

# On representation varieties of 3–manifold groups

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We prove universality theorems (“Murphy’s laws”) for representation varieties of fundamental groups of closed 3–dimensional manifolds. We show that germs of  $\mathrm{SL}(2)$ –representation schemes of such groups are essentially the same as germs of schemes over  $\mathbb{Q}$  of finite type.

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## 1 Introduction

In this paper we will prove that there are no restrictions on local geometry of representation schemes of 3–manifold groups to  $\mathrm{PO}(3)$  and  $\mathrm{SL}(2)$ . Note that both groups  $H = \mathrm{PO}(3)$  and  $H = \mathrm{SL}(2)$  are affine algebraic group schemes defined over  $\mathbb{Q}$ ; thus, for every finitely generated group  $\Gamma$ , the representation schemes

$$\mathrm{Hom}(\Gamma, H)$$

and character schemes

$$X(\Gamma, H) = \mathrm{Hom}(\Gamma, H) // H$$

are affine algebraic schemes over  $\mathbb{Q}$ . Our goal is to show that, *to some extent*, these are the only restrictions on local geometry of the representation and character schemes of fundamental groups of closed 3–manifolds. The universality theorem we thus obtain is one of many universality theorems about moduli spaces of geometric objects; see Mnëv [11], Richter-Gebert [15], Kapovich and Millson [6; 7; 8], Vakili [18], Payne [13], Rapinchuk [14].

Below is the precise formulation of our universality theorem. In what follows we use the notation  $G = \mathrm{PO}(3)$  and  $\tilde{G} = \mathrm{Spin}(3)$ .

**Theorem 1.1** *Let  $X \subset \mathbb{C}^N$  be an affine algebraic scheme over  $\mathbb{Q}$  and let  $x \in X$  be a rational point. Then there exist*

1. *an open subscheme  $X' \subset X$  containing  $x$ ,*
2. *a closed 3–dimensional manifold  $M$  with fundamental group  $\pi$ ,*

3. a representation  $\rho_0: \pi \rightarrow \text{PO}(3, \mathbb{R})$ , such that the image of  $\rho_0$  is dense in  $\text{PO}(3, \mathbb{R})$ ,
4. an open  $G$ -invariant subscheme  $R' \subset \text{Hom}(\pi, G)$ , whose set of real points contains  $\rho_0$  and a closed subscheme (over  $\mathbb{Q}$ )  $R'_c \subset R'$  that is a cross-section for the action

$$G \times R' \rightarrow R',$$

5. an isomorphism of schemes over  $\mathbb{Q}$

$$f: R'_c \rightarrow X' \times G^k \quad \text{with } f(\rho_0) = (x, 1),$$

for some  $k$ , and

6. an isomorphism  $F: R' \rightarrow X' \times G^{k+1}$  and the composition of isomorphisms

$$R' \cong R'_c \times G \cong X' \times G^{k+1}$$

(since  $R'_c$  is a cross-section).

**Remark 1.2** One can show that the same theorem holds for a homomorphism  $\rho_1$  whose image is a finite group with trivial centralizer in  $\text{PO}(3, \mathbb{R})$ .

**Theorem 1.1** is proven in [Section 6](#). In [Section 7](#) we prove various corollaries of our main theorem.

**Corollary 1.3** *With the notation of [Theorem 1.1](#), there exists an open embedding of schemes*

$$X' \times G^k \hookrightarrow X(\pi, G) = \text{Hom}(\pi, G) // G$$

which sends  $(x, 1)$  to  $[\rho_0]$ . In particular, the analytic germ  $(X \times \mathbb{C}^{3k}, x \times 0)$  is isomorphic to the analytic germ  $(X(\pi, G), [\rho_0])$ .

Since the groups  $\text{PSL}(2, \mathbb{C})$  and  $\text{PO}(3, \mathbb{C}) = G(\mathbb{C})$  are isomorphic, and

$$\tilde{G}(\mathbb{C}) = \text{Spin}(3, \mathbb{C}) \cong \text{SL}(2, \mathbb{C}),$$

the “universality theorem” for  $\text{PO}(3)$ -representations leads to the one for  $\text{SL}(2)$ -representations. In the next corollary,  $\pi$  and  $\rho_0$  are the 3-manifold group and its representation to  $\text{PO}(3, \mathbb{R})$  constructed in [Theorem 1.1](#) given  $X$  and  $x$ . We note that the action of  $\text{SL}(2)$  on  $\text{Hom}(\tilde{\pi}, \text{SL}(2))$  factors through an action of  $\text{PSL}(2)$ .

**Corollary 1.4** *Let  $X \subset \mathbb{C}^N$  be an affine algebraic scheme over  $\mathbb{Q}$  and  $x \in X$  be a rational point. Then there exist*

1. an open subscheme  $X' \subset X$  containing  $x$ ,
2. a closed 3–dimensional manifold  $\tilde{M}$  with fundamental group  $\tilde{\pi}$ ,
3. a representation  $\tilde{\rho}_0: \tilde{\pi} \rightarrow \text{SU}(2) < \text{SL}(2, \mathbb{C})$ , such that the image of  $\tilde{\rho}_0$  is dense in  $\text{SU}(2)$ ,
4. an open  $\text{SL}(2)$ –invariant subscheme  $\tilde{R}' \subset \text{Hom}(\tilde{\pi}, \text{SL}(2))$ , such that  $\tilde{\rho}_0 \in \tilde{R}'(\mathbb{C})$ ,
5. a cross-section  $\tilde{R}'_c$  for the action of  $\text{PSL}(2)$  on  $\tilde{R}'$ ,
6. a regular étale covering of schemes over  $\mathbb{C}$ , equivariant with respect to the action of  $\text{SL}(2)$ ,

$$\tilde{q}: \tilde{R}' \rightarrow R' \quad \text{with} \quad \tilde{q}(\tilde{\rho}_0) = \rho_0,$$

such that the deck-transformation group of this cover is isomorphic to  $\mathbb{Z}_2^{k+r}$ , for some  $k$  and  $r$ .

By combining part 6 with [Theorem 1.1](#), we also obtain an  $\text{SL}(2)$ –equivariant regular étale covering

$$\tilde{R}' \rightarrow X' \times \text{PSL}(2)^{k+1},$$

and the étale covering

$$\tilde{R}'_c \rightarrow X' \times \text{PSL}(2)^k$$

sending  $\tilde{\rho}_0$  to  $(x, 1)$ . The latter yields the regular étale covering

$$h: \tilde{R}' // \text{SL}(2) \rightarrow X' \times \text{PSL}(2)^k.$$

In particular, the morphisms  $\tilde{q}$  and  $F$  (from [Theorem 1.1](#)) and  $h$  (as above) yield isomorphisms of analytic germs

$$\begin{aligned} (\text{Hom}(\tilde{\pi}, \text{SL}(2)), \tilde{\rho}_0) &\rightarrow (X' \times \mathbb{C}^{3k+3}, x \times 0), \\ (X(\tilde{\pi}, \text{SL}(2)), [\tilde{\rho}_0]) &\rightarrow (X' \times \mathbb{C}^{3k}, x \times 0), \end{aligned}$$

for some  $k \geq 0$ . Thus, if the scheme  $X'$  is nonreduced at  $x$ , so are  $\text{Hom}(\tilde{\pi}, \text{SL}(2))$  and  $X(\tilde{\pi}, \text{SL}(2))$ .

**Remark 1.5** Despite our efforts, we were unable to replace an étale covering with an isomorphism in [Corollary 1.4](#). This is strangely reminiscent of the finite abelian coverings appearing in our universality theorem for planar linkages; see Kapovich and Millson [8]. Note that a relation between universality theorems for projective arrangements and spherical linkages was established in Kapovich and Millson [7], where a finite abelian covering appeared for essentially the same reason as in the present paper.

**Example 1.6** Pick a natural number  $\ell$ . Then there exists a closed 3–dimensional manifold  $\tilde{M}$ , an integer  $n$  and a representation  $\rho: \pi_1(\tilde{M}) \rightarrow \text{SU}(2)$  with dense image,

such that the completed local ring of the germ

$$X(\pi_1(\tilde{M}), \mathrm{SL}(2)), [\rho]$$

is isomorphic to the completion of the ring

$$\mathbb{C}[t, t_1, \dots, t_{3k}]/(t^\ell).$$

This shows that the representation and character schemes of 3-manifold groups can be nonreduced (at points of Zariski density), which is why we refrain from referring to these schemes as “varieties”, as is commonly done in the literature.

**Remark 1.7** Recently Igor Rapinchuk [14, Theorem 3] proved a universality theorem for character schemes of groups  $\Gamma$  satisfying Kazhdan’s property (T): it involves representations of such groups  $\Gamma$  into  $\mathrm{SL}(n, \mathbb{C})$ . Unlike the results in Kapovich and Millson [6] and this paper, Rapinchuk’s theorem applies to the entire character variety  $X^{\mathrm{red}}(\Gamma, \mathrm{SL}(n, \mathbb{C}))$  minus the trivial representation (which is an isolated point). In Rapinchuk’s theorem, the number  $n$  (and the group  $\Gamma$ ) depend on the given affine variety  $X$  over  $\mathbb{Q}$ .

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## 2 Preliminaries

### 2.1 Representation and character schemes

We will say that a subscheme  $Y \subset X$  is *clopen* if it is both closed and open. We will use the topologist’s notation

$$\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$$

for the cyclic group of order  $m$ . Given a subset  $S$  of a group  $G$  we will use the notation  $\langle\langle S \rangle\rangle$  for the normal closure of  $S$  in  $G$ .

Let  $G$  be an algebraic group scheme over a field  $k$  of characteristic zero (this will be the default assumption through the rest of the paper) with Lie algebra  $\mathfrak{g}$ . Let  $\Gamma$  be a finitely presented group with presentation

$$\langle s_1, \dots, s_p \mid r_1 = 1, \dots, r_q = 1 \rangle.$$

(In fact, one needs  $\Gamma$  only to be finitely generated, but all finitely generated groups in

this paper will be also finitely presented.) Every word  $w$  in the generators  $s_i$  and their inverses  $s_i^{-1}$ , for  $i = 1, \dots, p$ , defines a morphism

$$w: G^p \rightarrow G,$$

obtained by substituting elements  $g_1^{\pm 1}, \dots, g_p^{\pm 1} \in G$  in the word  $w$  for the letters  $s_1^{\pm 1}, \dots, s_p^{\pm 1}$ . We then obtain the *representation scheme*

$$\text{Hom}(\Gamma, G) = \{(g_1, \dots, g_p) \in G^p \mid r_j(g_1, \dots, g_p) = 1, j = 1, \dots, q\},$$

as every homomorphism  $\Gamma \rightarrow G$  is determined by its values on the generators of  $\Gamma$ . We will thus think of points of this scheme as homomorphisms  $\rho: \Gamma \rightarrow G$ . The representation scheme is known to be independent of the presentation of the group  $\Gamma$ . We refer the reader to [10; 17] for more details. We also refer the reader to [16; 17] for detailed discussion of character varieties/schemes and a survey of their applications to 3-dimensional topology.

We will frequently use the following two facts about representation schemes; see eg [17]:

1.  $\text{Hom}(\Gamma_1 \star \dots \star \Gamma_k, G) \cong \prod_{i=1}^k \text{Hom}(\Gamma_i, G)$ .
2. For each  $\rho \in \text{Hom}(\Gamma, G(\mathbf{k}))$  satisfying  $H^1(\Gamma, \mathfrak{g}_{\text{Ad } \rho}) = 0$ , the point  $\rho$  is a smooth point of the scheme  $\text{Hom}(\Gamma, G)$ . The  $G$ -orbit through  $\rho$  is open in  $\text{Hom}(\Gamma, G)$ .

In what follows we will use the simplified notation  $H^q(\Gamma, \text{Ad } \rho)$  instead of  $H^q(\Gamma, \mathfrak{g}_{\text{Ad } \rho})$ .

We assume from now on that the group  $G$  is affine; in particular,  $\text{Hom}(\Gamma, G)$  is also an affine scheme. The group  $G$  acts naturally on this scheme:

$$(g, \rho) \mapsto \rho^g \quad \text{where } \rho^g(\gamma) = g\rho(\gamma)g^{-1}.$$

Assuming, in addition, that  $G$  is reductive, we obtain the GIT quotient

$$X(\Gamma, G) = \text{Hom}(\Gamma, G) // G,$$

which is a scheme of finite type known as the *character scheme* (or, more commonly, as the *character variety*). However, as we will see, both representation and character schemes can be nonreduced, so we will avoid the traditional *representation/character variety* terminology.

We will use the notation

$$\text{Hom}^{\text{red}}(\Gamma, G) \quad \text{and} \quad X^{\text{red}}(\Gamma, G)$$

to denote the varieties which are the reductions of the schemes

$$\text{Hom}(\Gamma, G) \quad \text{and} \quad X(\Gamma, G).$$

Recall that for every  $\rho \in \text{Hom}(\Gamma, G)$ , the vector space of cocycles

$$Z^1(\Gamma, \text{Ad } \rho)$$

is isomorphic to the Zariski tangent space  $T_\rho \text{Hom}(\Gamma, G)$  and this isomorphism carries the subspace of coboundaries  $B^1(\Gamma, \text{Ad } \rho)$  to the tangent space of the  $G$ -orbit through  $\rho$ . Note, however, that  $H^1(\Gamma, \text{Ad } \rho)$  is *not* always isomorphic to the Zariski tangent space of  $[\rho] \in X(\Gamma, G)$ ; see [2, Section 6] as well as [17].

Suppose now that the group  $\Phi$  is finite. Then for every  $\rho \in \text{Hom}(\Phi, G)$ ,

$$H^1(\Phi, \text{Ad } \rho) = 0.$$

(Furthermore,  $H^i(\Phi, \text{Ad } \rho) = 0$  for  $i \geq 1$ .) In particular, the  $G$ -orbit of  $\rho$  is a clopen (closed and open) subscheme

$$\text{Hom}_\rho(\Phi, G) \subset \text{Hom}(\Phi, G).$$

This subscheme is isomorphic to the quotient  $G/\zeta_G(\rho(\Phi))$ , where  $\zeta_G(H)$  denotes the centralizer of the subgroup  $H$  in  $G$ . (Note that if  $\zeta_G(\rho(\Phi))$  equals the center of  $G$ , then the point  $[\rho] \in X(\Phi, G)$  is a reduced isolated point in the character scheme and the entire character scheme is smooth.) We obtain:

**Lemma 2.1** *For every finite group  $\Phi$  and connected affine group  $G$ , the scheme  $\text{Hom}(\Phi, G)$  is smooth and each of its irreducible components is  $G$ -homogeneous. These irreducible components are the open subschemes  $\text{Hom}_\rho(\Phi, G)$ . If the representation  $\rho$  is trivial, then  $\text{Hom}_\rho(\Phi, G)$  is a single point.*

The following lemma is also immediate:

**Lemma 2.2** *Let  $\phi: \Gamma \rightarrow \Gamma'$  be a group homomorphism. Then the pull-back map  $\phi^*(\rho) = \rho \circ \phi$  is a morphism of schemes*

$$\text{Hom}(\Gamma', G) \rightarrow \text{Hom}(\Gamma, G).$$

**Lemma 2.3** *Let  $\Gamma$  be a finitely presented group and let  $\Theta \subset \Gamma$  be a finite subset with the quotient group*

$$\Gamma' := \Gamma / \langle\langle \Theta \rangle\rangle.$$

*Let  $\phi: \Gamma \rightarrow \Gamma'$  denote the projection homomorphism. Then the pull-back morphism*

$$\phi^*: \text{Hom}(\Gamma', G) \rightarrow \text{Hom}_\Theta(\Gamma, G)$$

*is an isomorphism, where*

$$\text{Hom}_\Theta(\Gamma, G) \subset \text{Hom}(\Gamma, G)$$

is the closed subscheme defined by

$$\text{Hom}_\Theta(\Gamma, G) = \{\rho \in \text{Hom}(\Gamma, G) \mid \rho(\theta) = 1 \text{ for all } \theta \in \Theta\}.$$

**Proof** Given a finite presentation  $P$  of  $\Gamma$  let  $P'$  be the presentation of  $\Gamma'$  obtained from  $P$  by adding words representing elements of  $\Theta$  as the relators. Then the assertion follows immediately from the definition of the representation scheme of a group using a group presentation.  $\square$

**Corollary 2.4** *Suppose that every element  $\theta \in \Theta$  has finite order. Then the isomorphism  $\phi_*: \text{Hom}(\Gamma', G) \rightarrow \text{Hom}_\Theta(\Gamma, G)$  sends  $\text{Hom}(\Gamma', G)$  to the open subscheme  $\text{Hom}_\Theta(\Gamma, G) \subset \text{Hom}(\Gamma, G)$ .*

**Proof** Consider an element  $\theta \in \Theta$  and the trivial representation  $\rho_{0,\theta}: \langle \theta \rangle \rightarrow G$ . By Lemma 2.1, the singleton

$$\{\rho_{0,\theta}\} = \text{Hom}_{\rho_{0,\theta}}(\langle \theta \rangle, G) \subset \text{Hom}(\langle \theta \rangle, G)$$

is a reduced isolated point in the scheme  $\text{Hom}(\langle \theta \rangle, G)$ . In particular, this point is open in  $\text{Hom}(\langle \theta \rangle, G)$ . We have the pull-back morphism

$$\phi_\theta^*: \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\langle \theta \rangle, G),$$

induced by the inclusion homomorphism  $\phi_\theta: \langle \theta \rangle \hookrightarrow \Gamma$ . Therefore,

$$(\phi_\theta^*)^{-1}(\text{Hom}_{\rho_{0,\theta}}(\langle \theta \rangle, G)) \subset \text{Hom}(\Gamma, G)$$

is an open subscheme. Furthermore, by the definition of  $\text{Hom}_\Theta(\Gamma, G)$ ,

$$\text{Hom}_\Theta(\Gamma, G) = \bigcap_{\theta \in \Theta} (\phi_\theta^*)^{-1}(\text{Hom}_{\rho_{0,\theta}}(\langle \theta \rangle, G)).$$

(A homomorphism belongs to  $\text{Hom}_\Theta(\Gamma, G)$  if and only if it sends each  $\theta \in \Theta$  to  $1 \in G$ .) Therefore,  $\text{Hom}_\Theta(\Gamma, G)$  is also open in  $\text{Hom}(\Gamma, G)$ .  $\square$

## 2.2 Coxeter groups

We refer the reader to [3] for the basics of Coxeter groups.

Let  $\Delta$  be a finite simplicial graph with the vertex and edge sets denoted  $V = V(\Delta)$  and  $E = E(\Delta)$ , respectively. We will use the notation  $e = [v, w]$  for the edge of  $\Delta$  connecting  $v$  and  $w$ , if it exists. We assume also that we are given a function

$$m: E \rightarrow \mathbb{N} \quad \text{with } m(e) \geq 2 \text{ for all } e \in E$$

labeling the edges of  $\Delta$ . We will call the pair  $(\Delta, m)$  a *labeled graph* or a *Coxeter*

graph. Given a labeled graph, we define the associated Coxeter group  $\Gamma = \Gamma_\Delta$  by the presentation

$$\langle \{g_v\}_{v \in V} \mid g_v^2 = 1 \ \forall v \in V, \underbrace{g_v g_w \cdots}_{m(e) \text{ terms}} = \underbrace{g_w g_v \cdots}_{m(e) \text{ terms}} \ \forall v, w \in V \text{ for which } e = [v, w] \in E \rangle.$$

Alternatively, one can describe the relators of this group as  $g_v^2 = 1$  and

$$(g_v g_w)^m = 1$$

where  $m = m(e)$ ,  $e = [v, w]$ .

**Remark 2.5** Note that the notation we use here is different from the one in Lie theory, where two generators commute whenever the corresponding vertices are not connected by an edge. In our notation, every such pair of elements of  $\Gamma$  generates an infinite dihedral subgroup of  $\Gamma$ .

We also define the *canonical central extension*

$$(1) \quad 1 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{\Gamma} \xrightarrow{\eta} \Gamma \rightarrow 1$$

of the group  $\Gamma$ , with the *extended Coxeter group*  $\tilde{\Gamma} = \tilde{\Gamma}_\Delta$  given by the presentation

$$\langle z, \{g_v\}_{v \in V} \mid z^2 = 1, [g_v, z] = 1, g_v^2 = z \ \forall v \in V, \underbrace{g_v g_w \cdots}_{m(e) \text{ terms}} = z^{m(e)+1} \underbrace{g_w g_v \cdots}_{m(e) \text{ terms}} \ \forall v, w \in V \text{ for which } e = [v, w] \in E \rangle.$$

The number  $r = |V|$  (the cardinality of  $V$ ) is called the *rank* of  $\Gamma$  and  $\tilde{\Gamma}$ . We will refer to the generator  $z$  of the group  $\Gamma$  as *the central element* of  $\Gamma$ , even though, the center of  $\tilde{\Gamma}$  might be larger than  $\mathbb{Z}_2$ : this happens precisely when  $\Delta$  consists of a single vertex.

A subgraph  $\Sigma \subset \Delta$  is called *full* if for every pair of vertices  $v, w \in \Sigma$ , the edge  $[v, w]$  in  $\Delta$  also belongs to  $\Sigma$ . Every subgraph  $\Sigma \subset \Delta$  inherits labels from  $\Delta$ . For the new labeled graph (which we still denote  $\Sigma$ ), we have the natural homomorphism

$$\iota_\Sigma: \Gamma_\Sigma \rightarrow \Gamma_\Delta$$

sending each generator  $g_v \in \Gamma_\Sigma$ , where  $v \in V(\Sigma)$ , to the generator of  $\Gamma_\Delta$  with the same name. It is immediate that the homomorphism  $\iota_\Sigma$  lifts to a homomorphism

$$\tilde{\iota}_\Sigma: \tilde{\Gamma}_\Sigma \rightarrow \tilde{\Gamma}_\Delta$$

sending each  $g_v$  to itself ( $v \in V(\Sigma)$ ) and the central element  $z \in \tilde{\Gamma}_\Sigma$  to the central element  $z \in \tilde{\Gamma}_\Delta$ . We will use this construction in two special cases:



a.  $\Sigma := \Delta_\emptyset$  is the subgraph which has the same vertex set as  $\Gamma$ , but empty edge set. Then

$$\Gamma_\Sigma \cong F_r \quad \text{and} \quad \tilde{\Gamma}_\Sigma \cong F_r \times \mathbb{Z}_2,$$

where  $F_r$  is the free group on  $r$  generators.

b.  $\Sigma \subset \Delta$  is a full subgraph. In this case, the homomorphism  $\iota_\Sigma$  is injective; see eg [3, page 113]. It follows that the homomorphism  $\tilde{\iota}_\Sigma$  is injective as well.

For full subgraphs  $\Sigma \subset \Delta$ , the subgroups  $\iota_\Sigma(\Gamma_\Sigma) < \Gamma_\Delta$  and  $\tilde{\iota}_\Sigma(\tilde{\Gamma}_\Sigma) < \tilde{\Gamma}_\Delta$  are called *parabolic* subgroups of  $\Gamma_\Delta$  and  $\tilde{\Gamma}_\Delta$ , respectively. We say that a parabolic subgroup of  $\Gamma_\Delta$  or  $\tilde{\Gamma}_\Delta$  is *elementary*, if it is a finite parabolic subgroup of rank  $\leq 2$ . The latter requirement simply means that  $\Sigma$  consists of at most two vertices; the finiteness condition means that if  $\Sigma$  consists of two vertices, then these vertices are connected by an edge. We will refer to such subgraphs  $\Sigma$  as *elementary* as well.

**Example 2.6** 1. If  $\Delta$  consists of a single edge  $e$  labeled 2, then  $\Gamma_\Delta \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and

$$\tilde{\Gamma}_\Delta \cong Q_8,$$

the finite quaternion group.

2. If the edge  $e$  is labeled 4, then  $\Gamma_\Delta$  is the dihedral group  $I_2(4)$  of order 8; it admits an epimorphism

$$\Gamma_\Delta \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

with kernel the center of  $\Gamma_\Delta$ , which is generated by the involution  $g_v g_w g_v g_w$ .

### 3 Representations of Coxeter groups and extended Coxeter groups

In this section we prove some basic facts about representations of Coxeter and extended Coxeter groups to  $\text{PSL}(2, \mathbb{C})$  and  $\text{SL}(2, \mathbb{C})$ , respectively.

#### 3.1 Representations of elementary Coxeter groups

Recall that the quotient map

$$p: \tilde{G}(\mathbb{C}) = \text{SL}(2, \mathbb{C}) \rightarrow G(\mathbb{C}) = \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1\},$$

is a 2-fold covering. The extended Coxeter groups appear naturally in the context of lifting homomorphisms of Coxeter groups from  $\text{PSL}(2, \mathbb{C})$  to  $\text{SL}(2, \mathbb{C})$ .

Consider the labeled graph  $\Delta$ , consisting of two vertices  $v, w$  and the edge  $[v, w]$  labeled  $m \geq 2$ . The corresponding Coxeter group  $\Gamma_\Delta$  is a finite dihedral group, usually denoted  $I_2(m)$ . This group is isomorphic to the subgroup of the group of symmetries of a regular planar  $2m$ -gon, acting simply transitively on the set of edges of this polygon. Hence, this group embeds canonically (up to conjugation) into  $O(2)$  and thus into  $PO(3, \mathbb{R}) \cong SO(3, \mathbb{R}) < PSL(2, \mathbb{C})$ . If  $m$  is odd, then such a group of symmetries will lift isomorphically to a subgroup of  $SU(2) < SL(2, \mathbb{C})$ . In contrast, we will be interested (only) in the case when  $m$  is even; in fact, we will be using only Coxeter graphs with the labels  $m = 2$  and  $m = 4$  in this paper.

Below we will prove several lemmas about faithful representations of elementary Coxeter (and extended Coxeter) groups into  $G(\mathbb{C})$  (and  $\tilde{G}(\mathbb{C})$ ).

- Lemma 3.1**
1. *There exists, unique up to conjugation, a faithful representation  $\rho: \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow G(\mathbb{C})$ .*
  2. *There are no faithful representations  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \tilde{G}(\mathbb{C})$ .*

**Proof** 1. As the image of  $\rho$  is finite, it is conjugate to a subgroup of  $SO(3, \mathbb{R}) < G(\mathbb{C})$ . Part 1 now follows from the fact that the group  $SO(3, \mathbb{R})$  acts transitively on the set of pairs of orthogonal 1-dimensional subspaces of  $\mathbb{R}^3$  (these subspaces, in our setting, are fixed lines of the images of the direct factors of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  under  $\rho$ ).

2. Part 2 follows from the fact that any involution  $A \in SL(2, \mathbb{C})$  has both eigenvalues equal to  $-1$ , ie  $A$  equals  $-1 \in SL(2, \mathbb{C})$ . □

The next lemma and proposition generalize [Lemma 3.1](#) to representations of the dihedral group  $I_2(4)$ .

**Lemma 3.2** *All injective representations  $\rho: \Gamma = I_2(4) \rightarrow G(\mathbb{C})$  are conjugate to each other.*

**Proof** Since the group  $\Gamma$  is finite, its image in  $G(\mathbb{C})$  lies in a conjugate of the maximal compact subgroup  $SO(3, \mathbb{R}) < G(\mathbb{C})$ . Thus, we will assume that  $\rho(\Gamma)$  is contained in  $SO(3, \mathbb{R})$ . Since the product of the generating involutions  $\rho(g_v), \rho(g_w)$  of  $\rho(\Gamma)$  has order 4, the fixed lines of  $\rho(g_v), \rho(g_w)$  meet at the angle  $\frac{\pi}{4}$  in  $\mathbb{R}^3$ . Now, the assertion follows from the fact that  $SO(3, \mathbb{R})$  acts transitively on the set of 1-dimensional subspaces in  $\mathbb{R}^3$  meeting at the given angle. □

**Proposition 3.3** *Consider the dihedral group  $I_2(2m) = \Gamma_\Delta$  and its isomorphism*

$$\rho: \Gamma_\Delta \rightarrow \Gamma < G(\mathbb{C}).$$

*Then:*

1. For every choice of matrices  $\tilde{g}_u \in \tilde{G}(\mathbb{C})$  projecting to the generators  $\rho(g_u) \in \Gamma < G(\mathbb{C})$ , the map

$$g_u \rightarrow \tilde{g}_u \quad \text{for } u \in \{v, w\}$$

extends to a monomorphism  $\tilde{\rho}: \tilde{\Gamma}_\Delta \rightarrow \tilde{G}(\mathbb{C})$ .

2. The centralizer of the group  $\tilde{\rho}(\tilde{\Gamma}_\Delta)$  in  $\tilde{G}(\mathbb{C})$  equals the center of  $\tilde{G}(\mathbb{C})$ .

**Proof** The proof amounts to elementary linear algebra; we include the details for the sake of completeness. For notational convenience we will identify the isomorphic groups  $\Gamma$  and  $\Gamma_\Delta$ . After conjugating the subgroup  $\Gamma$  in  $G(\mathbb{C})$ , we can (and will) assume that  $\Gamma$  lies in the subgroup  $SO(3, \mathbb{R}) < G(\mathbb{C})$ . The orthogonal subgroup is covered by the unitary subgroup  $SU(2) < SL(2, \mathbb{C})$ . We let  $Z(SU(2)) \cong \mathbb{Z}_2$  denote the center of  $SU(2)$ ; this center consists of the matrices  $\pm 1$ .

We begin with several trivial observations. Since  $\rho$  is injective, the involutions  $g_v, g_w$  are distinct rotations in  $SO(3, \mathbb{R})$ . In particular, their fixed-point sets in  $\mathbb{C}P^1$  are pairwise disjoint. Suppose the elements  $\tilde{g}_u, \tilde{g}_v \in SU(2)$  project to  $g_v, g_w$ , respectively. Since the kernel of the covering  $\tilde{G} \rightarrow G$  is isomorphic to  $\mathbb{Z}_2$ , the unitary transformations  $\tilde{g}_v, \tilde{g}_w \in SU(2)$  have order at most 4:

$$\tilde{g}_u^2 \in Z(SU(2)) \quad \text{for } u \in \{v, w\}.$$

Note that the only involution in  $SU(2)$  is the matrix  $-1$ . Since  $\tilde{g}_u$  projects nontrivially to  $SO(3)$ , this matrix cannot be an involution. It follows that

$$\tilde{g}_u^2 = -1 \in SU(2) \quad \text{for } u \in \{v, w\}.$$

The eigenvalues of the matrices  $\tilde{g}_v, \tilde{g}_w$  have to be roots of unity of the order 4, which implies that the spectrum of each matrix  $\tilde{g}_u$ , where  $u \in \{v, w\}$ , equals  $\{i, -i\}$ .

We next claim that the eigenspaces of unitary transformations  $\tilde{g}_v, \tilde{g}_w$  are pairwise distinct. If not, then these matrices would be simultaneously diagonalizable, which would imply that their projections to  $PSL(2, \mathbb{C})$  are equal. (Two involutions in  $SO(3, \mathbb{R})$  which have same fixed-point sets have to be the same.)

Suppose now that  $A \in SL(2, \mathbb{C})$  is a matrix centralizing the subgroup  $\langle \tilde{g}_v, \tilde{g}_w \rangle$  generated by  $\tilde{g}_v, \tilde{g}_w$ . We claim that  $A$  is a scalar matrix, ie an element of the center of  $SL(2, \mathbb{C})$ . Indeed, since  $A$  commutes with both  $\tilde{g}_v, \tilde{g}_w$ , it has to preserve the eigenspaces of each matrix  $\tilde{g}_v, \tilde{g}_w$ . (Here we are using the fact that the eigenvalues of  $\tilde{g}_u$  are distinct for  $u \in \{v, w\}$ .) However, a nonscalar matrix in  $SL(2, \mathbb{C})$  cannot have three distinct eigenlines. Therefore,  $A$  is a scalar matrix. This implies the second claim of the lemma.

The generators  $g_v, g_w$  satisfy

$$t = (g_v g_w)^m = (g_w g_v)^m,$$

where  $t$  is an order-2 element, which belongs to the center of  $\Gamma$ . (In the geometric realization of  $\Gamma_\Sigma$  as a group of symmetries of a regular  $2m$ -gon, the element  $t$  corresponds to the order-2 rotation, the central symmetry of the polygon.)

If we had the relation

$$(\tilde{g}_v \tilde{g}_w)^m = (\tilde{g}_w \tilde{g}_v)^m,$$

it would result in the monomorphism

$$\alpha: \Gamma \rightarrow \text{SU}(2) \quad \text{with } \alpha(g_u) = \tilde{g}_u \text{ for } u \in \{v, w\},$$

lifting the embedding  $\rho: \Gamma \hookrightarrow \text{SO}(3, \mathbb{R})$ . The image of the center  $Z(\Gamma)$  of  $\Gamma$  would then be in the center of  $\alpha(\Gamma)$ , hence, as we noted above, in the center of  $\text{SU}(2)$ . Then, the composition  $\rho = p \circ \alpha$  would send  $Z(\Gamma)$  to 1, which is a contradiction.

This leaves us with the only possibility

$$(\tilde{g}_v \tilde{g}_w)^m = -(\tilde{g}_w \tilde{g}_v)^m.$$

To conclude, the map given by

$$g_v \mapsto \tilde{g}_v, \quad z \mapsto -1 \in \text{SL}(2, \mathbb{C}),$$

extends to an homomorphism  $\tilde{\Gamma}_\Delta \rightarrow p^{-1}(\Gamma)$ , sending the central element  $z \in \tilde{\Gamma}_\Delta$  to the matrix  $-1 \in \text{SL}(2, \mathbb{C})$ . Injectivity of this homomorphism follows from injectivity of the representation  $\Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ . □

### 3.2 Representations faithful on elementary subgroups

For a Coxeter group  $\Gamma = \Gamma_\Delta$  we define two subschemes

$$\text{Hom}_o(\Gamma, G) \subset \text{Hom}(\Gamma, G) \quad \text{and} \quad \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \subset \text{Hom}(\tilde{\Gamma}, \tilde{G}).$$

The former consists of homomorphisms which are injective on every elementary subgroup of  $\Gamma$ ; the latter consists of homomorphisms which are injective on every elementary subgroup of  $\tilde{\Gamma}$  and send  $z \in \tilde{\Gamma}$  to  $-1 \in \text{SL}(2, \mathbb{C})$ . (In fact, the requirement for  $z$  follows from faithfulness on elementary subgroups, except when  $\Delta$  has no edges.) Since elementary subgroups of  $\Gamma$  and  $\tilde{\Gamma}$  are finite, both  $\text{Hom}_o(\Gamma, G)$  and  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  are open subschemes of the respective representation schemes. We will see later on that these subschemes are also closed. For each

$$\rho \in \text{Hom}_o(\tilde{\Gamma}, G)$$

we have  $\rho(z) = 1$ , while for each  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  which projects to  $\rho \in \text{Hom}_o(\Gamma, G)$  we have  $\tilde{\rho}(z) = -1$ .

In the paper we will be using the labeled graph  $\Omega$  depicted in Figure 1: This graph has five vertices and nine edges. The edges left unlabeled in the figure all have the label 2. The vertices  $x, y$  are the only ones not connected to each other by an edge.

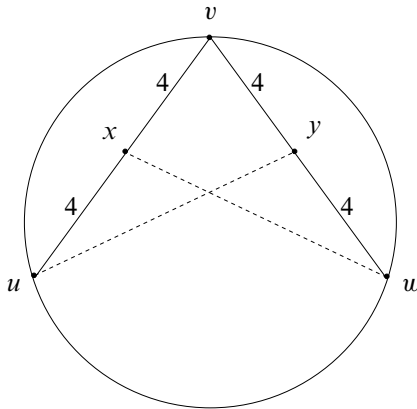


Figure 1: The graph  $\Omega$

In what follows, we will also use the subgraph  $\Upsilon \subset \Omega$ , which is the complete graph on the vertices  $u, v, w$ . The parabolic subgroup  $\Gamma_\Upsilon < \Gamma_\Omega$  is isomorphic to  $\mathbb{Z}_2^3$ . Since the group  $\Gamma_\Upsilon$  is finite, the representation scheme  $\text{Hom}(\Gamma_\Upsilon, G)$  is smooth.

**Lemma 3.4** *Each representation  $\rho \in \text{Hom}_o(\Gamma, G(\mathbb{C}))$  of the group  $\Gamma = \Gamma_\Upsilon$  satisfies:*

1. *The kernel of  $\rho$  is generated by the subgroup*

$$\langle g_u g_v g_w \rangle \cong \mathbb{Z}_2,$$

*and the image of  $\rho$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .*

2. *The centralizer of the abelian subgroup  $\rho(\Gamma) < G(\mathbb{C})$  in the group  $G(\mathbb{C})$  equals the subgroup  $\rho(\Gamma)$  itself.*
3.  *$\text{Hom}_o(\Gamma, G(\mathbb{C}))$  is the  $G(\mathbb{C})$ -orbit of a singleton  $\{\rho_\Upsilon\}$ .*

**Proof** This lemma is also elementary:

1. Consider a homomorphism  $\rho \in \text{Hom}_o(\Gamma, G(\mathbb{C}))$ . For each element  $\gamma \in \Gamma$  we let  $\text{Fix}(\gamma)$  denote the fixed-point set of  $\rho(\gamma)$  in  $\mathbb{C}\mathbb{P}^1$ . The condition that all three involutions  $\rho(g_u), \rho(g_v), \rho(g_w)$  are distinct implies that the three fixed-point sets  $\text{Fix}(g_u), \text{Fix}(g_v), \text{Fix}(g_w)$  are pairwise disjoint.

**Remark 3.5** If  $g_1, g_2$  are commuting involutions in  $\text{PSL}(2, \mathbb{C})$  with fixed-point sets  $\{\xi_1, \eta_1\}, \{\xi_2, \eta_2\}$ , respectively, and  $\xi_1 = \xi_2$ , then  $g_2 g_1 g_2^{-1} = g_1$  implies that

$$g_2(\{\xi_1, \eta_1\}) = \{\xi_1, \eta_1\}.$$

Since  $g_1(\xi_1) = \xi_1$ , it follows that  $g_2(\eta_1) = \eta_1$ . However, each involution in  $\text{PSL}(2, \mathbb{C})$  is determined by its fixed-point set. Therefore,  $g_1 = g_2$ .

Commutativity of  $\rho(\Gamma)$  implies that this group preserves the six-point set

$$F = \text{Fix}(g_u) \cup \text{Fix}(g_v) \cup \text{Fix}(g_w) \subset \mathbb{CP}^1.$$

The element  $\rho(g_u)$  fixes  $\text{Fix}(g_u)$ , of course, and defines nontrivial involutions of the other two fixed-point sets

$$\text{Fix}(g_v) \rightarrow \text{Fix}(g_v) \quad \text{and} \quad \text{Fix}(g_w) \rightarrow \text{Fix}(g_w).$$

The same applies to  $g_v$  and  $g_w$ . It follows that

$$\rho(g_u g_v)|_F = \rho(g_w)|_F.$$

Hence

$$\rho(g_u g_v) = \rho(g_w),$$

and thus

$$\langle g_u g_v g_w \rangle < \ker(\rho).$$

The equality of these subgroups of  $\Gamma$  follows from the condition that

$$\rho \in \text{Hom}_o(\Gamma, G(\mathbb{C})).$$

This establishes part 1 of the lemma.

2. To prove part 2, note that every  $g \in G$  centralizing  $\rho(\Gamma)$  has to preserve each of the sets  $\text{Fix}(g_u), \text{Fix}(g_v), \text{Fix}(g_w)$ . After composing  $g$  with elements of  $\rho(\Gamma)$ , we achieve that  $g$  fixes the set  $\text{Fix}(g_u) \cup \text{Fix}(g_v)$  pointwise. Therefore,  $g \in \rho(\Gamma)$ . This proves part 2 of the lemma.

3. To prove part 3 note that, by part 1, the pull-back morphism

$$\text{Hom}_o(\mathbb{Z}^2, G(\mathbb{C})) \rightarrow \text{Hom}_o(\Gamma, G(\mathbb{C}))$$

induced by the quotient homomorphism

$$1 \rightarrow \langle g_u g_v g_w \rangle \rightarrow \Gamma \rightarrow \mathbb{Z}^2$$

is surjective. Now, the claim follows from [Lemma 3.1](#). □

- Lemma 3.6**
1.  $\text{Hom}_o(\Gamma_\Omega, G(\mathbb{C}))$  is a single orbit  $G(\mathbb{C}) \cdot \rho_\Omega$ .
  2. The representation  $\rho_\Omega$  is infinitesimally rigid:  $H^1(\Gamma_\Omega, \mathfrak{sl}(2, \mathbb{C})) = 0$ .
  3. For each  $\rho \in \text{Hom}_o(\Gamma_\Omega, G(\mathbb{C}))$ , the adjoint action  $\text{Ad } \rho$  of  $\Gamma_\Omega$  on the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  has no nonzero fixed vectors.

**Proof** 1. Consider  $\rho \in \text{Hom}_o(\Gamma_\Omega, G)$ . In view of Lemma 3.4, we can assume that the restriction of  $\rho$  to the subgroup  $\Gamma_\Upsilon$  equals the representation  $\rho_\Upsilon$ . Consider now the dihedral subgroups

$$\langle g_u, g_x \rangle \quad \text{and} \quad \langle g_x, g_v \rangle$$

in the group  $\Gamma_\Omega$ . It follows from Lemma 3.2 that there are exactly two extensions (which are faithful on all elementary parabolic subgroups) of the representation

$$\rho|_{\langle g_u, g_v \rangle}$$

to the subgroup  $\langle g_u, g_x, g_v \rangle$ . We will denote these extensions  $\rho_+$  and  $\rho_-$ . For both extensions,  $\rho_\pm(g_x)$  lies in  $\text{SO}(3, \mathbb{R})$ , its fixed line in  $\mathbb{R}^3$  is contained in the span of the fixed lines of  $\rho_\Upsilon(g_u)$ ,  $\rho_\Upsilon(g_v)$ . This fixed line makes the angle  $\frac{\pi}{4}$  with the fixed lines of  $\rho_\Upsilon(g_u)$ ,  $\rho_\Upsilon(g_v)$  and is orthogonal to the fixed line of  $\rho(g_w)$ . These representations  $\rho_\pm$  are conjugate via the element  $\rho(g_v) \in \text{SO}(3)$ . Therefore, after such conjugation, we fix the value  $\rho(g_x)$ . We next repeat this argument for the subgroup of  $\Gamma_\Omega$  generated by

$$\{g_v, g_y, g_w\}.$$

Since conjugation via  $\rho(g_w)$  does not alter  $\rho(g_x)$ , we obtain the required uniqueness statement.

2. In what follows we will be using the fact that the adjoint representation of  $\text{PSL}(2, \mathbb{C})$  is isomorphic to the complexification  $V$  of the standard representation of  $\text{SO}(3, \mathbb{R})$  on  $\mathbb{R}^3$ . We will also use the notation  $V$  and  $\mathfrak{sl}(2, \mathbb{C})$  for the representation  $\text{Ad } \rho$  of the group  $\Gamma_\Omega$  (and its subgroups) in the notation for cocycles and cohomology groups. In particular, for each element  $a$  of

$$\{u, v, w, x, y\},$$

the fixed-point set of  $\text{Ad } \rho(g_a)$  is a line in  $V$ , which we will denote by  $V^a$ . An elementary but useful geometric observation is that

$$V^x \subset V^u \oplus V^v,$$

while

$$V = V^u \oplus V^v \oplus V^w = V^u \oplus V^x \oplus V^w.$$

Consider a cocycle  $\xi \in Z^1(\Gamma_\Omega, V)$ . Since  $\Gamma_\Upsilon$  is finite,  $H^1(\Gamma_\Upsilon, V) = 0$ . Since the restriction of  $\xi$  to the subgroup  $\Gamma_\Upsilon$  is a coboundary, by subtracting off a coboundary from  $\xi$ , we can assume that  $\xi$  vanishes in  $\Gamma_\Upsilon$ . Similarly, there exist  $\alpha, \beta \in V$  such that

$$\xi(h) = \alpha - \text{Ad } \rho(h)\alpha \quad \text{for all } h \in \langle x, u \rangle,$$

$$\xi(h) = \beta - \text{Ad } \rho(h)\beta \quad \text{for all } h \in \langle x, w \rangle.$$

It follows that  $\alpha \in V^u$  and  $\beta \in V^w$ . Moreover, by looking at the value  $\xi(x)$ , we see that

$$\alpha - \beta \in V^x.$$

Since the lines  $V^u, V^x, V^w$  also span  $V$ , it follows that  $\alpha = \beta = 0$ . Therefore,  $\xi(x) = 0$ . Similarly,  $\xi(y) = 0$  and thus  $\xi = 0$  on the entire group  $\Gamma_\Omega$ .

3. This follows from the fact that  $\rho(\Gamma_\Upsilon)$  has no nonzero fixed vectors in  $V = \text{sl}(2, \mathbb{C})$ . □

**Corollary 3.7** *The scheme  $\text{Hom}_o(\Gamma_\Omega, G)$  is smooth.*

From now on, we will be making the following assumption on the labeled graphs  $\Delta$  of Coxeter groups  $\Gamma$ :

**Assumption 3.8**    1. *Every label of the graph  $\Delta$  is even.*  
 2.  *$\Delta$  contains as a full subgraph the graph  $\Omega$  above.*

**Proposition 3.9**    1. *The schemes  $\text{Hom}_o(\Gamma, G)$  and  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  are clopen subschemes in  $\text{Hom}(\Gamma, G)$  and  $\text{Hom}(\tilde{\Gamma}, \tilde{G})$ , respectively.*  
 2. *There is a morphism of schemes  $q: \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G)$ , such that for every  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  and  $\rho = q(\tilde{\rho})$  we have*

$$p \circ \tilde{\rho} = \rho \circ \eta,$$

where  $\eta: \tilde{\Gamma} \rightarrow \Gamma$  is the quotient map from (1).

3. *The morphism  $q$  is a regular étale covering with the deck-group  $\mathbb{Z}_2^r$ , where  $r$  is the rank of  $\Gamma$ .*

**Proof** 1. We will give a proof for  $\text{Hom}_o(\Gamma, G)$ , since the other statement is similar. Consider an elementary subgroup  $\Gamma_\Sigma \subset \Gamma$ ; this subgroup is finite. In Lemma 2.1 we proved that each irreducible component of  $\text{Hom}(\Gamma_\Sigma, G)$  is a clopen subscheme of  $\text{Hom}(\Gamma_\Sigma, G)$ ; furthermore, each component is a single  $G$ -orbit of a representation  $\Gamma_\Sigma \rightarrow G$ . Then

$$\text{Hom}_o(\Gamma_\Sigma, G) = \text{Hom}(\Gamma_\Sigma, G) \setminus \bigcup_{\theta \in \Gamma_\Sigma - \{1\}} \text{Hom}_{(\theta)}(\Gamma_\Sigma, G)$$



is an open subscheme in  $\text{Hom}(\Gamma_\Sigma, G)$ . It is also closed since every subscheme removed was open.

For each elementary subgroup  $\Gamma_\Sigma < \Gamma$  and inclusion map  $\iota_\Sigma: \Gamma_\Sigma \rightarrow \Gamma$ , we have the pull-back morphism

$$\iota_\Sigma^*: \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma_\Sigma, G).$$

Then we have the finite intersection, taken over all elementary subgraphs  $\Sigma \subset \Delta$ ,

$$\text{Hom}_o(\Gamma, G) = \bigcap_{\Sigma} (\iota_\Sigma^*)^{-1}(\text{Hom}_o(\Gamma_\Sigma, G)).$$

Therefore,  $\text{Hom}_o(\Gamma, G) \subset \text{Hom}(\Gamma, G)$  is clopen as a finite intersection of clopen subschemes.

2. For each  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G}(\mathbb{C}))$ , the reduction modulo centers of  $\tilde{\Gamma}$  and  $\tilde{G}$  yields a homomorphism  $\rho \in \text{Hom}_o(\Gamma, G(\mathbb{C}))$ . We need to check that the map

$$q: \text{Hom}_o(\tilde{\Gamma}, \tilde{G}(\mathbb{C})) \rightarrow \text{Hom}_o(\Gamma, G(\mathbb{C})), \quad q(\tilde{\rho}) = \rho,$$

obtained in this fashion comes from a morphism of schemes. First, the composition  $\tilde{\rho} \mapsto p \circ \rho$  is clearly a morphism of schemes

$$\text{Hom}(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}(\tilde{\Gamma}, G).$$

For  $\Theta = \{z\}$ , we obtain an isomorphism of schemes

$$\text{Hom}_\Theta(\tilde{\Gamma}, G) \rightarrow \text{Hom}(\Gamma, G)$$

(see [Lemma 2.3](#)), and  $\text{Hom}_\Theta(\tilde{\Gamma}, G)$  contains the image of  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$ . Therefore,  $q$  is a composition of two morphisms.

We next verify surjectivity. Let  $\rho \in \text{Hom}_o(\Gamma, G)$ . Define  $\tilde{\rho}: \tilde{\Gamma} \rightarrow \tilde{G}$  by sending generators  $g_v$  to arbitrary elements of  $p^{-1}(\rho(g_v))$  and sending the central element  $z \in \tilde{\Gamma}$  to  $-1 \in \text{SL}(2, \mathbb{C})$ . In view of [Proposition 3.3](#), for each elementary subgroup  $\Gamma_\Sigma$  in  $\Gamma$ , the restriction of  $\tilde{\rho}$  to the generators of  $\tilde{\Gamma}_\Sigma$  extends to a faithful homomorphism  $\tilde{\rho}|_{\tilde{\Gamma}_\Sigma}$ .

Since all the relators of the group  $\tilde{\Gamma}$  come from elementary subgroups, it follows that our map of the generators of  $\tilde{\Gamma}$  to  $\text{SL}(2)$  extends to a homomorphism  $\tilde{\rho}: \tilde{\Gamma} \rightarrow \text{SL}(2)$ . This homomorphism belongs to  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  since it is faithful on each elementary subgroup.

Thus, we obtained a surjective morphism

$$q: \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G), \quad q(\tilde{\rho}) = \rho.$$

The group  $\mathbb{Z}_2$  is the group of automorphisms of the covering  $\tilde{G} \rightarrow G$ ; therefore, the product of  $r$  copies of  $\mathbb{Z}_2$  acts naturally on the product of  $r$  copies of  $\tilde{G}$  as the group of automorphisms of the (regular) cover

$$\hat{p} = p \times \cdots \times p: \prod_{i=1}^r \tilde{G} \rightarrow \prod_{i=1}^r G.$$

Since the rank  $r$  of the group  $\Gamma$  is the number of its generators  $g_v$ , we have the morphism

$$\prod_{i=1}^r \tilde{G} \cong \text{Hom}(F_r, \tilde{G}) \xrightarrow{\hat{p}} \prod_{i=1}^r G \cong \text{Hom}(F_r, G),$$

where  $F_r$  is the free group on  $r$  generators. We also have the commutative diagram

$$\begin{array}{ccc} \text{Hom}(\tilde{\Gamma}, \tilde{G}) & \xrightarrow{q} & \text{Hom}(\Gamma, G) \\ \downarrow & & \downarrow \\ \text{Hom}(F_r, \tilde{G}) & \xrightarrow{\hat{p}} & \text{Hom}(F_r, G) \end{array}$$

where the vertical arrows are the inclusions of representation schemes induced by the epimorphisms

$$F_r \rightarrow \tilde{\Gamma} \quad \text{and} \quad F_r \rightarrow \Gamma$$

sending the free generators of  $F_r$  to the generators  $g_r$  of the extended Coxeter and Coxeter groups. It is elementary and left to the reader to verify that the group  $\mathbb{Z}_2^r$  of automorphisms of  $\hat{p}$  preserves the subscheme  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$ . Therefore, this finite group acts simply transitively on the fibers of the morphism  $q$ .

3. It remains to show that the map  $q$  is étale, ie that it is an isomorphism of analytic germs at every point. Let  $\tilde{\rho}$  be in  $\text{Hom}_o(\tilde{\Gamma}, \text{SL}(2, \mathbb{C}))$  and set  $\rho := q(\tilde{\rho})$ .

Below is a proof which assumes the reader's familiarity with [1], where the theory of controlling differential graded Lie algebras for various deformation problems was developed.

In view of [1, Theorem 6.8], it suffices to verify that the differential graded Lie algebras controlling these germs are quasi-isomorphic. First, the Lie algebras of  $G$  and  $\tilde{G}$  are isomorphic under the covering  $p$ , which implies that the covering map  $p$  induces isomorphisms

$$H^i(\tilde{\Gamma}, \text{Ad} \circ \tilde{\rho}) \rightarrow H^i(\tilde{\Gamma}, \text{Ad} \circ p(\tilde{\rho})) \quad \text{for } i \geq 0.$$

Since the central subgroup  $\mathbb{Z}_2$  of  $\tilde{\Gamma}$  is finite,

$$H^i(\mathbb{Z}_2, \text{sl}(2, \mathbb{C})) = 0 \quad \text{for } i \geq 1.$$

Therefore, applying the Lyndon–Hochschild–Serre spectral sequence to the central extension (1), we obtain isomorphisms

$$(2) \quad H^i(\Gamma, \text{Ad} \circ \rho) \rightarrow H^i(\tilde{\Gamma}, \text{Ad} \circ \tilde{\rho}), \quad \text{where } \rho = q(\tilde{\rho}), \quad \text{for } i \geq 1.$$

(Actually, for  $i = 0$  both cohomology groups vanish, which implies that they too are isomorphic.) These isomorphisms ensure that the morphism

$$q: (\text{Hom}(\tilde{\Gamma}, \tilde{G}), \tilde{\rho}) \rightarrow (\text{Hom}(\Gamma, G), \rho)$$

is an isomorphism of germs. □

**Remark 3.10** Below is an alternative argument proving that  $q$  is étale, which does not rely upon differential graded Lie algebras. The morphism  $q$  is étale if and only if  $q$  induces bijections of sets of  $A$ -points of representation schemes for all local Artin  $\mathbb{C}$ -algebras  $A$ ; see [6, Theorem 2.2]. Let  $A$  be a local Artin  $\mathbb{C}$ -algebra and  $\epsilon: A \rightarrow \mathbb{C}$  be the quotient by the maximal ideal. Then we have natural bijections

$$\text{Hom}(\Gamma, G(A)) \cong \text{Hom}(\Gamma, G)(A) \quad \text{and} \quad \text{Hom}(\tilde{\Gamma}, \tilde{G}(A)) \cong \text{Hom}(\tilde{\Gamma}, \tilde{G})(A)$$

and the commutative diagram

$$\begin{array}{ccccc} \text{Hom}(\tilde{\Gamma}, \tilde{K}_A) & \rightarrow & \text{Hom}(\tilde{\Gamma}, \tilde{G}(A)) & \xrightarrow{\tilde{u}} & \text{Hom}(\tilde{\Gamma}, \tilde{G}(\mathbb{C})) \\ \downarrow q_{K_A} & & \downarrow q_A & & \downarrow q \\ \text{Hom}(\Gamma, K_A) & \rightarrow & \text{Hom}(\Gamma, G(A)) & \xrightarrow{u} & \text{Hom}(\Gamma, G(\mathbb{C})) \end{array}$$

where  $\tilde{K}_A$  and  $K_A$  are the respective kernels of the group homomorphisms

$$\tilde{G}(A) \rightarrow \tilde{G}(\mathbb{C}) \quad \text{and} \quad G(A) \rightarrow G(\mathbb{C})$$

induced by  $\epsilon: A \rightarrow \mathbb{C}$ . We observe that the group  $\tilde{K}_A$  is torsion-free and, since the covering map  $p: \tilde{G} \rightarrow G$  is étale, the induced map  $p_{K_A}: \tilde{K}_A \rightarrow K_A$  is an isomorphism. Let us prove that for each  $\tilde{\rho}: \tilde{\Gamma} \rightarrow \tilde{G}(\mathbb{C})$ , the restriction of  $q_A$  to  $\tilde{u}^{-1}(\tilde{\rho})$  is injective. Suppose that  $\tilde{\rho}_A, \tilde{\rho}'_A \in \tilde{u}^{-1}(\tilde{\rho})$  project via  $q_A$  to the same homomorphism  $\rho_A: \Gamma \rightarrow G$ . Then

$$\tilde{\rho}_A(\gamma) = \pm \tilde{\rho}'_A(\gamma) \quad \text{for each } \gamma \in \tilde{\Gamma}.$$

Assume that  $\tilde{\rho}_A(\gamma) = gk = -\tilde{\rho}'_A(\gamma) = -gk'$ , where  $g \in \text{SL}(2, \mathbb{C})$  and  $k, k' \in \tilde{K}_A$ . Then  $kk' = -1$  which contradicts the property that  $\tilde{K}_A$  is torsion-free. This proves injectivity. Lastly, we verify surjectivity of the restriction map

$$q_A|_{\tilde{u}^{-1}(\tilde{\rho})}: \tilde{u}^{-1}(\tilde{\rho}) \rightarrow \{\rho_A: \Gamma \rightarrow G(A) \mid u(\rho_A) = \rho = q(\tilde{\rho})\},$$

where  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G}(\mathbb{C}))$ . Let  $\rho_A \in \text{Hom}(\Gamma, G(A)) \in u^{-1}(\rho)$ , where  $q(\tilde{\rho}) = \rho$ . Given  $\tilde{g} \in \text{SL}(2, \mathbb{C})$ ,  $p(\tilde{g}) = g$  and  $gk \in \text{SL}(2, A)$ , where  $k \in K_A$ , we lift  $gk$  to the element  $\tilde{g}\tilde{k} = p_{K_A}^{-1}(k)$ . We apply this construction to the images of each generator  $g_v$  (of  $\Gamma$  and of  $\tilde{\Gamma}$ ) and the generator  $z$  of  $\tilde{\Gamma}$ , under the homomorphisms  $\tilde{\rho}$  and  $\rho_A$ . We leave it to the reader to verify that the resulting map of the generators of  $\tilde{\Gamma}$  to  $\text{SL}(2, A)$  defines a homomorphism  $\tilde{\rho}_A$  in  $\text{Hom}_o(\tilde{\Gamma}, \text{SL}(2, A))$ . By the construction,  $\tilde{u}(\tilde{\rho}_A) = \tilde{\rho}$  and  $q_A(\tilde{\rho}_A) = \rho_A$ .

### 3.3 Character schemes of representations faithful on elementary subgroups

In this section we extend the results of the previous section from representation schemes to character schemes.

**3.3.1 Stability** Given a reductive affine algebraic group  $H$  and a finitely generated group  $\Lambda$ , we have the algebraic action of the group  $H$  on the homomorphism scheme  $\text{Hom}(\Lambda, H)$  given by

$$(h, \rho) \mapsto \text{Inn}(h) \circ \rho,$$

where  $\text{Inn}(h)$  is the inner automorphism  $g \mapsto hgh^{-1}$  of the group  $H$ . Recall that the character scheme is defined as the Mumford quotient

$$X(\Lambda, H) = \text{Hom}(\Lambda, H) // H.$$

Geometrically speaking, the Mumford quotient is obtained by identifying the semisimple points  $\text{Hom}_{\text{ss}}(\Lambda, H)$  of the  $H$ -action by the *extended orbit equivalence relation*, while the restriction of the projection

$$\mu: \text{Hom}_{\text{ss}}(\Lambda, H) \rightarrow X(\Lambda, H)$$

to the stable locus  $\text{Hom}_{\text{st}}(\Lambda, H)$  (consisting of stable points) is just the quotient by the  $H$ -orbit equivalence. Hence the restriction of the projection to the stable locus has especially simple form. We will use the notation

$$\rho \mapsto [\rho]$$

for the projection  $\mu$ .

A *sufficient condition* for stability of representations  $\rho \in \text{Hom}(\Lambda, H)$  (under the  $H$ -action) in terms of the Zariski closure of  $\rho(\Lambda)$  in  $H$  was established in [4]:

**Theorem 3.11** *A representation  $\rho \in \text{Hom}(\Lambda, H)$  is semistable provided that the Zariski closure  $\overline{\rho(\Lambda)}$  is reductive. A representation is stable provided that the Zariski closure  $\overline{\rho(\Lambda)}$  is reductive and the centralizer  $Z_H(\rho(\Lambda))$  of the image of  $\rho$  is finite.*

In the case of representations into  $H = \text{PO}(3)$  and  $H = \text{Spin}(3)$ , the sufficient condition for stability amounts to requiring that the image of  $\rho$  is not contained in a Borel subgroup of  $H$ . Our next goal is to verify stability condition and identify centralizers of the images of representations in the context of Coxeter and extended Coxeter groups. Recall that we are using the notation  $G$  for  $\text{PSL}(2)$  and  $\tilde{G}$  for  $\text{SL}(2)$  (regarded as group schemes).

**Lemma 3.12** *Let  $\Gamma$  be a Coxeter group and  $\tilde{\Gamma}$  the corresponding extended Coxeter group, satisfying Assumption 3.8. Then for each  $\rho \in \text{Hom}_o(\Gamma, G)$  and  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  we have:*

1. *The representations  $\rho, \tilde{\rho}$  are stable points in Mumford’s sense.*
2. *The centralizers of the images of  $\rho, \tilde{\rho}$  equal the center of the target group.*

**Proof** Recall that we require the group  $\Gamma$  to contain a subgroup  $\Gamma_\Omega$ . It suffices to prove both 1 and 2 for the representations  $\rho \in \text{Hom}_o(\Gamma_\Omega, G)$ ,  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}_\Omega, \tilde{G})$ , since we have to verify that the image of the representation is not contained in a Borel subgroup and that its centralizer equals the center of the target group.

(i) First, we consider the case of representations  $\tilde{\rho}: \tilde{\Gamma} = \tilde{\Gamma}_\Omega \rightarrow \text{SL}(2, \mathbb{C})$ . We restrict our attention to the subgraph  $\Sigma \subset \Omega$ , which consists of two vertices  $x, y$  and the edge  $e = [x, y]$  labeled 4. Each representation  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  projects to a faithful representation

$$\rho: \Gamma_\Sigma \hookrightarrow \text{PSL}(2, \mathbb{C}).$$

By Proposition 3.3, the centralizer of the subgroup  $\tilde{\rho}(\tilde{\Gamma}_\Sigma)$  equals the center of  $\text{SL}(2, \mathbb{C})$ . Moreover, the images of the generators of  $\tilde{\Gamma}_\Sigma$  under  $\tilde{\rho}$  have distinct eigenlines. It follows that the subgroup  $\tilde{\rho}(\tilde{\gamma}_\Sigma)$  cannot have an invariant line in  $\mathbb{C}^2$ , thereby proving that  $\tilde{\rho}(\tilde{\Gamma})$  is not contained in a Borel subgroup of  $\text{SL}(2, \mathbb{C})$ . This proves parts 1 and 2 for representations to  $\text{SL}(2, \mathbb{C})$ .

(ii) Consider now representations  $\rho: \Gamma \rightarrow G$ . By the assumption,  $\rho$  sends distinct generators of  $\Gamma$  to distinct elements of  $G$ . It follows that the group  $\rho(\Gamma)$  cannot fix a point in  $\mathbb{CP}^1$ . In other words, the group  $\rho(\Gamma)$  is not contained in a Borel subgroup of  $G$ . This proves part 1.

To prove part 2, we will use subgroups  $\Gamma_\Upsilon$  and  $\Gamma_\Omega$  of the group  $\Gamma$ . Since  $\rho$  belongs to  $\text{Hom}_o(\Gamma, \text{PSL}(2, \mathbb{C}))$ , the centralizer of  $\rho(\Gamma_\Upsilon)$  in  $G$  equals the subgroup  $\rho(\Gamma_\Upsilon)$  itself (Lemma 3.4). On the other hand,  $\rho$  is faithful on the subgroups generated by  $\{g_u, g_x\}, \{g_v, g_y\}, \{g_w, g_z\}$ . Therefore,

$$[\rho(g_u), \rho(g_x)] \neq 1, \quad [\rho(g_v), \rho(g_y)] \neq 1 \quad \text{and} \quad [\rho(g_w), \rho(g_z)] \neq 1,$$

and hence the subgroup  $\rho(\Gamma)$  has trivial centralizer in  $\text{PSL}(2, \mathbb{C})$ . □

**3.3.2 Cross-sections** Let  $Y$  be a quasiasfine scheme and  $G \times Y \rightarrow Y$  be an algebraic group. Suppose that  $C \subset Y$  is a closed subscheme, such that the orbit map

$$G \times C \rightarrow Y$$

is an isomorphism. In particular,  $C$  projects isomorphically onto  $Y // G$ , since

$$(G \times C) // G \cong C.$$

Such a subscheme  $C$  is called a *cross-section* for the action of  $G$  on  $Y$ . We leave it to the reader to check that if  $A_c \subset A$  is a cross-section for the action of  $G$  on  $A$  and we have an action  $G \curvearrowright B$ , then  $A_c \times B$  is a cross-section for the product action on  $A \times B$ .

**Lemma 3.13** *Suppose that  $Y$  is a (quasiasfine) scheme of finite type,  $G \times Y \rightarrow Y$  is an (algebraic) action of an affine algebraic group,  $C \subset X$  is a cross-section for this action. Suppose that  $Y' \subset Y$  is a  $G$ -invariant subscheme. Then  $C' = Y' \cap C$  is also a cross-section for the action  $G \times Y' \rightarrow Y'$ .*

**Proof** We need to show that the orbit map  $G \times C' \rightarrow Y'$  is an isomorphism. It suffices to show that for each commutative ring  $A$ , the orbit map

$$\mu': G(A) \times C'(A) \rightarrow Y'(A)$$

of  $A$ -points is a bijection; see the [appendix](#). We have  $C'(A) = C(A) \cap Y(A)$ . Since the orbit map

$$\mu: G(A) \times C(A) \rightarrow Y(A)$$

is a bijection and  $Y'(A)$  is  $G(A)$ -invariant, it follows that  $\mu$  is a bijection. □

Note that if the scheme  $Y$  and its subscheme  $C \subset Y$  are both smooth then the condition that  $C$  is a cross-section for the action of  $G$  is easier to check: It suffices to verify that the set of complex points of  $C$  is a set-theoretic cross-section for the action of  $G$  on  $Y(\mathbb{C})$ . Indeed,  $\mu: G \times C \rightarrow Y$ , the restriction of the  $G$ -action on  $Y$ , is a morphism. Our hypothesis amounts to the assumption that  $\mu$  induces a bijection of  $\mathbb{C}$ -points. The fact that  $\mu$  is an isomorphism now follows, for instance, from the Zariski Main Theorem.

We now specialize to the case of representation schemes. Let  $\pi' = \pi/N$  be a finitely generated group (where  $\pi$  is a finitely generated group and  $N \triangleleft \pi$  is a normal subgroup),  $G$  be an affine algebraic group, and  $G \times \text{Hom}(\pi, G) \rightarrow \text{Hom}(\pi, G)$  be the action of  $G$  by conjugation on the representation scheme. We will assume that the scheme  $\text{Hom}(\pi, G)$  is smooth. Suppose that  $U \subset \text{Hom}(\pi, G)$  is a  $G$ -invariant open affine subscheme and  $U' = U \cap \text{Hom}(\pi', G)$ . We assume that  $U_c \subset U$  is a closed smooth subscheme. Then, in view of smoothness, the property that  $U_c$  is a cross-section for the  $G$ -action on  $U$

amounts to the condition that  $U_c(\mathbb{C})$  is a cross-section for the action of  $G(\mathbb{C})$  on  $U$ . If this is the case, then, according to [Lemma 3.13](#), the subscheme  $U'_c = U_c \cap U'$  is a cross-section for the action of  $G$  on  $U'$ .

**3.3.3 Cross-sections of representation schemes** We apply the above observations in two situations. First, suppose that  $\Gamma$  is a Coxeter group satisfying [Assumption 3.8](#); we let  $G = \text{PO}(3)$ . We have the identity embedding  $\iota_\Omega: \Gamma_\Omega \hookrightarrow \Gamma$  of the finite subgroup  $\Gamma_\Omega$ . Recall that, according to [Lemma 3.6](#),  $\text{Hom}_o(\Gamma_\Omega, G(\mathbb{C}))$  consists of a single  $G$ -orbit  $G(\mathbb{C}) \cdot \rho_\Omega$ . We then set

$$\text{Hom}_c(\Gamma, G) := (\iota_\Omega^*)^{-1}(\rho_\Omega).$$

The next lemma is an analogue of Corollary 12.11 in [\[6\]](#):

**Lemma 3.14** *The subscheme  $\text{Hom}_c(\Gamma, G)$  is a cross-section for the action  $G \curvearrowright \text{Hom}_o(\Gamma, G)$ . In particular,*

$$X_o(\Gamma, G) \cong \text{Hom}_c(\Gamma, G).$$

**Proof** We let  $\pi' = \Gamma$  and define the new group  $\pi$  as the Coxeter group whose Coxeter graph is obtained from the one of  $\Gamma$  by removing all the edges which are not in  $\Omega$ . Define

$$\pi_o := \underbrace{\mathbb{Z}_2 \star \cdots \star \mathbb{Z}_2}_{n \text{ times}}$$

with one free factor for each vertex of the Coxeter graph not contained in  $\Omega$ . We have

$$\text{Hom}_o(\pi_o) = \prod_{i=1}^n \text{Hom}_o(\mathbb{Z}_2, G).$$

Then

$$\pi \cong \pi_o \star \Gamma_\Omega \quad \text{and} \quad \text{Hom}_o(\pi, G) \cong \text{Hom}_o(\pi_o, G) \times \text{Hom}_o(\Gamma_\Omega, G).$$

The scheme  $U = \text{Hom}_o(\Gamma_\Omega, G)$  is smooth by [Lemma 3.6](#), while  $\text{Hom}_o(\mathbb{Z}_2, G)$  is smooth since  $\mathbb{Z}_2$  is finite; therefore, the representation scheme  $\text{Hom}_o(\pi, G)$  is smooth as well.

Clearly,  $\pi' = \pi/N$  for a normal subgroup  $N \triangleleft \pi$ . We again have the inclusion homomorphism  $\iota_\gamma: \Gamma_\Omega \rightarrow \pi$ ; the subscheme

$$C = \text{Hom}_c(\pi, G) := (\iota_\gamma^*)^{-1}(\rho_\Omega)$$

is smooth since it is naturally isomorphic to  $\text{Hom}(\pi_o, G)$ . The fact that  $C$  is a cross-section for the action of  $G$  on  $U$  follows immediately from [Lemma 3.6](#) and observations

following Lemma 3.13. Lastly, note that

$$U' = \text{Hom}_c(\Gamma, G) = U \cap \text{Hom}(\Gamma, G).$$

Now, the lemma follows from Lemma 3.13. □

The second situation when we apply our description of cross-sections is the one of representations of extended Coxeter groups  $\tilde{\Gamma}$  (again satisfying Assumption 3.8) to the group  $\tilde{G} \cong \text{SL}(2)$ . The group  $\tilde{G}$  does not act faithfully on  $\text{Hom}(\tilde{\Gamma}, \tilde{G})$ ; this action factors through the action of the group  $G = \text{PO}(3)$ .

Earlier, we defined the subscheme  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \subset \text{Hom}(\tilde{\Gamma}, \tilde{G})$ . Set

$$\text{Hom}_c(\tilde{\Gamma}, \tilde{G}) := q^{-1}(\text{Hom}_c(\Gamma, G)).$$

**Lemma 3.15**  $\text{Hom}_c(\tilde{\Gamma}, \tilde{G})$  is a cross-section for the action of  $G$  on  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$ .

**Proof** We let  $\pi' = \tilde{\Gamma}$ . Similarly to the proof of Lemma 3.14, we define the extended Coxeter group  $\pi$  by eliminating all the edges which are not in the subgraph  $\Omega$ . Then  $\pi'$  is isomorphic to a quotient of  $\pi$  and the same proof as in Lemma 3.14 goes through. □

**3.3.4 Character schemes** We let  $X_o(\Gamma, G)$  and  $X_o(\tilde{\Gamma}, \tilde{G})$  denote the projections of  $\text{Hom}_o(\Gamma, G)$  and  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  to the corresponding character schemes.

In view of Lemmata 3.14 and 3.15, the projections

$$\text{Hom}_o(\Gamma, G) \rightarrow X_o(\Gamma, G) \quad \text{and} \quad \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow X_o(\tilde{\Gamma}, \tilde{G})$$

are trivial principal fiber bundles with structure group  $G = \text{PSL}(2, \mathbb{C})$ : the center of the group  $\tilde{G}$  acts trivially on  $\text{Hom}(\tilde{\Gamma}, \tilde{G})$ . We record this as:

**Corollary 3.16** *There exist natural isomorphisms of germs*

$$\begin{aligned} (\text{Hom}_o(\tilde{\Gamma}, \tilde{G}), \tilde{\rho}) &\cong (\text{Hom}_c(\tilde{\Gamma}, \tilde{G}) \times G, \tilde{\rho} \times 1) \\ &\cong (X_o(\tilde{\Gamma}, \tilde{G}) \times G, [\tilde{\rho}] \times 1) \end{aligned}$$

and

$$\begin{aligned} (\text{Hom}_o(\Gamma, G), \rho) &\cong \text{Hom}_c(\Gamma, G) \times G, \rho \times 1 \\ &\cong (X_o(\Gamma, G) \times G, [\rho] \times 1). \end{aligned}$$

**3.3.5 Adding a free factor** Let  $F_k$  be the free group on  $k$  generators. For an arbitrary finitely generated group  $\Lambda$  and an algebraic group  $H$  we have an isomorphism of schemes

$$(3) \quad \text{Hom}(\Lambda \star F_k, H) \cong \text{Hom}(\Lambda, H) \times \text{Hom}(F_k, H) \cong \text{Hom}(\Lambda, H) \times H^k.$$



This isomorphism is  $H$ -equivariant; here and below the action of  $H$  is by conjugation on the left side and the diagonal action (by conjugations) on the product space on the right side.

We will use these isomorphisms in the following two special cases:  $\Lambda = \Gamma$ ,  $H = G$  and  $\Lambda = \tilde{\Gamma}$ ,  $H = \tilde{G}$ , where  $G = \text{PSL}(2)$ ,  $\tilde{G} = \text{SL}(2)$ , and  $\Gamma, \tilde{\Gamma}$  are Coxeter and extended Coxeter groups, respectively. Then the isomorphisms (3) for these groups allow us to define clopen subschemes

$$\text{Hom}_o(\Gamma \star F_k, G) \subset \text{Hom}(\Gamma \star F_k, G) \quad \text{and} \quad \text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \subset \text{Hom}(\tilde{\Gamma} \star F_k, \tilde{G})$$

as the images of

$$\text{Hom}_o(\Gamma, G) \times G^k \quad \text{and} \quad \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k,$$

respectively.

It follows from Lemmata 3.14 and 3.15 that  $\text{Hom}_c(\Gamma, G) \times G^k$  is a cross-section for the action of  $G$  on  $\text{Hom}_o(\Gamma, G) \times G^k$ , while  $\text{Hom}_c(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k$  is a cross-section for the action of  $G$  on  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k$ .

We thus obtain:

**Lemma 3.17**  $(\text{Hom}_o(\Gamma, G) \times G^k)/G \cong X_o(\Gamma, G) \times G^k.$

The étale covering  $q$  defined above yields, for each  $k$ , the étale covering

$$q_k: \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k \cong \text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma \star F_k, G) \cong \text{Hom}_o(\Gamma, G) \times G^k.$$

- Corollary 3.18**
1.  $X_o(\Gamma \star F_k, G) \cong X_o(\Gamma, G) \times G^k.$
  2.  $\text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \cong X_o(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k.$
  3. The covering  $q_k$  is étale.

## 4 Universality theorem of Panov and Petrunin

The proofs of Theorem 1.1 and its corollaries hinge upon two results, the first of which is the following:

**Theorem 4.1** (Panov–Petrunin universality theorem [12]) *Let  $\Gamma$  be a finitely presented group. Then there exists a closed 3-dimensional (nonorientable) hyperbolic orbifold  $O$  such that  $\pi_1(Y) \cong \Gamma$ , where  $Y$  is the underlying space of  $O$ . Furthermore,  $Y$  is a 3-dimensional pseudomanifold without boundary.*

**Remark 4.2** Examination of the proof in [12] shows that the orbifold  $O$  admits a hyperbolic manifold cover  $\tilde{O} \rightarrow O$  with deck-transformation group  $\mathbb{Z}_2^4$ .

The singular set of the pseudomanifold  $Y$  consists of singular points  $y_j$  for  $j=1, \dots, 2k$ , whose neighborhoods  $C_j$  in  $Y$  are cones over  $\mathbb{R}P^2$ . Note that, since  $\mathbb{R}P^2$  has Euler characteristic 1, the number of conical singularities has to be even. Observe also that one needs  $k \geq 1$  in this theorem, since fundamental groups of 3–dimensional manifolds are very restricted among finitely presented groups. For instance, there are no 3–manifolds  $M$  with  $\pi_1(M) \cong \mathbb{Z}^4$ ; therefore, for  $\Gamma \cong \mathbb{Z}^4$ , one cannot have  $k = 0$  in [Theorem 4.1](#).

**Problem 4.3** Does [Theorem 4.1](#) hold with  $k = 1$ ?

Given  $\Gamma$  and  $Y$  as in [Theorem 4.1](#), we will construct a closed (nonorientable) 3–dimensional manifold  $M = M_\Gamma$  as follows. (Formally speaking, this 3–manifold also depends on the choice of an orbifold  $O$  in [Theorem 4.1](#), which is very far from being unique, however, in order to simplify the notation, we will suppress this dependence).

Let  $O$  be a 3–dimensional orbifold as in [Theorem 4.1](#) and let  $Y$  be the underlying space of  $O$ . Let  $Y'$  be obtained by removing open cones  $C_j$  for  $j = 1, \dots, 2k$  from  $Y$ . Then  $Y'$  is a compact 3–dimensional manifold with  $2k$  boundary components each of which is a copy of the projective plane  $\mathbb{R}P^2$ . We let  $\theta_i$  denote the generator of the fundamental group of the projective plane  $P_i \cong \mathbb{R}P^2 \subset \partial M$ , which equals the boundary of the cone  $C_i$  and let  $\alpha_i: \langle \theta_i \rangle \rightarrow \langle \theta_{k+i} \rangle$  be the (unique) isomorphism. We will regard each  $\theta_i$  as an element of  $\pi_1(Y')$ . Set

$$\Theta := \{\theta_1, \dots, \theta_k\}.$$

Then

$$\Gamma = \pi_1(Y) = \pi_1(Y') / \langle\langle \theta_1, \dots, \theta_{2k} \rangle\rangle.$$

Next, let  $M$  be the closed 3–dimensional manifold obtained by attaching  $k$  copies of the product  $\mathbb{R}P^2 \times [0, 1]$  to  $Y'$  along the boundary projective planes, pairing the projective planes  $P_i$  and  $P_{i+k}$  for  $i = 1, \dots, k$ . Then  $\pi = \pi_1(M_\Gamma)$  is the iterated HNN extension

$$(((\pi_1(Y') \star_{\langle \alpha_1 \rangle}) \star_{\langle \alpha_2 \rangle}) \cdots) \star_{\langle \alpha_k \rangle}$$

of  $\pi_1(Y')$  with stable letters  $t_1, \dots, t_k$ .

Taking the quotient

$$(4) \quad \phi: \pi \rightarrow \pi / \langle\langle \Theta \rangle\rangle,$$

we therefore obtain the group  $\Gamma \star F_k$ , where  $F_k$  is the free group on  $k$  generators, projections of the stable letters  $t_i$  for  $i = 1, \dots, k$  in the above HNN extension. We let

$$\psi: \Gamma \star F_k \rightarrow \Gamma$$

denote the further projection to the first direct factor and set

$$(5) \quad \xi := \psi \circ \phi: \pi \rightarrow \Gamma.$$

Now, given an algebraic group  $H$ , we obtain

$$\text{Hom}_\Theta(\pi, H) = \phi^*(\text{Hom}(\Gamma \star F_k, H)),$$

a clopen subscheme in  $\text{Hom}(\pi, H)$  (see Corollary 2.4). The isomorphism

$$\text{Hom}(\Gamma, H) \times H^k \xrightarrow{\cong} \text{Hom}(\Gamma \star F_k, H) \xrightarrow{\phi^*} \text{Hom}_\Theta(\pi, H)$$

restricted to  $\text{Hom}(\Gamma, H) \times 1$  equals  $\xi^*$ . We thus obtain:

**Lemma 4.4** *For each open subscheme  $S \subset \text{Hom}(\Gamma, H)$ , there exists an open subscheme  $R \subset \text{Hom}(\pi, H)$  isomorphic to  $S \times H^k$  via the morphism  $\phi^*$ . Furthermore,  $R$  contains  $\xi^*(S)$ .*

**Proof** Take  $R := \phi^*(S \times H^k)$ , where we identify  $\text{Hom}(\Gamma, H) \times H^k$  with the representation scheme  $\text{Hom}(\Gamma \star F_k, H)$ . □

Note that, because of the  $H$ -equivariance of  $\phi^*$ , if  $S$  is  $H$ -invariant and there exists a cross-section  $S_c \subset S$  for the action  $H \curvearrowright S$ , then the pull-back  $R_c := \phi^*(S_c \times H^k)$  is a cross-section for  $H \curvearrowright R$ .

We will be using these results for Coxeter groups  $\Gamma$  as well as for the extended Coxeter groups  $\tilde{\Gamma}$ , with the group  $H$  given respectively by either  $H = G = \text{PSL}(2)$  or  $H = \tilde{G} = \text{SL}(2)$ . In order to simplify the notation, we will refer to the fundamental group  $\pi_1(M_\Gamma)$  as  $\pi$  and the fundamental group  $\pi_1(M_{\tilde{\Gamma}})$  as  $\tilde{\pi}$ . We will denote by  $\Theta$  the subset (defined above) of order-2 elements in  $\pi$  and by  $\tilde{\Theta}$  the similar subset in  $\tilde{\pi}$ .

We obtain a commutative diagram

$$\begin{array}{ccccc} \text{Hom}_{\tilde{\Theta}}(\tilde{\pi}, \tilde{G}) & \xrightarrow{\cong} & \text{Hom}(\tilde{\Gamma} \star F_k, \tilde{G}) & \xrightarrow{\cong} & \text{Hom}(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k \\ \hat{q} \downarrow & & \downarrow & & \downarrow qk \\ \text{Hom}_\Theta(\pi, G) & \xrightarrow{\cong} & \text{Hom}(\Gamma \star F_k, G) & \xrightarrow{\cong} & \text{Hom}(\Gamma, G) \times G^k \end{array}$$

where  $qk$  is an étale covering (see Section 3.3.5), and hence  $\hat{q}$  also is an étale covering. The groups of covering transformations for both are  $\mathbb{Z}_2^{k+r}$ , where  $r$  is the rank of the Coxeter group  $\Gamma$ .

We let

$$\text{Hom}_o(\pi, G) \subset \text{Hom}_\Theta(\pi, G) \quad \text{and} \quad \text{Hom}_o(\tilde{\pi}, \tilde{G}) \subset \text{Hom}_{\tilde{\Theta}}(\tilde{\pi}, \tilde{G})$$

be the subschemes which are the preimages of  $\text{Hom}_o(F_k \star \Gamma, G)$  and  $\text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G})$  under the isomorphisms

$$\text{Hom}_\Theta(\pi, G) \rightarrow \text{Hom}(\Gamma \star F_k, G) \quad \text{and} \quad \text{Hom}_{\tilde{\Theta}}(\tilde{\pi}, \tilde{G}) \rightarrow \text{Hom}(\tilde{\Gamma} \star F_k, \tilde{G}),$$

respectively. Note also that the covering  $q$  is equivariant with respect to the morphism  $p: \tilde{G} \rightarrow G$ , hence  $\hat{q}$  is equivariant as well. Because of this equivariance, if  $R_c \subset R \subset \text{Hom}_\Theta(\pi, G)$  is a cross-section, so is  $\tilde{R}_c := \hat{q}^{-1}(R_c) \subset \tilde{R} \subset \text{Hom}_{\tilde{\Theta}}(\tilde{\pi}, \tilde{G})$ . The cross-sections we will be using are

$$\text{Hom}_c(\pi, G) := R_c \subset R = \text{Hom}_o(\pi, G) \quad \text{and} \quad \text{Hom}_c(\tilde{\pi}, \tilde{G}) := \tilde{R}_c \subset \tilde{R} = \text{Hom}_o(\tilde{\pi}, \tilde{G}).$$

We obtain:

**Lemma 4.5** *Consider a representation  $\varphi \in \text{Hom}_o(\Gamma, G(\mathbb{C}))$ ; let  $\rho$  be its image in  $\text{Hom}_\Theta(\pi, G)$  and pick*

$$\tilde{\rho} \in \hat{q}^{-1}(\rho) \in \text{Hom}_o(\tilde{\pi}, \tilde{G}(\mathbb{C})).$$

Then we have isomorphisms of germs

$$(X(\tilde{\pi}, \tilde{G}), [\tilde{\rho}]) \cong (X(\pi, G), [\rho]) \cong (X_o(\Gamma, G) \times G^k, [\varphi] \times 1),$$

where  $1 = (1, \dots, 1) \in G^k$ .

**Proof** These isomorphisms follow from the fact that  $\hat{q}$  is an étale covering and the existence of the following cross-sections for the actions of  $G$ :

$$R_c = \text{Hom}_c(\pi, G) \subset \text{Hom}_o(\pi, G), \quad \text{Hom}_c(\Gamma, G) \times G^k \subset \text{Hom}_o(\Gamma, G) \times G^k$$

and

$$\tilde{R}_c = \text{Hom}_c(\tilde{\pi}, \tilde{G}) \subset \text{Hom}_o(\tilde{\pi}, \tilde{G}), \quad \text{Hom}_c(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k \subset \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \times \tilde{G}^k. \quad \square$$

## 5 A universality theorem for Coxeter groups

The second key ingredient we need is the following theorem, which is essentially contained in Kapovich and Millson [6]. Before stating the theorem we recall (Lemma 3.14) that the action  $G \curvearrowright \text{Hom}_o(\Gamma, G)$  has a cross-section  $\text{Hom}_c(\Gamma, G) \subset \text{Hom}_o(\Gamma, G)$ , ie  $\text{Hom}_o(\Gamma, G)$  is  $G$ -equivariantly isomorphic to the product  $X_o(\Gamma, G) \times G$ , where  $G$  acts trivially on  $X_o(\Gamma, G)$ . As always,  $G = \text{PO}(3)$ .

**Theorem 5.1** *Let  $X$  and  $x \in X$  be as in Theorem 1.1. Then there exists an open subscheme  $X' \subset X$  containing  $x$ , a finitely generated Coxeter group  $\Gamma$  (such that every edge of its graph  $\Delta$  has label 2 or 4) and a representation  $\rho_c: \Gamma \rightarrow \text{PO}(3, \mathbb{R})$  with dense*

image, such that  $X'$  is isomorphic to an open subscheme  $S' \subset X_o(\Gamma, G)$ . The representation  $\rho_c$  belongs to  $\text{Hom}_o(\Gamma, \text{PO}(3, \mathbb{R}))$ . Furthermore, under this isomorphism,  $x$  corresponds to  $[\rho_c]$ .

**Remark 5.2** Since  $\text{Hom}_o(\Gamma, G) \cong X_o(\Gamma, G) \times G$ , with  $\text{Hom}_c(\Gamma, G)$  containing  $\rho_c$  serving as a cross-section for the action  $G \curvearrowright \text{Hom}_o(\Gamma, G)$ , the preimage  $S'_o$  of  $S'$  in  $\text{Hom}_o(\Gamma, G)$  is isomorphic to  $S'_c \times G \cong S' \times G$ . Here and in what follows,  $S'_c \subset S'_o$  is the cross-section given by

$$S'_c := \text{Hom}_c(\Gamma, G) \cap S'_o.$$

Furthermore, as we saw in Section 2.2, the representation  $\rho_c$  lifts to a representation

$$\tilde{\rho}_c: \tilde{\Gamma} \rightarrow \text{SU}(2)$$

of the canonical central extension  $\tilde{\Gamma}$  of  $\Gamma$ .

Since the universality theorems proven in [6] are somewhat different from the one stated above, we outline the proof of Theorem 5.1. The main differences are that the results of [6] are about representations of Shephard and Artin groups rather than Coxeter groups. Furthermore, the representation to  $\text{PO}(3, \mathbb{R})$  constructed in [6] has finite image (which was important for [6]), although the image group does have trivial centralizer in  $\text{PO}(3, \mathbb{C})$ .

**Outline of proof of Theorem 5.1** The arguments below are minor modifications of the ones in [6].

**Step 1 (scheme-theoretic version of Mnëv universality theorem)** Without loss of generality, we may assume that the rational point  $x$  is the origin  $0$  in the affine space containing  $X$ . In [6] we first construct a *based projective arrangement*  $A$ , such that an open subscheme  $\text{BR}_0(A, \mathbb{P}^2)$  in the space of *based projective realizations*  $\text{BR}(A, \mathbb{P}^2)$ , is isomorphic to  $X$  as a scheme over  $\mathbb{Q}$ , and, moreover, the *geometrization* isomorphism

$$X \xrightarrow{\text{geo}} \text{BR}_0(A, \mathbb{P}^2)$$

sends  $x \in X$  to a based realization  $\psi_0: A \rightarrow \mathbb{P}^2$  whose image is the *standard triangle*. Furthermore, the images of the points and lines in  $A$  under  $\psi_0$  are real.

**Remark 5.3** Subsequently, a proof of this result was also given by Lafforgue in [9], who was apparently unaware of [6].

**Step 2** An arrangement  $A$  is a certain bipartite graph containing a subgraph  $T$  (the “base”) which is isomorphic to the incidence graph of the “standard triangle” (also known as “standard quadrangle”); see [6, Figure 7]. The subgraph  $T$  has five vertices

$v_{00}, v_x, v_y, c_{10}, v_{01}, v_{11}$  corresponding to the “points” of the standard triangle and six vertices  $l_x, l_d, l_y, l_{x1}, l_{y1}, l_\infty$  which correspond to the “lines” of the standard triangle. In [6, Section 11] we further modify the bipartite graph  $A$ : we make the identifications of vertices

$$v_{00} \sim l_\infty, \quad v_x \sim l_y \quad \text{and} \quad v_y \sim l_x,$$

and we also add to  $A$  the edges

$$[v_{10}, v_{00}] \quad \text{and} \quad [v_{01}, v_{00}].$$

We will use the upper-case notation  $V_{00} = \psi_0(v_{00}), V_x = \psi_0(v_x)$ , etc to denote vectors in  $\mathbb{C}^3$  which project to the images under  $\psi_0$  of the point-vertices of  $T$ . The choice of this vectors is not unique, of course; we assume that  $V_{00}, V_x, V_y$  form a basis and

$$(6) \quad V_{10} = V_{00} + V_x, \quad V_{01} = V_{00} + V_y \quad \text{and} \quad V_{11} = V_{00} + V_x + V_y.$$

This is possible because of the incidences in  $\psi_0(T)$ .

However, here, unlike in [6], we will not add the edge  $[v_{00}, v_{11}]$ . (The purpose of this edge in [6] was to ensure that certain representation of a Shephard group is finite.) We let  $A'$  denote the resulting graph (no longer bipartite). We assign labels to the edges of  $A'$  as follows: all edges are labeled 2 except for the two edges

$$[v_{10}, v_{00}] \quad \text{and} \quad [v_{01}, v_{00}],$$

which have the label 4. We then let  $\Gamma$  denote the Coxeter group corresponding to this labeled graph. We let  $T'$  denote the labeled subgraph of  $A'$ , whose vertices are the images of the vertices of the arrangement  $T$ .

The labeled graph  $\Omega$  as in Figure 1 embeds into  $T'$  via the map given by

$$v \mapsto v_{00}, \quad x \mapsto v_{10}, \quad y \mapsto v_{01}, \quad u \mapsto v_x, \quad w \mapsto v_y.$$

We equip the vector space  $\mathbb{C}^3$  with a nondegenerate bilinear form, so that:

1. All subspaces which appear in the image  $\psi_0(T) = \psi_0(A)$  are anisotropic (the bilinear form has nondegenerate restriction to these subspaces).
2. The vectors  $V_{00}, V_x, V_y \in \mathbb{C}^3$  are pairwise orthogonal and have unit norm.

We let  $\text{PO}(3)$  denote the projectivization of the orthogonal group  $O(3)$  preserving this bilinear form.

A realization  $\psi \in R(A, \mathbb{P}^2)$  is *anisotropic* if for each vertex  $v \in A$ , the image  $\psi(v)$  is an anisotropic subspace in  $\mathbb{C}^3$ . We will use the notation  $R_a(A, \mathbb{P}^2) \subset R(A, \mathbb{P}^2)$  and  $\text{BR}_a(A, \mathbb{P}^2) \subset \text{BR}(A, \mathbb{P}^2)$  for open schemes of anisotropic realizations and

anisotropic based realizations. By condition 1 on the inner product above,  $\text{BR}_a(A, \mathbb{P}^2)$  contains  $\psi_0$ .

To every anisotropic realization  $\psi \in R(A, \mathbb{P}^2)$ , we associate a representation of the group  $\Gamma$  by sending every generator  $g_v \in \Gamma$  to the isometric involution in  $\text{PO}(3)$  fixing the subspace  $\psi(v)$  in  $\mathbb{P}^2$ . As in [6], this map of generators of  $\Gamma$  to  $\text{PO}(3)$  defines a representation

$$\rho_\psi: \Gamma \rightarrow \text{PO}(3, \mathbb{C}).$$

We define

$$\rho_c := \rho_{\psi_0}.$$

By the construction, each representation  $\rho_\psi$  is faithful on elementary subgroups: For the edges  $[v, w]$  in  $A$  (where  $v$  is a point and  $w$  is a line), the incidence condition  $\psi(v) \in \psi(w)$  in  $\mathbb{P}^2$  forces the point reflection in  $\psi(v)$  be distinct from the line reflection in  $\psi(w)$ . For the edges

$$[v_{10}, v_{00}] \quad \text{and} \quad [v_{01}, v_{00}],$$

condition (6) forces the point reflections in  $\psi(v_{00}), \psi(v_{10}), \psi(v_{01})$  to be pairwise noncommuting and hence both subgroups

$$\rho_\psi(\langle g_{v_{00}}, g_{v_{10}} \rangle) < \text{PO}(3, \mathbb{C}) \quad \text{and} \quad \rho_\psi(\langle g_{v_{00}}, g_{v_{01}} \rangle) < \text{PO}(3, \mathbb{C})$$

are isomorphic to  $I_2(4)$ . We also note that

$$(7) \quad \rho_\psi|_{\Gamma_\Omega} = \rho_\Omega := \rho_{\psi_0}: \Gamma_\Omega \rightarrow \text{PO}(3, \mathbb{C}).$$

We thus obtain the *algebraization* morphism of schemes

$$\text{alg}: \text{BR}_a(A, \mathbb{P}^2) \rightarrow \text{Hom}(\Gamma, \text{PO}(3)) \quad \text{given by } \psi \mapsto \rho_\psi.$$

As in [6], the morphism  $\text{alg}$  is an isomorphism to its image. It follows from Lemma 3.14 and (7) that the subscheme

$$S_c := \text{alg}(\text{BR}_a(A, \mathbb{P}^2)) \subset \text{Hom}_c(\Gamma, \text{PO}(3)) \subset \text{Hom}_o(\Gamma, \text{PO}(3))$$

is a cross-section for the action of  $G$  on the  $G$ -orbit of  $S_c$ , which we denote

$$S \subset \text{Hom}_o(\Gamma, \text{PO}(3)).$$

Let  $\Sigma \subset A'$  denote the complete subgraph whose vertices are the vertices (points and lines) of the standard triangle in  $A$ , except for the vertex  $v_{11}$ . As in [6], the image under  $\rho_c$  of the corresponding parabolic Coxeter subgroup  $\Gamma_\Sigma \subset \Gamma$  is isomorphic to the finite Coxeter group  $B_3$  (the symmetry group of the regular octahedron) divided by the center  $\mathbb{Z}_2$ . Such a group is a maximal finite subgroup of  $\text{PO}(3, \mathbb{R})$ . However, the involution  $\rho_c(g_{v_{11}})$  does not belong to the group  $\rho_c(\Gamma_\Sigma)$  (this would be an order-2

rotation in the center of a face of the octahedron). Thus, the group  $\rho_c(\Gamma)$  has to be dense in  $\text{PO}(3, \mathbb{R})$ , as it contains (actually, is equal to) the dense subgroup  $\rho_c(\Gamma_{T'})$ . This is the only essential difference between the construction in this paper and in [6], where it was important for the group  $\rho_c(\Gamma)$  to be finite.

We let

$$\mu: \text{Hom}_o(\Gamma, G) \rightarrow X_o(\Gamma, G)$$

denote the restriction of the GIT quotient  $\text{Hom}(\Gamma, G) \rightarrow X(\Gamma, G)$ . Since  $\text{Hom}_c(\Gamma, G)$  is a cross-section for the  $G$ -action on  $\text{Hom}_o(\Gamma, G)$ , the morphism  $\mu$  is a trivial principal  $G$ -bundle.

**Theorem 5.4**  $\text{alg}: \text{BR}_a(A, \mathbb{P}^2) \rightarrow \text{Hom}_c(\Gamma, \text{PO}(3, \mathbb{C}))$  is an isomorphism.

**Proof** We will only sketch the proof since it follows closely the argument in [6, Theorem 12.14] and the latter is quite long. One verifies that  $\text{alg}$  induces a natural isomorphism of functors of points. For instance, over the complex numbers, each representation  $\rho \in \text{Hom}_c(\Gamma, \text{PO}(3, \mathbb{C}))$  gives rise to an anisotropic realization:  $\psi(v) \in \mathbb{P}^2(\mathbb{C})$  is the point fixed by  $\rho(g_v)$  (if  $v$  is a point-vertex) and  $\psi(v) \in \mathbb{P}^2(\mathbb{C})$  is the line fixed by  $\rho(g_v)$  (if  $v$  is a line-vertex).  $\square$

- Corollary 5.5**
1.  $S_c$  is a cross-section for the action of  $G$  on  $\text{Hom}_o(\Gamma, G)$ .
  2.  $\mu \circ \text{alg}: \text{BR}_a(A, \mathbb{P}^2) \rightarrow X_o(\Gamma, G)$  is an isomorphism.
  3.  $S = \mu \circ \text{alg}(\text{BR}_a(A, \mathbb{P}^2)) \subset X(\Gamma, G)$  is an open subscheme.

**Proof** Part 1 follows from the fact that

$$S_c = \text{alg}(\text{BR}_a(A, \mathbb{P}^2(\mathbb{C}))) = \text{Hom}_c(\Gamma, \text{PO}(3, \mathbb{C}))$$

and the latter is a cross-section for the  $G$ -action on  $\text{Hom}_o(\Gamma, G)$  (Lemma 3.14). Part 2 is immediate from Theorem 5.4 and part 1. Part 3 follows from the fact that  $X_o(\Gamma, G)$  is an open subscheme in  $X(\Gamma, G)$ .  $\square$

We define  $X' := \text{geo}^{-1}(\text{BR}_a(A, \mathbb{P}^2)) \subset X$ , an open subscheme in  $X$ . The composition of  $\text{geo}$ ,  $\text{alg}$  and  $\mu$

$$X \supset X' \xrightarrow{\text{geo}} \text{BR}_a(A, \mathbb{P}^2) \cap \text{BR}_0(A, \mathbb{P}^2) \xrightarrow{\text{alg}} S'_c \xrightarrow{\mu} S' \subset X_o(\Gamma, G),$$

where  $X' \subset X$  and  $S' \subset S \subset X_o(\Gamma, G)$  are open subschemes, yields an isomorphism

$$\kappa: X' \rightarrow S' \subset S.$$

The isomorphism  $\kappa$  sends the point  $x \in X'$  to  $[\rho_c] \in X_o(\Gamma, G)$ . This concludes the proof of Theorem 5.1.  $\square$



We let  $S'_o$  denote the preimage of  $S'$  in  $\text{Hom}_o(\Gamma, G)$ ; then

$$S'_c = S_c \cap S'_o \quad \text{and} \quad S'_o = G \cdot S'_c \cong S'_c \times G.$$

Inverting the isomorphism  $\kappa$  and multiplying with the identity map  $G \rightarrow G$ , we obtain:

**Corollary 5.6** *There exists isomorphisms of schemes over  $\mathbb{Q}$*

$$\omega: S'_o \rightarrow X' \times G \quad \text{and} \quad \kappa: S'_c \rightarrow X'$$

and a commutative diagram

$$\begin{array}{ccc} S'_o & \xrightarrow{\omega} & X' \times G \\ \downarrow & & \downarrow \\ S'_c & \xrightarrow{\kappa^{-1}} & X' \end{array}$$

where the vertical arrows are quotients by the  $G$ -action.

## 6 Proof of Theorem 1.1

We continue with notation introduced in the previous sections. Given an affine scheme  $X$  over  $\mathbb{Q}$  and a rational point  $x \in X$ , we use [Theorem 5.1](#) to construct a Coxeter group  $\Gamma$  and a representation  $\rho_c: \Gamma \rightarrow \text{PO}(3, \mathbb{R}) < \text{PO}(3, \mathbb{C})$ . Then, as in [Section 4](#), we will construct a closed 3-manifold  $M = M_\Gamma$  with fundamental group  $\pi$ , and a clopen subscheme  $\text{Hom}_o(\pi, G) \subset \text{Hom}(\pi, G)$  which is isomorphic to the product  $\text{Hom}_o(\Gamma, G) \times G^k$ . In [\(5\)](#) we defined an epimorphism

$$\xi: \pi \xrightarrow{\phi} \Gamma \star F_k \xrightarrow{\psi} \Gamma.$$

Set  $\rho_o := \xi^*(\rho_c) \in \text{Hom}(\pi, G)$ . The subgroup  $\rho_o(\pi) = \rho_c(\Gamma) < G(\mathbb{R})$  is dense according to [Theorem 5.1](#).

We next “convert” the open subscheme  $S'_o \subset \text{Hom}_o(\Gamma, G)$  (from the end of the previous section) to an open subscheme  $R' \subset \text{Hom}(\pi, G)$ . Namely, given  $S'_o$ , we let  $R'$  be the subscheme as in [Lemma 4.4](#), namely,  $\phi^*(S'_o \times G^k)$ .

By combining the isomorphism

$$R' \rightarrow S'_o \times G^k \subset \text{Hom}(\Gamma, G) \times G^k$$

with the isomorphism

$$\omega \times \text{id}: S'_o \times G^k \rightarrow X' \times G \times G^k$$

(where  $\omega$  is from [Corollary 5.6](#)), we obtain an isomorphism

$$f: R' \xrightarrow{\cong} S'_o \times G^k \xrightarrow{\cong} X' \times G \times G^k \subset X \times G^{k+1},$$

sending  $\rho_0 \in R'$  to

$$x' = x \times 1 \in X' \times G^{k+1}.$$

By the construction,  $R'$  is open in  $\text{Hom}(\pi, G)$  and  $X' \times G^{k+1}$  is open in  $X \times G^{k+1}$ .

The cross-section  $S'_c \subset S'_o \subset \text{Hom}_o(\Gamma, G)$  (see [Remark 5.2](#)) yields a cross-section  $R'_c \subset R' \subset \text{Hom}(\pi, G)$  for the action  $G \curvearrowright R'$ :

$$R'_c = \phi^*(\psi^*(S'_c) \times G^k).$$

This concludes the proof of [Theorem 1.1](#). □

## 7 Corollaries of Theorem 1.1

[Theorem 1.1](#) deals with representation schemes of 3–manifold groups to  $G = \text{PO}(3)$ ; we now consider the corresponding character schemes. Since  $R'_c \subset \text{Hom}_o(\pi, G)$  is a cross-section for the action of  $G$  on  $R'$ , part 5 of [Theorem 1.1](#) immediately implies [Corollary 1.3](#).

We next consider representations of 3–manifold groups to the group  $\tilde{G} = \text{SL}(2)$ ; we work over  $\mathbb{C}$  and thus identify  $\text{PSL}(2, \mathbb{C})$  with  $\text{PO}(3, \mathbb{C})$ .

Recall that, according to [Theorem 5.1](#) (and [Remark 5.2](#)), for every affine scheme  $X$  over  $\mathbb{Q}$  and a rational point  $x \in X$ , there exists an open subscheme  $X' \subset X$  containing  $x$ , a Coxeter group  $\Gamma$  an open subscheme  $S'_o \subset \text{Hom}(\Gamma, G)$ , and an isomorphism of schemes over  $\mathbb{C}$  (which is the identity on the  $G$ –factor)

$$S'_o \cong S'_c \times G \cong S' \times G \rightarrow X' \times G,$$

sending  $\rho_c \in S'_c$  to  $x \times 1$ . ( $S'_c$  is a certain cross-section for the action  $G \curvearrowright S'_o$ .)

Next, we consider representations of the corresponding extended Coxeter group  $\tilde{\Gamma}$ . [Proposition 3.9](#) gives us a  $G$ –equivariant regular étale covering

$$q: \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G)$$

with covering group  $\mathbb{Z}_2^{k+r}$ . Restricting to  $S'_o \subset \text{Hom}_o(\Gamma, G)$  we obtain a  $G$ –equivariant regular étale covering

$$q': \tilde{S}'_o \rightarrow S'_o \quad \text{where} \quad \tilde{S}'_o = q^{-1}(S'_o) \subset \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \quad \text{and} \quad q' = q|_{\tilde{S}'_o}.$$

We let  $\tilde{\rho}_c: \tilde{\Gamma} \rightarrow \tilde{G}(\mathbb{C})$  be a lift of  $\rho_c$ . The subscheme  $\tilde{S}'_o$  is open in  $\text{Hom}(\tilde{\Gamma}, \tilde{G})$ .

Next, as in [Section 4](#), given the group  $\tilde{\Gamma}$  we construct a closed 3–manifold  $\tilde{M}$  with fundamental group  $\tilde{\pi}$ . We obtain the epimorphism

$$\tilde{\xi}: \tilde{\pi} \rightarrow \tilde{\Gamma} \times F_k \rightarrow \tilde{\Gamma}$$

and set  $\tilde{\rho}_0 := \tilde{\xi}^*(\tilde{\rho}_c) \in \text{Hom}(\tilde{\pi}, \tilde{G})$ .

We have an étale covering

$$\hat{q}: \text{Hom}_{\tilde{\Theta}}(\tilde{\pi}, \tilde{G}) \rightarrow \text{Hom}_{\Theta}(\pi, G),$$

equivariant with respect to the action of  $G$ . Restricting to the open subscheme

$$\tilde{R} = \text{Hom}_o(\tilde{\pi}, \tilde{G}) \subset \text{Hom}_{\Theta}(\tilde{\pi}, \tilde{G}) \subset \text{Hom}(\tilde{\pi}, \tilde{G}),$$

$$\text{where } \tilde{R} = \hat{q}^{-1}(R) \quad \text{and} \quad R = \text{Hom}_o(\pi, G) \subset \text{Hom}_{\Theta}(\pi, G) \subset \text{Hom}(\pi, G),$$

we obtain the  $G$ -equivariant étale covering

$$\tilde{q}: \tilde{R} \rightarrow R$$

with the group  $\mathbb{Z}_2^{k+r}$  of covering transformations. According to Section 4 we also have cross-sections

$$R_c \subset R \quad \text{and} \quad \tilde{R}_c = \tilde{q}^{-1}(R_c) \subset \tilde{R}$$

for the  $G$ -actions on the schemes. The open subschemes that appear in Corollary 1.4 are smaller; we let

$$\tilde{R}' = \tilde{q}^{-1}(R'),$$

where  $R' \subset \text{Hom}(\pi, G)$  is the subscheme appearing in Theorem 1.1. The cross-sections for  $G$ -actions on these subschemes are  $\tilde{R}'_c$  and  $R'_c$ , respectively, where  $\tilde{R}'_c = \tilde{q}^{-1}(R'_c)$ . By the construction  $\tilde{\rho}_0$  belongs to  $\tilde{R}'_c(\mathbb{C})$ . This proves Corollary 1.4.  $\square$

## 8 Orbifold-group representations

Let  $\hat{\Gamma}$  be the fundamental group of the hyperbolic orbifold appearing in Theorem 4.1. This group contains cyclic subgroups  $\langle \theta_i \rangle \cong \mathbb{Z}_2$  for  $i = 1, \dots, 2k$  corresponding to the singular points  $y_i$ . The group  $\Gamma$  is the quotient

$$\hat{\Gamma} / \langle\langle \hat{\Theta} \rangle\rangle,$$

where  $\hat{\Theta} = \{\theta_1, \dots, \theta_{2k}\} \subset \hat{\Gamma}$ . Then for every algebraic group  $H$ ,

$$\text{Hom}(\Gamma, H) \cong \text{Hom}_{\hat{\Theta}}(\hat{\Gamma}, H),$$

and the latter is an open subscheme in  $\text{Hom}(\hat{\Gamma}, H)$  (see Corollary 2.4). Now, let  $\Gamma$  be a Coxeter group (as in Theorem 5.1) or its canonical central extension. In view of Theorems 4.1 and 5.1, we obtain:

**Corollary 8.1** *Theorem 1.1 and Corollaries 1.3 and 1.4 also hold for groups  $\pi$  which are fundamental groups of (nonorientable) 3-dimensional closed hyperbolic orbifolds.*

By passing to torsion-free subgroups of finite index of  $\pi$ , in view of [5, Theorem 5.1], we obtain new examples of fundamental groups of hyperbolic 3-manifolds and their representations to  $SO(3)$  and  $SU(2)$  with nonquadratic singularities of character varieties; see [5, Theorem 5.1], where it is proven that nonquadratic singularities of character schemes are inherited by finite index subgroups. (The first such examples were constructed in [5].)

**Question 8.2** Do Theorem 1.1 and Corollaries 1.3 and 1.4 also hold for groups  $\pi$  which are fundamental groups of 3-dimensional closed hyperbolic manifolds? Do they hold for 3-dimensional manifolds which are 3-dimensional (integer or rational) homology spheres?

### Appendix: Functor of points of affine schemes

The material of this section is standard; we include it for the sake of completeness. While the results follow easily from the Yoneda lemma, we will give a direct proof.

**Lemma A.1** *Let  $f: R \rightarrow S$  be a homomorphism of commutative rings, such that for every commutative ring  $A$  the induced map*

$$f_A^*: \text{Hom}(S, A) \rightarrow \text{Hom}(R, A)$$

*is a bijection. Then  $f$  is an isomorphism.*

**Proof** First, we take  $A = R$ . Since

$$f_R^*: \text{Hom}(S, R) \rightarrow \text{Hom}(R, R)$$

is a bijection, there exists  $g \in \text{Hom}(S, R)$  such that  $f_R^*(g) = \text{id}_R$ , ie

$$g \circ f = \text{id}_R,$$

the identity map. For general  $A$  we have the composition

$$\text{Hom}(R, A) \xrightarrow{g_A^*} \text{Hom}(S, A) \xrightarrow{f_A^*} \text{Hom}(R, A)$$

which satisfies  $f_A^* \circ g_A^* = (g \circ f)_A^* = \text{id}$ . Therefore,  $g_A^*$  is also a bijection.

Now, we take  $A = S$ . Since  $g_S^*$  is a bijection, there exists  $h \in \text{Hom}(R, S)$  such that

$$g^*(\text{id}_S) = h,$$

ie  $h \circ g = \text{id}_S$ . Thus

$$f = \text{id}_S \circ f = h \circ g \circ f = h \circ \text{id}_R = h.$$

Hence,  $f = h$  and the equations  $g \circ f = \text{id}_R$  and  $h \circ g = \text{id}_S$  show that  $g = f^{-1}$ .  $\square$

Suppose now that  $X$  and  $Y$  are affine schemes of finite type over a field  $k$ , ie schemes associated with quotient rings  $R = k[x_1, \dots, x_m]/I$  and  $Y = k[y_1, \dots, y_n]/J$ . For a commutative ring  $A$  the sets  $X(A)$  of  $A$ -points of  $X$  and  $Y(A)$  of  $Y$  are naturally identified with the sets of homomorphisms  $\text{Hom}(R, A)$  and  $\text{Hom}(S, A)$ .

**Corollary A.2** *If  $\phi: Y \rightarrow X$  is a morphism which induces isomorphisms of functors of  $A$ -points, ie bijections  $\phi_A: Y(A) \rightarrow X(A)$  for all commutative rings  $A$ , then  $\phi$  is an isomorphism of schemes.*

**Proof** Consider the ring homomorphism  $f: R \rightarrow S$  associated with  $\phi$ . The bijections  $\phi_A$  are identified with the bijections

$$f_A^*: \text{Hom}(S, A) \rightarrow \text{Hom}(R, A).$$

According to [Lemma A.1](#),  $f$  is a ring isomorphism. Hence,  $\phi$  is an isomorphism of schemes.  $\square$

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