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On the topological contents of η–invariants

ULRICH BUNKE

We discuss a universal bordism invariant obtained from the Atiyah–Patodi–Singer η–invariant from the analytic and homotopy-theoretic point of view. Classical invariants like the Adams e–invariant, ρ–invariants and String–bordism invariants are derived as special cases. The main results are a secondary index theorem about the coincidence of the analytic and topological constructions and intrinsic expressions for the bordism invariants.

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1 Introduction

The purpose of this work is to understand which topological information is encoded in the η–invariant, a spectral-geometric invariant introduced by Atiyah, Patodi and Singer [7] in the context of index theory for boundary value problems for Dirac operators. We are in particular interested in bordism invariants derived from the η–invariant. By now we know many examples; see eg Atiyah, Patodi and Singer [9], Bahri and Gilkey [12; 13], Bunke and Naumann [24], Deninger and Singhof [29], Jones and Westbury [42] and Crowley and Goette [27]. In the present paper we consider the universal structure behind these examples. We define a bordism invariant, which we call the universal η–invariant. We use Section 5 in order to review some of the known η–invariant-based bordism invariants. We put the emphasis on the demonstration of how they can be interpreted as special cases of our universal construction. Our construction also subsumes (by constructions similar to the one in Section 5.5) some, but not all, of the Kreck–Stolz-type or Eells–Kuiper-type invariants — see Kreck and Stolz [45] and Donnelly [30] — or the generalised Rochlin invariants of Miller and Lee [50]. The universal η–invariant does not seem to incorporate the invariant introduced by Kervaire and Milnor [44] in order to distinguish exotic spheres.

In the present paper we introduce and compare two versions of the universal η–invariant. The analytic version η^{an} given in Definition 3.6 is the bordism invariant which is derived from the appearance of the reduced η–invariant in the local index theorem for the Atiyah–Patodi–Singer boundary value problem by cancelling out the dependence on geometric data. The ideas for this construction are more or less standard and have been used previously in many special situations.

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The topological counterpart η^{top} introduced in Definition 2.3 is constructed by a simple homotopy-theoretic consideration using the interplay of $\mathbb{Q}/\mathbb{Z}$–bordism and K–theory.

While it is not so complicated to see that η^{an} is a bordism invariant, to understand its homotopy-theoretic meaning is slightly deeper. The bridge between analysis and topology is provided by our first main result, Theorem 3.7, stating that

$$\eta^{\mathrm{an}} = \eta^{\mathrm{top}}.$$

Its proof uses standard methods in index theory like the analytic picture of K–homology (see Baum and Douglas [15] and Kasparov [43]) the Atiyah–Patodi–Singer index theorem [7] and some ideas from $\mathbb{Z}/l\mathbb{Z}$–index theory (see Freed and Melrose [33]).

Bordism classes can be represented geometrically by manifolds with additional structures, called cycles (see Section 3.2 for details). It is then an interesting question how one can calculate the universal η–invariant or its specialisations in terms of the cycle. The definition of both the topological and the analytical version of the universal η–invariant involves the choice of a zero bordism of some multiple of the cycle. In applications it is often complicated to find such a zero bordism. It is a striking advantage of the analytic picture that it can be reorganised to an expression which only involves structures on the cycle itself. In special cases this has previously been exploited by Seade [56], Deninger and Singhof [29] (the case of the Adams e–invariant; see Section 5.1) and Bunke and Naumann [24] (to calculate String–bordism invariants; see Section 5.4).

We consider the intrinsic formula for η^{an} given in Theorem 4.19 as one of the main original contributions of the present paper. This formula is based on a new object, which we call a geometrisation. If a map $f\colon M \to BG$ classifies a principal G–bundle P, then a geometrisation of f is essentially given by a connection on P. In general, the notion of a geometrisation partially generalises the notion of a connection on the (in general nonexistent) principal bundle classified by the map $f\colon M \to B$ for an arbitrary space B. The details are slightly more complicated since we will take structures on the normal bundle into account.

Remark 1.1 In this paper we generally decided to work with complex K–theory. We think that there is a real version of the whole theory which can be obtained by replacing complex K–theory by real KO–theory and $B\mathrm{Spin}^c$ by $B\mathrm{Spin}$, and taking the real structures on the spinor bundles into account properly on the analytic side. The real version of the universal η–invariant would be slightly stronger than its complex counterpart, which loses some two-torsion classes. In order to recover the Adams e–invariant or the string bordism invariant (Bunke and Naumann [24]) completely as special cases of the universal η–invariant, we would need the real version.

Let us now describe the contents of the paper. In Section 2 we introduce the topological version η^{top} of the universal η–invariant and study its properties. Most interesting is probably the relation with the Adams spectral sequence (Theorem 2.7), which asserts that the universal η–invariant detects the first nontrivial subquotient of the bordism theory with respect to the K–theory-based Adams filtration.

The idea to use $\mathbb{Q}/\mathbb{Z}$–versions of homology theories to detect elements in the Adams spectral sequence is not new and goes back to Adams' construction of the e–invariant, which will be reviewed in Section 5.1. It is at the root of the chromatic approach to stable homotopy theory; see Miller, Ravenel and Wilson [51]. Our discussion of the topological version of the universal η–invariant can be viewed as a simple version of Laures [46; 47] and Behrens and Laures [16], where the main focus of these references is on the second (and higher) steps of the Adams filtration.

In Section 3 we introduce the analytic version η^{an} of the universal η–invariant and prove the secondary index theorem (Theorem 3.7), stating that $\eta^{\mathrm{an}} = \eta^{\mathrm{top}}$. Before we can define η^{an} we have to recall in Sections 3.2 and 3.3 some preliminary technical details concerning the relation of structures on the stable normal bundle as they come out of the Pontrjagin–Thom construction, and structures on the tangent bundle which will be used to do geometry and analysis.

Section 4 is devoted to geometrisations (Definition 4.5) and the intrinsic formula for η^{an} (Theorem 4.19).

Lastly, in Section 5 we discuss in detail various specialisations of the universal η–invariant. It contains mainly a review of known constructions and results with slight improvements or generalisations at some points (eg Corollary 5.12). In Propositions 5.13 and 5.14 we show how the usual geometric structures of Spin– and String–geometry (see Waldorf [60] for the latter) give rise to geometrisations which lead to the known intrinsic formulas for the corresponding bordism invariants. It has been the initial motivation for this work to understand the general principles behind the String–bordism invariants introduced by Bunke and Naumann [24]. It should be easy to adapt the arguments used here for the String $= MO\langle 8\rangle$–bordism case to bordism theories $MO\langle n\rangle$ associated to higher connected covers $BO\langle n\rangle$ of BO.

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2 The topological construction

2.1 Motivation

In this section we use complex K–theory in order to detect torsion elements in the homotopy groups $\pi_*(E)$ of a spectrum E. We introduce the topological version η^{top} of the universal η–invariant as a secondary version of the homomorphism

$$\epsilon\colon \pi_*(E) \to \pi_*(K \wedge E)$$

induced by the map of spectra

(1) $$\epsilon\colon E \simeq S \wedge E \xrightarrow{\mathtt{unit}\wedge\mathtt{id}} K \wedge E,$$

where "unit" is the unit of the ring spectrum K. The idea is to first lift the torsion element to the $\mathbb{Q}/\mathbb{Z}$–version $E\mathbb{Q}/\mathbb{Z}$ of E, then apply a similar map for E replaced by $E\mathbb{Q}/\mathbb{Z}$, and finally to detect the result via its evaluations against $K^0(E)$. This construction is a generalisation of the construction of the Adams e–invariant in the case of the sphere spectrum $E = S$.

In Section 2.2 we will give the construction of the invariant. In Section 2.3 we analyse its target in some detail. Finally, in Section 2.4 we understand completely in terms of the Adams spectral sequence which piece of $\pi_*(E)_{\mathrm{tors}}$ the universal η–invariant can detect.

2.2 The definition of η^{top}

For an abelian group A we let $\mathtt{M}A$ denote the Moore spectrum, which is characterised by its integral homology

$$\pi_n(H\mathbb{Z} \wedge \mathtt{M}A) \cong \begin{cases} A & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

More generally, for a spectrum E we have short exact sequences [20, (2.1)]

(2) $$0 \to \pi_n(E) \otimes A \to \pi_n(E \wedge \mathtt{M}A) \to \pi_{n-1}(E) * A \to 0$$

for all $n \in \mathbb{Z}$. To simplify the notation we abbreviate $EA := E \wedge \mathtt{M}A$. The equivalences $\mathtt{M}\mathbb{Z} \simeq S$ and $\mathtt{M}\mathbb{Q} \simeq H\mathbb{Q}$ induce equivalences $E\mathbb{Z} \simeq E$ and $E\mathbb{Q} \simeq E \wedge H\mathbb{Q}$.

The starting point for the construction of the topological version η^{top} of the universal η–invariant is the fibre sequence

(3) $$\mathtt{M}\mathbb{Z} \to \mathtt{M}\mathbb{Q} \to \mathtt{M}\mathbb{Q}/\mathbb{Z} \to \Sigma\mathtt{M}\mathbb{Z}$$

of Moore spectra. Smashing this sequence with the map $\epsilon\colon E \to K \wedge E$ (see (1)) we get the diagram

$$\begin{tikzcd}
\Sigma^{-1}E\mathbb{Q} \arrow[r, "\epsilon"] \arrow[d] \arrow[dr, dotted, "u"] & \Sigma^{-1}K \wedge E\mathbb{Q} \arrow[d] \\
{}^{\hat{x}}\Sigma^{-1}E\mathbb{Q}/\mathbb{Z} \arrow[r, "\epsilon"] \arrow[d] & {}^{\tilde{x}}\Sigma^{-1}K \wedge E\mathbb{Q}/\mathbb{Z} \arrow[d] \\
{}^{x}E \arrow[r, "\epsilon"] \arrow[d] & K \wedge E \arrow[d] \\
{}^{0}E\mathbb{Q} \arrow[r] & K \wedge E\mathbb{Q}
\end{tikzcd} \tag{4}$$

where we use the symbol ϵ also to denote other maps defined like (1) for different spectra in place of E.

Remark 2.1 If E is a spectrum and X is a space or spectrum, then for $n \in \mathbb{Z}$ we consider the cohomology $E^n(X)$ as a topological group, where a basis of neighbourhoods of zero is given by the kernels of restrictions along maps $Y \to X$ from finite CW–complexes or finite cell spectra Y. The topology on $E^n(X)$ is called the profinite topology. This should not be confused with the notion of a profinite group in algebra.

We have an evaluation pairing (ie just a bilinear map)

$$\langle -,-\rangle\colon \pi_n(\Sigma^{-1}K \wedge E\mathbb{Q}/\mathbb{Z}) \otimes K^0(E) \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \tag{5}$$

which sends $x \otimes \phi$ to the composition

$$\Sigma^{n+1}S \xrightarrow{x} K \wedge E \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \xrightarrow{\mathtt{id}_K \wedge \phi \wedge \mathtt{id}_{\mathtt{M}\mathbb{Q}/\mathbb{Z}}} K \wedge K \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \xrightarrow{\mu \wedge \mathtt{id}_{\mathtt{M}\mathbb{Q}/\mathbb{Z}}} K \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z},$$

where μ is the multiplication of the ring spectrum K. For a fixed class x the pairing is continuous in the second argument. In order to see this we use the fact that we can represent any spectrum E as a filtered colimit of finite spectra $E \simeq \mathtt{colim}_\alpha E_\alpha$. Since the smash product commutes with colimits we get an equivalence

$$K \wedge E \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \simeq \mathtt{colim}_\alpha K \wedge E_\alpha \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z}.$$

Since $\Sigma^{n+1}S$ is a finite spectrum, the map x has a factorisation over some stage of the colimit

$$\Sigma^{n+1}S \to K \wedge E_\alpha \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \to K \wedge E \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z}.$$

Now let $(\phi_i)_{i\in I}$ be a net in $K^0(E)$ converging to zero in the profinite topology. Then $\phi_i|_{E_\alpha} = 0$ for sufficiently large i. This immediately implies that $\langle x, \phi_i\rangle = 0$ for sufficiently large $i \in I$.

In the following the group $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$ has the discrete topology. The adjoint of the pairing is a homomorphism

$$\pi_n(\Sigma^{-1}K \wedge E\mathbb{Q}/\mathbb{Z}) \to \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})),$$

where $\mathtt{Hom}^{\mathrm{cont}}(-,-)$ stands for continuous homomorphisms. We let

$$U \subseteq \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \tag{6}$$

denote the subgroup given by the pairings with the elements in the image

$$\mathtt{image}(u\colon \pi_n(\Sigma^{-1}E\mathbb{Q}) \to \pi_n(\Sigma^{-1}K \wedge E\mathbb{Q}/\mathbb{Z}))$$

of the map u which can be read off from (4).

Definition 2.2 For every $n \in \mathbb{Z}$ we define the abelian group

$$Q_n(E) := \frac{\mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))}{U}. \tag{7}$$

The following definition uses a diagram chase of elements as indicated in (4).

Definition 2.3 The homotopy theoretic version of the universal η–invariant is the homomorphism

$$\eta^{\mathrm{top}}\colon \pi_n(E)_{\mathrm{tors}} \to Q_n(E)$$

defined by the following prescription: If $x \in \pi_n(E)_{\mathrm{tors}}$, then we choose a lift $\hat{x}$ in $\pi_n(\Sigma^{-1}E\mathbb{Q}/\mathbb{Z})$. We let $\eta^{\mathrm{top}}(x) \in Q_n(E)$ be the class represented by the homomorphism given by the pairing against $\epsilon(\hat{x})$.

We must show that η^{top} is well-defined. Indeed, the choice of $\hat{x}$ is unique up to elements which come from $\pi_n(\Sigma^{-1}E\mathbb{Q})$, but this ambiguity is taken care of by taking the quotient by U in the definition (7) of $Q_n(E)$.

The following lemma immediately follows from the definitions:

Lemma 2.4 *A map of spectra $E \to F$ naturally induces a commutative diagram:*

$$\begin{array}{ccc} \pi_n(E)_{\mathrm{tors}} & \xrightarrow{\eta^{\mathrm{top}}} & Q_n(E) \\ \downarrow & & \downarrow \\ \pi_n(F)_{\mathrm{tors}} & \xrightarrow{\eta^{\mathrm{top}}} & Q_n(F) \end{array}$$

2.3 Simplification of $Q_n(E)$

Because of its definition as a quotient it is difficult to define maps out of $Q_n(E)$. In this subsection we analyse the structure of this group and explain how one can detect its elements. We consider the ring spectrum

$$HP\mathbb{Q} := H\mathbb{Q}[b, b^{-1}],$$

where $\deg(b) = -2$. Additively it has a decomposition

$$HP\mathbb{Q} \simeq \bigvee_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q} \tag{8}$$

and for each $i \in \mathbb{Z}$ we consider the projection $p_{2i}\colon HP\mathbb{Q} \to \Sigma^{2i} H\mathbb{Q}$ to the corresponding component. We set $p_n := 0$ for odd n. The Chern character is an equivalence of ring spectra

$$\mathbf{ch}\colon K\mathbb{Q} \xrightarrow{\simeq} HP\mathbb{Q}.$$

The composition of the Chern character $\mathbf{ch}$ with the projection p_{n+1} gives a map of spectra whose kernel in degree zero homotopy will be denoted by

$$V_n := \mathtt{ker}(p_{n+1} \circ \mathbf{ch}\colon K^0(E) \to \mathbb{Q}). \tag{9}$$

Lemma 2.5 (1) *The restriction to $V_n \subseteq K^0(E)$ induces a well-defined map*

$$Q_n(E) \to \mathtt{Hom}^{\mathrm{cont}}(V_n, \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})). \tag{10}$$

(2) *This restriction map is an isomorphism if we assume that E is lower bounded and $\dim H\mathbb{Q}^{n+1}(E)$ is finite.*

Proof First we show that the restriction is well-defined. We must show that if $\phi \in V_n$ then the pairing of ϕ with $u(y) \in \pi_{n+1}(K \wedge E\mathbb{Q}/\mathbb{Z})$ vanishes for every $y \in \pi_{n+1}(E\mathbb{Q})$. This follows from the equality

$$\langle u(y), \phi\rangle = q\big(\big\langle \mathbf{ch}(\epsilon(y)), p_{n+1}(\mathbf{ch}(\phi))\big\rangle\big), \tag{11}$$

where

$$q\colon \pi_{n+1}(HP\mathbb{Q}) \xrightarrow{\mathbf{ch}^{-1}} \pi_{n+1}(K\mathbb{Q}) \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$$

and ϵ is as in (4). Note that $\mathbf{ch}(\epsilon(y))$ sits in the b–degree zero component of $HP\mathbb{Q}$.

We now show (2). We use the general fact that if $f\colon A \to V_n$ is a homomorphism of an abelian group into a $\mathbb{Q}$–vector space such that its image is finitely generated as an abelian group, then there exists a splitting $A \cong \mathtt{ker}(f) \oplus A'$. Indeed, in this case the image is free and hence projective.

Note that there exists an integer N (only depending on n and the lower bound of E) such that $p_{n+1}(\mathbf{ch}(K^0(E)))$ is contained in the image of $\frac{1}{N} H\mathbb{Z}^{n+1}(E) \to H\mathbb{Q}^{n+1}(E)$ and is therefore finitely generated as an abelian group since we assume that $H\mathbb{Q}^{n+1}(E)$ is finite-dimensional. We conclude that

$$K^0(E) \cong V_n \oplus V_n^c,$$

where $V_n^c \cong \mathtt{image}(p_{n+1} \circ \mathbf{ch})$ is a free abelian group. This immediately implies that (10) is surjective.

Any homomorphism $\phi \in \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$ can uniquely be decomposed as a sum of its restrictions to V_n and V_n^c. We claim that $U \cong \mathtt{Hom}(V_n^c, \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$, where U is as in (6). The claim implies that (10) is injective.

We can assume that n is odd. The claim follows from the surjectivity of the composition

$$\begin{aligned}\pi_n(\Sigma^{-1} E\mathbb{Q}) &\cong \mathtt{Hom}^{\mathrm{cont}}(H\mathbb{Q}^{n+1}(E), \mathbb{Q}) \\ &\cong \mathtt{Hom}^{\mathrm{cont}}(V_n^c, \pi_{n+1}(K\mathbb{Q})) \twoheadrightarrow \mathtt{Hom}^{\mathrm{cont}}(V_n^c, \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})),\end{aligned}$$

where we use the Chern character for the second equivalence and the fact that V_n^c is free in order to conclude the surjectivity of the last map. Note that in the present situation, continuity of homomorphisms is automatic by our finiteness assumption. □

The definition of η^{top} is based on first lifting the torsion element in the homotopy group of E to a $\mathbb{Q}/\mathbb{Z}$–homotopy class which is then paired with elements of K–theory. The pairing with torsion K–theory elements can be expressed in a dual way as a pairing of the original homotopy class with $\mathbb{Q}/\mathbb{Z}$–lifts of the K–theory elements. We now explain the details.

Assume that $\phi \in K^0(E)$ satisfies $\mathbf{ch}(\phi) = 0$. Then $\phi \in V_n$ and we get an evaluation

$$\mathtt{ev}_\phi \colon Q_n(E) \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}).$$

In view of the exact sequence

$$K\mathbb{Q}/\mathbb{Z}^{-1}(E) \xrightarrow{\partial} K^0(E) \xrightarrow{\mathbf{ch}} HP\mathbb{Q}^0(E),$$

we can choose $\widehat{\phi} \in K\mathbb{Q}/\mathbb{Z}^{-1}(E)$ such that $\partial\widehat{\phi} = \phi$. If we want to calculate $\mathtt{ev}_\phi(\eta^{\mathrm{top}}(x))$ then instead of lifting the class x to a $\mathbb{Q}/\mathbb{Z}$ class we can instead evaluate the class $\epsilon(x) \in \pi_n(K \wedge E)$ against the lift $\widehat{\phi}$. The following assertion follows easily from the definition of η^{top} and commutativity of the diagram:

$$\begin{tikzcd}
\Sigma^{-1}E\wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \arrow[r, "\phi\wedge \mathtt{id}_{\mathtt{M}\mathbb{Q}/\mathbb{Z}}"] \arrow[d, "\partial"] & \Sigma^{-1}K\mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
E \arrow[r, "\widehat{\phi}"] & \Sigma^{-1}K\mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Lemma 2.6 *For $x\in\pi_n(E)_{\mathrm{tors}}$ and $\widehat{\phi}\in K\mathbb{Q}/\mathbb{Z}^{-1}(E)$ we have*

$$\mathtt{ev}_\phi(\eta^{\mathrm{top}}(x)) = \langle\widehat{\phi},\epsilon(x)\rangle,$$

where $\phi := \partial\widehat{\phi}$.

In the present paper the spectrum E will often be a Thom spectrum MB associated to a map of spaces $B\to B\mathrm{Spin}^c$. The spectrum MB is K–oriented by

(12) $$\beta\colon MB\to M\mathrm{Spin}^c \xrightarrow{\mathrm{ABS}} K,$$

where ABS is the Atiyah–Bott–Shapiro orientation. In this case can use the Thom isomorphisms

$$\mathtt{Thom}^K\colon K^0(B)\xrightarrow{\cong} K^0(MB),\quad \mathtt{Thom}_{H\mathbb{Q}}\colon H\mathbb{Q}_{n+1}(MB)\xrightarrow{\cong} H\mathbb{Q}_{n+1}(B),$$

in order to express $Q_n(MB)$ in terms of B. We have an isomorphism

(13) $$Q_n(MB)\cong\frac{\mathtt{Hom}^{\mathrm{cont}}(K^0(B),\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))}{U'},$$

where U' is obtained by precomposing the elements in U (see (6)) with $\mathtt{Thom}^K$. We let $\mathbf{Td}\in HP\mathbb{Q}^0(B\mathrm{Spin}^c)$ be the universal Todd class and use the same symbol in order to denote the pull-back of this class to B. Then using (11), the isomorphism

$$\pi_{n+1}(MB\mathbb{Q})\cong H\mathbb{Q}_{n+1}(MB)\overset{\mathtt{Thom}_{H\mathbb{Q}}}{\cong} H\mathbb{Q}_{n+1}(B)$$

and the Riemann–Roch theorem we can describe U' as the space of homomorphisms given by

(14) $$K^0(B)\to\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}),\quad \phi\mapsto q\big(\langle y, p_{n+1}(\mathbf{Td}^{-1}\cup\mathbf{ch}(\phi))\rangle\big)$$

for all $y\in H\mathbb{Q}_{n+1}(B)$. The class $\mathbf{Td}$ is the Todd class of the stable universal Spin^c–bundle over $B\mathrm{Spin}^c$. It plays the role of the universal normal bundle. The Riemann–Roch formula contains the Todd class of the tangent bundle which is complementary to the normal bundle. This explains the appearance of the inverse of the Todd class in the formula above.

2.4 Relation with the Adams spectral sequence and K–localisation

In this subsection we will see that the topological η–invariant essentially detects the first line in the K–based Adams spectral sequence $(E_r^{*,*}(E), d_r)$ for E. We refer to [2; 54] for details on the Adams spectral sequence. Note that in the literature on the Adams spectral sequence it is frequently assumed that it is based on a connective spectrum. This assumption is important for the discussion of convergence. In our situation we consider the Adams spectral sequence based on the nonconnective spectrum K. This causes no problems since we are not interested in convergence questions.

We define the spectrum $\overline{K}$ by the cofibre sequence

$$\Sigma^{-1}\overline{K} \xrightarrow{\text{fib}} S \xrightarrow{\text{unit}} K \to \overline{K}$$

and form the Adams tower:

$$\begin{tikzcd}
\Sigma^{-2}E\wedge\overline{K}\wedge\overline{K} \arrow[d, "\mathtt{id}_E\wedge\mathtt{id}_{\overline{K}}\wedge\text{fib}"'] & \\
\Sigma^{-1}E\wedge\overline{K} \arrow[r, "\epsilon"] \arrow[d, "\mathtt{id}_E\wedge\text{fib}"'] & \Sigma^{-1}E\wedge\overline{K}\wedge K \arrow[ul, dotted] \\
E \arrow[r, "\epsilon"] & E\wedge K \arrow[ul, dotted]
\end{tikzcd} \tag{15}$$

The horizontal maps are induced by the unit of K as in (1) while the dotted arrows are degree-one maps which turn the triangles into fibre sequences. In the left column we have moved all shifts to the left. This tower gives rise to an Adams spectral sequence

$$(E_r^{*,*}(E), d_r)_{r\geq 1}$$

and a filtration $(\mathcal{F}^s\pi_*(E))_{s\geq 0}$ of the homotopy groups of E, where $\mathcal{F}^s\pi_*(E)\subseteq\pi_*(E)$ is defined as the image of $\pi_*(\Sigma^{-s}E\wedge\overline{K}^{\wedge s})$. We have a natural injective map

$$s\colon \mathrm{Gr}^1\pi_*(E)\to E_2^{1,*+1}(E).$$

Theorem 2.7 (1) *We have $\mathcal{F}^1\pi_*(E)\subseteq\pi_*(E)_{\mathrm{tors}}$.*

(2) *We have $\mathcal{F}^2\pi_*(E)\subseteq\mathtt{ker}(\eta^{\mathrm{top}})$. Hence η^{top} induces a map (denoted by the same symbol)*

$$\eta^{\mathrm{top}}\colon \mathrm{Gr}^1\pi_*(E)\to Q_*(E).$$

(3) *There exists a map*

$$\kappa\colon E_2^{1,*+1}(E)\to Q_*(E)$$

such that

$$\eta^{\mathrm{top}} = \kappa \circ s\colon \mathrm{Gr}^1 \pi_n(E) \to Q_n(E).$$

(4) *If $\pi_*(E \wedge K)$ is torsion-free, then*

$$\pi_*(E)_{\mathrm{tors}} = \mathcal{F}^1 \pi_*(E).$$

If n is odd, then the restriction

$$\kappa|_{E_2^{1,n+1}(E)_{\mathrm{tors}}}\colon E_2^{1,n+1}(E)_{\mathrm{tors}} \to Q_n(E)$$

of κ to the torsion subgroup is injective. Consequently, the map

$$\eta^{\mathrm{top}}\colon \mathrm{Gr}^1 \pi_n(E) \to Q_n(E)$$

induced by η^{top} is injective.

Proof We have a map

$$f\colon K \to K\mathbb{Q} \simeq \prod_{p \in \mathbb{Z}} \Sigma^{2p} H\mathbb{Q} \xrightarrow{p_0} H\mathbb{Q},$$

which we use to build the map between the K–based and the $H\mathbb{Q}$–based Adams towers:

$$\begin{tikzcd}
\Sigma^{-2} E \wedge \overline{K} \wedge \overline{K} \arrow[d] & & & \\
\Sigma^{-1} E \wedge \overline{K} \arrow[rr] \arrow[dr] \arrow[dd] & & \Sigma^{-1} E\mathbb{Q}/\mathbb{Z} \arrow[dr] \arrow[dd] & \\
& \Sigma^{-1} E \wedge \overline{K} \wedge K \arrow[rr] \arrow[uul, dotted] & & \Sigma^{-1} E\mathbb{Q}/\mathbb{Z} \wedge K \\
E \arrow[rr, equal] \arrow[dr] & & E \arrow[dr] & \\
& E \wedge K \arrow[rr, "\mathtt{id}_E \wedge f"] \arrow[uuul, dotted] & & E\mathbb{Q} \arrow[uuul, dotted]
\end{tikzcd}.$$

It connects a piece of the Adams tower of E with the basic diagram used in the definition of η^{top}.

Assertions (1) and (2) follow immediately from diagram chases. We now show (3). The map

$$E_1^{1,*+1}(E) = \pi_*(\Sigma^{-1} E \wedge \overline{K} \wedge K) \to \pi_*(\Sigma^{-1} E\mathbb{Q}/\mathbb{Z} \wedge K) \to Q_*(E)$$

annihilates the image of $\pi_*(E\wedge K)$, ie the image of the boundary map of the spectral sequence, and therefore factors through a map

$$\kappa\colon E_2^{1,*+1}(E)\to Q_n(E).$$

Assertion (3) again follows by a diagram chase.

We now show (4). By the additional assumption that $\pi_*(E\wedge K)$ is torsion-free, the map $\pi_*(E)\to\pi_*(E\wedge K)$ annihilates $\pi_*(E)_{\mathrm{tors}}$. In view of (1) we conclude the first assertion, that $\pi_*(E)_{\mathrm{tors}}=\mathcal{F}^1\pi_*(E)$. As in [24, Section 5.3] our additional assumption furthermore implies that we get a short exact sequence

$$(16)\qquad 0\to E_2^{0,n+1}(E)\otimes\mathbb{Q}/\mathbb{Z}\xrightarrow{j}E_2^{0,n+1}(E\mathbb{Q}/\mathbb{Z})\xrightarrow{\delta}E_2^{1,n+1}(E)_{\mathrm{tors}}\to 0.$$

Remark 2.8 For the convenience of the reader we recall the construction of (16). We start with the observation [4] that $\pi_0(K\wedge K)$ is a free $\mathbb{Z}$–module and

$$\pi_*(K\wedge K)\cong\pi_*(K)\otimes_{\mathbb{Z}}\pi_0(K\wedge K)$$

is a free $\pi_*(K)$–module. For any spectrum F we thus get an isomorphism of abelian groups

$$\pi_*(F\wedge K\wedge K)\cong\pi_*(F\wedge K)\otimes_{\mathbb{Z}}\pi_0(K\wedge K).$$

If $\pi_*(F\wedge K)$ is torsion-free, then so is $\pi_*(F\wedge K\wedge K)$. We now observe that the product of the ring spectrum K provides a split of the fibre sequence

$$\cdots\to F\wedge K\to F\wedge K\wedge K\to F\wedge K\wedge\bar{K}\to\cdots.$$

Consequently, $\pi_*(F\wedge K\wedge\bar{K})$ is a direct summand of a torsion-free abelian group $\pi_*(F\wedge K\wedge K)$ and hence torsion-free, too. We apply this argument to the spectra $F:=E\wedge K\wedge\bar{K}^{\wedge p}$ appearing in the Adams tower. Starting from our assumption that $\pi_*(E\wedge K)$ is torsion-free we conclude inductively that their homotopy groups are torsion-free. These homotopy groups assemble the first page $E_1^{*,*}(E)$, which therefore consists of torsion-free groups.

We now consider the K–based Adams spectral sequence $(E_r^{*,*}(E\mathbb{Q}/\mathbb{Z}),d_r)$. The Adams tower is obtained from (15) by smashing with $\mathtt{M}\mathbb{Q}/\mathbb{Z}$. The sequence (2) provides short exact sequences

$$0\to E_1^{s,t}(E)\otimes\mathbb{Q}/\mathbb{Z}\to E_1^{s,t}(E\mathbb{Q}/\mathbb{Z})\to E_1^{s,t-1}(E)*\mathbb{Q}/\mathbb{Z}\to 0$$

for all $s,t\in\mathbb{Z}$. Since $E_1^{*,*}(E)$ consists of torsion-free groups we get isomorphisms

$$(17)\qquad E_1^{s,t}(E)\otimes\mathbb{Q}/\mathbb{Z}\cong E_1^{s,t}(E\mathbb{Q}/\mathbb{Z}).$$

We now note that the first differential $d_1\colon E_1^{s,t} \to E_1^{s+1,t}$ is induced by the morphism

$$\Sigma^{-s} E \wedge K \wedge \bar{K}^{\wedge s} \to \Sigma^{-s-1} E \wedge K \wedge \bar{K}^{\wedge(s+1)}$$

of spectra, which exists before smashing with $\mathtt{M}\mathbb{Q}/\mathbb{Z}$ and taking homotopy groups. Therefore the isomorphism (17) is compatible with the differentials of $E_1^{*,*}(E)$ and $E_1^{*,*}(E\mathbb{Q}/\mathbb{Z})$. We conclude that the complex $(E_1^{*,*}(E\mathbb{Q}/\mathbb{Z}), d_1)$ is obtained from the complex of free abelian groups $(E_1^{*,*}(E), d_1)$ by tensoring with $\mathbb{Q}/\mathbb{Z}$. Hence we can apply the universal coefficient theorem, which gives short exact sequences

$$0 \to E_2^{s,t}(E) \otimes \mathbb{Q}/\mathbb{Z} \to E_2^{s,t}(E\mathbb{Q}/\mathbb{Z}) \to E_2^{s+1,t}(E) * \mathbb{Q}/\mathbb{Z} \to 0$$

for all $s, t \in \mathbb{Z}$. The desired sequence (16) is the special case $s = 0$ and $t = n+1$. ◁

By Pontrjagin duality, for odd n we have isomorphisms

$$\begin{aligned} \pi_{n+1}(E\mathbb{Q}/\mathbb{Z} \wedge K) &\cong \mathtt{Hom}^{\mathrm{cont}}(K^{n+1}(E), \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})). \end{aligned} \tag{18}$$

Remark 2.9 For completeness of the presentation we give the argument. The evaluation pairing (5) provides for every spectrum F a morphism

$$\pi_*(F\mathbb{Q}/\mathbb{Z} \wedge K) \to \mathtt{Hom}^{\mathrm{cont}}(K^*(F), \mathbb{Q}/\mathbb{Z}). \tag{19}$$

We claim that this is an isomorphism. The left-hand side preserves finite sums of spectra and sends fibre sequences to long exact sequences. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group the functor on the right-hand side has the same properties as long as the groups involved have the discrete topology. Since (19) is clearly an isomorphism for $F = S$ (the sphere spectrum) it is an isomorphism for all finite spectra.

We now notice that the left-hand side preserves filtered colimits. If (F_α) is a filtered system of finite spectra and $F := \mathtt{colim}_\alpha F_\alpha$, then by definition of the profinite topology on $K^*(F)$ and since $\mathbb{Q}/\mathbb{Z}$ is considered as a discrete abelian group we have

$$\mathtt{Hom}^{\mathrm{cont}}(K^*(F), \mathbb{Q}/\mathbb{Z}) \cong \mathtt{colim}_\alpha \mathtt{Hom}(K^*(F_\alpha), \mathbb{Q}/\mathbb{Z}).$$

So the right-hand side of (19) is compatible with filtered colimits of finite spectra, too. Consequently (19) is an isomorphism for all spectra F. ◁

We can now finish the argument for the second part of (4). Let $\gamma \in E_2^{1,n+1}(E)_{\mathrm{tors}}$ and assume that $\kappa(\gamma) = 0$. Since δ in (16) is surjective, we can write $\gamma = \delta(\beta)$ for some $\beta \in E_2^{0,n+1}(E\mathbb{Q}/\mathbb{Z})$. We have $\kappa(\gamma) = 0$ if and only if there exists $y \in \pi_{n+1}(E\mathbb{Q})$ which induces the same pairing with $K^0(E)$ as $a(\beta)$, where $a\colon E_2^{0,n+1}(E\mathbb{Q}/\mathbb{Z}) \to$

$\pi_{n+1}(E\mathbb{Q}/\mathbb{Z}\wedge K)$ is the inclusion of the kernel of the first differential. From the injectivity of (18) we conclude that

$$a(\beta)=a\big(j(q(\epsilon(y)))\big),$$

where $\epsilon(y)\in E_2^{0,n+1}(E\mathbb{Q})$ and

$$q\colon E_2^{0,n+1}(E\mathbb{Q})\cong E_2^{0,n+1}(E)\otimes\mathbb{Q}\to E_2^{0,n+1}(E)\otimes\mathbb{Q}/\mathbb{Z}.$$

Since a is injective this implies $\gamma=\delta\big(j(q(\epsilon(y)))\big)=0$. □

We let $E\to E_K$ denote the Bousfield localisation of the spectrum E at the complex K–theory spectrum. The following lemma shows that the universal η–variant factorises over the K–localisation.

Lemma 2.10 *We have a commuting diagram:*

$$\begin{tikzcd} \pi_n(E)_{\mathrm{tors}} \arrow[r, "\eta^{\mathrm{top}}"] \arrow[d] & Q_n(E) \arrow[d, "\cong"] \\ \pi_n(E_K)_{\mathrm{tors}} \arrow[r, "\eta^{\mathrm{top}}"] & Q_n(E_K) \end{tikzcd}$$

Proof Since $E\to E_K$ induces an isomorphism in K–theory we conclude that the right vertical map is an equivalence. The rest is Lemma 2.4. □

Remark 2.11 The principal ideas behind Theorem 2.7 are not new and go back to Adams [1]. Our approach here can be considered as a simple version of [46, Proposition 3.3.2; 47], where these references focus on the second and higher lines of the spectral sequence and filtration steps of the homotopy groups.

3 The spectral geometric construction

3.1 Motivation

In this section we define an analytic invariant η^{an} of torsion elements in the B–bordism theory. The analytic invariant will be derived from geometric and spectral geometric quantities associated to geometric cycles for bordism classes. The relation between the geometric and homotopy-theoretic picture of the bordism group is given by Thom–Pontrjagin construction; see [55, Chapter IV.7]. In Section 3.2 we give the details of the geometric picture of the B–bordism theory. Section 3.3 is devoted to some technical details on the transfer of Spin^c–structures from the normal bundle to the tangent bundle.

A reader with some experience with the Thom–Pontrjagin construction and Spin^c–structures may immediately proceed to the construction of η^{an} in Section 3.4. The final Section 3.5 of this part contains the proof of the theorem about the equality of the analytic and topological universal η–invariants.

3.2 Geometric cycles for B–bordism theory

We assume that we have chosen representatives of the homotopy types of the classifying spaces of the classical Lie groups, like $O(k)$, $\mathrm{Spin}^c(k)$, etc, and representatives of the homotopy classes of the usual maps connecting them, like $B\mathrm{Spin}^c(k) \to BO(k)$ or $\iota_k\colon BO(k) \to BO(k+1)$. We further choose universal euclidean bundles ξ_k on $BO(k)$ and isomorphisms

$$\iota_k^* \xi_{k+1} \cong \xi_k \oplus \underline{\mathbb{R}}_{BO(k)} \tag{20}$$

for all $k \in \mathbb{N}$, where for a vector space V we let $\underline{V}_M$ denote the trivial bundle on the space M with fibre V.

We consider a map of spaces $B \to B\mathrm{Spin}^c$ and the corresponding Thom spectrum MB.

Remark 3.1 For the convenience of the reader we briefly recall its construction. Up to homotopy equivalence we can assume that we have the following situation. We have a sequence of cofibrations between topological spaces

$$\cdots \to B\mathrm{Spin}^c(k-1) \to B\mathrm{Spin}^c(k) \to B\mathrm{Spin}^c(k+1) \to \cdots .$$

The homotopy type of the classifying space of Spin^c is then represented by the topological space

$$B\mathrm{Spin}^c := \mathtt{colim}_k B\mathrm{Spin}^c(k).$$

We can furthermore assume that we have a sequence of cofibrations

$$\cdots \to B(k-1) \to B(k) \to B(k+1) \to \cdots$$

and a homeomorphism

$$B \cong \mathtt{colim}_k B(k),$$

and that the map $B \to B\mathrm{Spin}^c$ is induced by a sequence of commuting diagrams:

$$\begin{tikzcd} B(k) \arrow[r, "\rho_k"] \arrow[d] & B\mathrm{Spin}^c(k) \arrow[d] \\ B(k+1) \arrow[r, "\rho_{k+1}"] & B\mathrm{Spin}^c(k+1) \end{tikzcd}$$

For every k we let $\xi_k^{\mathrm{Spin}^c}$ denote the pull-back of the universal bundle ξ_k along $B\mathrm{Spin}^c(k) \to BO(k)$. Then we form the Thom space

$$\mathtt{Th}(\rho_k^*\xi_k^{\mathrm{Spin}^c}) := D(\rho_k^*\xi_k^{\mathrm{Spin}^c})/S(\rho_k^*\xi_k^{\mathrm{Spin}^c}),$$

where D and S stand for the disc and sphere bundles. The identifications (20) induce maps

$$\Sigma\mathtt{Th}(\rho_k^*\xi_k^{\mathrm{Spin}^c}) \cong \mathtt{Th}\big((\rho_{k+1}^*\xi_{k+1}^{\mathrm{Spin}^c})|_{B_k}\big) \to \mathtt{Th}(\rho_k^*\xi_k^{\mathrm{Spin}^c}).$$

We now apply the suspension spectrum functor $\Sigma^{\infty-k} := \Sigma^{-k}\Sigma^{\infty}$ from pointed topological spaces to spectra in order to get the morphism of spectra

$$\Sigma^{\infty-k}\mathtt{Th}(\rho_k^*\xi_k^{\mathrm{Spin}^c}) \to \Sigma^{\infty-k-1}\mathtt{Th}(\rho_{k+1}^*\xi_{k+1}^{\mathrm{Spin}^c}).$$

The Thom spectrum is now defined as the (homotopy) colimit

$$MB := \mathtt{colim}_{k\in\mathbb{N}}\Sigma^{\infty-k}\mathtt{Th}(\rho_k^*\xi_k^{\mathrm{Spin}^c}). \qquad \triangleleft$$

For $n \in \mathbb{N}$ the homotopy group $\pi_n(MB)$ is called the n^{th} B–bordism group. The Pontrjagin–Thom construction provides an equivalent description of this group by cycles and relations. Cycles for elements in $\pi_n(MB)$ are pairs (M, f) consisting of a closed n–dimensional riemannian manifold M and a stable normal B–structure on the map $f\colon M \to B$. The additional data of a stable normal B structure is not written explicitly and fixes the relation between the map f and the tangent bundle of M. In the following we explain the details. Since M is compact there exists a factorisation

$$\begin{tikzcd} & & BO(k) \arrow[d] \\ M \arrow[rru, dotted, "\hat{f}"] \arrow[r, "f"] & B \arrow[r] & BO \end{tikzcd}$$

up to homotopy for a suitable integer k. We require that $\hat{f}^*\xi_k$ is a complement of the tangent bundle of M, ie that there exists an isomorphism

$$TM \oplus \hat{f}^*\xi_k \cong \underline{\mathbb{R}}_M^{n+k}. \tag{21}$$

Definition 3.2 A normal B–structure on f consists of the choice of $\hat{f}$ and the isomorphism (21).

There is an obvious notion of a stabilisation of a normal B–structure, which allows us to increase k. A stable normal B–structure is an equivalence class of normal B–structures under the relation generated by stabilisation.

The bordism group $\pi_n(MB)$ is the set of equivalence classes of cycles, where the equivalence relation is given by bordism, and the group structure is induced by the disjoint sum. A zero bordism of (M, f) is given by a pair (W, F) of similar data, where W is a compact $n+1$–dimensional riemannian manifold with boundary $\partial W \cong M$ and product structure and $F\colon W \to B$ carries a stable normal B–structure which extends the one on f. In detail this means the following: First of all the map $\widehat{F}$ extends $\widehat{f}$. The stable normal B–structure on W is represented by an isomorphism

$$TW \oplus \widehat{F}^*\xi_k \cong \underline{\mathbb{R}}_W^{n+1+k}. \tag{22}$$

The outgoing normal field of $TW|_{\partial W}$ provides an orthogonal decomposition

$$TW|_M \cong TM \oplus \underline{\mathbb{R}}_M. \tag{23}$$

It is here where we use the additional datum of the riemannian metrics in order to rigidify the choice of the unit normal vector field. We require that the isomorphism

$$TM \oplus \underline{\mathbb{R}}_M \oplus \widehat{f}^*\xi_k \overset{(23)}{\cong} TW|_M \oplus \widehat{f}^*\xi_k \overset{(22)|_M}{\cong} \underline{\mathbb{R}}_M^{n+1+k}$$

represents the stable normal B–structure on f.

3.3 Normal and tangential Spin^c–structures

Because of the factorisation $B \to B\mathrm{Spin}^c \to BO$, a normal B–structure induces a normal Spin^c–structure. As we will do geometry on the tangent bundle we must transfer normal Spin^c–structures to tangential Spin^c–structures. The homotopy-theoretic picture of this transition is explained in [24, Section 8] in the example of String–structures. In the following we describe its geometric counterpart.

Let $V \to M$ be an m–dimensional real vector bundle. Then a Spin^c–structure on V is a pair (P, κ), where $P \to M$ is a $\mathrm{Spin}^c(m)$–principal bundle and κ is an isomorphism of real vector bundles

$$\kappa\colon P \times_{\mathrm{Spin}^c(m)} \mathbb{R}^m \cong V.$$

With this definition a Spin^c–structure induces an euclidean metric and an orientation on V so that the oriented orthonormal frame bundle is $\mathrm{SO}(V) := P \times_{\mathrm{Spin}^c(m)} \mathrm{SO}(m)$.

The collection of all Spin^c–structures on the vector bundle V naturally forms a groupoid $\mathrm{Spin}^c(V)$. The objects of the groupoid $\mathrm{Spin}^c(V)$ are the Spin^c–structures (P, κ), and the morphisms $(P, \kappa) \to (P', \kappa')$ are isomorphisms of $\mathrm{Spin}^c(m)$–principal bundles $P \to P'$ which are compatible with the isomorphisms κ and κ'. This in particular implies that automorphisms of (P, κ) are given by the central action of $C^\infty(M, U(1))$ on P.

If we associate to any open subset $A \subseteq M$ the groupoid $\mathrm{Spin}^c(V|_A)$, then we obtain a sheaf of groupoids $\underline{\mathrm{Spin}^c(V)}$, which actually is a $U(1)$–banded gerbe. We refer to [21; 52; 36] for an introduction to gerbes. Isomorphism classes of $U(1)$–banded gerbes $\mathcal{G}$ are classified by their Dixmier–Douady classes $\mathtt{DD}(\mathcal{G}) \in H^3(M;\mathbb{Z})$. In particular, the Dixmier–Douday class of the Spin^c–gerbe $\underline{\mathrm{Spin}^c(V)}$ is the class

$$\mathtt{DD}(\underline{\mathrm{Spin}^c(V)}) = W_3(V) = \beta(w_2(V)) \in H^3(M;\mathbb{Z}),$$

where $w_2(V) \in H^2(M;\mathbb{Z}/2\mathbb{Z})$ is the second Stiefel–Whitney class and

$$\beta\colon H^2(M;\mathbb{Z}/2\mathbb{Z}) \to H^3(M;\mathbb{Z})$$

is the Bockstein operator [48, Theorem D2]. The groupoid $\mathrm{Spin}^c(V)$ is nonempty if and only if $W_3(V) = 0$, ie the class $W_3(V)$ is the obstruction against the existence of a Spin^c–structure on V. In the following we will simplify the notation and write P for the Spin^c–structure (P,κ).

Let $\mathcal{B}U(1)(M)$ denote the Picard groupoid (see [28]) of $U(1)$–principal bundles on M. Given an $U(1)$–principal bundle $E \in \mathcal{B}U(1)(M)$ and a Spin^c–structure $P \in \mathrm{Spin}^c(V)$, we can define a new Spin^c–structure $E \otimes P \in \mathrm{Spin}^c(V)$. A formula for this tensor product is given by (26) specialised to the case $n = 0$; see below. This construction defines a bifunctor

$$\mathcal{B}U(1)(M) \times \mathrm{Spin}^c(V) \to \mathrm{Spin}^c(V). \tag{24}$$

If $\mathrm{Spin}^c(V)$ is not empty, then the set of isomorphism classes of Spin^c–structures on V is a torsor over the group of isomorphism classes in $\mathcal{B}U(1)(M)$. Since the latter is canonically isomorphic to $H^2(M;\mathbb{Z})$ we get a simply transitive action of $H^2(M;\mathbb{Z})$ on $\mathrm{Spin}^c(V)/\mathrm{iso}$. Furthermore, we get natural isomorphisms

$$C^\infty(M,U(1)) \cong \mathtt{Aut}_{\mathcal{B}U(1)(M)}(E) \cong \mathtt{Aut}_{\mathrm{Spin}^c(V)}(E \otimes P) \cong \mathtt{Aut}_{\mathrm{Spin}^c(V)}(P).$$

The sum of two vector bundles with Spin^c–structures has a naturally induced Spin^c–structure. This is formalised as the natural bifunctor

$$\mathrm{Spin}^c(V) \times \mathrm{Spin}^c(U) \to \mathrm{Spin}^c(V \oplus U). \tag{25}$$

On the level of objects this bifunctor is given by

$$(P,Q) \mapsto P \otimes Q,$$

where the $\mathrm{Spin}^c(n+m)$–principal bundle

$$P \otimes Q := (P \times_M Q) \times_{(\mathrm{Spin}^c(n)\times \mathrm{Spin}^c(m))} \mathrm{Spin}^c(n+m) \tag{26}$$

is obtained from the $\mathrm{Spin}^c(n)\times\mathrm{Spin}^c(m)$–principal bundle $P\times_M Q$ by extension of structure groups along the upper horizontal map in the diagram:

$$\begin{array}{ccc} \mathrm{Spin}^c(n)\times\mathrm{Spin}^c(m) & \longrightarrow & \mathrm{Spin}^c(n+m) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n)\times\mathrm{SO}(m) & \longrightarrow & \mathrm{SO}(n+m) \end{array}$$

Here $n=\dim(V)$ and $m=\dim(U)$, and the compatibility with the lower part of this diagram is used to define the structure map $\kappa_{P\otimes Q}$ from κ_P and κ_Q. The bifunctor comes equipped with natural associativity constraints. We omit the details of the latter two aspects.

We set $\mathrm{Spin}^c(0):=U(1)$ and let 0_M denote the zero-dimensional vector bundle on M. Then we get an identification $\mathrm{Spin}^c(0_M)\cong\mathcal{B}U(1)(M)$, and for $n=0$ the bifunctor (25) specialises to (24). As a consequence of associativity the bifunctor (25) is compatible with the action (24) of $\mathcal{B}U(1)(M)$ in the sense that for $E\in\mathcal{B}U(1)(M)$ have natural isomorphisms

$$(E\otimes P)\otimes Q\cong E\otimes(P\otimes Q)\cong P\otimes(E\otimes Q). \tag{27}$$

The trivialised vector bundle $\underline{\mathbb{R}}^n_M$ has a preferred trivial Spin^c–structure $Q(n):=M\times\mathrm{Spin}^c(n)$. We can use this to produce a canonical equivalence of groupoids

$$\mathrm{Spin}^c(V)\cong\mathrm{Spin}^c(V\oplus\underline{\mathbb{R}}^n_M),\qquad P\mapsto P\otimes Q(n).$$

On the level of Spin^c–structures we speak of stabilisations.

Let us now consider a pair (M,f) of a compact n–dimensional riemannian manifold and a map $f\colon M\to B$ which admits a refinement to a stable normal B–structure. Then we can assume that f has a factorisation up to homotopy over $B\mathrm{Spin}^c(k)$ as in the diagram:

$$\begin{tikzcd} & & \xi_k^{\mathrm{Spin}^c} \arrow[r] \arrow[d] & \xi_k^{\mathrm{SO}} \arrow[r] \arrow[d] & \xi_k \arrow[d] \\ & & B\mathrm{Spin}^c(k) \arrow[r] \arrow[d] & B\mathrm{SO}(k) \arrow[r] \arrow[d] & BO(k) \arrow[d] \\ M \arrow[r, "f"'] \arrow[urr, dotted, "\tilde f"] & B \arrow[r] & B\mathrm{Spin}^c \arrow[r] & B\mathrm{SO} \arrow[r] & BO \end{tikzcd} \tag{28}$$

The map $\tilde f$ classifies a $\mathrm{Spin}^c(k)$–principal bundle $\tilde f^*Q_k\to M$, where $Q_k\to B\mathrm{Spin}^c(k)$ is the universal $\mathrm{Spin}^c(k)$–bundle. Note that $\tilde f^*Q_k\in\mathrm{Spin}^c(\tilde f^*\xi_k^{\mathrm{Spin}^c})$.

We let $\hat{f}\colon M \to BO(k)$ be induced by $\tilde{f}$, so that $\hat{f}^*\xi_k \cong \tilde{f}^*\xi_k^{\mathrm{Spin}^c}$. With these identifications the trivialisation (21) induces a bifunctor (25)

$$\mathrm{Spin}^c(TM) \times \mathrm{Spin}^c(\tilde{f}^*\xi_k^{\mathrm{Spin}^c}) \to \mathrm{Spin}^c(\underline{\mathbb{R}}_M^{n+k}).$$

Since $\mathcal{B}U(1)(M)$ acts simply transitively on isomorphisms classes, we conclude using (27) that there is a unique isomorphism class of Spin^c–structures $P \in \mathrm{Spin}^c(TM)$ such that

$$P \otimes \tilde{f}^* Q_k \cong Q(n+k). \tag{29}$$

One can further check that this isomorphism class only depends on the normal B–structure represented by f and not on its representative. This is the tangential Spin^c–structure determined by the normal Spin^c–structure.

For constructions which involve gluing or in the notion of a Spin^c–map we need a rigidified notion of a tangential Spin^c–structure.

Definition 3.3 Assume that we have fixed a normal B–structure in terms of the factorisation $\tilde{f}$ and the isomorphism (21). Then we define a tangential Spin^c–structure as a pair of a Spin^c–structure $P \in \mathrm{Spin}^c(TM)$ together with a choice of an isomorphism in (29).

There are many tangential Spin^c–structures associated to a normal Spin^c–structure, but the main point is that two of them are isomorphic by a canonical isomorphism.

Let $h\colon M \to W$ be a smooth map and assume that we are given oriented euclidean vector bundles $V_M \to M$ and $V_W \to W$ together with an isomorphism

$$V_M \oplus \underline{\mathbb{R}}_M^k \cong h^* V_W \oplus \underline{\mathbb{R}}_M^l. \tag{30}$$

Assume further that we are given Spin^c–structures P_M and P_W in $\mathrm{Spin}^c(V_M)$ and $\mathrm{Spin}^c(V_W)$. If the Spin^c–structures P_M and h^*P_W are stably isomorphic, then we can define the following notion:

Definition 3.4 A refinement of h to a Spin^c–map is a choice of an isomorphism

$$h^*P_W \otimes Q(l) \cong P_M \otimes Q(k) \tag{31}$$

in $\mathrm{Spin}^c(V_M \oplus \underline{\mathbb{R}}_M^k)$ (this uses (30)). The equivalence class of a Spin^c–map under stabilisation will be called a stable Spin^c–map.

Being a Spin^c–map is an additional datum, not just a property of the map. Observe that we can compose stable Spin^c–maps in a natural way.

We now assume that (W, F) is a zero bordism of (M, f). We choose a representative of the normal B–structure on W involving the factorisation $\tilde{F}\colon W \to B\mathrm{Spin}^c(k)$ as in (28). On M we take the induced factorisation $\tilde{f} := \tilde{F}|_M$. Then we have a natural decomposition of oriented euclidean vector bundles

$$TW|_M \cong TM \oplus \underline{\mathbb{R}}_M, \tag{32}$$

where we trivialise the normal bundle by the outgoing unit normal vector field. Assume now that we have chosen tangential Spin^c–structures $P(TM)$ and $P(TW)$ on M and W, respectively (see Definition 3.3). In this situation we get a natural refinement of the inclusion $M \to W$ to a Spin^c–map. This refinement is distinguished by the condition that the diagram in $\mathrm{Spin}^c(\underline{\mathbb{R}}_M^{n+1+k})$

$$\begin{tikzcd}
P(TM) \otimes Q(1) \otimes \tilde{f}^* Q_k \arrow[r, "\cong"] \arrow[d, "\cong"] & P(TW)|_M \otimes \tilde{f}^* Q_k \arrow[d, "\cong"] \\
Q(n+1+k) \arrow[r, equal] & Q(n+1+k)
\end{tikzcd} \tag{33}$$

commutes. Here the upper corners are interpreted in $\mathrm{Spin}^c(\underline{\mathbb{R}}_M^{n+1+k})$ using the normal B–structures on M or W, respectively. The vertical morphisms are given by the tangential Spin^c–structures; see (29). Finally, the upper horizontal isomorphism uses (32) and fixes the refinement of the inclusion $M \to W$ to a Spin^c–map.

3.4 The definition of η^{an}

We consider a cycle (M, f) for a class $x = [M, f] \in \pi_n(MB)$ and assume in addition that x is torsion. Then there exists a nonzero integer $l \in \mathbb{N}$ such that $lx = 0$. We can thus find a zero bordism (W, F) of the disjoint union of l copies of (M, f), which we denote by $l(M, f)$.

We will define $\eta^{\mathrm{an}}(x) \in Q_n(B)$ in terms of a collection of indices of associated $\mathbb{Z}/l\mathbb{Z}$–index problems [33]. In order to formulate these index problems and to express the indices in terms of geometric and spectral invariants we must choose appropriate geometric structures.

We choose a tangential Spin^c–structure $(P, \kappa) \in \mathrm{Spin}^c(TM)$ related to the stable normal Spin^c–structure on f. A connection $\tilde{\nabla}^{TM}$ on P induces via κ a connection on TM. We say that $\tilde{\nabla}^{TM}$ is a Spin^c–extension of the Levi-Civita connection on M if it induces the Levi-Civita connection $\nabla^{TM,\mathrm{LC}}$ on TM.

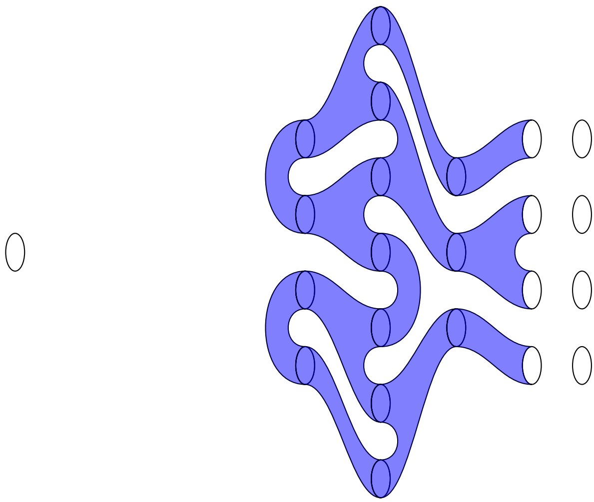

Figure 1: A picture of M and the zero bordism W of $4M$

The group $\mathrm{Spin}^c(n)$ has a distinguished unitary representation called the spinor representation Δ^n. For even n its dimension is $2^{n/2}$, and it has a decomposition $\Delta^n \cong \Delta^{n,+} \oplus \Delta^{n,-}$. It is related to the odd-dimensional case by $\Delta^{n,+}|_{\mathrm{Spin}^c(n-1)} \cong \Delta^{n-1}$.

The bundle $S(TM) := P \times_{\mathrm{Spin}^c(n)} \Delta^n \to M$ is called the spinor bundle of M. Given a Spin^c–extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection on M, the spinor bundle carries the structure of a Dirac bundle. We thus obtain the Spin^c–Dirac operator $\not{D}_M$ which acts on sections of $S(TM)$. Standard references for these constructions are [17, Chapter 3; 48, Appendix D].

If we are given a class $\phi \in K^0(B)$, then we can choose a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle $V \to M$ whose K–theory class satisfies $[V] = f^*\phi \in K^0(M)$. We choose a hermitian metric h^V and a metric connection ∇^V which preserve the grading. The triple $\boldsymbol{V} := (V, h^V, \nabla^V)$ will then be called a geometric vector bundle. We let $\not{D}_M \otimes \boldsymbol{V}$ be the Dirac operator twisted by $\boldsymbol{V}$. It acts on sections of $S(TM) \otimes V$.

We now assume that $n = \dim(M)$ is odd. The η–invariant [7] of the twisted Dirac operator

$$\eta(\not{D}_M \otimes \boldsymbol{V}) \in \mathbb{R}$$

is defined as the value at $z = 0$ of the meromorphic continuation of the η–function function

$$\eta(\not{D}_M \otimes \boldsymbol{V})(z) := \mathtt{Tr}_s\big(|\not{D}_M \otimes \boldsymbol{V}|^{-z}\mathtt{sign}(\not{D}_M \otimes \boldsymbol{V})\big),$$

where $\mathtt{Tr}_s$ is the supertrace with respect to the grading of V. Note that the trace exists if $\mathrm{Re}(z) > n$, and that the meromorphic continuation of the η–function is regular at $z = 0$ by the results of [7]. The η–invariant depends on the geometry of M and V in a possibly discontinuous way, with jumps when eigenvalues of $\not{D}_M \otimes \boldsymbol{V}$ cross zero. In order to get a quantity which depends continuously on the geometry, one usually considers the reduced η–invariant, for which we will use the symbol ξ:

$$\xi(\not{D}_M \otimes \boldsymbol{V}) := \left[\tfrac{1}{2}(\eta(\not{D}_M \otimes \boldsymbol{V}) + \dim \mathtt{ker}(\not{D}_M \otimes \boldsymbol{V}))\right] \in \mathbb{R}/\mathbb{Z}. \tag{34}$$

In an appropriate model of $\pi_n(MB\,\mathbb{Q}/\mathbb{Z})$ the zero bordism (W, F) of $l(M, f)$ geometrically represents the lift of x to a class

$$\hat{x} = [W, F] \in \pi_{n+1}(MB\,\mathbb{Q}/\mathbb{Z}),$$

using the notation of the diagram (4). We refer to Lemma 3.8 for more details. We choose a Spin^c–connection $\tilde{\nabla}^{TW}$ extending the Levi-Civita connection on TW which extends the connection on the boundary of W induced by $\tilde{\nabla}^{TM}$.

We can choose a compact subspace $B_c \subseteq B$ which contains the image of F. Given $\phi \in K^0(B)$ we choose a $\mathbb{Z}/2\mathbb{Z}$–graded complex vector bundle $V_c \to B_c$ such that $[V_c] = \phi|_{B_c}$ in $K^0(B_c)$. If we now take $V := f^* V_c$, then we have $[V] = f^*\phi$. The bundle $U := F^* V_c$ extends the bundle induced by V on ∂W to W. As above we choose a metric h^V and a metric connection ∇^V. This induces corresponding geometric structures on $U|_{\partial W}$. We then choose a hermitian metric h^U and a metric connection ∇^U on U which extend the already-given data on the boundary. In this way we get a geometric bundle $\boldsymbol{U} := (U, h^U, \nabla^U)$.

We can now form the Atiyah–Patodi–Singer boundary value problem for $\not{D}_W \otimes \boldsymbol{U}$. The analytic details of that boundary value problem are not important for our present purpose, so we refer to [7] for a precise description. We only have to know that it produces a Fredholm operator $(\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}}$ which has a well-defined index

$$\mathtt{index}((\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}}) \in \mathbb{Z},$$

and that the following index formula, proved in [7], holds true:

$$\mathtt{index}((\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}}) = \int_W p_{n+1}(\mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U)) - l\xi(\not{D}_M \otimes \boldsymbol{V}). \tag{35}$$

In this formula the closed form $\mathbf{Td}(\tilde{\nabla}^{TW}) \in \Omega^0(W)[b, b^{-1}]$ is the Chern–Weil representative determined by the universal class $\mathbf{Td} \in HP\,\mathbb{Q}^0(B\mathrm{Spin}^c)$ and the connection $\tilde{\nabla}^{TM}$. Similarly, the form $\mathbf{ch}(\nabla^U) \in \Omega^0(W)[b, b^{-1}]$ is the Chern–Weil representative determined by the class $\mathbf{ch} \in HP\,\mathbb{Q}^0(BU)$ and the connection ∇^U. Note that we use powers of b in order to shift the higher form-degree components to total degree zero. The projection p_{n+1} projects on the b–degree $\frac{1}{2}(n+1)$ part, hence produces a form of degree $n+1$.

We consider the element

$$e := \left[\frac{\mathtt{index}(\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}}}{l}\right] \in \mathbb{Q}/\mathbb{Z}. \tag{36}$$

Equivalently, by the index theorem (35) and (34) we can write

$$e = \left[\frac{1}{l}\int_W p_{n+1}(\mathbf{Td}(\widetilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U))\right] - \xi(\not{D}_M \otimes \boldsymbol{V}) \tag{37}$$

if we interpret this equality in $\mathbb{R}/\mathbb{Z}$. The quantity e can be interpreted as a $\mathbb{Z}/l\mathbb{Z}$–index in the sense of [33]. In the following proposition we state how the number e depends on the data.

Proposition 3.5 (1) *The value of e does not depend on the choices of the geometric structures on M and W.*

(2) *The value of e only depends on the K–theory class ϕ. This dependence is additive and determines an element $\tilde{e} \in \mathtt{Hom}^{\mathrm{cont}}(K^0(B), \mathbb{Q}/\mathbb{Z})$.*

(3) *The class $[\tilde{e}] \in Q_n(MB)$ of this homomorphism (using the presentation (13)) does not depend on l or the choice of the zero bordism of (W, F).*

(4) *The element $[\tilde{e}] \in Q_n(MB)$ described in (3) only depends on the bordism class x. This dependence is additive, so we obtain a well-defined homomorphism*

$$\eta^{\mathrm{an}}\colon \pi_n(MB)_{\mathrm{tors}} \to Q_n(MB).$$

Proof On the one hand, we have $e \in \frac{1}{l}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z}$. On the other hand, we know that the right-hand side of (37) depends continuously on the geometric data. This shows that e does not depend on the geometric structures at all since two choices of geometric structures can be connected by a family. This proves (1).

The element e depends additively on the bundle V_c. It therefore only depends on the class $\phi|_{B_c} := [V_c] \in K^0(B_c)$. The construction thus induces a homomorphism $\tilde{e} \in \mathtt{Hom}(K^0(B), \mathbb{Q}/\mathbb{Z})$. Since it factors over the restriction along the map $B_c \hookrightarrow B$ and B_c is compact, this homomorphism is continuous. We use the identification

$$\mathbb{Q}/\mathbb{Z} \cong \pi_0(K\mathbb{Q}/\mathbb{Z}) \xrightarrow{b^{-(n+1)/2}} \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$$

in order to interpret $\tilde{e}$ as an element of $\mathtt{Hom}^{\mathrm{cont}}(K^0(B), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$. This shows (2).

Assume that we have a second zero bordism (W', F') of $l'(M, f)$ yielding a homomorphism $\tilde{e}' \in \mathtt{Hom}^{\mathrm{cont}}(K^0(B), \mathbb{Q}/\mathbb{Z})$. Then by gluing along boundary components we

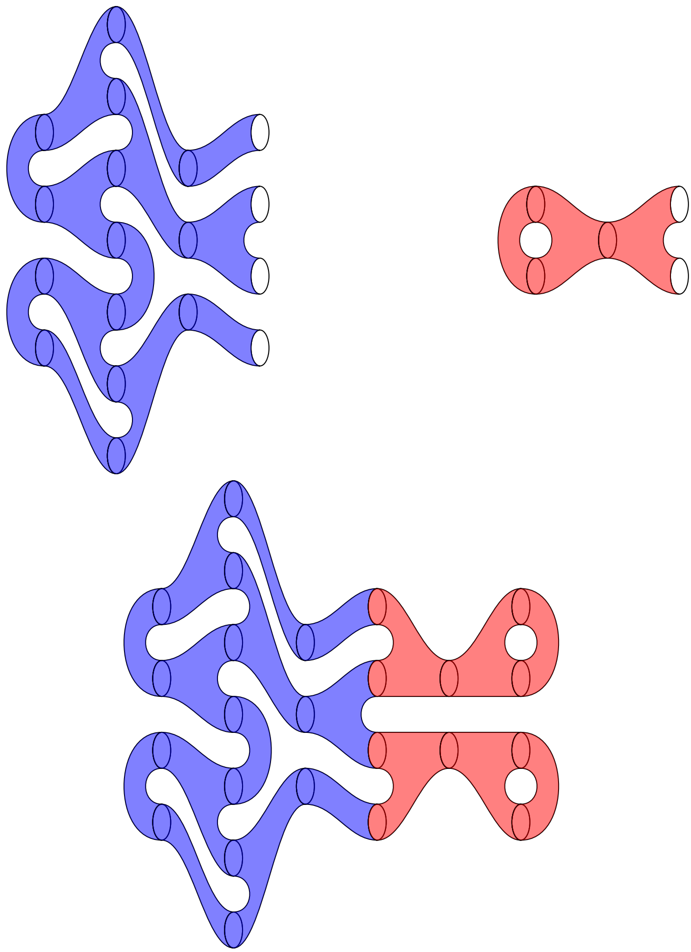

Figure 2: Pictures of W, W' and $\frac{1}{2}\widetilde{W}$ with $l = 4$ and $l' = 2$

can form the closed riemannian $n+1$–dimensional B–manifold $\widetilde{W} := l'W \cup_{ll'M} lW'$, which comes with a map $\widetilde{F}\colon \widetilde{W} \to B$. The latter has a natural refinement to a stable normal B–structure which restricts to the given stable normal B–structures on W and W'.

Note that the tangential Spin^c–structures (P,κ) and (P',κ') come with isomorphisms of the type (29). Compatibility with these fixes the morphism which we have to use to glue P with P'. In this way we get a tangential Spin^c–structure on $\widetilde{W}$. The triple $(\widetilde{W},\widetilde{F})$ is thus a cycle for a class

$$y := [\widetilde{W},\widetilde{F}] \in \pi_{n+1}(MB).$$

Then for $\phi \in K^0(B)$ we get from the right-hand side of (37) that

$$\tilde{e}(\phi) - \tilde{e}'(\phi) = \left[\frac{1}{ll'}\langle \mathbf{Td}(T\tilde{W}) \cup \tilde{F}^*\mathbf{ch}(\phi), [\tilde{W}]\rangle\right].$$

Since $\tilde{F}^*\mathbf{Td}^{-1} = \mathbf{Td}(T\tilde{W})$ this is exactly the formula (14) for the evaluation of $\epsilon\left(\frac{1}{ll'}y\right) \in \pi_{n+1}(K \wedge MB\mathbb{Q})$ against $\mathtt{Thom}^K(\phi) \in K^0(MB)$. Therefore, the class $[\tilde{e}] \in Q_n(MX)$ is independent of the choice of l and the zero bordism (W, F). This finishes the verification of (3).

We observe that the map which associates to (M, f) the class $[\tilde{e}] \in Q_n(MB)$ is additive under disjoint unions. Moreover, if (M, f) itself is zero bordant, ie we can find (W, F) as above with $l = 1$, then $[\tilde{e}] = 0$. It follows that the construction above uniquely descends to a homomorphism

$$\eta^{\mathrm{an}}\colon \pi_n(MB)_{\mathrm{tors}} \to Q_n(MB). \tag{38}$$ □

Let us collect the essentials of this construction in the following definition:

Definition 3.6 We define $\eta^{\mathrm{an}} := 0$ for even n. For odd n we define the homomorphism

$$\eta^{\mathrm{an}}\colon \pi_n(MB)_{\mathrm{tors}} \to Q_n(MB)$$

by the following prescription: If $x \in \pi_n(MB)_{\mathrm{tors}}$ is represented by (M, f), then we choose a zero bordism (W, F) of $l(M, f)$ for a suitable nonzero $l \in \mathbb{N}$. We choose a Spin^c–geometry for W such that the restrictions to the l copies of M in the boundary of W are again pairwise isomorphic.

We choose a compact subspace $B_c \subseteq B$ which contains the image of F. If $\phi \in K^0(B)$, then we choose a bundle $V_c \to B_c$ such that $[V_c] = \phi|_{B_c}$. We take $U := F^*V_c$ and choose a geometry $\boldsymbol{U} = (U, h^U, \nabla^U)$ such that the restrictions to the l copies of M in the boundary are pairwise isomorphic. Then $\eta^{\mathrm{an}}(x) \in Q_n(MB)$ is represented by the homomorphism

$$K^0(B) \to \mathbb{Q}/\mathbb{Z} \cong \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}), \quad \phi \mapsto \left[\frac{1}{l}\mathtt{index}((\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}})\right]. \tag{39}$$

3.5 The secondary index theorem

In Definitions 2.3 and 3.6 we have described homomorphisms

$$\eta^{\mathrm{top}}\colon \pi_n(MB)_{\mathrm{tors}} \to Q_n(MB), \quad \eta^{\mathrm{an}}\colon \pi_n(MB)_{\mathrm{tors}} \to Q_n(MB).$$

Both constructions follow a common idea. Given a torsion element $x \in \pi_n(MB)_{\mathrm{tors}}$, in a first step we choose a lift $\hat{x} \in \pi_n(MB\mathbb{Q}/\mathbb{Z})$ or, respectively, a geometric representative of such a lift. The homotopy theoretic invariant $\eta^{\mathrm{top}}(x)$ is represented by the homomorphism $K^0(MB) \to \mathbb{Q}/\mathbb{Z}$ induced by this lift via the homotopy-theoretic pairing between K–homology and cohomology. The analytic variant $\eta^{\mathrm{an}}(x)$ is represented by a homomorphism, which this time is obtained from a suitable family of Atiyah–Patodi–Singer index problems on the geometric representative of the lift $\hat{x}$. Because of these coincidences it is very natural to expect that the following theorem holds true:

Theorem 3.7 $$\eta^{\mathrm{an}} = \eta^{\mathrm{top}}.$$

Proof An obvious option is to apply the $\mathbb{Z}/l\mathbb{Z}$–index theorem [33] directly to η^{an} in order to express it in homotopy-theoretic terms. In this paper we decided to take a different path. It is interesting since it explains in greater detail the sense in which the homotopy-theoretic construction of η^{top} and the geometric or analytic constructions involved in η^{an} correspond to each other. Our bridge between analysis and topology will be the identification of homotopy-theoretic K–homology with the analytic picture [15] and the ordinary Atiyah–Singer index theorem for elliptic operators [11] and, respectively, its local form described in [17, Chapter IV].

Some ideas of our proof of Theorem 3.7, in particular the usage of Moore spaces, are taken from [33] and the proof of the $\mathbb{R}/\mathbb{Z}$–index theorem [9, Theorem 5.3].

We start with a description of Moore spaces for cyclic groups $\mathbb{Z}/l\mathbb{Z}$ for $l \in \mathbb{Z}$. Let $S^1 \to S^1$ be the l–fold covering of the pointed circle. Its mapping cylinder Z_l and mapping cone C_l fit into the cofibre sequence of pointed spaces

$$S^1 \to Z_l \to C_l \xrightarrow{\partial} \Sigma S^1 \to \cdots . \tag{40}$$

Note that the shifted suspension spectrum $\Sigma^{\infty-1} C_l$ is then a model for the Moore spectrum $\mathtt{M}\mathbb{Z}/l\mathbb{Z}$ discussed in Section 2.2. Further note that the inclusion of the cylinder basis $S^1 \to Z_l$ is a homotopy equivalence. Hence we have equivalences $\Sigma^{\infty-1} Z_l \simeq \Sigma^{\infty-1} S^1 \simeq \mathtt{M}\mathbb{Z}$. Applying the functor $\Sigma^{\infty-1}$ to the sequence (40) and using these identifications we get the fibre sequence

$$\mathtt{M}\mathbb{Z} \xrightarrow{l} \mathtt{M}\mathbb{Z} \to \mathtt{M}\mathbb{Z}/l\mathbb{Z} \xrightarrow{\partial} \Sigma\mathtt{M}\mathbb{Z} \tag{41}$$

of Moore spectra. We use the Moore spectra $\mathtt{M}\mathbb{Z}/l\mathbb{Z}$ and the sequence (41) as approximations for $\mathtt{M}\mathbb{Q}/\mathbb{Z}$ and (3) by spectra with finite skeleta in the sense that

$$\mathtt{M}\mathbb{Q}/\mathbb{Z} \simeq \mathtt{hocolim}_l \mathtt{M}\mathbb{Z}/l\mathbb{Z}.$$

The connecting maps for the system of Moore spectra $(\mathtt{M}\mathbb{Z}/l\mathbb{Z})_{l\in\mathbb{N}}$ are fixed by their compatibility with the inclusions

$$\mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \quad [n] \mapsto \left[\frac{n}{l}\right]. \tag{42}$$

Smashing the sequence (41) with MB and taking homotopy groups, we get a long exact sequence of abelian groups

$$\cdots \to \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z}) \xrightarrow{\partial} \pi_n(MB) \xrightarrow{l} \pi_n(MB) \to \pi_n(MB\mathbb{Z}/l\mathbb{Z}) \to \cdots . \tag{43}$$

The spectrum $MB \wedge X$ is related to Thom spectra (see Remark 3.1) by a fibre sequence

$$MB \to M(B \times X) \xrightarrow{\iota} MB \wedge X \to \Sigma MB,$$

where we use the structure map $B \times X \xrightarrow{\mathrm{pr}} B \to BO$ in order to define the Thom spectrum $M(B \times X)$ and the map $MB \to M(B \times X)$ is induced by the base point of X. We further have an equivalence of spectra

$$\Sigma MB\mathbb{Z}/l\mathbb{Z} \cong MB \wedge C_l. \tag{44}$$

We use the notation (F, G) in order to write maps from M to $B \times C_l$. In the following we construct a cycle $(\widetilde{W}, (\widetilde{F}, \widetilde{G}))$ (see Section 3.2 for notation) for a class in $\pi_{n+2}(M(B \times C_l))$ such that

$$\iota[\widetilde{W}, (\widetilde{F}, \widetilde{G})] = \hat{x} \in \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z})$$

under the identification (44), where $\widetilde{F}\colon \widetilde{W} \to B$ is the underlying map of a B–structure and $\widetilde{G}\colon \widetilde{W} \to C_l$. We will obtain $(\widetilde{W}, (\widetilde{F}, \widetilde{G}))$ from the zero bordism (W, F) found in Section 3.4.

The details are as follows. We consider a two-sphere S^2_l with l holes.

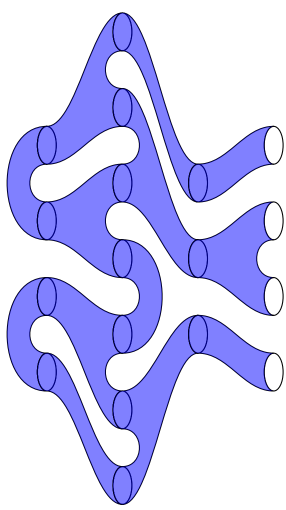

Figure 3: A picture of $S^1 \times W$

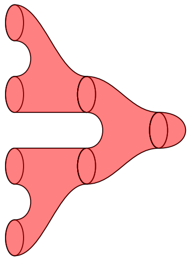

Figure 4: A picture of $S^2_4 \times M$

More precisely we let $S^2_l \subset S^2$ be the compact submanifold with boundary $\partial S^2_l \cong \bigsqcup_{i=1}^l S^1$ obtained by deleting the interiors of l disjoint discs from S^2. We equip S^2_l with a riemannian metric with product structure such that all boundary components are isometric to the standard S^1. The identification of the boundary with the l copies of S^1 is fixed such that it preserves the natural orientations. We now have an identification

$$\partial(S^1 \times W) \cong l(S^1 \times M) \cong \partial(S^2_l \times M).$$

We let

$$\tilde{W} := (S^1 \times W) \cup_{l(S^1 \times M)} (S^2_l \times M) \tag{45}$$

be the manifold obtained by gluing along the boundary.

We define $\tilde{F}\colon \tilde{W} \to B$ so that it restricts to

$$S^1 \times W \xrightarrow{\mathrm{pr}_W} W \xrightarrow{F} B, \quad S^2_l \times M \xrightarrow{\mathrm{pr}_M} M \xrightarrow{f} B.$$

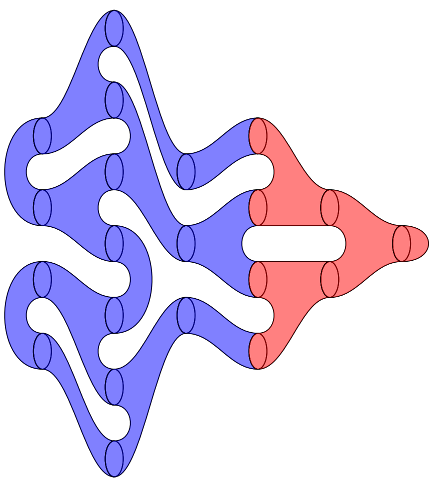

Figure 5: A picture of $\tilde{W}$

We must refine the map $\widetilde{F}$ to a normal B–structure. We start with the usual normal framing

$$TS^2 \oplus \underline{\mathbb{R}}_{S^2} \cong \underline{\mathbb{R}}^3_{S^2}$$

of S^2. If we take $k=1$ and let f and $\hat{f}$ in (21) be the constant maps, then we can interpret this isomorphism as a normal B–structure, too. By restriction we obtain a normal B–structure on S^2_l. Furthermore, the construction explained at the end of Section 3.2 provides normal B–structures

$$TS^1 \oplus \underline{\mathbb{R}}^2_{S^1} \cong \underline{\mathbb{R}}^3_{S^1}$$

on the l copies of S^1 in the boundary of S^2_l which are isomorphic to each other. We are given a normal B–structure

$$TW \oplus \hat{F}^*\xi_k \cong \underline{\mathbb{R}}^{n+1+k}_W.$$

We therefore get an induced normal B–structure on the product $S^1 \times W$ refining $\widetilde{F}|_{S^1\times W}$:

$$T(S^1\times W) \oplus \hat{F}'^{,*}\xi_{k+2} \cong TS^1 \oplus TW \oplus \underline{\mathbb{R}}^2_{S^1\times W} \oplus (\hat{F}^* \circ \mathtt{pr}_W)^*\xi_k \cong \underline{\mathbb{R}}^{n+k+4}_{S^1\times W},$$

where $\hat{F}'$ is the two-fold stabilisation of $\hat{F}\circ \mathtt{pr}_W$. In a similar manner, using the induced normal B structures on the copies of M in the boundary of W we get a normal B–structure

$$T(S^2_l \times M) \oplus \hat{f}^*\xi_{k+1} \cong \underline{\mathbb{R}}^{n+k+4}_{S^2_l\times M}$$

on the product $S^2_l \times M$ which refines $\widetilde{F}|_{S^2_l\times M}$. These isomorphisms coincide over the locus of gluing $l(S^1\times M)$. Hence we get a refinement of $\widetilde{F}$ to a normal B–structure.

We now consider the map

$$S^1\times W \xrightarrow{\mathtt{pr}_{S^1}} S^1 \xrightarrow{i} C_l, \tag{46}$$

where $i\colon S^1 \to C_l$ is the identification of S^1 with the basis of the mapping cone. Note that the map

$$\bigsqcup_{j=1}^{l} i\colon \partial S^2_l \cong \bigsqcup_{j=1}^{l} S^1 \to C_l$$

can be extended to a map

$$g\colon S^2_l \to C_l. \tag{47}$$

We can and will restrict the choice of g so that it is smooth on the preimage of a neighbourhood U of the cone basis $\partial C_l \subset C_l$ and regular values of g in the interior $U \setminus \partial C_l$ have exactly one preimage. The restriction of the map (46) to

$\partial(S^1 \times W) \cong l(S^1 \times M)$ thus has an extension across the other part $S^2_l \times M$ of $\widetilde{W}$, given by

$$S^2_l \times M \xrightarrow{\mathrm{pr}_{S^2_l}} S^2_l \xrightarrow{g} C_l .$$

Altogether we obtain the map $\widetilde{G}\colon \widetilde{W} \to C_l$. The cycle $(\widetilde{W}, (\widetilde{F}, \widetilde{G}))$ represents a class in $\pi_{n+2}(M(B \times C_l))$ and we consider its image

$$\hat{x} := \iota[\widetilde{W}, (\widetilde{F}, \widetilde{G})] \in \pi_{n+2}(MB \wedge C_l) \overset{(44)}{\cong} \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z}).$$

Let $\partial\colon \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z}) \to \pi_n(MB)$ be the boundary as in (43).

Lemma 3.8 $$\partial\hat{x} = x.$$

Proof The boundary operator ∂ in the lemma is induced by the map denoted by the same symbol in (40),

$$\partial\colon C_l \xrightarrow{p} \Sigma S^1 \cong S^2,$$

where p is the projection which contracts the cone basis to a point. Therefore, $\partial\hat{x} \in \pi_{n+2}(MB \wedge S^2)$ is represented by $(\widetilde{W}, (\widetilde{F}, p \circ \widetilde{G}))$. We must show that it corresponds to x under the suspension isomorphism

$$\pi_n(MB) \cong \pi_{n+2}(MB \wedge S^2).$$

To this end we invert the suspension isomorphism in the geometric picture. This inverse is of course given by taking the inverse image of a regular point in S^2 of the corresponding component $p \circ \widetilde{G}$ of the structure map. If we take the inverse image of a point in the neighbourhood $U \setminus \partial C_l$ mentioned above, we exactly recover the representative (M, f) of x. □

The construction of η^{top} involves the K–homology of a based space Y defined homotopy-theoretically as $\pi_*(K \wedge Y)$. It is equivalent to the analytic picture introduced in [15]. The analytic K–homology is subsumed in the more general bivariant KK–theory (see [43] and the textbook [18]), which allows us to treat K–homology and cohomology on equal footing. Of particular importance for our purpose is that the product in KK–theory provides a description of the $\cap$–product between K–homology and K–theory which easily compares with the operation of twisting Dirac operators.

The unit of K–theory induces the map (compare with (1))

$$\epsilon\colon \pi_{n+2}(MB \wedge C_l) \to \pi_{n+2}(K \wedge MB \wedge C_l). \tag{48}$$

We use the Thom isomorphism for MB in K–homology

$$\mathtt{Thom}_K\colon \pi_{n+2}(K \wedge MB \wedge C_l) \xrightarrow{\sim} \pi_{n+2}(K \wedge B_+ \wedge C_l). \tag{49}$$

Finally we use KK–theory in order to represent this K–homology of a pointed space analytically. For the moment we assume that X and B are compact. This is no real restriction since we are calculating with a finite number of cycles at a time and their structure maps can only hit compact parts of the spaces B and X. For a compact based space Y we let $C(Y)$ denote the C^*–algebra of continuous $\mathbb{C}$–valued functions which vanish on the base point. Then, by the equivalence between homotopy-theoretic and analytic K–homology [15], we have an isomorphism

$$\pi_{n+2}(K \wedge B_+ \wedge C_l) \cong KK_{n+2}(C(B_+ \wedge C_l), \mathbb{C}). \tag{50}$$

The Spin^c–extension of the Levi-Civita connection on W together with the standard Spin^c–geometry of S^1 induce a corresponding product Spin^c–extension of the Levi-Civita connection on $S^1 \times W$. The Spin^c–geometry on S^1 also induces such a geometry on the boundary $\partial S^2_l \cong l S^1$, which we extend to S^2_l, again with a product structure. We get a corresponding product Spin^c–extension of the Levi-Civita connection on $S^2_l \times M$. These geometric structures glue nicely and give a Spin^c–extension of the Levi-Civita connection on $\widetilde{W}$. We let $\not{D}_{\widetilde{W}}$ denote the corresponding Dirac operator. It acts on the complex spinor bundle $S(T\widetilde{W})$. The Hilbert space $L^2(\widetilde{W}, S(\widetilde{W}))$ of square integrable sections of this bundle carries an action ρ of the C^*–algebra $C(\widetilde{W}_+)$ of continuous functions on $\widetilde{W}$ by multiplication. The triple

$$(\not{D}_{\widetilde{W}}) := (L^2(\widetilde{W}, S(\widetilde{W})), \not{D}_{\widetilde{W}}, \rho)$$

is an unbounded Kasparov module for the pair of C^*–algebras $(C(\widetilde{W}_+), \mathbb{C})$ and represents a class

$$[\not{D}_{\widetilde{W}}] \in KK_{n+2}(C(\widetilde{W}_+), \mathbb{C}).$$

The map $(\widetilde{F}, \widetilde{G})$ induces a homomorphism of C^*–algebras

$$(\widetilde{F}, \widetilde{G})^* \colon C(B_+ \wedge C_l) \to C(\widetilde{W}_+),$$

which in turn induces the push-forward in analytic K–homology in the statement of the following lemma:

Lemma 3.9 *The image of the class $\hat{x} \in \pi_{n+2}(MB \wedge C_l)$ under the composition of the unit (48), Thom isomorphism (49) and the identification (50) is given by*

$$(\widetilde{F}, \widetilde{G})_*[\not{D}_{\widetilde{W}}] \in KK_{n+2}(C(B_+ \wedge C_l), \mathbb{C}).$$

Proof The image of $\hat{x}$ under the unit and Thom isomorphism is given

$$\mathtt{Thom}_K(\epsilon(\hat{x})) = (\beta \wedge \mathtt{id}_{B_+ \wedge C_l})((\Delta \wedge \mathtt{id}_{C_l})(\hat{x})) \in \pi_{n+2}(K \wedge B_+ \wedge C_l),$$

where $\Delta\colon MB \to MB \wedge B_+$ is the Thom diagonal and $\beta\colon MB \to K$ is the K–orientation of MB given by (12). We have

$$(\Delta \wedge \mathtt{id}_{C_l})(\hat{x}) = \iota[\widetilde{W}, (\widetilde{F}, (\widetilde{F}, \widetilde{G}))] \in \pi_{n+2}(MB \wedge B_+ \wedge C_l).$$

Formally we can view this as the push-forward of the B–bordism fundamental class of $\widetilde{W}$ along the map $(\widetilde{F}, \widetilde{G})$. Its image under the K–orientation $(\beta \wedge \mathtt{id}_{B_+ \wedge C_l})$ is then the push-forward of the K–theory fundamental class of $\widetilde{W}$ associated to the Spin^c–structure along this map. In the analytic picture of K–homology the K–theory fundamental class of $\widetilde{W}$ is represented by the Spin^c–Dirac operator. Hence it is equal to $[\not{D}_{\widetilde{W}}]$. We thus get

$$(\beta \wedge \mathtt{id}_{B_+ \wedge C_l})(\iota[\widetilde{W}, (\widetilde{F}, (\widetilde{F}, \widetilde{G}))]) = (\widetilde{F}, \widetilde{G})_*[\not{D}_{\widetilde{W}}]. \qquad \square$$

We let $\phi \in K^0(B)$. The pairing on the left-hand side in the following calculation in $\pi_{n+2}(K \wedge C_l) \cong \mathbb{Z}/l\mathbb{Z}$ is reminiscent to the evaluation occurring in the definition of η^{top}:

$$\begin{aligned} \langle \mathtt{Thom}^K(\phi), \epsilon(\hat{x}) \rangle &= \langle \phi, \mathtt{Thom}_K(\epsilon(\hat{x})) \rangle \\ &= \langle \phi, (\widetilde{F}, \widetilde{G})_*[\not{D}_{\widetilde{W}}] \rangle \quad \text{by Lemma 3.9} \\ &= \widetilde{G}_*([\not{D}_{\widetilde{W}}] \cap \widetilde{F}^*\phi). \end{aligned}$$

We choose a geometric bundle $\widetilde{\boldsymbol{V}}$ whose underlying K–theory class is equal to $\widetilde{F}^*\phi$. The restriction of the map $\widetilde{F}$ to the part $S^1 \times W \subset \widetilde{W}$ factors over the projection to W and $F\colon W \to B$. Hence we can assume that the restriction of $\widetilde{\boldsymbol{V}}$ to $S^1 \times W \subset \widetilde{W}$ is isomorphic to the pull-back of the bundle $\boldsymbol{U}$ on W if we allow some stabilisation of $\widetilde{\boldsymbol{V}}$ and $\boldsymbol{U}$.

In the KK–picture, the $\cap$–product

$$[\not{D}_{\widetilde{W}}] \cap \widetilde{F}^*\phi \in KK_{n+2}(C(\widetilde{W}_+), \mathbb{C})$$

is realised by the unbounded Kasparov module $(L^2(\widetilde{W}, S(\widetilde{W}) \otimes \widetilde{V}), \not{D}_{\widetilde{W}} \otimes \widetilde{\boldsymbol{V}}, \rho)$ associated to the twisted Dirac operator $\not{D}_{\widetilde{W}} \otimes \widetilde{\boldsymbol{V}}$, where ρ again denotes the action of $C(\widetilde{W}_+)$ on $L^2(\widetilde{W}, S(\widetilde{W}) \otimes \widetilde{V})$ by multiplication. Hence we have

$$[\not{D}_{\widetilde{W}} \otimes \widetilde{\boldsymbol{V}}] = [\not{D}_{\widetilde{W}}] \cap \widetilde{F}^*\phi.$$

We conclude that $\eta^{\mathrm{top}}(x) \in Q_n(MB)$ is represented by the map

$$\begin{aligned} K^0(B) \to KK_{n+2}(C(C_l), \mathbb{C}) &\cong \pi_{n+1}(K\mathbb{Z}/l\mathbb{Z}) \subset \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}), \\ \phi \mapsto \widetilde{G}_*[\not{D}_{\widetilde{W}} &\otimes \widetilde{\boldsymbol{V}}], \end{aligned} \tag{51}$$

where the last inclusion is induced by (42).

Next we want to calculate the element in $\mathbb{Z}/l\mathbb{Z}$ given by $\tilde{G}_*([\not{D}_{\tilde{W}} \otimes \tilde{V}])$. Since the usual index theorem [11] calculates integral indices, we have to construct and calculate an integral representative of this $\mathbb{Z}/l\mathbb{Z}$–valued index. The inclusion of the cone base $i\colon S^1 \to C_l$ induces a surjective map

$$\mathbb{Z} \cong \pi_{n+2}(K \wedge S^1) \to \pi_{n+2}(K \wedge C_l) \cong \mathbb{Z}/l\mathbb{Z}.$$

We try to construct a lift of $\tilde{G}_*([\not{D}_{\tilde{W}} \otimes \tilde{V}])$ to $\pi_{n+2}(K \wedge S^1)$ by providing a factorisation γ as in the diagram:

$$\begin{tikzcd} & S^1 \arrow[d, "i"] \\ \tilde{W} \arrow[ur, dotted, "\gamma"] \arrow[r, "\tilde{G}"] & C_l \end{tikzcd}$$

For our given representative such a factorisation does not exist in general. The idea is to modify the representative without changing its $\mathbb{Z}/l\mathbb{Z}$–valued index so that this lift exists for the modified cycle.

Note that M is a closed odd-dimensional manifold. The Dirac operator $\not{D}_M \otimes V$ is selfadjoint. We can find a selfadjoint smoothing operator Q on $L^2(M, S(M) \otimes V)$ such that $\not{D}_M \otimes V + Q$ is invertible. In [23] such a perturbation was called a taming. As described in this reference, a taming can be lifted to the product $S^2_l \times M$ and also to a collar neighbourhood $l(S^1 \times (-\epsilon, 0] \times M) \cong Z \subset S^1 \times W$ of $\partial(S^1 \times W)$. This lift is a selfadjoint operator $\bar{Q}$ on $L^2(Z \cup_{l(S^1 \times M)} S^2_l \times M, S(\tilde{W}) \otimes \tilde{V})$ which is an integral operator along M and local in the remaining directions. Let $\chi\colon \tilde{W} \to [0,1]$ be a cut-off function which is supported on $Z \cup_{l(S^1 \times M)} S^2_l \times M$, is equal to one in a neighbourhood of the subset $S^2_l \times M$, and only depends on the normal variable near $\partial(S^1 \times W)$. We define the extension $\tilde{Q} := \chi \bar{Q} \chi$ of $\bar{Q}$ to all of $\tilde{W}$. Note that $\tilde{Q}$ commutes with the image of $\tilde{G}^*(C(C_l))$. Adding $\tilde{Q}$ to $\not{D}_W \otimes \tilde{V}$ gives a relatively compact perturbation. Therefore we have

$$\tilde{G}_*[\not{D}_{\tilde{W}} \otimes \tilde{V}] = \tilde{G}_*[\not{D}_{\tilde{W}} \otimes \tilde{V} + \tilde{Q}]$$

in $KK_{n+2}(C(C_l), \mathbb{C})$. On the part $S^2_l \times M \subset \tilde{W}$, the perturbed operator $\not{D}_W \otimes \tilde{V} + \tilde{Q}$ is invertible along the fibres of the projection to S^2_l.

On $S^1 \times (0, \infty)$, we fix a warped product riemannian metric of the form $g := dt^2 + f(t) g^{TS^1}$, where $f|_{[0,1]} \equiv 1$ and $f(t) = t$ for $t \in (2, \infty)$. The precise form is not important, but we need that $\mathtt{lim}_{t\to\infty} f(t) = \infty$. We further choose a Spin^c–extension of this geometry. We define the manifold

$$\tilde{\tilde{W}} := S^1 \times W \cup_{l(S^1 \times M)} l(S^1 \times [0, \infty) \times M).$$

Its Spin^c–geometry is given as the product of the geometries of W and S^1 on the left-hand side, and by the product of the geometries on lM and $S^1 \times (0,\infty)$ on the right-hand side. In a similar manner we define the geometric bundle $\tilde{\tilde{V}}$ on $\tilde{\tilde{W}}$ by a cylindrical extension of $\tilde{V}|_{S^1\times W}$. We define an operator $\tilde{\tilde{Q}}$ similarly to $\tilde{Q}$ by lifting Q to the cylinder $S^1\times[0,\infty)\times M$ and cutting off in the interior of $S^1\times W$. Finally we let $\tilde{\tilde{G}}\colon \tilde{\tilde{W}}\to C_l$ be given by G on $S^1\times W$ and the radially constant extension of $(\tilde{G})|_{\partial(S^1\times W)}$ to the cylinder $l([0,\infty)\times S^1\times M)$.

To every complete riemannian manifold (N,g) we associate the commutative C^*–algebra $C_g(N)$ defined as the closure in the sup–norm of the algebra $C_g^\infty(N)$ of all bounded smooth functions f on N such that $|df|\in C_0(N)$. Note that if N is compact, then $C_g(N)=C(N)$.

The operator $\not{D}_{\tilde{\tilde{W}}}\otimes\tilde{\tilde{V}}+\tilde{\tilde{Q}}$ is invertible along the fibre M of the projection from the cylindrical end of $\tilde{\tilde{W}}$ to $S^1\times[0,\infty)$. Therefore it is invertible at infinity in the sense of [22, Assumption 1]. The arguments given [22, Section 1] show that $\big(L^2(\tilde{\tilde{W}},S(T\tilde{\tilde{W}})\otimes\tilde{\tilde{V}}),\not{D}_{\tilde{\tilde{W}}}\otimes\tilde{\tilde{V}}+\tilde{\tilde{Q}},\tilde{\tilde{\rho}}\big)$ is an unbounded Kasparov module over $C_g(\tilde{\tilde{W}})$. We write $[\not{D}_{\tilde{\tilde{W}}}\otimes g+\tilde{\tilde{Q}}]\in KK_{n+2}(C_g(\tilde{\tilde{W}}),\mathbb{C})$ for its class. Because of the choice of the warped product metric we have the homomorphism of C^*–algebras $\tilde{\tilde{G}}^*\colon C(C_l)\to C_g(\tilde{\tilde{W}})$, so that the class $\tilde{\tilde{G}}_*[\not{D}_{\tilde{\tilde{W}}}\otimes\tilde{\tilde{V}}+\tilde{\tilde{Q}}]\in KK_{n+2}(C(C_l),\mathbb{C})$ is well-defined.

The operators $\not{D}_W\otimes\tilde{V}+\tilde{Q}$ and $\not{D}_{\tilde{\tilde{W}}}\otimes g+\tilde{\tilde{Q}}$ coincide on $S^1\times W$ and are invertible along the fibres M outside of this submanifold of $\tilde{W}$ and $\tilde{\tilde{W}}$.

We let N_1 be the double of $lS^1\times[0,\infty)\times M$ which carries an obvious geometric bundle V_1, taming Q_1, and admits a map $G_1\colon N_1\to C_l$. The associated K–theory class $[N_1]:=[\not{D}_{N_1}\otimes V_1+Q_1]\in KK_{n+2}(C_g(N_1),\mathbb{C})$ vanishes since $\not{D}_{N_1}\otimes V_1+Q_1$ is invertible. Similarly we define N_2, a map $G_2\colon N_2\to C_l$ and a trivial class $[\not{D}_N\otimes V_2+Q_2]\in KK_{n+2}(C_g(N_2),\mathbb{C})$ by attaching the (reflected) warped product $lS^1\times[0,\infty)\times M$ to $S^2_l\times M$.

In this situation, which is schematically pictured below, we can apply a relative index theorem (the proof of [22, Theorem 1.14] extends since all constructions there are compatible with the action of the C^*–algebra $C(C_l)$) in order to get the second equality in $KK_{n+2}(C(C_l),\mathbb{C})$:

$$\begin{aligned}\tilde{G}_*[\not{D}_{\tilde{W}}\otimes\tilde{V}] &= \tilde{G}_*[\not{D}_{\tilde{W}}\otimes\tilde{V}]+G_{1,*}[N_1]\\ &= \tilde{\tilde{G}}_*[\not{D}_{\tilde{\tilde{W}}}\otimes g+\tilde{\tilde{Q}}]+G_{2,*}[N_2]=\tilde{\tilde{G}}_*[\not{D}_{\tilde{\tilde{W}}}\otimes g+\tilde{\tilde{Q}}].\end{aligned}$$

Note the factorisation $\tilde{\tilde{G}}\colon\tilde{\tilde{W}}\xrightarrow{\mathrm{pr}_{S^1}}S^1\xrightarrow{i}C_l$, where the last map is the embedding of

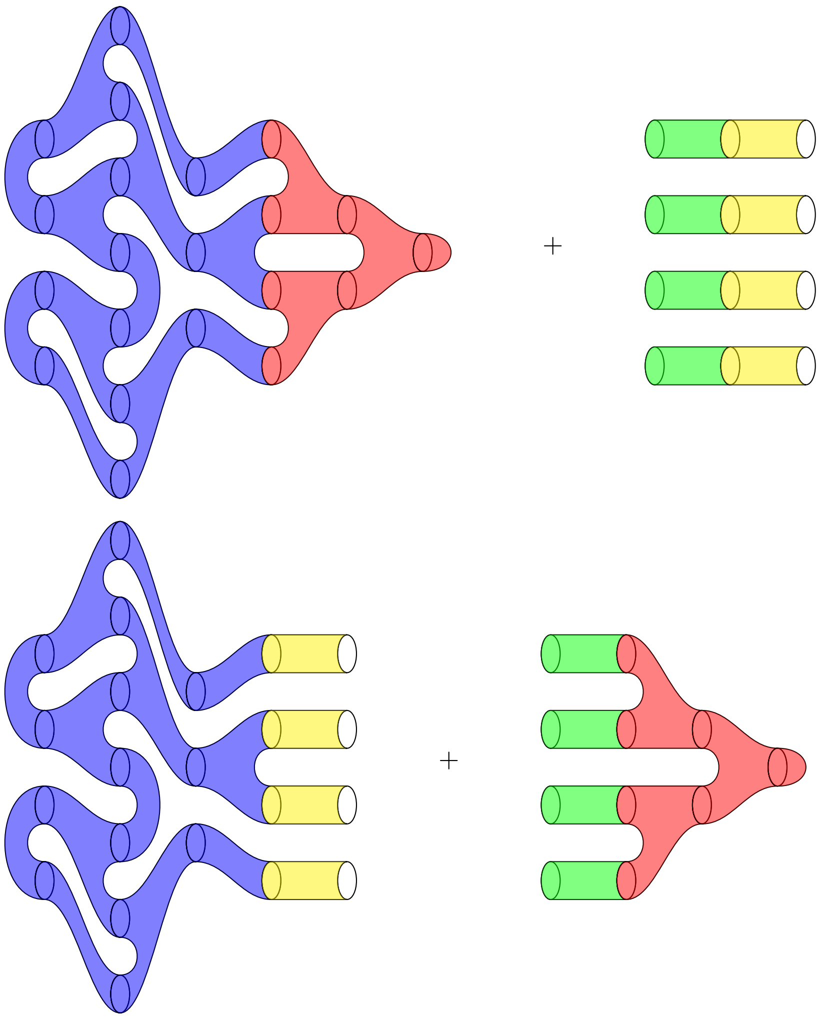

Figure 6: A picture of the relative index theorem. The operator is invertible on the parts which are not blue. The index of the operator associated to the upper picture is the index of its left part $\widetilde{W}$. The index is preserved under cut-and paste as indicated. The index of the operator associated to the lower picture is again the index of the left part $\widetilde{\widetilde{W}}$.

the cone basis. Therefore

$$\mathtt{pr}_{S^1*}[\not{D}_{\widetilde{\widetilde{W}}} \otimes g + \widetilde{\widetilde{Q}}] \in KK_{n+2}(C(S^1), \mathbb{C}) \cong \pi_{n+2}(K \wedge S^1) \cong \mathbb{Z} \tag{52}$$

represents the desired integral lift. We reduce the calculation of this integer to a calculation of a Fredholm index by suspending once more. Let $\boldsymbol{L}$ be a geometric line bundle on $S^1 \times S^1$ such that $c_1(L) \in H^2(S^1 \times S^1; \mathbb{Z})$ is a generator and the sign is fixed with

$$\mathtt{index}(\not{D}_{S^1 \times S^1} \otimes \boldsymbol{L}) = 1. \tag{53}$$

We define $\tilde{\tilde{\tilde{W}}} := S^1 \times \tilde{\tilde{W}}$ with the product Spin^c–geometry. Then the desired integer (52) is the index of the Fredholm operator

$$\not{D}_{\tilde{\tilde{\tilde{W}}}} \otimes \mathtt{pr}_{\tilde{\tilde{W}}} \tilde{\tilde{\boldsymbol{V}}} \otimes \mathtt{pr}^*_{S^1 \times S^1} \boldsymbol{L} + \tilde{\tilde{\tilde{Q}}},$$

where $\tilde{\tilde{\tilde{Q}}}$ is the taming induced by $\tilde{\tilde{Q}}$. In order to see this note that the identification $KK_1(C(S^1), \mathbb{C}) \xrightarrow{\cong} \mathbb{Z}$ is given by the iterated Kasparov product

$$x \mapsto [\boldsymbol{L}] \otimes_{C(S^1 \times S^1)} ([\not{D}_{S^1}] \otimes_{\mathbb{C}} x) \in KK_2(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z},$$

where $[\not{D}_{S^1}] \in KK_1(C(S^1), \mathbb{C})$ is the class of the standard Spin^c–Dirac operator on S^1 and $[\boldsymbol{L}] \in KK_0(\mathbb{C}, C(S^1 \times S^1))$ is the class represented by the line bundle $\boldsymbol{L}$. We now have

$$\mathtt{pr}_{S^1 \times S^1 *}[\not{D}_{\tilde{\tilde{\tilde{W}}}} \otimes \mathtt{pr}_{\tilde{\tilde{W}}} \tilde{\tilde{\boldsymbol{V}}} + \tilde{\tilde{\tilde{Q}}}] = [\not{D}_{S^1}] \otimes_{\mathbb{C}} \mathtt{pr}_{S^1 *}[\not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\boldsymbol{V}}} + \tilde{\tilde{Q}}]$$

in $KK_2(C(S^1 \times S^1), \mathbb{C})$ and therefore, using the relation of twisting Dirac operators and Kasparov products,

$$[\not{D}_{\tilde{\tilde{\tilde{W}}}} \otimes \mathtt{pr}_{\tilde{\tilde{W}}} \tilde{\tilde{\boldsymbol{V}}} \otimes \mathtt{pr}^*_{S^1 \times S^1} \boldsymbol{L} + \tilde{\tilde{\tilde{Q}}}] = [\boldsymbol{L}] \otimes_{C(S^1 \times S^1)} \mathtt{pr}_{S^1 \times S^1 *}[\not{D}_{\tilde{\tilde{\tilde{W}}}} \otimes \mathtt{pr}_{\tilde{\tilde{W}}} \tilde{\tilde{\boldsymbol{V}}} + \tilde{\tilde{\tilde{Q}}}]$$

in $KK_2(\mathbb{C}, \mathbb{C})$.

We deform the warped product metric on the end of $\tilde{\tilde{\tilde{W}}}$ to a product metric. This produces a continuous family of Fredholm operators, and therefore does not change the index. After this deformation, we see that $\tilde{\tilde{\tilde{W}}} \cong S^1 \times S^1 \times \widehat{W}$ geometrically, where $\widehat{W} := W \cup_{\partial W} l([0, \infty) \times M)$ carries the cylindrical extension of the geometry of W. Similarly we let $\widehat{\boldsymbol{U}}$ be the geometric bundle on $\widehat{W}$ obtained by the cylindrical extension of $\boldsymbol{U}$. Then the resulting operator represents the product

$$[\not{D}_{\widehat{W}} \otimes \widehat{\boldsymbol{U}} + \widehat{Q}] \otimes_{\mathbb{C}} [\not{D}_{S^1 \times S^1} \otimes \boldsymbol{L}],$$

where $\widehat{Q}$ is the taming $\widehat{W}$ uniquely determined by the property that its lift is the taming $\tilde{\tilde{\tilde{Q}}}$. Because of (53) the integer (52) is equal to $\mathtt{index}(\not{D}_{\widehat{W}} \otimes \widehat{\boldsymbol{U}} + \widehat{Q})$.

We now calculate this index. The index theory for these kinds of perturbations of Dirac operators has been developed in [23]. In the language of this reference the operator Q defines a taming $(M \otimes \boldsymbol{V})_{\mathrm{t}}$ of the geometric manifold $M \otimes \boldsymbol{V}$ and a boundary taming $(W \otimes \boldsymbol{U})_{\mathrm{bt}}$ of the geometric manifold $W \otimes \boldsymbol{U}$. The following equality holds true by definition of the right-hand side: $\mathtt{index}(\not{D}_{\hat{W}} \otimes \hat{\boldsymbol{U}} + \hat{Q}) = \mathtt{index}((W \otimes \boldsymbol{U})_{\mathrm{bt}})$. The index theorem [23, Theorem 4.18] gives

$$\mathtt{index}((W \otimes \boldsymbol{U})_{\mathrm{bt}}) = \int_W p_{n+1}(\mathbf{Td}(\widetilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U)) - l\eta((M \otimes \boldsymbol{V})_{\mathrm{t}}).$$

In $\mathbb{R}/\mathbb{Z}$ we have $[\eta((M \otimes \boldsymbol{U})_{\mathrm{t}})] = \xi(D_M \otimes \boldsymbol{V})$. Hence, by comparison with (35), we get the equality in $\mathbb{Q}/\mathbb{Z}$

$$\left[\frac{1}{l}\mathtt{index}((W \otimes \boldsymbol{U})_{\mathrm{bt}})\right] = \left[\frac{1}{l}\mathtt{index}((\not{D}_W \otimes \boldsymbol{U})_{\mathrm{APS}})\right].$$

In view of the construction of η^{an}, in particular of (39), we see that the map (51) also represents $\eta^{\mathrm{an}}(x)$. This finishes the proof of Theorem 3.7. □

4 An intrinsic formula

4.1 Motivation

In a typical situation for the theory of the present paper, one is given a geometric representative (M, f) for a torsion class $x = [M, f] \in \pi_n(MB)$ and wants to calculate the universal η–invariants $\eta^{\mathrm{top}}(x) = \eta^{\mathrm{an}}(x) \in Q_n(MB)$. The expressions for the universal η–invariant that we have at our disposal at the moment share the disadvantage that one has to find a lift $\hat{x} \in \pi_{n+1}(MB\,\mathbb{Q}/\mathbb{Z})$ or a geometric zero bordism (W, F) of l copies of (M, f) explicitly. It is at this point where differential and spectral geometry helps. In the present section we develop a generalisation of Chern–Weil theory which is designed to finally obtain formulas for the universal η–invariant which are intrinsic in the cycle (M, f).

The main new object is the notion of a geometrisation of $(M, f, \widetilde{\nabla})$, which is defined in Definition 4.5. It involves differential K–theory, which is reviewed in Section 4.2. In Section 4.3 we show the existence of geometrisations and study their functorial properties. In Section 4.4 we introduce a special class of geometrisations, which we call good. In contrast to general geometrisations they have the property that they extend over zero bordisms. The main result is the intrinsic formula for the universal η–invariant formulated in Theorem 4.19.

4.2 Review of differential K–theory

The definition of a geometrisation utilises differential K–theory. We refer to [32; 38; 25] for constructions and further information. In the following we review the basic structures which, by [26], uniquely characterise differential K–theory. Differential K–theory is a five-tuple

$$(\widehat{K}, I, R, a, \textstyle\int)$$

of the following objects: The first entry is a contravariant functor

$$\widehat{K}\colon \text{smooth manifolds} \to \mathbb{Z}/2\mathbb{Z}\text{–graded commutative rings.}$$

The remaining entries are natural transformations between functors. The domains and ranges of the first three are given by

$$I\colon \widehat{K} \to K, \quad R\colon \widehat{K} \to \Omega P_{\mathrm{cl}}, \quad a\colon \Omega P/\mathtt{im}(d)[1] \to \widehat{K}. \tag{54}$$

Here the evaluation of ΩP at M is the graded vector space $\Omega P(M) := \Omega(M)[b, b^{-1}]$ of two-periodic smooth real differential forms on M, which carries a differential d. By $\Omega P_{\mathrm{cl}}(M) \subseteq \Omega P(M)$ we mean its subspace of closed forms. The transformations R and I preserve the ring structures while a is just additive. These transformations are compatible in the sense that for every manifold M the following differential cohomology diagram commutes:

$$\begin{tikzcd}
& \Omega P^{*-1}(M)/\mathtt{im}(d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega P^*_{\mathrm{cl}}(M) \arrow[dr, "\mathrm{Rham}"] & \\
HP\mathbb{R}^{*-1}(M) \arrow[ur] \arrow[dr] & & \widehat{K}^*(M) \arrow[ur, "R"] \arrow[dr, "I"] & & HP\mathbb{R}^*(M) \\
& K\mathbb{R}/\mathbb{Z}^{*-1}(M) \arrow[rr, "\mathrm{Bockstein}"] \arrow[ur] & & K^*(M_+) \arrow[ur, "\mathbf{ch}"] &
\end{tikzcd}$$

Here we define the spectrum $HP\mathbb{R}$ representing periodic real cohomology similarly as $HP\mathbb{Q}$ in (8). Furthermore, for $\alpha \in \Omega P^*(M)/\mathtt{im}(d)$ and $x \in \widehat{K}^*(M)$ we have the identity

$$a(\alpha) \cup x = a(\alpha \wedge R(x)). \tag{55}$$

The flat part of differential K–theory is defined as the kernel of the curvature transformation R. It is canonically isomorphic to $\mathbb{R}/\mathbb{Z}$–K–theory (with a shift):

$$\widehat{K}^*_{\mathrm{flat}}(M) := \mathtt{ker}(R\colon \widehat{K}^*(M) \to \Omega P^*_{\mathrm{cl}}(M)) \cong K\mathbb{R}/\mathbb{Z}^{*-1}(M). \tag{56}$$

The sequence

$$K^{*-1}(M) \xrightarrow{\mathbf{ch}} \Omega P^{*-1}(M)/\mathtt{im}(d) \xrightarrow{a} \widehat{K}^*(M) \xrightarrow{I} K^*(M) \to 0 \tag{57}$$

is exact. The integration is a natural (in M) transformation

$$\int : \hat{K}^*(S^1 \times M) \to \hat{K}^{*-1}(M),$$

whose existence and compatibility with the other structures fixes the odd part of the differential extension uniquely up to unique isomorphism, as discussed in [26]. Since we do not need the integration in the present paper we will not write out the long list of these compatibilities explicitly.

Differential K–theory is not homotopy-invariant. The deviation from homotopy-invariance is quantified by the homotopy formula. If $\hat{x} \in \hat{K}^*([0,1] \times M)$, then it states that

$$i_1^*\hat{x} - i_0^*\hat{x} = a\Big(\int_{[0,1]\times M/M} R(\hat{x})\Big). \tag{58}$$

Let $\boldsymbol{V} = (V, h^V, \nabla^V)$ be a geometric bundle on manifold M, where h^V is a hermitian metric which is preserved by the connection ∇^V. Then we have a natural class

$$[\boldsymbol{V}] \in \hat{K}^0(M). \tag{59}$$

This class is in fact tautological in the model [25] in view of [25, Section 2.1.4]. It satisfies

$$I([\boldsymbol{V}]) = [V] \in K^0(M), \quad R([\boldsymbol{V}]) = \mathbf{ch}(\nabla^V) \in \Omega P^0_{\mathrm{cl}}(M), \tag{60}$$

where

$$\mathbf{ch}(\nabla^V) := \mathtt{Tr}\Big(\exp\Big(-\frac{bR^\nabla}{2\pi i}\Big)\Big)$$

is the normalised Chern character form.

Remark 4.1 If we replace $HP\mathbb{R}$ and ΩP by $HP\mathbb{C}$ and $\Omega P \otimes \mathbb{C}$, then we get a complex version $\hat{K}_{\mathbb{C}}$ of differential K–theory with similar properties. A complex vector bundle with connection $\boldsymbol{V} = (V, \nabla^V)$ on a manifold M, where ∇^V is not necessarily hermitian, gives a class $[\boldsymbol{V}] \in \hat{K}^0_{\mathbb{C}}(M)$ such that analogues of the equalities (60) still hold true. For the flat part we get the equivalence

$$\hat{K}^0_{\mathbb{C},\mathrm{flat}}(M) \cong K\mathbb{C}/\mathbb{Z}^{-1}(M).$$

4.3 Geometrisations

Let M be a compact manifold equipped with a map $f: M \to B$. At the moment we do not require any connection of f with the tangent bundle. Later, the manifold M will be either a part of a cycle for a B–bordism class or an approximation of the space B. In order to cover both cases at the same time we use the following set-up. We can assume that f has a factorisation over $\tilde{f}\colon M \to B\mathrm{Spin}^c(k)$ as in (28) which classifies a $\mathrm{Spin}^c(k)$–bundle $\tilde{f}^*Q_k \in \mathrm{Spin}^c(\tilde{f}^*\xi_k^{\mathrm{Spin}^c})$ on M. The role of the tangent bundle is taken by the choice of a complementary Spin^c–bundle. In detail, we choose an l–dimensional oriented euclidean vector bundle $\eta \to M$ for some $l \geq 0$ together with an orientation-preserving isomorphism of euclidean vector bundles

$$\eta \oplus \tilde{f}^*\xi_k^{\mathrm{Spin}^c} \cong \underline{\mathbb{R}}_M^{l+k}. \tag{61}$$

Then we choose a Spin^c–structure $P \in \mathrm{Spin}^c(\eta)$ together with an isomorphism

$$P \otimes \tilde{f}_k^* Q_k \cong Q(l+k), \tag{62}$$

where we use the isomorphism (61) in order view the left- and right-hand sides in the same groupoid $\mathrm{Spin}^c(\underline{\mathbb{R}}_M^{l+k})$ (see Section 3.3 for details).

We choose a connection $\tilde{\nabla}$ on P and get an induced Todd form $\mathbf{Td}(\tilde{\nabla}) \in \Omega P^0_{\mathrm{cl}}(M)$ which represents the class $f^*\mathbf{Td}^{-1} \in HP\mathbb{Q}^0(M)$. We must take the inverse here since P is the complement of $\tilde{f}_k^* Q_k$ by (62).

We now consider a continuous homomorphism

$$\mathcal{G}\colon K^0(B) \to \hat{K}^0(M),$$

where the domain has the topology described in Remark 2.1 and the target is discrete.

Definition 4.2 A cohomological character for $\mathcal{G}$ is a continuous homomorphism

$$c_{\mathcal{G}}\colon HP\mathbb{Q}^0(B) \to \Omega P^0_{\mathrm{cl}}(M)$$

such that the following diagram commutes:

$$\begin{tikzcd} K^0(B) \arrow[r, "\mathcal{G}"] \arrow[d, "\mathbf{Td}^{-1}\cup\mathbf{ch}(\cdot)"] & \hat{K}^0(M) \arrow[d, "\mathbf{Td}(\tilde{\nabla})\wedge R(\cdot)"] \\ HP\mathbb{Q}^0(B) \arrow[r, "c_{\mathcal{G}}"] & \Omega P^0_{\mathrm{cl}}(M) \end{tikzcd}$$

Lemma 4.3 *Given $\mathcal{G}$ there exists a cohomological character $c_{\mathcal{G}}$. If B is compact, then it is unique.*

Proof Since $\mathcal{G}$ is continuous there exists a map $r\colon A \to B$ from a finite CW–complex A such that $\mathcal{G}$ factors over the quotient $K^0(B)/\mathtt{ker}(r^*_{K^0})$. For a space X with a map to B we write

$$HP\mathbb{Q}^0(X)_0 := \overline{\mathtt{im}\big(\mathbf{Td}^{-1}\cup\mathbf{ch}(\cdot)\colon K^0(X)\otimes\mathbb{Q} \to HP\mathbb{Q}^0(X)\big)}, \tag{63}$$

where we use $\mathbf{Td}$ also as a symbol for the pull-back of the Todd class via $X \to B \to B\mathrm{Spin}^c$. We get the diagram:

$$\begin{tikzcd}
K^0(B) \arrow[r] \arrow[rr, bend left, "\mathcal{G}"] \arrow[d, "\mathbf{Td}^{-1}\cup\mathbf{ch}(\cdot)"] & K^0(B)/\mathtt{ker}(r^*_{K^0}) \arrow[r, "\bar{\mathcal{G}}"] \arrow[d, "\phi"] & \hat{K}^0(M) \arrow[d, "\mathbf{Td}(\tilde{\nabla})\wedge R(\cdot)"] \\
HP\mathbb{Q}^0(B)_0 \arrow[r, "p_0"] \arrow[d] & HP\mathbb{Q}^0(B)_0/(\mathtt{ker}(r^*_{HP\mathbb{Q}^0}) \cap HP\mathbb{Q}^0(B)_0) \arrow[d] & \Omega P^0_{\mathrm{cl}}(M) \\
HP\mathbb{Q}^0(B) \arrow[r, "p"] \arrow[rru, dotted, bend right, "c_{\mathcal{G}}"'] & HP\mathbb{Q}^0(B)/\mathtt{ker}(r^*_{HP\mathbb{Q}^0}) \arrow[u, dashed, bend right, "\mathtt{pr}"'] &
\end{tikzcd}$$

The map ϕ induces an isomorphism after tensoring with $\mathbb{Q}$. In order to see this we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & K^0(B)\otimes\mathbb{Q}/\mathtt{ker}(r^*_{K^0\otimes\mathbb{Q}}) \arrow[r] \arrow[d, "\phi\otimes\mathbb{Q}"] & K^0(A)\otimes\mathbb{Q} \arrow[d, "(r^*_{HP\mathbb{Q}^0}\mathbf{Td}^{-1}\wedge\mathbf{ch}(\cdot))\otimes\mathbb{Q}", "\cong"'] \\
0 \arrow[r] & HP\mathbb{Q}^0(B)_0/(\mathtt{ker}(r^*_{HP\mathbb{Q}^0}) \cap HP\mathbb{Q}^0(B)_0) \arrow[r] & HP\mathbb{Q}^0(A)
\end{tikzcd}$$

with exact horizontal lines. We immediately see that $\phi\otimes\mathbb{Q}$ is injective. On the other hand, by definition (63) the image of $(\mathbf{Td}^{-1}\cup\mathbf{ch}(\cdot))\otimes\mathbb{Q}\colon K^0(B)\otimes\mathbb{Q}\to HP\mathbb{Q}^0(B)$ is dense in $HP\mathbb{Q}^0(B)_0$. Note that the quotient $HP\mathbb{Q}^0(B)/\mathtt{ker}(r^*_{HP\mathbb{Q}^0})$ carries the discrete topology. Therefore the composition

$$K^0(B)\otimes\mathbb{Q} \xrightarrow{(\mathbf{Td}^{-1}\cup\mathbf{ch}(\cdot))\otimes\mathbb{Q}} HP\mathbb{Q}^0(B)_0 \to HP\mathbb{Q}^0(B)_0/(\mathtt{ker}(r^*_{HP\mathbb{Q}^0})\cap HP\mathbb{Q}^0(B)_0)$$

is surjective. This implies that $\phi\otimes\mathbb{Q}$ is surjective, too.

After choosing some linear projection

$$\mathtt{pr}\colon HP\mathbb{Q}^0(B)/\mathtt{ker}(r^*_{HP\mathbb{Q}^0}) \to HP\mathbb{Q}^0(B)_0/(\mathtt{ker}(r^*_{HP\mathbb{Q}^0}) \cap HP\mathbb{Q}^0(B)_0)$$

we can define a cohomological character by

$$c_{\mathcal{G}}(x) := \mathbf{Td}(\tilde{\nabla}) \wedge ((R\circ\bar{\mathcal{G}})\otimes\mathbb{Q})\circ(\phi\otimes\mathbb{Q})^{-1}\big(\mathtt{pr}(p(x))\big), \quad x\in HP\mathbb{Q}^0(B).$$

Since it factorises over p it is continuous.

If B is compact, then $HP\mathbb{Q}^0(B)_0 = HP\mathbb{Q}^0(B)$ and this implies uniqueness of $c_{\mathcal{G}}$. □

Example 4.4 (Völkl) If B is not compact, then a cohomological character is not necessarily unique. Consider $B := K(\mathbb{Z},4)$ and assume that M is compact. By [5, Theorem II] the reduced K–theory group $\widetilde{K}^0(B)$ is nonzero, but consists of phantom classes and therefore has the indiscrete topology. Consequently, a continuous map $\mathcal{G}\colon \widetilde{K}^0(B) \to K^0(M)$ must be trivial. Furthermore, we have $HP\mathbb{Q}(B)_0 = b^0\mathbb{Q}$. Let $\mathcal{G} := 0$. In this case any continuous homomorphisms $HP\mathbb{Q}^0(B) \to \Omega P^0_{\mathrm{cl}}(M)$ vanishing on $b^0\mathbb{Q}$ can serve as a cohomological character $c_{\mathcal{G}}$. Note that $HP\mathbb{Q}^0(B) \cong \mathbb{Q}[\![q]\!]$ with $q := b^2 u$ for the canonical class $u \in H^4(K(\mathbb{Z},4),\mathbb{Q})$, so there are many such homomorphisms.

We say that the cohomological character $c_{\mathcal{G}}$ preserves degree if it preserves the decompositions

$$HP\mathbb{Q}^0(B) \cong \prod_{k\in\mathbb{Z}} b^k H^{2k}(B;\mathbb{Q}), \quad \Omega P^0_{\mathrm{cl}}(M) \cong \prod_{k\in\mathbb{Z}} b^k \Omega^{2k}_{\mathrm{cl}}(M).$$

Definition 4.5 A geometrisation of $(M, f, \widetilde{\nabla})$ is a continuous homomorphism

$$\mathcal{G}\colon K^0(B) \to \widehat{K}^0(M)$$

such that the diagram

$$\begin{tikzcd} & \widehat{K}^0(M) \arrow[d, "I"] \\ K^0(B) \arrow[ur, dotted, "\mathcal{G}"] \arrow[r, "f^*"] & K^0(M) \end{tikzcd} \tag{64}$$

commutes, and which admits a degree-preserving cohomological character.

Example 4.6 The notion of a geometrisation generalises the notion of a connection. This is demonstrated in Proposition 5.13 for the case $B = B\mathrm{Spin}$. Here we will discuss another example, where we put $B := B\mathrm{Spin}^c \times B\Gamma$ for some compact Lie group Γ and $B \to B\mathrm{Spin}^c$ is the projection. We write maps to $B\mathrm{Spin}^c \times B\Gamma$ as pairs (f, g).

Let us assume that we already have a geometrisation $\mathcal{G}^0$ of $(M, f, \widetilde{\nabla})$ with a degree-preserving cohomological character $c_{\mathcal{G}^0}$. Its existence is guaranteed by Proposition 4.8. The map $g\colon M \to B\Gamma$ classifies a Γ–principal bundle $R \to M$. We choose a connection ∇^R on R.

Lemma 4.7 *There exists a natural geometrisation $\mathcal{G}$ of $(M, (f, g), \widetilde{\nabla})$ associated to this data.*

Proof The completion theorem [10] gives an isomorphism $K^0(B\Gamma)\cong R(\Gamma)^\wedge_{I_\Gamma}$ of topological groups, where $I_\Gamma\subseteq R(\Gamma)$ is the dimension-ideal of the integral representation ring. We consider a representation $\sigma\colon \Gamma\to U(m_\sigma)$ which represents an element $[\sigma]\in K^0(B\Gamma)$. The associated complex vector bundle $V_\sigma:=R\times_{\Gamma,\sigma}\mathbb{C}^{m_\sigma}$ on M then represents the element $[V_\sigma]=f^*[\sigma]\in K^0(M)$. This bundle comes with a hermitian metric h^{V_σ} and a metric connection ∇^{V_σ} induced by ∇^R. We therefore get a geometric bundle $\boldsymbol{V}_\sigma:=(V_\sigma,\nabla^{V_\sigma},\nabla^{V_\sigma})$. It represents the class $[\boldsymbol{V}_\sigma]\in\widehat{K}^0(M)$ such that $[V_\sigma]=I([\boldsymbol{V}_\sigma])$ in $K^0(M)$; see (59). Let $\phi\in K^0(B\mathrm{Spin}^c)$. Then we get the element $\phi\times[\sigma]\in K^0(B\mathrm{Spin}^c\times B\Gamma)=K^0(B)$. We define

$$\mathcal{G}(\phi\times[\sigma]):=\mathcal{G}^0(\phi)\cup[\boldsymbol{V}_\sigma].$$

By linear extension this construction defines the map $\mathcal{G}$ on a dense subgroup of $K^0(B)$.

We now show that the map $\mathcal{G}$ extends by continuity to all of $K^0(B)$ and defines a geometrisation of $(M,(f,g),\widetilde{\nabla})$. Indeed, the map $R(\Gamma)\to\widehat{K}^0(M)$ induced by $\sigma\mapsto[\boldsymbol{V}_\sigma]$ is multiplicative and annihilates I_Γ^{2n+1}, where $n:=\dim(M)$. Therefore, since $\mathcal{G}^0$ is continuous, the map $\mathcal{G}$ is continuous as well. We now use the fact that $HP\mathbb{Q}^0(B\Gamma)$ is topologically generated by the classes $\mathbf{ch}([\sigma])$ for $\sigma\in R(\Gamma)$. We let $c_\Gamma\colon HP\mathbb{Q}^0(B\Gamma)\to\Omega P^0_{\mathrm{cl}}(M)$ be the unique continuous map such that $\mathbf{ch}(\nabla^{V_\sigma})=c_\Gamma(\mathbf{ch}([\sigma]))$. Note that c_Γ preserves degree. Since the cohomological character $c_{\mathcal{G}^0}$ preserves degree, we can take $c_{\mathcal{G}}:=c_{\mathcal{G}^0}\wedge c_\Gamma$ of $\mathcal{G}$ as a degree-preserving cohomological character for $\mathcal{G}$. □

The geometrisation $\mathcal{G}$ allows us to recover the Chern character form of ∇^{V_σ} by

$$\mathbf{ch}(\nabla^{V_\sigma})=\mathbf{Td}(\widetilde{\nabla})^{-1}\wedge R(\mathcal{G}(1\otimes[\sigma])).$$

It also allows us to partially recover transgressions, as we will explain in the following. If $\nabla^{R\prime}$ is a second connection on R and $\mathcal{G}'$ is the associated geometrisation, then

$$\mathcal{G}'(1\otimes[\sigma])-\mathcal{G}(1\otimes[\sigma])=a(\mathbf{Td}(\widetilde{\nabla})\wedge\widetilde{\mathbf{ch}}(\nabla^{V_\sigma\prime},\nabla^{V_\sigma})).$$

Here $\widetilde{\mathbf{ch}}(\nabla^{V_\sigma\prime},\nabla^{V_\sigma})\in\Omega P^{-1}(M)$ denotes the transgression form which satisfies

$$d\,\widetilde{\mathbf{ch}}(\nabla^{V_\sigma\prime},\nabla^{V_\sigma})=\mathbf{ch}(\nabla^{V_\sigma\prime})-\mathbf{ch}(\nabla^{V_\sigma}),$$

and a is as in (54). This concludes Example 4.6. ◁

The following Proposition 4.8 asserts that geometrisations exist. Its proof uses the functoriality of geometrisations in the space B. Consider a map ϕ over $B\mathrm{Spin}^c$, ie a homotopy commutative diagram:

$$\begin{tikzcd} B' \arrow[rr, "\phi"] \arrow[dr] & & B \arrow[dl] \\ & B\mathrm{Spin}^c & \end{tikzcd}$$

Given a geometrisation $\mathcal{G}$ of $(M, \phi \circ f, \tilde{\nabla})$ we get a geometrisation

(65) $$\phi_* \mathcal{G} := \mathcal{G} \circ \phi^*$$

of $(M, f, \tilde{\nabla})$.

Note that our standing assumption is that M is compact.

Proposition 4.8 *Given $(M, f, \tilde{\nabla})$ there exists a geometrisation.*

Proof Since M is compact, the map f factors over a compact subspace of B. In view of the functoriality of the geometrisation (65) we can assume that B is compact. Then $K^0(B)$ is a finitely generated abelian group. We choose a decomposition

$$K^0(B) \cong A_{\mathrm{tors}} \oplus A_{\mathrm{free}}$$

into a torsion and a free part. We write

$$A_{\mathrm{tors}} := \bigoplus_{y \in I} y\mathbb{Z}/\mathtt{ord}(y)\mathbb{Z}$$

for some set of generators $I \subset A_{\mathrm{tors}}$. For all $y \in I$, using the exactness at the right end of (57), we choose $\tilde{y}_0 \in \hat{K}^0(M)$ such that $I(\tilde{y}_0) = f^* y$. Then $\mathtt{ord}(y)\tilde{y}_0 = a(\omega_y)$ for some $\omega_y \in \Omega P^{-1}(M)/\mathtt{im}(d)$, again by (57). We define

$$\tilde{y} := \tilde{y}_0 - a\Big(\frac{1}{\mathtt{ord}(y)} \omega_y\Big).$$

Then $\mathtt{ord}(y)\tilde{y} = 0$ and we can define $\mathcal{G}|_{A_{\mathrm{tors}}} \colon A_{\mathrm{tors}} \to \hat{K}^0(M)$ so that $\mathcal{G}(y) = \tilde{y}$ for all $y \in I$. Since $\mathbf{Td}^{-1} \wedge \mathbf{ch}$ vanishes on A_{tors} and $\mathcal{G}|_{A_{\mathrm{tors}}}$ maps to flat classes it is clear that the cohomological character of this part of $\mathcal{G}$ preserves degree.

We now come to the free part. We choose a basis $J \subset A_{\mathrm{free}}$ and classes $\tilde{z}_0(z) \in \hat{K}^0(M)$ such that $I(\tilde{z}_0(z)) = f^* z$ for all $z \in J$. We further choose a basis $J' \subset A_{\mathrm{free}} \otimes \mathbb{Q}$ such that $\{\mathbf{Td}^{-1} \wedge \mathbf{ch}(z')\}_{z' \in J'}$ is a homogeneous basis with respect to the decomposition

$$HP\mathbb{Q}^0(B) \cong \bigoplus_{m \in \mathbb{Z}} b^m H^{2m}(B; \mathbb{Q}).$$

We define the even integers $n_{z'} := \deg(\mathbf{Td}^{-1} \wedge \mathbf{ch}(z'))$ for all $z' \in J'$. Then there exists an invertible rational (J, J')–indexed matrix A such that $z = \sum_{z' \in J'} A_{zz'} z'$ for all $z \in J$. We now can choose forms $\alpha_{z'} \in \Omega P^{-1}(M)/\mathtt{im}(d)$ for all $z' \in J'$ such that

$$\sum_{z \in J} A^{-1}_{z'z} \mathbf{Td}(\tilde{\nabla}) \wedge R(\tilde{z}_0(z)) - d\alpha_{z'} \in b^{n_{z'}/2} \Omega^{n_{z'}}_{\mathrm{cl}}(M) \subseteq \Omega P^0_{\mathrm{cl}}(M)$$

for all $z' \in J'$. We define

$$\mathcal{G}|_{A_{\mathrm{free}}} \colon A_{\mathrm{free}} \to \hat{K}^0(M)$$

by linear extension, with

$$\mathcal{G}(z) = \tilde{z}_0(z) - a\Bigl(\mathbf{Td}(\tilde{\nabla})^{-1} \wedge \sum_{z' \in J'} A_{zz'} \alpha_{z'}\Bigr).$$

By construction its (uniquely determined) cohomological character preserves degree. □

Geometrisations can be pulled back along stable Spin^c–maps over B. In detail, the construction goes as follows: Let (M', f') be a compact manifold with a map $f' \colon M' \to B$. We consider a smooth map $h \colon M' \to M$ such that $f \circ h$ is homotopic to f'. We can then choose a stable isomorphism of complementary bundles

$$\eta' \oplus \underline{\mathbb{R}}^s_{M'} \cong h^* \eta \oplus \underline{\mathbb{R}}^t_{M'}. \tag{66}$$

We refine h to a Spin^c–map (Definition 3.4) by choosing an isomorphism

$$P' \otimes Q(s) \cong h^* P \otimes Q(t). \tag{67}$$

Assume now that we have connections $\tilde{\nabla}$ on P and $\tilde{\nabla}'$ on P'. They induce connections on the stabilisations $P \otimes Q(t)$ and $P' \otimes Q(s)$. We thus can define the transgression

$$\widetilde{\mathbf{Td}}(h^* \tilde{\nabla}, \tilde{\nabla}') \in \Omega P^{-1}(M')/\mathtt{im}(d),$$

where we use the isomorphism (67) in order to compare the stabilisation of $h^* \tilde{\nabla}$ with that of $\tilde{\nabla}'$ on the same bundle. The transgression satisfies

$$d\widetilde{\mathbf{Td}}(h^* \tilde{\nabla}, \tilde{\nabla}') = h^* \mathbf{Td}(\tilde{\nabla}) - \mathbf{Td}(\tilde{\nabla}').$$

Let $\mathcal{G}$ be a geometrisation of $(M, f, \tilde{\nabla})$.

Lemma 4.9 *If $h \colon M' \to M$ is a* Spin^c*–map between compact manifolds, then there exists a construction of a pull-back $\mathcal{G}' := h^* \mathcal{G}$ of $(M', f', \tilde{\nabla}')$. This pull-back only depends on the joint stable homotopy class of the isomorphisms (66) and (67) and is functorial under compositions.*

Proof By our assumptions the equivalence class

(68) $$\beta := \widetilde{\mathbf{Td}}(h^*\tilde{\nabla}, \tilde{\nabla}') \wedge \mathbf{Td}(\tilde{\nabla}')^{-1} \in \Omega P^{-1}(M')/\mathtt{im}(d)$$

of forms is well-defined. It satisfies

(69) $$d\beta = h^*\mathbf{Td}(\tilde{\nabla}) \wedge \mathbf{Td}(\tilde{\nabla}')^{-1} - 1.$$

We define the pull-back $\mathcal{G}' := h^*\mathcal{G}$ by

(70) $$\mathcal{G}'(y) := h^*\mathcal{G}(y) + a(\beta \wedge h^* R(\mathcal{G}(y))), \quad y \in K^0(B),$$

where a and R belong to the structure maps of differential K–theory. We have, by construction,

$$\mathbf{Td}(\tilde{\nabla}') \wedge R(\mathcal{G}'(y)) = h^*\big(\mathbf{Td}(\tilde{\nabla}) \wedge R(\mathcal{G}(y))\big)$$

and hence can take $c_{\mathcal{G}'} := h^* c_{\mathcal{G}}$ as a cohomological character for $\mathcal{G}'$. Since the cohomological character $c_{\mathcal{G}}$ preserves degree, so does the cohomological character of $\mathcal{G}'$.

We show that the pull-back is functorial. We consider a second triple $(M'', f'', \tilde{\nabla}'')$ with a Spin^c–map $h'\colon M'' \to M'$ and the associated transgression form β'. Then we have, for the iterated pull-back,

$$\begin{aligned}\mathcal{G}''(y) &= h'^*(h^*(\mathcal{G}(y))) + h'^*\big(a\big(\beta \wedge h^*R(\mathcal{G}(y))\big)\big) + a(\beta' \wedge h'^*R(\mathcal{G}'(y)))\\ &= (h \circ h'^*)^*(\mathcal{G}(y)) + a\Big(h'^*\beta \wedge h'^*\big(h^*R(\mathcal{G}(y))\big) + \beta' \wedge h'^*\big(h^*R(\mathcal{G}(y))\big)\\ &\qquad\qquad + \beta' \wedge h'^* d\beta \wedge h'^*\big(h^*R(\mathcal{G}(y))\big)\Big).\end{aligned}$$

Let $\tilde{\beta}$ be the transgression form for the composition $h \circ h'$ of Spin^c–maps over X. Then we must show that

$$\tilde{\beta} - (h'^*\beta + \beta' + h'^*\beta' \wedge d\beta) \in \mathtt{im}(d).$$

This follows from

$$d(h'^*\beta + \beta' + h'^*\beta' \wedge d\beta) = h'^*h^*\mathbf{Td}(\tilde{\nabla})^{-1} \wedge \mathbf{Td}(\tilde{\nabla}'') - 1 = d\tilde{\beta}$$

and the fact that all these forms are defined by transgressions and the contractibility of the space of connections. The assertion about homotopy-invariance easily follows from the homotopy formula (58) for differential K–theory. □

Note that the form β is determined up to closed forms by (69). The refinement of the map h to a Spin^c–map is necessary in order to rigidify the choice of β up to exact forms by (68) by constructing it via transgression.

Remark 4.10 The identity of M refines to a Spin^c–map in a natural way by choosing the identity in (67). The pull-back of geometrisations for the identity of M can be used to transfer a geometrisation defined for one choice of the connection $\tilde{\nabla}$ to a second choice. This allows us to define a notion of geometrisation which is independent of the choice of the connection. This could play a role if one wants to classify geometrisations. We will not pursue that goal in the present paper.

4.4 Good geometrisations

Assume that (W,F) is a zero bordism of the n–dimensional cycle (M,f). We fix tangential Spin^c–structures $P(TW)\in\mathrm{Spin}^c(TW)$ and $P(TM)\in\mathrm{Spin}^c(TM)$ associated to the normal Spin^c–structure on W and M; see Definition 3.3. As explained in Section 3.3, the diagram (33) fixes a natural isomorphism of Spin^c–structures

$$P(TM)\otimes Q(1)\cong P(TW)_{|M}, \tag{71}$$

which turns the inclusion

$$i\colon M\to W$$

into a Spin^c–map.

We choose a Spin^c–extension of the Levi-Civita connection $\tilde{\nabla}^{TW}$ on W with product structure and a Spin^c–extension of the Levi-Civita connection $\tilde{\nabla}^{TM}$ on M such that the isomorphism (71) preserves the connections. In this situation the form (68) is trivial. Assume now that we have a geometrisation of $(W,F,\tilde{\nabla}^{TW})$. Then we can define the restriction $\mathcal{G}_{\partial W}:=(\mathcal{G}_W)_{|\partial W}$ as in Lemma 4.9. It is given by

$$\mathcal{G}_{\partial W}(\phi)=\mathcal{G}_W(\phi)_{|\partial W},\quad \phi\in K^0(B). \tag{72}$$

In general we do not expect that a given geometrisation $\mathcal{G}_M$ of $(M,f,\tilde{\nabla}^{TM})$ can be obtained by restricting a geometrisation $\mathcal{G}_W$ of $(W,F,\tilde{\nabla}^{TW})$. In this respect, geometrisations are more rigid than connections.

Example 4.11 Here is a very simple example showing that geometrisations do not always extend to zero bordisms. We consider the case $B=*$ and let (S^3,f) be a cycle for $\pi_3(S)$. Here $S=MB$ is the sphere spectrum and $\pi_*(S)$ is identified with framed bordism groups. We choose a normal framing of S^3 that extends over D^4, so that the framed bordism class $[S^3,f]$ is trivial. Furthermore we equip S^3 with its standard riemannian metric.

We know by Proposition 4.8 that $(S^3,f,\tilde{\nabla}^{TS^3})$ admits geometrisations. Let us choose some geometrisation $\mathcal{G}_0$. If it does not extend over the disc D^4, then we have found the desired example. Otherwise assume that it extends.

We have $K^0(B) \cong K^0(*) \cong \mathbb{Z}$, so a geometrisation is fixed by the image of 1 in $\widehat{K}^0(S^3)$. Let $\omega \in \Omega^3(S^3)$ be some form. Then we can define a new geometrisation $\mathcal{G}_\omega$ of $(S^3, f, \widetilde{\nabla}^{TS^3})$ by

$$\mathcal{G}_\omega(1) := \mathcal{G}_0(1) + a(b^2\omega).$$

We claim that $\mathcal{G}_\omega$ extends to D^4 if and only if $\int_{S^3} \omega \in \mathbb{Z}$. For our present purpose the "only if" part is relevant. We leave the other direction as an exercise. So let us assume that $\mathcal{G}_\omega$ extends, too. Let $\widetilde{\mathcal{G}}_0$ and $\widetilde{\mathcal{G}}_\omega$ denote the extensions of $\mathcal{G}_0$ and $\mathcal{G}_\omega$ to D^4. By (57) and the defining relation (64) there exists a form $\widetilde{\omega} \in \Omega^3(D^4)$ such that

$$\widetilde{\mathcal{G}}_\omega(1) - \widetilde{\mathcal{G}}_0(1) = a(b^2\widetilde{\omega}).$$

Since the cohomological characters of the geometrisations preserve the b–degree and Todd forms are invertible, we have the equality $c_{\widetilde{\mathcal{G}}_\omega}(1) = c_{\widetilde{\mathcal{G}}_0}(1)$. Consequently

$$b^2 d\widetilde{\omega} = c_{\widetilde{\mathcal{G}}_\omega}(1) - c_{\widetilde{\mathcal{G}}_0}(1) = 0.$$

It follows from the exact sequence (57) that there is a class $u \in K^{-1}(S^3)$ such that

$$b^2\widetilde{\omega}|_{S^3} - b^2\omega = \mathbf{ch}_3(u).$$

Using that $\widetilde{\omega}$ is closed we get

$$\int_{S^3} b^2\omega = -\int_{S^3} b^2\mathbf{ch}_3(u) \in b^2\mathbb{Z}.$$

This concludes Example 4.11. $\triangleleft$

In order to deal with the problem of nonextendability of geometrisations appropriately we introduce the notion of a k–good geometrisation. If $\mathcal{G}_M$ is k–good with $k \geq \dim(M) + 1$, then it will extend to zero bordisms.

For $k \in \mathbb{N}$ we define the notion of k–good geometrisations constructively. We use the symbol M_u in order to denote a smooth manifold (hence the letter M) which approximates the space B. It will carry a kind of universal geometrisation which will be pulled back to various manifolds mapping to M_u. We attach the subscript u (for universal) to the symbols for objects living over M_u.

We consider a map $f_u\colon M_u \to B$ from a smooth compact manifold M_u, a lift $\widetilde{f}_u$ (as in (28)) together with a choice of a complementary bundle η_u, a connection $\widetilde{\nabla}_u$ on $P_u \in \mathrm{Spin}^c(\eta_u)$ and a geometrisation $\mathcal{G}_u$ of $(M_u, f_u, \widetilde{\nabla}_u)$. If the map $f\colon M \to B$ has a factorisation up to homotopy through a smooth map $h\colon M \to M_u$, then we can refine h to a stable Spin^c–map since, after stabilisation, there exists an isomorphism between $h^*\eta_u$ and TM.

Definition 4.12 The geometrisation $\mathcal{G}_M$ is called k–good if $\mathcal{G}_M = h^*\mathcal{G}_u$ for some choices as above such that f_u is k–connected. We say that $\mathcal{G}_M$ is good if it is $\dim(M)+1$–good.

Remark 4.13 We consider a sequence $(M_{u,i}, f_{u,i}, \tilde{\nabla}^{u,i})$ for $i \in \mathbb{N}$ of data as above together with Spin^c–maps $h_{u,i}\colon M_{u,i} \to M_{u,i+1}$ for all $i \in \mathbb{N}$ such that $f_{u,i+1} \circ h_{u,i} \sim f_{u,i}$ and $f_{u,i}$ is i–connected. A family of geometrisations $\mathcal{G}_{u,i}$ of $(M_{u,i}, f_{u,i}, \tilde{\nabla}^{u,i})$ for all $i \in \mathbb{N}$ such that $h^*_{u,i}\mathcal{G}_{u,i+1} = \mathcal{G}_{u,i}$ will be called a universal geometrisation. Universal geometrisations are constructed and classified in the thesis of M Völkl [59].

Let us fix a universal geometrisation. A geometrisation of $(M, f, \tilde{\nabla}^{TM})$ will be called very good (relative to the chosen universal geometrisation) if it is isomorphic to $h^*\mathcal{G}_{u,i}$ for some $i \in \mathbb{N}$ and Spin^c–map $h\colon M \to M_{u,i}$ such that $f_{u,i} \circ h \sim f$. Note that such a geometrisation is l–good for every $l \in \mathbb{N}$.

Lemma 4.14 *If B has the homotopy type of a CW–complex with finite skeleta, then for every triple $(M, f, \tilde{\nabla}^{TM})$ and every $k \in \mathbb{N}$ there exists a k–good geometrisation.*

Proof By the assumption, for every $k \in \mathbb{N}$ we can find a compact manifold M_u and a map $f_u\colon M_u \to B$ such that $f_u\colon M_u \to B$ is k–connected. We choose complementary data $\tilde{f}_u$, η_u, P_u and $\tilde{\nabla}^u$ as above. By Proposition 4.8 there exists a geometrisation $\mathcal{G}_u$ of the triple $(M_u, f_u, \tilde{\nabla}^u)$. Given (M, f) with $\dim(M) \le k-1$ there exists a factorisation up to homotopy

$$\begin{tikzcd} & M_u \arrow[d, "f_u"] \\ M \arrow[ur, dotted, "h"] \arrow[r, "f"'] & B \end{tikzcd} \tag{73}$$

and a refinement of h to a Spin^c–map. Then $\mathcal{G}_M := h^*\mathcal{G}_u$ is a k–good geometrisation. □

Lemma 4.15 *Let $\mathcal{G}_M$ be a good geometrisation of $(M, f, \tilde{\nabla}^{TM})$. If (W, F, G) is a zero bordism of (M, f) with connection $\tilde{\nabla}^{TW}$, then there exists a geometrisation $\mathcal{G}_W$ of $(W, F, \tilde{\nabla}^{TW})$ which restricts to $\mathcal{G}_M$.*

Proof Since $f_u\colon M_u \to B$ is an $n+1$–equivalence and $\dim(W) = n+1$ we can extend the factorisation (73) to a factorisation:

$$\begin{tikzcd} & M_u \arrow[d, "f_u"] \\ W \arrow[ur, dotted, "H"] \arrow[r, "F"'] & B \end{tikzcd} \tag{74}$$

There exists a refinement of H to a stable Spin^c–map such that $H\circ i = h$ in the sense of Spin^c–maps. Then we can define the pull-back $\mathcal{G}_W := H^*\mathcal{G}_u$ and get $(\mathcal{G}_W)|_M = \mathcal{G}_M$ by the functoriality of the pull-back. □

Remark 4.16 Lemma 4.14 does not imply the existence of very good geometrisations (see Remark 4.13). Furthermore, observe the following potentially bad behaviour with respect to disjoint unions: Let $(M_i, f_i, \tilde{\nabla}^{TM_i})$ for $i = 0, 1$ be two geometric cycles with k–good geometrisations $\mathcal{G}_i$. Then we can form the disjoint union $(M, f, \tilde{\nabla}^{TM}) := (M_0, f_0, \tilde{\nabla}^{TM_0}) \sqcup (M_1, f_1, \tilde{\nabla}^{TM_1})$, which carries a geometrisation $\mathcal{G}$ naturally induced by $\mathcal{G}_i$. It is not clear that this geometrisation is k–good. For this it would good to know that the geometrisations $\mathcal{G}_i$ are pulled back from the *same* geometrisation $\mathcal{G}_u$ by different maps. This is the motivation behind the notion of a very good geometrisation. If the two geometrisations are very good (with respect to the same universal geometrisation), then the geometrisation $\mathcal{G}$ above is again very good (with respect to the universal geometrisation). In Remark 4.20 below we further explain why very good geometrisations might be interesting.

4.5 An intrinsic formula for η^{an}

The main goal of the present subsection is to give an intrinsic formula for $\eta^{\mathrm{an}}(x)$ which only involves structures on the cycle (M, f) for $x \in \pi_n(MB)_{\mathrm{tors}}$.

The geometric and analytic terms in the formula (37) for $\eta^{\mathrm{an}}(x)$ separately have values in $\mathbb{R}/\mathbb{Z}$; only their sum belongs to $\mathbb{Q}/\mathbb{Z}$. In order to deal with these terms separately it is useful to use a real version $Q_n^{\mathbb{R}}(E)$ of the group $Q_n(E)$. We start with introducing this group. We further show that there is no loss of information when passing to the real version. We let (compare with (6))

$$U^{\mathbb{R}} \subseteq \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{R}/\mathbb{Z})) \tag{75}$$

be the subgroup given by evaluations against elements in $\pi_{n+1}(E\mathbb{R})$ and define

$$Q_n^{\mathbb{R}}(E) := \frac{\mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{R}/\mathbb{Z}))}{U^{\mathbb{R}}}. \tag{76}$$

The inclusion $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \to \pi_{n+1}(K\mathbb{R}/\mathbb{Z})$ induces a map

$$i_{\mathbb{R}}\colon Q_n(E) \to Q_n^{\mathbb{R}}(E).$$

Lemma 4.17 *The map $i_{\mathbb{R}}\colon Q_n(E) \to Q_n^{\mathbb{R}}(E)$ is injective.*

Proof Let $\kappa \in Q_n(E)$ be represented by $\widehat{\kappa} \in \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$. Since κ is continuous it factors over a finitely generated quotient of $K^0(E)$. Hence there exists $N \in \mathbb{N}$ such that $N\widehat{\kappa}$ vanishes.

Assume now that $i_{\mathbb{R}}(\kappa) = 0$. Then there exists $w \in \pi_{n+1}(E\mathbb{R})$ such that $\widehat{\kappa}(\phi) = [\langle w, \phi\rangle] \in \pi_{n+1}(K\mathbb{R}/\mathbb{Z})$ for all $\phi \in K^0(E)$. Since $\pi_{n+1}(E\mathbb{R}) \cong \pi_{n+1}(E) \otimes \mathbb{R}$ (see (2)) there exists a finite subset $I \subset \pi_{n+1}(E)$ and a map $\lambda\colon I \to \mathbb{R}$ such that $w = \sum_{v\in I} \lambda(v)v$. We have $\widehat{\kappa}(\phi) = \sum_{v\in I}[\lambda(v)\langle\phi, v\rangle]$, where here $\langle\phi, v\rangle \in \pi_{n+1}(K)$. For $v \in I$ we define $\widehat{v} \in \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K))$ by $\widehat{v}(\phi) := \langle\phi, v\rangle$. The set $\{\widehat{v} \mid v \in I\}$ generates a free abelian subgroup $A \subseteq \mathtt{Hom}^{\mathrm{cont}}(K^0(E), \pi_{n+1}(K))$. We can choose a minimal subset $J \subseteq I$ which generates a subgroup of A of full rank. Then there exists a suitable map $\mu\colon J \to \mathbb{R}$ such that $\widehat{\kappa}(\phi) = \sum_{v\in J}[\mu(v)\widehat{v}(\phi)]$ for all $\phi \in K^0(E)$.

The image of $K^0(E) \to \mathtt{Hom}(A, \pi_{n+1}(K))$ has full rank. Hence for every $v \in J$ there exists $\phi_v \in K^0(E)$ such that $\widehat{v}(\phi_v) \neq 0$ and $\widehat{v}'(\phi_v) = 0$ for all $v' \in J$ with $v' \neq v$. It follows that $\widehat{\kappa}(\phi_v) = [\mu(v)\widehat{v}(\phi_v)]$. Since $0 = N\widehat{\kappa}(\phi_v) = [N\mu(v)\widehat{v}(\phi_v)]$ it follows that $\mu(v) \in \mathbb{Q}$. We set $w_{\mathbb{Q}} := \sum_{v\in J} \mu(v)v \in \pi_{n+1}(E) \otimes \mathbb{Q} \cong \pi_{n+1}(E\mathbb{Q})$. Then we have $\widehat{\kappa}(\phi) = [\langle\phi, w_{\mathbb{Q}}\rangle]$ for all $\phi \in K^0(E)$. This shows that $\widehat{\kappa} \in U$ and $\kappa = 0$. □

Let $x \in \pi_n(MB)_{\mathrm{tors}}$ be an l–torsion element and (M, f) be a cycle for x. We choose a Spin^c–extension $\widetilde{\nabla}^{TM}$ of the Levi-Civita connection on M. We further assume that we have a good geometrisation $\mathcal{G}_M$ of $(M, f, \widetilde{\nabla}^{TM})$ (see Definition 4.12) with a choice of a degree-preserving cohomological character $c_{\mathcal{G}}$. If B has finite skeleta, then its existence is guaranteed by Lemma 4.14.

For every ϕ in $K^0(B)$ we choose a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle $V_\phi \to M$ such that $[V_\phi] = f^*\phi$ in $K^0(M)$. We furthermore choose a hermitian metric h^{V_ϕ} and a metric connection ∇^{V_ϕ} so that we get the geometric bundle $\boldsymbol{V}_\phi = (V_\phi, h^{V_\phi}, \nabla^{V_\phi})$. It represents a differential K–theory class $[\boldsymbol{V}_\phi] \in \widehat{K}^0(M)$ such that $I([\boldsymbol{V}_\phi]) = [V_\phi] = f^*\phi = I(\mathcal{G}_M(\phi))$. By the exactness of (57) we get a uniquely determined element

$$\gamma_\phi \in \Omega P^{-1}(M)/\mathtt{im}(\mathbf{ch})$$

such that

$$\mathcal{G}_M(\phi) - [\boldsymbol{V}_\phi] = a(\gamma_\phi). \tag{77}$$

Definition 4.18 We will refer to γ_ϕ as the correction form associated to ϕ.

Theorem 4.19 *The element*

$$i_{\mathbb{R}}(\eta^{\mathrm{an}}([M, f])) \in Q_n^{\mathbb{R}}(MB)$$

is represented by the homomorphism

(78) $$K^0(MB) \overset{\mathtt{Thom}^K}{\cong} K^0(B) \to \mathbb{R}/\mathbb{Z}, \quad \phi \mapsto \left[-\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi\right] - \xi(\not{D}_M \otimes \boldsymbol{V}_\phi).$$

Proof The integral in formula (78) belongs to the group $\mathbb{R}[b, b^{-1}]^{-n-1}$, which will be identified with $\mathbb{R}$ using the generator $b^{-(n+1)/2}$. First note that, despite the fact that γ_ϕ is only defined up the image of $\mathbf{ch}$: $K^{-1}(M) \to HP\mathbb{Q}^{-1}(M)$, the class

$$\left[\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi\right] \in \mathbb{R}/\mathbb{Z}$$

is well-defined. Indeed, we have $\langle [M], \mathbf{Td}(TM) \cup \mathbf{ch}(\psi)\rangle \in \mathbb{Z}$ for all $\psi \in K^{-1}(M)$ by the odd version of the Atiyah–Singer index theorem.

We now argue that the right-hand side of (78) indeed defines a homomorphism. To this end we observe, using the variation formulas for the η–invariant and the homotopy formula for the differential K–theory class $[\boldsymbol{V}_\phi]$, that

$$\left[-\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi\right] - \xi(\not{D}_M \otimes \boldsymbol{V}_\phi) \in \mathbb{R}/\mathbb{Z}$$

does not depend on the choice of the geometry of V_ϕ. Using the invariance of this term under stabilisation of $\boldsymbol{V}_\phi$ with geometric bundles of the form $\boldsymbol{V} \oplus \boldsymbol{V}^{\mathrm{op}}$ we see that it only depends on the K–theory class ϕ and is clearly additive.

In contrast to the notation in Section 3.4 we write U_ϕ instead of U in order to indicate the dependence of ϕ. We use (37) in order to express the right-hand side of (39) as

$$\left[\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^{U_\phi})\right] - \xi(\not{D}_M \otimes \boldsymbol{V}_\phi).$$

The whole idea is now to turn the integral over W into an integral over M. To this end we assume by Lemma 4.15 that the good geometrisation $\mathcal{G}_M$ has an extension $\mathcal{G}_W$ to W.

The geometric bundle $\boldsymbol{U}_\phi$ extends the geometric bundle $\boldsymbol{V}_\phi$ across W. We let $\gamma_\phi^W \in \Omega P^{-1}(W)/\mathtt{im}(\mathbf{ch})$ be the correction form defined by

$$\mathcal{G}_W(\phi) - [\boldsymbol{U}_\phi] = a(\gamma_\phi^W).$$

By (72) we conclude that $(\gamma^W_\phi)_{|\partial W}$ coincides up to the image of **ch** with γ_ϕ on all copies of M. We now use Stokes' theorem in order to rewrite

$$\begin{aligned}\left[\frac{1}{l}\int_W \mathbf{Td}(\tilde\nabla^{TW})\wedge\mathbf{ch}(\nabla^{U_\phi})\right] &= \left[\frac{1}{l}\int_W \mathbf{Td}(\tilde\nabla^{TW})\wedge R(\mathcal{G}_W(\phi)) - \frac{1}{l}\int_W \mathbf{Td}(\tilde\nabla^{TW})\wedge d\gamma^W_\phi\right]\\ &= \left[\frac{1}{l}\int_W \mathbf{Td}(\tilde\nabla^{TW})\wedge R(\mathcal{G}_W(\phi))\right] - \left[\int_M \mathbf{Td}(\tilde\nabla^{TM})\wedge\gamma_\phi\right].\end{aligned}$$

We want to show that the homomorphism

$$\kappa\colon K^0(B)\to\mathbb{R}/\mathbb{Z},\quad \phi\mapsto\left[\frac{1}{l}\int_W \mathbf{Td}(\tilde\nabla^{TW})\wedge R(\mathcal{G}_W(\phi))\right],$$

belongs to $U^{\mathbb{R}}$. The integrand of the integral over W can be expressed in terms of the cohomological character $c_{\mathcal{G}_W}$ of $\mathcal{G}_W$. Therefore κ has a factorisation as

$$K^0(B)\xrightarrow{\mathbf{Td}^{-1}\cup\mathbf{ch}} HP\mathbb{Q}^0(B)\xrightarrow{c_{\mathcal{G}_W}}\Omega P^0_{\mathrm{cl}}(W)\xrightarrow{\frac{1}{l}\int_W}\mathbb{R}\xrightarrow{[\cdot]_{\mathbb{R}/\mathbb{Z}}}\mathbb{R}/\mathbb{Z}.$$

Since the cohomological character $c_{\mathcal{G}_W}$ preserves degree and $\frac{1}{l}\int_W$ factorises over the degree $n+1$ part, the homomorphism κ actually factorises over

$$K^0(B)\xrightarrow{p_{n+1}(\mathbf{Td}^{-1}\cup\mathbf{ch})} H\mathbb{Q}^{n+1}(B)\xrightarrow{(c_{\mathcal{G}_W})_{|H\mathbb{Q}^{n+1}(B)}}\Omega^{n+1}_{\mathrm{cl}}(W)\xrightarrow{\frac{1}{l}\int_W}\mathbb{R}\xrightarrow{[\cdot]_{\mathbb{R}/\mathbb{Z}}}\mathbb{R}/\mathbb{Z}.$$

Since $c_{\mathcal{G}_W}$ is continuous, the composition

$$H\mathbb{Q}^{n+1}(B)\xrightarrow{(c_{\mathcal{G}_W})_{|H\mathbb{Q}^{n+1}(B)}}\Omega^{n+1}_{\mathrm{cl}}(W)\xrightarrow{\frac{1}{l}\int_W}\mathbb{R}$$

is continuous and therefore given by the pairing against an element of $H\mathbb{R}_{n+1}(B)$. We now use the $\mathbb{R}$–version of (14) in order to conclude that $\kappa\in U^{\mathbb{R}}$. Therefore $i_{\mathbb{R}}(\eta^{\mathrm{an}}([M,f]))$ is also represented by the map (78). □

Remark 4.20 Let us mention the following aspect of the intrinsic formula (78), which is not completely understood at the moment. For the intrinsic formula to make sense we do not need the zero bordism (W,F) of l–copies of (M,f). Therefore, formula (78) provides an element $\eta^{\mathrm{intrinsic}}_{\mathcal{G}}(M,f,\tilde\nabla^{TM})\in Q^{\mathbb{R}}_n(MB)$. One can check that $\eta^{\mathrm{intrinsic}}_{\mathcal{G}}(M,f,\tilde\nabla^{TM})=0$ if the data $(M,f,\tilde\nabla^{TM},\mathcal{G}_M)$ extends to a zero bordism. Note that in general two good geometrisations of $(M,f,\tilde\nabla^{TM})$ cannot be connected over the cylinder, but this is possible if both are very good with respect to the same universal geometrisation. More generally, if we work with a very good geometrisation associated to a fixed universal geometrisation (see Remark 4.13), then the problem with disjoint unions mentioned in Remark 4.16 disappears, too. Therefore, if we fix a universal geometrisation, then we would get a homomorphism

$$\eta_{\mathcal{G}}^{\text{intrinsic}}\colon \pi_n(MB) \to Q_n^{\mathbb{R}}(MB)$$

which restricts to $i_{\mathbb{R}} \circ \eta^{\text{top}} = i_{\mathbb{R}} \circ \eta^{\text{an}}$ on $\pi_n(MB)_{\text{tors}}$. In general we do not know the topological content of this extension of the universal η–invariant. It might be related to the effect observed in Remark 5.22.

5 Examples

5.1 Adams' e–invariant

We consider the example $B = *$. The associated Thom spectrum is the sphere spectrum $S \simeq MB$. By Serre's theorem [57] the homotopy groups $\pi_n(S)$ are finite for $n \geq 1$ (we refer to [54] for more details about their structure). Therefore the universal η–invariant is defined on all of $\pi_n(S)$.

We have an identification $K^0(S) \cong \mathbb{Z}$. Furthermore, since $\pi_{n+1}(S\mathbb{Q}) \cong 0$ for $n \geq 0$, the group U defined in (6) is trivial. From now on let $n \in \mathbb{N}$ be odd. After identifying $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ we obtain the identification

$$Q_n(S) \cong \mathtt{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

given by evaluation against $1 \in \mathbb{Z}$. For every odd $n \in \mathbb{N}$ the universal η–invariant is thus interpreted as a homomorphism

$$\eta\colon \pi_n(S) \to \mathbb{Q}/\mathbb{Z}. \tag{79}$$

The universal η–invariant essentially coincides with Adams' e–invariant [1]

$$e^{\text{Adams}}\colon \pi_n(S) \to \mathbb{Q}/\mathbb{Z},$$

which was introduced in order to detect the image of the J–homomorphism

$$J\colon KO_{n+1} \to \pi_n^S. \tag{80}$$

We will see below that we have the relation

$$\eta = e_{\mathbb{C}}^{\text{Adams}} := \begin{cases} e^{\text{Adams}} & \text{if } \frac{n-1}{2} \text{ is even,} \\ 2e^{\text{Adams}} & \text{if } \frac{n-1}{2} \text{ is odd.} \end{cases} \tag{81}$$

We consider the Adams filtration of $\pi_*(S)$ associated to the K–theory based Adams spectral sequence (see eg [2] or Section 2.4). It follows from Theorem 2.7(4) that the universal η–invariant (79) induces an injection

$$\eta^{\text{top}}\colon \mathrm{Gr}^1\pi_n(S) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

(recall that n is assumed to be odd). The relation of the Adams e–invariant to spectral geometry has first been observed in [9]. The spectral geometric calculation of the Adams e–invariant interprets $\pi_n(S)$ as a framed bordism group. It has the favourable property that it provides an intrinsic formula for e^{Adams}, a fact which has been successfully exploited eg in [29; 56].

The goal of the following discussion is to derive, using Theorems 4.19 and 3.7, the intrinsic formula (83) for η^{top} and compare it with the formula [8, Theorem 4.14] for $e^{\mathrm{Adams}}_{\mathbb{C}}$. This is finally our argument for (81).

We let (M, f) be a cycle for $\pi_n(S)$ as in Section 3.2, where the constant map $f\colon M \to *$ is refined to a stable framing $TM \oplus \underline{\mathbb{R}}^k_M \cong \underline{\mathbb{R}}^{n+k}_M$ of the tangent bundle. A tangential Spin^c–structure is now given by a trivialisation

$$P(TM) \otimes Q(k) \cong Q(n+k). \tag{82}$$

We will in fact assume that $P(TM)$ comes from a Spin–structure. In this case the Levi-Civita connection induces a canonical Spin^c–connection $\tilde{\nabla}^{TM}$.

In order to apply Theorem 4.19 we must first choose a good geometrisation of $(M, f, \tilde{\nabla}^{TM})$. We will use the notation of Section 4.4. We can choose the manifold M_u to be a point. The datum $(M_u, f_u, \tilde{\nabla})$ (where $f_u\colon M_u \to *$ is constant and $\tilde{\nabla}$ is the trivial connection on the trivial bundle P_u) has a unique geometrisation $\mathcal{G}_u$. Let $h\colon M \to M_u$ be the constant map. The Spin^c–bundle h^*P_u is trivial. Hence the given trivialisation (82) refines h to a Spin^c–map. Using this refinement we define the geometrisation $\mathcal{G} := h^*\mathcal{G}_u$, which turns out to be l–good for every $l \in \mathbb{N}$.

For $1 \in K^0(*) \cong \mathbb{Z}$ we have $\mathcal{G}_u(1) = 1 \in \hat{K}^0(M_u)$. We now use Lemma 4.9 in order to calculate $\mathcal{G}(1) \in \hat{K}^0(M)$. By (70) we have

$$\mathcal{G}(1) = 1 + a\left(\frac{\tilde{\mathbf{Td}}(\tilde{\nabla}^{\mathrm{triv}}, \tilde{\nabla}^{TM})}{\mathbf{Td}(\tilde{\nabla}^{TM})}\right),$$

where ∇^{triv} is the connection on $P(TM) \otimes Q(k)$ induced by the trivialisation (82). Let V_1 be the trivial one-dimensional geometric bundle on M. Then $[V_1] = 1 \in \hat{K}^0(M)$ and in view of (77) we must take the correction form

$$\gamma_1 := \frac{\tilde{\mathbf{Td}}(\tilde{\nabla}^{\mathrm{triv}}, \tilde{\nabla}^{TM})}{\mathbf{Td}(\tilde{\nabla}^{TM})}.$$

We now specialise Theorem 4.19 to the present situation and obtain the intrinsic formula

$$i_{\mathbb{R}}\big(\eta^{\mathrm{an}}([M, f])\big)(1) = \left[-\int_M \tilde{\mathbf{Td}}(\tilde{\nabla}^{\mathrm{triv}}, \tilde{\nabla}^{TM})\right] - \xi(\not{D}_M) \in \mathbb{R}/\mathbb{Z}. \tag{83}$$

This formula directly compares with the formula for $e^{\mathrm{Adams}}_{\mathbb{C}}([M,f])$ derived by specialising [8, Theorem 4.14].

5.2 ρ–invariants and the index theorem for flat bundles

We consider a closed odd-dimensional Spin^c–manifold M. The geometry on M is given by a riemannian metric and the Spin^c–extension of the Levi-Civita connection. If $\boldsymbol{V} = (V, \nabla^V, h^V)$ is a flat hermitian vector bundle of dimension k on M, then the difference of reduced η–invariants

$$\rho(\not{D}_M, \boldsymbol{V}) := \xi(\not{D}_M \otimes \boldsymbol{V}) - k\xi(\not{D}_M) \tag{84}$$

is invariant under variations of the geometry of M. The ρ–invariant is thus a differential topological invariant of the Spin^c–manifold M with a flat hermitian bundle $\boldsymbol{V}$.

The ρ–invariant is a classical example of a topological invariant derived from the η–invariant which has been studied a lot. For example, it has been used successfully to detect elements in Spin^c–bordism groups of classifying spaces of finite cyclic groups [12; 13]. We refer to these references for examples of explicit calculations of ρ–invariants.

The precise homotopy-theoretic description of ρ–invariants is given by the index theorem for flat bundles [9, Theorem 5.3]. The goal of the following discussion is to explain the connection of the relation $\eta^{\mathrm{an}} = \eta^{\mathrm{top}}$ shown in Theorem 3.7 with the index theorem for flat bundles. Roughly speaking, this goes as follows. The index theorem for flat bundles is about the pairing of the K–homology class represented by the Spin^c–Dirac operator with the torsion K–cohomology classes obtained from the flat bundle, while our index theorem considers the pairing of a torsion K–homology class with K–theory classes. Clearly the overlap is when both classes are torsion.

We first translate the data of the Spin^c–manifold M of odd dimension n with a flat hermitian bundle $\boldsymbol{V}$ into the bordism picture. Let $U(k)^\delta$ denote the unitary group equipped with the discrete topology. Its classifying space $BU(k)^\delta$ is universal for flat hermitian vector bundles of dimension k. We consider the bordism group based on the Thom spectrum of

$$B := B\mathrm{Spin}^c \times BU(k)^\delta \xrightarrow{\mathrm{pr}} B\mathrm{Spin}^c.$$

We let (M,f) be a cycle for $\pi_n(M\mathrm{Spin}^c)$ and consider in addition a map $g\colon M \to BU(k)^\delta$ which classifies a flat bundle $\boldsymbol{V}$. In this way we get a class

$$[M,(f,g)] \in \pi_n(MB).$$

We assume that this class is torsion in order to apply the universal η–invariant.

We consider the K–theory class $\lambda_k \in K^0(BU(k)^\delta)$ of the universal $\mathbb{C}^k$–bundle on $BU(k)^\delta$ and the projection

$$q\colon B\mathrm{Spin}^c \times BU(k)^\delta \to BU(k)^\delta. \tag{85}$$

Since $\mathbf{ch}(\lambda_k - k) = 0$, the evaluation against the difference $q^*\lambda_k - k$ provides a well-defined homomorphism

$$\mathtt{ev}_{q^*\lambda_k - k}\colon Q_n(MB) \to \mathbb{Q}/\mathbb{Z}.$$

Lemma 5.1 *We assume that $[M,(f,g)] \in \pi_n(MB)$ is a torsion class. Then we have*

$$\mathtt{ev}_{q^*\lambda_k - k}\big(\eta^{\mathrm{an}}\big([M,(f,g)]\big)\big) = \rho(\not{D}_M, \boldsymbol{V}). \tag{86}$$

Proof We are going to use the notation introduced in Section 3.4. As an intermediate step we choose, for a suitable nonvanishing integer l, a zero bordism $(W,(F,G))$ of the union of l copies of the cycle $(M,(f,g))$ with Spin^c–geometry. The geometric bundle $\boldsymbol{U}$ is then the flat hermitian bundle classified by F, and we have, by Definition 3.6,

$$\mathtt{ev}_{q^*\lambda_k - k}\big(\eta^{\mathrm{an}}\big([M,(f,g)]\big)\big) = \Big[\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \boldsymbol{U})\Big] - \Big[\frac{k}{l}\mathtt{index}(\not{D}_W)\Big].$$

If we use (37) instead, then we express this evaluation in terms of an integral over local data on W and the reduced η–invariants. Because $\mathbf{ch}(\nabla^U) = k$, the local contributions cancel out and we are left with

$$\mathtt{ev}_{q^*\lambda_k - k}\big(\eta^{\mathrm{an}}\big([M,(f,g)]\big)\big) = \rho(\not{D}_M, \boldsymbol{V}). \qquad \square$$

We now calculate the topological version of the universal η–invariant explicitly. We again assume that $x = [M,(f,g)] \in \pi_n(MB)$ is a torsion element and let $\tilde{x}$ in $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z} \wedge MB)$ be as in (4). By definition of η^{top} we get the equality

$$\mathtt{ev}_{q^*\lambda_k - k}\big(\eta^{\mathrm{top}}\big([M,(f,g)]\big)\big) = \langle \mathtt{Thom}^K(q^*\lambda_k - k), \tilde{x}\rangle. \tag{87}$$

The right-hand side of (86) is the analytic side of the index theorem for flat bundles in [9, Theorem 5.3]. The topological side of the index theorem for flat bundles [9, Theorem 5.3] is not given as the pairing of a $K\mathbb{Q}/\mathbb{Z}$–homology class with a K–theory class, but rather by a pairing between a K–homology class and a $K\mathbb{R}/\mathbb{Z}$–cohomology class. In the following we rewrite the right-hand side of (87) in this way.

There exists a class $\Lambda_k \in K\mathbb{R}/\mathbb{Z}^{-1}(BU(k)^\delta)$ with $\partial\Lambda_k = \lambda_k - k$, where ∂ is the Bockstein operator $\partial\colon K\mathbb{R}/\mathbb{Z}^{-1}(BU(k)^\delta) \to K^0(BU(k)^\delta)$, and which is uniquely characterised by the following property: if $h\colon N \to BU(k)^\delta$ is a map from a smooth manifold N, then $h^*(\Lambda_k) = [\boldsymbol{U}] - k$, where $\boldsymbol{U}$ is the unitary flat bundle classified by h. In order to interpret this equality we employ the identification $K\mathbb{R}/\mathbb{Z}^{-1}(N) \cong \widehat{K}^0_{\mathrm{flat}}(N)$ given by (56).

Recall that q denotes the projection (85). We use Lemma 2.6 for the first equality in the chain

$$\begin{aligned}\mathtt{ev}_{q^*\lambda_k-k}(\eta^{\mathrm{top}}(x)) &= \langle \mathtt{Thom}^{K\mathbb{R}/\mathbb{Z}}(q^*\Lambda_k), \epsilon(x)\rangle \\ &= \langle q^*\Lambda_k, \mathtt{Thom}_K(\epsilon(x))\rangle = \langle [V]-k, [M]_K\rangle,\end{aligned}$$

where $[M]_K$ denotes the K–theory fundamental class of the Spin^c–manifold M. The right-hand side of this equality is the topological side of the index theorem for flat bundles of [9, Theorem 5.3]. The following corollary now immediately follows from the equality $\eta^{\mathrm{top}} = \eta^{\mathrm{an}}$.

Corollary 5.2 *Let $n \in \mathbb{N}$ be odd and M be a closed n–dimensional* Spin^c *–manifold with a flat hermitian k–dimensional vector bundle V. We assume in addition that the corresponding class $[M,(f,g)] \in \pi_n\big(M(B\mathrm{Spin}^c \times BU(k)^\delta)\big)$ is torsion. Then we have the following equality in $\mathbb{R}/\mathbb{Z}$:*

$$\rho(\not{D}_M, V) = \langle [V]-k, [M]_K\rangle. \tag{88}$$

In this way Theorem 4.19 implies a special case of [9, Theorem 5.3]. Let us remark that, by [9, Theorem 5.3], the equality (88) holds true without the additional assumption that $[M,(f,g)]$ is a torsion class.

5.3 Algebraic K–theory

In this subsection we use the universal η–invariant in order detect algebraic K–theory classes of $\mathbb{C}$. We will observe that for odd $n \in \mathbb{N}$ the well-known homomorphism

$$\varepsilon\colon K_n^{\mathrm{alg}}(\mathbb{C}) \to \mathbb{Q}/\mathbb{Z}$$

(see (94)) can be obtained from an appropriate evaluation of the universal η–invariant. Our main result is Theorem 5.5, which provides a formula for ε in terms of geometric cycles for K–theory classes. We will explain how the results of [53; 42] can be interpreted as constructions with the universal η–invariant.

For $n \in \mathbb{N}$ the algebraic K–theory groups $K_n^{\mathrm{alg}}(\mathbb{C})$ of the field $\mathbb{C}$ are defined as the homotopy groups of the connective algebraic K–theory spectrum $K^{\mathrm{alg}}(\mathbb{C})$. This spectrum is connected with classifying spaces through Quillen's $+$ construction (see [3, Chapter 3] for a detailed description)

$$p\colon B\mathrm{GL}(\mathbb{C}^\delta) \to B\mathrm{GL}(\mathbb{C}^\delta)^+ \tag{89}$$

by the equivalence

$$\Omega^\infty K^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{Z} \times B\mathrm{GL}(\mathbb{C}^\delta)^+. \tag{90}$$

There exists a class

$$\Lambda_0 \in K\mathbb{R}/\mathbb{Z}^{-1}(B\mathrm{GL}(\mathbb{C}^\delta)^+)$$

which is uniquely characterised by the following property: if $k \in \mathbb{N}$ and $g_k\colon N \to B\mathrm{GL}(k, \mathbb{C}^\delta)$ is a map from a smooth manifold, then

$$\iota(p \circ i_k \circ g_k)^* \Lambda_0 = [\boldsymbol{V}] - k - a\big(i\,\mathrm{Im}(\widetilde{\mathbf{ch}}(\nabla^{V,u}, \nabla^V))\big) \in K\mathbb{C}/\mathbb{Z}^{-1}(N),$$

where on the right-hand side we use the identification of $K\mathbb{C}/\mathbb{Z}^{-1}(N)$ with the flat part of the complex version $\widehat{K}^0_{\mathbb{C}}(N)$ of differential K–theory mentioned in Remark 4.1, $i_k\colon B\mathrm{GL}(k, \mathbb{C}^\delta) \to B\mathrm{GL}(\mathbb{C}^\delta)$ is the canonical map, $\boldsymbol{V} = (V, \nabla^V)$ is the flat complex vector bundle of dimension k classified by g_k, $\nabla^{V,u}$ is some choice of unitarisable connection on V and $\iota\colon K\mathbb{R}/\mathbb{Z}^{-1} \to K\mathbb{C}/\mathbb{Z}^{-1}$ is the canonical map. The difference $[\boldsymbol{V}] - k$ is a flat class in $\widehat{K}^0_{\mathbb{C}}(N)$, but in general it is not real since ∇^V is not unitary. The additional correction term is added in order to obtain a real class, which could also be written in the manifestly real form

$$[\boldsymbol{V}] - k - a\big(i\,\mathrm{Im}(\widetilde{\mathbf{ch}}(\nabla^{V,u}, \nabla^V))\big) = [\boldsymbol{V}_u] - k - a\big(\mathrm{Re}(\widetilde{\mathbf{ch}}(\nabla^{V,u}, \nabla^V))\big), \tag{91}$$

where $\boldsymbol{V}_u := (V, \nabla^{V,u})$. One checks, using the relation in $\widehat{K}^0_{\mathbb{C}}(N)$

$$[V, \nabla] - [V, \nabla'] = a(\widetilde{\mathbf{ch}}(\nabla, \nabla')),$$

that the class (91) does not depend on the choice of $\nabla^{V,u}$.

We further define

$$\Theta_0 := \partial\Lambda_0 \in K^0(B\mathrm{GL}(\mathbb{C}^\delta)^+),$$

where $\partial\colon K\mathbb{R}/\mathbb{Z}^{-1} \to K^0$ is the Bockstein operator.

We will consider the Thom spectrum MB associated to the projection

$$B := B\mathrm{Spin}^c \times B\mathrm{GL}(\mathbb{C}^\delta)^+ \to B\mathrm{Spin}^c.$$

Furthermore we let $q\colon B \to B\mathrm{GL}(\mathbb{C}^\delta)^+$ be the projection. Let $n \in \mathbb{N}$ be odd. Since $\mathbf{ch}(\Theta_0) = 0$, by Lemma 2.5 the second map in the following definition of ε_0 is well-defined:

$$\varepsilon_0\colon \pi_n(MB)_{\mathrm{tors}} \xrightarrow{\eta^{\mathrm{top}}} Q_n(MB) \xrightarrow{\mathrm{ev}_{q^*\Theta_0}} \mathbb{Q}/\mathbb{Z}.$$

We consider a cycle (M, f) for $\pi_n(M\mathrm{Spin}^c)$ and a flat complex vector bundle $\boldsymbol{V}$ on M of dimension k. It is classified by a map $g_k\colon M \to B\mathrm{GL}(k, \mathbb{C}^\delta)$, and we consider the induced map

$$g_0 := p \circ i_k \circ g_k\colon M \to B\mathrm{GL}(\mathbb{C}^\delta)^+.$$

Then $(M, (f, g_0))$ is a cycle for $\pi_n(MB)$.

Theorem 5.3 *We assume that* $[M,(f,g_0)]\in\pi_n(MB)_{\mathrm{tors}}$. *If* V *carries a flat hermitian metric, then*

$$\varepsilon_0\big([M,(f,g_0)]\big)=\rho(\not{D}_M,V). \tag{92}$$

In general we have

$$\varepsilon_0\big([M,(f,g_0)]\big)=\langle [M]_K, g_0^*\Lambda_0\rangle. \tag{93}$$

Proof We first prove the general case (93) by the chain of equalities

$$\begin{aligned}\varepsilon_0([M,(f,g_0)]) &:= \mathtt{ev}_{q^*\Theta_0}\big(\eta^{\mathrm{top}}\big([M,(f,g_0)]\big)\big)\\ &=\big\langle \mathtt{Thom}^{K\mathbb{R}/\mathbb{Z}}(q^*\Lambda_0),\epsilon\big([M,f,g_0]\big)\big\rangle\\ &=\langle g_0^*\Lambda_0,[M]_K\rangle.\end{aligned}$$

In the second equality we use the $\mathbb{R}/\mathbb{Z}$–analogue of Lemma 2.6.

In the unitary case we observe that $g_0^*\Lambda_0=[V]-k$. The equality (92) now follows from (88) and the chain of equalities

$$\varepsilon_0\big([M,(f,g_0)]\big)\overset{(93)}{=}\langle g_0^*\Lambda_0,[M]_K\rangle=\langle [V]-k,[M]_K\rangle\overset{(88)}{=}\rho(\not{D}_M,V).$$

Note that in the present paper we have shown (88) under the assumption that $[M,(f,g_k^u)]$ is a torsion class in $\pi_n\big(M(B\mathrm{Spin}^c_+\times BU(k)^\delta)\big)$, where $g_k^u\colon M\to BU(k)^\delta$ classifies the hermitian flat bundle V. Since this might not be the case in general, we have to appeal to the proof of this formula (88) without such an assumption given in [9, Theorem 5.3]. $\square$

We have a canonical map of spectra

$$\Theta\colon K^{\mathrm{alg}}(\mathbb{C})\to K.$$

Assume again that $n\in\mathbb{N}$ is odd. Since $\mathbf{ch}\circ\Theta=\dim$ is concentrated in degree zero, by Lemma 2.5 the evaluation in the second map of the following definition is well-defined:

$$\varepsilon\colon K_n^{\mathrm{alg}}(\mathbb{C})_{\mathrm{tors}}\xrightarrow{\eta^{\mathrm{top}}}Q_n(K^{\mathrm{alg}}(\mathbb{C}))\xrightarrow{\mathtt{ev}_{q^*\Theta}}\mathbb{Q}/\mathbb{Z}. \tag{94}$$

We know from Suslin [58, Theorem 4.9] that the map ε induces an isomorphism

$$K_n^{\mathrm{alg}}(\mathbb{C})_{\mathrm{tors}}\cong\begin{cases}\mathbb{Q}/\mathbb{Z} & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even},\end{cases} \tag{95}$$

and that the kernel of ε is a uniquely divisible group.

Our goal is to provide a formula for ε using geometric cycles for algebraic K–theory classes of $\mathbb{C}$. Let (M,f) be a cycle for $\pi_n(S)$, where f stands for the constant map

$M \to *$ refined by a stable normal framing. Furthermore, let V be a flat complex vector bundle of dimension k classified by a map $g_k\colon M \to B\mathrm{GL}(k, \mathbb{C}^\delta)$. Using the equivalence (90) we define

$$g := (\{0\} \times \mathtt{id}) \circ p \circ i_k \circ g_k\colon M \to \Omega^\infty K^{\mathrm{alg}}(\mathbb{C}).$$

In this way we get a class

$$[M, (f, g)] \in \pi_n(\Sigma^\infty_+ \Omega^\infty K^{\mathrm{alg}}(\mathbb{C})),$$

where $\Sigma^\infty_+ W$ denotes the suspension spectrum of a space W. Employing the canonical map

$$u\colon \Sigma^\infty_+ \Omega^\infty K^{\mathrm{alg}}(\mathbb{C}) \to K^{\mathrm{alg}}(\mathbb{C}),$$

we can form the class

$$u_*[M, (f, g)] \in K^{\mathrm{alg}}_n(\mathbb{C}).$$

Remark 5.4 Every element $x \in K^{\mathrm{alg}}_n(\mathbb{C})_{\mathrm{tors}}$ can be represented geometrically in this way. More precisely there exists $(M, (f, g))$ as above such that $x = u_*[M, (f, g)]$ and $[M, (f, g)] \in \pi_n(\Sigma^\infty_+ \Omega^\infty K^{\mathrm{alg}}(\mathbb{C}))_{\mathrm{tors}}$. In order to see this, first represent x by a map $\gamma\colon S^n \to \Omega^\infty K^{\mathrm{alg}}(\mathbb{C})$. Using the standard normal framing of S^n, this map represents a torsion class $[S^n, (f, \gamma)] \in \pi_n(\Sigma^\infty_+ \Omega^\infty K^{\mathrm{alg}}(\mathbb{C}))_{\mathrm{tors}}$. Since $\Sigma^\infty_+(p)$ is an equivalence by the universal property of the $+$ construction, there exists a class $z \in \pi_n(\Sigma^\infty_+ B\mathrm{GL}(\mathbb{C}^\delta))_{\mathrm{tors}}$ such that $p_*(z) = [S^n, (f, \gamma)]$. Finally, the class z can be represented in the form $z = [M, (f, i_k \circ g_k)]$ for some $k \in \mathbb{N}$. Using this data we get $u_*[M, (f, g)] = x$.

Theorem 5.5 *We assume that $[M, (f, g)] \in \pi_n(\Sigma^\infty_+ \Omega^\infty K^{\mathrm{alg}}(\mathbb{C}))_{\mathrm{tors}}$. If V carries a flat hermitian metric, then*

$$\varepsilon\big(u_*[M, (f, g)]\big) = \rho(\not{D}_M, V) + k e^{\mathrm{Adams}}_{\mathbb{C}}([M, f]). \tag{96}$$

In general we have

$$\varepsilon\big(u_*[M, (f, g)]\big) = \langle g_0^* \Lambda_0, [M]_K \rangle + k e^{\mathrm{Adams}}_{\mathbb{C}}([M, f]). \tag{97}$$

Proof We again start with the general case (97). We calculate

$$\begin{aligned} \varepsilon\big(u_*[M, (f, g)]\big) &= \mathtt{ev}_\Theta\big(\eta^{\mathrm{top}}\big(u_*[M, (f, g)]\big)\big) \\ &= \mathtt{ev}_{u^*\Theta}\big(\eta^{\mathrm{top}}\big([M, (f, g)]\big)\big) \\ &= \mathtt{ev}_{i_k^* u^* \Theta}\big(\eta^{\mathrm{top}}\big([M, (f, g_0)]\big)\big) \\ &= \mathtt{ev}_{\Theta_0 + k}\big(\eta^{\mathrm{top}}\big([M, (f, g_0)]\big)\big). \end{aligned} \tag{98}$$

We can now use the unstable case (93) and (81) in order to deduce

$$\varepsilon\big(u_*([M,(f,g)])\big) = \langle g_0^*\Lambda_0, [M]_K\rangle + ke_{\mathbb{C}}^{\mathrm{Adams}}([M,f]).$$

The unitary case (96) now follows from (97) and (92). □

Remark 5.6 We now show how one can deduce a special case of [42, Theorem A] from (97). In [42] an algebraic K–theory class is constructed from a homology sphere M of dimension n and a representation $\alpha\colon \pi_1(M)\to \mathrm{GL}(k,\mathbb{C}^\delta)$. One gets an induced map

$$\tilde{g}\colon M \xrightarrow{\alpha} B\mathrm{GL}(k,\mathbb{C}^\delta) \xrightarrow{i_k} B\mathrm{GL}(\mathbb{C}^\delta),$$

to which Quillen's $+$ construction is applied. The fundamental group of a homology sphere is perfect, so the $+$ construction M^+ of M is homotopy equivalent to a simply connected homology sphere, hence to S^n. Thus we get a map

$$g^+\colon S^n \simeq M^+ \xrightarrow{\tilde{g}^+} B\mathrm{GL}(\mathbb{C}^\delta)^+$$

which represents a class $[g^+]\in K_n^{\mathrm{alg}}(\mathbb{C})$. The homology sphere M admits a stable normal framing (see eg [44] or [31, Lemma 1]) which refines the constant map $f\colon M\to *$. We further use the composition $g\colon M\to M^+ \xrightarrow{\tilde{g}^+} B\mathrm{GL}(\mathbb{C}^\delta)^+$ in order to define the class $[M,(f,g)]\in \pi_n(\Sigma^\infty_+\Omega^\infty K^{\mathrm{alg}}(\mathbb{C})_+)$ such that $u_*[M,(f,g)] = [g^+]$.

We consider the map of fibre sequences

$$\begin{tikzcd}
\Sigma^{-1}K \arrow[r] \arrow[d] & \Sigma^{-1}HP\mathbb{C} \arrow[d] \\
K^{\mathrm{rel}}(\mathbb{C}) \arrow[r, dotted] \arrow[d] & \Sigma^{-1}K\mathbb{C}/\mathbb{Z} \arrow[d] \\
K^{\mathrm{alg}}(\mathbb{C}) \arrow[r, "\Theta"] \arrow[d, "\Theta"] & K \arrow[d, "\mathbf{ch}"] \\
K \arrow[r, "\mathbf{ch}"] & HP\mathbb{C}
\end{tikzcd}$$

(the left column defines the relative K–theory spectrum $K^{\mathrm{rel}}(\mathbb{C})$). For odd $n\in\mathbb{N}$ the dotted horizontal arrow induces a map

$$(99)\qquad e\colon K_n^{\mathrm{alg}}(\mathbb{C}) \cong \frac{\pi_n(K^{\mathrm{rel}}(\mathbb{C}))}{\mathtt{im}\big(\pi_n(\Sigma^{-1}K)\to\pi_n(K^{\mathrm{rel}}(\mathbb{C}))\big)} \to \pi_n(\Sigma^{-1}K\mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}.$$

Let V^{flat} be the flat vector bundle determined by the representation α. The statement of [42, Theorem A] is the equality

$$e([g^{+}]) = \rho_{\mathbb{C}}(\not{D}_M, V^{\mathrm{flat}}) \tag{100}$$

in $\mathbb{C}/\mathbb{Z}$. The subscript $\mathbb{C}$ on the right-hand side indicates a complex version of the ρ–invariant defined for flat vector bundles without requiring a flat hermitian metric. We have

$$\mathrm{Re}(\rho_{\mathbb{C}}(\not{D}_M, V^{\mathrm{flat}})) = \langle g^*\Lambda_0, [M]_K\rangle.$$

Note that $(\mathbb{C}/\mathbb{Z})_{\mathrm{tors}} = \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$. If $[g^{+}] \in K_n^{\mathrm{alg}}(\mathbb{C})$ is a torsion class, then using Lemma 2.6 we get the identification

$$e([g^{+}]) = \varepsilon([g^{+}]) - k e_{\mathbb{C}}^{\mathrm{Adams}}([M, f]) \in \mathbb{R}/\mathbb{Z},$$

by an argument which is similar to the calculation (98). Hence if $[g^{+}]$ is a torsion class, then the real part of (100) is equivalent to (97).

Remark 5.7 Let us comment on the fact that Adams' e–invariant appears on the right-hand sides of (96) and (97). Note that $K^{\mathrm{alg}}(\mathbb{R})$ is a ring spectrum with unit $\epsilon_{K^{\mathrm{alg}}(\mathbb{R})}\colon S \to K^{\mathrm{alg}}(\mathbb{R})$. The unit induces a homomorphism $\pi_*(S) \to K_n^{\mathrm{alg}}(\mathbb{R})$. Since the image $\mathtt{im}(J)$ of the J–homomorphism (80) is a well-known summand of $\pi_*(S)$, it was an interesting question to determine its image under the unit $\epsilon_{K^{\mathrm{alg}}(\mathbb{R})}$. Let us consider the case

$$\frac{\mathbb{Z}}{(B_m/4m)\mathbb{Z}} \cong \mathtt{im}(J)_{4m-1} \subseteq \pi_{4m-1}(S)$$

for $m \in \mathbb{N}$. In [53] it has been shown that this piece maps injectively to algebraic K–theory. This was deduced from the following two facts:

(1) $\mathtt{im}(J)_{4m-1}$ is detected by (the real version of) Adams' e–invariant

$$e\colon \pi_{4m-1}(S) \to \mathbb{Q}/\mathbb{Z}.$$

(2) The e–invariant has a factorisation over the analogue

$$K_n^{\mathrm{alg}}(\mathbb{R})(\cdot) \to \pi_n(\Sigma^{-1} KO\mathbb{C}/\mathbb{Z})$$

of the homomorphism (99).

The complex case of this factorisation is easily seen from Theorem 5.5. In fact the elements in $\mathtt{im}(J)_{4m-1}$ can be represented by cycles of the form $u_*[S^{4m-1}, (f, g)]$, where f carries a normal framing obtained from the standard framing by twisting with an element of $\pi_{4m-1}(O)$, and g is obtained from the classifying map of the trivial one-dimensional bundle V. Then we get, from (96) in Theorem 5.5,

$$\varepsilon\big(u_*[S^{4m-1}, (f, g)]\big) = e_{\mathbb{C}}^{\mathrm{Adams}}([S^{4m-1}, f]).$$

5.4 String–bordism

In this subsection we describe the connection of the present paper with constructions in [24]. In this reference we introduced an invariant b^{an} of elements of the bordism group $\pi_{4m-1}(M\mathrm{String})$ using a formula which shares a lot of similarities with the intrinsic formula (78) for η^{an}. We will recall the definition of the complex version $b^{\mathrm{an}}_{\mathbb{C}}$ during the proof of Theorem 5.9.

The spectrum of topological modular forms tmf has been constructed by Miller, Goerss and Hopkins, and in an alternative way by Lurie; see the survey [49]. It is related to K–theory and String–bordism by a factorisation of the Witten genus

$$\begin{tikzcd} M\mathrm{String} \arrow[r, "\sigma_{\mathrm{AHR}}"] \arrow[rr, bend left, dotted, "\sigma_{\mathrm{Witten}}"] \arrow[rrr, bend left=50, dotted, "\sigma^{\mathbb{C}}_{\mathrm{Witten}}"] & \mathrm{tmf} \arrow[r, "W"] \arrow[rr, bend right, dotted, "W_{\mathbb{C}}"'] & KO[[q]] \arrow[r] & K[[q]] \end{tikzcd} \tag{101}$$

where σ_{AHR} is the String–orientation[1] of tmf constructed by Ando, Hopkins and Rezk [6]. One of the interesting features of the restriction of b^{an} to the kernel of the map to Spin–bordism is that it has a factorisation over σ_{AHR}. Since b^{an} is calculable in interesting cases it can be used to detect the tmf–class represented by a closed String–manifold.

Our goal here is Theorem 5.9 which gives the precise relation between b^{an} and the universal η–invariant. In Corollary 5.12 we show one of the conjectures stated in [24], asserting that the factorisation of b^{an} over topological modular forms holds true on the whole String–bordism group, ie we get rid of the restriction to the kernel to the Spin–bordism. Formally our proof is complete in dimensions $8m-1$, while in dimensions $8m-5$ we lose some two-torsion since in the present paper we work with complex K–theory instead of real K–theory. We strongly believe that the relevant part of the theory has a real version which does prove the case in dimension $8m-5$ completely, too.

In Theorem 5.14 we show how the riemannian geometry on a String–manifold together with a geometric string structure give rise to a geometrisation, and we derive the corresponding intrinsic formula for b^{an}. We consider the proof of Theorem 5.14 as a model for many other situations where a construction of a geometrisation is required.

[1] A good alternative name would be tmf–orientation of $M\mathrm{String}$, but this name appears so in [6].

We start with recalling the definition of String–bordism. The homotopy type of the space $B\mathrm{String}$ is defined as a stage in the Whitehead tower of BO:

$$
\begin{tikzcd}
B\mathrm{String} \arrow[d, "p"] & \\
B\mathrm{Spin} \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & K(\mathbb{Z},4) \\
B\mathrm{SO} \arrow[r, "w_2"] \arrow[d] & K(\mathbb{Z}/2\mathbb{Z},2) \\
BO \arrow[r, "w_1"] & K(\mathbb{Z}/2\mathbb{Z},1)
\end{tikzcd}
$$

The space $B\mathrm{String} \simeq BO\langle 8\rangle$ is just a low instance of a whole tower of higher connected coverings $BO\langle n\rangle$ of the classifying space BO. Starting with $B\mathrm{String}$, these higher spaces are no longer associated to classical families of compact Lie groups.

In Proposition 5.13 we demonstrate that a connection on a Spin–principal bundle gives rise to a geometrisation. While the connection on the principal bundle allows us to define connections on all associated vector bundles, the geometrisation partially keeps this information in terms of the differential K–theory classes represented by these vector bundles with connections. The geometrisation associated to a geometric String structure in this sense replaces the theory of connections on the nonexisting principal bundle with structure group String. We think that the methods used in the case of $B\mathrm{String} \simeq BO\langle 8\rangle$ can easily be adapted to the higher stages $BO\langle n\rangle$.

We let $M\mathrm{String}$ be the Thom spectrum associated to the map $B\mathrm{String} \to B\mathrm{Spin} \xrightarrow{\mathrm{can}} B\mathrm{Spin}^c$. The String–bordism spectrum $M\mathrm{String}$ is rationally even (see [40; 41; 39] for more calculations), so that for $m \in \mathbb{N}$ and $n = 4m-1$ the group $\pi_n(M\mathrm{String})$ is torsion. Hence the universal η–invariant gives a map

$$\eta^{\mathrm{top}} = \eta^{\mathrm{an}} \colon \pi_n(M\mathrm{String}) \to Q_n(M\mathrm{String}).$$

We will show that we can obtain b^{an} from the universal η–invariant by defining an interesting homomorphism out of the universal target $Q_n(M\mathrm{String})$. It involves evaluations against a collection of elements $R_k(\lambda^{\mathrm{String}}) \in K^0(B\mathrm{String})$ for all $k \geq 0$. It is useful to organise this collection in a formal power series

$$R(\lambda^{\mathrm{String}}) := \sum_{k\geq 0} q^k R_k(\lambda^{\mathrm{String}}) \in K[[q]]^0(B\mathrm{String}),$$

which we will describe in the following. By $K[\![q]\!]$ we denote the multiplicative cohomology theory (or the corresponding spectrum) which associates to a space Y the ring

$$K[\![q]\!]^*(Y) := K^*(Y)[\![q]\!]$$

of formal power series with coefficients in $K^*(Y)$. The following constructions with real vector bundles are standard in the theory of the Witten genus (106); compare eg with [35; 24]. Given a real vector bundle $V \to Y$ we consider the element $R(V) \in K[\![q]\!]^0(Y)$ defined by

$$R(V) := \sum_{k=0}^{\infty} R_k(V) q^k, \tag{102}$$

where $R_k(V)$ is the K–theory class of the virtual bundle given by the coefficient in front of q^k in the expansion of

$$\prod_{k\geq 1}(1-q^k)^{\dim(V)} \bigotimes_{k\geq 1} \mathtt{Sym}_{q^k}(V \otimes_{\mathbb{R}} \mathbb{C}), \tag{103}$$

where

$$\mathtt{Sym}_p(V) := \bigoplus_{l\geq 0} p^l \,\mathtt{Sym}^l(V).$$

The prefactor $\prod_{k\geq 1}(1-q^k)^{\dim(V)}$ normalises the power series $R(V)$ so that $R(\underline{\mathbb{R}}^n_Y)=1$ for all $n \in \mathbb{N}$.[2] Setting $q = e^{2\pi i z}$, it can be expressed in terms of the Dedekind eta function $\eta(z)$:

$$\prod_{k\geq 1}(1-q^k)^{\dim(V)} = q^{-\dim(V)/24}\eta(z)^{\dim(V)}.$$

The transformation $V \mapsto R(V)$ is exponential, ie for two bundles V and W on Y it satisfies

$$R(V \oplus W) = R(V) \cup R(W).$$

Moreover, it has values in the group of multiplicative units $K[\![q]\!]^0(Y)^\times$, since the power series (102) starts with 1. In view of the universal property of KO^0 it therefore extends to a natural transformation

$$R\colon KO^0(Y) \to K[\![q]\!]^0(Y)^\times.$$

Such exponential transformations are an essential ingredient of the construction of K–theoretic multiplicative genera, of which the following construction of the Witten

[2] The formula (103) corrects a mistake in [24, (18)], which has an erroneous factor 2 in the exponent of the normalising factor. The author thanks the referee for pointing this out.

genus is a special instance. The composition

$$B\mathrm{String} \to BO \xrightarrow{x \mapsto (0,x)} \mathbb{Z} \times BO \simeq \Omega^\infty KO$$

classifies the universal class $\theta^{\mathrm{String}} \in KO^0(B\mathrm{String})$. We fix $n = 4m-1$ and let

$$\lambda_n^{\mathrm{String}} := n - \theta^{\mathrm{String}} \in KO^0(B\mathrm{String}). \tag{104}$$

If (M, f) is a cycle for $\pi_n(M\mathrm{String})$, then we have

$$[TM] + 1 = f^* \lambda_{n+1}^{\mathrm{String}} \tag{105}$$

in $KO^0(M)$. We have well-defined classes $R_k(\lambda_{n+1}^{\mathrm{String}}) \in K^0(B\mathrm{String})$ for all $k \geq 0$ and therefore $R(\lambda_{n+1}^{\mathrm{String}}) \in K[\![q]\!]^0(B\mathrm{String})$. With this notation, the Witten genus

$$\sigma^{\mathbb{C}}_{\mathrm{Witten}} \colon \pi_{n+1}(M\mathrm{String}) \to \pi_{n+1}(K[\![q]\!])$$

satisfies

$$\sigma^{\mathbb{C}}_{\mathrm{Witten}}(x) = \big\langle \mathtt{Thom}^K(R(\lambda_{n+1}^{\mathrm{String}})), \epsilon(x) \big\rangle. \tag{106}$$

We use the superscript $\mathbb{C}$ in order to indicate that we work with the image of the $KO[\![q]\!]$–valued Witten genus in complex K–theory $K[\![q]\!]$.

We organise the evaluations of $Q_n(M\mathrm{String})$ against the family of classes $R_k(\lambda_{n+1}^{\mathrm{String}})$ into a formal power series and define a homomorphism

$$W \colon \mathtt{Hom}(K^0(B\mathrm{String}), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}[\![q]\!] := \prod_{k \geq 0} \mathbb{Q}/\mathbb{Z}\, q^k \tag{107}$$

by

$$W(\phi) := \sum_{k \geq 0} \mathtt{ev}_{R_k(\lambda_{n+1}^{\mathrm{String}})}(\phi) q^k.$$

The homomorphism (107) does not yet factorise over $Q_n(M\mathrm{String})$ since it does not vanish on the subgroup U' appearing in (13). In order to get such a factorisation we must replace the target $\mathbb{Q}/\mathbb{Z}[\![q]\!]$ of W by the quotient by a subgroup which contains $W(U')$. This subgroup will be defined using modular forms.

Modular forms of weight $2m$ are holomorphic sections of the m^{th} power of the canonical bundle of the upper half plane which are invariant under $\mathrm{SL}(2,\mathbb{Z})$, and which satisfy a growth condition. For example, the Dedekind η–function gives rise to a modular form $\eta(z)^{24} dz^{12}$ of weight 24. If $f(z)dz^m$ is such a modular form, then the holomorphic function $z \mapsto f(z)$ is invariant under the transformation $z \mapsto z+1$. If we set $q = e^{2\pi i z}$, then we have a Fourier expansion $f(q) = \sum_{k \geq 0} q^k f_k$ with $f_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. Note that negative q–powers are excluded by the growth condition. By definition, the q–expansion of the modular form is the formal power series $\sum_{k=0}^{\infty} f_k q^k$. For every $m \in \mathbb{Z}$ the space $\mathcal{M}_{2m}$ of modular forms of weight $2m$ is finite-dimensional (and its dimension explicitly known). We refer to [35] for a detailed introduction to modular forms.

We let $\mathcal{M}^R_{2m} \subseteq \mathcal{M}_{2m}$ denote the space of modular forms for $\mathrm{SL}(2,\mathbb{Z})$ of weight $2m$ whose q–expansion have coefficients in the subring $R \subseteq \mathbb{C}$. In particular, we let

$$\mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!] \subseteq \mathbb{Q}[\![q]\!]$$

be the finite-dimensional vector space of q–expansions of rational modular forms $\mathcal{M}^{\mathbb{Q}}_{2m}$ of weight $2m$. Its image in $\mathbb{Q}/\mathbb{Z}[\![q]\!]$ will be denoted by $\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!]}$. We define

$$T_{2m} := \frac{\mathbb{Q}/\mathbb{Z}[\![q]\!]}{\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!]}}. \tag{108}$$

Up to the replacement of $\mathbb{Q}$ by $\mathbb{R}$ this is exactly the group defined in [24, Definition 1.1].

Lemma 5.8 *The composition of (107) with the projection to the quotient (108) induces a well-defined map*

$$\overline{W}\colon Q_{4m-1}(M\mathrm{String}) \to T_{2m}.$$

Proof We must show that under this composition the subgroup U defined in (6) is mapped to $\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!]}$. By (106) we have for $y \in \pi_{n+1}(M\mathrm{String})$ that

$$\langle \mathtt{Thom}^K(R(\lambda^{\mathrm{String}}_{n+1})), \epsilon(y)\rangle = \sigma^{\mathbb{C}}_{\mathrm{Witten}}(y) \in \pi_{n+1}(K[\![q]\!]) \cong \mathbb{Z}[\![q]\!].$$

We now use the fact that the Witten genus has values in $\mathcal{M}^{\mathbb{Z}}_{2m}[\![q]\!] \subset \mathbb{Z}[\![q]\!]$. More generally, for $y \in \pi_{n+1}(M\mathrm{String}\mathbb{Q})$ we get

$$\langle \mathtt{Thom}^K(R(\lambda^{\mathrm{String}}_{n+1})), \epsilon(y)\rangle \in \mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!].$$

This shows that $\overline{W}(U') \subseteq \overline{\mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!]}$. □

In [24, Section 3.3] we have constructed homomorphisms

$$b^{\mathrm{an}}\colon \pi_{4m-1}(M\mathrm{String}) \to T_{2m}, \quad b^{\mathrm{top}}\colon A_{4m-1} \to T_{2m},$$

where

$$A_{4m-1} := \mathtt{ker}(\pi_{4m-1}(M\mathrm{String}) \to \pi_{4m-1}(M\mathrm{Spin})).$$

Since in the present paper we work we complex K–theory as opposed to real K–theory in [24, Section 3.3], we define

$$b^*_{\mathbb{C}} := \begin{cases} b^* & \text{if } m \text{ even},\\ 2b^* & \text{if } m \text{ odd},\end{cases} \qquad \text{for } * \in \{\mathrm{an}, \mathrm{top}, \mathrm{tmf}\}, \tag{109}$$

where the tmf–versions will be introduced in (115) and (114) below.

Theorem 5.9 *We have the equalities*

$$\overline{W}\circ\eta^{\mathrm{top}}|_{A_{4m-1}}=b^{\mathrm{top}}_{\mathbb{C}}\quad\textit{and}\quad \overline{W}\circ\eta^{\mathrm{an}}=b^{\mathrm{an}}_{\mathbb{C}}.$$

Proof We extend the map $M\mathrm{String}\to M\mathrm{Spin}$ to a fibre sequence

$$\Sigma^{-1}M\mathrm{Spin}\to\mathcal{A}\to M\mathrm{String}\to M\mathrm{Spin},$$

which defines the spectrum $\mathcal{A}$. The smash product of the fibre sequence with the fibre sequence (3) yields the following quadratic diagram:

$$\begin{tikzcd}
\Sigma^{-2}M\mathrm{Spin}\mathbb{Q} \arrow[r]\arrow[d] & \Sigma^{-1}\mathcal{A}\mathbb{Q} \arrow[r]\arrow[d] & \Sigma^{-1}M\mathrm{String}\mathbb{Q} \arrow[r]\arrow[d] & {}^{\hat{w}}\Sigma^{-1}M\mathrm{Spin}\mathbb{Q}\arrow[d]\\
\Sigma^{-2}M\mathrm{Spin}\mathbb{Q}/\mathbb{Z} \arrow[r]\arrow[d] & \Sigma^{-1}\mathcal{A}\mathbb{Q}/\mathbb{Z} \arrow[r]\arrow[d] & {}^{\hat{x}}\Sigma^{-1}M\mathrm{String}\mathbb{Q}/\mathbb{Z} \arrow[r]\arrow[d] & {}^{\hat{y}}\Sigma^{-1}M\mathrm{Spin}\mathbb{Q}/\mathbb{Z}\arrow[d]\\
\Sigma^{-1}M\mathrm{Spin} \arrow[r]\arrow[d] & {}^{\hat{z}}\mathcal{A} \arrow[r]\arrow[d] & {}^{x}M\mathrm{String} \arrow[r]\arrow[d] & {}^{0}M\mathrm{Spin}\arrow[d]\\
{}^{-\hat{w}}\Sigma^{-1}M\mathrm{Spin}\mathbb{Q} \arrow[r] & {}^{\tilde{z}}\mathcal{A}\mathbb{Q} \arrow[r] & {}^{0}M\mathrm{String}\mathbb{Q} \arrow[r] & M\mathrm{Spin}\mathbb{Q}
\end{tikzcd}$$

We start with $x\in A_{4m-1}\subseteq\pi_{4m-1}(M\mathrm{String})$. This element goes to zero if it is mapped to the right or down. The class $\overline{W}(\eta^{\mathrm{top}}(x))$ is represented by the power series

$$\sum_{k\geq 0}\langle\mathtt{Thom}^K(R_k(\lambda^{\mathrm{String}}_{4m})),\epsilon(\hat{x})\rangle q^k\in\mathbb{Q}/\mathbb{Z}[\![q]\!]. \tag{110}$$

Note that we can define classes θ^{Spin} and $\lambda^{\mathrm{Spin}}_n:=n-\theta^{\mathrm{Spin}}\in KO^0(B\mathrm{Spin})$ analogously to (104). Then we have equalities of the evaluations

$$\begin{aligned}\langle\mathtt{Thom}^K(R_k(\lambda^{\mathrm{String}}_{4m})),\epsilon(\hat{x})\rangle&=\langle\mathtt{Thom}^K(R_k(\lambda^{\mathrm{Spin}}_{4m})),\epsilon(\hat{y})\rangle\\&=[\langle\mathtt{Thom}^K(R_k(\lambda^{\mathrm{Spin}}_{4m})),\epsilon(\hat{w})\rangle]\in\mathbb{Q}/\mathbb{Z},\end{aligned}\tag{111}$$

where the elements $\hat{y}$ and $\hat{w}$ are images and lifts as indicated in the above diagram, and where we use the compatibility of the K–theory Thom isomorphisms for $M\mathrm{String}$ and $M\mathrm{Spin}$. In the construction of b^{top} in [24, Section 4.1] we go the other way. We first lift x to $\hat{z}$, which maps to $\tilde{z}$, which is then again lifted to $\Sigma^{-1}M\mathrm{Spin}\mathbb{Q}$. It is a general fact of such a diagram chase in a product of fibre sequences that, modulo the obvious ambiguities, this element in the lower left corner is the negative of $\hat{w}$ from the upper right corner. By the definition of b^{top} in [24, Section 4.1] we see that $b^{\mathrm{top}}_{\mathbb{C}}(x)$ is represented by

$$\sum_{k\geq 0}\big[\big\langle \mathtt{Thom}^K(R_k(\lambda_{4m}^{\mathrm{Spin}})),\epsilon(\widehat{w})\big\rangle\big]q^k\in\mathbb{Q}/\mathbb{Z}[\![q]\!].$$

Combining this with (110) and (111) we see that

$$\overline{W}\circ\eta^{\mathrm{top}}|_{A_{4m-1}}=b^{\mathrm{top}}_{\mathbb{C}}.$$

This proves the first assertion of Theorem 5.9.

We now show the second. Let $x=[M,f]\in\pi_n(M\mathrm{String})$ be an l–torsion element represented by the cycle (M,f) and (W,F) be a zero bordism of the union of l copies of (M,f). The Spin^c–structures come from Spin–structures, so the Levi-Civita connections have canonical Spin^c–extensions $\widetilde{\nabla}^{TM}$ and $\widetilde{\nabla}^{TW}$. In view of (105) the class $f^*R(\lambda_{4m}^{\mathrm{String}})\in K[\![q]\!]^0(M)$ can be represented by a formal power series of $\mathbb{Z}/2\mathbb{Z}$–graded bundles $R(TM\oplus 1)$ associated to the tangent bundle. The riemannian metric and the Levi-Civita connection turn TM into a geometric bundle. The construction of $R(TM\oplus 1)$ therefore produces a formal power series of geometric bundles $\boldsymbol{R}(TM\oplus 1)$.

The construction of b^{an} involves the choice of a geometric String–structure α on M. This notion has been introduced in [60]. Since the complete definition of the notion of a geometric string structure is quite complicated and involves concepts not relevant for the present paper we refrain from repeating it here. We must know that geometric string structures are geometric refinements of string structures on real vector bundles with connection and Spin–structures. They behave as flexibly as connections. In particular, geometric string structures can be glued using partitions of unity. Most importantly, a geometric string structure produces a form $H_\alpha\in\Omega^3(M)$ with the property that

$$2dH_\alpha=p_1(\nabla^{TM,\mathrm{LC}}). \tag{112}$$

This form is not arbitrary. It is a form-level reflection of the trivialisation (an additional datum) of $\frac{1}{2}(p_1)$ determined by the string structure.

In the following we use characteristic forms associated to certain power series

$$\widetilde{\Phi},\Phi,\Theta\in\mathbb{Q}[\![q]\!][b,b^{-1}][\![b^2p_1,b^4p_2,\dots]\!].$$

We refer to [24, Section 3.3] or (122) for an explicit definition. In the notation of the latter we have

$$\Phi:=\Phi_{R(\lambda_{4m}^{\mathrm{Spin}})},\quad \widetilde{\Phi}:=\widetilde{\Phi}_{R(\lambda_{4m}^{\mathrm{Spin}})},\quad \Theta:=\Phi-p_1\widetilde{\Phi}.$$

In the present paper we distribute the powers of b so that Φ and Θ have total degree zero and $\widetilde{\Phi}$ has total degree -4. The notation $\widetilde{\Phi}(\widetilde{\nabla}^{TM})$ is as in (123). We start with the representative of $b^{\mathrm{an}}_{\mathbb{C}}(x)$ given in [24, Definition 4.1] by

$$\left[2\int_M H_\alpha \wedge \tilde{\Phi}(\tilde{\nabla}^{TM})\right] - \xi(\not{D}_M \otimes \boldsymbol{R}(TM \oplus 1)) \in \mathbb{R}/\mathbb{Z}[\![q]\!], \tag{113}$$

where here and below we ignore the power b^{2m}. We use the APS index formula (35) in order to express the reduced η–invariant appearing in (113). Using the equality

$$\Phi(\tilde{\nabla}^{TW}) = \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \mathbf{ch}(\nabla^{\boldsymbol{R}(TM\oplus 1)}),$$

we get

$$(113) = \left[2\int_M H_\alpha \wedge \tilde{\Phi}(\tilde{\nabla}^{TM}) - \frac{1}{l}\int_W \Phi(\tilde{\nabla}^{TW})\right] + \left[\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \boldsymbol{R}(TW))_{\mathrm{APS}}\right].$$

We now use Stokes' theorem and the relation (112) in order to calculate

$$\begin{aligned} 2\int_M H_\alpha \wedge \tilde{\Phi}(\tilde{\nabla}^{TM}) - \frac{1}{l}\int_W \Phi(\tilde{\nabla}^{TW}) &= \frac{1}{l}\int_W (p_1(\nabla^{TW}) \wedge \tilde{\Phi}(\tilde{\nabla}^{TW}) - \Phi(\tilde{\nabla}^{TW})) \\ &= \frac{1}{l}\int_W \Theta(\tilde{\nabla}^{TW}) \in \mathcal{M}^{\mathbb{R}}_{2m}[\![q]\!]. \end{aligned}$$

For the last inclusion we use the crucial fact that

$$p_{4m}(\Theta) \in \mathcal{M}^{\mathbb{Q}}_{2m}[\![q]\!][b^2p_1, b^4p_2, \dots] \subset \mathbb{Q}[\![q]\!][\![b^2p_1, b^4p_2, \dots]\!];$$

see [24, Section 3.3], where p_{4m} on the left is not a Pontrjagin class but the projection to the respective factor in (8). Therefore

$$\left[\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \boldsymbol{R}(TW))_{\mathrm{APS}}\right] \in \mathbb{R}/\mathbb{Z}[\![q]\!]$$

is a representative of $b^{\mathrm{an}}_{\mathbb{C}}(x) \in T_{2m}$, too. But in view of Definition 3.6 and the construction of $\overline{W}$ this is also a representative of $\overline{W}(\eta^{\mathrm{an}}(x)) \in T_{2m}$. This shows

$$\overline{W} \circ \eta^{\mathrm{an}} = b^{\mathrm{an}}_{\mathbb{C}}. \qquad \square$$

As a consequence of the equality $\eta^{\mathrm{an}} = \eta^{\mathrm{top}}$ shown in Theorem 3.7 we get another proof of [24, Theorem 2.2].

Corollary 5.10 $$b^{\mathrm{an}}_{\mathbb{C}}|_{A_{4m-1}} = b^{\mathrm{top}}_{\mathbb{C}}.$$

We now recall from [24, Section 4.3] the construction of the homomorphism

$$b^{\mathrm{tmf}}\colon \pi_{4m-1}(\mathrm{tmf}) \to T_{2m},$$

which is very similar to that of η^{top}. Note that $\pi_{4m-1}(\mathrm{tmf})$ is a torsion group (see [37; 14] for more calculations of $\pi_*(\mathrm{tmf})$). Therefore an element $y \in \pi_{4m-1}(\mathrm{tmf})$ can be lifted to an element $\hat{y} \in \pi_{4m}(\mathrm{tmf}\mathbb{Q}/\mathbb{Z})$. Then

$$b^{\mathrm{tmf}}(y) := [W(\hat{y})] \in T_{2m}, \tag{114}$$

where $[W(\hat{y})]$ denotes the class in T_{2m} of the element $W(\hat{y}) \in \pi_{4m}(KO[\![q]\!]\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[\![q]\!]$ and W is as in (101). The complex version $b^{\mathrm{tmf}}_{\mathbb{C}}$ of b^{tmf} is defined similarly by

$$b^{\mathrm{tmf}}_{\mathbb{C}}(y) := [W_{\mathbb{C}}(\hat{y})] \in T_{2m}, \tag{115}$$

or alternatively by (109).

Proposition 5.11 *We have the equality*

$$b^{\mathrm{tmf}}_{\mathbb{C}} \circ \sigma_{\mathrm{AHR}} = \overline{W} \circ \eta^{\mathrm{top}}\colon \pi_{4m-1}(M\mathrm{String}) \to T_{2m}.$$

Proof If $x \in \pi_{4m-1}(M\mathrm{String})$ and $\hat{x} \in \pi_{4m}(M\mathrm{String}\mathbb{Q}/\mathbb{Z})$ is a lift, then

$$\big\langle \mathtt{Thom}^K(R(\lambda^{\mathrm{String}}_{4m})), \epsilon(\hat{x})\big\rangle \in \mathbb{Q}/\mathbb{Z}[\![q]\!]$$

represents $\overline{W} \circ \eta^{\mathrm{top}}(x)$. We have already seen in the proof of Lemma 5.8 that this expression is equal to the Witten genus (extended to $\mathbb{Q}/\mathbb{Z}$–theory)

$$\big\langle \mathtt{Thom}^K(R(\lambda^{\mathrm{String}}_{4m})), \epsilon(\hat{x})\big\rangle = \sigma^{\mathbb{C}}_{\mathrm{Witten}}(\hat{x}).$$

The Witten genus (see (101)) can now be decomposed as

$$\sigma^{\mathbb{C}}_{\mathrm{Witten}}(\hat{x}) = W_{\mathbb{C}}(\sigma_{\mathrm{AHR}}(\hat{x})).$$

We can take $\sigma_{\mathrm{AHR}}(\hat{x}) \in \pi_{4m}(\mathrm{tmf}\mathbb{Q}/\mathbb{Z})$ as the lift of $\sigma_{\mathrm{AHR}}(x) \in \pi_{4m}(\mathrm{tmf})$ such that $W_{\mathbb{C}}(\sigma_{\mathrm{AHR}}(\hat{x}))$ represents $b^{\mathrm{tmf}}_{\mathbb{C}}(\sigma_{\mathrm{AHR}}(x))$. Hence we can conclude that

$$b^{\mathrm{tmf}}_{\mathbb{C}} \circ \sigma_{\mathrm{AHR}}(x) = \overline{W} \circ \eta^{\mathrm{top}}(x). \qquad \square$$

Using Theorem 3.7, stating the equality $\eta^{\mathrm{an}} = \eta^{\mathrm{top}}$, and $\overline{W} \circ \eta^{\mathrm{an}} = b^{\mathrm{an}}_{\mathbb{C}}$, given by Theorem 5.9, we get:

Corollary 5.12 $$b^{\mathrm{an}}_{\mathbb{C}} = b^{\mathrm{tmf}}_{\mathbb{C}} \circ \sigma_{\mathrm{AHR}}.$$

This proves the complex version of Conjecture 3 in [24, Section 1.5]. In fact, for even m there is no difference between the real and complex case, but in the case of odd m the complex version implies the real version up to two-torsion, which was known before. We believe that a real version of the present theory would prove the conjecture completely.

The formula for b^{an} given in [24, Section 3.3] and reproduced here as (113) is an intrinsic formula which uses the notion of a geometric String–structure [60]. In the following we show that a geometric String–structure gives rise to a good geometrisation $\mathcal{G}^{\mathrm{String}}$ of $(M, f, \tilde{\nabla}^{TM})$ such that the intrinsic formula Theorem 4.19 specialises to the one for b^{an}. Since String–structures refine Spin–structures we start with the construction of a geometrisation for a Spin–structure.

We consider a cycle (M, f) for $\pi_n(M\mathrm{Spin})$. We are going to use a version of Section 3.3 for Spin–structures. If $V \to M$ is a real euclidean oriented vector bundle, then the Spin–gerbe $\underline{\mathrm{Spin}(V)}$ of V associates to each open subset $A \subseteq M$ the groupoid $\mathrm{Spin}(V|_A)$ of Spin–structures on the restriction of V to A. This gerbe has the band $\mathbb{Z}/2\mathbb{Z}$, and its isomorphism class is classified by the Dixmier–Douady class

$$\mathtt{DD}(\underline{\mathrm{Spin}(V)}) = w_2(V) \in H^2(M; \mathbb{Z}/2\mathbb{Z}),$$

the second Stiefel–Whitney class of V.

We choose a tangential Spin–structure on TM given by a Spin–structure $P \in \mathrm{Spin}(TM)$ together with a trivialisation

$$P \otimes \tilde{f}^* Q_k^{\mathrm{Spin}} \cong Q(n+k), \tag{116}$$

where $\tilde{f}\colon M \to B\mathrm{Spin}(k)$ is some factorisation of f and $Q_k^{\mathrm{Spin}} \to B\mathrm{Spin}(k)$ is the universal Spin–bundle. It naturally induces a tangential Spin^c–structure by extension of structure groups along $\mathrm{Spin}(l) \to \mathrm{Spin}^c(l)$ (see [48, Example D.5]).

The Levi-Civita connection gives rise to a connection ∇^{TM} on P, which in turn has a natural Spin^c–extension $\tilde{\nabla}^{TM}$. We furthermore choose a connection $\nabla^k := \nabla^{\tilde{f}^* Q_k^{\mathrm{Spin}}}$ on $\tilde{f}^* Q_k^{\mathrm{Spin}}$ and let $\tilde{\nabla}^k$ be its Spin^c–extension.

Proposition 5.13 *There exists a geometrisation $\mathcal{G}^{\mathrm{Spin}}$ of $(M, f, \tilde{\nabla}^{TM})$ which is l–good for every $l \in \mathbb{N}$.*

Proof The connections $\tilde{\nabla}^{TM}$ and $\tilde{\nabla}^k$ together induce a Spin^c–extension $\tilde{\nabla}^{TM} \otimes \tilde{\nabla}^k$ of the connection $\nabla^{TM} \otimes \nabla^k$ on $P \otimes \tilde{f}^* Q_k^{\mathrm{Spin}}$, which can be compared with a trivial connection using the isomorphism (116). Therefore, the transgression form $\widetilde{\mathbf{Td}}(\tilde{\nabla}^{TM} \otimes \tilde{\nabla}^k, \nabla^{\mathrm{triv}}) \in \Omega P^{-1}(M)/\mathtt{im}(d)$ is defined and satisfies

$$d\,\widetilde{\mathbf{Td}}(\tilde{\nabla}^{TM} \otimes \tilde{\nabla}^k, \nabla^{\mathrm{triv}}) = \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \mathbf{Td}(\tilde{\nabla}^k) - 1.$$

We define the form of total degree -1

$$\mu := \mathbf{Td}(\tilde{\nabla}^k)^{-1} \wedge \widetilde{\mathbf{Td}}(\tilde{\nabla}^{TM} \otimes \tilde{\nabla}^k, \nabla^{\mathrm{triv}}). \tag{117}$$

For any space or spectrum Y we let $\bar{K}^*(Y)$ denote the completion of the topological group $K^*(Y)$ equipped with the profinite topology (see [19, Definition 4.9] or Remark 2.1). Recall that the process of completing may add additional limit elements and then takes the quotient by the subgroup of elements which cannot be separated from zero. We have an equivalence $B\mathrm{Spin} \simeq \mathtt{hocolim}_l B\mathrm{Spin}(l)$ and therefore $\bar{K}^*(B\mathrm{Spin}) \cong \mathtt{lim}_l \bar{K}^*(B\mathrm{Spin}(l))$. The completion theorem [10] gives

$$K^*(B\mathrm{Spin}(k)) = \bar{K}^*(B\mathrm{Spin}(k)) \cong R(\mathrm{Spin}(k))^{\wedge}_{I_{\mathrm{Spin}(k)}}.$$

We therefore get the following description of the completion of the K–theory of $B\mathrm{Spin}$:

$$\bar{K}^*(B\mathrm{Spin}) \cong \mathtt{lim}_l K^*(B\mathrm{Spin}(k+l)) \cong \mathtt{lim}_l R(\mathrm{Spin}(k+l))^{\wedge}_{I_{\mathrm{Spin}(k+l)}}.$$

We fix an integer $l \geq 0$ and form the l–fold stabilisation $\tilde{f}^* Q_k^{\mathrm{Spin}} \otimes Q(l)$ of $\tilde{f}^* Q_k^{\mathrm{Spin}}$. This stabilisation is a $\mathrm{Spin}(k+l)$–principal bundle with the connection $\nabla^k \otimes \nabla^{Q(l)}$. Given a representation ρ of $\mathrm{Spin}(k+l)$ we define a geometric bundle $\boldsymbol{V}_\rho$ as an associated bundle to $\tilde{f}^* Q_k^{\mathrm{Spin}} \otimes Q(l)$. We define

$$G_l(\rho) := [\boldsymbol{V}_\rho] - a(\mu \wedge \mathbf{ch}(\nabla^{V_\rho})) \in \hat{K}^0(M).$$

We have chosen the form μ in (117) so that the following equality holds true in $\Omega P^0_{\mathrm{cl}}(M)$:

$$\mathbf{Td}(\tilde{\nabla}^{TM}) \wedge R(G(\rho)) = \mathbf{Td}(\tilde{\nabla}^k)^{-1} \wedge \mathbf{ch}(\nabla^{V_\rho}). \tag{118}$$

The map $\rho \mapsto G_l(\rho)$ extends to a map $G_l\colon R(\mathrm{Spin}(k+l)) \to \hat{K}^0(M)$ by linearity. This extension annihilates the $2n+1^{\mathrm{st}}$ power $I^{2n+1}_{\mathrm{Spin}(k+l)} \subseteq R(\mathrm{Spin}(k+l))$ of the dimension ideal. In order to see this note that if $\rho \in I^p_{\mathrm{Spin}(k+l)}$ and $2p > n$, then we have $\mathbf{ch}(\nabla^{V_\rho}) = 0$. For those ρ we have $G_l(\rho) = [\boldsymbol{V}_\rho]$, and this class is flat. If $p > n$, then we have $[V_\rho] = 0$, so $G_l(\rho) = a(\omega)$ for some $\omega \in HP\mathbb{R}^{-1}(M)$. The product of a flat class with a class of this form vanishes by (55). Hence $G_l(\rho) = 0$ if $p > 2n$. The map G_l thus further extends by continuity to a map

$$\mathcal{G}^{\mathrm{Spin}(k+l)}\colon K^0(B\mathrm{Spin}(k+l)) \to \hat{K}^0(M).$$

One now checks that for $l \geq 1$ we have

$$G_l(\rho) = G_{l-1}(\rho|_{\mathrm{Spin}(k+l-1)}).$$

In this way the maps $\mathcal{G}^{\mathrm{Spin}(k+l)}$ for the various l are compatible. We consider the continuous map

$$\mathcal{G}^{\mathrm{Spin}}\colon \bar{K}^0(B\mathrm{Spin}) \to K^0(B\mathrm{Spin}(k)) \xrightarrow{\mathcal{G}^{\mathrm{Spin}(k)}} \hat{K}^0(M).$$

We now show that $\mathcal{G}^{\mathrm{Spin}}$ and $\mathcal{G}^{\mathrm{Spin}(k+l)}$ are geometrisations. To this end we must show that they admit degree-preserving cohomological characters. By their compatibility it suffices to consider $\mathcal{G}^{\mathrm{Spin}(k)}$. It follows from (118) and $\mathbf{Td}(\tilde{\nabla}^{TM}) = \hat{\boldsymbol{A}}(\nabla^{TM})$ (since $\tilde{\nabla}^{TM}$ is induced from a Spin–connection ∇^{TM}) that $\mathbf{Td}(\tilde{\nabla}^{TM}) \wedge R(G_0(\rho))$ is the Chern–Weil representative of the class $\hat{\boldsymbol{A}}^{-1} \cup \mathbf{ch}([\rho]) \in HP\mathbb{Q}^0(B\mathrm{Spin}(k))$ associated to the connection ∇^k, where $[\rho] \in K^0(B\mathrm{Spin}(k))$ is the class represented by ρ. Note that

$$H^*(B\mathrm{Spin}(k);\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots, p_{r_k}] \tag{119}$$

is the polynomial ring generated by the universal Pontrjagin classes. For the cohomological character

$$c_{\mathcal{G}^{\mathrm{Spin}(k)}}\colon HP\mathbb{Q}^0(B\mathrm{Spin}(k)) \to \Omega P^0_{\mathrm{cl}}(M)$$

of $\mathcal{G}^{\mathrm{Spin}(k)}$ we choose the map which sends the class $b^{2i}p_i \in HP\mathbb{Q}^0(B\mathrm{Spin}(k))$ to the corresponding characteristic form $b^{2i}p_i(\nabla^k) \in \Omega P^0_{\mathrm{cl}}(M)$. This map clearly preserves degrees.

We now show that $\mathcal{G}^{\mathrm{Spin}}$ is l–good for every $l \in \mathbb{N}$. By an inspection of the construction of $\mathcal{G}^{\mathrm{Spin}}$ we observe that the connection of the map $f\colon M \to B$ with the stable normal bundle of M has not been used. This map can be arbitrary if we replace TM by some complement $\eta \to M$ of $\hat{f}^*\xi_k$ as in Section 4.3 and choose some connection ∇^η of the associated complementary Spin–principal bundle $P^{\mathrm{Spin}} \in \mathrm{Spin}(\eta)$ in place of ∇^{TM}. We obtain the Spin^c–bundle P with connection $\tilde{\nabla}^\eta$ which replaces $\tilde{\nabla}^{TM}$ by extension of the structure group.

Let $l \in \mathbb{N}$ be such that $l \geq n+1$. We choose an l–connected approximation $f_u\colon M_u \to B\mathrm{Spin}$ such that there is a factorisation of f over a closed embedding $h\colon M \to M_u$. As in Section 4.4 we obtain a natural refinement of h to a Spin^c–map. Since h is a closed embedding we can choose the connections $\tilde{\nabla}^u$ on P_u and $\nabla^{k,u}$ on $\tilde{f}_u^* Q_k^{\mathrm{Spin}}$ so that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably and $h^*\nabla^{k,u} = \nabla^k$. We now define $\mathcal{G}_u^{\mathrm{Spin}}$ as above. Then, by construction, $\mathcal{G}^{\mathrm{Spin}} = h^*\mathcal{G}_u^{\mathrm{Spin}}$ since the correction forms (68) vanishes. Since l can be taken arbitrarily large we see that $\mathcal{G}^{\mathrm{Spin}}$ is l–good for every $l \in \mathbb{N}$. □

Note that the geometrisations $\mathcal{G}^{\mathrm{Spin}(k+l)}$ constructed in the proof of Proposition 5.13 are l–good for every $l \in \mathbb{N}$, too.

Let $p\colon B\mathrm{String} \to B\mathrm{Spin}$ be the natural map. We consider a cycle (M, f) for $\pi_n(M\mathrm{String})$. Then $(M, p \circ f)$ naturally becomes a cycle for $\pi_n(M\mathrm{Spin})$.

Theorem 5.14 *A choice of a geometric string structure α on $(\tilde{f}^* Q_k^{\mathrm{Spin}}, \nabla^k)$ naturally determines a geometrisation $\mathcal{G}^{\mathrm{String}}$ of $(M, f, \tilde{\nabla}^{TM})$ which is l–good for every $l \in \mathbb{N}$. For $\phi \in \bar{K}^0(B\mathrm{Spin})$ it is given by*

$$\mathcal{G}^{\mathrm{String}}(p^*\phi) = \mathcal{G}^{\mathrm{Spin}}(\phi) - a(\nu_\phi)$$

with

$$\nu_\phi := \mathbf{Td}(\tilde{\nabla}^{TM})^{-1} \wedge 2H_\alpha \wedge \tilde{\Phi}_\phi(\nabla^m) \in \Omega P^{-1}(M).$$

Here $\mathcal{G}^{\mathrm{Spin}}$ is as in Proposition 5.13, $\tilde{\Phi}_\phi(\nabla^m)$ is defined in (123) and $H_\alpha \in \Omega^3(M)$ is the three-form given by the geometric string structure.

Proof We fix an integer $l \geq n+1$. For a sufficiently large integer $m \in \mathbb{N}$ with $m \geq 3$ we can assume that the map f has a factorisation

$$M \xrightarrow{f_m} B\mathrm{String}(m) \xrightarrow{\iota_m} B\mathrm{String}$$

such that ι_m is l–connected. We have a fibre sequence

$$K(\mathbb{Z},3) \to B\mathrm{String}(m) \xrightarrow{p} B\mathrm{Spin}(m) \to K(\mathbb{Z},4).$$

By [5] the reduced K–theory group $\tilde{K}^*(K(\mathbb{Z},3))$ is uniquely divisible and consists of phantom classes, ie classes which vanish when pulled back to finite CW–complexes. This suggests the following proposition, which is probably well known, but we could not find a reference for it.

Proposition 5.15 *For $m \geq 3$ the pull-back*

$$p^*\colon \bar{K}^*(B\mathrm{Spin}(m)) \to \bar{K}^*(B\mathrm{String}(m)) \tag{120}$$

along the projection $p\colon B\mathrm{String}(m) \to B\mathrm{Spin}(m)$ represents $\bar{K}^*(B\mathrm{String}(m))$ as a completion of $\bar{K}^*(B\mathrm{Spin}(m))$ with respect to the topology induced on $\bar{K}^*(B\mathrm{Spin}(m))$ via p^* by the profinite topology of $\bar{K}^*(B\mathrm{String}(m))$.

The map (120) is continuous but is not expected to be a homeomorphism. The topology induced via this map on $\bar{K}^*(B\mathrm{Spin}(m))$ is expected to be smaller than the original locally finite topology, so that this group is no longer complete in this induced topology.

We defer the proof of Proposition 5.15 to Section 6. and continue with the construction of a geometrisation $\mathcal{G}^{\mathrm{String}}$ which is l–good. We choose an $l+1$–connected approximation $f_u\colon M_u \to B\mathrm{String}(m)$, so we can factorise $f_m\colon M \to B\mathrm{String}(m)$ over the closed embedding $h\colon M \to M_u$. As in Section 4.4 we obtain a natural refinement of h to a Spin^c–map. Since h is a closed embedding we can choose the connections $\tilde{\nabla}^u$ on P_u and $\nabla^{m,u}$ on $f_u^* Q_m^{\mathrm{Spin}}$ such that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably and $h^*\nabla^{m,u} = \nabla^m$.

Geometric string structures behave as flexibly as connections and metrics [60]. We can therefore assume that there is a geometric string structure α_u on $(f_u^* Q_m^{\mathrm{Spin}}, \nabla^{m,u})$ which restricts to the geometric string structure α on M.

In the proof of Proposition 5.13 we have constructed a geometrisation

$$\mathcal{G}_u^{\mathrm{Spin}(m)}\colon \bar{K}^0(B\mathrm{Spin}(m)) \to \hat{K}^0(M_u).$$

of $(M_u, p \circ f_u, \nabla^{m,u})$. As a first approximation we define

$$\mathcal{G}_{u,0}^{\mathrm{String}(m)}\colon \bar{K}^0(B\mathrm{Spin}(m)) \to \hat{K}^0(M_u)$$

by

$$\mathcal{G}_{u,0}^{\mathrm{String}(m)}(\phi) := \mathcal{G}_u^{\mathrm{Spin}(m)}(\phi) \quad \text{for } \phi \in \bar{K}^0(B\mathrm{Spin}(m)).$$

In view of Proposition 5.15 the homomorphism $\mathcal{G}_{u,0}^{\mathrm{String}(m)}$ is defined on a group which defines $\bar{K}^0(B\mathrm{String}(m))$ by completion with respect to a certain topology. If $\mathcal{G}_{u,0}^{\mathrm{String}(m)}$ were continuous with respect to this topology it would descend to a continuous homomorphism defined on $\bar{K}^0(B\mathrm{String}(m))$ and would admit a degree-preserving cohomological character.

Note that via p^* we can identify

$$H\mathbb{Q}^*(B\mathrm{String}(m);\mathbb{Q}) \cong \mathbb{Q}[\![p_2, \dots, p_{r_m}]\!]$$

with the quotient ring of $H^*(B\mathrm{Spin}(m);\mathbb{Q})$ given in (119) by setting $p_1 = 0$. The contribution of $p_1(\tilde{\nabla}^u)$ to the curvature of $\mathcal{G}_{u,0}^{\mathrm{String}(m)}$ obstructs the existence of a degree-preserving cohomological character; see Remark 5.17 below. The idea is now to kill this contribution by a correction term given by a geometric string structure α_u on $(\tilde{f}_u^* Q_m^{\mathrm{Spin}}, \nabla^{m,u})$. The geometric string structure provides the form $H_{\alpha_u} \in \Omega^3(M_u)$ with the property that $2dH_{\alpha_u} = p_1(\nabla^{m,u})$; see (112).

For a formal power series

$$\Lambda \in \mathbb{Q}[b, b^{-1}][\![b^2 p_1, b^4 p_2, \dots]\!]$$

we define a new formal power series

$$\tilde{\Lambda} := \frac{\Lambda - i_{p_1=0}\Lambda}{p_1} \in \mathbb{Q}[b, b^{-1}][\![b^2 p_1, b^4 p_2, \dots]\!]. \tag{121}$$

In other words, the power series $\tilde{\Lambda}$ is p_1^{-1} times the sum of those monomials of Λ which contain p_1. Since the periodic rational cohomology of any space Y is complete, ie we have $HP\mathbb{Q}^*(Y) \cong \overline{HP\mathbb{Q}}^*(Y)$, the Chern character factorises over the completion of K–theory as $\mathbf{ch}\colon \bar{K}^0(B\mathrm{Spin}(m)) \to HP\mathbb{Q}^0(B\mathrm{Spin}(m))$. Let $\phi \in \bar{K}^0(B\mathrm{Spin}(m))$. Then we define

$$\Phi_\phi := \mathbf{Td}^{-1} \cup \mathbf{ch}(\phi) \in HP\mathbb{Q}^0(B\mathrm{Spin}(m)) \cong \mathbb{Q}[\![b^2p_1, b^4p_2, \dots, b^{2r_m}p_{r_m}]\!] \tag{122}$$

and obtain $\tilde{\Phi}_\phi$ as described above. We define the form

$$\nu_{u,\phi} := \mathbf{Td}(\tilde{\nabla}^u)^{-1} \wedge 2H_{\alpha_u} \wedge \tilde{\Phi}_\phi(\nabla^{m,u}) \in \Omega P^{-1}(M_u),$$

where we use the abbreviation

$$\tilde{\Phi}_\phi(\nabla^{m,u}) := \tilde{\Phi}_\phi(p_1(\nabla^{m,u}), p_2(\nabla^{m,u}), \dots). \tag{123}$$

We can now define the map

$$\mathcal{G}_u^{\mathrm{String}(m)}\colon \bar{K}^0(B\mathrm{Spin}(m)) \to \hat{K}^0(M_u)$$

by the prescription

$$\mathcal{G}_u^{\mathrm{String}(m)}(\phi) := \mathcal{G}_{u,0}^{\mathrm{String}(m)}(\phi) - a(\nu_{u,\phi}).$$

We further define

$$\mathcal{G}^{\mathrm{String}(m)} := h^*\mathcal{G}_u^{\mathrm{String}(m)}.$$

Unfortunately we cannot verify directly that $\mathcal{G}_u^{\mathrm{String}(m)}$ is continuous, but we have the following lemma:

Lemma 5.16 *The map*

$$\mathcal{G}^{\mathrm{String}(m)}\colon \bar{K}^0(B\mathrm{Spin}(m)) \to \hat{K}^0(M)$$

extends by continuity to $\bar{K}^0(B\mathrm{String}(m))$. The continuous extension (for which we use the same symbol $\mathcal{G}^{\mathrm{String}(m)}$) *admits the degree-preserving cohomological character given by $b^{2i}p_i \mapsto b^{2i}p_i(\nabla^m)$ for all $i \in \{2, \dots, r_m\}$ and is therefore a geometrisation of $(M, f, \tilde{\nabla}^{TM})$.*

Proof $B\mathrm{String}(2m)$ has finite skeleta, so the profinite topology on $\bar{K}^*(B\mathrm{String}(2m))$ has a countable basis of neighbourhoods of zero, so we can check continuity of homomorphisms using sequences converging to zero. Let us consider a sequence (ϕ_k) in $\bar{K}^0(B\mathrm{Spin}(m))$ such that $p^*\phi_k \to 0$ in the profinite topology of $\bar{K}^0(B\mathrm{String}(m))$ as $k \to \infty$. We must show that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\mathcal{G}^{\mathrm{String}(m)}(\phi_k) = 0$. Let $t\colon N \to B\mathrm{String}(m)$ be a compact $\dim(M_u)+1$–connected approximation. We can choose $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $t^*p^*\phi_k = 0$ and $t^*p^*(\mathbf{Td}^{-1} \wedge \mathbf{ch}(\phi_k)) = 0$. Since the pull-back $t^*\colon H^*(B\mathrm{String}(m); \mathbb{Q}) \to H^*(N; \mathbb{Q})$ is injective in degrees $\leq \dim(M_u)$ we see that Φ_{ϕ_k} is a formal power series of terms of homogeneity $\geq \dim(M_u) + 1$ (here we count the topological degree, ie disregard the degree of b). Consequently, $i_{p_1=0}\Phi_{\phi_k}$ is a formal power series of terms of homogeneity $\geq \dim(M_u)+1$.

We now calculate, using (118) and (121), that for $\phi \in \bar{K}^0(B\mathrm{Spin}(m))$

$$\begin{aligned}(124)\quad \mathbf{Td}(\tilde{\nabla}^u)\wedge R(\mathcal{G}_u^{\mathrm{String}(m)}(\phi)) &\\ &= \mathbf{Td}(\tilde{\nabla}^u)\wedge R(\mathcal{G}_u^{\mathrm{Spin}(m)}(\phi)) - p_1(\tilde{\nabla}^{m,u})\wedge\tilde{\Phi}_\phi(\nabla^{m,u})\\ &= \Phi_\phi(\nabla^{m,u}) - p_1(\tilde{\nabla}^{m,u})\wedge\tilde{\Phi}_\phi(\nabla^{m,u})\\ &= (i_{p_1=0}\Phi_\phi)(\nabla^{m,u}).\end{aligned}$$

We conclude that $\mathbf{Td}(\tilde{\nabla}^u)\wedge R(\mathcal{G}_u^{\mathrm{String}(m)}(\phi_k))=0$ for $k\geq k_0$ since $(i_{p_1=0}\Phi_\phi)(\nabla^{m,u})$ would be a form on M_u of a degree which exceeds the dimension. Hence for $k\geq k_0$ the class $\mathcal{G}_u^{\mathrm{String}(m)}(\phi_k)$ is flat and in the kernel of $I\colon \hat{K}^0(M_u)\to K^0(M_u)$. We conclude that

$$\mathcal{G}_u^{\mathrm{String}(m)}(\phi_k)\in HP\mathbb{R}^{-1}(M_u)/\mathtt{im}(\mathbf{ch}).$$

An $l+1$–connected map induces an isomorphism in ordinary cohomology in degrees at most l. Since $B\mathrm{String}(m)$ is rationally even, the odd-dimensional real cohomology of the $l+1$–connected approximation M_u is concentrated in degrees $\geq l+1$. Since $\dim(M)\leq l$ the restriction $h^*\colon HP\mathbb{R}^{-1}(M_u)\to HP\mathbb{R}^{-1}(M)$ is trivial. This implies that $\mathcal{G}^{\mathrm{String}(m)}(\phi_k)=h^*\mathcal{G}_u^{\mathrm{String}(m)}(\phi_k)=0$ for all $k\geq k_0$.

The assertion about the cohomological character follows from the relation

$$\mathbf{Td}(\tilde{\nabla}^{TM})\wedge R(\mathcal{G}^{\mathrm{String}(m)}(\phi)) = (i_{p_1=0}\Phi_\phi)(\nabla^m)$$

derived from (124). This finishes the proof of Lemma 5.16. □

Remark 5.17 We can now explain better why the contribution of the first Pontrjagin form needs to be killed. We have

$$\mathbf{Td}(\tilde{\nabla}^u)\wedge R(\mathcal{G}_{u,0}^{\mathrm{String}(m)}(\phi)) = \Phi_\phi(\nabla^{m,u}).$$

The polynomial Φ_ϕ contains monomials which contain p_1 and therefore does not belong to

$$HP\mathbb{Q}^0(B\mathrm{String}(m))\cong\mathbb{Q}[[b^4p_2,\dots,b^{2r_m}p_{r_m}]].$$

Since a general connection $\nabla^{m,u}$ has a nonvanishing first Pontrjagin form, the homomorphism

$$K^0(B\mathrm{Spin}(m))\to\Omega P^0_{\mathrm{cl}}(M_u),\quad \phi\mapsto\Phi_\phi(\nabla^{m,u}),$$

has no factorisation over

$$K^0(B\mathrm{Spin}(m))\to K^0(B\mathrm{String}(m))\to HP\mathbb{Q}^0(B\mathrm{String}(m)).$$

If we replace Φ_ϕ by $i_{p_1=0}\Phi_\phi$, then $i_{p_1=0}\Phi_\phi \in \mathbb{Q}[\![b^4 p_2, \dots, b^{2r_m} p_{r_m}]\!]$ and the desired factorisation exists.

To show that this geometrisation $\mathcal{G}^{\mathrm{String}(m)}$ is l–good we must show that $\mathcal{G}_u^{\mathrm{String}(m)}$ is continuous itself. To this end we argue similarly by representing this geometrisation as a pull-back from a $\dim(M_u)+1$–connected approximation of $B\mathrm{String}(m)$.

We now define the geometrisation $\mathcal{G}^{\mathrm{String}}$ of $[M, f, \widetilde{\nabla}^{TM}]$ by

$$\mathcal{G}^{\mathrm{String}} := \iota_{m,*}\mathcal{G}^{\mathrm{String}(m)}.$$

Since ι_m is l–connected, this geometrisation is l–good. If $m' \in \mathbb{N}$ satisfies $m \le m'$, then it follows from the compatibility of the family of geometrisations $(\mathcal{G}_u^{\mathrm{Spin}(m')})_{m' \ge m}$ that

$$\mathcal{G}^{\mathrm{String}(m')}(\phi) = \mathcal{G}^{\mathrm{String}(m)}(\phi|_{B\mathrm{String}(m)}), \quad \phi \in \overline{K}^0(B\mathrm{String}(m')).$$

Therefore $\mathcal{G}^{\mathrm{String}}$ does not depend on the choice of m. In particular, it is l–good for all l. This finishes the proof of Theorem 5.14. □

We now specialise Theorem 4.19 in order to derive an intrinsic formula for

$$b^{\mathrm{an}}([M, f]) = \overline{W} \circ \eta^{\mathrm{an}}([M, f]) \in T_{2m}.$$

The connection ∇^k on the $\mathrm{Spin}(k)$–principal bundle $\tilde{f}^* Q_k^{\mathrm{Spin}} \to M$ turns the real vector bundle $\tilde{f}^*\xi_k^{\mathrm{String}}$ into a geometric bundle $\boldsymbol{N}_k$. It is a geometric representative of the stable normal bundle of M, hence the notation. We have

$$R([TM]+1) = R(n+1+k-[\tilde{f}^*\xi_k^{\mathrm{String}}]) \in K[\![q]\!]^0(M).$$

Therefore we get an interpretation of $\boldsymbol{R}(n+1+k-\boldsymbol{N}_k)$ as a virtual geometric representative of $R([TM]+1)$ (which differs from $\boldsymbol{R}(TM+1)$ used in (113) since we work with the geometry on the normal bundle). By construction we have

$$\mathcal{G}^{\mathrm{String}}(\lambda_{n+1}^{\mathrm{String}}) = [\boldsymbol{R}(n+1+k-\boldsymbol{N}_k)] + a(\nu_{R(\lambda_{n+1}^{\mathrm{Spin}})}) \in \widehat{K}^0(M)[\![q]\!].$$

In other words, the correction form for $[\lambda_{n+1}^{\mathrm{String}}] \in K[\![q]\!]^0(B\mathrm{String})$ is given by

$$\gamma_{R(\lambda_{n+1}^{\mathrm{String}})} = \nu_{R(\lambda_{n+1}^{\mathrm{Spin}})} = \mathbf{Td}(\widetilde{\nabla}^{TM})^{-1} \wedge 2H_\alpha \wedge \widetilde{\Phi}_{R(\lambda_{n+1}^{\mathrm{Spin}})}(\nabla^k) \in \Omega P^{-1}(M)[\![q]\!].$$

By Theorem 4.19 the composition $\overline{W} \circ \eta^{\mathrm{top}}([M, f]) \in T_{2m}$ is now represented by the formal power series

$$\left[-\int_M 2H_\alpha \wedge \widetilde{\Phi}_{R(\lambda_{n+1}^{\mathrm{Spin}})}(\nabla^k)\right] - \xi(\not{D}_M \otimes \boldsymbol{R}(n+1+k-\boldsymbol{N}_k)) \in \mathbb{R}/\mathbb{Z}[\![q]\!].$$

This is the version of (113) using the normal bundle geometry on the twisting bundles.

5.5 The Crowley–Goette invariants

In this subsection we want to show how some of the Eells–Kuiper- or Kreck–Stolz-type invariants from geometric topology can be understood from the point of view of the universal η–invariant. Our approach can be described schematically as follows. In a first step, called preconstruction, we translate the geometric topology data into elements of a suitable bordism group MB. We then apply the universal η–invariant. Finally, the desired invariant is obtained by a suitable evaluation against classes in $K^0(MB)$.

In detail, we will consider the example of the Crowley–Goette invariant recently introduced in [27] for S^3–principal bundles on certain $n=4m-1$–dimensional manifolds. We start with recalling the definitions from [27]. Since in the present paper we decided to work with Spin^c–bordism and complex Dirac operators, we will define the variant $t^{\mathbb{C}}_M$ which coincides with the Crowley–Goette invariant for even m and is its double for odd m. Let S^3 be the group of unit quaternions and BS^3 be its classifying space. The set of homotopy classes $[M, \mathrm{BS}^3]$ is in natural bijection with the set of isomorphism classes of S^3–principal bundles on M denoted in [27] by $\mathtt{Bun}(M)$.

Let M be a closed n–dimensional Spin–manifold such that $H^3(M;\mathbb{Q})=0$ and $H^4(M;\mathbb{Q})=0$. Then the Crowley–Goette invariant is defined as a certain function

$$t_M\colon \mathtt{Bun}(M)\to \mathbb{Q}/\mathbb{Z}.$$

In the following we recall the intrinsic formula [27, (1.9)] for t_M. Note that

$$HP\mathbb{Q}^*(\mathrm{BS}^3)\cong \mathbb{Q}[b,b^{-1}][\![b^2c_2]\!],$$

and by the completion theorem [10] we have the isomorphism

$$K^0(\mathrm{BS}^3)\cong R(S^3)^\wedge_{I_{S^3}}.$$

We let ρ be the representation of S^3 by left-multiplication on $\mathbb{H}\cong\mathbb{C}^2$. This ρ gives rise to a class $[\rho]\in K^0(\mathrm{BS}^3)$ and a power series $\mathbf{ch}([\rho])\in \mathbb{Q}[b,b^{-1}][\![b^2c_2]\!]^0$ of total degree zero. There exists a unique power series $\tilde\Phi\in \mathbb{Q}[b,b^{-1}][\![b^2c_2]\!]^{-4}$ of total degree -4 such that $2-\mathbf{ch}([\rho])=c_2\,\tilde\Phi$.

Let $\tilde g\in \mathtt{Bun}(M)$ and $R\to M$ be a S^3–bundle classified by $\tilde g$. We choose a connection ∇^R on R. For every unitary representation (λ, V_λ) of S^3 we let $E_\lambda:=P\times_{S^3,\lambda}V_\lambda$ be the vector bundle associated to R and λ. It comes with a natural hermitian metric h^{E_λ}. The connection ∇^R induces a connection ∇^{E_λ} which preserves h^{E_λ}. In this way we get a geometric bundle $\boldsymbol{E}_\lambda:=(E_\lambda,h^{E_\lambda},\nabla^{E_\lambda})$. By our assumptions on the rational cohomology of M the Chern–Weil representative $c_2(\nabla^R)$ of c_2 is exact, and there exists a unique element $\hat c_2(\nabla^R)\in\Omega^3(M)/\mathtt{im}(d)$ such that $d\hat c_2(\nabla^R)=c_2(\nabla^R)$. We define $\tilde\Phi(\nabla^R)\in \Omega P^{-4}_{\mathrm{cl}}(M)$ by replacing c_2 by $c_2(\nabla^R)$ in the power series $\tilde\Phi$.

We choose a riemannian metric on M which induces the Levi-Civita connection on TM. Furthermore we choose the Spin^c–structure induced by the Spin–structure. We then get a natural Spin^c–extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection. The complex version

$$t_M^{\mathbb{C}}\colon \mathtt{Bun}(M) \to \mathbb{R}/\mathbb{Z}$$

of t_M is now given by [27, (1.9)]:

$$t_M^{\mathbb{C}}(\tilde{g}) := \left[\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \hat{c}_2(\nabla^R) \wedge \tilde{\Phi}(\nabla^R)\right] - 2\xi(\not{D}_M) + \xi(\not{D}_M \otimes \boldsymbol{E}_\rho) \in \mathbb{R}/\mathbb{Z}. \tag{125}$$

To be precise, the value of the integral belongs to $\mathbb{R}[b, b^{-1}]^{-n-1}$, which will be identified with $\mathbb{R}$ using powers of the generator b.

In order to relate the Crowley–Goette invariant to the universal η–invariant we are led to consider a bordism theory of Spin^c–manifolds with S^3–bundles with rationally trivial second Chern class. This bordism theory is constructed homotopy-theoretically as follows: We have a fibre sequence

$$K(\mathbb{Q}, 3) \to K(\mathbb{Q}/\mathbb{Z}, 3) \xrightarrow{\partial} K(\mathbb{Z}, 4) \to K(\mathbb{Q}, 4)$$

of Eilenberg–Mac Lane spaces. We define the space X by the following homotopy pull-back:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q"] & K(\mathbb{Q}/\mathbb{Z}, 3) \arrow[d, "\partial"] \\ \mathrm{BS}^3 \arrow[r, "c_2"] & K(\mathbb{Z}, 4) \end{tikzcd} \tag{126}$$

Since c_2 is a rational isomorphism and $K(\mathbb{Q}/\mathbb{Z}, 3)$ is rationally trivial, we see that the space X is rationally contractible.

We consider the Thom spectrum MB associated to the projection

$$B := B\mathrm{Spin}^c \times X \to B\mathrm{Spin}^c.$$

We conclude that, for $n = 4m - 1$,

$$\pi_n(MB) \otimes \mathbb{Q} \cong \pi_n(M\mathrm{Spin}^c \wedge X_+) \otimes \mathbb{Q} \cong \pi_n(M\mathrm{Spin}^c\mathbb{Q} \wedge X_+) \cong \pi_n(M\mathrm{Spin}^c\mathbb{Q}) \cong 0.$$

It follows that

$$\pi_n(MB)_{\mathrm{tors}} = \pi_n(MB),$$

so the universal η–invariant is defined on the whole Spin^c–bordism group of X:

$$\eta^{\mathrm{top}} = \eta^{\mathrm{an}}\colon \pi_n(MB) \to Q_n(MB).$$

The preconstruction for the Crowley–Goette invariant is a map

$$s_M\colon \mathtt{Bun}(M) \to \pi_n(MB)$$

given by the following lemma:

Lemma 5.18 *A pair $((M,f),\tilde{g})$ of a cycle (M,f) for $\pi_n(M\mathrm{Spin}^c)$ with $H^3(M;\mathbb{Q})$ and $H^4(M;\mathbb{Q})$ both trivial, and a map $\tilde{g}\in \mathtt{Bun}(M)$ naturally gives rise to a class $s_M(\tilde{g}) := [M,(f,g)] \in \pi_n(MB)$.*

Proof The main point is to show that $\tilde{g}\colon M \to \mathrm{BS}^3$ has a natural lift to $g\colon M \to X$ in the diagram (126). The rationalisation of $\tilde{g}^*c_2$ vanishes, so there exists a class $\hat{c}_2 \in H^3(M;\mathbb{Q}/\mathbb{Z})$ such that $\partial\hat{c}_2 = \tilde{g}^*c_2$. This lift is unique up to the image of a rational class of degree 3, hence unique by our assumption. The map $\tilde{g}$ and the lift $\hat{c}_2\colon M \to K(\mathbb{Q}/\mathbb{Z},3)$ together determine the lift $g\colon M \to X$. □

Our next task is to determine the element in $K^0(MB)$ at which we want to evaluate. To this end we calculate the K–theory $K^*(X)$. We have a fibration

$$X \xrightarrow{q} \mathrm{BS}^3$$

with fibre $K(\mathbb{Q},3)$. Note that $\mathbb{Q}$ is a countable abelian group. Furthermore, the space BS^3 has a CW–structure with finite skeleta $(\mathrm{BS}^3_k)_{k\geq 0}$ and $\mathtt{lim}^1_k K^*(\mathrm{BS}^3_k)=0$ by [10]. We can apply Proposition 6.1 and see that

$$q^*\colon \overline{K}^*(\mathrm{BS}^3) \to \overline{K}^*(X) \tag{127}$$

represents $\overline{K}^*(X)$ as a completion of $\overline{K}^*(\mathrm{BS}^3)$ with respect to the topology induced from the profinite topology on $\overline{K}^*(X)$ via q^*. The domain of this map can be calculated by using the completion theorem [10]. The element $2-\rho$ of the representation ring $R(S^3)$ generates the dimension ideal I_{S^3}. If we let $A := 2-[\rho] \in \overline{K}^0(\mathrm{BS}^3)$, then we have

$$\overline{K}^*(\mathrm{BS}^3) \cong \mathbb{Z}[\![A]\!].$$

Since X is rationally contractible we have

$$\mathbf{ch}^*(q^*A) = q^*\mathbf{ch}(A) = 0.$$

Let $p\colon B\mathrm{Spin}^c \times X \to X$ be the projection. By Lemma 2.5 the evaluation

$$\mathtt{ev}_{p^*q^*A}\colon Q_n(MB) \to \mathbb{Q}/\mathbb{Z}$$

is well-defined. We define

$$\varepsilon = \mathtt{ev}_{p^*q^*A} \circ \eta^{\mathrm{top}}\colon \pi_n(MB) \to \mathbb{Q}/\mathbb{Z}.$$

The following proposition clarifies the relation between $t^{\mathbb{C}}_M$ and the universal η–invariant:

Proposition 5.19 *If (M, f) is a cycle for $\pi_n(M\mathrm{Spin}^c)$ which satisfies $H^3(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0$, then we have the relation*

$$t^{\mathbb{C}}_M = \varepsilon \circ s_M \colon \mathtt{Bun}(M) \to \mathbb{Q}/\mathbb{Z}.$$

Proof It is an instructive exercise in the use of geometrisations to derive an intrinsic formula for the composition $\varepsilon \circ s_M$ which can be compared with the formula (125) for $t^{\mathbb{C}}_M$. In a first step we must approximate the space X by spaces with finite skeleta. Note that we can write (compare with (42) for the connecting maps)

$$K(\mathbb{Q}/\mathbb{Z}, 3) := \mathtt{hocolim}_l K(\mathbb{Z}/l\mathbb{Z}, 3).$$

If we define X_l by the pull-back

$$\begin{tikzcd} X_l \arrow[r] \arrow[d, "q_l"] & K(\mathbb{Z}/l\mathbb{Z},3) \arrow[d, "\partial_l"] & \\ \mathrm{BS}^3 \arrow[r, "c_2"] & K(\mathbb{Z},4) \arrow[r, "l"] & K(\mathbb{Z},4) \end{tikzcd} \tag{128}$$

then we get connecting maps $X_l \to X_{l'}$ if $l \,|\, l'$ and

$$X \simeq \mathtt{hocolim}_l X_l, \quad \pi_n(MB) = \mathtt{colim}_l \pi_n(MB_l),$$

where the Thom spectrum MB_l is associated to the projection $B_l := B\mathrm{Spin}^c \times X_l \to B\mathrm{Spin}^c$. The main advantage of X_l is that it has finite skeleta.

We consider a cycle (M, f) for $\pi_n(M\mathrm{Spin}^c)$ and an auxiliary map $g\colon M \to X$. We can assume that g has a factorisation

$$g\colon M \xrightarrow{g_l} X_l \to X$$

for some l. We choose a Spin^c–extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection. We are going to construct an l–good geometrisation for $(M, (f, g_l), \tilde{\nabla}^{TM})$ using similar ideas as in the String–bordism case Theorem 5.14. We first take m sufficiently large such that the canonical map $\iota_m\colon B\mathrm{Spin}^c(m) \to B\mathrm{Spin}^c$ is $\max\{l, n+1, 4\}$–connected. We choose a compact $\max(l, n+1, 4)$–connected approximation

$$(f_u, g_u)\colon M_u \to B\mathrm{Spin}^c(m) \times X_l$$

such that the map (f, g_l) factorises over a closed embedding $h\colon M \to M_u$. It is here where we use the property that X_l has finite skeleta which ensures that we can find a compact approximation M_u. Note that for M_u we allow compact manifolds with boundary. We require a closed embedding since below we want to extend geometric structures defined on M to M_u. Working with compatible geometric structures on M and M_u avoids more complicated formulas involving correction terms β as in (70). This embedding condition can easily be satisfied since, if necessary, we can enlarge M_u by forming a product with a high-dimensional disc and smoothing out the corners.

We choose the $\mathrm{Spin}^c(m)$–connection $\tilde{\nabla}^u$ as in Section 4.4. The map h has a refinement to a Spin^c–map and we can assume that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably.

The composition $q_l \circ g_u\colon M_u \to \mathrm{BS}^3$ classifies an S^3–principal bundle $R_u \to M_u$ on which we choose a connection ∇^{R_u}. We can assume that $R \cong h^* R_u$ with connection $\nabla^R = h^*\nabla^{R_u}$.

We let

$$\tilde{\mathcal{G}}_u\colon K^0(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3) \to \hat{K}^0(M_u)$$

denote the geometrisation of $(M_u, (f_u, q_l \circ g), \tilde{\nabla}^u)$ which was constructed by the method of Lemma 4.7 from a geometrisation $\mathcal{G}^0 := \mathcal{G}^{\mathrm{Spin}^c(m)}$ which is the analogue of $\mathcal{G}^{\mathrm{Spin}(m)}$ in the proof of Proposition 5.13.

We have a fibration

$$B\mathrm{Spin}^c(m)\times X_l \xrightarrow{(\mathtt{id},q_l)} B\mathrm{Spin}^c(m)\times \mathrm{BS}^3 \tag{129}$$

with fibre $K(\mathbb{Z},3)$. We can again apply Proposition 6.1 and conclude that

$$(\mathtt{id}, q_l)^*\colon \bar{K}^*(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3) \to \bar{K}^*(B\mathrm{Spin}^c(m)\times X_l) \tag{130}$$

represents $\bar{K}^*(B\mathrm{Spin}^c(m)\times X_l)$ as a completion of $\bar{K}^*(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3)$. We define

$$\mathcal{G}_{u,0}\colon \bar{K}^*(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3) \to \hat{K}^0(M_u)$$

by

$$\mathcal{G}_{u,0}(\phi) := \tilde{\mathcal{G}}_u(\phi) \in \hat{K}^0(M_u) \quad \text{for } \phi \in \bar{K}^*(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3).$$

Similarly as in the string bordism case (see the explanation after the statement of Proposition 5.15 and Remark 5.17) the contribution of the exact form $c_2(\nabla^{R_u})$ obstructs the existence of a degree-preserving cohomological character for $\mathcal{G}_{u,0}$. We must kill the contribution of $c_2(\nabla^{R_u})$ to the curvature of $\mathcal{G}_{u,0}(\phi)$. Note that $q_l^* c_2 \in H^4(X_l;\mathbb{Z})$ is l–torsion. Hence we can choose a form $\alpha_u \in \Omega^3(M_u)/\mathtt{im}(d)$ such that $d\alpha_u = c_2(\nabla^{R_u})$. By an easy application of Serre's spectral sequence to the fibration (129) we see that

$$p^*\colon H^*(B\mathrm{Spin}^c(m);\mathbb{Q}) \to H^*(B\mathrm{Spin}^c(m)\times X_l;\mathbb{Q})$$

is an isomorphism. Since $H^*(B\mathrm{Spin}^c(m);\mathbb{Q})$ is concentrated in even degrees the odd-dimensional cohomology of M_u is concentrated in degrees $\geq n+1$. In particular we see that α_u is uniquely determined. Moreover, the restriction

$$h^*\colon HP\mathbb{R}^{-1}(M_u) \to HP\mathbb{R}^{-1}(M) \tag{131}$$

is trivial.

We have

$$HP\,\mathbb{Q}^*(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3) \cong \mathbb{Q}[b,b^{-1}][\![bc_1, b^2p_1, b^4p_2\dots, b^{2r_m}p_{r_m}, b^2c_2]\!],$$
$$HP\,\mathbb{Q}^*(B\mathrm{Spin}^c(m)\times X_l) \cong \mathbb{Q}[b,b^{-1}][\![bc_1, b^2p_1, b^4p_2\dots, b^{2r_m}p_{r_m}]\!],$$

where c_1 and the Pontrjagin classes come from $B\mathrm{Spin}^c$ and c_2 is pulled back from BS^3. The pull-back $(\mathtt{id}, q_l)^*$ is the quotient map defined by setting $c_2=0$. For ϕ in $K^0(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3)$ we define the formal power series

$$\Phi_\phi := \mathbf{Td}^{-1}\cup \mathbf{ch}(\phi) \in \mathbb{Q}[b,b^{-1}][\![bc_1, b^2p_1, b^4p_2\dots, b^2c_2]\!]^0$$

and set

$$\widetilde{\Phi}_\phi := \frac{\Phi_\phi - i_{c_2=0}\Phi_\phi}{c_2} \in \mathbb{Q}[b,b^{-1}][\![bc_1, b^2p_1, b^4p_2\dots, b^2c_2]\!]^{-4}.$$

For $\phi\in \overline{K}^0(B\mathrm{Spin}^c(m)\times \mathrm{BS}^3)$ we now define

$$\mathcal{G}_u(\phi) := \mathcal{G}_{u,0}(\phi) - a(\alpha_u \wedge \mathbf{Td}(\widetilde{\nabla}^u)^{-1}\wedge \widetilde{\Phi}_\phi(\nabla^u, \nabla^{R_u})),$$

where $\widetilde{\Phi}_\phi(\nabla^u,\nabla^{R_u})\in \Omega P^{-4}(M)$ is obtained from $\widetilde{\Phi}_\phi$ by replacing the generators c_1, p_i and c_2 by their corresponding Chern–Weil representatives $c_1(\widetilde{\nabla}^u)$, $p_i(\widetilde{\nabla}^u)$ and $c_2(\nabla^{R_u})$. We calculate, similarly as in (124), that

$$\mathbf{Td}(\widetilde{\nabla}^u)\wedge R(\mathcal{G}_u(\phi)) = i_{c_2=0}\Phi_\phi(\nabla^u,\nabla^{R_u}). \tag{132}$$

We now define

$$\mathcal{G}\colon \overline{K}^*(B\mathrm{Spin}^c\times \mathrm{BS}^3)\to \widehat{K}^0(M)$$

by

$$\mathcal{G}(\phi) := (\iota_m\times \mathtt{id}_{X_l})_* h^*\mathcal{G}_u(\phi).$$

We claim that $\mathcal{G}$ extends by continuity to a good geometrisation of $(M,(f,g_l),\widetilde{\nabla}^{TM})$. The argument is very similar to that of Lemma 5.16. We first show continuity. If (ϕ_k) is a sequence in $\overline{K}^0(B\mathrm{Spin}^c\times \mathrm{BS}^3)$ with $(\mathtt{id},q_l)^*\phi_k\to 0$ as $k\to\infty$ in the profinite topology, then we can find a $k_0\in\mathbb{N}$ such that

$$\mathcal{G}_u(\phi_k)\in HP\mathbb{R}^{-1}(M_u)/\mathtt{im}(\mathbf{ch})$$

for all $k\geq k_0$. It follows from the vanishing of the map h^* in (131) that $h^*\mathcal{G}_u(\phi_k)=0$.

Because of (132), a degree-preserving cohomological character of $\mathcal{G}$ is given by

$$bc_1 \mapsto bc_1(\widetilde{\nabla}^{TM}), \quad b^{2i} p_i \mapsto b^{2i} p_i(\widetilde{\nabla}^{TM}).$$

It follows that $\mathcal{G}$ is a geometrisation. In order to see that it is l–good we show that $\mathcal{G}_u$ itself is continuous using a similar argument based on a $\dim(M_u)+1$–connected approximation of $B\mathrm{Spin}^c(m) \times X_l$.

We can now apply Theorem 4.19 in order to derive a formula for $\varepsilon\big([M,(f,g_l)]\big) \in \mathbb{R}/\mathbb{Z}$. We can take $\hat{c}_2(\nabla^R) := h^*\alpha_u$ and have $\widetilde{\Phi}_A = \mathbf{Td}^{-1}\widetilde{\Phi}$. We have, by construction,

$$\mathcal{G}((\mathtt{id}, q_l)^* A) = [2 - \boldsymbol{E}_\rho] - a(\hat{c}_2(\nabla^R) \wedge \widetilde{\Phi}(\nabla^R)),$$

hence the correction form (Definition 4.18) is given by

$$\gamma_{(\mathtt{id},q_l)^*A} = -\hat{c}_2 \wedge \widetilde{\Phi}(\nabla^R)\ .$$

It follows from (78) that

$$\begin{aligned}\mathtt{ev}_{(\mathtt{id},q_l)^*A}&\big(\eta^{\mathrm{an}}\big([M,(f,g_l)]\big)\big)\\ &= \left[\int_M \mathbf{Td}(\widetilde{\nabla}^{TM}) \wedge \hat{c}_2(\nabla^R) \wedge \widetilde{\Phi}(\nabla^R)\right] - 2\xi(\not{D}_M) + \xi(\not{D}_M \otimes \boldsymbol{E}).\end{aligned}$$

This is exactly the formula (125) for $t_M^{\mathbb{C}}(\tilde{g}) \in \mathbb{R}/\mathbb{Z}$. Proposition 5.19 now follows from Lemma 2.4, which gives the first equality in the chain

$$\mathtt{ev}_{(\mathtt{id},q_l)^*A}\big(\eta^{\mathrm{an}}\big([M,(f,g_l)]\big)\big) = \mathtt{ev}_{(\mathtt{id},q)^*A}\big(\eta^{\mathrm{an}}\big([M,(f,g)]\big)\big) = \varepsilon(s_M(\tilde{g})). \qquad \square$$

The paper [27] provides a lot of interesting explicit calculations. Our general point of view is probably not of much help here. But it is useful to understand structural results like the relation with the Adams e–invariant [27, Proposition 1.11]. This is what we are going to explain now. We define the space Y by extending the diagram (126) by another cartesian square

$$\begin{tikzcd} Y \arrow[r, "H"] \arrow[d, "r"] & X \arrow[r] \arrow[d, "q"] & K(\mathbb{Q}/\mathbb{Z},3) \arrow[d, "\partial"] \\ S^4 \arrow[r, "h"] & \mathrm{BS}^3 \arrow[r, "c_2"] & K(\mathbb{Z},4) \end{tikzcd} \tag{133}$$

where h generates $\pi_4(\mathrm{BS}^3)$ with $h^*c_2 \in H^4(S^4;\mathbb{Z})$ the positive orientation class. We use the Serre spectral sequence in order to calculate the rational cohomology of Y:

$$H^k(Y;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 0, 7, \\ 0 & \text{if } k \notin \{0,7\}. \end{cases}$$

This implies

(134) $$\pi_{4m-1}(S \wedge Y_+)_{\mathrm{tors}} = \pi_{4m-1}(S \wedge Y_+)$$

for $m \geq 3$.

From now on we assume that $m \geq 2$. By Lemma 2.4 we get the commutativity of the squares (except the lower right, which will be explained below) of the following diagram:

(135)
$$\begin{tikzcd}
\pi_{4m-1}(M\mathrm{Spin}^c \wedge X_+) \arrow[r, "\eta^{\mathrm{top}}"] \arrow[rr, bend left, dotted, "\varepsilon"] & Q_{4m-1}(M\mathrm{Spin}^c \wedge X_+) \arrow[r, "\mathtt{ev}_{p^*q^*A}"] & \mathbb{Q}/\mathbb{Z} \\
\pi_{4m-1}(S \wedge Y_+)_{\mathrm{tors}} \arrow[u, "(\epsilon_{M\mathrm{Spin}^c} \wedge H)_*"] \arrow[r, "\eta^{\mathrm{top}}"] \arrow[d, "(\mathtt{id} \wedge r)_*"'] & Q_{4m-1}(S \wedge Y_+) \arrow[u, "(i_*, H)"'] \arrow[r, "\mathtt{ev}_{r^*h_+^*A}"] \arrow[d, "(\mathtt{id}, r)"'] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal] \arrow[d, equal] \\
\pi_{4m-1}(S \wedge S^4_+) \arrow[r, "\eta^{\mathrm{top}}"] \arrow[d, "w_*"'] & Q_{4m-1}(S \wedge S^4_+) \arrow[r, "\mathtt{ev}_{h_+^*A}"] \arrow[d, "w_*"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
\pi_{4m-1}(S \wedge S^4) \arrow[r, "\eta^{\mathrm{top}}"] \arrow[d, "\cong"'] & Q_{4m-1}(S \wedge S^4) \arrow[r, "\mathtt{ev}_{h^*A}"] \arrow[d, "\cong"'] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
\pi_{4m-5}(S) \arrow[r, "\eta^{\mathrm{top}}"] \arrow[rr, bend right, dotted, "e^{\mathrm{Adams}}_{\mathbb{C}}"'] & Q_{4m-5}(S) \arrow[r, "\mathtt{ev}_1"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

We need the condition $m \geq 2$ in order to have well-defined evaluations $\mathtt{ev}_{h_+^*A}$, $\mathtt{ev}_{h^*A}$ and $\mathtt{ev}_1$. The map w_* is induced by the map $w\colon S^4_+ \to S^4$ which is the identity on S^4 and maps the extra base point to the base point of S^4. In terms of the canonical decomposition

$$K^0(S^4_+) \cong K^0(S^4) \oplus \mathbb{Z}$$

we have $w^* = (\mathtt{id}_{K^0(S^4)}, \dim)$. This map induces

$$w_*\colon Q_{4m-1}(S \wedge S^4_+) \to Q_{4m-1}(S \wedge S^4).$$

We use the symbol $h_+\colon S^4_+ \to \mathrm{BS}^3$ for the map induced by h, which maps the extra base point to a base point of BS^3. The lower left vertical map is the suspension isomorphism. The lower middle vertical isomorphism is again induced by suspension and the Bott isomorphism

$$K^0(S^4) \cong K^{-4}(S^0) \xrightarrow{b^{-2}} K^0(S^0).$$

In order to see that the lower right square commutes note that this isomorphism maps h^*A to 1. This follows from

$$\mathbf{ch}(2-A) = bc_2 + O(b^3)$$

and the fact that $c_2 \in H^4(S^4;\mathbb{Z})$ is the suspension of $1 \in H^0(*;\mathbb{Z})$. The composition of the lower two arrows is the definition (79) of the complex version of the Adams e–invariant.

We conclude that

$$\varepsilon \circ (\epsilon_{M\mathrm{Spin}^c} \wedge H) = e^{\mathrm{Adams}}_{\mathbb{C}} \circ w_* \,. \tag{136}$$

The same argument as for Lemma 5.18 gives:

Lemma 5.20 *A pair $((M,f),\tilde{g})$ of a cycle for $\pi_{4m-1}(S)$ such that $H^3(M;\mathbb{Q})=0$ and $H^4(M;\mathbb{Q})=0$ and a map $\tilde{g} \in [M,S^4]$ gives naturally rise to a class $[M,(f,g)]$ in $\pi_{4m-1}(S \wedge Y_+)$.*

If M satisfies the assumption of the lemma, then we have a preconstruction map

$$\tilde{s}_M\colon [M,S^4] \to \pi_{4m-1}(S \wedge Y_+)$$

and conclude from Proposition 5.19, (136) and (134) that, for $m \geq 3$ (or $m=2$ with $[M,(f,g)] \in \pi_7(S \wedge Y_+)$ a torsion class),

$$t^{\mathbb{C}}_M = e^{\mathrm{Adams}}_{\mathbb{C}} \circ w_* \circ \tilde{s}_M\colon [M,S^4] \to \mathbb{Q}/\mathbb{Z}.$$

This is [27, Proposition 1.11] if one takes the following geometric description of the composition $w_* \circ \tilde{s}_M(\tilde{g})$ into account. First of all we have $\tilde{s}_M(\tilde{g}) = [M,(f,g)]$, where $g\colon M \to Y$ is the lift of $\tilde{g}\colon M \to S^4$. Then $w_*([M,(f,g)]) = [M,(f,\tilde{g})] - [M,\mathrm{const}]$ is in $\pi_{4m-1}(S \wedge S^4)$. The geometric representative of the 4–fold desuspension of this class is the stably normally framed manifold obtained by taking the preimage $Y := \tilde{g}^{-1}(s)$ of a regular point $s \in S^4$ of $\tilde{g}$.

Corollary 5.21 [27, Proposition 1.11][3] *We assume that $m \geq 2$. Let $(M,(f,\tilde{g}))$ be as in Lemma 5.20. If $m=2$, then in addition we assume that $[M,(f,g)] \in \pi_7(S \wedge Y_+)$ is a torsion class. Then we have*

$$t^{\mathbb{C}}_M(h \circ \tilde{g}) = e^{\mathrm{Adams}}_{\mathbb{C}}([Y,f']),$$

where Y is the preimage $Y := \tilde{g}^{-1}(s)$ of a regular point $s \in S^4$ of $\tilde{g}$ with its induced normal framing (and f' is the constant map).

[3] The e–invariant used in the present paper is the negative of the e–invariant in the conventions of [27, Proposition 1.11]. This accounts for the different sign.

Remark 5.22 In the following we consider the case $m=2$ and discuss what happens if we drop the assumption that $[M,(f,g)]\in\pi_{4m-1}(S\wedge Y_+)$ is a torsion element. We consider the Hopf fibration $\tilde{g}\colon S^7\to S^4$. By Lemma 5.20 we get an element $[S^7,(f,g)]\in\pi_7(S\wedge Y_+)$. This element is not torsion.

If it were a torsion element, then by Corollary 5.21 we would have $t^{\mathbb{C}}_{S^7}(h\circ\tilde{g})=e^{\mathrm{Adams}}_{\mathbb{C}}([Y,f'])$, where (Y,f') is a Hopf fibre with the induced framing. It has been shown in [27, Example 3.5] that $t^{\mathbb{C}}_{S^7}(h\circ\tilde{g})=0$. On the other hand, since the Hopf fibration generates the stable homotopy group $\pi_3(S)\cong\mathbb{Z}/24\mathbb{Z}$ which is detected completely by e^{Adams} we know that $e^{\mathrm{Adams}}_{\mathbb{C}}([Y,f'])\in\mathbb{Q}/\mathbb{Z}$ has order 12, and in particular is nontrivial.

Now
$$\mathtt{ev}_{h_+^*A}\big(\eta^{\mathrm{top}}\big((\mathtt{id}\wedge r)_*[S^7,(f,g)]\big)\big)=e^{\mathrm{Adams}}_{\mathbb{C}}([Y,f'])\neq 0$$
is a nontrivial torsion class of order 12. On the other hand,
$$\mathtt{ev}_{p^*q^*A}\big(\eta^{\mathrm{top}}\big((\epsilon_{M\mathrm{Spin}^c}\wedge H)_*[S^7,(f,g)]\big)\big)=t^{\mathbb{C}}_{S^7}(h\circ\tilde{g})=0.$$
We see that the upper half of the diagram (135) no longer commutes if we delete the subscript $(\cdot)_{\mathrm{tors}}$ in the second line.

6 The K–theory of $K(\pi,n)$–bundles

The goal of this section is to give a proof of Proposition 5.15. It uses some theory which we develop in greater generality for the purpose of applications in other places. The main result is Proposition 6.1.

Let Z be a space with an increasing filtration
$$\cdots\subseteq Z_k\subseteq Z_{k+1}\subseteq\cdots,\quad k\in\mathbb{N},$$
such that $Z\simeq\mathtt{hocolim}_{k\in\mathbb{N}}Z_k$. Then we consider the decreasing filtration
$$F^kK^*(Z):=\mathtt{ker}(K^*(Z)\to K^*(Z_{k-1}))$$
of the K–theory group $K^*(Z)$, and we write $\mathrm{Gr}^k_F(K^*(Z))$ for its associated subquotients. The filtration $(F^kK^*(Z))_{k\in\mathbb{N}}$ induces a topology on $K^*(Z)$, and we write ${}^F\overline{K}^*(Z)$ for the Hausdorff completion of $K^*(Z)$ with respect to this topology.

If Z' is a second space with increasing filtration $(Z'_k)_{k\in\mathbb{N}}$ and $Z\to Z'$ is a filtration-preserving map, then the pull-back $K^*(Z')\to K^*(Z)$ is filtration-preserving, continuous and induces a continuous map ${}^F\overline{K}^*(Z')\to{}^F\overline{K}^*(Z)$.

If the Z_k for $k \in \mathbb{N}$ are finite CW–complexes, then the induced topology on $K^*(Z)$ is the profinite topology. For a general filtration of Z the profinite topology is always contained in the topology associated to the filtration and we have a continuous map ${}^F\bar{K}^*(Z) \to \bar{K}^*(Z)$.

The filtration of Z induces a spectral sequence (E_r, d_r) with $E_1^{s,t} \cong K^{t+s}(Z_s/Z_{s-1})$. If the filtration of Z is the skeletal filtration, then this is the Atiyah–Hirzebruch spectral sequence, whose second page is given by $E_2^{s,t} \cong H^s(Z, \pi_{-t}(K))$.

If we fix $(s,t) \in \mathbb{N} \times \mathbb{Z}$, then for $r \geq s$ the group $E_r^{s,t}$ does not receive differentials. Hence we can consider the sequence of inclusions of groups

$$E_\infty^{s,t} := \bigcap_{r \geq s} E_r^{s,t} \subseteq \cdots \subseteq E_{r+2}^{s,t} \subseteq E_{r+1}^{s,t} \subseteq E_r^{s,t}.$$

For every $k \in \mathbb{N}$ there is a natural map

$$\sigma_k\colon \operatorname{Gr}_F^k K^{t+k}(Z) \to E_\infty^{k,t}. \tag{137}$$

We say that the spectral sequence converges strongly if (137) is an isomorphism for all $k \in \mathbb{N}$ and $t \in \mathbb{Z}$. This is the case, for example, if the spectral sequence degenerates at a finite stage.

Proposition 6.1 *Let X be a CW–complex with an increasing filtration $(X_k)_{k \in \mathbb{N}}$ by finite subcomplexes X_k such that $X = \bigcup_k X_k$. We further fix an integer $n \geq 3$ and a countable abelian group π, and consider a fibration $p\colon Y \to X$ with fibre $K(\pi, n)$. Let $(Y_k)_{k \in \mathbb{N}}$ be the induced filtration of Y.*

(1) *The projection $p\colon Y \to X$ induces a continuous injective map*

$$\bar{K}^*(X) \to {}^F\bar{K}^*(Y). \tag{138}$$

(2) *If $\mathtt{lim}_k^1 K^*(X_k) = 0$, then the image of the composition*

$$\bar{K}^*(X) \to {}^F\bar{K}^*(Y) \to \bar{K}^*(Y) \tag{139}$$

is dense.

Remark 6.2 If π is a countable torsion group, then [5, Proposition 4.7] gives the stronger statement that $p^*\colon K^*(X) \to K^*(Y)$ is an isomorphism. In this case we can even assume that $n \geq 2$, and drop the assumption that X has finite skeleta. ◁

Remark 6.3 In order to understand what is going on here we recall the much simpler dual result concerning K–homology. It is known by [5] that the inclusion of the base point induces an isomorphism $K_*(*) \to K_*(K(\pi, n))$ for $n \geq 3$. Hence the Serre spectral sequence for K–homology for the fibration $p\colon Y \to X$ degenerates at the second page and we can conclude that $p_*\colon K_*(Y) \to K_*(X)$ is an isomorphism. The two difficulties in the case of cohomology are that the Serre spectral sequence only calculates the graded components of a certain infinite filtration of the K–theory, and that the K–theory of the fibres does not vanish but consists of phantom classes.

Proof of Proposition 5.15 assuming Proposition 6.1 Proposition 5.15 follows from the combination of the two assertions of Proposition 6.1 if we take $n := 3$, $\pi := \mathbb{Z}$ and $p\colon B\mathrm{String}(m) \to B\mathrm{Spin}(m)$ (note that $m \geq 3$). The homotopy type $B\mathrm{Spin}(m)$ admits a cell structure with finite skeleta. Moreover, it has been shown in [10, Section 2] that $\mathtt{lim}^1_k K^*(B\mathrm{Spin}(m)_k) = 0$. $\square$

In order to show Proposition 6.1 we first need some preparations about divisible groups. Let A be some abelian group. Then we define its subgroup

$$A_{\mathrm{div}} := \{a \in A \mid \forall n \in \mathbb{N}\ \exists a' \in A\ \ a = na'\}$$

of divisible elements. We consider the exact sequence

$$0 \to A_{\mathrm{div}} \to A \to \bar{A} \to 0.$$

Since a divisible group is injective, this sequence is split. Hence we have a noncanonical decomposition

$$A \cong A_{\mathrm{div}} \oplus \bar{A}.$$

This implies that $\bar{A}_{\mathrm{div}} = 0$. We now consider a short exact sequence of groups

$$0 \to A \to B \to C \to 0$$

together with a homomorphism $B \to X$, where X is a finitely generated abelian group.

Lemma 6.4 *If $c \in C_{\mathrm{div}}$, then we can find a lift $b \in B$ of c whose image in X vanishes.*

Proof We consider the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_{\mathrm{div}} \arrow[r] \arrow[d] & B_{\mathrm{div}} \arrow[r] \arrow[d] & C_{\mathrm{div}} \arrow[d] \\
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
\bar{A} \arrow[r] \arrow[d] & \bar{B} \arrow[r] \arrow[d] & \bar{C} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

with vertical exact sequences. The snake lemma gives an isomorphism

$$\frac{C_{\mathrm{div}}}{\mathtt{im}(B_{\mathrm{div}} \to C_{\mathrm{div}})} \cong \frac{\mathtt{ker}(\bar{B} \to \bar{C})}{\mathtt{im}(\bar{A} \to \bar{B})}.$$

This shows that the group on the right-hand side is divisible. Since any map from a divisible group to a finitely generated group is trivial, we have a factorisation $\bar{B} \to X$ of the map $B \to X$. We thus get a homomorphism

$$\frac{\mathtt{ker}(\bar{B} \to \bar{C})}{\mathtt{im}(\bar{A} \to \bar{B})} \to \frac{X}{Y},$$

where $Y \subseteq X$ is the image of $\bar{A} \to \bar{B} \to X$. The quotient X/Y is still finitely generated, and this implies that this map is trivial since its domain is divisible.

We now choose a preimage $b_0 \in B$ of c. Its image in $\bar{b}_0 \in \bar{B}$ then vanishes when mapped to $\bar{C}$, so it represents a class in $\mathtt{ker}(\bar{B} \to \bar{C})/\mathtt{im}(\bar{A} \to \bar{B})$. The image of this class in X/Y vanishes, so there exists $\bar{a} \in \bar{A}$ such that the image of $\bar{b}_0 - \bar{a}$ in X vanishes. We choose some lift $a \in A$ of $\bar{a}$. Then the image of $b := b_0 - a$ in X vanishes. Moreover, b is a lift of c, too. □

Proposition 6.5 *Assume that X is a finite CW–complex and $p\colon Y \to X$ is a fibration with fibre $K(\pi, n)$ for $n \geq 3$ and π a countable abelian group. Then the map*

$$p^*\colon K(X) \to K^*(Y) \tag{140}$$

is injective and has dense range.

Proof Let

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{\dim(X)} = X$$

be the filtration of X given by the cellular structure. It induces a filtration

$$\varnothing = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{\dim(X)} = Y$$

by taking preimages. These filtrations induce spectral sequences $E(\mathtt{id})$ and $E(p)$ which both degenerate at the $\dim(X)^{\text{th}}$ page and strongly converge. Here $E(\mathtt{id})$ is the Atiyah–Hirzebruch spectral sequence for X, and $E(p)$ is the Serre spectral sequence for the fibration p.

We consider the map of fibrations

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"] & X \arrow[d, "\mathtt{id}"] \\ X \arrow[r] & X \end{tikzcd}$$

which induces a morphism of spectral sequences

$$p^*\colon E(\mathtt{id}) \to E(p).$$

Lemma 6.6 *The image $p^* E(\mathtt{id})$ is a direct summand of $E(p)$.*

Proof The K–theory of the fibres of p form a local system of abelian groups $\mathcal{K}^*(K(\pi,n))$ on X with fibres isomorphic to $K^*(K(\pi,n))$. We have

$$E_1^{s,t}(p) \cong C^s\bigl(X, \mathcal{K}^{t-s}(K(\pi,n))\bigr),$$

where $C^s(X,\mathcal{A})$ denotes the cellular cochains of X with coefficients in the local system of abelian groups $\mathcal{A}$. We have an exact sequence

$$0 \to E_1^{s,t}(p)_{\mathrm{div}} \to E_1^{s,t}(p) \to \bar{E}_1^{s,t}(p) \to 0.$$

Since $\widetilde{K}^*(K(\pi,n))$ is uniquely divisible by [5, Theorem II], the composition

$$E_1^{s,t}(\mathtt{id}) \xrightarrow{p^*} E_1^{s,t}(p) \to \bar{E}_1^{s,t}(p)$$

is an isomorphism. Moreover, since p^* is a chain map and there are no nontrivial maps from a divisible group to a finitely generated group, we have a decomposition of chain complexes

$$E_1^{*,*}(p) \cong E_1^{*,*}(p)_{\mathrm{div}} \oplus p^* E_1^{*,*}(\mathtt{id}).$$

We now use that $E_1^{*,*}(p)_{\mathrm{div}}$ is actually uniquely divisible. Furthermore, kernels and images of homomorphisms between uniquely divisible groups, and also quotients of uniquely divisible groups, are again uniquely divisible. We hence obtain a corresponding decomposition of the higher pages and a decomposition of the whole spectral sequence as

$$E(p) \cong E(p)_{\mathrm{div}} \oplus p^* E(\mathtt{id}). \qquad \square$$

Lemma 6.7 *The map (140) is injective.*

Proof Assume that $\phi \in K^*(X)$ is such that $p^*(\phi) = 0$. We are going to show that $\phi \in F^k K^*(X)$ for all $k \geq -1$. For $k \geq \dim(X)+1$ this then implies that $\phi = 0$.

We clearly have $\phi \in F^{-1}K^*(X) = K^*(X)$. Assume by induction that $\phi \in F^k K^*(X)$. Then ϕ represents an element $\sigma_k(\phi) \in E_\infty^{k,*}(\mathtt{id})$. We observe that the image of $\sigma_k(\phi)$ in $E_\infty^{k,*}(p)$ and in particular its component in $p^* E_\infty^{k,*}(\mathtt{id})$ vanishes. By Lemma 6.6 we have $\sigma_k(\phi) = 0$. This implies $\phi \in F^{k+1}K^*(X)$. $\square$

Lemma 6.8 *The range of the map (140) is dense.*

Proof We fix an element $\phi \in K^*(Y)$. We must show that we can approximate this element in the profinite topology of $K^*(Y)$ by elements in the image of $p^*\colon K^*(X) \to K^*(Y)$. Let $t\colon T \to Y$ be a map from a finite CW–complex. Then $\mathtt{ker}(t^*) \subseteq K^*(Y)$ is some neighbourhood of zero. We must find an element $\psi \in K^*(X)$ such that

$$\phi - p^*\psi \in \mathtt{ker}(t^*). \tag{141}$$

To this end we use the following lemma:

Lemma 6.9 *If $\hat{\phi} \in F^k K^*(Y)$, then there exists $\hat{\psi} \in K^*(X)$ and $\rho \in \mathtt{ker}(t^*)$ such that $\hat{\phi} - p^*\hat{\psi} - \rho \in F^{k+1}K^*(Y)$.*

Assuming this lemma and using that $\phi \in F^{-1}K^*(Y)$ we obtain the desired $\psi \in K^*(X)$ by a finite iteration. Thus Lemma 6.8 follows from Lemma 6.9. □

Proof of Lemma 6.9 The element $\hat{\phi}$ gives rise to an element $u := \sigma_k(\hat{\phi}) \in E_\infty^{k,*}(p)$ which we can decompose as $u = v \oplus p^*w$ with $v \in E_\infty^{k,*}(p)_{\mathrm{div}}$ and $w \in E_\infty^{k,*}(\mathtt{id})$. We let $\hat{\psi} \in F^k K^*(X)$ be an element which represents w. We apply Lemma 6.4 to the exact sequence

$$0 \to F^{k+1}K^*(Y) \to F^k K^*(Y) \to E_\infty^{k,*}(p) \to 0$$

and the map $t^*\colon F^k K^*(Y) \to K^*(T)$. By Lemma 6.4, we can find an element $\rho \in F^k K^*(Y) \cap \mathtt{ker}(t^*)$ which represents v. Then we have $\hat{\phi} - p^*\hat{\psi} - \rho \in F^{k+1}K^*(Y)$, as required. □

Proposition 6.5 now follows from Lemmas 6.7 and 6.8. □

Proof of Proposition 6.1 Let

$$\varnothing = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$$

be the filtration of Y induced by taking the preimages of the subcomplexes X_k along $p\colon Y \to X$. By construction the map p preserves filtrations and hence (138) is continuous.

We first show that (138) is injective. Note that the filtration $(F^k K^*(X))_{k \geq 0}$ induces the profinite topology. Let $\phi \in K^*(X)$ be such that $p^*\phi \in \bigcap_{k \geq 0} F^k K^*(Y)$. It suffices to show that this implies $\phi \in F^k K^*(X)$ for all $k \geq 0$, since then ϕ represents the zero element in $\bar{K}^*(X)$.

We clearly have $\phi \in F^{-1}K^*(X)$. We now assume by induction that $\phi \in F^k K^*(X)$. We can apply Proposition 6.5 to the fibration $p_k\colon Y_k \to X_k$. Since

$$p_k^*(\phi_{|X_k}) = (p^*\phi)_{|Y_k} = 0$$

we conclude that $\phi|_{X_k}=0$ and hence $\phi\in F^{k+1}K^*(X)$.

We now show that the image of (139) is dense. Let $\phi\in K^*(Y)$. We must approximate ϕ by elements in the image of (139). Let $t\colon T\to Y$ be a map from a finite CW–complex. Then $\mathtt{ker}(t^*)\subseteq K^*(Y)$ is some neighbourhood of zero. We must find an element $\psi\in K^*(X)$ such that

$$\phi-p^*\psi\in\mathtt{ker}(t^*). \tag{142}$$

Since T is finite there exists a number $k\in\mathbb{N}$ for which there is a factorisation

$$t\colon T\xrightarrow{t_k}Y_k\xrightarrow{\iota_k}Y.$$

It suffices to find an element $\psi\in K^*(X)$ such that

$$\iota_k^*\phi-\iota_k^*p^*\psi\in\mathtt{ker}(t_k^*). \tag{143}$$

Then indeed $\phi-p^*\psi\in\mathtt{ker}(t^*)$.

For every $r\in\mathbb{N}$ we apply Proposition 6.5 to the fibration $p_{k+r}\colon Y_{k+r}\to X_{k+r}$ in order to see that p_{k+r}^* induces an isomorphism

$$\frac{K^*(X_{k+r})}{\mathtt{ker}((p_{k+r}\circ t_{k+r})^*)}\xrightarrow{\sim}\frac{K^*(Y_{k+r})}{\mathtt{ker}(t_{k+r}^*)}.$$

We have a projective system of exact sequences indexed by $r\in\mathbb{N}$,

$$0\to\mathtt{ker}((p_{k+r}\circ t_{k+r})^*)\to K^*(X_{k+r})\to\frac{K^*(Y_{k+r})}{\mathtt{ker}(t_{k+r}^*)}\to 0.$$

If we take the limit, identify

$$\frac{\mathtt{lim}_r K^*(X_{k+r})}{\mathtt{lim}_r\mathtt{ker}((p_{k+r}\circ t_{k+r})^*)}\cong\frac{\bar{K}^*(X)}{\mathtt{ker}((p\circ t)^*)}$$

and use the assumption that $\mathtt{lim}_r^1 K^*(X_{k+r})=0$, then we get the exact sequence

$$0\to\frac{\bar{K}^*(X)}{\mathtt{ker}((p\circ t)^*)}\to\mathtt{lim}_r\frac{K^*(Y_{k+r})}{\mathtt{ker}(t_{k+r}^*)}\to\mathtt{lim}_r^1\mathtt{ker}((p_{k+r}\circ t_{k+r})^*)\to 0.$$

Note that the graded groups

$$\mathtt{lim}_r\frac{K^*(Y_{k+r})}{\mathtt{ker}(t_{k+r}^*)}\cong\bigcap_{r\in\mathbb{N}}\mathtt{im}(t_{k+r}^*)\quad\text{and}\quad\mathtt{ker}((p_{k+r}\circ t_{k+r})^*)$$

are of finite type, since they are subgroups of the groups of finite type $K^*(T)$ and $K^*(X_{k+r})$, respectively.

Fact 6.10 [34] *If $(G_r)_{r\in\mathbb{N}}$ is a projective system of countable abelian groups, then either $\mathtt{lim}^1_r G_r = 0$ or $\mathtt{lim}^1_r G_r$ is uncountable.*

Since $\mathtt{lim}^1_r \mathtt{ker}((p_{k+r}\circ t_{k+r})^*)$ is a quotient of a countable group we conclude that this group is actually trivial. The pull-back along the family of maps $(p\circ i_{k+r})_{r\in\mathbb{N}}$ induces an isomorphism

$$\frac{\overline{K}^*(X)}{\mathtt{ker}((p\circ t)^*)} \xrightarrow{\sim} \mathtt{lim}_r \frac{K^*(Y_{k+r})}{\mathtt{ker}(t^*_{k+r})}.$$

The element ϕ represents a class $[\phi]\in \mathtt{lim}_r K^*(Y_{k+r})/\mathtt{ker}(t^*_{k+r})$. Hence we can find a class $[\psi]\in \overline{K}^*(X)/\mathtt{ker}((p\circ t)^*)$ which is mapped to $[\phi]$ under the isomorphism above. The elements ϕ and ψ satisfy the relation (143). This finishes the proof of Proposition 6.1. □

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Fakultät für Mathematik, Universität Regensburg
D-93040 Regensburg, Germany

`ulrich.bunke@mathematik.uni-regensburg.de`

Proposed: Ralph Cohen — Received: 8 November 2013
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