickland in Theorem 8.4 can be used to prove the theorem for the additive total power operation applied to finite abelian groups. To extend this to all finite groups, we use a modification of the fact that $E_n(BG)$ rationally embeds under the restriction map into $\prod_{A \subseteq G} E_n(BA)$, where the product is over all abelian subgroups

of G. Finally, an inductive argument using character theory extends the result from the additive total power operation to the total power operation.

Proposition 9.2 *For a finite abelian group A the diagram*

$$\begin{tikzcd} E_n(BA) \arrow[r, "\mathbb{P}_{p^k}/J"] \arrow[d] & E_n(BA \wr \Sigma_{p^k})/J \arrow[d] \\ \mathrm{Cl}_n(A, C_0) \arrow[r, "\mathbb{P}^{\phi}_{p^k}/J"] & \mathrm{Cl}_n(A \wr \Sigma_{p^k}, C_0)/J \end{tikzcd} \tag{8}$$

commutes.

Proof By Proposition 6.2, the terminal object in the square is the product

$$\mathrm{Cl}_n(A \wr \Sigma_{p^k}, C_0)/J \cong \prod_{\mathrm{Sub}_{p^k}(\mathbb{T}, A)} C_0.$$

Thus it suffices to fix an element $(H, [\alpha]) \in \mathrm{Sub}_{p^k}(\mathbb{T}, A)$ and prove the result for the factor corresponding to $(H, [\alpha])$. Since $\mathbb{P}^{\phi}_{p^k}/J$ is the rationalization of a map between products of the Drinfel'd ring D_∞ by Remark 6.5, it suffices to replace C_0 with D_∞. For R a complete local ring, a map $D_\infty \to R$ out of the factor corresponding to $(H, [\alpha])$ is the data

$$(H \subset \mathbb{T},\ \alpha\colon \mathbb{L}_H \to A,\ \mathbb{T} \hookrightarrow R \otimes_{E_n} \mathbb{G}). \tag{9}$$

We are suppressing the data of the Lubin–Tate moduli problem. Note that, since A is abelian, $\alpha = [\alpha]$ and that we may replace α by

$$\alpha^*\colon A^* \to \mathbb{T}/H.$$

Let $\mathbb{L}' = \mathbb{Z}_p^t$ and $\mathbb{T}' = (\mathbb{L}')^*$, where t is greater than or equal to the number of generators of A. Let $\mathbb{L}' \twoheadrightarrow A$ be a surjection. By the algebrogeometric description of $E_n(BA \wr \Sigma_{p^k})/J$ in [20, Theorem 7.11], the right vertical map in (8) sends the data in (9) to the pullback:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathbb{G} \oplus \mathbb{T}' \arrow[d] \\ A^* \arrow[r] & (R \otimes_{E_n} \mathbb{G})/H \oplus \mathbb{T}' \end{tikzcd}$$

The top horizontal map in (8) sends this data to the composite

$$A^* \to (R \otimes_{E_n} \mathbb{G})/H \cong R\,{}^{Q_H}\!\otimes_{E_n} \mathbb{G} \tag{10}$$

by Proposition 8.3.

Going around the other way first sends the data to the pair of composites

$$(A^* \xrightarrow{\alpha^*} \mathbb{T}/H \xrightarrow{\psi_H} \mathbb{T},\ \mathbb{T} \xrightarrow{\psi_H^{-1}} \mathbb{T}/H \to R^{Q_H} \otimes_{E_n} \mathbb{G})$$

and then composes them to give

$$A^* \to R^{Q_H} \otimes_{E_n} \mathbb{G},$$

which is the same as the map in (10). □

Lemma 9.3 *There is an embedding*

$$p^{-1}E_n(BG \wr \Sigma_{p^k})/J \hookrightarrow \prod_{A \subseteq G} p^{-1}E_n(BA \wr \Sigma_{p^k})/J.$$

Proof Since C_0 is a faithfully flat $p^{-1}E_n$–algebra it suffices to check this on class functions. Thus the claim is equivalent to there being a surjection of sets

$$\coprod_{A \subseteq G} \mathrm{Sub}_{p^k}(\mathbb{T}, A) \to \mathrm{Sub}_{p^k}(\mathbb{T}, G).$$

This is a surjection because any map $\mathbb{L}_H \to G$ factors through its image, which is an abelian subgroup of G. □

Proposition 9.4 *For G a finite group, the diagram*

$$\begin{array}{ccc} E_n(BG) & \xrightarrow{\mathbb{P}_{p^k}/J} & E_n(BG \wr \Sigma_{p^k})/J \\ \downarrow & & \downarrow \\ \mathrm{Cl}_n(G, C_0) & \xrightarrow{\mathbb{P}^{\phi}_{p^k}/J} & \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0)/J \end{array}$$

commutes.

Proof The map from the top arrow to the bottom arrow factors through the rationalization because $\mathbb{P}_{p^k}/J$ is a ring map and C_0 is a rational algebra. It suffices to consider the following cube:

$$\begin{tikzcd}
 & p^{-1}E_n(BG) \arrow[rr] \arrow[dl, "\mathbb{P}_{p^k}/J"'] \arrow[dd] & & \prod_{A\subseteq G} p^{-1}E_n(BA) \arrow[dl, "\prod \mathbb{P}_{p^k}/J"'] \arrow[dd] \\
p^{-1}E_n(BG \wr \Sigma_{p^k})/J \arrow[rr] \arrow[dd] & & \prod_{A\subseteq G} p^{-1}E_n(BA \wr \Sigma_{p^k})/J \arrow[dd] & \\
 & \mathrm{Cl}_n(G, C_0) \arrow[rr] \arrow[dl, "\mathbb{P}^{\phi}_{p^k}/J"'] & & \prod_{A\subseteq G} \mathrm{Cl}_n(A, C_0) \arrow[dl, "\prod \mathbb{P}^{\phi}_{p^k}/J"'] \\
\mathrm{Cl}_n(G \wr \Sigma_{p^k})/J \arrow[rr] & & \prod_{A\subseteq G} \mathrm{Cl}_n(A \wr \Sigma_{p^k})/J. &
\end{tikzcd}$$

The top and bottom squares commute by naturality, the back and front squares commute by character theory [12, Theorems C and D], and the right square commutes by Proposition 9.2. Now, since the horizontal maps are injections, the left square must commute. □

Proposition 9.5 *For all $k > 0$ and any finite group G, there is an injection*

$$\mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0) \hookrightarrow (\mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0)/J) \times \mathrm{Cl}_n(G \wr \Sigma^p_{p^{k-1}}, C_0),$$

where the map to the left factor is the quotient and the map to the right factor is given by restriction.

Proof It is just a matter of checking this on conjugacy classes. Every element of $\mathrm{Sub}_{p^k}(\mathbb{T}, G) \subseteq \mathrm{Sum}_{p^k}(\mathbb{T}, G)$ is hit by the left factor. By Proposition 3.6, the map to the other factor is induced by the map

$$\prod_{l=1}^{p} \mathrm{Sum}_{p^{k-1}}(\mathbb{T}, G) \twoheadrightarrow \mathrm{Sum}_{p^k}(\mathbb{T}, G) \setminus \mathrm{Sub}_{p^k}(\mathbb{T}, G)$$

defined by

$$\prod_{l=1}^{p} \bigoplus_{i=1}^{j_l} (H_{i,l}, [\alpha_{i,l}]) \mapsto \bigoplus_{l=1}^{p} \bigoplus_{i=1}^{j_l} (H_{i,l}, [\alpha_{i,l}]),$$

which is clearly surjective. Note that it is not an isomorphism as it sends an ordered collection of sums to their (unordered) sum. □

The following proposition is the base case of an induction on k in the proof of Theorem 9.1:

Proposition 9.6 *There is a commutative diagram:*

$$\begin{tikzcd} E_n(BG) \arrow[r, "\mathbb{P}_p"] \arrow[d, "\chi"'] & E_n(BG \wr \Sigma_p) \arrow[d, "\chi"] \\ \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_p^{\phi}"] & \mathrm{Cl}_n(G \wr \Sigma_p, C_0) \end{tikzcd}$$

Proof Consider the following diagram:

$$\begin{tikzcd} E_n(BG) \arrow[r, "\mathbb{P}_p"] \arrow[d] & E_n(BG \wr \Sigma_p) \arrow[r] \arrow[d] & E_n(BG^p) \times (E_n(BG \wr \Sigma_p)/J) \arrow[d] \\ \mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}_p^{\phi}"] & \mathrm{Cl}_n(G \wr \Sigma_p, C_0) \arrow[r] & \mathrm{Cl}_n(G^p, C_0) \times (\mathrm{Cl}_n(G \wr \Sigma_p, C_0)/J) \end{tikzcd} \tag{11}$$

The right vertical arrow is just the product of character maps. First we show that the outer rectangle

$$\begin{tikzcd} E_n(BG) \arrow[r] \arrow[d] & E_n(BG^p) \times (E_n(BG \wr \Sigma_p)/J) \arrow[d] \\ \mathrm{Cl}_n(G, C_0) \arrow[r] & \mathrm{Cl}_n(G^p, C_0) \times (\mathrm{Cl}_n(G \wr \Sigma_p, C_0)/J) \end{tikzcd} \tag{12}$$

commutes. It commutes for the right factor by Proposition 9.4.

The commutativity of the left factor is proven as follows. It is a result of [12] that the image of $E_n(BG)$ is in the $\mathrm{Aut}(\mathbb{T}) \cong \mathrm{GL}_n(\mathbb{Z}_p)$ invariants of $\mathrm{Cl}_n(G, C_0)$, where the $\mathrm{Aut}(\mathbb{T})$–action is the action induced by Proposition 4.1. Recall that, for $f \in \mathrm{Cl}_n(G, C_0)$, the action takes the form

$$f^{\phi_e}([\alpha]) = \phi_e^* f([\alpha \psi_e^*]),$$

where $\psi_e = \phi_e$ since it factors the identity map:

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi_e"] \arrow[dr, "\mathrm{id}=q_e"'] & \mathbb{T} \\ & \mathbb{T}/e = \mathbb{T} \arrow[u, "\psi_e"'] \end{tikzcd}$$

Next note that the inclusion $G^p \hookrightarrow G \wr \Sigma_p$ induces the map on conjugacy classes sending

$$([\alpha_i])_{i=1\dots p} \mapsto \bigoplus_{i=1\dots p} (e, [\alpha_i]).$$

Thus the composite of the bottom arrows sends

$$f \mapsto \bigotimes_i f^{\phi_e}([\alpha_i]).$$

For $x \in E_n(BG)$, the element $\chi(x)$ is fixed by this action, so $\chi(x)$ maps to $\bigotimes_i \chi(x) = \chi\big(\bigotimes_i x\big)$. Now by Corollary 7.3 for $k = 1$, the square in (12) commutes.

Finally, the right square in (11) commutes by naturality of the character map. Since the bottom right arrow is an injection by Proposition 9.5, the left square must commute as well. □

Now we finish the induction.

Proposition 9.7 *The following diagram commutes:*

$$\begin{tikzcd}
E_n(BG) \arrow[r, "\mathbb{P}_{p^k}"] \arrow[d, "\chi"'] & E_n(BG \wr \Sigma_{p^k}) \arrow[d, "\chi"] \\
\mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}^{\phi}_{p^k}"] & \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0)
\end{tikzcd}$$

Proof Consider the following diagram:

$$\begin{tikzcd}
E_n(BG) \arrow[r, "\mathbb{P}_{p^k}"] \arrow[d] & E_n(BG \wr \Sigma_{p^k}) \arrow[r] \arrow[d] & \begin{array}{l} E_n(BG \wr \Sigma^p_{p^{k-1}}) \times \\ \quad (E_n(BG) \otimes_{E_n} E_n(\Sigma_{p^k})/I) \end{array} \arrow[d] \\
\mathrm{Cl}_n(G, C_0) \arrow[r, "\mathbb{P}^{\phi}_{p^k}"] & \mathrm{Cl}_n(G \wr \Sigma_{p^k}, C_0) \arrow[r] & \begin{array}{l} \mathrm{Cl}_n(G \wr \Sigma^p_{p^{k-1}}, C_0) \times \\ \quad (\mathrm{Cl}_n(G, C_0) \otimes_{C_0} \mathrm{Cl}_n(\Sigma_{p^k}, C_0)/I) \end{array}
\end{tikzcd}$$

The outer rectangle commutes because we understand the maps to each factor in the right lower corner. The map to the left factor is determined by induction and Corollary 7.3. The map to the right factor commutes by Proposition 9.4. The bottom right map is an injection by Proposition 9.5. Thus the left square must commute. □

Finally, we finish the proof of Theorem 9.1.

Proof of Theorem 9.1 Let $\sum_j a_j p^j$ be the p–adic expansion of m. Then the inclusion of groups

$$\prod_j \Sigma^{a_j}_{p^j} \to \Sigma_m$$

induces a commutative square

$$\begin{tikzcd} E_n(BG \wr \Sigma_m) \arrow[r] \arrow[d] & E_n\big(\prod_j BG \wr \Sigma_{p^j}^{a_j}\big) \arrow[d] \\ \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \arrow[r, hook] & \mathrm{Cl}_n\big(\prod_j G \wr \Sigma_{p^j}^{a_j}, C_0\big) \end{tikzcd}$$

in which the bottom arrow is an injection by Lemma 3.8. By Proposition 7.2(3), the composite

$$E_n(BG) \xrightarrow{\mathbb{P}_m} E_n(BG \wr \Sigma_m) \to E_n\left(\prod_j BG \wr \Sigma_{p^j}^{a_j}\right)$$

is the external tensor product $\bigotimes_j \mathbb{P}_{p^j}^{\otimes a_j}$. Consider the following diagram:

$$\begin{tikzcd} E_n(BG) \arrow[r] \arrow[d] & E_n(BG \wr \Sigma_m) \arrow[r] \arrow[d] & E_n\big(\prod_j BG \wr \Sigma_{p^j}^{a_j}\big) \arrow[d] \\ \mathrm{Cl}_n(G, C_0) \arrow[r] & \mathrm{Cl}_n(G \wr \Sigma_m, C_0) \arrow[r, hook] & \mathrm{Cl}_n\big(\prod_j G \wr \Sigma_{p^j}^{a_j}, C_0\big) \end{tikzcd}$$

By Proposition 9.7, we know that the outer rectangle commutes, and the right square commutes by naturality of the character map. This implies that the left square commutes. □

Example 9.8 In [3, Proposition 3.6.1], Ando constructs Adams operations for Morava E–theory. Reformulating his construction in terms of the power operation for E_n shows that the Adams operations are the composite

$$\psi^{p^k}\colon E_n(BG) \xrightarrow{P_{p^k n}/I} E_n(BG) \otimes_{E_n} E_n(B\Sigma_{p^{kn}})/I \to E_n(BG),$$

where the last map is induced by the map

$$E_n(B\Sigma_{p^{kn}})/I \to E_n,$$

picking out the subgroup $\mathbb{G}[p^k] \subset \mathbb{G}$. Example 6.7 computes the same composite on class functions. As a special case of Theorem 9.1, we have a commutative diagram:

$$\begin{tikzcd} E_n(BG) \arrow[r, "\psi^{p^k}"] \arrow[d, "\chi"'] & E_n(BG) \arrow[d, "\chi"] \\ \mathrm{Cl}_n(G, C_0) \arrow[r, "\psi^{p^k}"] & \mathrm{Cl}_n(G, C_0) \end{tikzcd}$$

When $\kappa \subset \mathbb{F}_{p^n}$ and $\mathbb{G}$ is the universal deformation of the Honda formal group, this gives a generalization of the well-known formula from representation theory stating that, for a representation ρ,

$$\chi(\psi^m(\rho))(g) = \chi(\rho)(g^m).$$

10 The rational total power operation

Recall that there is an action of $\mathrm{Aut}(\mathbb{T})$ on $\mathrm{Cl}_n(G, C_0)$. It is [12, Theorem C] that there is a canonical isomorphism

$$p^{-1}E_n(BG) \cong \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}.$$

In this section we prove that, for any section ϕ, the map $\mathbb{P}_m^\phi$ sends $\mathrm{Aut}(\mathbb{T})$–invariants to $\mathrm{Aut}(\mathbb{T})$–invariants and that the restriction of $\mathbb{P}_m^\phi$ to the $\mathrm{Aut}(\mathbb{T})$–invariants is independent of the choice of ϕ. The resulting "rational total power operation" is a global power functor.

Theorem 10.1 *For all finite groups G and any section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$, the function*

$$\mathbb{P}_m^\phi\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G \wr \Sigma_m, C_0)$$

sends $\mathrm{Aut}(\mathbb{T})$–invariants to $\mathrm{Aut}(\mathbb{T})$–invariants. By restricting $\mathbb{P}_m^\phi$ to the $\mathrm{Aut}(\mathbb{T})$–invariants, this gives rise to a commutative diagram:

$$\begin{tikzcd} E_n(BG) \arrow[r, "\mathbb{P}_m"] \arrow[d] & E_n(BG \wr \Sigma_m) \arrow[d] \\ p^{-1}E_n(BG) \arrow[r, "\mathbb{P}_m^\phi"'] & p^{-1}E_n(BG \wr \Sigma_m) \end{tikzcd}$$

Proof By [12, Theorem C] and Theorem 9.1, it is enough to show that the multiplicative natural transformation

$$\mathbb{P}_m^\phi\colon \mathrm{Cl}_n(G, C_0) \to \mathrm{Cl}_n(G \wr \Sigma_m, C_0)$$

restricts to the $\mathrm{Aut}(\mathbb{T})$–fixed points of both sides. To this end, recall from Corollary 3.12 that the action of $\sigma \in \mathrm{Aut}(\mathbb{T})$ on $\bigoplus_i (H_i, [\alpha_i]) \in \mathrm{Sum}_m(\mathbb{T}, G)$ is given by

$$\sigma\colon \bigoplus_i (H_i, [\alpha_i]) \mapsto \bigoplus_i (\sigma H_i, [\alpha_i \sigma^*_{|\mathbb{L}_{H_i}}]), \tag{13}$$

with notation as in the following commutative diagram:

$$\begin{tikzcd} \mathbb{L}_{\sigma H_i} \arrow[r] \arrow[d, "\sigma^*_{|\mathbb{L}_{\sigma H_i}}"'] & \mathbb{L} \arrow[d, "\sigma^*"] \\ \mathbb{L}_{H_i} \arrow[r] & \mathbb{L} \end{tikzcd}$$

Fix an automorphism $\sigma \in \mathrm{Aut}(\mathbb{T})$ and an invariant element $f \in \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$. Since ϕ_{H_i} and $\phi_{\sigma H_i}\sigma$ have the same kernel, there exists a unique isomorphism $\gamma_i \in \mathrm{Aut}(\mathbb{T})$ making the diagram

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\sigma"] \arrow[d, "\phi_{H_i}"'] & \mathbb{T} \arrow[d, "\phi_{\sigma H_i}"] \\ \mathbb{T} \arrow[r, "\cong", "\gamma_i"'] & \mathbb{T} \end{tikzcd}$$

commute. Upon dualizing and using the identities $\phi_{H_i} = \psi_{H_i} \circ q_{H_i}$ and $\phi_{\sigma H_i} = \psi_{\sigma H_i} \circ q_{\sigma H_i}$, we see that the inner squares and triangles in the next diagram commute:

$$\begin{tikzcd} \mathbb{L}_{H_i} \arrow[rr, leftarrow, "\sigma^*_{|\mathbb{L}_{\sigma H_i}}"] \arrow[dr, "q^*_{H_i}"] & & \mathbb{L}_{\sigma H_i} \arrow[dl, "q^*_{\sigma H_i}"'] \\ & \mathbb{L} & \mathbb{L} \arrow[l, "\sigma^*"'] \\ \mathbb{L} \arrow[uur, "\psi^*_{H_i}"] \arrow[u, "\phi^*_{H_i}\phi^*_{\sigma H_i}"'] & & \mathbb{L} \arrow[uul, "\psi^*_{\sigma H_i}"'] \arrow[u] \arrow[ll, "\gamma^*"] \end{tikzcd}$$

Therefore, the outer diagram commutes as well and hence gives

$$\sigma^*_{|\mathbb{L}_{\sigma H_i}} \psi^*_{\sigma H_i} = \psi^*_{H_i} \gamma^*. \tag{14}$$

Now we can check that $\mathbb{P}^{\phi}_m(f)$ is invariant under the action of $\mathrm{Aut}(\mathbb{T})$:

$$\begin{aligned} \mathbb{P}^{\phi}_m(f)^{\sigma}\Big(\bigoplus_i (H_i, [\alpha_i])\Big) &= \sigma^* \prod_i \mathbb{P}^{\phi}_m(f)((\sigma H_i, [\alpha_i \sigma^*_{|\mathbb{L}_{\sigma H_i}}])) \quad \text{by (13)} \\ &= \prod_i \sigma^* \phi^*_{\sigma H_i} f([\alpha_i \sigma^*_{|\mathbb{L}_{\sigma H_i}} \psi^*_{\sigma H_i}]) \\ &= \prod_i \phi^*_{H_i} \gamma_i^* f([\alpha_i \psi^*_{H_i} \gamma_i^*]) \quad \text{by (14)} \end{aligned}$$

$$= \prod_i \phi^*_{H_i} f([\alpha_i \psi^*_{H_i}]) \qquad \text{as } f \in \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$$

$$= \prod_i f^{\phi_{H_i}}([\alpha_i])$$

$$= \mathbb{P}^\phi_m(f)\Big(\bigoplus_i (H_i, [\alpha_i])\Big). \qquad \square$$

Corollary 10.2 *The restriction of $\mathbb{P}^\phi_m$ to the* $\mathrm{Aut}(\mathbb{T})$*–invariants*

$$\mathbb{P}^\phi_m\colon p^{-1}E_n(BG) \to p^{-1}E_n(BG \wr \Sigma_m)$$

is independent of the chosen section ϕ.

First proof As in the proof of Theorem 9.1, we can reduce to the case $m = p^k$. By naturality and Lemma 9.3, it furthermore suffices to prove this for abelian groups, since $p^{-1}E_n(BG)$ embeds into the product of the rationalized E–cohomology of the abelian subgroups of G. We have two maps

$$p^{-1}E_n(BA) \to p^{-1}E_n(BA \wr \Sigma_{p^k})/J;$$

the first is the rationalization $\mathbb{Q} \otimes \mathbb{P}_{p^k}/J$ of $\mathbb{P}_{p^k}/J$ and the second is $\mathbb{P}^\phi_{p^k}/J$ restricted to the $\mathrm{Aut}(\mathbb{T})$–fixed points. Because $E_n(BA)$ is a finitely generated free E_n–module, we may choose a basis of $E_n(BA)$, which thus gives a basis for $p^{-1}E_n(BA)$. By Theorem 10.1 both maps send the basis elements to the same elements of the codomain, thus the maps are the same.

We get the full result by induction. We use the embedding

$$p^{-1}E_n(BA \wr \Sigma_{p^k}) \hookrightarrow p^{-1}E_n(BA \wr \Sigma_{p^k})/J \times p^{-1}E_n(BA \wr \Sigma^p_{p^{k-1}})$$

and induct on k. The base case is clear and the induction follows by considering the large diagram and right diagram just as in Proposition 9.7. $\square$

Second proof We give a second proof of the corollary which is intrinsic to the construction of $\mathbb{P}^\phi_m$ and in particular does not rely on properties of Morava E–theory. To this end, consider two sections ϕ, $\phi' \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$ with associated isomorphisms $\psi_H, \psi'_H\colon \mathbb{T}/H \xrightarrow{\cong} \mathbb{T}$ for $H \subset \mathbb{T}$ as in Section 2. For a fixed $\bigoplus_i (H_i, [\alpha_i]) \in \mathrm{Sum}_m(\mathbb{T}, G)$, take $\gamma_{H_i} \in \mathrm{Aut}(\mathbb{T})$ to be the unique automorphism

making the following diagram commute:

$$\begin{tikzcd} \mathbb{T} \arrow[r, "="] \arrow[d, "\phi'_{H_i}"'] & \mathbb{T} \arrow[d, "\phi_{H_i}"] \\ \mathbb{T} \arrow[r, "\cong", "\gamma_{H_i}"'] & \mathbb{T} \end{tikzcd} \tag{15}$$

As in the proof of Theorem 10.1 with $\sigma = \mathrm{id}$, we see that, for any $f \in \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$,

$$\begin{aligned} \mathbb{P}_m^{\phi}(f)\Big(\bigoplus_i (H_i, [\alpha_i])\Big) &= \prod_i \phi_{H_i}^* f([\alpha_i \psi_{H_i}^*]) \\ &= \prod_i (\phi'_{H_i})^* \gamma_{H_i}^* f\big([\alpha_i (\psi'_{H_i})^* \gamma_{H_i}^*]\big) \quad \text{by (15)} \\ &= \prod_i (\phi'_{H_i})^* f\big([\alpha_i (\psi'_{H_i})^*]\big) \qquad \text{as } f \in \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})} \\ &= \mathbb{P}_m^{\phi'}(f)\Big(\bigoplus_i (H_i, [\alpha_i])\Big), \end{aligned}$$

hence $\mathbb{P}_m^{\phi} = \mathbb{P}_m^{\phi'}$ on $p^{-1}E_n(BG) \cong \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$. □

It follows that $\mathbb{Q} \otimes \mathbb{P}_{p^k}/J = \mathbb{P}_{p^k}^{\phi}/J$ after restricting $\mathbb{P}_m^{\phi}/J$ to $p^{-1}E_n(BG)$. Therefore, the following definition makes sense:

Definition 10.3 For any section ϕ, let

$$\mathbb{P}_m^{\mathbb{Q}}\colon p^{-1}E_n(BG) \to p^{-1}E_n(BG \wr \Sigma_m)$$

be the restriction of $\mathbb{P}_m^{\phi}$ to $p^{-1}E_n(BG)$. We will call this the rational total power operation.

Theorem 10.4 *The rational total power operation $\mathbb{P}_m^{\mathbb{Q}}$ is a global power functor.*

Proof We must show that the diagram

$$\begin{tikzcd} p^{-1}E_n(BG) \arrow[r, "\mathbb{P}_m^{\mathbb{Q}}"] \arrow[d, "\mathbb{P}_s^{\mathbb{Q}}"'] & p^{-1}E_n(BG \wr \Sigma_m) \arrow[d, "\nabla"] \\ p^{-1}E_n(BG \wr \Sigma_s) \arrow[r, "\mathbb{P}_t^{\mathbb{Q}}"'] & p^{-1}E_n(BG \wr \Sigma_s \wr \Sigma_t) \end{tikzcd}$$

commutes, where $st = m$ and ∇ is induced by the natural inclusion.

This is a computation that follows the lines of the proof of Proposition 6.9. In Proposition 6.9 we work with a power section and here we work with an $\mathrm{Aut}(\mathbb{T})$–invariant class functions. We will use the notation of Proposition 6.9.

Choose a section $\phi \in \Gamma(\mathrm{Sub}(\mathbb{T}), \mathrm{Isog}(\mathbb{T}))$. Since $\phi_{T_{i,j}}$ and $\phi_{T_{i,j}/H_i}\phi_{H_i}$ have the same kernel, there exists an automorphism $\sigma \in \mathrm{Aut}(\mathbb{T})$ such that

$$\phi_{T_{i,j}} = \sigma\phi_{T_{i,j}/H_i}\phi_{H_i}.$$

Diagram (5) implies that

$$\psi^*_{T_{i,j}} = \psi^*_{H_i}|_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}\psi^*_{T_{i,j}/H_i}\sigma^*.$$

Now assume that $f \in \mathrm{Cl}_n(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$. The power section structure of ϕ in the proof of Proposition 6.9 is used in going from line 7 to line 8 in the sequence of equalities. Up to line 7 we have the same sequence of equalities, which give

$$\begin{aligned}\mathbb{P}^\phi_t(\mathbb{P}^\phi_s(f))&\Big(\bigoplus_i\Big(H_i, \bigoplus_j(K_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big)\Big)\\ &= \prod_i\prod_j \phi^*_{H_i}\phi^*_{T_{i,j}/H_i} f\big([(\alpha_{H_i})_{K_{i,j}}\psi^*_{H_i}|_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}\psi^*_{T_{i,j}/H_i}]\big).\end{aligned}$$

Now we use the fact that f is $\mathrm{Aut}(\mathbb{T})$–invariant to get

$$\begin{aligned}&\prod_i\prod_j \phi^*_{H_i}\phi^*_{T_{i,j}/H_i} f\big([(\alpha_{H_i})_{K_{i,j}}\psi^*_{H_i}|_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}\psi^*_{T_{i,j}/H_i}]\big)\\ &\qquad= \prod_i\prod_j \phi^*_{H_i}\phi^*_{T_{i,j}/H_i}\sigma^* f\big([(\alpha_{H_i})_{K_{i,j}}\psi^*_{H_i}|_{\mathbb{L}_{\psi_{H_i}(K_{i,j})}}\psi^*_{T_{i,j}/H_i}\sigma^*]\big)\\ &\qquad= \prod_i\prod_j \phi^*_{T_{i,j}} f\big([(\alpha_{H_i})_{K_{i,j}}\psi^*_{T_{i,j}}]\big)\\ &\qquad= \mathbb{P}^\phi_m(f)\Big(\bigoplus_{i,j}(T_{i,j}, [(\alpha_{H_i})_{K_{i,j}}])\Big).\end{aligned}$$

□

In particular, this generalizes a result of Ganter [10, Proposition 4.12] to arbitrary heights and answers a question that she poses after the proof of this result.

Corollary 10.5 *The character map*

$$\chi\colon E_n(B-) \to p^{-1}E_n(B-) \cong \mathrm{Cl}_n(-, C_0)^{\mathrm{Aut}(\mathbb{T})}$$

is a map of global power functors.

Example 10.6 We give an example of how these theorems may be used. Proposition 8.3 describes Ando, Hopkins and Strickland's algebrogeometric interpretation of the additive power operation

$$P_{p^k}/I\colon E_n \to E_n(B\Sigma_{p^k})/I.$$

Let

$$L \subset E_n(B\Sigma_{p^k} \wr \Sigma_{p^h})$$

be the ideal generated by the image of the transfer along the maps

$$(\Sigma_i \times \Sigma_j) \wr \Sigma_{p^h} \to \Sigma_{p^k} \wr \Sigma_{p^h},$$

where $i,\ j > 0$ and $i + j = p^k$, as well as the maps

$$\Sigma_{p^k} \wr (\Sigma_i \times \Sigma_j) \to \Sigma_{p^k} \wr \Sigma_{p^h},$$

where $i, j > 0$ and $i + j = p^h$. It is a folklore result of Rezk (now proved by Nelson [14]) that

$$E_n(B\Sigma_{p^k} \wr \Sigma_{p^h})/L$$

is finitely generated free as an E_n–module and corepresents the scheme $\mathrm{Sub}_{p^k,p^h}(\mathbb{G})$ of flags of subgroup schemes $H_0 \subseteq H_1 \subset \mathbb{G}$, where $|H_0| = p^k$ and $|H_1/H_0| = p^h$.

By considering the power operation $\mathbb{P}_{p^h}$ applied to the transfer along $\Sigma_i \times \Sigma_j \subset \Sigma_{p^k}$ for $i,\ j > 0$ and $i + j = p^k$, there is a commutative ring map

$$E_n(B\Sigma_{p^k})/I \xrightarrow{P_{p^h}/L} E_n(B\Sigma_{p^k} \wr \Sigma_{p^h})/L.$$

It is also natural to consider the composite

$$E_n \xrightarrow{P_{p^k}/I} E_n(B\Sigma_{p^k})/I \xrightarrow{P_{p^h}/L} E_n(B\Sigma_{p^k} \wr \Sigma_{p^h})/L.$$

Both P_{p^h}/L and $P_{p^h}/L \circ P_{p^k}/I$ can be understood algebrogeometrically by using Theorem 9.1.

There is a natural map of formal schemes from flags of subgroups to subgroups

$$Z\colon \mathrm{Sub}_{p^k,p^h}(\mathbb{G}) \to \mathrm{Sub}_{p^h}(\mathbb{G}), \quad (H_0 \subseteq H_1 \subset \mathbb{G}) \mapsto (H_1/H_0 \subset \mathbb{G}/H_0),$$

and a map

$$z\colon \mathrm{Sub}_{p^k,p^h}(\mathbb{G}) \to \mathrm{Sub}_{p^h}(\mathbb{G}) \to \mathrm{Spf}(E_n), \quad (H_0 \subseteq H_1 \subset \mathbb{G}) \mapsto \mathbb{G}/H_1.$$

Theorem 9.1 gives a way to see that the algebrogeometric maps and the power operations agree. Using Theorem 10.4, it is easy to check that both maps make the following

diagram commute:

$$\begin{tikzcd} E_n \arrow[r, shift left, "P_{p^h}/L \circ P_{p^k}/I"] \arrow[r, shift right, "z^*"'] \arrow[d] & E_n(B\Sigma_{p^k} \wr \Sigma_{p^h})/L \arrow[d] \\ C_0 \arrow[r] & \mathrm{Cl}_n(B\Sigma_{p^k} \wr \Sigma_{p^h}, C_0)/L \end{tikzcd}$$

Since the vertical maps are injective, this implies that $P_{p^h}/L \circ P_{p^k}/I$ and z^* are the same map.

References

[1] **J F Adams**, *Maps between classifying spaces, II*, Invent. Math. 49 (1978) 1–65 MR

[2] **J F Adams**, **M F Atiyah**, *K–theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. 17 (1966) 31–38 MR

[3] **M Ando**, *Isogenies of formal group laws and power operations in the cohomology theories E_n*, Duke Math. J. 79 (1995) 423–485 MR

[4] **M Ando**, **M J Hopkins**, **N P Strickland**, *The sigma orientation is an H_∞ map*, Amer. J. Math. 126 (2004) 247–334 MR

[5] **T Barthel**, **N Stapleton**, *Centralizers in good groups are good*, Algebr. Geom. Topol. 16 (2016) 1453–1472 MR

[6] **R R Bruner**, **J P May**, **J E McClure**, **M Steinberger**, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics 1176, Springer, Berlin (1986) MR

[7] **H Carayol**, *Nonabelian Lubin–Tate theory*, from "Automorphic forms, Shimura varieties, and L–functions, II" (L Clozel, J S Milne, editors), Perspect. Math. 11, Academic Press, Boston (1990) 15–39 MR

[8] **E S Devinatz**, **M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47 MR

[9] **V G Drinfel'd**, *Elliptic modules*, Mat. Sb. 94(136) (1974) 594–627 MR In Russian; translated in Math. USSR-Sb. 23 (1974) 561–592

[10] **N Ganter**, *Global Mackey functors with operations and n–special lambda rings*, preprint (2013) arXiv

[11] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from "Structured ring spectra" (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR

[12] **M J Hopkins**, **N J Kuhn**, **D C Ravenel**, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. 13 (2000) 553–594 MR

[13] **J Lubin**, **J Tate**, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966) 49–59 MR

[14] **P Nelson**, *On the Morava E–theory of wreath products of symmetric groups*, in preparation

[15] **C Rezk**, *Notes on the Hopkins–Miller theorem*, from "Homotopy theory via algebraic geometry and group representations" (M Mahowald, S Priddy, editors), Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 313–366 MR

[16] **C Rezk**, *The units of a ring spectrum and a logarithmic cohomology operation*, J. Amer. Math. Soc. 19 (2006) 969–1014 MR

[17] **C Rezk**, *Power operations for Morava E-theory of height* 2 *at the prime* 2, preprint (2008) arXiv

[18] **C Rezk**, *Power operations in Morava E–theory: structure and calculations*, preprint (2013) Available at `http://www.math.uiuc.edu/~rezk/power-ops-ht-2.pdf`

[19] **J Rognes**, *Galois extensions of structured ring spectra: stably dualizable groups*, Mem. Amer. Math. Soc. 898, Amer. Math. Soc., Providence, RI (2008) MR

[20] **T M Schlank**, **N Stapleton**, *A transchromatic proof of Strickland's theorem*, Adv. Math. 285 (2015) 1415–1447 MR

[21] **B Schuster**, *Morava K–theory of groups of order* 32, Algebr. Geom. Topol. 11 (2011) 503–521 MR

[22] **N Stapleton**, *An introduction to HKR character theory*, preprint (2013) arXiv

[23] **N Stapleton**, *Transchromatic generalized character maps*, Algebr. Geom. Topol. 13 (2013) 171–203 MR

[24] **N P Strickland**, *Finite subgroups of formal groups*, J. Pure Appl. Algebra 121 (1997) 161–208 MR

[25] **N P Strickland**, *Morava E–theory of symmetric groups*, Topology 37 (1998) 757–779 MR

[26] **A V Zelevinsky**, *Representations of finite classical groups: a Hopf algebra approach*, Lecture Notes in Mathematics 869, Springer, Berlin (1981) MR

[27] **Y Zhu**, *The power operation structure on Morava E–theory of height* 2 *at the prime* 3, Algebr. Geom. Topol. 14 (2014) 953–977 MR

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen, Denmark
Fakultät für Mathematik, Universität Regensburg
D-93040 Regensburg, Germany

`tbarthel@math.ku.dk, nat.j.stapleton@gmail.com`

`http://people.mpim-bonn.mpg.de/tbarthel/,`
`http://guests.mpim-bonn.mpg.de/nstapleton/`

Proposed: Haynes Miller
Seconded: Mark Behrens, Jesper Grodal

Received: 26 February 2015
Revised: 26 September 2016