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A very special EPW sextic and two IHS fourfolds

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We show that the Hilbert scheme of two points on the Vinberg K3 surface has a two-to-one map onto a very symmetric EPW sextic Y in $\mathbb{P}^5$. The fourfold Y is singular along 60 planes, 20 of which form a complete family of incident planes. This solves a problem of Morin and O'Grady and establishes that 20 is the maximal cardinality of such a family of planes. Next, we show that this Hilbert scheme is birationally isomorphic to the Kummer-type IHS fourfold X_0 constructed by Donten-Bury and Wiśniewski [*On 81 symplectic resolutions of a 4–dimensional quotient by a group of order 32*, preprint (2014)]. We find that X_0 is also related to the Debarre–Varley abelian fourfold.

14D06, 14J35, 14J70, 14K12, 14M07; 14J50, 14J28

1 Introduction

By an irreducible holomorphic symplectic (IHS) fourfold we mean a compact, 4–dimensional, simply connected Kähler manifold with trivial canonical bundle which admits a unique (up to a constant) closed nondegenerate holomorphic 2–form and is not a product of two manifolds; see Beauville [2]. There are only two known families of such fourfolds: the 21–dimensional family of deformations of the Hilbert scheme of two points on a K3 surface (we say that elements of this family are *of* $\mathrm{K3}^{[2]}$*–type*) having $b_2 = 23$ and the 5–dimensional family of deformations of the Hilbert scheme of three points summing to zero on an abelian surface with $b_2 = 7$.

A well-known family of projective IHS fourfolds of $\mathrm{K3}^{[2]}$–type is the family of double EPW sextics found by O'Grady in [22]. Recall that an EPW sextic is a special sextic hypersurface in $\mathbb{P}^5$ constructed by Eisenbud, Popescu and Walter in [12]. It arises from a choice of a subspace $A \subset \bigwedge^3\mathbb{C}^6$, Lagrangian with respect to the symplectic

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form η (unique up to choice of volume form on $\bigwedge^6\mathbb{C}^6$) on $\bigwedge^3\mathbb{C}^6$ given by the wedge product. More precisely the EPW sextic associated to A is

$$Y_A = \{[v] \in \mathbb{P}(\mathbb{C}^6) \mid \dim(A \cap (v \wedge \textstyle\bigwedge^2\mathbb{C}^6)) \geq 1\}.$$

Such EPW sextics appear as quotients of polarized IHS fourfolds of $\mathrm{K3}^{[2]}$–type by an antisymplectic involution.

In this paper, we investigate birational models of a very special IHS fourfold of K3–type. The fourfold is obtained as a Hilbert scheme of two points on a special K3 surface studied by Vinberg. We find out, in particular, that on one hand the fourfold is birational to a double EPW sextic (see Proposition 1.1) and on the other it is birational to the Kummer-type IHS fourfold constructed by Donten-Bury and Wiśniewski in [11] as a desingularization of a quotient of an abelian fourfold by a group action (see Theorem 1.3).

To introduce our fourfold, let us denote by S the K3 surface which is the desingularization of the double covering of the Del Pezzo surface of degree 5, denoted by S_5, branched over the union of its ten lines. This surface was studied by Vinberg in [27] as one of two "most algebraic K3 surfaces". The starting point of our investigation is the following proposition.

Proposition 1.1 *There exists a series of flops from $S^{[2]}$ to a fourfold $\overline{S^{[2]}}$ and a generically two-to-one morphism $\overline{S^{[2]}} \to Y$ to an EPW sextic $Y \subset \mathbb{P}^5$ which is singular along 60 planes.*

This EPW sextic Y is a very symmetric and natural sextic related to many classical objects in algebraic geometry. First, it is invariant with respect to the action of the symmetric group Σ_6 acting on $\mathbb{P}^5$ by permutation of the coordinates. We prove in Section 5 that there are 16 hyperplanes in $\mathbb{P}^5$ which are tangent to Y along Segre cubics (see Dolgachev [9] for the discussion about such cubics) such that the 15 planes contained in each such cubic are singular planes on Y. Moreover, Y admits 16 singular points of multiplicity 4 whose tangent cone is the cone over the Igusa quartic. Furthermore, $Y \subset \mathbb{P}^5$ is projectively self-dual. Finally, the EPW sextic Y gives rise to a maximal configuration of 20 incident planes in $\mathbb{P}^5$, thus solving a classical problem proposed by Morin and reformulated by O'Grady.

The problem of O'Grady addresses the question of finding *complete families of incident planes* in $\mathbb{P}^5$, ie configuration of planes in $\mathbb{P}^5$ intersecting each other and such that no planes outside of this set intersect all of the planes in the set. In 1930 Ugo Morin classified in [20] all complete irreducible families of incident planes in $\mathbb{P}^5$. In the same paper he acknowledged that the classification of complete finite sets of incident planes presents essential difficulties. The latter problem was readdressed by Dolgachev and Markushevich who announced in [10] having found, using the geometry of the Fano model of an Enriques surface, a description of some families of ten incident planes. Moreover, they found an explicit description of a complete family of 13 incident planes. Then O'Grady in [22] proved, using the results of Ferretti, that for $10 \leq k \leq 16$ there exists a $(20-k)$–dimensional moduli space of complete families of incident planes of cardinality k. Moreover, he proved that the maximal cardinality of a finite complete family of planes is between 10 and 20. Then O'Grady asked: what is the maximal cardinality of a finite family of incident planes in $\mathbb{P}^5$?

Finding families of incident planes is in fact strictly related to EPW sextics. Indeed from O'Grady [24, Claim 3.2], the set of points in $G(3,\mathbb{C}^6)$ corresponding to any complete family $\mathcal{F}$ of incident planes spans a space $\mathbb{P}(A_{\mathcal{F}}) \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)$ with $A_{\mathcal{F}}$ Lagrangian with respect to η. In particular, a finite complete family $\mathcal{F}$ of incident planes gives rise to an EPW sextic $Y_{A_{\mathcal{F}}}$. The configuration of planes is then contained in the singular locus of the EPW sextic. It follows that the construction of a finite complete family of incident planes amounts to finding a special EPW sextic. In our case, it turns out that a suitable subset of 20 of the 60 singular planes of the EPW sextic Y is a complete family of incident planes of cardinality 20. Note that the configuration of these 20 planes is probably rigid. We hence infer the following answer to O'Grady's problem:

Theorem 1.2 *There exists a complete family of* 20 *incident planes in* $\mathbb{P}^5$*. So* 20 *is the maximal possible cardinality of a finite complete family of incident planes in* $\mathbb{P}^5$*.*

The outline of the first part of the paper is as follows. In Section 2.1 we construct a rational map $S^{[2]} \to \mathbb{P}^5$ and find its image. The flops of Proposition 1.1 are described in Section 3; see Proposition 3.3. Then, the detailed proof of Theorem 1.2 is given in Section 2.3. In Section 5 further geometric properties of the EPW sextic are studied.

The second aim of the paper is to establish a relation between $S^{[2]}$ and the Kummer-type IHS fourfold constructed by Donten-Bury and Wiśniewski in [11] as a desingularization of an abelian fourfold by a group action. We shall see that these manifolds are in fact related through the sextic Y.

Let us recall the construction from [11]. Bellamy and Schedler in [3] showed that a certain action of the group $G := Q_8 \times_{\mathbb{Z}_2} D_8$, of order 32, on $\mathbb{C}^4$ admits a symplectic resolution. This action can be given by matrices with coefficients in $\mathbb{Z}[i]$. Thus we get an action of G on E^4, where E is the elliptic curve $E := \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ with complex multiplication by $\mathbb{Z}[i]$. Donten-Bury and Wiśniewski [11, Section 6] showed that there is an inclusion $G \subset \mathrm{Aut}(E^4)$ such that the quotient E^4/G admits a symplectic resolution X_0 (not uniquely defined) which is an IHS fourfold with second Betti number given by $b_2(X_0) = 23$. One of the goals of our paper is to show that X_0 is of $\mathrm{K3}^{[2]}$–type. We shall see that it is in fact birational to the Hilbert scheme of two points on a K3 surface.

To relate E^4/G to $S^{[2]}$ and Y, we find a morphism $E^4 \to \mathbb{P}^5$ which factors over the group $(G, i) \subset \mathrm{Aut}(E^4)$ generated by G and diagonal scalar multiplication by i. This morphism is proven to be the quotient map $E^4 \to E^4/(G, i)$ and $E^4/(G, i)$ is shown to be a sextic hypersurface Y' in $\mathbb{P}^5$. After finding special geometrical properties of Y' (singular planes, tangent hyperplanes, etc), we are finally able in Section 6 to show that actually $Y = Y'$. We then conclude that both $\overline{S^{[2]}}$ and E^4/G admit a two-to-one morphism to the same sextic Y and have the same ramification locus. We also obtain:

Theorem 1.3 *There exists a birational contraction $\overline{S^{[2]}} \to E^4/G$. In particular X_0 is deformation equivalent to the Hilbert scheme of two points on a* K3 *surface.*

We describe explicitly, in Remark 6.17, the exceptional locus of the contraction $\overline{S^{[2]}} \to E^4/G$ over the quotient singularity of type $\mathbb{C}^4/G$ which was studied in [3]; see also [11].

To obtain the map from E^4 to $\mathbb{P}^5$, we show first in Section 4 that there is a unique G–invariant principal polarization H on E^4. This is not the product polarization, but (E^4, H) is a principally polarized abelian fourfold with the maximal number of ten vanishing theta nulls found in Varley [26] and Debarre [6]. The singular locus of any theta divisor of (E^4, H) consists of exactly ten ODPs. We study in Section 4.3 the automorphism group of (E^4, H) and we deduce that the configuration of the 16 G–fixed points in E^4 and the singular points on the 16 G–invariant theta divisors is a $(16, 6)$ (actually a complementary $(16, 10)$) configuration. The map $E^4 \to Y$ is given by a subsystem of $|2D|$ where D is a G–invariant theta divisor (see Remark 5.6).

It is well known that the K3 surface S is a desingularization of a quotient of E^2 by a diagonal action of $\mathbb{Z}_4$ generated by $(i, -i)$, so it is maybe not so surprising that E^4/G and $S^{[2]}$ are closely related (see also van Geemen and Top [13] for another description of S). In fact, there is a lattice theoretical argument proving this (see Proposition 6.14). However, the groups involved in the quotient maps $(E^2)^2 \to S^{[2]}$ and in $E^4 \to E^4/G$ are different and we do not know of a more direct method to obtain Theorem 1.3. In any case, the sextic Y is clearly of independent interest.

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Σ_n	n^{th} permutation group
$S_5 \subset \mathbb{P}^5$	degree-5 Del Pezzo surface via anticanonical embedding
$\mathbb{P}^2_r$	blow-up of $\mathbb{P}^2$ at r general points; for example, $S_5 \cong \mathbb{P}^2_4$
$\alpha\colon \Sigma_5 \to \mathrm{Aut}(S_5)$	action of Σ_5 on S_5; second wedge of standard representation
$S' = \Pi \cap Q \subset \mathbb{P}^6$	intersection of the cone Π over S_5 with vertex P and a special quadric Q
$\rho\colon \mathbb{P}^6 \supset S' \to S_5 \subset \mathbb{P}^5$	projection from the vertex of Π
$C = Q \cap S_5 \subset S_5$	ramification of ρ; union of ten (-1)–curves on S_5
$\phi\colon S \to S'$	resolution of 15 A_1 singularities, where S is a Vinberg K3 surface
$\nu\colon S \to S_5$	composition $\rho \circ \phi$
$e_1, \dots, e_{15}$	(-2)–curves on S which are contracted by ϕ
$l_1, \dots, l_{10}$	(-2)–curves on S in strict transform of branching of ρ
$S^{[2]} = \mathrm{Hilb}^2(S)$	Hilbert scheme of subschemes of S of length 2
$\mu\colon \mathrm{DivS} \to \mathrm{DivS}^{[2]}$	$\mu(C)$ consists of cycles having nonempty intersection with $C \subset S$
$\phi^{[2]}\colon S^{[2]} \dashrightarrow S'^{[2]}$	push-forward map
$\rho^{[2]}\colon S'^{[2]} \dashrightarrow Y \subset \mathbb{P}^5$	maps a pair of points on S' to the intersection of their span with $\mathbb{P}^5$
$g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$	composition $\rho^{[2]} \circ \phi^{[2]}$
$Y \subset \mathbb{P}^5$	image of g; hypersurface of degree 6; EPW sextic
$\beta\colon \Sigma_5 \to \mathrm{Aut}(\mathbb{P}^5)$	action compatible with α and g
$\overline{g}\colon \overline{S^{[2]}} \to Y \subset \mathbb{P}^5$	small modification of g which is a regular morphism
$F_{ij} \subset S^{[2]}$	$F_{ij} = \{\{p,q\} \in S^{[2]} : p \in e_i,\ q \in e_j\} \cong \mathbb{P}^1 \times \mathbb{P}^1$
$E_{ij} \subset S^{[2]}$	$E_{ij} = \{\{p,q\} \in S^{[2]} : p \in e_i,\ q \in l_j\} \cong \mathbb{P}^2_2$
$L_{ij} \subset \overline{S^{[2]}}$	strict transform of $L'_{ij} = \{\{p,q\} \in S^{[2]} : p \in l_i, q \in l_j\} \subset S^{[2]}$
$\overline{g}\colon \overline{S^{[2]}} \xrightarrow{c} Z \xrightarrow{f} Y$	Stein factorization

Table 1: Notation

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2 The Hilbert scheme of two points on the Vinberg K3 surface

In this section we study the birational geometry of $S^{[2]} := \mathrm{Hilb}^2(S)$, where S is the (unique) K3 surface with transcendental lattice T_S isomorphic to the rank two lattice

$\bigl(\mathbb{Z}^2, q = 2(x^2+y^2)\bigr)$, studied by Vinberg in [27]; see also [13]. Moreover, we exhibit an explicit (rational) two-to-one map $g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$ whose image is an EPW sextic Y and we find the equation defining Y.

2.1 The K3 surface S

The surface S is the desingularization of the double cover of $\mathbb{P}^2$ branched along the union of six lines. An equation for the branch curve is

$$xyz(x-y)(x-z)(y-z) = 0. \tag{2-1}$$

The six lines defined by (2-1) meet three at a time in the points

$$p_1 := (1:0:0), \quad p_2 := (0:1:0), \quad p_3 := (0:0:1), \quad p_4 := (1:1:1),$$

and two at a time in the points

$$(0:1:1), \quad (1:0:1), \quad (1:1:0).$$

Blowing up the four triple points $p_1, \dots, p_4$ in $\mathbb{P}^2$ we obtain a Del Pezzo surface S_5. The strict transform of the six lines, together with the four exceptional divisors of the birational map $S_5 \to \mathbb{P}^2$, are the ten (-1)–curves on S_5. The reduced divisor B in the Del Pezzo surface S_5, whose irreducible components are the ten (-1)–curves has only ordinary double points, has arithmetic genus 6 and lies in $|-2K_{S_5}|$. Hence, B is even in the Picard group of S_5, so it is the branch locus of a degree-two map $\rho\colon S' \to S_5$.

The linear system $|-2K_{S_5}|$ embeds S_5 as an intersection of five quadrics $q_i' = 0$, for $i = 1, \dots, 5$, in $\mathbb{P}^5$. There is a further quadric $q_0' = 0$ in $\mathbb{P}^5$ which cuts out the divisor B on S_5. The K3 surface S thus has a (nodal) model S' as the intersection of the cone Π over S_5 with vertex $P = (0:\cdots:0:1) \in \mathbb{P}^6$ (so Π is defined by the five quadrics $q_i' = 0$ in $\mathbb{P}^6$) and the quadric Q defined by $y_6^2 = q_0'$ (see the proof of Proposition 2.1 below); that is, we have a contraction map

$$\phi\colon S \to S' := \Pi \cap Q \quad (\subset \mathbb{P}^6).$$

The degree-two map $\rho\colon S' \to S_5$ is simply the projection from P to the hyperplane $y_6 = 0$.

The divisor B in S_5 has $4 \cdot 3 + 3 = 15$ ordinary double points, and after blowing up these points, we get a surface S_5' on which the strict transforms of the six lines and four exceptional curves of the first blow-up are disjoint. The K3 surface S is the double cover of S_5' branched over these ten curves; the inverse images of the 15 rational curves over double points of B are rational curves in S. The map $\phi\colon S \to S'$ is the contraction of these 15 rational curves. The intersection diagram of the 25 curves

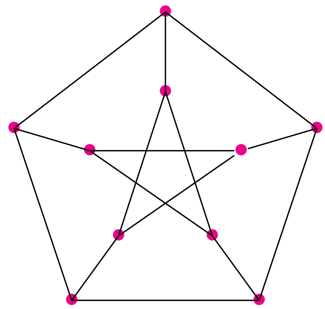

Figure 1

on S above is the Petersen graph in Figure 1. In this diagram, the dots • correspond to the ten curves from $\mathcal{T}$ and the edges to the 15 exceptional curves of $\phi\colon S \to S'$.

Let C be the pull-back of B to S along the composition $S \to S' \to S_5$. Then $C^2 = 10$ and the linear system given by this curve is the contraction map $\phi = \phi_C\colon S \to S' \subset \mathbb{P}^6$.

2.2 The map $g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$

Let $S^{[2]}$ be the Hilbert scheme of two points on S. We define a rational map g (see [21, Section 4.3]) that on an open part of $S^{[2]}$ is a composition of rational maps:

$$g\colon S^{[2]} \overset{\phi^{[2]}}{\dashrightarrow} (S')^{[2]} \overset{\rho^{[2]}}{\dashrightarrow} Y \subset \mathbb{P}^5.$$

Here the map $\phi^{[2]}$ is the map naturally induced by ϕ on the Hilbert scheme. By $(S')^{[2]}$ we denote the Hilbert scheme of two points in the smooth part of S'. The rational map $\rho^{[2]}$ is induced by mapping $\{p, q\} \in (S')^2$ to the hyperplane of those quadrics in the ideal of $S' \subset \mathbb{P}^5$ which contain the line spanned by p and q. The base locus of the map g will be studied in Section 3. We shall see in Proposition 3.3 that the map g is given by a complete linear system.

To describe the image Y of g, we need the following symmetric functions in six variables $Z_0, \dots, Z_5$:

$$\begin{aligned}
P_6 &:= Z_0^6 + Z_1^6 + Z_2^6 + Z_3^6 + Z_4^6 + Z_5^6,\\
P_{42} &:= Z_0^4 Z_1^2 + Z_0^4 Z_2^2 + \cdots + Z_4^2 Z_5^4,\\
P_{222} &:= Z_0^2 Z_1^2 Z_2^2 + Z_0^2 Z_1^2 Z_3^2 + \cdots + Z_3^2 Z_4^2 Z_5^2,\\
P_{111111} &:= Z_0 Z_1 Z_2 Z_3 Z_4 Z_5.
\end{aligned}$$

The polynomials P_{42} and P_{222} have 30 and 20 terms, respectively.

Proposition 2.1 *The image Y of $S^{[2]}$ under $g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$ is the sextic $F_6 = 0$ where, with the notation above,*

$$F_6 = P_6 - P_{42} + 2P_{222} - 16P_{111111}.$$

Proof The anticanonical map from S_5 to $\mathbb{P}^5$ is given by the linear system of cubics in $\mathbb{P}^2$ which pass through the points $p_1, \dots, p_4$. A basis for these cubics is given by

$$\begin{aligned} &y_0 = x_0^2x_1 - x_0x_1x_2, \quad y_1 = x_0^2x_2 - x_0x_1x_2, \quad y_2 = x_0x_1^2 - x_0x_1x_2, \\ &y_3 = x_0x_2^2 - x_0x_1x_2, \quad y_4 = x_1^2x_2 - x_0x_1x_2, \quad y_5 = x_1x_2^2 - x_0x_1x_2. \end{aligned}$$

The image of S_5 in $\mathbb{P}^5$ is defined by the following five quadratic forms:

$$\begin{aligned} q_1' &= y_0y_3 + y_1y_2 - y_2y_5 - y_3y_4, \\ q_2' &= y_0y_4 + y_1y_2 - y_1y_5 - y_3y_4, \\ q_3' &= y_0y_5 + y_1y_2 - y_1y_5 - y_2y_5 - y_3y_4, \\ q_4' &= y_1y_4 - y_1y_5 - y_3y_4, \\ q_5' &= y_2y_3 - y_2y_5 - y_3y_4. \end{aligned}$$

Moreover, the quadratic form

$$q_0' := y_1y_2 - y_3y_4$$

cuts out the union of the ten lines on S_5. Thus the image S' of the K3 surface S in $\mathbb{P}^6$ under the map ϕ is defined by the five quadrics which cut out the image of S_5 and the quadric with equation $y_6^2 = q_0'$, where now $y_0, \dots, y_5, y_6$ are the homogeneous coordinates on $\mathbb{P}^6$. We define $q_6' := q_0' - y_6^2$. In order to get the very symmetric polynomial F_6 we need the following change of basis on the space of these quadrics:

$$\begin{pmatrix} q_1 \\ q_2 \\ \\ \vdots \\ \\ q_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & -2 & -1 \\ -1 & 1 & 0 & 0 & 2 & -1 \\ 1 & -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & -1 \\ 1 & 1 & -2 & 0 & 0 & -1 \\ 1 & 1 & 0 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \\ \\ \vdots \\ \\ q_6' \end{pmatrix}.$$

The map $g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$ maps $\{p,q\} \in S^{[2]}$ to the hyperplane of quadrics in the ideal of $\phi(S)$ which contains the line spanned by $\phi(p)$ and $\phi(q)$. If $\phi(p) = (y_0 : \cdots : y_6)$ and $\phi(q) = (z_0 : \cdots : z_6)$ then we can write

$$q_i(\lambda y + \mu z) = \lambda^2 q_i(y) + \mu^2 q_i(z) + 2\lambda\mu Q_i(y,z),$$

where Q_i is the symmetric bilinear form given by the polarization of q_i. The map g is thus induced by the rational map

$$\widetilde{g}\colon S\times S\to\mathbb{P}^5 \quad \text{given by } (p,q)\mapsto(\cdots:Q_i(\phi(p),\phi(q)):\cdots)_{1\le i\le 6}.$$

It is now easy to verify that the polynomial F_6 vanishes on the image of g: one only needs to check that $F_6(\dots,Q_i(y,z),\dots)=0$ on $S\times S$ where $y,z\in\phi(S)$, so one substitutes the cubic polynomials in x_0,x_1,x_2 for $y_0,\dots,y_5$, and similarly for $z_0,\dots,z_5$, but now with polynomials in u_0,u_1,u_2 (coordinates for another copy of $\mathbb{P}^2$) and one uses that $y_6^2=y_1y_2-y_3y_4$ and $z_6^2=z_1z_2-z_3z_4$.

For later use we notice that a general line contained in the cone Π over S_5 (defined by $q_1'=\cdots=q_5'=0$) and passing through its vertex P cuts S' in two points $\phi(p)$ and $\phi(q)$. Using the change of basis, we find that (p,q) maps to the point $(-1:\cdots:-1)=(1:\cdots:1)$ in Y. □

Remark 2.2 Note that the equation of the image of g can be found in a theoretical way using the results from Section 3 and from Section 6.

2.3 The sextic Y is a special EPW sextic

We will show that the degree-six fourfold $Y\subset\mathbb{P}^5$, which is the image of $S^{[2]}$, is an EPW sextic; see [23]. The singular locus of a general EPW sextic is a surface of degree 40. The sextic Y is (very) special in the sense that its singular locus has degree 60; in fact it is the union of 60 planes. The double cover of an EPW sextic along the singular locus is an IHS fourfold. Forty of the 60 singular planes in Y are in the branch locus of the map $g\colon S^{[2]}\to Y$. The other "extra" 20 planes must then be a set of incident planes. To identify the planes in the branch locus, we use that the symmetric group Σ_5 acts on the Del Pezzo surface S_5 and that this action lifts to the action of a group $\widetilde{\Sigma}_5$ on the K3 surface S. This group then also acts on $S^{[2]}$ and we show that g is an equivariant map, where $\widetilde{\Sigma}_5$ acts through a subgroup $\beta\Sigma_5$, isomorphic to Σ_5, of Σ_6 on Y. Knowing the 20 incident planes then allows us to find explicitly a Lagrangian subspace $A\subset\bigwedge^3\mathbb{C}^6$ such that $Y=Y_A$ in the EPW construction.

The first part of the following lemma is well known.

Lemma 2.3 *The symmetric group Σ_5 acts as group of automorphisms on S_5, the permutations of the points $p_1,\dots,p_4$ induce the elements in $\Sigma_4\subset\Sigma_5$ and the Cremona transformation on p_1, p_2, p_3 induces the transposition (45).*

These automorphisms of $S_5\subset\mathbb{P}^5$, where the embedding is given by the cubics from the proof of Proposition 2.1, are induced by projective transformations which map

$y := (y_0 : y_1 : \cdots : y_5)$ *to*

$$\alpha_{34}(y) := (y_0 - y_1 + y_4 : -y_1 + y_3 - y_5 : y_2 - y_4 + y_5 : -y_5 : -y_3 - y_4 + y_5 : -y_3),$$

and the maps α_{12}, α_{23} *and* α_{45} *permute the coordinates* y_i *as*

$$\alpha_{12}\colon (02)(14)(35), \quad \alpha_{23}\colon (01)(23)(45), \quad \alpha_{45}\colon (05)(14)(23).$$

The map $g\colon S^{[2]} \to \mathbb{P}^5$ *is equivariant for the action of* Σ_5, *where the action of* $\sigma \in \Sigma_5$ *on* $\mathbb{P}^5$ *is given by the permutation* β_σ *of the projective coordinates* $Z_0, \dots, Z_5$*:*

$$\beta_{12}\colon (03)(14)(25), \quad \beta_{23}\colon (01)(24)(35), \quad \beta_{34}\colon (05)(14)(23), \quad \beta_{45}\colon (01)(25)(34).$$

Proof These are straightforward verifications. The permutation of the points p_3 and p_4 is given by $(x : y : z) \mapsto (x - z : y - z : -z)$ and now one computes the action on the six cubics from the proof of Proposition 2.1. The Cremona transformation is induced by $(x : y : z) \mapsto (x^{-1} : y^{-1} : z^{-1})$. This is substituted in the cubics and next one multiplies them by $(xyz)^2$.

Interchanging for example x and y, the equation of the branch curve (2-1) changes sign, and the same happens for any other transposition in Σ_5. Thus to lift the action to $S' \subset \mathbb{P}^6$ one must map $y_6 \mapsto i y_6$ (with $i^2 = -1$). Finally, one considers the induced action on the quadratic forms $q_1, \dots, q_6$ in the variables $y_0, \dots, y_6$. These q_i are the coordinate functions Z_{i-1}. □

To describe the singular locus of Y and the action of Σ_6 on the irreducible components, we introduce the following notation. Let $\{\{i,j\},\{k,l\},\{m,n\}\}$ be a partition of $\{0,\dots,5\}$, so $\{i,j,k,l,m,n\} = \{0,\dots,5\}$. Notice that there are 15 such partitions. Then, with three choices of sign, we define planes in $\mathbb{P}^5$ by

$$V_{i\pm j,k\pm l,m\pm n}: \quad Z_i \pm Z_j = Z_k \pm Z_l = Z_m \pm Z_n = 0 \qquad (\subset \mathbb{P}^5).$$

Notice that besides being symmetric in the variables $Z_0, \dots, Z_5$, the polynomial F_6 which defines Y is also invariant under the change of sign of an even number of variables.

Proposition 2.4 *The singular locus of* Y *is the union of* 60 *planes. There are two* Σ_6*–orbits, of lengths* 15 *and* 45*, respectively, of these planes, and they are the orbits of*

$$V_{0-1,2-3,4-5} \quad \text{and} \quad V_{0+1,2+3,4-5},$$

respectively. Let $\beta\Sigma_5 \subset \Sigma_6$ *be the subgroup, isomorphic to* Σ_5, *generated by the permutations* β_{12}, β_{23}, β_{34} *and* β_{45} *from Lemma 2.3. Then* $\beta\Sigma_5$ *has four orbits on* *the set of* 60 *singular planes of* Y. *They are the orbits of*

$$V_{0-1,2-3,4-5}, \quad V_{0-1,2-4,3-5}, \quad V_{0+1,2+3,4-5} \quad \text{and} \quad V_{0+1,2+4,3-5},$$

and these orbits have lengths 5, 10, 15 *and* 30, *respectively.*

Proof A Magma [5] computation gives the irreducible components of the singular locus of Y and the rest are straightforward verifications (done with Magma as well). □

A plane $V \subset \mathbb{P}^5$ is the projectivization of a linear subspace $\widetilde{V} \subset \mathbb{C}^6$ of dimension 3 and it is determined by the line $\bigwedge^3 \widetilde{V} \subset \bigwedge^3 \mathbb{C}^6$ (equivalently, by the point in the Grassmannian $\mathrm{Gr}(3,6) \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)$). The 20–dimensional vector space $\bigwedge^3 \mathbb{C}^6$ has a natural symplectic form given by $(\omega, \theta) := \omega \wedge \theta$ $(\in \bigwedge^6 \mathbb{C}^6 = \mathbb{C})$, where we fix a basis of $\bigwedge^6 \mathbb{C}^6$. Two planes $V, W \subset \mathbb{P}^5$ are incident if and only if $(\bigwedge^3 \widetilde{V}) \wedge (\bigwedge^3 \widetilde{W}) = 0$. A set of planes is called a set of incident planes if any plane of this set has a nonempty intersection with each of the other planes in the set. In particular, a set of incident planes determines an isotropic subspace in $\bigwedge^3 \mathbb{C}^6$. The following result, together with Corollary 2.7 solves the problem of O'Grady from [24].

Proposition 2.5 (proof of Theorem 1.2) *The union of the two $\beta\Sigma_5$–orbits of the planes $V_{0-1,2-3,4-5}$ and $V_{0+1,2+3,4-5}$ consists of the* 20 *planes $V_{0\pm1,2\pm3,4\pm5}$, $V_{0\pm2,1\pm4,3\pm5}$, $V_{0\pm3,1\pm5,2\pm4}$, $V_{0\pm4,1\pm3,2\pm5}$ and $V_{0\pm5,1\pm2,3\pm4}$, all with an odd number of $-$ signs. This is a set of 20 incident planes. The span of $\bigwedge^3 V_{0\pm1,2\pm3,4\pm5}$, $\bigwedge^3 V_{0\pm2,1\pm4,3\pm5}$, $\bigwedge^3 V_{0\pm3,1\pm5,2\pm4}$, $\bigwedge^3 V_{0\pm4,1\pm3,2\pm5}$ and $\bigwedge^3 V_{0\pm5,1\pm2,3\pm4}$ (all with an odd number of $-$ signs) in $\bigwedge^3 \mathbb{C}^6$ is a Lagrangian subspace A of $\bigwedge^3 \mathbb{C}^6$.*

Proof This is again a (Magma) computation. The set of 20 planes is easy to find. For example, $V := V_{0+1,2+4,3-5}$ is in this set. It has a basis $e_0 - e_1$, $e_2 - e_4$, $e_3 + e_5$, where $e_0, \dots, e_5$ denotes the standard basis of $\mathbb{C}^6$. Thus the line $\bigwedge^3 \widetilde{V}_{0+1,2+4,3-5}$ is spanned by the vector

$$-e_0 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_3 - e_0 \wedge e_3 \wedge e_4 + e_1 \wedge e_3 \wedge e_4 \\ - e_0 \wedge e_2 \wedge e_5 + e_1 \wedge e_2 \wedge e_5 + e_0 \wedge e_4 \wedge e_5 - e_1 \wedge e_4 \wedge e_5.$$

Now one verifies that these planes are indeed incident and span a 10–dimensional, hence Lagrangian, subspace A. □

A general Lagrangian subspace $A \subset \bigwedge^3 \mathbb{C}^6$ determines a sextic hypersurface $Y_A \subset \mathbb{P}^5$, the EPW sextic defined by A, as follows. Let $v \in \mathbb{P}^5$ and let $\widetilde{v} \in \mathbb{C}^6$ be a representative. Then

$$F_v := \{\omega \in \textstyle\bigwedge^3 \mathbb{C}^6 : \widetilde{v} \wedge \omega = 0\}$$

is a Lagrangian subspace of $\bigwedge^3\mathbb{C}^6$. Define

$$Y_A[k] := \{v \in \mathbb{P}^5 : \dim(F_v \cap A) \geq k\}.$$

Then the EPW sextic Y_A is defined as $Y_A = Y_A[1]$. The sextic Y_A is singular along the surface $Y_A[2]$ of degree 40 and along the planes $W \subset \mathbb{P}^5$ such that $\bigwedge^3\widetilde{W} \in A$ [24, Proposition 3.3].

Proposition 2.6 *Let $A \subset \bigwedge^3\mathbb{C}^6$ be the Lagrangian subspace from Proposition 2.5. Then the EPW sextic Y_A is the sextic Y from Proposition 2.1. Its singular locus consists of the surface $Y_A[2]$, which is the union of the 40 planes in the two $\beta\Sigma_5$–orbits of lengths 10 and 30, and the 20 incident planes from Proposition 2.5.*

Proof Let $\omega_1, \dots, \omega_{10}$ be a basis of A and let $\theta_1, \dots, \theta_{15}$ be a basis of $\bigwedge^4\mathbb{C}^6$. For each basis vector e_i of $\mathbb{C}^6$ one computes the matrix M_i of the linear map $A \to \bigwedge^4\mathbb{C}^6$ defined by $\omega \mapsto e_i \wedge \omega$. Then $v = (v_0 : \dots : v_5) \in Y_A$ exactly when the matrix $M_v := v_0 M_0 + \dots + v_5 M_5$ has rank at most 9, and hence all 10×10 minors of M_v must be zero. Since Y_A is either a sextic or is identically zero, it suffices to factor a minor to find a sextic polynomial defining Y_A.

A convenient basis for the Lagrangian subspace A consists of the elements (up to sign) in the $\beta\Sigma_5$–orbit of $e_0 \wedge e_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_5$. We used the standard basis $e_i \wedge e_j \wedge e_k \wedge e_l$ for $\bigwedge^4\mathbb{C}^6$. The submatrix obtained from the ten basis elements $e_0 \wedge e_i \wedge e_j \wedge e_k$, with $i, j, k \in \{1, \dots, 5\}$ and $i < j < k$, has nonzero determinant and one finds the polynomial F_6 as an irreducible factor. We already found the singular locus of Y and all planes corresponding to points in $\mathbb{P}A$, and hence $Y_A[2]$ is the union of the planes as in the proposition. □

Corollary 2.7 *The set of 20 incident planes in Proposition 2.5 is a complete set of incident planes.*

Proof Any plane incident to all the 20 incident planes in Proposition 2.5 corresponds to a point of intersection of $\mathbb{P}(A) \cap \mathrm{Gr}(3, 6)$. Since Y is an EPW sextic, such a plane is in the singular locus of Y [24, Proposition 3.3]. But none of the remaining 40 planes in the singular locus of Y corresponds to a point of A, so the set of 20 planes is complete. □

Remark 2.8 (compare with Remark 6.8) We find, for the Lagrangian space A from Proposition 2.5, that $Y_A[3] = Y_A[4]$. Moreover, $Y_A[4]$ consists of 16 points and for each $v \in Y_A[4]$ the corresponding linear space $\mathbb{P}(F_v \cap A)$ cuts $G(3, 6) \subset \mathbb{P}(\bigwedge^3\mathbb{C}^6)$ in five points. One of these points is $(1 : \dots : 1) \in Y_A[4]$ (see the end of the proof of Proposition 2.1). This is the point where five planes of type $V_{i-j,k-l,n-m}$ (with three $-$ signs) meet.

Remark 2.9 Given one of five partitions $\{\{i,j\},\{k,l\},\{n,m\}\}$, used to describe 20 incident planes in Proposition 2.5, and choosing one of the three pairs in this partition, one produces a point of incidence of another 5–tuple of planes. For example, for a partition $\{\{0,2\},\{1,4\},\{3,5\}\}$ we choose the second pair $\{1,4\}$ and take the point $(1:-1:1:1:-1:1)$ (minus sign with first and fourth coordinates). Then this point is contained in the five planes $V_{0+1,2-3,4+5}$, $V_{0-3,1+5,2+4}$, $V_{0+4,1+3,2-5}$, $V_{0-5,1+2,3+4}$ and $V_{0-2,1-4,3-5}$. This way we get $16 = 1+5\cdot 3$ points in $Y_A[4]$, each of the points contained in a 5–tuple of planes coming from the five different partitions from Proposition 2.5.

Similarly one finds 30 points where planes associated to the same partition meet pairwise. For example, for a partition $\{\{0,2\},\{1,4\},\{3,5\}\}$, we choose the second pair. Then $V_{0-2,1-4,3-5}$ meets $V_{0+2,1-4,3+5}$ at $(0:1:0:0:1:0)$ while $V_{0-2,1+4,3+5}$ meets $V_{0+2,1+4,3-5}$ at $(0:1:0:0:-1:0)$. This shows that the planes from Proposition 2.5 meet pairwise indeed.

Thus the set of 20 planes is divided into 5 subsets of 4 planes in a natural way: a subset corresponds to a partition $\{\{i,j\},\{k,l\},\{n,m\}\}$ and there are four choices of signs such that the number of $-$ signs is odd. Each subset of four planes contains 16 points in $Y_A[4]$, on each plane there are four of these points. Any two planes corresponding to different partitions meet in one of the 16 points.

Remark 2.10 Note we can reconstruct the K3 surface $S' \subset \mathbb{P}^6$ starting from A; see [23, Section 4.2]. Indeed, if we let $v \in Y_A[4]$, then the dual space $\mathbb{P}(F_v \cap A)^* \subset \mathbb{P}(F_v)^*$ is a 5–dimensional linear space contained in $\mathbb{P}^9$. The projective space $\mathbb{P}(F_v)^*$ naturally contains a Grassmannian $G(2,5)$ that cuts $\mathbb{P}(F_v \cap A)^*$ along a smooth Del Pezzo surface S_5 of degree 5. From [23, (4.1.5)] we know that there is a nondegenerate quadratic form on $\mathbb{P}(F_v \cap A)^* \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)^*$ (induced by $A \subset \bigwedge^3 \mathbb{C}^6$). We find that the corresponding quadric cuts $S_5 \subset \mathbb{P}^5$ along ten lines. The K3 surface S' is the double cover of S_5 branched along these lines. Note that $S' \subset \mathbb{P}^6$ is a nongeneric K3 surface of degree 10.

3 The resolution of the map $S^{[2]} \dashrightarrow Y$

Let $S^{[2]}$ be the Hilbert scheme of two points on S. In this section we analyze the rational map $g\colon S^{[2]} \dashrightarrow Y$ defined in Section 2.2. In Section 3.1 we present a sequence of flops that resolves the indeterminacy of this map; in Proposition 3.3 we obtain a morphism $\overline{g}\colon \overline{S^{[2]}} \to Y$ such that $S^{[2]}$ and $\overline{S^{[2]}}$ differ by Mukai flops. In Section 3.2 we describe the ramification locus of $\overline{g}$; we need it to obtain the explicit desingularization in Theorem 1.3. A consequence of our construction is the description (see Remark 6.17) of a symplectic resolution of the singular point $\mathbb{C}^4/G$ discussed in [3], ie the fiber of $\overline{g}$ over points from $Y_A[4]$.

3.1 Flops

In order to resolve the map g we need to perform birational transformations such that the divisor $g^*(\mathcal{O}_{\mathbb{P}^5}(1))$ becomes nef. Let us first describe this divisor.

Let $\mu\colon H^2(S,\mathbb{Z}) \to H^2(S^{[2]},\mathbb{Z})$ be the natural morphism of cohomology groups. For (the class of) a curve $C \subset S$, its image $\mu(C)$ is the class of the divisor with support $\{\{p,q\} : p \in C, q \in S\} \in S^{[2]}$. We denote by $\Delta_S^{[2]}$ the diagonal divisor on $S^{[2]}$. The isomorphism of $H^2(S^{[2]},\mathbb{Z})$, with the lattice S given in (6-1), will be fixed so that $\mu(H^2(S,\mathbb{Z})) = \Lambda_{\mathrm{K3}}$ and $\Delta_S^{[2]} = 2\eta$.

Proposition 3.1 *We have* $g^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{S^{[2]}}(\mu(C) - \Delta_S^{[2]})$.

Proof See [23, Section 4]. □

Let us describe a sequence of Mukai flops that resolves the map g. Recall that in Section 2.2 we defined the map g such that on an open part of $S^{[2]}$ it can be described as the composition

$$g\colon S^{[2]} \overset{\phi^{[2]}}{\dashrightarrow} (S')^{[2]} \overset{\rho^{[2]}}{\dashrightarrow} Y \subset \mathbb{P}^5$$

of rational maps. By $(S')^{[2]}$ we understand the Hilbert scheme of two points in the smooth part of S'. We shall see that there are two sources of indeterminacy for the map g: the first is the presence of lines on S' and the second are the nodes of S'. In order to understand more precisely the map g we need to understand the geometry of $S' \subset \mathbb{P}^6(y_0,\dots,y_6)$.

We take a Del Pezzo surface S_5 of degree 5 contained in the hyperplane $W = \mathbb{P}^5 \subset \mathbb{P}^6$ defined by $y_6 = 0$. Let Π be the cone over S_5 in $\mathbb{P}^6$ with vertex $P = (0:\cdots:0:1)$ and let Q be the quadric with equation $y_6^2 = q'(y_0,\dots,y_5)$ as in Section 2.1. Then Q intersects S_5 along the union of the ten exceptional lines on S_5, we have $S' = \Pi \cap Q$ and S' is singular exactly at the 15 points of intersection of these lines. The projection $\mathbb{P}^6 \to \mathbb{P}^5$ with center P induces a two-to-one *morphism* $\rho\colon S' \to S_5$ ramified along the sum of the ten lines on $S_5 \subset W$. Since the projection by ρ of a line in S' is a line in S_5, we infer that there are exactly ten lines on $S' \subset \mathbb{P}^6$. Denote the set of strict transforms of those lines on S by $\mathcal{T} = \{l_1,\dots,l_{10}\}$ and the 15 exceptional curves on S by $e_1,\dots,e_{15}$.

From the definition of the map g in the proof of Proposition 2.1 it follows that g is well defined on $S^{[2]}$ except possibly in $z \in S^{[2]}$ with $z = \{p,q\}$ such that:

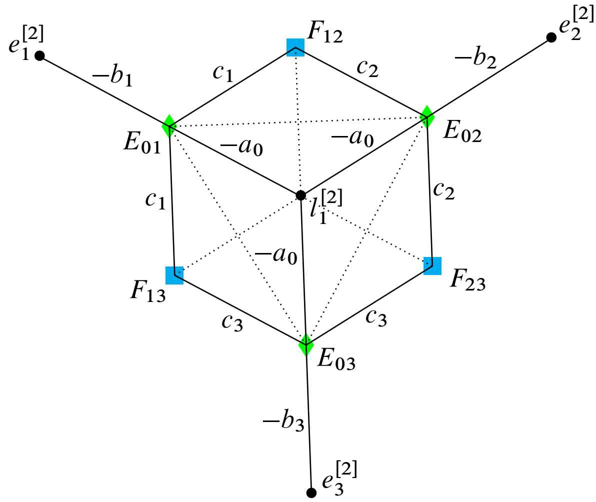

Figure 2

(1) Both $p \in l_i$ and $q \in l_i$, ie $z \in l_i^{[2]}$; we have 10 such planes.

(2) Both $p \in e_i$ and $q \in e_i$, ie $z \in e_i^{[2]}$; we have 15 such planes.

(3) $p \in e_i$ and $q \in l_j$, where e_i and l_j intersect. We find a surface denoted by E_{ij} that parametrizes the closure of this set of reduced subschemes $z = \{p, q\}$. We obtain 30 surfaces (because there are three exceptional curves cutting a given line) in $S^{[2]}$, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in one point,

(4) $p \in e_i$ and $q \in e_j$ with e_i and e_j mapping to two distinct points in one of the lines from $\mathcal{T}$. We obtain 30 surfaces $F_{ij} = \{\{p, q\} \in S^{[2]} : p \in e_i, q \in e_j\}$, each of which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{K} \subset S^{[2]}$ be the union of the 85 surfaces of types (1), (2), (3) and (4) described above. Note that the indeterminacy locus of g is contained in $\mathcal{K}$. We now perform a sequence of flops to obtain a fourfold on which the transform of the map g is a morphism. Let us analyze the flops locally on $S^{[2]}$ around the surface $l_1^{[2]}$.

A node of the trivalent Petersen graph corresponding to the (-2)–curve l_1 meets three other (-2)–curves e_1, e_2 and e_3 (see Figure 1). This gives an initial configuration of ten surfaces in $S^{[2]}$,

$$l_1^{[2]}, \quad e_1^{[2]}, \quad e_2^{[2]}, \quad e_3^{[2]}, \quad F_{12}, \quad F_{23}, \quad F_{13}, \quad E_{01}, \quad E_{02} \quad \text{and} \quad E_{03},$$

which can be described as in Figure 2.

In the diagram we use the following notation to describe the types of the surfaces:

● $= \mathbb{P}^2$, ■ $= \mathbb{P}^1 \times \mathbb{P}^1$ and ◆ $= \mathbb{P}^2_2$ is $\mathbb{P}^2$ blown-up in two different points. In subsequent diagrams we will also use ▼ $= \mathbb{P}^2_1$ and ✻ $= \mathbb{P}^2_3$ to denote the blow-up of $\mathbb{P}^2$ at one and three (noncollinear) points, respectively.

Solid line edges of the diagram are intersections of surfaces along curves; dotted line edges denote intersections in points. The solid lines edges will be labeled by the classes of curves in $\mathrm{Hilb}^2(S)$.

Given a curve C on S we have a divisor $\mu(C)$ in $\mathrm{Hilb}^2(S)$ consisting of schemes whose support has nonempty intersection with C. The "diagonal" divisor $\Delta_S^{[2]}$ in $S^{[2]}$ is the exceptional divisor of the resolution of singularities $S^{[2]} \to (S \times S)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$–action interchanges the factors. Outside the divisor $\Delta_S^{[2]}$ the divisor $\mu(C)$ is isomorphic to the complement of $C \times C$ in $C \times S$. By $[C]$ we will denote the class of the curve $C \times \{s\}$ where $s \notin C$.

In what follows $c_0 = [l_1]$ and $c_i = [e_i]$, for $i = 1, 2, 3$, and d is the class of a fiber in the blow-up of the diagonal, that is, in $\Delta_S^{[2]}$. We have the following intersection rules:

- $\mu(C) \cdot c_0 = 1$ and $\mu(C) \cdot c_i = \mu(C) \cdot d = 0$ for $i > 0$,
- $\Delta_S^{[2]} \cdot c_i = 0$ for $i \geq 0$ and $\Delta_S^{[2]} \cdot d = -2$.

To spare notation in diagrams we set

$$h = \sum_{i \geq 0} c_i - d \quad \text{and} \quad a_0 = d - c_0,$$

and for $j > 0$ we set

$$\begin{aligned} b_j &= d - c_j, & f_j &= d - c_0 - c_j, \\ g_j &= h - c_j = \sum_{i \geq 0} c_i - d - c_j, & v_j &= \sum_{i \geq 0} c_i - c_0 - c_j. \end{aligned}$$

Then for $j > 0$ we have

$$\begin{aligned} -\mu(C) \cdot f_j = \mu(C) \cdot g_j &= \mu(C) \cdot h = 1, \\ \Delta_S^{[2]} \cdot f_j = \Delta_S^{[2]} \cdot b_j &= -2, \\ \Delta_S^{[2]} \cdot g_j = \Delta_S^{[2]} \cdot h &= 1, \\ \mu(C) \cdot b_j = \mu(C) \cdot v_j &= \Delta_S^{[2]} \cdot v_j = 0. \end{aligned}$$

We start the process of flopping. Note that the lines contained in $e_i^{[2]}$ and $l_j^{[2]}$ have negative intersection with $\mu(C) - \Delta_S^{[2]}$. Note also that among the surfaces from $\mathcal{K}$ only the surfaces $e_i^{[2]}$ and $l_j^{[2]}$ are planes so we have to start the flopping procedure with them. Flopping the $e_i^{[2]}$, for $i = 1, 2, 3$, each isomorphic to $\mathbb{P}^2$, outside the hexagon we get

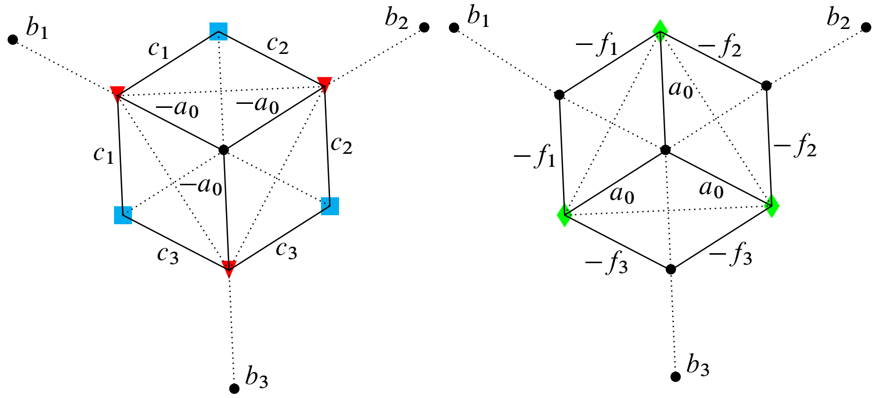

Figure 3

the picture on the left-hand side of Figure 3 and then flopping $l_1^{[2]}$, the $\mathbb{P}^2$ in the center, we get the picture on the right-hand side. We suppress labeling the surfaces and label only the classes of curves in the surfaces, notably those which are in the intersections of them.

The left-hand side of Figure 4 presents the result of flopping the three copies of $\mathbb{P}^2$ at the perimeter of the hexagon. Note that the copies of $\mathbb{P}^2$ outside the hexagon are blown up twice because they are on the link of two such hexagons. The resulting copies of $\mathbb{P}^2_2$ have a common point with the central surface. They have three (-1)–curves whose classes are in f_i, f_j' and $b_j - f_j - f_j'$ where f_j' is the class of the curve coming from the configuration of the adjacent node of the Petersen graph. The point of intersection of this surface with the central $\mathbb{P}^2_3$ lies on the curve whose class is f_j. The right-hand side of Figure 4 is obtained by the subsequent flopping of the other three copies of $\mathbb{P}^2$ at the perimeter. In this step the surfaces outside the hexagon are not affected.

Now we flop at the central surface (see Figure 5).

In Figure 5, ❋ denotes $\mathbb{P}^2_2$ blown-up at two points at two nonmeeting (-1)–curves, which then become (-2)–curves whose classes are in $v_j = f_j + h = \sum_i c_i - c_0 - c_j$, and v_j', respectively. This surface has also five (-1)–curves with classes $b_j - f_j - f_j'$, h and $b_j - f_j - h$ as well as h' and $b_j' - f_j' - h'$.

Lemma 3.2 *The divisor $\mu(C) - \Delta_S^{[2]}$ is nef on this configuration of surfaces. It is $\mathcal{O}(1)$ on the central $\mathbb{P}^2$, it is trivial on copies of $\mathbb{P}^1 \times \mathbb{P}^1$ (■ in Figure 5) and defines a ruling on the remaining six surfaces. Thus some multiple of $\mu(C) - \Delta_S^{[2]}$ defines the contraction of the configuration of these surfaces to a configuration of lines on the image of the plane $\overline{l_0^{[2]}}$.*

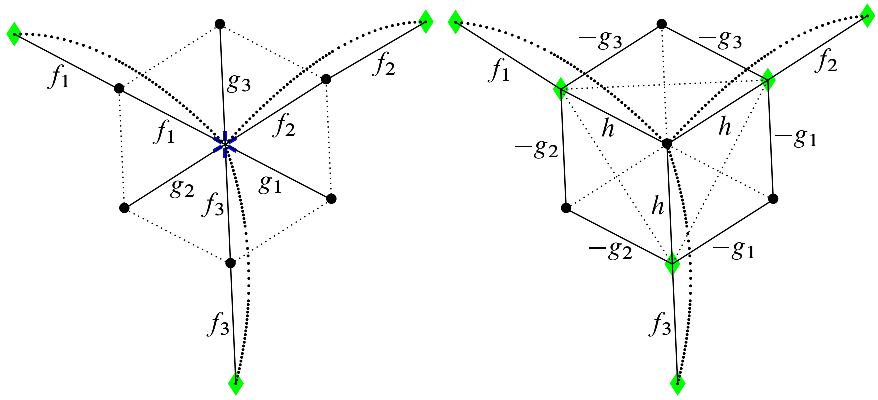

Figure 4

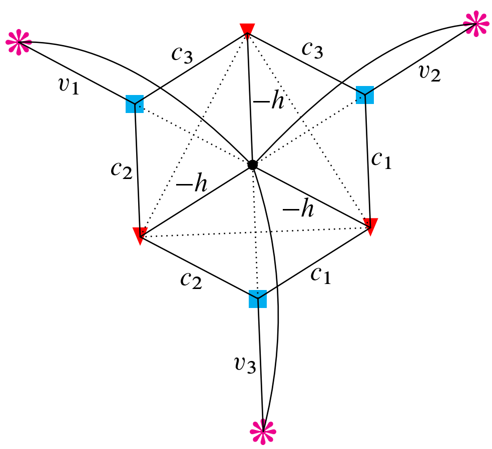

Figure 5

Proof We check that $(\mu(C)-\Delta_S^{[2]})\cdot(-h)=1$ while the intersection of $\mu(C)-\Delta_S^{[2]}$ with each of c_j, v_j, f_j+h, b_j-f_j-h and $b_j-f_j-f_j'$ is zero. $\square$

Summary We constructed above a sequence of Mukai flops of copies of $\mathbb{P}^2$ determined by the following classes of 1–cycles: (1) b_j, (2) a_i, (3) f_{ij}, (4) g_{ij}, (5) h_i. Here i is among indices parametrizing vertices and j is among indices parametrizing edges of the Petersen graph. After flopping those classes $\mu(C)-\Delta_S^{[2]}$ becomes nef (on the configuration of the strict transforms of the surfaces in question). The fourfold $\overline{S^{[2]}}$ is obtained from $S^{[2]}$ by performing this sequence of Mukai flops in the five families of surfaces and $\overline{g}\colon \overline{S^{[2]}}\to Y$ is the map obtained from $g\colon S^{[2]}\dashrightarrow Y$. We infer the following:

Proposition 3.3 *The strict transform of the complete linear system $|\mu(C) - \Delta_S^{[2]}|$ on $\overline{S^{[2]}}$ is big and nef and defines a morphism $\overline{g}\colon \overline{S^{[2]}} \to Y \subset \mathbb{P}^5$ that resolves the indeterminacy of the map g. In particular g is defined by the complete linear system $|\mu(C) - \Delta_S^{[2]}|$ and the set $\mathcal{K}$ is the indeterminacy locus of g.*

Proof From Proposition 3.1 the map g is defined by a 5–dimensional linear subsystem of $|\mu(C) - \Delta_S^{[2]}|$.

We are flopping $S^{[2]}$ step by step as described in Figures 2–5. Also, we saw how the classes of curves change after our flops. At each step we are doing Mukai flops of a set of disjoint copies of $\mathbb{P}^2$ corresponding to a curve class that has negative intersection with the strict transform of $\mu(C) - \Delta_S^{[2]}$. By Lemma 3.2 we deduce that the proper transform H of $\mu(C) - \Delta_S^{[2]}$ on $\overline{S^{[2]}}$ is big and nef.

The Beauville degree satisfies $q(H) = q(\mu(C) - \Delta^{[2]}) = q(\mu(C)) - q(2\eta) = 10 - 8 = 2$. So, in particular, $\chi(\mathcal{O}_{\overline{S^{[2]}}}(H)) = 6$, and thus $h^0(\mathcal{O}_{\overline{S^{[2]}}}(H)) = 6$. It follows that the map $\overline{g}\colon \overline{S^{[2]}} \to \mathbb{P}^5$ defined by $|H|$ is a morphism. Since $\overline{S^{[2]}}$ and $S^{[2]}$ are isomorphic in codimension 1 the morphism $\overline{g}\colon S^{[2]} \to Y \subset \mathbb{P}^5$ gives the resolution of g. It follows also that the map $g\colon S^{[2]} \dashrightarrow \mathbb{P}^5$ is given by the complete linear system $|\mu(C) - \Delta_S^{[2]}|$ on $S^{[2]}$.

The set $\mathcal{K}$ is in the base locus because it is covered by curves with negative intersection with $\mu(C) - \Delta_S^{[2]}$. Outside $\mathcal{K}$ there are no base points since the map $\rho^{[2]}$, so g, is well defined there. □

3.2 The structure of the map $\overline{S^{[2]}} \to Y$

Let $S^{[2]}$ be the Hilbert scheme of two points on S.

In this subsection we describe technical results needed in the proof of Theorem 1.3. Our aim is to give a description of the ramification locus of the map $\overline{g}$. Our plan is to first describe the ramification of the map g and then to look how this ramification locus transforms by flops, described in Section 3.1, transforming g to $\overline{g}$.

We consider the Stein factorization

$$\overline{g}\colon \overline{S^{[2]}} \xrightarrow{c} Z \xrightarrow{f} Y \subset \mathbb{P}^5.$$

Let us first identify 20 divisors $B_1, \dots, B_{20}$ on $\overline{S^{[2]}}$ which are contracted by c to singular surfaces on Z. Then we identify the ramification locus of f as the image by c of 40 surfaces that are the strict transforms of some surfaces from $S^{[2]}$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.4 *The covering involution of $f\colon Z \to Y$ is induced by the map $\rho\colon S \to S_5$, ie if $\rho(x_1) = \rho(x_2)$ and $\rho(y_1) = \rho(y_2)$, such that $x_1 \neq x_2$ and $y_1 \neq y_2$, are generic points on S, then we have $f(c((x_1, y_1))) = f(c((x_2, y_2)))$.*

Proof It suffices to study the involution induced by $\rho^{[2]}\colon (S')^{[2]} \to Y$. The linear system of quadrics containing S' is generated by the 5–dimensional system of quadrics containing the cone Π and the quadric Q. For a pair $\{p, q\} \in (S')^{[2]}$ consider another pair $(p', q') \in (S')^{[2]}$ such that p' (resp. q') is the second point of intersection of the line Pp (resp. Pq) with Q, since the quadrics containing Π have the property: when they vanish on the line pq then they vanish on the line $p'q'$. Moreover, Q vanishes at the points p, q, p' and q', and so the proof is finished. □

We can now describe the ramification locus of the map $\overline{g}$. Let $L_{ij} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $i \neq j$ be the strict transform on $\overline{S^{[2]}}$ of the surface $L'_{ij} = \{\{p, q\} \in S^{[2]} : p \in l_i, q \in l_j\}$ (where l_i for $i = 1, \dots, 10$ are the strict transforms of the lines from S' on S). Note that L'_{ij} is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and that there are 45 such surfaces.

Corollary 3.5 *The branch locus of the map $\overline{g}$ consists of 40 surfaces on $\overline{S^{[2]}}$. Each such surface is an element of one of the following sets:*

- *the set of 30 surfaces $L_{ij} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ from $S^{[2]}$ for $i \neq j$ such that the lines $\nu(l_i)$ and $\nu(l_j)$ do not intersect,*
- *the set of ten planes $\overline{l_i^{[2]}} \subset \overline{S^{[2]}}$ which are the strict transforms of the ten planes $l_i^{[2]} \subset S^{[2]}$.*

Proof The surfaces L'_{ij} are invariant with respect to $\rho^{[2]}$, so it is enough to show that they are not contracted by $\overline{g}$. First observe that L_{ij} is isomorphic to $\mathbb{P}^2$ blown-up at two points. Indeed, consider the $\mathbb{P}^3$ which is the span of two disjoint lines $\nu(l_i)$ and $\nu(l_j)$ on S_5. It cuts S_5 along three lines $\nu(l_i)$, $\nu(l_j)$ and $\nu(l_k)$ such that $\nu(l_k)$ cuts $\nu(l_i)$ in one point $\nu(A_i)$ and $\nu(l_j)$ in one point $\nu(A_j)$. We can deduce that the restriction of the map $S^{[2]} \dashrightarrow \overline{S^{[2]}}$ to L_{ij} is the blow-up of the point $(A_i, A_j) \in L_{ij}$ (corresponding to the intersection with the line $e_i e_j$).

By Lemma 3.2 the strict transform of the system $|\mu(C) - \Delta_S^{[2]}|$ restricted to the plane $\overline{l_i^{[2]}}$ is the system $\mathcal{O}_{\mathbb{P}^2}(1)$. Since $\overline{l_i^{[2]}}$ maps through c to a plane we infer that $\overline{l_i^{[2]}}$ is in the ramification locus. □

Remark 3.6 The 15 surfaces L_{ij} that correspond to two intersecting lines are mapped to points by $\overline{g}$. Indeed, in the case when $\nu(l_i)$ and $\nu(l_j)$ intersect, a quadric containing S' either cuts the plane spanned by $\nu(l_i)$ and $\nu(l_j)$ along these two lines or it contains this plane.

Let us identify the exceptional divisors on $\overline{S^{[2]}}$ that are mapped to surfaces on Z by c. The exceptional divisors B_i will be the strict transforms on $\overline{S^{[2]}}$ of divisors $B'_i \subset S^{[2]}$ defined in the following way. Fifteen of them are easy to describe. Let $B'_i = \{\{p,q\} \in S^{[2]} : p \in e_i\}$, where e_i is one of the 15 curves contracted by the map $S \to S'$. Then these divisors are already contracted by $\phi^{[2]}\colon S^{[2]} \to (S')^{[2]}$.

Let us find the remaining five divisors $B'_{16}, \dots, B'_{20}$. There are five pencils of conics on $S_5 \subset \mathbb{P}^5$ (if $S_5 \to \mathbb{P}^2$ is the blow-up in four points the pencils correspond to lines passing through these points and the conics through the four points). These pencils induce five elliptic pencils $\lambda_1, \dots, \lambda_5$ on S. We define the divisors B'_{15+i} by

$$B'_{15+i} = \{\{p,q\} \in S^{[2]} : \text{there exists } K \in \lambda_i \text{ such that } p \in K, q \in K\}.$$

Each of the divisors B'_{15+i} has a fibration $B'_{15+i} \to \mathbb{P}^1$ with fibers of type $K^{[2]}$.

Denote by $B_1, \dots, B_{20}$ the strict transforms of $B'_1, \dots, B'_{20}$ on $\overline{S^{[2]}}$. We know from Proposition 2.6 that Y is singular along 60 planes. We shall see that the 20 of them described in Proposition 2.5 are the images of $B_1, \dots, B_{20}$. Note that in Proposition 4.8 we prove that the images of $B_1, \dots, B_{20}$ on Z are singular K3 surfaces with normalization being the Vinberg K3 surface.

Proposition 3.7 *The divisors B_i for $i = 1, \dots, 20$ are contracted through c to surfaces in Z such that the images of the B_i on Z intersect each other. A general fiber of c in the divisors $B_1, \dots, B_{15}$ is a curve of type c_i, with $i > 0$, as defined in Section 3.1. Moreover, there are no other divisors on $S^{[2]}$ that are contracted to surfaces by g.*

Proof Let u be the strict transform on S of a conic in $S_5 \subset \mathbb{P}^5$. It is enough to prove that the surface $u^{[2]} \subset S^{[2]}$ maps to a line in Z. First, the curve u is contained in $\mathbb{P}^3_u$ (spanned by P and the plane spanned by the conic in S_5). The curve u is the intersection of two quadrics: Q and the cone with vertex P. When u is chosen generically, the quadric $Q \cap \mathbb{P}^3_u$ is smooth and thus is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Each line on Q cuts S' in two points such that the two rulings on Q define two curves on $S^{[2]}$. By the description of the map $\rho^{[2]}$ we see that both these curves are contracted by g to the same point on Y.

Finally, we find explicitly points in the intersections of each pair B_i, B_j of the divisors for $0 \le i < j \le 15$.

In order to prove that there are no more contracted divisors, it is enough to observe that each such divisor maps to a plane in the singular locus of $Y \subset \mathbb{P}^5$. On the other hand, we know that Y is an EPW sextic and from [24] the contracted divisors B_i for $i = 1, \dots, 20$ are mapped to a maximal set of incident planes. There are no more contracted divisors since the image of such a divisor would be a plane which intersects all the 20 planes above. □

4 A Kummer-type IHS and the Debarre–Varley ppav

The IHS fourfold X_0 constructed by Donten-Bury and Wiśniewski in [11] is a quotient of a principally polarized abelian fourfold (ppav) which we study in detail in this section. In Theorem 1.3 we show that X_0 is birationally equivalent to $S^{[2]}$.

4.1 Polarization

The variety X_0 is constructed as a desingularization of a quotient of the form E^4/G where $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is the elliptic curve with complex multiplication by $\mathbb{Z}[i]$ and $G \cong Q_8 \times_{\mathbb{Z}_2} D_8$ is a subgroup of $\mathrm{Aut}(E^4)$.

Recall that the action of G on E^4 is given by the matrices T_j for $j = 0, \dots, 4$, listed in Section 4.3 below; see [11, Section 6.B]. They satisfy the relations $T_j^2 = I$, $T_j T_k = -T_k T_j$, and thus $(T_j T_k)^2 = -I$ for $j \neq k$.

The abelian fourfold E^4 also has as automorphisms

$$i\colon E^4 \to E^4 \quad \text{given by } (x, y, z, t) \mapsto (ix, iy, iz, it) \qquad \text{and} \qquad (-1) := i^2.$$

Let us find a polarization $H \in \mathrm{NS}(E^4)$ on the abelian fourfold $E^4 = \mathbb{C}^4/\Lambda$ which is invariant with respect to G, where $\Lambda = \mathbb{Z}[i]^4$. By [4, Section 2.2] this is equivalent to finding a G–invariant Hermitian 4×4 matrix H with coefficients in $\mathbb{Z}[i]$.

Proposition 4.1 *Any G–invariant Hermitian matrix with coefficients in $\mathbb{Z}[i]$ has the following shape, where $a \in \mathbb{Z}$:*

$$H_a := a \begin{pmatrix} 2 & 0 & 1 & 1+i \\ 0 & 2 & 1+i & -i \\ 1 & 1-i & 2 & 0 \\ 1-i & i & 0 & 2 \end{pmatrix}.$$

Proof The G–invariant Hermitian matrices satisfy equations $T_j \cdot H \cdot \overline{T}_j^t = H$, for each $0 \le j \le 4$, where $\overline{T}_j^t$ is the transpose of the complex conjugate of T_j. Notice that these are linear equations in the coefficients of H. □

We then find that $H := H_1$ is positive definite and since $\det H = 1$, it defines a principal polarization on E^4.

It turns out that the principally polarized abelian variety (E^4, H) was known before; see [6; 26]. In fact, Debarre in [6] proved that there exists a unique indecomposable ppav (A_{10}, L) of dimension 4 that is not a hyperelliptic Jacobian and admits the maximal number of ten vanishing theta-constants (points of order two on a symmetric theta divisor which are singular with even multiplicity).

Proposition 4.2 *The abelian fourfold (E^4, H) is isomorphic as a ppav to the Debarre–Varley ppav (A_{10}, Θ). In particular, the singular locus of the corresponding theta divisor consists of ten ordinary double points and if theta divisor is chosen to be symmetric, these ODPs are two-torsion points of E^4.*

Proof The real part of H defines a $\mathbb{Z}$–valued quadratic form on $\mathbb{Z}^8 = \mathbb{Z}[i]^4$ and one finds that it is a positive definite even unimodular quadratic form. Hence, in a suitable basis, it must be the quadratic form associated to the root system E_8. Now the proposition follows from the construction in [6, Section 5]. □

4.2 Invariant line bundles

We will show that there are exactly 16 G–invariant line bundles on E^4 whose first Chern class is the alternating form $E = \operatorname{Im} H$. First we consider the fixed points of the action of G on E^4.

Lemma 4.3 *The subgroup $(E^4)^G$ of points of E^4 which are fixed by G is isomorphic to $(\mathbb{Z}_2)^4$. The 16 fixed points are (a_1, a_2, a_3, a_4) where a_j is either 0 or $(1+i)/2$. Moreover, these points are also the fixed points of the automorphism i of E^4.*

Proof As $(T_i T_j)^2 = -I$ if $i \neq j$, a fixed point of G is a point $x \in E^4$ with $-x = x$, that is, $2x = 0$. Now one checks that of the $2^8 = 256$ two-torsion points in E^4 only the 16 points given in the lemma are fixed by G. Similarly, the fixed points for i must be two-torsion points and one finds the same 16 points. □

The line bundles on an abelian variety $\mathbb{C}^d/\Lambda$ with a given first Chern class E and a $\mathbb{Z}$–valued alternating form on Λ, are parametrized by semicharacters, ie by maps $\alpha\colon \Lambda \to \mathbb{C}_1$ with $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$, the circle group, satisfying

$$\alpha(x+y) = \alpha(x)\alpha(y)e(x,y)$$

where $e(x,y) := \exp(\pi i E(x,y))$; see [4, Section 2.2]. Notice that a semicharacter is completely determined by its values on a $\mathbb{Z}$–basis of Λ.

The line bundle L_α defined by α is G–invariant if and only if for each $g \in G$ we have $\alpha(g(x)) = \alpha(x)$, and it is symmetric, so $(-1)^* L_\alpha \cong L_\alpha$, if and only if $\alpha(\Lambda) \subset \{\pm 1\}$.

The semicharacter of a symmetric line bundle factors over $\Lambda/2\Lambda$ and this group is naturally isomorphic to the group of two-torsion points on the abelian variety.

Proposition 4.4 *There are exactly* 16 *G–invariant line bundles on E^4 whose first Chern class is the alternating form* $\operatorname{Im} H$*. These* 16 *line bundles are symmetric and are also invariant under the automorphism i of E^4. The corresponding semicharacters $\alpha\colon E[2]^4 \to \{\pm 1\}$ are exactly those for which $\alpha(x) = 1$ for all $x \in (E^4)^G$.*

Proof Since $-I \in G$, any G–invariant line bundle is symmetric. Hence its semicharacter takes values in $\{\pm 1\}$. To find such semicharacters with $\alpha(T_j(x)) = \alpha(x)$ for all $x \in \Lambda$ and each $0 \le j \le 4$, we use the $\mathbb{Z}$–basis of $\Lambda = \mathbb{Z}[i]^4$ given by eight vectors $(1,0,0,0), (i,0,0,0), \ldots, (0,0,0,i)$. By computations one finds that the G–invariant semicharacters are those that have the values $(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4)$, with $a_i \in \{\pm 1\}$, on these basis vectors. In particular, there are 16 of these and they satisfy $\alpha(ix) = \alpha(x)$.

Then it is easy to check that for $x_1 = (1+i,0,0,0), \ldots, x_4 = (0,0,0,1+i)$ one has $e(x_i, x_j) = +1$, for $1 \le i, j \le 4$, hence the Weil pairing is trivial on $(E^4)^G$. One also easily verifies that these 16 G–invariant semicharacters satisfy $\alpha(x_i) = 1$, for $i = 1, \ldots, 4$, and hence $\alpha(x) = 1$ for all $x \in (E^4)^G$. Conversely, a semicharacter with values in $\{\pm 1\}$ which is trivial on $(E^4)^G$ is completely determined by its values on the four vectors $(1,0,0,0), \ldots, (0,0,0,1)$. Thus there are exactly 16 such semicharacters and, using the results above, we conclude that these are the semicharacters of the G–invariant line bundles with first Chern class $\operatorname{Im} H$. □

Since (E^4, H) is a principally polarized abelian variety and H is G–invariant, the isomorphism $\lambda_H\colon E^4 \to \operatorname{Pic}^0(E^4)$ induced by H is a bijection between the fixed points of G in E^4 and the G–invariant line bundles with trivial first Chern class. Tensoring a G–invariant line bundle L having $c_1(L) = \operatorname{Im} H$ with a G–invariant line bundle in $\operatorname{Pic}^0(E^4)$, one obtains again a G–invariant line bundle with first Chern class $\operatorname{Im} H$. Conversely, if M is G–invariant and $c_1(M) = \operatorname{Im} H$, then $L \otimes M^{-1}$ has trivial first Chern class and is G–invariant.

A G–invariant line bundle L with first Chern class $\operatorname{Im} H$ defines a principal polarization, and hence $\dim H^0(E^4, L) = 1$. We denote by D the corresponding theta divisor, that is, the unique effective divisor D in E^4 such that $L = \mathcal{O}(D)$. We refer to these 16 theta divisors as the G–invariant theta divisors; each has a singular locus consisting of ten ODPs by Debarre's results in [6]. Moreover, if D is a G–invariant theta divisor and $p \in (E^4)^G$, then $D + p = t_p^* D$ also is a G–invariant theta divisor.

4.3 The automorphism group

To find the configuration of the G–invariant points and divisors, we will use the action of the automorphism group G_{DV} of the ppav (E^4, H).

In [6, Section 5] one finds that after choosing (any) $J \in W(E_8)$ with $J^2 = -1$, one obtains an isomorphism between the root lattice of E_8 and the lattice Λ defining A_{DV} such that J corresponds to the multiplication by i. The automorphism group $G_{\mathrm{DV}} := \mathrm{Aut}(E^4, H)$ is the subgroup of the Weyl group $W(E_8)$ of elements which commute with J. The group G_{DV} has order $46080 = 2^6 \cdot (6!)$. It has a normal subgroup $\widetilde{G} := G \times_{\mathbb{Z}_2} \mathbb{Z}/4\mathbb{Z}$ of order 2^6, which is the group (G, i) generated by G and i. The quotient group $G_{\mathrm{DV}}/\widetilde{G}$ is isomorphic to the symmetric group S_6. The isomorphism between the root lattice of E_8 and $\Lambda = \mathbb{Z}[i]^4 \subset \mathbb{C}^4$ defines a 4–dimensional complex representation of G_{DV}. The invariant theory of this group was studied by Maschke [19]. The representation of G_{DV} on the alternating forms on $\mathbb{C}^4$ permutes, up to scalar factors, a certain basis of six alternating forms. This provides the surjective homomorphism $G_{\mathrm{DV}} \to S_6$:

$$0 \to \widetilde{G} = (G, i) \to G_{\mathrm{DV}} \to S_6 \to 0.$$

The unitary group of the hermitian form $H = H_1$ from Proposition 4.1 is

$$U(H) := \{M \in \mathrm{GL}(4, \mathbb{Z}[i]) : MH\overline{M}^t = H\}.$$

By definition, it is the subgroup of $\mathrm{Aut}(E^4)$, fixing $0 \in E^4$, which preserves the polarization defined by H. In particular, $U(H) \cong G_{\mathrm{DV}}$ and $\sharp U(H) = 46080$.

The group G, which is a subgroup of $U(H)$, is generated by the five matrices given by $T_j := N_5^j T_0 N_5^{-j}$ with $j = 0, 1, \dots, 4$, where

$$T_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1+i & 1 & 0 \\ 1-i & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad N_5 := \begin{pmatrix} 1 & -1 & 0 & -i \\ -i & 0 & i & i \\ -i & -1 & 1+i & 0 \\ 1 & -1+i & 0 & -1-i \end{pmatrix}.$$

In particular, N_5 normalizes G and we verified that $N_5 \in U(H)$. The following matrices are also in $U(H)$:

$$N_{01} := \begin{pmatrix} 1+i & 0 & -1-i & -i \\ 0 & i+1 & -i & 0 \\ 0 & 1 & -1-i & 0 \\ 1 & -1+i & 0 & -1-i \end{pmatrix} \quad \text{and} \quad N_{45} := \begin{pmatrix} 0 & 1 & 0 & i \\ 0 & 1+i & -i & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1+i \end{pmatrix}.$$

One has $N_{01} T_0 N_{01}^{-1} = T_1$ and conjugation by N_{01} fixes T_2 and T_3 and maps T_4 to $-T_4$. The matrices N_5 and N_{01} generate a subgroup N_G of order $7680 = 64 \cdot 120$

of $U(H)$ which contains the subgroup G as well as multiplication by i, and the quotient $N_G/(G,i)$ is isomorphic to the symmetric group S_5.

The matrices N_5, N_{01} and N_{45} generate the group $U(H)$:

$$U(H) = \langle N_5, N_{01}, N_{45} \rangle.$$

One has $N_{45} T_j N_{45}^{-1} = -i T_j T_4$ for $j = 0,1,2,3$ and $N_{45} T_4 N_{45}^{-1} = T_4$. The isomorphism of $U(H)/(G,i)$ with the symmetric group S_6 can be seen from the action of $U(H)$ on the alternating forms on $\mathbb{C}^4$. Let

$$E_5 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_4 := N_{45} E_5 N_{45}^t = \begin{pmatrix} 0 & 1-i & -i & 1+i \\ -1+i & 0 & 0 & i \\ i & 0 & 0 & 1+i \\ -1-i & -i & -1-i & 0 \end{pmatrix}.$$

Then we have $g E_5 g^t = E_5$ for all $g \in G$ and the $U(H)$–orbit of E_5 consists of $6 = \sharp U(H)/\sharp(G,i)$ matrices, up to sign, one of which is E_4. One has $T_j E_4 T_j^t = -E_4$ unless $j = 4$ in which case one finds $T_4 E_4 T_4^t = E_4$.

The group $U(H)$ permutes the 16 (G,i)–invariant divisors and there are at least two orbits, since such a divisor may or may not contain $0 \in E^4$. Each of the ten G–invariant divisors which contain $0 \in E^4$ has a node in 0 and thus has a tangent cone which is given by a quadratic form on $T_0 E^4 = \mathbb{C}^4$. These ten quadratic forms are fixed, up to a scalar multiple, by G and they are permuted, up to a scalar multiple, by $U(H)$. We define two symmetric matrices

$$q_{012} := \begin{pmatrix} 2 & 0 & 1 & 1-i \\ 0 & 2i & 1+i & -1 \\ 1 & 1+i & 2 & 0 \\ 1-i & -1 & 0 & -2i \end{pmatrix} \quad \text{and} \quad q_{013} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2i & 1+i & -1 \\ 1 & 1+i & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

One has $T_j q_{012} T_j^t = \epsilon_j q_{012}$ with $\epsilon_j = +1$ for $j = 0,1,2$ and -1 for $j = 3,4$. Thus q_{012} is a common eigenvector for the group G of the space of 4×4 symmetric matrices on $\mathbb{C}^4$. The other nine eigenvectors can be obtained as $N_5^j q_{012} (N_5^j)^t$ and $N_5^j q_{013} (N_5^j)^t$ where $j = 0, 1, \ldots, 4$. These two N_5–orbits form one $U(H)$–orbit, since $N_{01} N_5 q_{012} (N_{01} N_5)^t = N_5^2 q_{013} (N_5^2)^t$.

Thus the corresponding quadrics are the ten tangent cones to the G–invariant divisors passing through $0 \in E^4$.

The subgroup $U(H)_{012}$ of $U(H)$ which fixes q_{012} up to a scalar multiple has index 10 in $U(H)$. We checked that it can be generated by three elements:

$$U(H)_{012} = \langle N_{01}, N_{45}, N_f \rangle \quad \text{where } N_f := \begin{pmatrix} i & 0 & 0 & 0 \\ 1-i & i & 0 & -2 \\ -i & 0 & i & -1+i \\ 1+i & 0 & 0 & -i \end{pmatrix}.$$

Proposition 4.5 *Let $D := D_{012}$ be the G–invariant divisor with $0 \in D$ and with tangent cone defined by q_{012}. Then the ten ODPs on D are the points in $(E^4)^G$ which are* not *in the following list of six points in $(E^4)^G$:*

$$p_1 := [1,0,0,0], \qquad p_3 := [0,0,1,0], \qquad p_1 + p_3,$$
$$p_2 := [0,1,0,0], \qquad p_4 := [0,0,0,1], \qquad p_2 + p_4,$$

where $[a_1,a_2,a_3,a_4] = \frac{1}{2}(1+i)(a_1,a_2,a_3,a_4)$ in $E^4 = (\mathbb{C}/\mathbb{Z}[i])^4$.

Proof The ten ODPs of D are two-torsion points of E^4, and 0 is one of them. Thus nine of them are nonzero and the automorphism i of E^4, which maps D into itself, permutes these nine ODPs. As $i^2 = -1$, which is the identity on the two-torsion points, at least one ODP is an i–fixed point and hence, by Lemma 4.3, it is also a fixed point of G. We checked that $U(H)_{012}$ has three orbits on the fixed points of G in E^4: they are 0, the six points listed above and the remaining nine points of $(E^4)^G$. If the orbit of six consists of ODPs, then there remain $10-1-6=3$ ODPs to be identified, but again one of these three must be an i–fixed point and then using the $U(H)$–action we get eight more ODPs, which contradicts that D has only ten ODPs. Thus the ten nodes all lie in $(E^4)^G$ and there are two $U(H)$–orbits, one of length 1 and one of length 9. $\square$

The divisors $2(D+p)$ in E^4, with $p \in E^4[2]$, are all linearly equivalent by the theorem of the square and the linear system $|2D|$ has dimension $2^4 - 1 = 15$. We will prove in Proposition 5.4 that the span of the 16 divisors $2(D+p)$ where p runs over $(E^4)^G$ is 5–dimensional. Here we give an estimate for the span that will be used to deduce this fact.

Proposition 4.6 *The 16 divisors $2(D+p)$, where p runs over $(E^4)^G$, span a subspace of dimension at least 5 in $|2D|$.*

Proof We give a list of six points $q_1, \dots, q_6$ and six points $r_1, \dots, r_6$, all in $(E^4)^G$, such that $r_1 \in D + q_i$ for $i \geq 2$ but $r_1 \notin D + q_1$, and similarly such that $r_i \in D + q_j$ if $i < j$ but $r_i \notin D + q_i$, which proves the proposition. Here $D = D_{012}$ and the six points in $(E^4)^G$ which are not in $D \cap (E^4)^G$ are listed in Proposition 4.5. Notice that $r_i \in D + q_i$ if and only if $r_i - q_i \in D$. We take, with the notation from Proposition 4.5, the points q_i to be

$$0, \quad p_3, \quad p_1 + p_2 + p_4, \quad p_1, \quad p_2 + p_3, \quad p_1 + p_2 + p_3 + p_4,$$

whereas the points r_i are

$$p_4, \quad p_2 + p_3 + p_4, \quad p_1 + p_2, \quad p_1 + p_2 + p_4, \quad p_1 + p_2 + p_3, \quad p_1 + p_3 + p_4.$$

This concludes the proof. □

What emerges from these results is a configuration of 16 points, those of $(E^4)^G$, and the 16 G–invariant theta divisors, which are the $D + p$ with $p \in (E^4)^G$. Each divisor contains exactly ten of the points. A principally polarized abelian surface A also defines a $(16, 6)$–configuration (see [4, Section 10.2]), consisting of the points of $A[2] \cong (\mathbb{Z}_2)^4$ and the six points (with multiplicity one) on each of the 16 symmetric theta divisors which can be written as $\Theta_A + p$ for $p \in A[2]$ for a(ny) symmetric theta divisor Θ_A on A. In case $A = E_1 \times E_2$ is a product of two elliptic curves with the product polarization, one can take $\Theta_A = E_1 \times \{0\} + \{0\} \times E_2$. This divisor contains the six points $(q_1, 0)$ and $(0, q_2)$ with multiplicity one, where $q_i \in E_i[2] - \{0\}$ (and it contains $0 = (0, 0)$, but with multiplicity two). So if we use the basis of $(E^4)^G$ from Proposition 4.5, (with second and third coordinates permuted) we see that there is an isomorphism $(E[2]^4)^G \cong A[2]$ such that the first set of three points not on D is mapped to $E_1[2] - \{0\}$ and the second set is mapped to $E_2[2] - \{0\}$. Using translations we then find that the configuration defined by the G–invariant theta divisors of E^4 in the group $(E^4)^G$ is also a $(16, 6)$–configuration.

Corollary 4.7 *The intersection of two distinct G–invariant divisors contains exactly six points from $(E^4)^G$, and thus there are exactly two points in $(E^4)^G$ not contained in their union.*

Proof This is a well-known property of the $(16, 6)$–configuration and can also be checked by making an incidence table (as in [15, page 787, Figure 21]) of the G–invariant divisors and the points in $(E^4)^G$. □

4.4 The fixed points of G

We consider the fixed points of the action of the groups G and (G, i) on E^4. As a consequence we will describe the singular locus of E^4/G and $E^4/(G, i)$. Recall from [11, Section 6.B] that the action of G on E^4 has the following sets of points where the isotropy group is not trivial:

- 16 fixed points with isotropy G,
- 240 points with isotropy $\mathbb{Z}_2 \oplus \mathbb{Z}_2$,
- 40 surfaces with isotropy $\mathbb{Z}_2$.

The $16 + 240 = 256 = 2^8$ isolated points with nontrivial stabilizer are exactly the two-torsion points in E^4. Let us now describe the fixed surfaces more precisely. The five generators T_i of G and also the $T_{i+5} := -T_i$ for $i = 0, \dots, 4$ are symplectic reflections, ie they have exactly two eigenvalues different from 1. When acting on E^4, each of these ten symplectic reflections T_i fixes four disjoint surfaces isomorphic to $E \times E$. Denote them by $K_1^i, \dots, K_4^i$ for $i = 0, \dots, 9$.

Since T_i and $-T_i$ are conjugate in G, after reordering $K_1^i, \dots, K_4^i$ we may assume that K_j^i and K_j^{i+5} have the same image in E^4/G for $j = 0, \dots, 4$. Let us fix one such pair of surfaces K and K' fixed by symplectic reflections $T, -T \in G$, respectively. The action of G restricted to K is the action of $N(T)/\langle T \rangle \simeq Q_8$, described in [11, Section 6.A]. The quotient $Z := E^2/Q_8$ has three A_1 singularities and four D_4 singularities and was studied in [11].

This means that on K there are three orbits of four points each and four points that are fixed by the action of Q_8 (so also of (G, i)).

Proposition 4.8 *The singular locus of E^4/G is made up of* 20 *singular surfaces $L_1, \dots, L_{20}$. Each of these surfaces is a* K3 *surface isomorphic to $Z = E^2/Q_8$.*

Proof A singular point of E^4/G is the image of a point of E^4 with nontrivial isotropy. We observe that the union of the surfaces K_j^i contains all such points. As we saw before, the 40 fixed surfaces K_j^i are mapped through $\eta\colon E^4 \to E^4/G$ to 20 surfaces. It can be checked in local coordinates that the image $\eta(K) = \eta(K') = L \in E^4/G$ is normal. It is easy to see that the map $K \to L \in E^4/G$ factors through Z. Looking at the orbits of G we conclude that the map $Z \to L$ is a bijection. □

Remark 4.9 We shall see in Section 6 that the surfaces $L_1, \dots, L_{20}$ are mapped through the quotient map $E^4/G \to E^4/(G, i)$ to planes contained in the singular locus. In fact, after proving that $E^4/(G, i) = Y \subset \mathbb{P}^5$ we will see that the above planes will be the 20 incident planes considered in Proposition 2.5.

Since $L_1, \dots, L_{20}$ are images of K_j^i for $i = 0, \dots, 5$ and $j = 1, \dots, 4$, we obtain a division of this set into 5 subsets of 4 planes. It follows from the description above that the configuration of their intersection points is as described in Remark 2.9. In particular, each subset contains 16 points with isotropy group G, and any two planes from different subsets meet in one of the 16 points. Note also that the remaining three surfaces fixed by $-T$ cut K in three orbits of four elements for the action of Q_8. We observe that a Q_8–orbit of four points on K is a part of a G–orbit with eight points on E^4 and, more precisely, given a four element orbit on K there is a four element orbit on K' forming together an eight element orbit of G.

The group (G,i) has 30 symplectic reflections, since the 20 elements $\pm i T_j T_k$ (for $0 \le j < k \le 4$) also are reflections. Each of these reflections has four fixed surfaces on E^4. We denote by $\mathcal{F}$ the set of the $30 \cdot 4 = 120$ fixed surfaces in E^4 of the 30 symplectic involutions in (G,i).

Lemma 4.10 *The fourfold $E^4/(G,i)$ is singular along 60 surfaces. We denote by $\mathcal{S}$ the set of these surfaces. The surfaces from $\mathcal{S}$ are the images of the 120 surfaces in $\mathcal{F}$.*

Proof We know that $E^4/(G,i)$ is singular along the image of points with nongeneric isotropy; these are the surfaces from $\mathcal{F}$. (Note that all points whose isotropy group contains an element, which is not a symplectic reflection, is already a 2–torsion point.) A symplectic involution $T \in (G,i)$ is conjugate to $-T$, and hence the four surfaces fixed by T and the four surfaces fixed by $-T$ map to the same four surfaces in $E^4/(G,i)$. It follows that $E^4/(G,i)$ is singular along the 60 surfaces which are the images of the surfaces from $\mathcal{F}$. □

Lemma 4.11 *The ramification locus of the map $E^4/G \to E^4/(G,i)$ consists of 40 surfaces. They are contained in the set $\mathcal{S}$ of 60 singular surfaces in $E^4/(G,i)$ and are characterized by the fact that they are not the images of the singular surfaces from E^4/G.*

Proof We saw in Proposition 4.8 that the images of 40 surfaces from $\mathcal{F}$ map to the singular locus of E^4/G. Since we can write these 40 surfaces explicitly, it is easy to check that the actions of G and (G,i) are different on them. It follows that the 20 singular surfaces of E^4/G cannot be in the branch locus of $E^4/G \to E^4/(G,i)$. The remaining 40 singular surfaces of E^4/G are actually fixed by an involution from (G,i). □

Lemma 4.12 *Each surface from $\mathcal{S}$ contains four points, which are the images of the points from $(E^4)^G$ on $E^4/(G,i)$. Each of the divisors $D+p \subset E^4$ for $p \in (E^4)^G$ contains 15 sets of four points such that each such set of four points is contained in two of the fixed surfaces from $\mathcal{F}$.*

Proof From the proof of Proposition 4.5 we know which points from $(E^4)^G$ are contained in $D+p$. Now it is a straightforward verification with Magma. □

5 The morphism $E^4/G \to Y' \subset \mathbb{P}^5$

We find a line bundle on E^4/G which gives a two-to-one morphism to a sextic hypersurface $Y' \subset \mathbb{P}^5$. Then, in Section 6 we will show that $Y' = Y$ and give the proof of Theorem 1.3.

5.1 The linear system $|\Delta|$ on E^4/G

Let $D \subset E^4$ be the G–invariant theta divisor as in Proposition 4.5 and $L = \mathcal{O}_{E^4}(D)$. In this section we show that the image of the G–invariant theta divisor D is not a Cartier divisor in E^4/G, but twice the image of D defines a Cartier divisor Δ on E^4/G. As $-1 \in G$, diagonal multiplication by i on E^4 induces an involution on E^4/G. The group $\mathrm{Aut}(E^4, H) = U(H)$ induces an action of the symmetric group on Σ_6 on $|\Delta|$ and this allows us to show that Δ gives a morphism of degree 2 that induces an isomorphism of $E^4/(G,i)$ with a sextic $Y' \subset \mathbb{P}^5$; see Proposition 5.11.

By the Riemann–Roch theorem for abelian varieties, $h^0(2D) = 16$, and by the theorem of the square, $2(D+p) \in |2D|$ for all $p \in (E^4)^G$ since $2p = 0$.

Consider the morphism λ given by the global sections of the invertible sheaf L^2. Since L^2 is not a product polarization and is symmetric of type $(2,2,2,2)$, we infer that λ is a two-to-one morphism equal to the quotient morphism $E^4 \to E^4/(-1)$ and that $E^4/(-1) \subset \mathbb{P}H^0(L^2) = \mathbb{P}^{15}$.

Lemma 5.1 *Assume that $G \subset \mathrm{GL}(n, \mathbb{Z}[i])$ is any finite group containing $-I$. Then the quotient E^n/G has trivial fundamental group.*

Proof The method which we use to compute the fundamental group of this quotient is well known and often applied in mathematical physics articles, which usually mention [7] as the main reference for this topic. It boils down to checking which elements of a certain extension of the action of G to the action of $\Lambda_E^n \rtimes G$ on $\mathbb{C}^n$, where Λ_E is a lattice such that $\mathbb{C}/\Lambda_E \simeq E$, have fixed points. In particular, if there is an $A \in G$ such that $I - A$ has maximal rank, eg $A = -I$, then there are so many elements with fixed points that the fundamental group must be trivial. □

Corollary 5.2 *The symplectic desingularization X_0 of E^4/G is simply connected: $\pi_1(X_0) = 0$.*

Proof By Lemma 5.1, $\pi_1(E^4/G)$ is trivial, and by [18, Theorem 7.8(a)] the resolution does not change the fundamental group. □

In this section, η is the quotient map $\eta\colon E^4 \to E^4/G$. Recall that, by [11, Corollary 6.4], the symplectic resolution X_0 of E^4/G has $b_2(X_0) = 23$.

Proposition 5.3 *The Picard group of E^4/G has rank one and has no torsion. The divisor $\eta(D)$ (with the reduced structure) is Weil but not Cartier whereas $\Delta = 2\eta(D)$ is Cartier, ample and generates the Picard group of E^4/G. Moreover, $\Delta^4 = 12$ and $h^0(\Delta) = 6$.*

Proof Since $\pi_1(X_0) = 0$, then also $H^1(X_0, \mathcal{O}_{X_0}) = 0$ and, from the exponential sequence, $\mathrm{Pic}(X_0) \subseteq H^2(X_0, \mathbb{Z})$. Then from the universal coefficient theorem we get an exact sequence

$$0 \to \mathrm{Ext}(H_1(X_0, \mathbb{Z}), \mathbb{Z}) \to H^2(X_0, \mathbb{Z}) \to \mathrm{Hom}(H_2(X_0, \mathbb{Z}), \mathbb{Z}) \to 0.$$

The triviality of $\pi_1(X_0) = 0$ implies that the first term of this sequence is 0. Thus $\mathrm{Pic}(X_0) \subseteq \mathrm{Hom}(H_2(X_0, \mathbb{Z}), \mathbb{Z})$, which is torsion-free.

It follows from [11, Proposition 6.2] that the symplectic resolution $X_0 \to E^4/G$ contracts at least 20 independent divisors on X_0. On the other hand, since $b_2(X_0) = 23$ and $h^{2,0}(X_0) = 1$, the Picard rank of X_0 is at most 21. Thus the Picard group of E^4/G has rank at most 1.

We claim that E^4/G is 2–factorial. It is enough to prove this locally. Since E^4/G has only quotient singularities we see that the only nonfactorial singularities it admits are isomorphic to the quotient singularity of $\mathbb{C}^4/G$. In [11, Lemma 2.10] it was shown that $\mathrm{Cl}(\mathbb{C}^4/G) = \mathrm{Ab}(G) = (\mathbb{Z}_2)^4$, and hence the claim follows.

It follows that $2\eta(D)$ is a Cartier divisor, necessarily ample from the Nakai–Moishezon criterion. The pull-back of Δ on E^4 is a divisor from the system $|2D|$, so we can compute the self-intersection of $(2\eta(D))^4$. Consider the pull-back $\overline{\Delta}$ of Δ by the map $X_0 \to E^4/G$. Since Δ is ample we infer that $\overline{\Delta}$ is big and nef. Observe that the proof of [17, Proposition 2.1] can be adapted for big and nef divisors. It follows that $\Delta^4 = 12k^2$ for some $k \in \mathbb{Z}$. As $(2D)^4 = 2^4(4!) = 2^5 \cdot 12$ and $\sharp G = 2^5$ it follows that $\Delta^4 = 12$. By [11, Proposition 6.2], $b_2(X_0) = 23$, and thus we can repeat the arguments from [17, Proposition 2.1] to show that $h^0(\Delta) = h^0(\overline{\Delta}) = 6$.

Since the self-intersection of a big and nef divisor should be a multiple of 12, we infer that Δ is the generator of the Picard group. In order to see that $\eta(D)$ is not Cartier, it is enough to observe that otherwise the pull-back of $\eta(D)$ on the IHS fourfold X_0 would be a big and nef divisor with self-intersection $\frac{12}{16}$. □

To understand the map from E^4/G to $\mathbb{P}^5$ provided by the linear system $|\Delta|$, where Δ is the ample generator of the Picard group of E^4/G, we study the pull-back of $|\Delta|$ to E^4.

Proposition 5.4 *The linear system $|\Delta|$ on E^4/G is generated by the 16 divisors $2\eta(D+p)$ for $p \in (E^4)^G$.*

Proof As in the proof of Proposition 5.3, these divisors $2\eta(D+p)$ are Cartier and are invariant with respect to G. Hence they have self-intersection $(2D)^4/\sharp G = 12$. Since we know that the Picard group of E^4/G is $\mathbb{Z}\Delta$, with $\Delta^4 = 12$, we conclude that the divisors $2\eta(D+p)$ with $p \in (E^4)^G$ are in the linear system $|\Delta|$.

Notice that $\langle 2\eta(D+p) : p \in (E^4)^G \rangle \subset |\Delta| \cong \mathbb{P}^5$ and that the pull-back map η^* maps $\langle 2\eta(D+p) : p \in (E^4)^G \rangle$ linearly into the subsystem $\langle 2(D+p) : p \in (E^4)^G \rangle \subset |2D|$ on E^4. Proposition 4.6 asserts that this subsystem of $|2D|$ has dimension at least 5, hence also $\dim\langle 2\eta(D+p) : p \in (E^4)^G \rangle = 5 = \dim|\Delta|$, which concludes the proof. □

Denote by $f\colon E^4/G \to \mathbb{P}^5$ the map given by $|\Delta|$.

Proposition 5.5 *The linear system $|\Delta|$ on E^4/G is base-point-free and the morphism $f\colon E^4/G \to \mathbb{P}^5$ defined by $|\Delta|$ factors through the quotient map $h\colon E^4/G \to E^4/(G,i)$.*

Proof Using Proposition 5.4, we have to show that the subsystem

$$\langle 2(\eta(D+p)) : p \in (E^4)^G \rangle$$

is base-point-free on E^4/G. Since the divisors $D+p$ for $p \in (E^4)^G$ are G–invariant, it is enough to prove that the linear subsystem

$$\langle 2(D+p) : p \in (E^4)^G \rangle \subset |2D|$$

is base-point-free on E^4.

We identify $H^0(2D) := H^0\big(E^4, \mathcal{O}_{E^4}(2D)\big)$ with the vector space of rational functions on E^4 with poles of order at most two along D. As $i(D) = D$, we have an endomorphism i^* of $H^0(2D) = \{f \in \mathbb{C}(E^4) : (f) + 2D > 0\}$:

$$i^*\colon H^0(2D) \to H^0(2D) \quad \text{given by } f \mapsto f \circ i.$$

As $i^2 = -1$ and all functions f in $H^0(2D)$ are even (so $f(-x) = f(x)$ in particular), the map i^* is an involution. Let $H^0(2D) = H^0_+ \oplus H^0_-$ be the eigenspace decomposition into the even and the odd part.

Since D is effective, we have the constant function $1 \in H^0_+$. Let $p \in (E^4)^G$ such that $p \neq 0$. Then $2(D+p) \in |2D|$, so there is a rational function $f_p \in H^0(2D)$ such that $(f_p) = 2(D+p) - 2D$. As $i(D+p) = D+p$, we have $i^* f_p = \pm f_p$ for some choice of sign. From Corollary 4.7 we infer that there is a $q \in (E^4)^G$ which is not on D and also not on D_p. Then f_p has no pole and is not zero in q, so

$f_p(q) \in \mathbb{C} - \{0\}$. As $i(q) = q$, it follows that $(i^* f_p)(q) = f_p(i(q)) = f_p(q)$, and hence $i^* f_p = f_p$. Therefore $f_p \in H^0_+$, for all $p \in (E^4)^G$, and the map defined by $|\Delta|$ factors through $E^4/(G, i)$.

Next, we show that H^0_+ is spanned by these f_p. For each $p \in (E^4)^G$ there is a G–invariant divisor $D + q$ such that $p \notin D + q$. This implies that the fibers $\mathcal{O}(2D)_p := \mathcal{O}(2D)/\mathfrak{m}_p\mathcal{O}(2D)$ are generated by global sections in H^0_+ and one finds that the lift i^* of i to $\mathcal{O}(2D)$ we consider acts as $+1$ on the fibers over the fixed points $(E^4)^i = (E^4)^G$ of i. The Atiyah–Bott Lefschetz formula [1, Theorem 4.12] states

$$\sum_{j=0}^{4} \operatorname{Tr}\big(i^* | H^j(E^4, \mathcal{O}(2D))\big) = \sum_{p \in (E^4)^i} \frac{\operatorname{Tr}\big(i_p^* \colon \mathcal{O}(2D)_p \to \mathcal{O}(2D)_p\big)}{\det(I - (\mathrm{d}i)_p)}.$$

Since $H^j(\mathcal{O}(2D)) = 0$ for $j > 0$ and since, for all p, we have $\operatorname{Tr}(i_p^*) = +1$ and $\det(I - (\mathrm{d}i)_p) = (1 - i)^4 = -4$, we find that $\dim H^0_+ - \dim H^0_- = -4$, and hence $\dim H^0_+ = 6$ and $\dim H^0_- = 10$. Thus

$$\mathbb{P}(H^0_+) = \langle 2(D + p) : p \in (E^4)^G \rangle.$$

By Wirtinger duality, the map λ defined by $|2D|$ can be identified with the map

$$\lambda' \colon E^4 \ni x \longmapsto (D + x) + (D - x) \in \mathbb{P}\big(H^0(E^4, \mathcal{O}_{E^4}(2D))\big).$$

In particular, $x \in E^4$ is a base point of the map defined by $\mathbb{P}(H^0_+)$ exactly when $(D + x) + (D - x)$ is the divisor of a global section f_x with $f_x \in H^0_-$. But then $i^* f_x = -f_x$ so $i^*(D + x) + i^*(D - x) = (D + x) + (D - x)$ and thus $x = \pm i(x)$, so $x = i(x)$ or $x = i^3(x)$, and hence $x \in (E^4)^i = (E^4)^G$. But we already found that $f_x \in H^0_+$ for such a fixed point. We conclude there are no base points for the system $\mathbb{P}H^0_+$ and thus $|\Delta|$ also is base-point-free. $\square$

Remark 5.6 We have the commutative diagram

$$\begin{tikzcd} E^4 \arrow[r, "\eta"] \arrow[d, "|2D|"'] & E^4/G \arrow[r, "h"] \arrow[d, "|\Delta|"'] & E^4/(G, i) \arrow[d, "m"'] \\ \mathbb{P}^{15} \arrow[r, "\theta"] & \mathbb{P}^5 \arrow[r, "="] & \mathbb{P}^5 \end{tikzcd}$$

where θ is the linear projection from the i^*–eigenspace $\mathbb{P}H^0_-$.

The action of $U(H)$ on E^4, which fixes the polarization and which commutes with the automorphism i of E^4, induces an action of $U(H)$ on the projective space $\mathbb{P}H^0_+ \cong |\Delta|$. As the subgroup (G, i) of $U(H)$ acts trivially on this space, we get an induced action of the symmetric group $\Sigma_6 = U(H)/(G, i)$ on $\mathbb{P}^5$.

5.2 The tangent cone

First we study the image of the tangent space to E^4 at the neutral element $0 \in E^4$ under the quotient map $h \circ \eta\colon E^4 \to E^4/(G,i) =: Z'$. Since $h \circ \eta$ is a quotient map and the image point $p \in Z'$ of 0 is singular on Z', the image of $T_0 E^4$ via the induced quotient is the tangent cone $C_p Z'$ to Z' at p.

Proposition 5.7 *The tangent cone to the image of 0 to $Z' = E^4/(G,i)$ is isomorphic to the cone over the Igusa quartic $\mathcal{I}_4$ in $\mathbb{P}^4$. It is defined by the two equations*

$$y_1 + \cdots + y_6 = 0 \quad \textit{and} \quad \mathrm{Ig}_y := (y_1^2 + \cdots + y_6^2)^2 - 4(y_1^4 + \cdots + y_6^4) = 0$$

in $\mathbb{C}^6$.

The images of the tangent cones in 0 of the ten (G,i)–invariant divisors with first Chern class H and passing through 0 are the ten hyperplane sections defined by

$$y_a + y_b + y_c = y_d + y_e + y_f = 0 \quad \textit{where } \{a,b,c,d,e,f\} = \{1,2,\dots,6\}.$$

The group $U(H)$ acts on $C_p Z'$ through its quotient $U(H)/(G,i) \cong \Sigma_6$ as the standard representation, ie it simply permutes the six variables $y_1,\dots,y_6$.

Proof Note that the tangent cone to $Z' = E^4/(G,i)$ is the spectrum of the ring of (G,i)–invariants on $T_0 E^4$. Choosing a suitable basis of $T_0 E^4 \cong \mathbb{C}^4$, the action of G on $T_0 E$ is given by the simpler matrices from [11, Section 2.C]. The ring of invariants is generated by the following five polynomials in four variables:

$$\begin{aligned} &p_0 := t_0^4 + t_1^4 + t_2^4 + t_3^4, \quad p_1 := 2(t_0^2 t_1^2 + t_2^2 t_3^2), \quad p_2 := 2(t_0^2 t_2^2 + t_1^2 t_3^2), \\ &p_3 := 2(t_0^2 t_3^2 + t_1^2 t_2^2), \qquad p_4 := 4 t_0 t_1 t_2 t_3. \end{aligned}$$

So $\mathbb{C}[t_0,\dots,t_3]^{(G,i)} \cong \mathbb{C}[p_0,\dots,p_4]$. Introducing the polynomial ring in five variables $\mathbb{C}[P_0,\dots,P_4]$, we have that the homomorphism $\mathbb{C}[P_0,\dots,P_4] \to \mathbb{C}[t_0,\dots,t_3]^{(G,i)}$ which sends $P_i \mapsto p_i$ is surjective, and its kernel is generated by the quartic polynomial

$$\mathrm{Ig}_P := P_1^2 P_2^2 + P_1^2 P_3^2 + P_2^2 P_3^2 + (-P_0^2 + P_1^2 + P_2^2 + P_3^2 - P_4^2) P_4^2 - 2 P_0 P_1 P_2 P_3.$$

Hence $C_p Z' \cong \mathrm{Spec}(\mathbb{C}[P_0,\dots,P_4]/(\mathrm{Ig}_4))$; more concretely, $C_p Z'$ is isomorphic to the image of $\mathbb{C}^4$ in $\mathbb{C}^5$ by the map defined by the five p_i.

The zero locus of the quartic polynomial Ig_P in $\mathbb{P}^4$ is known as the Igusa quartic $\mathcal{I}_4$. The group $U(H)/(G,i) \cong \Sigma_6$ acts on $\mathcal{I}_4$. To make this action visible, we define the following six linear combinations of the p_i:

$$\begin{aligned} y_1 &:= p_0 + 3p_4, & y_2 &:= p_0 - 3p_4, \\ y_3 &:= \tfrac{1}{2}(-p_0 + 3p_1 + 3p_2 + 3p_3), & y_4 &:= \tfrac{1}{2}(-p_0 + 3p_1 - 3p_2 - 3p_3), \\ y_5 &:= \tfrac{1}{2}(-p_0 - 3p_1 + 3p_2 - 3p_3), & y_6 &:= \tfrac{1}{2}(-p_0 - 3p_1 - 3p_2 + 3p_3). \end{aligned}$$

One easily verifies that

$$y_1 + \cdots + y_6 = 0.$$

Let

$$\mathrm{Ig}_y := (y_1^2 + \cdots + y_6^2)^2 - 4(y_1^4 + \cdots + y_6^4).$$

Then, after replacing p_j by P_j in the definition of the y_i, the polynomial Ig_y becomes the polynomial Ig_P up to a scalar multiple. Thus Σ_6 acts on $\mathcal{I}_4$ by permuting the coordinates y_j (and explicit computations show that this is indeed the action induced by $U(H)$).

There are ten (G,i)–invariant divisors which contain 0, and 0 is a node on these divisors. The tangent cone to such a divisor is then a (G,i)–invariant quadratic hypersurface in T_0E^4. There are exactly ten such hypersurfaces and they are permuted by the action of $U(H)$; explicit equations can be found in [11, (3.13)]. It is easy to check that the image of such a quadric in $\mathbb{C}^5$ is the intersection of $\mathcal{I}_4$ with one of the following hyperplanes:

$$y_a + y_b + y_c = y_d + y_e + y_f = 0 \quad \text{where } \{a,b,c,d,e,f\} = \{1,2\ldots,6\}.$$

As $\sum y_i = 0$, the equation $y_a + y_b + y_c = 0$ implies the equation $y_d + y_e + y_f = 0$. For example, $y_0 + y_1 + y_2 = 0$ has preimage defined by $p_0 + p_1 + p_2 + p_3 = 0$ and this is $(t_0^2 + t_1^2 + t_2^2 + t_3^2)^2$, which is (up to a scalar multiple and replacing t_j by x_j) the quadratic form ϕ_{14} in [11, (3.13)]. The fact that we find the squares of the quadratic forms corresponds to the fact that these ten hyperplanes are tangent to the Igusa quartic. □

Recall that Y' is the image of the map $f\colon E^4/G \to \mathbb{P}^5$ defined by the system $|\Delta|$.

Corollary 5.8 *The tangent cone to Y' at the image of 0 is a cone over the Igusa quartic I_4.*

Proof If $D + p$ is a (G,i)–invariant divisor passing through 0 on E^4, then we have $2(D + p) \in |2D|$ by the theorem of the square and this divisor lies in the pull-back by the quotient map of the 5–dimensional linear system defining the map $f\colon E^4/G \to \mathbb{P}^5$. It is easy to check that the squares of the ten quadratic forms defining the tangent cones to these divisors span the space spanned by the generators $p_0, \dots, p_4$ of the (G,i)–invariants on T_0E^4. Hence the tangent cone to $Z' = E^4/(G,i)$ in the image of 0 is embedded in $\mathbb{P}^5$ by the map $m\colon E/(G,i) \to \mathbb{P}^5$ induced by f (see Proposition 5.5), ie the tangent cone to Y' is a cone over the Igusa quartic I_4. □

We infer also that the considered action of Σ_6 on $\mathbb{P}^5$ that preserves Y' is given by the permutation of coordinates:

Corollary 5.9 *The action of the group $\Sigma_6 = U(H)/(G,i)$ on $\mathbb{P}^5 = \mathbb{P}(|\Delta|)$ induced by the action of $U(H)$ on E^4 (see Remark 5.6) is the action by permutation of coordinates.*

Proof We use the fact that C_pZ' spans $\mathbb{C}^5$, that $\mathbb{C}^5$ is dense in $\mathbb{P}^5$ and the description from Proposition 5.7 of the action of Σ_6 on C_pZ'. □

Remark 5.10 One can use the action of Σ_6 to provide the following approach to the proof that $Y' = Y$ that is a possible alternative to the proof we give in Section 6. From the explicit description of the generators of $U(H)$ and of the 16 G–fixed points in E^4, one finds that $(E^4)^G$ consists of two $U(H)$–orbits, one is $\{0\}$ and the other has 15 elements. For $q \in (E^4)^G$ such that $q \neq 0$, the line spanned by $p = f(0)$ and $f(q)$ in $\mathbb{P}^5$ intersects $\mathbb{C}^5 \subset \mathbb{P}^5$ in a linear subspace of dimension one. The 15 lines in $\mathbb{C}^5$ we obtain in this way form one Σ_6–orbit. This suffices to identify these lines, and with some additional effort, we can then find the images of the remaining 15 points in $(E^4)^G$. Using the fact that these map to singular points on Y, with quartic tangent cones, and the description of the images of the (G,i)–invariant divisors from Proposition 5.7, we would then show that $Y' = Y$ with Y as in Proposition 2.1. It is interesting to look at the above action on the Igusa quartic from the point of view of [25, page 254].

Let us now pass to the proof of our main result in this section:

Proposition 5.11 *The image Y' of the morphism $f\colon E^4/G \to \mathbb{P}^5$ is a sextic hypersurface $Y' \subset \mathbb{P}^5$. Moreover, the morphism f is the quotient map of the involution induced by i acting on E^4/G, and hence $Y' \cong E^4/(G,i)$.*

Proof We saw that Δ is ample, $\Delta^4 = 12$ and $|\Delta|$ is a base-point-free linear system. First, it follows that $\dim f(E^4/G) \subset \mathbb{P}^5 = 4$ and the degree of this image divides 12. By Proposition 4.4 the morphism $f\colon E^4/G \to \mathbb{P}^5$ factors through the quotient by the involution i. We infer a factorization of the map f:

$$f\colon E^4/G \xrightarrow{h} E^4/(G,i) \xrightarrow{m} Y' \subset \mathbb{P}^5.$$

Our aim is first to show that the image Y' of m is a sextic, and next that m is an isomorphism.

Since the tangent cone $C_p Y'$ is the image of $C_p Z'$ by the embedding induced by m it has degree four, by Proposition 5.7. It follows that the degree of $Y' \subset \mathbb{P}^5$ is higher than four (clearly $Y' \neq C_p Y'$). Since $\Delta^4 = 12$ and f has degree at least two, we infer that Y' is a sextic hypersurface in $\mathbb{P}^5$.

It follows from the adjunction formula for the birational morphism m that the sextic $Y' \subset \mathbb{P}^5$ is normal. Indeed, we have $\omega_{E^4/(G,i)} = m^*(\omega_{Y'}) - C$, where C is the conductor divisor supported on the nonnormal locus of Y'. Since Y' is a sextic, $\omega_{Y'}$ is trivial, and thus C is the zero divisor and Y' is normal. This implies also that m is an isomorphism because Δ is ample and base-point-free, so f cannot contract any curve. □

Remark 5.12 The double cover $E^4/G \to E^4/(G,i)$ is determined by the sheaf $f_*\mathcal{O}_{E^4/G} = \mathcal{O}_{E^4/(G,i)} \oplus \mathcal{G}$ such that $E^4/G = \mathrm{Spec}_{E^4/(G,i)}(\mathcal{G} \oplus \mathcal{O}_{E^4/(G,i)})$. It can be shown that $\mathcal{G}$ is a symmetric sheaf such that $\mathcal{G}(3)$ is globally generated and fits in the exact sequence

$$0 \to \Omega^3_{\mathbb{P}^5}(3) \to W \otimes \mathcal{O}_{\mathbb{P}^5} \to \mathcal{G}(3) \to 0.$$

This gives another proof (without using Section 2.3) that $Y' \subset \mathbb{P}^5$ is an EPW sextic; see [12]. Note that the sheaf $\mathcal{G}$ can be seen as a kind of Casnati–Catanese sheaf, however it has a complicated local structure around the 16 most singular points of $E^4/(G,i)$.

We now obtain some further information on the geometry of Y' which will be used in the next section to prove that $Y' = Y$.

Corollary 5.13 *The sextic $Y' \subset \mathbb{P}^5$ is singular along 60 planes.*

Proof It follows from Lemma 4.10 that the fourfold $E^4/(G,i)$ is singular along 60 surfaces.

Note that all the 60 singular surfaces of $E^4/(G,i)$ are isomorphic to each other (since the group generated by $(E^4)^G$ (acting by translations) and $U(H)$ acts transitively on the corresponding 120 surfaces in E^4). On the other hand it is known that a surface section of $Y' \subset \mathbb{P}^5$ admits no more than 65 nodes (recall that $E^4/(G,i)$ has transversal A_1 singularities along generic points on the singular surfaces). It follows that all the singular surfaces of $Y' \subset \mathbb{P}^5$ are planes. □

Recall that there exists a unique, up to projective isomorphism, normal cubic hypersurface in $\mathbb{P}^4$ with ten isolated singularities (that have to be ordinary double points). It is called the Segre cubic. See [8] for many beautiful classical facts about this threefold. The Segre cubic in $\mathbb{P}^5(x_0,\dots,x_5)$ can be defined by the following equations:

$$x_0+\dots+x_5=0, \qquad x_0^3+\dots+x_5^3=0. \tag{5-1}$$

The action of the permutation group Σ_6 that permutes the variables on $\mathbb{P}^5$ preserves the above cubic. It is singular at the points in the orbit of the point $(1,1,1,-1,-1,-1)$ under the action of Σ_6. This cubic contains also 15 planes in the Σ_6–orbit of the plane defined by the equations $x_0+x_1=0$, $x_2+x_3=0$ and $x_4+x_5=0$. There are exactly 15 hyperplanes cutting the cubic along the sum of three planes; these hyperplanes are defined by the equations $x_i+x_j=0$ for $0\le i<j\le 5$.

Corollary 5.14 *The sextic $Y'\subset\mathbb{P}^5$ is tangent to 16 hyperplanes along Segre cubics such that the singular points of each cubic are the points from the set $f(\eta((E^4)^G))=\mathcal{R}$. The sextic $Y'\subset\mathbb{P}^5$ is singular along the 15 planes contained in each of these cubics. Moreover, the intersection of $Y'\subset\mathbb{P}^5$ with two of the hyperplanes defined by the divisors $D+p$ for $p\in(E^4)^G-0$ is a union of three planes. Each singular plane in $Y'\subset\mathbb{P}^5$ is contained in four tangent hyperplanes.*

Proof We shall show that each divisor $D+p$ for $p\in(E^4)^G$ maps to a cubic that is singular at ten isolated points. It is known that only the Segre cubic has this property.

It is enough to give a proof of our statements for D. From Proposition 5.3 we infer that the image of the divisor D in $E^4/(G,i)\subset\mathbb{P}^5$ is contained in a hyperplane $K\subset\mathbb{P}^5$ that is tangent to the sextic.

The 120 fixed surfaces from $\mathcal{F}$ map to 60 singular surfaces in $E^4/(G,i)$, and hence they are the singular planes on $Y'\subset\mathbb{P}^5$. Since D contains exactly ten of the 16 points in $(E^4)^G$, its image $K\cap Y$ contains ten of the 16 points from $\mathcal{R}$. Each fixed surface contains four fixed points from the set $\mathcal{R}$ and we claim that the images of these points on $Y'\subset\mathbb{P}^5$ are noncollinear (so a singular plane on Y' is spanned by the four points of $\mathcal{R}$ contained in it).

In fact, suppose that the images of the four points of $\mathcal{R}$ in a plane are collinear. Choosing points p and q from these four points, we can find another fixed surface from the set $\mathcal{F}$ containing only these two of the four points. The corresponding planes cut along a line spanned by the images of p and q. Now this line has to contain the remaining two points. This is a contradiction with our choice of the other fixed surface, so the claim follows.

It follows that the 30 surfaces considered in Lemma 4.12, containing four points from the ten contained in $\mathcal{R}_0 = K \cap \mathcal{R}$, map to 15 planes, each spanned by the images of those points; hence these 30 surfaces are contained in K.

As D is (G, i)–invariant, we see that the reflections generating this group act on D in such a way that they fix 30 surfaces. It follows that the image of D can only be singular at isolated points, ie at the images of singular points of D. Since through each point in $(E^4)^G$ there are two fixed surfaces from $\mathcal{F}$ contained in D intersecting only at this point, these surfaces map to two planes in K intersecting only at a point from $\mathcal{R}_0$, so this point must be singular on the cubic threefold. We deduce that the cubic is (only) singular at the ten points in $\mathcal{R}_0$ and hence it must be the Segre cubic.

It is known that there are exactly 15 planes contained in the Segre cubic. On the other hand, by Lemma 4.12, the 30 surfaces from $\mathcal{F}$ that are contained in D map to 15 planes contained in the cubic; hence the planes in the cubic are the images of these surfaces.

The intersection of two (G, i)–invariant divisors maps to the intersection of two tangent hyperplanes to $Y' \subset \mathbb{P}^5$. In particular, it maps to the intersection of a Segre cubic threefold with a hyperplane, and hence it consists of at most three planes. Thus two (G, i)–invariant divisors cut each other along at most six surfaces from $\mathcal{F}$. On the other hand, given two such divisors, we easily find, using Proposition 4.5, three sets of four points contained in a given surface from $\mathcal{F}$, each one contained in both of these divisors. We show similarly that any fixed surface is contained in four (G, i)–invariant divisors. □

Remark 5.15 From the incidence of the 120 fixed surfaces from $\mathcal{F}$ we deduce that each plane in the singular locus of $Y' \subset \mathbb{P}^5$ cuts 12 of the remaining planes from this locus along six lines (such that three planes pass through one line).

6 The proof that $Y = Y'$

We proved in Corollary 5.9 that the image $Y' = f(E^4/G) \subset \mathbb{P}^5$ is invariant under the action of Σ_6 by the permutation of coordinates. Moreover, from Corollary 5.14 it is tangent to 16 hyperplanes. In this section we show that such a Σ_6–invariant sextic $Y' \subset \mathbb{P}^5$ can be easily reconstructed from the Σ_6–invariant set of 16 hyperplanes tangent to it. This allows us to show that $Y = Y'$. Then, in Section 6.2, we prove Theorem 1.3.

6.1 The equation of the sextic

We start by classifying sets of 16 hyperplanes which are invariant under the action of Σ_6, which acts by permutations of the coordinates on $\mathbb{P}^5$. Let $t \in \mathbb{C}$ and $0 \le i, j, k \le 5$ be distinct indices. We consider (families of) hyperplanes

- H defined by $x_0 + \dots + x_5 = 0$,
- H_i^t defined by $x_i + t(x_0 + \dots + x_5) = 0$,
- $H_{i,j}^t$ defined by $x_i + x_j + t(x_0 + \dots + x_5) = 0$,
- $H_{i,j,k}$ defined by $x_i + x_j + x_k - \frac{1}{2}(x_0 + \dots + x_5) = 0$.

Lemma 6.1 *There are exactly two one-parameter families of Σ_6–invariant sets of 16 hyperplanes with the following property: every such set determines 60 planes, such that each plane is contained in four of these hyperplanes (this is one of the properties of Y' from Corollary 5.14). They are*

$$\mathcal{H}_1^t = \{H\} \cup \{H_{i,j}^t : 0 \le i, j \le 5\}$$

and

$$\mathcal{H}_2^t = \{H_{i,j,k} : 0 \le i, j, k \le 5\} \cup \{H_i^t : 0 \le i \le 5\}.$$

Proof Consider the action of Σ_6 on a hyperplane with equation $a_0x_0 + \dots + a_5x_5 = 0$. If its orbit has ≤ 16 elements, then the coefficients $\{a_0, \dots, a_5\}$ can take only two different values. That is, it must be one of H, H_i^t, $H_{i,j}^t$, $H_{i,j,k}$ for some i, j, k and t. The lengths of Σ_6–orbits of hyperplanes of these types are, respectively, 1, 6, 15 and 10. Thus, to obtain an invariant set of cardinality 16, we have two possibilities: to take the union of the orbits of 1 and 15 elements or of the orbits of 6 and 10 elements.

In both cases it is easy to check that for any $t \in \mathbb{C}$ there are the required sets of 60 planes. They are intersections of the following sets of four hyperplanes (the indices for each set are different):

(1a) H_{i_1,i_2}^t, H_{i_2,i_3}^t, H_{i_3,i_4}^t, H_{i_4,i_1}^t (there are 45 planes of this type),

(1b) H, H_{i_1,j_j}^t, H_{i_2,j_2}^t, H_{i_3,j_3}^t (there are 15 such planes);

(2a) H_{i,j_1,j_2}, H_{i,j_2,j_3}, H_{i,j_3,j_4}, H_{i,j_4,j_0} (there are 15 such planes),

(2b) H_{i,j,k_1}, H_{i,j,k_2}, $H_{k_1}^t$, $H_{k_2}^t$ (there are 45 such planes). □

We want to find all polynomials $f \in \mathbb{C}[x_0, \dots, x_5]$ such that the corresponding sextic hypersurface Y_f satisfies the following:

- Y_f is invariant with respect to the Σ_6–action by permutations of coordinates.
- The 16 hyperplanes from the configuration $\mathcal{H}_1^t$ or $\mathcal{H}_2^t$, for some $t \in \mathbb{C}$, are tangent to Y_f along cubics.
- If the intersection of four of these 16 hyperplanes is a plane, then this plane is contained in the singular locus of Y_f.

The fact that Y_f is Σ_6–invariant means that f is symmetric or antisymmetric under the action of Σ_6. In any case, f is invariant with respect to the alternating group. But the invariants of the alternating group are generated by those of the symmetric group and the polynomial $\prod_{i<j}(x_i - x_j)$, which has degree 15. Hence f must be a symmetric polynomial and thus f is a linear combination of the symmetric polynomials $P_{j_1\ldots j_k} = \sum_{i_1,\ldots,i_k} x_{i_1}^{j_1} \cdots x_{i_k}^{j_k}$ such that $i_1, \ldots, i_k \in \{0, \ldots, 5\}$ are pairwise different and $j_1 \geq \cdots \geq j_k$ with $j_1 + \cdots + j_k = 6$. Thus f is an element of an 11–dimensional vector space of polynomials. Now we shall determine all possible sets of coefficients in

$$f(x_0, \ldots, x_5) = \sum_{j_1+\cdots+j_k=6} a_{j_1\ldots j_k} P_{j_1\ldots j_k}(x_0, \ldots, x_5).$$

The following lemma, which is easy to verify, shows that in all but some exceptional cases it suffices to consider the case $t = -\frac{1}{2}$.

Lemma 6.2 *We define linear maps on $\mathbb{C}^6$ which commute with the Σ_6–action by*

$$N_1^t := -(t+1)M_6 + (6t+2)\,\mathrm{Id}_6 \quad \text{and} \quad N_2^t := -(t+1)M_6 + (6t+1)\,\mathrm{Id}_6,$$

where M_6 is the 6×6 matrix with all entries equal to 1 and Id_6 is the 6×6 identity matrix.

Then N_i^t induces an isomorphism on $\mathbb{P}^5$ that maps $\mathcal{H}_i^t$ to $\mathcal{H}_i^{-1}$ for $i = 1, 2$, except when $i = 1$, $t = -\frac{1}{3}$ or $i = 2$, $t = -\frac{1}{6}$.

We consider the restrictions on the coefficients $a_{j_1\ldots j_k}$ coming from the assumption that f and all its partial derivatives vanish along one plane of type (1a) and one of type (1b) or, respectively, one of type (2a) and one of type (2b). From the symmetry of f it follows that Y_f is then also singular along all 60 planes. To find all sequences $(a_{j_1\ldots j_k})$ satisfying these conditions, one just has to compute the kernel of the matrix whose entries are $P_{j_1\ldots j_k}$ and its partial derivatives restricted to the two chosen planes. We obtain a unique solution, up to a scalar multiple. Notice that in the second case we did find the polynomial defining Y from Proposition 2.1:

Proposition 6.3 *For the first case, there is a unique sextic*

$$Y^\vee = Y_{F_6^\vee}: \quad F_6^\vee = P_6 - P_{42} + 2P_{222} + 16P_{111111}$$

which satisfies the conditions for the 16 *hyperplanes* $\mathcal{H}_1^t$ *with* $t = -\frac{1}{2}$.

For the second case, there is a unique sextic

$$Y = Y_{F_6}: \quad F_6 = P_6 - P_{42} + 2P_{222} - 16P_{111111}$$

which satisfies the conditions for the 16 *hyperplanes* $\mathcal{H}_2^t$ *with* $t = -\frac{1}{2}$.

Corollary 6.4 *The hypersurfaces* $Y^\vee \subset \mathbb{P}^5$ *and* $Y \subset \mathbb{P}^5$ *are isomorphic.*

Proof The only thing to observe is that $F_6(-x_0, x_1, \dots, x_5) = F_6^\vee(x_0, x_1, \dots, x_5)$, so changing the sign of an odd number of variables interchanges the two cases. □

Corollary 6.5 *The EPW sextic hypersurfaces* $Y \subset \mathbb{P}^5$ *(the image of* $S^{[2]}$*) and* $Y' \subset \mathbb{P}^5$ *(the image of* E^4/G*) are isomorphic.*

Proof By Proposition 2.1 it is enough to prove that $Y' \subset \mathbb{P}^5$ is defined by the polynomial F_6 or $F_6^\vee$. It follows from Corollary 5.14 that the equation defining $Y' \subset \mathbb{P}^5$ satisfies the conditions satisfied by F_6 and $F_6^\vee$. We conclude by Proposition 6.3. □

Remark 6.6 For $t = -\frac{1}{3}$ the corresponding sextic is the square of a cubic which is singular in ten lines. For $t = -\frac{1}{6}$ the corresponding sextic is singular along 60 planes intersecting in one point. This gives us isotrivial degenerations of our EPW sextic Y.

It turns out that $Y^\vee$ and Y are related in one more way.

Proposition 6.7 *The sextics* $Y^\vee$ *and* Y *in* $\mathbb{P}^5$ *are projectively dual to each other.*

Proof Substituting the gradient $(\dots, \partial F_6/\partial x_i, \dots)$ of the equation defining Y in the polynomial $F_6^\vee$ defining $Y^\vee$, one finds, using eg Macaulay2 [14], the product of F_6 with another polynomial. Hence the dual of Y is $Y^\vee$. In particular, the 16 hyperplanes are mapped to the 16 points with singularity $\mathbb{C}^4/(G, i)$, and the Segre cubics in these hyperplanes are contracted to points. (Notice that the Segre cubic and the Igusa quartic are projectively dual threefolds in $\mathbb{P}^4$.) □

Remark 6.8 The images of the 16 points in $(E^4)^G$ in $Y' \cong Y$ are the points in the orbit of the point $p_0 := (1 : 1 : \dots : 1)$ under the action of the group which changes an even number of signs (this action is induced by the action of $(E^4)^G$ on E^4 by translation) and the group Σ_6 (which is induced by the action of $U(H)$ on E^4). In fact, these points are the singular points on the Segre cubics that are tangent hyperplane sections of Y. The point p_0 was also identified with the image of a surface in $S^{[2]}$ in the proof of Proposition 2.1; see also Remark 6.17. In particular these 16 points in Y are the set $Y_A[4]$ where $Y = Y_A$; see Remark 2.8.

To describe the incident planes, we need the following combinatorial description of the 60 singular planes of Y':

Remark 6.9 A partition $\{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}\}$ of $\{0, \dots, 5\}$, defining a plane of type (1b), determines in a natural way three sequences (i_1, i_2, j_1, j_2), (i_1, i_3, j_1, j_3), (i_2, i_3, j_2, j_3), which correspond to planes of type (1a). Note that the orders of pairs and indices in pairs do not matter, ie if we change them, we still get a sequence determining the same plane. This way we obtain a natural subdivision of the set of 60 planes into subsets of cardinality 4: each consists of a plane of type (1b) given by a partition of $\{0, \dots, 5\}$ and three planes of type (1a) described by sequences determined by this partition. We show below that the set of 20 incident planes consists of 5 such subsets; see Remark 2.9 and Section 4.4.

The following is a nice exercise:

Lemma 6.10 *The pairs of planes intersecting in a point are as follows:*

- *Two planes of type* (1b) *intersect in a point if and only if the corresponding partitions do not have any common component.*
- *Two planes of type* (1a) *intersect in a point if and only if the corresponding sequences come from the same partition (as described in Remark 6.9) or from two partitions with empty intersection.*
- *A plane of type* (1a) *intersects a plane of type* (1b) *in a point if and only if the sequence representing the first plane comes from a partition which represents the second one or which has empty intersection with this partition.*

Proposition 6.11 *There are exactly six possible choices of the set of* 20 *incident planes in the sextic* $Y_t^\vee$. *They are all obtained from the one in Proposition 2.5 by the* Σ_6*–action. The stabilizer of such a configuration is isomorphic to* Σ_5.

Proof Assume first that there are at least 16 planes of type (1a) in such a set. Then their corresponding sequences must come from at least six different partitions. But by Lemma 6.10 some two of these partitions must have a common element, so we would have two planes which intersect along a line or do not intersect at all. If they intersect along a line, then by [24, Proposition 2.2] this configuration is contained in an infinite family of incident planes. Hence we may restrict to the case where they intersect in a point.

Hence we may assume that there are five planes of type (1b) in the chosen set. By Lemma 6.10 the corresponding partitions do not have a common component; thus their union consists of all possible pairs of indices. Hence there is no other plane of type (1b). Again by Lemma 6.10, the only possible planes of type (1a) which can appear in the set are those represented by sequences which come from partitions corresponding to the chosen planes of type (1b). There are 15 of them, so the configuration can be completed in a unique way. □

Remark 6.12 It is worth noticing that six possible choices of the set of 20 incident planes in Y correspond to six possible choices of a 32-element subgroup isomorphic to G inside (G,i).

We are ready for the proof of our theorem.

6.2 Proof of Theorem 1.3

Let $\overline{S^{[2]}} \to Z \to Y$ be the Stein factorization of the two-to-one morphism $\overline{g}$ constructed above. From Proposition 5.11 and Corollary 6.5 we have a finite two-to-one morphism $E^4/G \to Y$. Our aim is to show that Z is isomorphic to E^4/G by proving that the two double covers $Z \to Y \leftarrow E^4/G$ are the same. First we shall show that the ramification loci of the morphisms are the same.

The sextic Y is singular along 60 planes. In Lemma 4.11 we already identified 40 of them that are in the ramification locus of $E^4/G \to Y$. We already showed in Corollary 3.5 that the ramification locus $Z \to Y$ also consists of 40 planes. From Proposition 3.7 the remaining 20 singular planes of Y are the images of the singular surfaces on Z that are incident. It follows that the images of the above singular surfaces from Z are incident planes on $Y \subset \mathbb{P}^5$. From Proposition 6.11 we infer that the choices of the 20 incident planes differ by a projective transformation fixing $Y \subset \mathbb{P}^5$. Thus the ramification loci of the maps $Z \to Y \leftarrow E^4/G$ are the same.

Finally, consider the coverings $Z \to Y$ and $E^4/G \to Y$. The two maps have the same ramification locus; moreover, outside the singular locus of Y both maps are étale covers. Since $\overline{S^{[2]}}$ is simply connected we infer that the fundamental group satisfies $\pi_1(Y - \mathrm{Sing}(Y)) = \mathbb{Z}_2$. From the uniqueness of integral closures this is enough to conclude that Z is isomorphic to E^4/G (they are both universal covers in codimension 1).

Remark 6.13 In the proof of Theorem 1.3 (in Proposition 5.3) we use the results from [11] about the existence of a symplectic desingularization $X_0 \to E^4/G$. It is an interesting problem to compare the manifolds X_0 and $\overline{S^{[2]}}$.

6.3 Final remarks

It follows from Proposition 5.11 that X_0 is of $\mathrm{K3}^{[2]}$–type as a double cover of an EPW sextic. Knowing this we have a direct, lattice theoretical proof that X_0 is birationally isomorphic to $S^{[2]}$. This result is weaker than Theorem 1.3, but the proof is much shorter!

Recall that the second integral cohomology group of a $\mathrm{K3}^{[2]}$–type IHS fourfold, with the Beauville–Bogolomov form, is isomorphic to the lattice Γ that is an orthogonal direct sum

(6-1) $$\Gamma := \Lambda_{\mathrm{K3}} \oplus \mathbb{Z}\xi \quad \text{where } \Lambda_{\mathrm{K3}} \cong E_8(-1)^2 \oplus U^3 \text{ and } \xi^2 = -2.$$

The following result was shown to us by G Mongardi.

Proposition 6.14 *The IHS fourfolds X_0, the desingularization of E^4/G, and $S^{[2]}$ are birationally isomorphic.*

Proof From the construction of X_0 we know that the 23–dimensional vector space $H^2(X,\mathbb{Q})$ has a 21–dimensional subspace spanned by the class of the divisor Δ and 20 exceptional divisors which map to the singular surfaces in E^4/G. Each class in this subspace is invariant under the action of i^*, where $i \in \mathrm{Aut}(X_0)$ is the covering involution for the map $X_0 \to Y$. As the holomorphic 2–form on X_0 does not descend to Y, we see that $i^* = -1$ on a complementary 2–dimensional subspace. Thus $H^2(X_0,\mathbb{Z})$ contains, with finite index, the direct sum of the i^*–invariant and antiinvariant sublattices, which are the Picard group of X_0 and the transcendental lattice T, respectively. The lattice $H^2(X_0,\mathbb{Z}) \cong \Gamma$ is not unimodular, but we can embed it in an even unimodular lattice as follows. Let $\widetilde{\Gamma} \subset \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Q}\eta$, with $\eta^2 = 2$, be the lattice generated by Γ and $e_1 := (\xi + \eta)/2$. Let $e_2 := e_1 - \xi \in \widetilde{\Gamma}$. Then e_1, e_2 generate a hyperbolic plane U (so $e_1^2 = e_2^2 = 0$ and $e_1 e_2 = 1$) and $\widetilde{\Gamma} = \Lambda_{\mathrm{K3}} \oplus U$.

Since the discriminant group of Γ is $\mathbb{Z}_2$ and i^* acts trivially on it, it extends to an isometry j of $\widetilde{\Gamma}$ with $j(\eta) = \eta$. Then the sublattice of j–antiinvariants in $\widetilde{\Gamma}$ has rank two and is isometric to T. As j is an involution, the discriminant of T is a power of two, and as the rank of T is two it is either 1, 2 or 4. Since T is positive definite and even, we must then have that T has discriminant 4 and that $T \cong \big(\mathbb{Z}^2, q = 2(x^2+y^2)\big)$.

On the other hand, the K3 surface S has the same transcendental lattice. (This can also be seen with a similar argument: in $\mathrm{Pic}(S)$ we have the pull-back of the classes, of a general line in $\mathbb{P}^2$, of the strict transforms of the four exceptional divisors of the triple points and of the 15 exceptional divisors in the second blow-up; these 20 classes are all invariant under the covering involution on S and hence, again, T_S is an even lattice of rank two with discriminant 1, 2 or 4.) Thus the transcendental lattice T_2 of $S^{[2]}$ is also isomorphic to T.

Notice that $\Gamma = \eta^{\perp}$ in $\widetilde{\Gamma}$ and that the sublattice $\widetilde{T}$ of $\widetilde{\Gamma}$ spanned by T and η is isomorphic to $\big(\mathbb{Z}^3, 2(x^2+y^2+z^2)\big)$. A well-known result of Nikulin implies that the embedding of $\widetilde{T}$ in the even unimodular lattice $\widetilde{\Gamma}$ is unique up to isometry. In particular, we may assume it lies in three copies of U, the first two in Λ_{K3}, the last spanned by e_1 and e_2. From this one deduces that there is an isometry between the lattices $H^2(X_0,\mathbb{Z})$ and $H^2(S^{[2]},\mathbb{Z})$ which restricts to an isometry on the transcendental lattices. Hence by the Torelli theorem for IHS it follows that X_0 and $S^{[2]}$ are birationally isomorphic. □

Remark 6.15 We expect that $\mathbb{Z}_2^5 \ltimes \Sigma_5$ is the group of automorphisms of $\overline{S^{[2]}} = X_0$. Indeed, the group of linear automorphisms of the EPW sextic $Y \subset \mathbb{P}^5$ is $\Sigma_6 \rtimes \mathbb{Z}_2^4$ (the permutations of coordinates and the change of an even number of signs). The linear automorphisms that preserve the 20 ramification planes of $X_0 \to Y$ form the group $\mathbb{Z}_2^4 \ltimes \Sigma_5$ (see Proposition 6.11) and they lift to symplectic automorphisms of X_0. Moreover, the covering involution $X_0 \to Y$ is antisymplectic. Note that $\mathbb{Z}_2^4 \ltimes \Sigma_5$ is one of the maximal groups of symplectic automorphisms of IHS fourfolds of K3–type found in [16, Theorem 5.1]. Note also that the automorphism group of S, and hence of $S^{[2]}$, is infinite.

Remark 6.16 It is natural to consider the map $(S')^{[2]} \dashrightarrow S_5^{[2]}$ induced by the double cover $\rho\colon S' \to S_5$ (see Lemma 3.4). We can deduce that we have the diagram

$$\begin{tikzcd} (S')^{[2]} \arrow[r, dashed, "z"] \arrow[d, dashed, "\rho^{[2]}"'] & S_5^{[2]} \arrow[d, dashed, "\pi"] \\ \mathbb{P}^5 \supset Y \arrow[r, dashed, "z'"] & \mathbb{P}^4 \end{tikzcd}$$

of rational maps such that the bottom map is the central projection with center a singular point on Y of type $\mathbb{C}^4/(G,i)$. By Proposition 5.7 the center of projection is of multiplicity 4 on Y. It follows that the maps $z'\colon Y \to \mathbb{P}^4$ and z are generically two-to-one. Moreover, the image by π of a generic point $\{p,q\} \in S_5^{[2]}$ is the hyperplane of quadrics containing $S_5 \subset \mathbb{P}^5$ and vanishing along the line $\langle p,q\rangle \subset \mathbb{P}^5$. It is easy to see that the double cover z' is ramified along ten hyperplanes which are the images of the divisor $\mu(l_i) \subset S^{[2]}$. It can be shown that these ten hyperplanes form the configuration of the ten hyperplanes tangent to the Igusa quartic along quadrics (see Proposition 5.7). It follows that the EPW sextic Y can be constructed as a partial resolution of the double cover of $\mathbb{P}^4$ ramified along the configuration of these ten hyperplanes.

Remark 6.17 Let us describe how from our picture we obtain a description of a symplectic resolution of the singularity $\mathbb{C}^4/G$ considered in [3]. We will use the notation from Section 3. We constructed a resolution of singularities

$$\overline{g}\colon \overline{S^{[2]}} \to E^4/(G,i).$$

The idea now is to look locally at this map around the singular point $\mathbb{C}^4/(G,i)$. Looking at the Stein factorization $\overline{S^{[2]}} \to E^4/G \to Y$ of $\overline{g}$ we see that we need to describe exceptional sets on $\overline{S^{[2]}}$ that map to one of the 16 singular points of type $\mathbb{C}^4/G$ on E^4/G.

We shall use the geometric definition of the map g. Recall that the nodal K3 surface $S' \subset \Pi \subset \mathbb{P}^6$ is contained in the cone Π with vertex P. A general line in this cone passing through P cuts S' in two points. On the other hand Π is contained in a 4–dimensional system Q of quadrics. Denote the set of points in $(S')^{[2]}$ that correspond to lines cutting S' in two points and contained in Π by $A \subset (S')^{[2]}$. From the definition of the map $\rho^{[2]}$, the set $A \subset (S')^{[2]}$ maps to the point $(1:\cdots:1) \in Y$ corresponding to Q (see the end of the proof of Proposition 2.1).

Note for each line $\phi(l_1) \subset S'$ determined by $l_1 \subset S$ we have a plane P_{l_1} in $\mathbb{P}^5$ spanned by P and $\phi(l_1)$. The rational curve l_1 cuts three exceptional curves e_1, e_2 and e_3 in points. All the lines on the plane P_{l_1} are tangent to S' and they determine a surface E_{l_1} in $S^{[2]}$. The surfaces $e_1^{[2]}$, $e_2^{[2]}$ and $e_3^{[2]}$ intersect E_{l_1} in points and they also intersect two of the indeterminacy loci E_{13}, E_{12} and E_{11} of type (3) along lines. Moreover, E_{l_1} is isomorphic to $\mathbb{F}_4$. It follows that the strict transform of E_{l_1} on $\overline{S}^{[2]}$ is isomorphic to $\mathbb{P}^2$.

The surfaces $l_1^{[2]}$ and E_{l_1} intersect on $S^{[2]}$ along a conic curve c. The proper transform of this conic curve c before the last flop is a line contained in the proper transform of $l_1^{[2]}$ (the Cremona transformation of $l_1^{[2]}$ with center at three points on c transforms c to a line). Before the last flop the proper transform of the surface E_{l_1} will be $\mathbb{F}_1$ (ie $\mathbb{F}_4$

transformed by three elementary transformations). After the last flop the proper transform of the above line will be contracted such that the proper transform of E_{l_1} will be a plane.

Finally, consider also the del Pezzo surface A of degree 5 contained in $S^{[2]}$, corresponding to the rays of the cone Π. The strict transform on $\overline{S^{[2]}}$ it is still a del Pezzo surface of degree 5. These 11 surfaces form the exceptional set of the resolution of the singular point $\mathbb{C}^4/G$; see [11].

Remark 6.18 Some results of the present paper can be neatly illustrated with the combinatorics related to the Petersen graph.

First, let us recall that the graph describes the incidence of (-1)–curves on $\mathbb{P}^2_4$ with 10 vertices standing for the curves and edges for their intersection. Using this notation we can encode five conic pencils on $S_5 = \mathbb{P}^2_4$: each pencil contains three reducible fibers which can be represented by three edges in this graph (two lines plus their point of intersection). Three double edges in Figure 6 stand for such a pencil. Thus the 15 edges of the graph are divided into five classes; every edge in each class shares no common adjacent edge with another edge in the same class. The five conic pencils on S_5 give five distinguished elliptic pencils on the Vinberg K3 surface S; see Section 2.1.

It is well known — see the first part of Lemma 2.3 — that $S_5 \subset \mathbb{P}^5$ admits an action of the permutation group Σ_5 which yields an action of Σ_5 on the set of (-1)–curves on S_5. In fact, we can identify the (-1)–curves with transpositions and the action of Σ_5 is then by conjugation. This is depicted in Figure 6 by assigning to each vertex of the Petersen graph a pair from the set $\{a, \dots, e\}$.

There are exactly two nonconjugate embeddings $\Sigma_5 \hookrightarrow \Sigma_6$: apart from the standard one (coming from the embedding $\{0, \dots, 4\} \hookrightarrow \{0, \dots, 5\}$), there is an embedding β described in the second part of Lemma 2.3. The embedding β assigns to a pair in a 5–element set (a transposition in Σ_5) a partition of a 6–element set into 3 pairs.

We label the edges of the Petersen graph by pairs in $\{0, \dots, 5\}$ so that the three edges stemming from a given vertex give the respective partition (see Figure 6); for example: $(a\,b) \mapsto (03)(14)(25)$, $(b\,c) \mapsto (01)(24)(35)$, $(c\,d) \mapsto (05)(14)(23)$, $(d\,e) \mapsto (01)(25)(34)$. Compare this with Lemma 2.3. This way, out of 15 partitions of the set $\{0, \dots, 5\}$, ten can be associated to vertices of the Petersen graph. As a result, the remaining five partitions come from the five triples of edges which are associated to conic pencils on S_5 or distinguished elliptic pencils on S. In Figure 6, they are the following: $(01)(23)(45)$, $(02)(14)(35)$, $(03)(15)(24)$, $(04)(13)(25)$, $(05)(12)(34)$. These are exactly the partitions which occur in Proposition 2.5; there

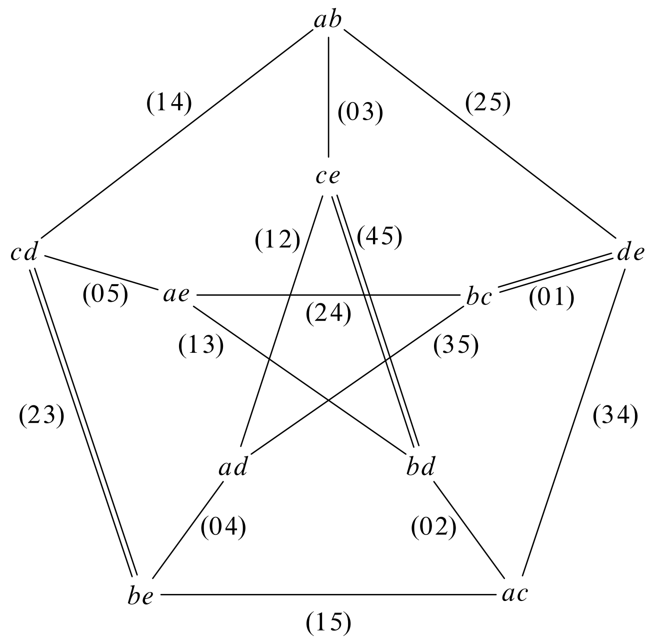

Figure 6

they define the 20 incident planes: each partition defines $3+1$ planes, which depend on the number of $\pm$ signs. Similarly, the divisors B_i in Proposition 3.7 can be divided into five classes related to five distinguished elliptic fibrations of S, each of the classes containing $3+1$ divisors.

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