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Geometry & Topology 21 (2017) 1275–1283

Presentation complexes with the fixed point property

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We prove that there exists a compact two-dimensional polyhedron with the fixed point property and even Euler characteristic. This answers a question posed by R H Bing in 1969. We also settle a second question by Bing regarding the homotopy invariance of the fixed point property in low dimensions.

55M20, 57M05, 57M20

1 Introduction

In his influential article "The elusive fixed point property" [3], R H Bing stated twelve questions. Since then eight of these questions have been answered; see Hagopian [12]. In this paper we answer Questions 1 and 8.

Recall that a space X is said to have the *fixed point property* if every map $f\colon X \to X$ has a fixed point. By a *polyhedron* we mean a space homeomorphic to the geometric realization of a simplicial complex. Motivated by an example of W Lopez [15], Bing [3] asked the following question.

Question 1.1 (Bing's Question 1) *Is there a compact two-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

This question was studied by Barmak and Sadofschi Costa [2] and Waggoner [18]. In [2] it is proved that the answer is negative if we restrict ourselves to spaces with abelian fundamental group. In Corollary 2.7 below we show that there exists a compact two-dimensional polyhedron with the fixed point property and Euler characteristic equal to 2 whose fundamental group is nonabelian of order 243. This settles Question 1.1 affirmatively.

The example constructed by Lopez [15] shows that the fixed point property is not a homotopy invariant for polyhedra of dimension 17. The smallest dimension n for which the fixed point property fails to be a homotopy invariant coincides with the smallest n such that there is an n–dimensional compact polyhedron without the fixed point property which collapses by an elementary collapse to a complex with the fixed point property. This follows from the next result, due to Jiang:

Published: 17 March 2017 DOI: 10.2140/gt.2017.21.1275

Theorem 1.2 [13, Theorem 7.1] *In the category of compact connected polyhedra without global separating points, the fixed point property is a homotopy-type invariant.*

Moreover, if $X \simeq Y$ are compact connected polyhedra such that Y lacks the fixed point property and X does not have global separating points, then X lacks the fixed point property.

Recall that a point x in a connected polyhedron X is said to be a *global separating point* if $X - \{x\}$ is not connected.

In Theorem 2.12 below we show that there exists a compact polyhedron of dimension 2 without the fixed point property which collapses to a polyhedron with the fixed point property. This settles a second question by Bing:

Question 1.3 (Bing's Question 8) *What is the smallest number n such that there exists an n–dimensional polyhedron X with the fixed point property, and a disk D such that $X \cap D$ is an arc and $X \cup D$ does not have the fixed point property?*

According to C L Hagopian [12], Bing conjectured that the answer to Question 1.3 is two. Theorem 2.12 proves this conjecture.

Acknowledgment I am grateful to Jonathan Barmak, without whose advice and suggestions this paper would not have been possible. I would like to thank the referee for helpful comments and suggestions.

2 Bing groups

By *invariant factors* of a finitely generated abelian group A we mean the nonnegative integers $d_1 \mid \cdots \mid d_k$ such that $A = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$. If $\mathcal{P}$ is a presentation of a group, the presentation complex of $\mathcal{P}$ will be denoted by $X_{\mathcal{P}}$. Presentation complexes are in fact polyhedra. If a group G has finite abelianization and is presented by a presentation $\mathcal{P}$ with g generators and r relators, then $r - g$ is at least the number of invariant factors of the second homology group $H_2(G)$ of G. If this lower bound is attained for $\mathcal{P}$, then the presentation is said to be *efficient*. Equivalently, a presentation $\mathcal{P}$ of a group G with finite abelianization is efficient if the rank of $H_2(X_{\mathcal{P}})$ is the number of invariant factors of $H_2(G)$. A group G is said to be *efficient* if it admits an efficient presentation. If R is a principal ideal domain, the trace of an endomorphism ϕ of a free R–module of finite rank is denoted by $\mathrm{tr}(\phi) \in R$.

Definition 2.1 Let G be a finitely presentable group and let $d_1 \mid \cdots \mid d_k$ be the invariant factors of $H_2(G)$. We say that G is a *Bing group* if $H_1(G)$ is finite and for every endomorphism $\phi\colon G \to G$ we have $\operatorname{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$.

The above definition makes sense unless $H_2(G) = 0$. If G is a finitely presentable group such that $H_1(G)$ is finite and $H_2(G) = 0$, we make the convention that G is a Bing group.

Theorem 2.2 *If $\mathcal{P}$ is an efficient presentation of a Bing group G then $X_{\mathcal{P}}$ has the fixed point property.*

Proof Let $X = X_{\mathcal{P}}$ and $f\colon X \to X$ be a map. If $H_2(G) = 0$, X is rationally acyclic, so f has a fixed point. Therefore we may assume $H_2(G) \neq 0$. Let $d_1 \mid \cdots \mid d_k$ be the invariant factors of $H_2(G)$. There is a $K(G,1)$ space Y with $X = Y^2$. Now f extends to a map $\bar{f}\colon Y \to Y$.

In the following commutative diagram, the horizontal arrows, induced by the inclusion $i\colon X \hookrightarrow Y$, are epimorphisms:

$$\begin{tikzcd} H_2(X) \arrow[r, two heads, "i_*"] \arrow[d, "f_*"'] & H_2(Y) \arrow[d, "\bar{f}_*"] \\ H_2(X) \arrow[r, two heads, "i_*"'] & H_2(Y) \end{tikzcd}$$

Since $\mathcal{P}$ is efficient, the rank of $H_2(X)$ equals the number of invariant factors of $H_2(Y)$. Therefore the horizontal arrows in the following commutative diagram are isomorphisms:

$$\begin{tikzcd} H_2(X) \otimes \mathbb{Z}_{d_1} \arrow[r, "i_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}", "\approx"'] \arrow[d, "f_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}"'] & H_2(Y) \otimes \mathbb{Z}_{d_1} \arrow[d, "\bar{f}_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}"] \\ H_2(X) \otimes \mathbb{Z}_{d_1} \arrow[r, "\approx", "i_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}"'] & H_2(Y) \otimes \mathbb{Z}_{d_1} \end{tikzcd}$$

Now $\operatorname{tr}(f_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) = \operatorname{tr}(\bar{f}_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$ since G is a Bing group. Here we are using the natural isomorphism $H_2(BG) \approx H_2(G)$ of [16, Theorem 5.1.27]. Recall that every map $BG \to BG$ is induced, up to homotopy, by an endomorphism $G \to G$.

Finally we obtain $\operatorname{tr}(f_*) \neq -1$ in $\mathbb{Z}$, since tensoring with $\mathbb{Z}_{d_1}$ reduces the trace modulo d_1. So $L(f) \neq 0$ and, by the Lefschetz fixed point theorem, f has a fixed point. $\square$

Example 2.3 Efficient Bing groups with trivial second homology are easy to find (for example $\mathbb{Z}_n$ or any other finite group with deficiency zero). But the presentation complexes we get in this way are rationally acyclic, therefore have Euler characteristic 1. Aside from cyclic groups, abelian groups are not Bing groups (this follows from [2, Theorem 4.6]).

Example 2.4 If G is a group, we consider the action $\mathrm{Aut}(G) \curvearrowright H_2(G)$. Moreover, if $\phi \in \mathrm{Inn}(G)$ then $H_2(\phi)$ is the identity morphism. So there is an induced action $\mathrm{Out}(G) \curvearrowright H_2(G)$. When G is a finite simple group, every endomorphism $\phi\colon G \to G$ is either trivial or an automorphism. For the trivial morphism ϕ we have $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) = 0$. Therefore for a finite simple group G, understanding the action $\mathrm{Out}(G) \curvearrowright H_2(G)$ suffices to determine if G is a Bing group. Using the classification of the finite simple groups [9, Table I, page 8–9] and the description of this action [10, Theorems 6.1.4, 6.3.1 and 2.5.12] we can prove that the only finite simple groups with nontrivial second homology that are also Bing groups are the groups $D_{2m}(q)$ for $m > 2$ and q odd. The smallest of these groups is $D_6(3)$, a group of order $67628447006087702382529609728\,00$. Simple groups of order at most 5000000 are efficient, except perhaps $C_2(4)$ [6; 7]. However, it is not known if A_n is efficient for all n [6]. It is known that $D_{2m}(q)$ has deficiency at most 24 [11, Theorem 10.1]. Since $H_2(D_{2m}(q)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, if these groups turn out to be efficient, they would give examples of two dimensional polyhedra with the fixed point property and Euler characteristic equal to 3. To answer Question 1.1 we will need another source of Bing groups.

Proposition 2.5 *The group G presented by*

$$\mathcal{P} = \langle x, y \mid x^3,\ xyx^{-1}yxy^{-1}x^{-1}y^{-1},\ x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

is a finite group of order 3^5. We have $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. Moreover G is a Bing group.

Proof We will need the following GAP [17] program, which uses the packages HAP [8] and SONATA [1]:

```
LoadPackage("HAP");;
LoadPackage("SONATA");;
F:=FreeGroup(2);;
G:= F/[F.1^3, F.1*F.2*F.1^-1*F.2*F.1*F.2^-1*F.1^-1*F.2^-1,
F.1^-1*F.2^-4*F.1^-1*F.2^2*F.1^-1*F.2^-1];;
Order(G);
```

```
G:=SmallGroup(IdGroup(G));;
R:=ResolutionFiniteGroup(G,3);;
Homology(TensorWithIntegers(R),2);
Set(List(Endomorphisms(G),
f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
```

The program prints three lines. The first one contains the order of G, the second one is a list with the invariant factors of $H_2(G)$ and the third one is a list with the endomorphisms of $H_2(G)$ that are induced by an endomorphism of G. The output is:

```
243
[ 3 ]
[ [ f1 ] -> [ <identity ...> ], [ f1 ] -> [ f1 ] ]
```

Therefore $|G| = 243$ and $H_2(G) = \mathbb{Z}_3$. The third line of the output says that there are only two endomorphisms of $H_2(G)$ that are induced by an endomorphism of G. The first endomorphism maps the generator `f1` of $H_2(G) = \mathbb{Z}_3$ to $0 \in H_2(G)$, so it is the zero morphism. The second endomorphism maps `f1` to `f1`, so it is the identity morphism of $H_2(G)$. From this we conclude that, after tensoring with $\mathbb{Z}_3$, the traces of these endomorphisms are 0 and 1, proving that G is a Bing group. □

Remark 2.6 Using GAP, it is easy to show that the group G in the previous proposition is a semidirect product $(\mathbb{Z}_9 \oplus \mathbb{Z}_9) \rtimes \mathbb{Z}_3$. The action of $\mathbb{Z}_3$ in $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ is multiplication by $\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}$.

By Theorem 2.2 and Proposition 2.5 we have:

Corollary 2.7 *The complex $X_{\mathcal{P}}$ associated to the presentation*

$$\mathcal{P} = \langle x, y \mid x^3,\ xyx^{-1}yxy^{-1}x^{-1}y^{-1},\ x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

has the fixed point property. Moreover, $\chi(X_{\mathcal{P}}) = 2$.

Borsuk proved that a polyhedron with nontrivial first rational homology group retracts to S^1 [4, Théorème 30; 5, Korollar 11]. Therefore a two-dimensional polyhedron with the fixed point property has positive Euler characteristic.

Corollary 2.8 *There are compact 2–dimensional polyhedra with the fixed point property and Euler characteristic equal to any positive integer n.*

Proof For $n = 1$ this is immediate. For $n > 1$ take a wedge of $n - 1$ copies of the space $X_{\mathcal{P}}$ of Corollary 2.7. □

To prove Theorem 2.12 we will need another efficient Bing group:

Proposition 2.9 *The group H presented by $\mathcal{Q} = \langle x, y \mid x^4, y^4, (xy)^2, (x^{-1}y)^2 \rangle$ is a finite group of order 2^4. We have $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\mathcal{Q}$ is efficient. Moreover, H is a Bing group.*

Proof As above we will use a GAP program:

```
LoadPackage("HAP");;
LoadPackage("SONATA");;
F:=FreeGroup(2);;
H:= F/[F.1^4, F.2^4, (F.1*F.2)^2, (F.1^-1*F.2)^2];;
Order(H);
H:=SmallGroup(IdGroup(H));;
R:=ResolutionFiniteGroup(H,3);;
Homology(TensorWithIntegers(R),2);
Set(List(Endomorphisms(H),
f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
```

The program produces the following output:

```
16
[ 2, 2 ]
[ [ f1, f2 ] -> [ <identity ...>, <identity ...> ],
[ f1, f2 ] -> [ f1, f2 ],[ f1, f2 ] -> [ f1^-1*f2^-1, f2^-1 ] ]
```

The first line says that $|H| = 16$. The second line says that $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Finally, the last two lines say that there are only three endomorphisms of $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ that are induced by an endomorphism of H. The first of these endomorphisms maps both generators `f1` and `f2` to $0 \in H_2(H)$, so it is the zero morphism. The second one maps `f1` to `f1` and `f2` to `f2`, so it is the identity morphism. The third endomorphism maps `f1` to `f1^-1*f2^-1=f1*f2` and `f2` to `f2^-1=f2`. So in the basis given by `f1` and `f2` it is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. From this we see that, after tensoring with $\mathbb{Z}_2$, the trace of each of these endomorphisms is 0. Therefore H is a Bing group. □

Remark 2.10 The group H in the previous proposition is a semidirect product $(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \rtimes \mathbb{Z}_2$. The action of $\mathbb{Z}_2$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

We will also need the following:

Proposition 2.11 [2, Proposition 3.3] *Let X be a compact connected 2–dimensional polyhedron. The following are equivalent:*

- *X is homotopy equivalent to a polyhedron Y having S^2 as a retract.*
- *The number of invariant factors of $H_2(\pi_1(X))$ is strictly smaller than the rank of $H_2(X)$.*

Now we will show that the answer to Question 1.3 is 2:

Theorem 2.12 *There is a compact 2–dimensional polyhedron Y without the fixed point property and such that the polyhedron X, obtained from Y by an elementary collapse of dimension 2, has the fixed point property.*

Proof Let $\mathcal{P}$ and $\mathcal{Q}$ be the presentations of Propositions 2.5 and 2.9. By Theorem 2.2, $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$ have the fixed point property, so $X = X_{\mathcal{P}} \vee X_{\mathcal{Q}}$ also has the fixed point property. Since neither $X_{\mathcal{P}}$ nor $X_{\mathcal{Q}}$ have global separating points, by adding a 2–simplex, we can turn X into a polyhedron Y, without global separating points and such that, by collapsing that 2–simplex, we obtain X. We have $H_2(\pi_1(Y)) = H_2(\pi_1(X_{\mathcal{P}}) * \pi_1(X_{\mathcal{Q}})) = H_2(\pi_1(X_{\mathcal{P}})) \oplus H_2(\pi_1(X_{\mathcal{Q}})) = \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and $\mathrm{rk}(H_2(Y)) = 3$. By Proposition 2.11 and Theorem 1.2, Y does not have the fixed point property. $\square$

Let $\Sigma_2(X)$ denote the image of the Hurewicz homomorphism $h\colon \pi_2(X) \to H_2(X)$. Then we have an exact sequence

$$0 \to \Sigma_2(X) \to H_2(X) \to H_2(\pi_1(X)) \to 0.$$

Lemma 2.13 [14, Lemma 1.4] *Let X and Y be compact, connected, 2–dimensional CW–complexes. If $f\colon X \to Y$ is a map and $\delta\colon H_2(X) \to \Sigma_2(Y)$ is any homomorphism, there is a map $g\colon X \to Y$ such that $\pi_1(f) = \pi_1(g)$ and $H_2(g) = H_2(f) + \delta$.*

If X is a compact, connected 2–dimensional complex with fundamental group G, we say that X has *minimum Euler characteristic* if any other such complex has Euler characteristic greater than or equal to $\chi(X)$.

Theorem 2.14 *Let X be a compact, connected, 2–dimensional polyhedron and let G be its fundamental group. Suppose that G is not Bing, or that G is not efficient or that X does not have minimum Euler characteristic. Then there is a map $f\colon X \to X$ with $L(f) = 0$.*

Proof If $H_1(G)$ is not finite, X retracts to S^1, so X has a self–map f with Lefschetz number zero. Therefore we may assume that $H_1(G)$ is finite. Let $d_1, \dots, d_n$ be the invariant factors of $H_2(G)$. Consider the inclusion $\iota\colon \Sigma_2 X \to H_2(X)$. Let m be the rank of $H_2(X)$ and let k be the rank of $\Sigma_2(X)$. We consider the Smith normal form of ι. Let $\alpha_1 \mid \cdots \mid \alpha_k$ be the numbers on the diagonal and let $\{e_1, \dots, e_m\}$ be the basis of $H_2(X)$. Since ι is injective, α_i is nonzero for $i = 1, \dots, k$. By the short exact sequence above we have $H_2(G) = \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_k} \oplus \mathbb{Z}^{m-k}$. Note that some of the first α_i may be equal to 1. But in any case (if $k > 0$) we have $\alpha_1 \mid d_1$.

Suppose G is not Bing. Then there is an endomorphism $\phi\colon G \to G$ such that $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) = -1$ in $\mathbb{Z}_{d_1}$. Let $\widetilde{f}\colon X \to X$ be a map inducing ϕ on fundamental groups. We have $\mathrm{tr}(H_2(\widetilde{f})) \equiv -1 \mod d_1$. If $d_1 = 0$ we are done. Otherwise, $k > 0$ and since $\alpha_1 \mid d_1$ there is $c \in \mathbb{Z}$ such that $\mathrm{tr}(H_2(\widetilde{f})) + c\alpha_1 = -1$. Define $\delta\colon H_2(X) \to \Sigma_2(X)$ by $\delta(e_1) = c\alpha_1 e_1$ and $\delta(e_j) = 0$ if $1 < j \leq m$. Now, using Lemma 2.13 we get a map $f\colon X \to X$ with $\mathrm{tr}(H_2(f)) = \mathrm{tr}(H_2(\widetilde{f})) + \mathrm{tr}(\delta) = -1$, therefore $L(f) = 0$.

Now suppose G is not efficient or X does not have minimum Euler characteristic. Then $m > n$, so we must have $k > 0$ and $\alpha_1 = 1$. By the argument above we get a map $f\colon X \to X$ with $L(f) = 0$. Alternatively, in this case we could use Proposition 2.11. □

The previous result can be seen as a converse to Theorem 2.2. Notice that this is not enough to conclude that X does not have the fixed point property. To do that we would need to find a map f with Nielsen number 0.

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Received: 15 February 2016
Revised: 1 March 2016