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Growth and order of automorphisms of free groups and free Burnside groups

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We prove that an outer automorphism of the free group is exponentially growing if and only if it induces an outer automorphism of infinite order of free Burnside groups with sufficiently large odd exponent.

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1 Introduction

Let n be an integer. A group G has exponent n if for all $g \in G$, $g^n = 1$. In 1902, W Burnside [7] asked the following question. Is a finitely generated group with finite exponent necessarily finite? In order to study this question, the natural object to look at is the free Burnside group of rank r and exponent n. It is defined to be quotient of the free group $\boldsymbol{F}_r$ of rank r by the (normal) subgroup $\boldsymbol{F}_r^n$ generated by the n^{th} power of all elements. We denote it by $\boldsymbol{B}_r(n)$. Every finitely generated group with finite exponent is a quotient of a free Burnside group.

For a long time, hardly anything was known about free Burnside groups. It was only proved that $\boldsymbol{B}_r(n)$ was finite for some small exponents: $n = 2$, Burnside [7]; $n = 3$, Burnside [7] and Levi and van der Waerden [24]; $n = 4$, Sanov [35]; and $n = 6$, Hall [21]. In 1968, P S Novikov and S I Adian [29; 30; 31] achieved a breakthrough by providing the first examples of infinite Burnside groups. More precisely, they proved the following theorem. Assume that r is at least 2 and n is an odd exponent larger than or equal to 4381; then the free Burnside group of rank r and exponent n is infinite.

This result has been improved in many directions. Adian [1] decreased the bound on the exponent. A Y Ol'shanskiĭ [32] obtained a similar statement using a diagrammatical approach of small cancellation theory. The case of even exponents has been solved by S V Ivanov [22] and I G Lysënok [26]. More recently, T Delzant and M Gromov [17] gave an alternative proof of the infiniteness of Burnside groups. To sharpen our understanding of Burnside groups, we would like to study the symmetries of $\boldsymbol{B}_r(n)$. This leads us to the outer automorphism group of $\boldsymbol{B}_r(n)$.

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The subgroup $\boldsymbol{F}_r^n$ is characteristic. Hence the projection $\boldsymbol{F}_r \twoheadrightarrow \boldsymbol{B}_r(n)$ induces a natural homomorphism $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$. This map is neither one-to-one nor onto. However, it provides numerous examples of automorphisms of Burnside groups. For instance, the first author [13] proved that for sufficiently large odd exponents, the image of $\mathrm{Out}(\boldsymbol{F}_r)$ in $\mathrm{Out}(\boldsymbol{B}_r(n))$ contains free subgroups of arbitrary rank and free abelian subgroups of rank $\lfloor \frac{r}{2} \rfloor$. In this article, we are interested in the following question.

Question Which (outer) automorphism of $\boldsymbol{F}_r$ induces an (outer) automorphism of infinite order of $\boldsymbol{B}_r(n)$?

Let $\boldsymbol{G}$ be a finitely generated group endowed with the word-metric. Given $g \in \boldsymbol{G}$, the length $\|g\|$ of its conjugacy class is the length of the smallest word over the generators which represents an element conjugated to g. Given an outer automorphism Φ of $\boldsymbol{G}$, one says that

- Φ is *exponentially growing* if there exist $g \in \boldsymbol{G}$ and $\lambda > 1$ such that for all integers k, $\|\Phi^k(g)\| \geqslant \lambda^k$,
- Φ is *polynomially growing* if for every $g \in \boldsymbol{G}$, there is a polynomial P such that for all integers k, $\|\Phi^k(g)\| \leqslant P(k)$.

The word-metrics relative to two finite generating sets are bi-Lipschitz equivalent. Therefore, the asymptotic behavior of $\|\Phi^k(g)\|$ does not depend on the choice of generators. Automorphisms of free groups are either exponentially or polynomially growing; see Bestvina and Handel [6] and Bestvina, Feighn and Handel [3]. See also Levitt [25].

We study here the map $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$. Our main theorem states that an automorphism of $\boldsymbol{F}_r$ has exponential growth if and only if it induces an automorphism of infinite order of $\boldsymbol{B}_r(n)$ for sufficiently large exponents n. From the viewpoint of $\mathrm{Out}(\boldsymbol{F}_r)$, this result provides an unexpected characterization of the growth of automorphisms of free groups. At the level of Burnside groups, it completely describes the automorphisms of $\boldsymbol{F}_r$ that induce automorphisms of infinite order of some Burnside groups.

Remark 1.1 Since $\boldsymbol{B}_r(n)$ is a torsion group, every inner automorphism of $\boldsymbol{B}_r(n)$ has finite order. Therefore, an automorphism $\varphi \in \mathrm{Aut}(\boldsymbol{B}_r(n))$ has finite order if and only if its outer class does also. Hence, for our purpose, we can equivalently work with $\mathrm{Out}(\boldsymbol{B}_r(n))$ or $\mathrm{Aut}(\boldsymbol{B}_r(n))$.

The first examples of automorphisms of $\boldsymbol{B}_r(n)$ with infinite order were given by E A Cherepanov [8]. In particular, he proved that the automorphism φ of $\boldsymbol{F}(a,b)$ given by $\varphi(a) = ab$ and $\varphi(b) = a$ (also called the Fibonacci morphism) induces an automorphism of infinite order of $\boldsymbol{B}_2(n)$ for all odd integers $n \geqslant 665$. In [13], the first author provides a large class of automorphisms with the same property.

Theorem 1.2 (Coulon [13, Theorem 1.3]) *Let φ be a hyperbolic automorphism of $\boldsymbol{F_r}$ (ie the semidirect product $\boldsymbol{F_r} \rtimes_\varphi \mathbb{Z}$ is word-hyperbolic). There exists an integer n_0 such that for all odd exponents $n \geqslant n_0$, the automorphism φ induces an element of infinite order of $\mathrm{Out}(\boldsymbol{B_r}(n))$.*

The Fibonacci morphism φ used by Cherepanov is not hyperbolic. Indeed φ fixes the commutator $[a,b] = aba^{-1}b^{-1}$. Hence the semidirect product $\boldsymbol{F_2} \rtimes_\varphi \mathbb{Z}$ contains a copy of $\mathbb{Z}^2$ which is an obstruction to being hyperbolic. This observation has a more general topological interpretation. Indeed, any automorphism φ of $\boldsymbol{F_2}$ can be represented by a homeomorphism f of the punctured torus (if φ is the Fibonacci morphism, then f is even pseudo-Anosov). This map f necessarily preserves the boundary component of the torus — which corresponds to the commutator $[a,b]$. Hence the mapping torus induced by f contains an embedded torus. Its fundamental group $\boldsymbol{F_2} \rtimes_\varphi \mathbb{Z}$ is therefore not hyperbolic.

Nevertheless, like hyperbolic automorphisms, the Fibonacci morphism is exponentially growing. On the other hand, we also know that a polynomially growing automorphism of $\boldsymbol{F_r}$ induces an automorphism of finite order of $\boldsymbol{B_r}(n)$ for every exponent n [13]. It suggests a link between the growth of an automorphism of $\boldsymbol{F_r}$ and its order as automorphism of $\boldsymbol{B_r}(n)$. More precisely, we prove the following statement.

Theorem 1.3 *Let $\Phi \in \mathrm{Out}(\boldsymbol{F_r})$ be an outer automorphism of $\boldsymbol{F_r}$. The following assertions are equivalent:*

(1) *Φ has exponential growth;*

(2) *there exists $n \in \mathbb{N}$ such that Φ induces an outer automorphism of $\boldsymbol{B_r}(n)$ of infinite order;*

(3) *there exist $\kappa, n_0 \in \mathbb{N}$ such that for all odd integers $n \geqslant n_0$, the automorphism Φ induces an outer automorphism of $\boldsymbol{B_r}(\kappa n)$ of infinite order.*

Remark In this article, we adopt the following convention. The notation $\mathbb{N}$ stands for the set of nonnegative integers, whereas $\mathbb{N}^*$ represents $\mathbb{N} \setminus \{0\}$.

In the statement of Theorem 1.3, (3) $\Longrightarrow$ (2) is easy whereas (2) $\Longrightarrow$ (1) follows from the work of the first author [13, Theorem 1.6]. The new result of this article is the implication (1) $\Longrightarrow$ (3). Before sketching this proof, let us have a look at the arguments used by Cherepanov [8]. The proof of the infiniteness of $\boldsymbol{B_r}(n)$ by Novikov and Adian is based on the following important fact [1].

Proposition 1.4 *Let w be a reduced word of $\boldsymbol{F_r}$. If w does not contain a subword of the form u^{16}, then w induces a nontrivial element of $\boldsymbol{B_r}(n)$ for all odd exponents $n \geqslant 655$.*

In particular, two distinct reduced words without an 8^{th} power define distinct elements of $\boldsymbol{B}_r(n)$. Compute now the orbit of b under the automorphism φ of $\boldsymbol{F}(a,b)$ defined by $\varphi(a)=ab$ and $\varphi(b)=a$. It leads to the following sequence of words:

$$\begin{aligned} &\varphi^1(b)=a, && \varphi^5(b)=abaababa,\\ &\varphi^2(b)=ab, && \varphi^6(b)=abaababaabaab,\\ &\varphi^3(b)=aba, && \varphi^7(b)=abaababaabaababaababa,\\ &\varphi^4(b)=abaab, && \quad\vdots \end{aligned}$$

None of these words contains a 4^{th} power; see Karhumäki [23]. Therefore, they induce pairwise distinct elements of $\boldsymbol{B}_r(n)$. In particular, φ seen as an automorphism of $\boldsymbol{B}_r(n)$, has infinite order.

This argument can be generalized for any exponentially growing automorphism of $\boldsymbol{F}_2$ using an appropriate train track representative. However, it does not work anymore in higher rank. Consider, for instance, the exponentially growing automorphism ψ of $\boldsymbol{F}(a,b,c,d)$ defined by $\psi(a)=a$, $\psi(b)=ba$, $\psi(c)=cbcd$ and $\psi(d)=c$. As previously, we compute the orbit of d under ψ:

$$\begin{aligned} &\psi^1(d)=c,\\ &\psi^2(d)=c\mathbf{b}cd,\\ &\psi^3(d)=cbcd\mathbf{ba}cbcdc,\\ &\psi^4(d)=cbcdbacbcdc\mathbf{ba}^2cbcdbacbcdccbcd,\\ &\psi^5(d)=cbcdbacbcdcba^2cbcdbacbcdc^2bcd\mathbf{ba}^3cbcdbacbcdcba^2\dots\\ &\qquad\qquad\qquad\qquad\qquad\qquad\dots cbcdbacbcdc^2bcdcbcdbacbcdc. \end{aligned}$$

This orbit is exponentially growing. Note that if $\psi^p(d)$ contains a subword ba^m then $\psi^{p+1}(d)$ contains ba^{m+1}. Hence as p tends to infinity, $\psi^p(d)$ contains arbitrarily large powers of a. This cannot be avoided by choosing the orbit of another element. Proposition 1.4 is no more sufficient to tell us whether or not the $\psi^p(d)$ are pairwise distinct in $\boldsymbol{B}_r(n)$. Therefore, we need a more accurate criterion to distinguish two different elements of $\boldsymbol{B}_r(n)$. This is done using elementary moves.

Let $n\in\mathbb{N}$ and $\xi\in\mathbb{R}_+$. An (n,ξ)*–elementary move* consists in replacing a reduced word of the form $pu^ms\in\boldsymbol{F}_r$ by the reduced representative of $pu^{m-n}s$, provided m is an integer larger than $\frac{n}{2}-\xi$. The word u is called the *support* of the elementary move. Note that an elementary move may increase the length of the word.

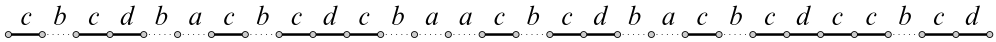

Figure 1: The yellow-red decomposition of $\psi^4(d)$

Theorem 1.5 (Coulon [12]) *There exist integers n_1 and ξ such that for all odd exponents $n \geqslant n_1$, we have the following property. Let w and w' be two reduced words of $\boldsymbol{F}_r$. If w and w' define the same element of $\boldsymbol{B}_r(n)$, then there are two sequences of (n,ξ)–elementary moves which respectively send w and w' to the same word.*

Remark As will be detailed in Section 6.1, this statement is a direct application of the main theorem of Coulon [12]. Its proof relies on the geometric approach of the Burnside problem developed by Delzant and Gromov [17]. Although Theorem 1.5 is not explicitly mentioned in their articles, it should be possible to deduce an analogue statement from the work of Adian [1] and Ol'shanskiĭ [32]. For the convenience of the reader who would be more familiar with Ol'shanskiĭ's techniques, these analogies and differences are discussed in Section 6.1 and in the Appendix.

Thanks to this tool, we can now explain using the example ψ how the implication (1) $\Longrightarrow$ (3) of Theorem 1.3 works. We need to understand the effect of elementary moves on a word $\psi^p(d)$. To that end, we assign colors to the letters. Let us say that a and b are *yellow* letters (dotted lines on Figure 1) whereas c and d are *red* letters (thick lines on the figure). The word $\psi^p(d)$ is the concatenation of maximal yellow and red subwords. To any word w over the alphabet $\{a,b,c,d\}$ we associate its red part $\mathrm{Red}(w)$ obtained by removing from w all the yellow letters. We start with two observations, one on the red words, the other on the yellow ones.

Red words We claim that the support of elementary moves that can be performed on $\psi^p(d)$ only contains yellow letters. Since the orbit of d grows exponentially, one can prove that $\mathrm{Red}(\psi^p(d))$ does not contain large powers. More precisely, there is an integer n_2 such that for all $p \in \mathbb{N}$, the word $\mathrm{Red}(\psi^p(d))$ does not contain any n_2^{th} power; see Proposition 5.11. This fact can be interpreted in terms of dynamical properties of the attracting laminations associated to the automorphism ψ. Let $n > 2n_2 + 2\xi$. Assume now that the support u of an (n,ξ)–elementary move performed on $\psi^p(d)$ contains a red letter. By definition, there exists $m > n_2$ such that u^m is a subword of $\psi^p(d)$. In particular, $\mathrm{Red}(u)^m$ is a subword of $\mathrm{Red}(\psi^p(d))$, which contradicts the definition of n_2. It follows from this remark that the support of any (n,ξ)–elementary move with $n > 2n_2 + 2\xi$ only contains yellow letters.

Yellow words We now claim that elementary moves with yellow support cannot send a maximal yellow subword of $\psi^p(d)$ to the empty word. This fact is important for the

Word before the elementary move:

Word after the elementary move:

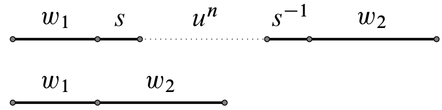

Figure 2: An elementary move collapsing red letters

following reason. We explained that the support of an elementary move performed on $\psi^p(d)$ only contains yellow letters. Such a move could change the red part of $\psi^p(d)$, though. It could indeed completely collapse a maximal yellow subword and thus affect the red letters; see Figure 2.

To prove this second claim, we look at the yellow subwords of $\psi^p(d)$. Notice that the image by ψ of a yellow word is still a yellow word. On the contrary, the image of a red word may contain yellow subwords. Indeed b is a subword of $\psi(c)$. Actually, the yellow subwords of $\psi^p(d)$ can be sorted in two categories: the words that consist in the single letter b which appear as a subword of $\psi(c)$ and the ones which arise as the images by ψ of yellow subwords of $\psi^{p-1}(d)$. In particular, all the maximal yellow subwords of $\psi^p(d)$ belong to the orbit under ψ of b. Consequently, there is an integer n_3 such that for every odd integer $n > n_3$, none of them becomes trivial in $\boldsymbol{B}_r(n)$. In particular, no sequence of (n,ξ)–elementary moves sends a maximal yellow subword of $\psi^p(d)$ to the empty word.

We can now argue by contradiction. Let $n > \max\{n_1, 2n_2 + 2\xi, n_3\}$ be an odd integer. Assume that ψ induces an automorphism of finite order of $\boldsymbol{B}_r(n)$. In particular, there exists $p \in \mathbb{N}^*$ such that $\psi^p(d)$ and d have the same image in $\boldsymbol{B}_r(n)$. It follows from Theorem 1.5 that a sequence of (n,ξ)–elementary moves sends $\psi^p(d)$ to d. We claim that performing (n,ξ)–elementary moves on $\psi^p(d)$ does not change its red part. Indeed, $n > 2n_2 + 2\xi$; thus these moves will only change the yellow subwords of $\psi^p(d)$. Moreover, since $n > n_3$, none of the yellow words can completely disappear. In particular, the red word $\mathrm{Red}(\psi^p(d))$ associated to $\psi^p(d)$ should be exactly d. This is a contradiction.

The proof for an arbitrary exponentially growing automorphism of $\boldsymbol{F}_r$ follows the same ideas. One has to replace the words in a, b, c, d by paths in an appropriate relative train track. This leads to a technical difficulty, though. The red and yellow paths that we want to consider do not necessarily represent elements of the free groups. This problem is handled in Sections 4.2 and 5. There we use subtle aspects of the machinery of train-tracks to show that the red words do not contain large powers (Proposition 5.11). In particular, we need to pass to a finite-index subgroup of $\boldsymbol{F}_r$. This operation actually ensures at the same time that no yellow subpath will be removed by elementary moves (see the prior discussion). Beside this fact, the main ingredients are the ones described above.

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2 Primitive matrices and substitutions

In this section, we summarize a few properties about primitive integer matrices and substitutions on an alphabet that will be useful later.

2.1 Primitive matrices

A square matrix M of size ℓ whose entries are nonnegative integers is *irreducible* if for each $i, j \in \{1, \dots, \ell\}$, there exists $p \in \mathbb{N}$ such that the (i, j)–entry of M^p is not zero. It is *primitive* when there exists $p \in \mathbb{N}$ such that any entry of M^p is not zero.

The Perron–Frobenius theorem for an irreducible matrix M with nonnegative integer entries states that there exists a unique dominant eigenvalue $\lambda \geqslant 1$ of M associated to an eigenvector with positive coordinates (see for instance Seneta's book [36]). This λ is called the *Perron–Frobenius-eigenvalue* (or simply *PF-eigenvalue*) of M. In addition, if $\lambda = 1$, then M is a transitive permutation matrix.

2.2 Primitive substitutions

Let $\mathcal{A} = \{a_1, \dots, a_\ell\}$ be a finite alphabet. The free monoid generated by $\mathcal{A}$ is denoted by $\mathcal{A}^*$. We write 1 for the empty word, also called the *trivial word*. An infinite word is an element of $\mathcal{A}^{\mathbb{N}}$. Let $m \in \mathbb{N}^*$. A word $w \in \mathcal{A}^*$ is an m^{th} *power* if there exists a nontrivial word $u \in \mathcal{A}^*$ such that $w = u^m$. A nontrivial word $w \in \mathcal{A}^*$ is *primitive* if it is not an m^{th} power with m at least 2 (ie if $w = u^m$, then $u = w$ and $m = 1$). A word $w \in \mathcal{A}^*$ (or an infinite word $w \in \mathcal{A}^{\mathbb{N}}$) *contains an* m^{th} *power*, if there exists a word $u \in \mathcal{A}^* \setminus \{1\}$ such that u^m is a subword of w. The *shift* is the map $S\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ which sends $(w_i)_{i \in \mathbb{N}}$ to $(w_{i+1})_{i \in \mathbb{N}}$. An infinite word w is said to be *shift-periodic* if there exists $q \in \mathbb{N}^*$ such that $S^q(w) = w$. If u stands for the word $w_0 w_1 \cdots w_{q-1}$, then we write $w = u^\infty$. Roughly speaking, it means that w is the infinite power of u.

An endomorphism of the free monoid $\mathcal{A}^*$ is called a *substitution* defined on $\mathcal{A}$. Such a substitution σ is indeed completely determined by the images $\sigma(a) \in \mathcal{A}^*$ of all the letters $a \in \mathcal{A}$. Moreover, it naturally extends to a map $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. The *matrix* M

of a substitution σ is a square matrix of size ℓ whose (i,j)–entry is the number of occurrences of the letter a_i in the word $\sigma(a_j)$. The substitution σ is said to be *primitive* when M is primitive.

Proposition 2.1 *Let a be a letter of $\mathcal{A}$. Let σ be a primitive substitution on $\mathcal{A}$ such that a is a prefix of $\sigma(a)$.*

(i) *The sequence $(\sigma^p(a))$ converges for the prefix topology to an infinite word $\sigma^\infty(a)$ fixed by σ.*

(ii) *If $\sigma^\infty(a)$ is not shift-periodic, then there exists an integer $m \geqslant 2$ such that for all $p \in \mathbb{N}$, the word $\sigma^p(a)$ does not contain an m^{th} power.*

(iii) *If there exists a nontrivial primitive word u such that $\sigma^\infty(a) = u^\infty$, then there exists an integer $q \geqslant 2$ such that $\sigma(u) = u^q$.*

Remark 2.2 The case covered by Proposition 2.1(iii) is not vacuous. Consider for instance the substitution defined on $\mathcal{A} = \{a,b,c\}$ by $\sigma(a) = ab$, $\sigma(b) = c$ and $\sigma(c) = abc$. The transition matrix M of σ and its square are

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

In particular, σ is primitive. However, $(\sigma^n(a))$ converges to the infinite shift-periodic word $(abc)^\infty$.

To prove Proposition 2.1, we use the following results due to B Mossé.

Proposition 2.3 (Mossé [27, Théorème 2.4]) *Let σ be a primitive substitution on a finite alphabet $\mathcal{A}$. Let $u \in \mathcal{A}^{\mathbb{N}}$ be an infinite word fixed by σ. Then either*

(i) *u is shift-periodic, or*

(ii) *there exists an integer $m \geqslant 2$ such that u does not contain an m^{th} power.* □

Lemma 2.4 (Mossé [27, Proposition 2.3]) *Let $u \in \mathcal{A}^*$ be a primitive word. Let $m \geqslant 2$ be an integer. If uwu is subword of u^m, then there exists an integer $p \geqslant 0$ such that $w = u^p$.* □

Proof of Proposition 2.1 By assumption, a is a prefix of $\sigma(a)$. Thus there exists $w \in \mathcal{A}^*$ such that $\sigma(a) = aw$. Since σ is primitive, w is not trivial. For every $p \in \mathbb{N}^*$, it follows that $\sigma^p(a)$ is exactly the word

$$\sigma^p(a) = aw\sigma(w)\sigma^2(w)\cdots\sigma^{p-1}(w).$$

In particular, $\sigma^p(a)$ is a prefix of $\sigma^{p+1}(a)$. Therefore, $(\sigma^p(a))$ converges to an infinite word $\sigma^\infty(a)$ fixed by σ:

$$\sigma^\infty(a) = aw\sigma(w)\sigma^2(w)\cdots\sigma^p(w)\cdots,$$

which proves (i). Assume now that this infinite word is not shift-periodic. According to Proposition 2.3, there exists $m \geqslant 2$ such that $\sigma^\infty(a)$ does not contain an m^{th} power. The same holds for the prefixes of $\sigma^\infty(a)$, in particular for all $\sigma^p(a)$, which proves (ii).

Finally, assume that $\sigma^\infty(a) = u^\infty$, where u is a nontrivial primitive word. Since $\sigma^\infty(a)$ is fixed by σ, we obtain that $u^\infty = \sigma(u)^\infty$. In particular, u is a prefix of $\sigma^\infty(u)$. The substitution σ being primitive, $\sigma(u)$ is not shorter than u. We derive that there exists $w_0 \in \mathcal{A}^*$ such that $\sigma(u) = uw_0$. Hence $u^\infty = (uw_0)^\infty$. Lemma 2.4 shows that there exists $p \in \mathbb{N}$ satisfying $w_0 = u^p$. Thus $\sigma(u) = u^{p+1}$. Remember that a is a prefix of u. Hence u can be written $u = au'$. It follows that the length of $\sigma(u) = aw\sigma(u')$ is larger than that of u. Thus $p+1 \geqslant 2$, which proves (iii). □

3 Train-tracks and automorphisms of free groups

In this section, we recollect some material about relative train-track maps. Details can be found in [6], where they have been introduced by Bestvina and Handel. There exist several improvements of relative train-track maps, and we will use here (very few of the) improved relative train-track maps introduced by Bestvina, Feighn and Handel in [4].

3.1 Paths and circuits

The graphs that we consider are metric graphs with oriented edges. By metric graph, we mean a graph equipped with a path metric. If e is an edge of a graph G, then e^{-1} stands for the edge with the reverse orientation. The pair $\{e, e^{-1}\}$ is the *unoriented edge associated to* e (or e^{-1}). By abuse of notation, we will just say the *unoriented edge* e for the pair $\{e, e^{-1}\}$. Let $\Theta\colon \mathcal{E} \to \mathcal{E}$ be the map defined by $\Theta(e) = e^{-1}$. Sometimes, it will be useful to consider a subset $\vec{\mathcal{E}}$ of $\mathcal{E}$ such that $\vec{\mathcal{E}}$ and $\Theta(\vec{\mathcal{E}})$ give rise to a partition of $\mathcal{E}$ (ie we choose a preferred oriented edge for each unoriented edge). We call such a set $\vec{\mathcal{E}}$ a *preferred set of oriented edges for* G.

A *path* in a graph G is a continuous locally injective map $\alpha\colon I \to G$, where $I = [a,b]$ is a segment of $\mathbb{R}$. The *initial point* of α is $\alpha(a)$ and its *terminal point* is $\alpha(b)$; both $\alpha(a)$ and $\alpha(b)$ are the *endpoints* of α. We do not make any distinction between two paths which differ by an orientation-preserving homeomorphism between their domains.

A path is *trivial* if its domain is a point. When the endpoints of α are vertices, α can be viewed as a *path of edges*, ie a concatenation of edges $\alpha = e_1 \cdots e_p$, where the e_i are edges of G such that the terminal vertex of e_i is the initial vertex of e_{i+1} and $e_i \neq e_{i+1}^{-1}$. A *circuit* in G is a continuous locally injective map of an oriented circle into G. We do not make any distinction between two circuits which differ by an orientation-preserving homeomorphism between their domains. A circuit can be viewed as a cyclically ordered sequence of edges without backtracking. If α is a path or a circuit, we denote by α^{-1} the path or circuit, with the reverse orientation.

A continuous map $\alpha\colon I \to G$, where I is segment in $\mathbb{R}$, is homotopic relative to the endpoints to a unique path denoted by $[\alpha]$. A nonhomotopically trivial continuous map $\alpha\colon S^1 \to G$ is homotopic to a unique circuit denoted by $[\alpha]$.

3.2 Topological representatives

Marked graphs and topological representatives Let $r \geqslant 2$. We denote by R_r the *rose* of rank r. It is a graph with one vertex $\star$ and r unoriented edges. The fundamental group $\pi_1(R_r, \star)$ is the free group $\boldsymbol{F}_r$, with basis given by a preferred set of oriented edges. A *marked graph* (G, τ) (often simply denoted by G) is a connected metric graph G having no vertex of valence 1, equipped with a homotopy equivalence $\tau\colon R_r \to G$. This homotopy equivalence τ gives an identification of the fundamental group $\pi_1(G, \tau(\star))$ with $\boldsymbol{F}_r$, well defined up to an inner automorphism. A *topological representative* of an outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ is a homotopy equivalence $f\colon G \to G$ of a marked graph (G, τ) such that

- f takes vertices to vertices and edges to paths of edges,
- $\tau^{-} \circ f \circ \tau\colon R_r \to R_r$ induces Φ on $\boldsymbol{F}_r = \pi_1(R_r, \star)$, where τ^{-} is a homotopy inverse of τ.

In particular, the restriction of f to an open edge is locally injective.

Induced map on paths and circuits If α is a path or a circuit in G, one defines $f_\#(\alpha)$ as being equal to $[f(\alpha)]$.

Legal turns For any edge e of G, we let $Df(e)$ denote the first edge of $f(e)$. A *turn* is a pair of edges (e_1, e_2) of G which have the same initial vertex. The turn (e_1, e_2) is *degenerate* if $e_1 = e_2$, and nondegenerate otherwise. A turn (e_1, e_2) is *legal* if $((Df)^p(e_1), (Df)^p(e_2))$ is nondegenerate for all $p \in \mathbb{N}$; otherwise, the turn is *illegal*.

3.3 Lifts

Let $f\colon G \to G$ be a topological representative of $\Phi \in \mathrm{Out}(\boldsymbol{F_r})$. Let $\widetilde{G}$ be the universal cover of G. The theory of covering spaces gives a one-to-one correspondence between the set of the lifts of f to $\widetilde{G}$ and the set of automorphisms in the outer class Φ. More precisely, a lift $\widetilde{f}$ of f is in correspondence with the automorphism $\varphi \in \Phi$ if

$$\widetilde{f} \circ g = \varphi(g) \circ \widetilde{f} \quad \text{for all } g \in \boldsymbol{F_r}, \tag{1}$$

where the elements of $\boldsymbol{F_r}$ are viewed as deck transformations of $\widetilde{G}$.

3.4 Invariant filtrations and transition matrices

Let $f\colon G \to G$ be a topological representative of $\Phi \in \mathrm{Out}(\boldsymbol{F_r})$.

Filtration, strata and k–legal paths A *filtration* of a topological representative $f\colon G \to G$ is a strictly increasing sequence of f–invariant subgraphs $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$. The *stratum of height* k denoted by H_k is the closure of $G_k \setminus G_{k-1}$. The edges *of height* k are the edges of H_k. A path *of height* k is a path in G_k which crosses H_k nontrivially; ie its intersection with H_k contains a nontrivial path. A path (of edges) α is k*–legal* if it is a path of G_k and for all subpaths e_1e_2 of α with e_1, e_2 edges of height k, the turn (e_1^{-1}, e_2) is legal.

Transition matrices A *transition matrix* M_k is associated to the stratum H_k. We choose a preferred set of oriented edges $\vec{\mathcal{E}} = \{e_1, \dots, e_\ell\}$ for H_k (where ℓ is the number of unoriented edges of H_k). The transition matrix M_k of H_k is a square matrix of size ℓ whose (i,j)–entry is the number of times the edge e_i or the reverse edge e_i^{-1} occur in the path $f(e_j)$.

The stratum H_k is *irreducible* when its transition matrix M_k is irreducible. Let λ_k be the PF-eigenvalue of M_k; see Section 2.1. If $\lambda_k > 1$, then H_k is called an *exponential stratum*. If M_k is primitive, H_k is said to be *aperiodic*. When the stratum H_k is irreducible and $\lambda_k = 1$, H_k is called a *nonexponential stratum*. When M_k is the zero matrix, the stratum H_k is called a *zero stratum*.

Remark 3.1 Given a topological representative $f\colon G \to G$ and an invariant filtration for f, up to refining the filtration, one can always suppose that any stratum is of one of three possible types: exponential, nonexponential or zero. Moreover, up to replacing Φ by a positive power of Φ, one can assume that Φ admits a topological representative $f\colon G \to G$ with the following properties [4]:

- each exponential stratum is aperiodic,
- each nonexponential stratum H_k consists of a single edge e, and that $f(e) = eu$ where u is loop in G_{k-1} based at the endpoint of e.

3.5 A quick review on relative train-track maps

Relative train-track maps A topological representative $f\colon G \to G$ of an outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F_r})$ with a filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ is a *relative train-track map* (RTT) if for every exponential stratum H_k,

- (RTT-i) Df maps the set of edges of height k to itself (in particular, each turn consisting of an edge of height k and one of height less than k is legal);
- (RTT-ii) if α is a nontrivial path with endpoints in $H_k \cap G_{k-1}$, then $f_\#(\alpha)$ is a nontrivial path with endpoints in $H_k \cap G_{k-1}$;
- (RTT-iii) for each k–legal path α, the path $f_\#(\alpha)$ is k–legal.

In particular, an edge e of an exponential stratum H_k is k–legal. Theorem 5.12 in [6] ensures that any outer automorphism Φ of $\boldsymbol{F_r}$ can be represented by an RTT f. By replacing Φ by a positive power of Φ if necessary, one can suppose that Φ satisfies Remark 3.1. In addition, we can ask that all the images of vertices are fixed by f (see Theorem 5.1.5 in [4]). We sum up these facts in the following theorem.

Theorem 3.2 (Bestvina and Handel [6], Bestvina, Feighn and Handel [4]) *Let Φ be an outer automorphism of $\boldsymbol{F_r}$. There exists $p \geqslant 1$ such that Φ^p has a topological representative $f\colon G \to G$ which is an RTT, with the properties that*

- *for all vertices v of G, $f(v)$ is fixed by f,*
- *every exponential stratum of f is aperiodic,*
- *each nonexponential stratum H_k consists of a single edge e, and that $f(e) = eu$, where u is loop in G_{k-1} based at the endpoint of e.*

Splittings Let $f\colon G \to G$ be a topological representative. A *splitting* of a path or a circuit α is a decomposition of α as a concatenation of subpaths $\alpha = \alpha_1\alpha_2\cdots\alpha_q$ (with $q \geqslant 1$ if α is a circuit, and $q \geqslant 2$ if α is a path) such that for all $p \geqslant 0$, $f_\#^p(\alpha) = f_\#^p(\alpha_1) f_\#^p(\alpha_2)\cdots f_\#^p(\alpha_q)$. In that case, one writes $\alpha = \alpha_1 \cdot \alpha_2 \cdots \alpha_q$, and $\alpha_1, \alpha_2, \dots, \alpha_q$ are called the *terms* of the splitting. A basic, but important, property of RTT is given by the following lemma.

Lemma 3.3 (Bestvina and Handel [6, Lemma 5.8]) *Let $f\colon G \to G$ be an RTT. If H_k is an exponential stratum, and if α is a k–legal path, then the decomposition of α as maximal subpaths in H_k or in G_{k-1} is a splitting*

$$\alpha = \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdots \alpha_{q-1} \cdot \beta_{q-1} \cdot \alpha_q,$$

where the α_i are paths in H_k and the β_i are paths in G_{k-1}, all nontrivial (except possibly α_1 and α_q).

3.6 Growth of automorphisms of free groups

As explained in [3, page 219], the growth of an outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ can be detected on an RTT representative [6]. For our purpose, we will use the following observations.

Remark 3.4 [6; 3] Let $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$.

(1) Φ has either polynomial growth or exponential growth.

(2) Moreover, Φ has exponential growth if and only if one (hence any) RTT $f\colon G \to G$ representing Φ has at least one exponential stratum.

A detailed discussion about the growth of a conjugacy class under iteration of an outer automorphism can be found in [25].

4 Reductions of Theorem 1.3

In this section, we explain how to reduce our main theorem to an easier statement. First, note that given an outer automorphism Φ of the free group, Φ has exponential (polynomial) growth if and only if for every $p \in \mathbb{N}^*$, so does Φ^p. In particular, to prove Theorem 1.3, Φ can be replaced by some positive power of Φ. It will be advantageous to do so, since it allows us to use relative train-track maps with better properties; see Theorem 3.2. We now discuss three reductions.

(1) The first focuses on polynomially growing automorphisms of $\boldsymbol{F}_r$; see Section 4.1. We explain that such an automorphism always induces a finite-order automorphism of Burnside groups. Thus it is sufficient to look at exponentially growing automorphisms.

(2) The second reduction is rather technical. Let $f\colon G \to G$ be an RTT of an exponentially growing automorphism Φ of $\boldsymbol{F}_r$ and H an exponential stratum. The image under f of an edge e in H consists of edges of H and paths contained in lower strata. Later we will need that for every $p \in \mathbb{N}$, maximal subpaths of $f_\sharp^p(e)$ contained in the lower strata are not loops. In Section 4.2, we show that up to passing to a finite-index subgroup, we can always assume that our RTT satisfies this property.

(3) The RTT of an exponentially growing automorphism may contain several exponential strata. In Section 4.3, we prove that it is sufficient to consider automorphisms whose RTT has only one exponential stratum, which is also the top one.

4.1 Polynomially growing automorphisms

Arguing by induction on the rank r of $\boldsymbol{F}_r$, the first author handled the case of polynomially growing automorphisms.

Proposition 4.1 (Coulon [13, Theorem 1.6]) *If $\Phi \in \operatorname{Out}(\boldsymbol{F}_r)$ is polynomially growing, then Φ induces an outer automorphism of finite order of $\boldsymbol{B}_r(n)$ for all positive integers n.*

Remark 4.2 The same proof actually gives a quantitative bound for the order of Φ in $\operatorname{Out}(\boldsymbol{B}_r(n))$. If Φ is an outer polynomially growing automorphism of $\boldsymbol{F}_r$, then $\Phi^{p(r,n)}$ induces a trivial outer automorphism of $\boldsymbol{B}_r(n)$, where

$$p(r,n) = n^{2(2^{r-1}-1)}.$$

Example 4.3 A particular case of polynomially growing automorphisms is given by the automorphisms of $\boldsymbol{F}_2$ induced by a Dehn-twist on a punctured torus. For instance, the automorphism φ defined by $\varphi(a) = a$ and $\varphi(b) = ba$. Here φ^n is trivial in $\operatorname{Aut}(\boldsymbol{B}_r(n))$.

In view of Remark 3.4 (1) and Proposition 4.1, we see that Theorem 1.3 is a consequence of the following proposition.

Proposition 4.4 *If $\Phi \in \operatorname{Out}(\boldsymbol{F}_r)$ has exponential growth, then there exist $\kappa, n_0 \in \mathbb{N}$ such that for all odd integers $n \geqslant n_0$, the automorphism Φ induces an outer automorphism of $\boldsymbol{B}_r(\kappa n)$ of infinite order.*

In the next section, we discuss a second reduction and prove that Proposition 4.4 is a consequence of Proposition 4.8.

4.2 Passing to a finite-index subgroup

Let Φ be an exponentially growing outer automorphism of $\boldsymbol{F}_r$. By replacing Φ by a power of Φ if necessary, we can assume that Φ is represented by an RTT $f\colon G \to G$ with a filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$, satisfying the properties of Theorem 3.2. We denote by H_k the stratum of height k. Let e be an edge of an exponential stratum H_k. According to Lemma 3.3, $f(e)$ can be split as follows:

$$f(e) = \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdots \alpha_{q-1} \cdot \beta_{q-1} \cdot \alpha_q,$$

where the α_i are nontrivial paths contained in H_k and the β_i are nontrivial paths contained in G_{k-1}. We denote by $\mathcal{P}_e$ the set $\{\beta_1, \dots, \beta_{q-1}\}$. Let $\mathcal{P}$ be the union of $\mathcal{P}_e$ for all edges e belonging to an exponential stratum. Note that $\mathcal{P}$ is finite.

Let $p \in \mathbb{N}$ and e be an edge of the exponential stratum H_k. Recall that the image under f of an edge of H_k starts and ends by an edge of H_k (property (RTT-i) of relative train-track maps). Thus if β is a maximal subpath of $f_\#^p(e)$ contained in G_{k-1}, then it is the image by some (possibly trivial) power of $f_\#$ of a path in $\mathcal{P}$. Moreover, we assumed that the image by f of any vertex of G is fixed by f. Hence if β is also a loop, there exists a path β' in $\mathcal{P} \cup f_\#(\mathcal{P})$ which is a loop such that β is the image of β' by a (possibly trivial) power of $f_\#$. Since $\boldsymbol{F_r}$ is residually finite, there exists a finite-index normal subgroup $\boldsymbol{H}$ of $\boldsymbol{F_r}$ with the following property. For every path $\beta \in \mathcal{P} \cup f_\#(\mathcal{P})$, if β is a loop, then the conjugacy class of $\boldsymbol{F_r}$ that it represents does not intersect $\boldsymbol{H}$.

Recall that $\widetilde{G}$ stands for the universal cover of G. Let us fix a base point x_0 in G. The fundamental group $\boldsymbol{F_r} = \pi_1(G, x_0)$ can therefore be identified with the deck transformation group acting on the left on $\widetilde{G}$. We fix a lift $\tilde{f}\colon \widetilde{G} \to \widetilde{G}$ of f. It determines an automorphism φ in the outer class of Φ such that for every $g \in \boldsymbol{F_r}$,

$$\varphi(g) \circ \tilde{f} = \tilde{f} \circ g. \tag{2}$$

There are only finitely many subgroups of $\boldsymbol{F_r}$ of a given index. Thus there exists an integer q such that $\varphi^q(\boldsymbol{H}) = \boldsymbol{H}$. Consequently, the intersection $\boldsymbol{L} = \bigcap_{p\in\mathbb{Z}} \varphi^p(\boldsymbol{H})$ is also a normal finite-index subgroup of $\boldsymbol{F_r}$. By definition, $\boldsymbol{L}$ is invariant by both φ and φ^{-1}. It directly follows that φ induces an automorphism of $\boldsymbol{L}$.

We now denote by κ the index of $\boldsymbol{L}$ in $\boldsymbol{F_r}$. Let $\widehat{G}$ be the space $\widehat{G} = \boldsymbol{L}\backslash\widetilde{G}$ and $\rho\colon \widehat{G} \to G$ the natural projection induced by $\widetilde{G} \to G$. The group $\boldsymbol{F_r}$ still acts on the left on $\widehat{G}$ and $\boldsymbol{L}$ is the kernel of this action. The map $\tilde{f}$ induces a map $\hat{f}\colon \widehat{G} \to \widehat{G}$ such that $\rho \circ \hat{f} = f \circ \rho$. Moreover, according to (2), for every $g \in \boldsymbol{F_r}$,

$$\varphi(g) \circ \hat{f} = \hat{f} \circ g. \tag{3}$$

Lemma 4.5 *The map $\hat{f}\colon \widehat{G} \to \widehat{G}$ admits a filtration which makes $\hat{f}$ an RTT representing the outer class of φ restricted to $\boldsymbol{L}$. Moreover, for every exponential stratum $\widehat{H}$ of $\widehat{G}$, there exists an exponential stratum H_k of G such that*

1. *$\widehat{H}$ is contained in $\rho^{-1}(H_k)$, and*
2. *$\hat{f}$ sends $\widehat{H}$ into $\widehat{H} \cup \rho^{-1}(G_{k-1})$.*

Proof We observe that, by construction, $\hat{f}\colon \widehat{G} \to \widehat{G}$ is a topological representative of φ restricted to $\boldsymbol{L}$, and $\varnothing = \rho^{-1}(G_0) \subset \rho^{-1}(G_1) \subset \cdots \subset \rho^{-1}(G_m) = \widehat{G}$ is an invariant filtration for $\hat{f}$. We are going to define a finer filtration $(\widehat{G}_{k,j})$ where the pairs (k, j) are endowed with the lexicographical order such that for every $k \in \{1, \dots, m\}$, we have

$$\rho^{-1}(G_{k-1}) \subset \widehat{G}_{k,1} \subset \widehat{G}_{k,2} \subset \cdots \subset \widehat{G}_{k,s} = \rho^{-1}(G_k).$$

Let $k \in \{0, \dots, m\}$. We focus on the stratum H_k of height k of G. We distinguish three cases.

(1) If H_k is a zero stratum, we just put $\widehat{G}_{k,1} = \rho^{-1}(G_k)$. Since $\rho \circ \widehat{f} = f \circ \rho$, we have $\widehat{f}(\widehat{G}_{k,1}) \subseteq \rho^{-1}(G_{k-1})$. Therefore, the associated stratum is a zero stratum.

(2) If H_k is a nonexponential stratum, it consists of a single edge e with $f(e) = eu$ where u is a loop in G_{k-1}. Then $\rho^{-1}(e)$ is a collection of κ edges: $\widehat{e}_1, \dots, \widehat{e}_\kappa$ (recall that κ is the index of $\boldsymbol{L}$ in $\boldsymbol{F}_r$). Since $\rho \circ \widehat{f} = f \circ \rho$, the map $\widehat{f}$ induces a permutation σ of $\{1, \dots, \kappa\}$ with the following property. For every $j \in \{1, \dots, \kappa\}$, we have $\widehat{f}(\widehat{e}_j) = \widehat{e}_{\sigma(j)}\widehat{u}_j$, where $\widehat{u}_j$ is a path in $\rho^{-1}(G_{k-1})$. We let $\widehat{G}_{k,1} = \rho^{-1}(G_k)$. Note that $\widehat{f}$ leaves $\widehat{G}_{k,1}$ invariant. The corresponding stratum is the closure of $\rho^{-1}(G_k) \setminus \rho^{-1}(G_{k-1})$. Its transition matrix is just the permutation matrix associated to σ. In particular, it is a nonexponential stratum.

(3) Assume now that H_k is an exponential stratum. We define a binary relation on $\rho^{-1}(H_k)$. Given two edges $\widehat{e}_1$ and $\widehat{e}_2$, we say that $\widehat{e}_1 \sim \widehat{e}_2$ if there exists $p \in \mathbb{N}$ such that $\widehat{e}_1$ or $\widehat{e}_1^{-1}$ is an edge of $\widehat{f}^p(\widehat{e}_2)$. This relation is reflexive and transitive. We claim that it is an equivalence relation. Let $\widehat{e}_1$ and $\widehat{e}_2$ be two edges of $\rho^{-1}(H_k)$ such that $\widehat{e}_1 \sim \widehat{e}_2$. We want to prove that $\widehat{e}_2 \sim \widehat{e}_1$. By definition of our relation, there exists $p \in \mathbb{N}$ such that $\widehat{e}_1$ or $\widehat{e}_1^{-1}$ is an edge of $\widehat{f}^p(\widehat{e}_2)$. For simplicity, we assume that $\widehat{e}_1$ belongs to $\widehat{f}^p(\widehat{e}_2)$. The other case works in the same way. We write $e_1 = \rho(\widehat{e}_1)$ and $e_2 = \rho(\widehat{e}_2)$ for their respective images in G. Since the stratum H_k is aperiodic, there exists $q \in \mathbb{N}$ such that e_2 or e_2^{-1} is an edge of $f^q(e_1)$. For simplicity, we assume that e_2 is an edge $f^q(e_1)$. Since $\rho \circ \widehat{f} = f \circ \rho$, there exists a preimage of e_2 in $\widehat{G}$ which is an edge of $\widehat{f}^q(\widehat{e}_1)$. Thus there exists $u \in \boldsymbol{F}_r$ such that $u \cdot \widehat{e}_2$ is an edge of $\widehat{f}^q(\widehat{e}_1)$. We now prove by induction that $u_\ell \cdot \widehat{e}_2$ is an edge of $\widehat{f}^{\ell(p+q)+q}(\widehat{e}_1)$ for every $\ell \in \mathbb{N}$, where

$$u_\ell = \varphi^{\ell(p+q)}(u) \cdots \varphi^{p+q}(u)u.$$

If $\ell = 0$, then the statement follows from the definition of u. Assume that it is true for $\ell \in \mathbb{N}$; ie $u_\ell \cdot \widehat{e}_2$ is an edge of $\widehat{f}^{\ell(p+q)+q}(\widehat{e}_1)$. Using (3) we get that $\varphi^p(u_\ell) \cdot \widehat{f}^p(\widehat{e}_2) = \widehat{f}^p(u_\ell\widehat{e}_2)$ is a subpath of $\widehat{f}^{(\ell+1)(p+q)}(\widehat{e}_1)$. In particular, $\varphi^p(u_\ell) \cdot \widehat{e}_1$ lies in $\widehat{f}^{(\ell+1)(p+q)}(\widehat{e}_1)$. With a similar argument, we get that $\varphi^{p+q}(u_\ell)u \cdot \widehat{e}_2$ lies in $\widehat{f}^{(\ell+1)(p+q)+q}(\widehat{e}_1)$. However,

$$\varphi^{p+q}(u_\ell)u = \varphi^{(\ell+1)(p+q)}(u) \cdots \varphi^{p+q}(u)u = u_{\ell+1}.$$

Thus the statement holds for $\ell + 1$, which completes the proof of the induction.

Since $\boldsymbol{L}$ has finite index in $\boldsymbol{F}_r$, there exist $\ell \in \mathbb{N}$ and $t \in \mathbb{N}^*$ such that u_ℓ and $u_{\ell+t}$ are in the same $\boldsymbol{L}$–coset, ie $u_{\ell+t}u_\ell^{-1} \in \boldsymbol{L}$. However,

$$u_{\ell+t}u_\ell^{-1} = \varphi^{(\ell+t)(p+q)}(u) \cdots \varphi^{(\ell+1)(p+q)}(u) = \varphi^{(\ell+1)(p+q)}(u_{t-1}).$$

Since $\boldsymbol{L}$ is φ–invariant, we derive that u_{t-1} belongs to $\boldsymbol{L}$. Recall that $\boldsymbol{L}$ is the kernel of the action of $\boldsymbol{F}_r$ on $\widehat{G}$; hence $u_{t-1} \cdot \hat{e}_2 = \hat{e}_2$. On the other hand, $u_{t-1} \cdot \hat{e}_2$ is an edge of $\hat{f}^{(t-1)(p+q)+q}(\hat{e}_1)$. Consequently, $\hat{e}_2 \sim \hat{e}_1$, which completes the proof of our claim.

We denote by $\widehat{H}_{k,1}, \dots, \widehat{H}_{k,s}$ the equivalence classes for the relation $\sim$. For every $j \in \{1, \dots, s\}$, we put $\widehat{G}_{k,j} = \widehat{H}_{k,1} \cup \cdots \cup \widehat{H}_{k,j} \cup \rho^{-1}(G_{k-1})$. By construction, the filtration

$$\rho^{-1}(G_{k-1}) \subset \widehat{G}_{k,1} \subset \cdots \subset \widehat{G}_{k,s} = \rho^{-1}(G_k)$$

is $\hat{f}$–invariant. Moreover, the strata $\widehat{H}_{k,1}, \dots, \widehat{H}_{k,s}$ associated to this filtration are irreducible. We claim that they are exponential. Let $j \in \{1, \dots, s\}$. Let $M_{k,j}$ be the transition matrix of $\widehat{H}_{k,j}$. It is known that if the PF-eigenvalue of $M_{k,j}$ is 1, then $M_{k,j}$ is a permutation matrix. Thus there exists an edge $\hat{e} \in \widehat{H}_{k,j}$ and a positive integer p such that $\hat{e}$ is the only edge of $\rho^{-1}(H_k)$ in $\hat{f}^p(\hat{e})$. In particular, if e stands for $e = \rho(\hat{e})$, we get that e is the only edge of H_k in $f^p(e)$. This contradicts the fact that H is aperiodic. Hence the PF-eigenvalue of $M_{k,j}$ is larger than 1, and the stratum $\widehat{H}_{k,j}$ is exponential.

Finally, recall that f satisfies properties (RTT-i)–(RTT-iii). It follows from $\rho \circ \hat{f} = f \circ \rho$ that $\hat{f}$ also satisfies these properties. □

Lemma 4.6 *For every edge $\hat{e}$ in an exponential stratum $\widehat{H}$ of $\widehat{G}$, for every $p \in \mathbb{N}$, every maximal subpath of $\hat{f}^p_{\#}(\hat{e})$ that does not cross $\widehat{H}$ is not a loop.*

Proof Let $\hat{e}$ be an edge of an exponential stratum $\widehat{H}$ of $\widehat{G}$. Let $p \in \mathbb{N}$. Let $\hat{\beta}$ be a maximal subpath of $\hat{f}^p_{\#}(\hat{e})$ that does not cross $\widehat{H}$. By Lemma 4.5, there exists $k \in \{1, \dots, m\}$ such that H_k is an exponential stratum of G, $\widehat{H}$ is contained in $\rho^{-1}(H_k)$ and $\hat{f}(\widehat{H})$ is a subset of $\widehat{H} \cup \rho^{-1}(G_{k-1})$. We denote by e the image of $\hat{e}$ by ρ. It belongs to H_k. Since ρ is a continuous locally injective map, $\hat{f}^p_{\#}(\hat{e})$ is a lift of $f^p_{\#}(e)$. It follows that $\beta = \rho(\hat{\beta})$ is a maximal subpath of $f^p_{\#}(e)$ contained in G_{k-1}. If β is not a loop, neither is $\hat{\beta}$. Therefore, we can assume that β is a loop in G. By construction of $\mathcal{P}$, there exists a loop β' in $\mathcal{P} \cup f_{\#}(\mathcal{P})$ such that β is the image of β' by some power of $f_{\#}$. However, by definition, the conjugacy class of $\boldsymbol{F}_r$ represented by β' does not intersect $\boldsymbol{L} \subset \boldsymbol{H}$. Since $\boldsymbol{L}$ is φ–invariant, neither does the conjugacy class of $\boldsymbol{F}_r$ represented by β. Thus its lift $\hat{\beta}$ in $\widehat{G}$ cannot be a loop. □

Lemma 4.7 *Let n be an integer. Recall that κ is the index of $\boldsymbol{L}$ in $\boldsymbol{F}_r$. If Φ induces an outer automorphism of finite order of $\boldsymbol{B}_r(\kappa n)$, then its restriction to $\boldsymbol{L}$ induces an outer automorphism of finite order of $\boldsymbol{L}/\boldsymbol{L}^n$.*

Proof According to Remark 1.1, the image of φ in $\mathrm{Aut}(\boldsymbol{B}_r(\kappa n))$ has finite order. Hence there exists $p \in \mathbb{N}$ such that for every $g \in \boldsymbol{F}_r$, the element $\varphi^p(g)g^{-1}$ belongs to $\boldsymbol{F}_r^{\kappa n}$. However, $\boldsymbol{L}$ has index κ in $\boldsymbol{F}_r$. It follows that g^κ lies in $\boldsymbol{L}$ for every $g \in \boldsymbol{F}_r$. In particular, $\boldsymbol{F}_r^{\kappa n}$ is a subset of $\boldsymbol{L}^n$. Consequently, $\varphi^p(g)g^{-1}$ belongs to $\boldsymbol{L}^n$ for every $g \in \boldsymbol{L}$. It exactly means that, as an automorphism of $\boldsymbol{L}/\boldsymbol{L}^n$, φ^p is trivial. Hence the restriction of Φ to $\boldsymbol{L}$ induces an automorphism of finite order of $\boldsymbol{L}/\boldsymbol{L}^n$. □

Proposition 4.4 becomes a consequence of the following result.

Proposition 4.8 *Let $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ be an outer automorphism represented by an RTT $f\colon G \to G$. Assume that for every edge e in an exponential stratum H, for every $p \in \mathbb{N}$, every maximal subpath of $f_\#^p(e)$ that does not cross H is not a loop. Then there exists $n_0 \in \mathbb{N}$ such that for all odd integers $n \geqslant n_0$, the automorphism Φ induces an outer automorphism of $\boldsymbol{B}_r(n)$ of infinite order.*

In the next section, we discuss a third reduction and prove that Proposition 4.8 is a consequence of Proposition 4.11.

4.3 Automorphisms with only one exponential stratum

The following lemma is proved by the first author in [13] using the structure of free products.

Lemma 4.9 (Coulon [13, Lemma 1.9]) *Let n be an integer. Let φ be an automorphism of $\boldsymbol{F}_r$ which stabilizes a free factor $\boldsymbol{H}$. We assume that φ induces an automorphism of finite order of $\boldsymbol{B}_r(n)$. Then, the restriction of φ to $\boldsymbol{H}$ also induces an automorphism of finite order of $\boldsymbol{H}/\boldsymbol{H}^n$.*

Let $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ be an exponentially growing outer automorphism, and let $f\colon G \to G$ be an RTT representing Φ with a filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$. By Remark 3.4 (2), f has at least one exponential stratum. We assume that f satisfies the additional assumption of Proposition 4.8; ie for every edge e in an exponential stratum H, for every $p \in \mathbb{N}$, every maximal subpath of $f_\#^p(e)$ that does not cross H is not a loop. By replacing Φ by a power of Φ if necessary, we can assume that the exponential strata of the RTT are aperiodic. Note that this operation does not affect the graph G. However, one might need to refine the filtration of the RTT. In particular, the RTT does not necessarily satisfy the additional assumption of Proposition 4.8 anymore. Nevertheless, for every edge e in the *lowest* exponential stratum H_k, for every $p \in \mathbb{N}$, every maximal subpath of $f_\#^p(e)$ that does not cross H_k is not a loop.

Let G' be the connected component of the graph G_k which contains H_k. We assert that G' is f–invariant, ie $f(G') \subseteq G'$. Indeed, $G' \cap f(G')$ is nonempty (it contains H_k) and $f(G')$ is connected. Let $\boldsymbol{H}$ be the free factor of $\boldsymbol{F}_r$ defined by $G' \subseteq G$, and let $\Psi \in \mathrm{Out}(\boldsymbol{H})$ be the outer automorphism induced by the restriction $f' = f|_{G'}\colon G' \to G'$. We note that $f|_{G'}\colon G' \to G'$ is an RTT representing Ψ, which has exactly one exponential stratum, namely H_k, which is aperiodic and the top stratum of f'. In particular, Ψ has exponential growth.

Lemma 4.10 *If Ψ induces an outer automorphism of $\boldsymbol{H}/\boldsymbol{H}^n$ of infinite order, then Φ also induces an outer automorphism of $\boldsymbol{B}_r(n)$ of infinite order.*

Proof There exists an automorphism φ in the class of Φ which stabilizes $\boldsymbol{H}$. Assume that Φ induces an outer automorphism of $\boldsymbol{B}_r(n)$ of finite order. In particular, the image of φ in $\mathrm{Aut}(\boldsymbol{B}_r(n))$ has finite order; see Remark 1.1. It follows from the previous lemma that the restriction to $\boldsymbol{H}$ of φ (and thus Ψ) induces an automorphism (outer automorphism) of finite order of $\boldsymbol{H}/\boldsymbol{H}^n$. $\square$

It follows from our discussion that Proposition 4.8 is a consequence of the following statement.

Proposition 4.11 *Let $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ be an outer automorphism represented by an RTT $f\colon G \to G$ with exactly one exponential stratum H, which is aperiodic and the top stratum of f. Assume that for every edge e in H, for every $p \in \mathbb{N}$, every maximal subpath of $f_\#^p(e)$ that does not cross H is not a loop. Then there exists $n_0 \in \mathbb{N}$ such that for all odd integers $n \geqslant n_0$, the automorphism Φ induces an outer automorphism of $\boldsymbol{B}_r(n)$ of infinite order.*

We have seen that Theorem 1.3 can be deduced from Proposition 4.11. The latter will be proved in Sections 5 and 6.

5 Tracking powers

The next two sections are dedicated to the proof of Proposition 4.11. As we explained in the introduction, the goal is to understand to what extent a periodic path can appear in the orbit of a circuit under the iteration of the train-track map. This is the purpose of this section.

The general strategy is the following. We consider an outer automorphism Φ represented by an RTT $f\colon G \to G$ with a single exponential stratum H which is aperiodic. Then, we fix an edge $e_\bullet$ in H. For every $p \in \mathbb{N}$, we look at the path obtained by removing from $f_\#^p(e_\bullet)$ all the edges which are not in H. This sequence can be interpreted as the orbit of $e_\bullet$ under a substitution over the set of oriented edges of H; see Lemma 5.1. It follows from the aperiodicity of H that this substitution is primitive. Therefore, we would like to apply Proposition 2.1. We need to rule out first the case of an infinite shift-periodic word, though (Proposition 2.1 (ii)). The dynamic of the substitution is not sufficient to conclude here. Remark 2.2 provides indeed an example of a primitive substitution σ with an infinite shift-periodic fixed point. However, the particularity of this example is that σ does not represent an automorphism of $\boldsymbol{F}_3$. Our proof (see Proposition 5.2) strongly uses the fact that the substitution we are looking at comes from an automorphism of the free group.

From now on, Φ denotes an outer automorphism of $\boldsymbol{F}_r$ which can be represented by an RTT $f\colon G \to G$ with exactly one exponential stratum H. Moreover, H is aperiodic and the top stratum of f. We denote by $\mathcal{E}$ the set of all the oriented edges of H. In addition, we assume that for every $e \in \mathcal{E}$, for every $p \in \mathbb{N}$, every maximal subpath of $f_\#^p(e)$ that does not cross H is not a loop.

By replacing Φ by a power of Φ if necessary, we can assume that $f(v)$ is fixed by f for every vertex v of G, and that there exists $e_\bullet \in \mathcal{E}$ such that $Df(e_\bullet) = e_\bullet$. Note that this operation does not affect the graph G or the exponential stratum. In particular, H is still the only exponential stratum of f. It is aperiodic and the top stratum. By choice of $e_\bullet$, we have that f fixes the initial vertex x_0 of $e_\bullet$. Thus it naturally defines an automorphism $\varphi \in \mathrm{Aut}(\pi_1(G, x_0))$ in the outer class Φ: if g is an element of $\pi_1(G, x_0)$ represented by a loop α based at x_0, then $\varphi(g)$ is the homotopy class of $f(\alpha)$ (relative to x_0).

5.1 The yellow-red decomposition

We refer to the edges of H as *red edges* and to the edges of $G \setminus H$ as *yellow edges*. Recall that $\widetilde{G}$ denotes the universal cover of G. An edge of $\widetilde{G}$ can be labeled by the edge of G of which it is the lift. In particular, its color is given by the color of its label.

A k–legal path of G (where k is the height of H) will be call a *red-legal* path. A path (in G or in $\widetilde{G}$) is a *yellow path* if it only crosses yellow edges. *Red paths* are defined in the same way. Any path α (in G or in $\widetilde{G}$) can be decomposed as a concatenation of maximal yellow and red subpaths: $\alpha = \alpha_1 \cdots \alpha_q$, where α_i $(1 \leqslant i \leqslant q)$ is a nontrivial subpath of α which is either yellow or red, and α_i and α_{i+1} have not the same color for all $i \in \{1, \dots, q-1\}$. According to Lemma 3.3, this decomposition is a splitting of α.

The red word associated to a path We associate to any path of edges α in G or $\widetilde{G}$ a word $\mathrm{Red}(\alpha)$ over the alphabet $\mathcal{E}$. As a path of edges, α is labeled by a word over the alphabet that consists of all oriented edges of G. The word $\mathrm{Red}(\alpha)$ is obtained from this word by removing all the letters corresponding to yellow edges. We stress on the fact that if α is a reduced path, then $\mathrm{Red}(\alpha)$ is not, in general, a reduced word.

5.2 The induced substitution on red edges

Definition and first properties We associate to the RTT f a substitution σ on $\mathcal{E}$ called the *induced substitution*. It is defined as

$$\sigma(e) = \mathrm{Red}(f(e)) \quad \text{for every } e \in \mathcal{E}.$$

Lemma 5.1 *Let α be a red-legal path in G. For all $p \in \mathbb{N}$, we have*

$$\mathrm{Red}(f_\#^p(\alpha)) = \sigma^p(\mathrm{Red}(\alpha)).$$

Proof We consider a decomposition of α as $\alpha = \alpha_1 e_1 \alpha_2 e_2 \cdots \alpha_q e_q \alpha_{q+1}$ where each $e_i \in \mathcal{E}$ is a red edge, and each α_i is a (possibly trivial) yellow subpath. In particular, $\mathrm{Red}(\alpha) = e_1 e_2 \cdots e_q$. The path α being red-legal, Lemma 3.3 leads to

$$\begin{aligned} f_\#(\alpha) &= f_\#(\alpha_1 e_1 \alpha_2 e_2 \cdots \alpha_q e_q \alpha_{q+1}) \\ &= f_\#(\alpha_1) f(e_1) f_\#(\alpha_2) f(e_2) \cdots f_\#(\alpha_q) f(e_q) f_\#(\alpha_{q+1}). \end{aligned}$$

However, f sends yellow edges to yellow paths. We deduce that

$$\begin{aligned} \mathrm{Red}(f_\#(\alpha)) &= \mathrm{Red}(f(e_1))\,\mathrm{Red}(f(e_2)) \cdots \mathrm{Red}(f(e_q)) \\ &= \sigma(e_1)\sigma(e_2)\cdots\sigma(e_q) = \sigma(e_1 e_2 \cdots e_q) = \sigma(\mathrm{Red}(\alpha)). \end{aligned}$$

The image by $f_\#$ of a red-legal path is still a red-legal path. Therefore, for all $p \in \mathbb{N}$,

$$\mathrm{Red}(f_\#^{p+1}(\alpha)) = \mathrm{Red}(f_\#(f_\#^p(\alpha))) = \sigma(\mathrm{Red}(f_\#^p(\alpha))).$$

The result follows by induction on p. □

Primitivity of the induced substitution The material of this paragraph is widely inspired by the work of P Arnoux et al [2, Section 3]. Recall that $\Theta\colon \mathcal{E} \to \mathcal{E}$ is the map which sends e to e^{-1}. We extend Θ to the free monoid $\mathcal{E}^*$ in the following way. Let w be an element of $\mathcal{E}^*$. By definition, it can be written $w = e_1 e_2 \cdots e_q$ where $e_i \in \mathcal{E}$. We put $\Theta(w) = e_q^{-1} \cdots e_2^{-1} e_1^{-1}$. It defines an involution of $\mathcal{E}^*$ called the *flip map*. Moreover, we observe that $\sigma \circ \Theta(e) = \Theta \circ \sigma(e)$ for all edges $e \in \mathcal{E}$. Thus σ and Θ commute on $\mathcal{E}^*$. The substitution σ is said to be *orientable* with respect to a subset $\vec{\mathcal{E}}$ of $\mathcal{E}$ if

(i) $\vec{\mathcal{E}}$ and $\Theta(\vec{\mathcal{E}})$ make a partition of $\mathcal{E}$,

(ii) $\sigma(\vec{\mathcal{E}}) \subset \vec{\mathcal{E}}^*$.

Note that (i) just says that $\vec{\mathcal{E}}$ is a preferred set of oriented edges for H. In that case, σ induces a substitution of $\vec{\mathcal{E}}^*$, that we still denote by σ.

By assumption, the red stratum H of f is aperiodic. In other words, its transition matrix M is primitive. Applying [2, Proposition 3.7], we know that either

- σ is not orientable, and then σ is a primitive substitution on the alphabet $\mathcal{E}$, or
- there exists a subset $\vec{\mathcal{E}}$ of $\mathcal{E}$ such that σ is orientable with respect to $\vec{\mathcal{E}}$, and then σ induces a primitive substitution on the alphabet $\vec{\mathcal{E}}$.

Thus in both cases, there exists a subset $\mathcal{E}_\bullet$ of $\mathcal{E}$ containing $e_\bullet$ such that $\sigma(\mathcal{E}_\bullet) \subset \mathcal{E}_\bullet^*$, and the substitution $\sigma\colon \mathcal{E}_\bullet^* \to \mathcal{E}_\bullet^*$ is primitive.

5.3 A red word without large powers

The infinite red word $\sigma^\infty(e_\bullet)$ Recall that $e_\bullet$ is a red edge of $\mathcal{E}$ that has been chosen in such a way that $Df(e_\bullet) = e_\bullet$. Because the red stratum is aperiodic, $f(e_\bullet) = e_\bullet \cdot \alpha$ where $\mathrm{Red}(\alpha)$ is nontrivial. In particular, $e_\bullet$ is a prefix of $\sigma(e_\bullet)$. According to Proposition 2.1 the sequence $(\sigma^p(e_\bullet))$ converges to an infinite word $\sigma^\infty(e_\bullet)$ of $\mathcal{E}_\bullet^{\mathbb{N}}$. Note that $f(e_\bullet) = e_\bullet \cdot \alpha$ is a splitting. Hence for every $p \in \mathbb{N}$,

$$f_\#^p(e_\bullet) = e_\bullet \cdot \alpha \cdot f_\#(\alpha) \cdots f_\#^{p-1}(\alpha).$$

Hence $(f_\#^p(e_\bullet))$ also converges to an infinite path

$$f_\#^\infty(e_\bullet) = e_\bullet \cdot \alpha \cdot f_\#(\alpha) \cdots f_\#^p(\alpha) \cdots .$$

Proposition 5.2 *The infinite word $\sigma^\infty(e_\bullet)$ is not shift periodic.*

This proof combines a dynamical argument (σ is a primitive substitution) and a group theoretical one (φ is an automorphism of $\boldsymbol{F}_r$). Let us sketch first the main steps. We assume that the proposition is false. This means that if we restrict our attention to the red edges, the path $f_\#^\infty(e_\bullet)$ is periodic. We construct from G a colored graph Γ on which $f_\#^\infty(e_\bullet)$ coils up. More precisely, its fundamental group $\boldsymbol{H}$ can be decomposed as a free product $\boldsymbol{H} = \boldsymbol{L} * \langle h \rangle$, where $\boldsymbol{L}$ is generated by conjugates of yellow loops, and h is represented by a loop $\hat{\gamma}$ with the following property. If we collapse all the yellow edges of Γ, we obtain a simple (red) loop which is exactly the image of $\hat{\gamma}$ by the same operation. Moreover, this red loop is the period of the red word associated to $f_\#^\infty(e_\bullet)$; see Figure 3. We show that the RTT f induces a homotopy equivalence $\hat{f}\colon \Gamma \to \Gamma$ that catches two conflicting features of Φ:

1. (1) Since the stratum H is exponential, $\hat{f}$ should increase the length of the red word associated to $\hat{\gamma}$; see Proposition 5.7.
2. (2) The yellow components of G are invariant under f. It follows that the automorphism of $\boldsymbol{H}$ induced by $\hat{f}$ sends h to $gh^{\pm 1}$, where g belongs to the normal subgroup generated by $\boldsymbol{L}$.

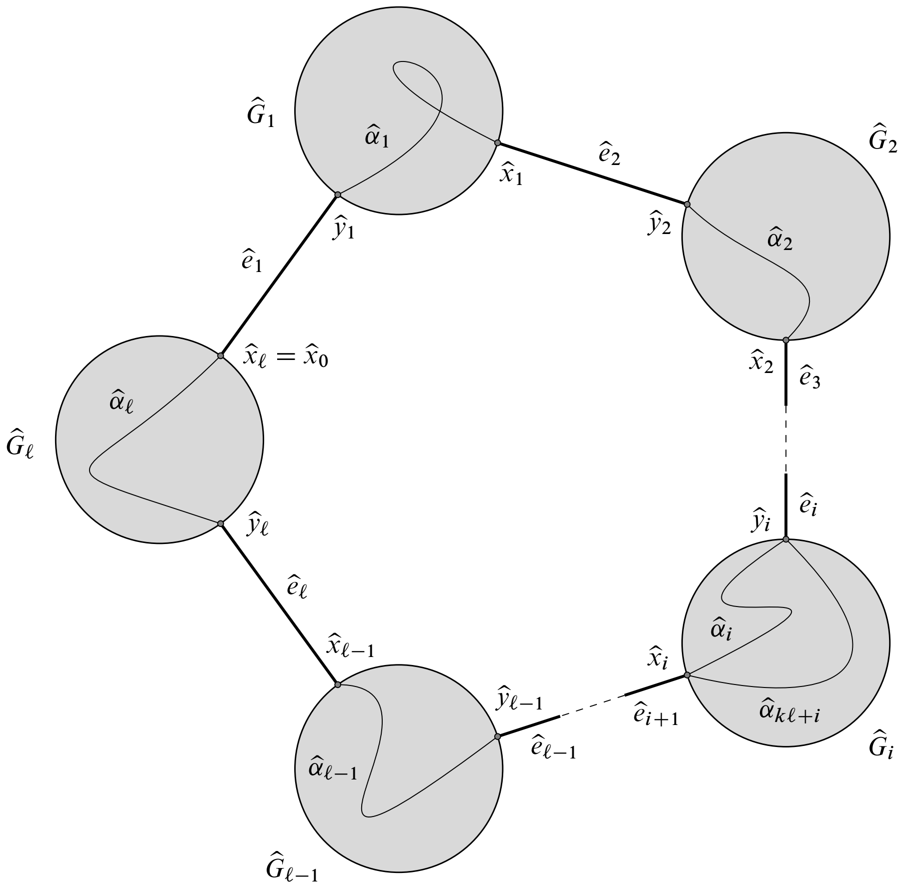

Figure 3: The graph Γ

The key fact is that these two properties can be observed in the abelianization of $\boldsymbol{H}$, which leads to a contradiction.

Proof of Proposition 5.2 Assume that $\sigma^\infty(e_\bullet)$ is shift-periodic. Recall that σ is primitive as a substitution of $\mathcal{E}_\bullet^*$. Proposition 2.1 implies that there exist an integer $q \geqslant 2$ and a primitive word $u = e_1 e_2 \cdots e_\ell$ of $\mathcal{E}_\bullet^*$ such that $\sigma^\infty(e_\bullet) = u^\infty$ and $\sigma(u) = u^q$. Notice that $e_1 = e_\bullet$. This means, in particular, that the infinite path $f_\#^\infty(e_\bullet)$ is obtained as a concatenation

$$f_\#^\infty(e_\bullet) = \gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_k \cdots$$

of loops

$$\gamma_k = e_1 \cdot \alpha_{k\ell+1} \cdot e_2 \cdot \alpha_{k\ell+2} \cdots e_\ell \cdot \alpha_{(k+1)\ell},$$

where the α_i are (possibly trivial) yellow paths. Moreover, if α_i is nontrivial, then it is a maximal yellow subpath of the image by a power of $f_\#$ of a red edge. By assumption, none of them is a loop. Recall that x_0 is the initial point of $e_1 = e_\bullet$. For every $i \in \{1, \dots, \ell\}$, we have that y_i and x_i respectively stand for the initial and the terminal points of α_i. In particular, $x_0 = x_\ell$. We now focus on the path $\gamma = \gamma_0$:

$$\gamma = e_1 \cdot \alpha_1 \cdot e_2 \cdot \alpha_2 \cdots e_\ell \cdot \alpha_\ell.$$

Lemma 5.3 *The path $f_\#(\gamma)$ is exactly $\gamma_0 \cdots \gamma_{q-1}$. In particular, it is an initial subpath of the infinite path $f_\#^\infty(e_\bullet)$.*

Proof By construction, there exists $p \in \mathbb{N}$ such that $\gamma_0 \cdot \gamma_1$ is a proper initial subpath of $f_\#^p(e_\bullet)$. Moreover, the terminal point of $\gamma_0\gamma_1$, which is also the initial vertex of the red edge $e_1 = e_\bullet$, is a splitting point of the yellow-red splitting of $f_\#^p(e_\bullet)$. Thus $f_\#(\gamma_0\gamma_1) = f_\#(\gamma_0) \cdot f_\#(\gamma_1)$ is an initial subpath of $f_\#^{p+1}(e_\bullet)$, hence of $f_\#^\infty(e_\bullet)$. However, by Lemma 5.1,

$$\mathrm{Red}(f_\#(\gamma_0)) = \sigma(\mathrm{Red}(\gamma_0)) = \sigma(u) = u^q = \mathrm{Red}(\gamma_0 \cdots \gamma_{q-1}).$$

It follows that there exists a subpath α' of $\alpha_{q\ell}$ such that

$$f_\#(\gamma_0) = [\gamma_0 \cdots \gamma_{q-2}][e_1\alpha_{(q-1)\ell+1}e_2\alpha_{(q-1)\ell+2} \cdots e_\ell\alpha'].$$

On the other hand, γ_1 and thus $f_\#(\gamma_1)$ starts with the red edge $e_1 = e_\bullet$. Since $f_\#(\gamma_0) \cdot f_\#(\gamma_1)$ is an initial subpath of $f_\#^{p+1}(e_\bullet)$, the path α' is necessarily the whole $\alpha_{q\ell}$. Consequently, $f_\#(\gamma) = \gamma_0 \cdots \gamma_{q-1}$. □

The graph Γ and the loop $\hat{\gamma}$ Let $i \in \{1, \dots, \ell\}$. We define $\widehat{G}_i$ to be a copy of the largest connected yellow subgraph of G containing y_i. We denote by $\hat{\alpha}_i$ (respectively $\hat{y}_i$ and $\hat{x}_i$) the path α_i (respectively the vertices y_i and x_i) viewed as a path of $\widehat{G}_i$ (respectively as vertices of $\widehat{G}_i$).

We now construct a graph Γ as follows. We start with the disjoint union of the $\widehat{G}_i$ for $i \in \{1, \dots, \ell\}$. Then for every $i \in \{1, \dots, \ell\}$, we add an oriented edge $\hat{e}_i$ whose initial and terminal points are respectively $\hat{x}_{i-1}$ and $\hat{y}_i$. The reverse edge $\hat{e}_i^{-1}$ is attached accordingly. In this process, we think about the indices i as elements of $\mathbb{Z}/\ell\mathbb{Z}$. In particular, $\hat{x}_0$ should be understood as the point $\hat{x}_\ell$ of $\widehat{G}_\ell$. We denote by Γ the graph obtained in this way; see Figure 3. Let ρ be the graph morphism $\rho\colon \Gamma \to G$ such that for every $i \in \{1, \dots, \ell\}$, we have that $\rho(\hat{e}_i) = e_i$ and the restriction of ρ to $\widehat{G}_i$ is the natural embedding $\widehat{G}_i \hookrightarrow G$. We color the edges of Γ by the color of their images under ρ. In other words, the edges $\hat{e}_i$ are red, whereas the edges of the subgraphs $\widehat{G}_i$ are yellow. By construction, the loop $\hat{\gamma}$ defined below is a lift of γ in Γ:

$$\hat{\gamma} = \hat{e}_1\hat{\alpha}_1\hat{e}_2\hat{\alpha}_2 \cdots \hat{e}_\ell\hat{\alpha}_\ell.$$

The subgroup $\boldsymbol{H}$ We denote by $\boldsymbol{H}$ the fundamental group $\pi_1(\Gamma, \hat{x}_0)$. Let us choose a maximal tree T_i in each $\widehat{G}_i$. The union T defined below is a maximal tree of Γ:

$$T = \left(\bigcup_{i=1}^{\ell} T_i\right) \cup \left(\bigcup_{i=1}^{\ell-1} \hat{e}_i\right).$$

For every edge e of Γ not in T, we write β_e for the path contained in T starting at $\hat{x}_0$ and ending at the initial vertex of e. We define h_e as the element of $\boldsymbol{H}$ represented by $\beta_e e \beta_{e^{-1}}^{-1}$. Let $i \in \{1, \dots, \ell\}$. For each unoriented edge of $\widehat{G}_i \setminus T_i$, we chose one of the two corresponding oriented edges. We denote then by $\mathcal{F}_i$ the preferred set of oriented edges obtained in this way. We write $\mathcal{F}$ for the union

$$\mathcal{F} = \bigcup_{i=1}^{\ell} \mathcal{F}_i.$$

Lemma 5.4 *Let h be the element of $\boldsymbol{H}$ represented by $\hat{\gamma}$. The family $\mathcal{B}$ obtained by taking the union of $(h_e)_{e\in\mathcal{F}}$ and $\{h\}$ is a free basis of $\boldsymbol{H}$.*

Proof It follows from the definition of $\mathcal{F}$ that the family $(h_e)_{e\in\mathcal{F}\cup\{\hat{e}_\ell\}}$ is a free basis of $\boldsymbol{H}$. By construction of Γ, we have $h = g \cdot h_{\hat{e}_\ell}$ where g is a product of some h_e with $e \in \mathcal{F} \cup \mathcal{F}^{-1}$. This implies that the family $(h_e)_{e\in\mathcal{F}}$ together with h forms a free basis of $\boldsymbol{H}$. □

Let $k \in \mathbb{N}$ and $i \in \{1, \dots, \ell\}$. The path $\alpha_{k\ell+i}$ and α_i have the same endpoints, namely y_i (the terminal point of e_i) and x_i (the initial point of e_{i+1}). In particular, they are contained in the same maximal yellow connected component of G. We denote by $\hat{\alpha}_{k\ell+i}$ the copy in $\widehat{G}_i$ of $\alpha_{k\ell+i}$; see Figure 3. We put

$$\hat{\gamma}_k = \hat{e}_1\hat{\alpha}_{k\ell+1}\hat{e}_2\hat{\alpha}_{k\ell+2}\cdots\hat{e}_\ell\hat{\alpha}_{(k+1)\ell}.$$

By construction, $\hat{\gamma}_k$ is a loop of Γ based at $\hat{x}_0$ lifting γ_k (ie $\rho \circ \hat{\gamma}_k = \gamma_k$).

Lemma 5.5 *Let h be the element of $\boldsymbol{H}$ represented by $\hat{\gamma}$. Let $k \in \mathbb{N}$. There exists g in the normal subgroup generated by $(h_e)_{e\in\mathcal{F}}$ such that the element of $\boldsymbol{H}$ represented by the loop $\hat{\gamma}_k$ is gh.*

Proof It follows from the equality

$$\begin{aligned}\hat{\gamma}_k &= \big[\hat{e}_1(\hat{\alpha}_{k\ell+1}\hat{\alpha}_1^{-1})\hat{e}_1^{-1}\big]\big[\hat{e}_1\hat{\alpha}_1\hat{e}_2(\hat{\alpha}_{k\ell+2}\hat{\alpha}_2^{-1})\hat{e}_2^{-1}\hat{\alpha}_1^{-1}\hat{e}_1^{-1}\big]\cdots \\ &\cdots\big[\hat{e}_1\hat{\alpha}_1\hat{e}_2\cdots\hat{e}_{\ell-1}(\hat{\alpha}_{(k+1)\ell-1}\hat{\alpha}_{\ell-1}^{-1})\hat{e}_{\ell-1}^{-1}\cdots\hat{e}_2^{-1}\hat{\alpha}_1^{-1}\hat{e}_1^{-1}\big]\big[\hat{\gamma}(\hat{\alpha}_\ell^{-1}\hat{\alpha}_{(k+1)\ell})\hat{\gamma}^{-1}\big]\hat{\gamma}.\end{aligned}$$ □

Lemma 5.6 *The map $\rho\colon \Gamma \to G$ is locally injective.*

Proof We prove this lemma by contradiction. Let $\hat{e}$ and $\hat{e}'$ be two distinct edges of Γ with the same initial vertex $\hat{v}$. Suppose that $\rho(\hat{e}) = \rho(\hat{e}')$. There exists $i \in \{1, \dots, \ell\}$ such that $\hat{v}$ is a vertex of $\hat{G}_i$. By construction, ρ preserves the color of the edges, thus $\hat{e}$ and $\hat{e}'$ necessarily have the same color. We distinguish two cases. Assume first that $\hat{e}$ and $\hat{e}'$ are both yellow edges. Recall that the restriction of ρ to $\hat{G}_i$ is the inclusion $\hat{G}_i \hookrightarrow G$. Thus $\hat{e} = \hat{e}'$, a contradiction. Assume now that $\hat{e}$ and $\hat{e}'$ are red. By construction of Γ, at most two red edges have an initial vertex in $\hat{G}_i$. Without loss of generality, we can assume that $\hat{e}^{-1} = \hat{e}_i$ and $\hat{e}' = \hat{e}_{i+1}$ (as previously, if $i = \ell$ then $\hat{e}_{i+1}$ corresponds to $\hat{e}_1$). Then $\rho(\hat{v})$ is the terminal vertex y_i of e_i and the initial vertex x_i of e_{i+1}. Thus the yellow path α_i is either trivial or a loop of G. By assumption, it cannot be a loop; thus α_i is trivial and $e_{i+1} = \rho(\hat{e}') = \rho(\hat{e}) = e_i^{-1}$. This contradicts the fact that γ is a path. Consequently, ρ is locally injective. □

If follows from the lemma that ρ induces an embedding ρ_* from $\boldsymbol{H}$ into $\pi_1(G, x_0)$. From now on, we identify $\boldsymbol{H}$ with its image in $\pi_1(G, x_0)$.

The automorphism induced on $\boldsymbol{H}$ Recall that φ is the automorphism of $\pi_1(G, x_0)$ in the outer class Φ induced by f. We now prove that φ induces an automorphism of $\boldsymbol{H}$. To that end, we lift the RTT f into a map $\hat{f}\colon \Gamma \to \Gamma$.

Proposition 5.7 *There exists a continuous map $\hat{f}\colon \Gamma \to \Gamma$ satisfying the following:*

(1) $f \circ \rho = \rho \circ \hat{f}$,

(2) *$\hat{f}(\hat{\gamma})$ is homotopic relative to its endpoints to $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$.*

Proof The map $\hat{f}\colon \Gamma \to \Gamma$ is built step by step. Let us first define some auxiliary objects that will be needed during the construction. Let Γ_ℓ be the graph obtained from Γ by disconnecting $\hat{e}_1$ from $\hat{G}_\ell$ at $\hat{x}_0$; see Figure 4. It comes with a natural map $\Gamma_\ell \to \Gamma$ which is a local isometry. For simplicity, we use the same notation for the paths of Γ_ℓ and their images in Γ. For instance, $\hat{\gamma}$ can be seen as a subpath of Γ_ℓ. Similarly, we still denote by ρ the locally injective map $\rho\colon \Gamma_\ell \to G$. For every $i \in \{1, \dots, \ell\}$, we denote by Γ_i the subgraph of Γ_ℓ consisting of the red edges $\hat{e}_1, \dots, \hat{e}_i$ and the yellow graphs $\hat{G}_1, \dots, \hat{G}_i$. By convention, we put $\Gamma_0 = \{\hat{x}_0\}$.

Let $i \in \{1, \dots, \ell\}$. The path γ can be split as follows:

$$\gamma = (e_1\alpha_1 \cdots e_i\alpha_i) \cdot (e_{i+1}\alpha_{i+1} \cdots e_\ell\alpha_\ell).$$

Therefore, $f_\#(e_1\alpha_1 \cdots e_i\alpha_i)$ is an initial subpath of $f_\#(\gamma) = \gamma_0 \cdots \gamma_{q-1}$; see Lemma 5.3. However, the path $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$ is, by construction, the unique lift of $\gamma_0 \cdots \gamma_{q-1}$ in Γ

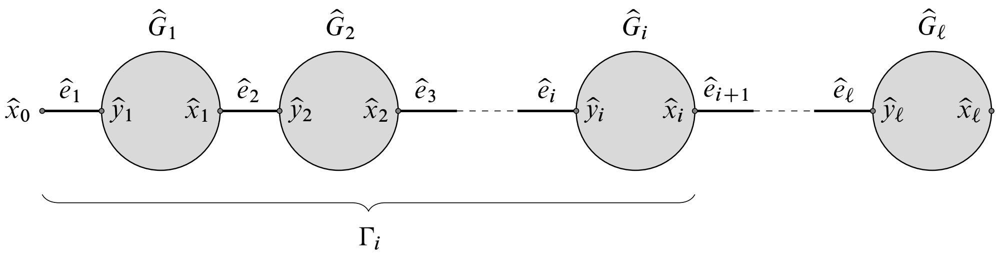

Figure 4: The graph Γ_ℓ

starting at $\hat{x}_0$. We denote by $\widehat{\beta}_i$ the initial subpath of $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$ such that $\rho \circ \widehat{\beta}_i = f_\#(e_1\alpha_1 \cdots e_i\alpha_i)$. In particular, $\widehat{\beta}_\ell = \hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$. By convention, we define $\widehat{\beta}_0$ to be the trivial path equal to $\hat{x}_0$. We begin with the following claim whose proof is by induction on i.

Claim *For every $i \in \{0, \dots, \ell\}$, there exists a continuous map $\widehat{f}_i\colon \Gamma_i \to \Gamma$ satisfying the following:*

1. $f \circ \rho = \rho \circ \widehat{f}_i$,
2. $\widehat{f}_i(\hat{e}_1\hat{\alpha}_1 \cdots \hat{e}_i\hat{\alpha}_i)$ *is homotopic relative to its endpoints to* $\widehat{\beta}_i$.

The base of induction By assumption, f fixes the vertex x_0. We put $\widehat{f}_0(\hat{x}_0) = \hat{x}_0$; hence the claim holds for $i = 0$.

The inductive step Assume now that the claim holds for $i \in \{0, \dots, \ell-1\}$. Our goal is to extend $\widehat{f}_i$ into a map $\widehat{f}_{i+1}\colon \Gamma_{i+1} \to \Gamma$. To that end, we need to define the restriction of $\widehat{f}_{i+1}$ to $\hat{e}_{i+1}$ and $\widehat{G}_{i+1}$. We start with the following observation: $\widehat{f}_i(\hat{x}_i)$ is exactly the terminal point of $\widehat{\beta}_i$. Indeed $\hat{x}_i$ is the terminal point of $\hat{\alpha}_i$, hence of $\hat{e}_1\hat{\alpha}_1 \cdots \hat{e}_i\hat{\alpha}_i$. According to the induction hypothesis, $\widehat{f}_i(\hat{e}_1\hat{\alpha}_1 \cdots \hat{e}_i\hat{\alpha}_i)$ is homotopic relative to its endpoints to $\widehat{\beta}_i$. In particular, they have the same terminal point, namely $\widehat{f}_i(\hat{x}_i)$.

Let us now focus on $\hat{e}_{i+1}$. By construction, the path γ splits as follows:

$$\gamma = (e_1\alpha_1 \cdots e_i\alpha_i) \cdot e_{i+1} \cdot \alpha_{i+1} \cdot (e_{i+2}\alpha_{i+2} \cdots e_\ell\alpha_\ell).$$

Therefore, we have

$$f_\#(\gamma) = f_\#(e_1\alpha_1 \cdots e_i\alpha_i) \cdot f(e_{i+1}) \cdot f_\#(\alpha_{i+1}) \cdot f_\#(e_{i+2}\alpha_{i+2} \cdots e_\ell\alpha_\ell).$$

In particular, $f(e_{i+1})$ is a subpath of $f_\#(\gamma)$. As we explained before, $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$ is the (unique) lift of $f_\#(\gamma)$ in Γ starting at $\hat{x}_0$. Moreover, $\widehat{\beta}_i$ is the initial path of $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$ lifting $f_\#(e_1\alpha_1 \cdots e_i\alpha_i)$. We denote by $\hat{\nu}$ the subpath of $\hat{\gamma}_0 \cdots \hat{\gamma}_{q-1}$ lifting $f(e_{i+1})$ whose initial point is the terminal point of $\widehat{\beta}_i$. As we noticed above the initial point of $\widehat{\nu}$ (ie the terminal point of $\widehat{\beta}_i$) is exactly $\widehat{f}_i(\widehat{x}_i)$. Consequently, we can extend $\widehat{f}_i\colon \Gamma_i \to \Gamma$ to a continuous map $\widehat{f}_{i+1}\colon \Gamma_i \cup \widehat{e}_{i+1} \to \Gamma$ by sending $\widehat{e}_{i+1}$ to $\widehat{\nu}$.

The next step is to define the map $\widehat{f}_{i+1}$ on $\widehat{G}_{i+1}$. Since e_{i+1} is a red edge, its image under f starts and ends by a red edge. In particular, there exists $j \in \{1, \dots, \ell\}$ such that the last edge of $\widehat{\nu}$ is $\widehat{e}_j$. It follows that f maps y_{i+1} (the terminal point of e_{i+1}) to y_j (the terminal point of e_j). On the other hand, f is continuous and sends yellow edges to yellow paths. Therefore, it maps the largest yellow connected component of G containing y_{i+1} to the largest yellow connected component of G containing y_j. It provides a continuous map from $\widehat{f}_{i+1}\colon \widehat{G}_{i+1} \to \widehat{G}_j$ such that $\widehat{f}_{i+1}(\widehat{y}_{i+1}) = \widehat{y}_j$ and $\rho \circ \widehat{f}_{i+1} = f \circ \rho$. This completes the construction for $i+1$. We end the proof of the claim with the following lemma.

Lemma 5.8 *The path $\widehat{f}_{i+1}(\widehat{e}_1\widehat{\alpha}_1 \cdots \widehat{e}_{i+1}\widehat{\alpha}_{i+1})$ is homotopic relative to its endpoints to $\widehat{\beta}_{i+1}$.*

Proof By construction,

$$\widehat{f}_{i+1}(\widehat{e}_1\widehat{\alpha}_1 \cdots \widehat{e}_{i+1}\widehat{\alpha}_{i+1}) = \widehat{f}_i(\widehat{e}_1\widehat{\alpha}_1 \cdots \widehat{e}_i\widehat{\alpha}_i)\, \widehat{f}_{i+1}(\widehat{e}_{i+1})\, \widehat{f}_{i+1}(\widehat{\alpha}_{i+1}).$$

In particular, it is homotopic relative to its endpoints to $\widehat{\beta}_i\, \widehat{f}_{i+1}(\widehat{e}_{i+1})\, \widehat{f}_{i+1}(\widehat{\alpha}_{i+1})$. By construction, we also have that $\widehat{\beta}_i\, \widehat{f}(\widehat{e}_i) = \widehat{\beta}_i\widehat{\nu}$ is the initial path at $\widehat{x}_0$ of $\widehat{\gamma}_0 \cdots \widehat{\gamma}_{q-1}$ lifting $f_\#(e_1\alpha_1 \cdots e_i\alpha_i e_{i+1})$. Note also that $\widehat{\beta}_i\widehat{\nu}$ ends where $\widehat{f}_{i+1}(\widehat{\alpha}_{i+1})$ starts, namely at the point $\widehat{f}_{i+1}(\widehat{y}_{i+1}) = \widehat{y}_j$. Thus it is sufficient to prove that $\widehat{f}_{i+1}(\widehat{\alpha}_{i+1})$ is homotopic relative to its endpoints to the lift starting at $\widehat{f}_{i+1}(\widehat{y}_{i+1})$ of $f_\#(\alpha_{i+1})$. However, these last paths all belong to $\widehat{G}_j$. Moreover, the restriction of ρ to $\widehat{G}_j$ is the natural embedding $\widehat{G}_j \hookrightarrow G$. The conclusion follows then from the fact that $f(\alpha_{i+1})$ and $f_\#(\alpha_{i+1})$ are homotopic relative to their endpoints in the yellow connected component of G to which they belong. □

Lemma 5.9 *The map $\widehat{f}_\ell\colon \Gamma_\ell \to \Gamma$ induces a continuous map $\widehat{f}\colon \Gamma \to \Gamma$ such that $f \circ \rho = \rho \circ \widehat{f}$.*

Proof By definition, Γ is obtained from Γ_ℓ by attaching the initial point $\widehat{x}_0$ of $\widehat{e}_1$ to the point $\widehat{x}_\ell$ of $\widehat{G}_\ell$. Therefore, it is sufficient to prove that $\widehat{f}_\ell(\widehat{x}_\ell) = \widehat{f}_\ell(\widehat{x}_0)$. It follows from the first step of the construction that $\widehat{f}_\ell(\widehat{x}_0) = \widehat{x}_0$. On the other hand, $\widehat{f}_\ell(\widehat{\gamma})$ and $\widehat{\beta}_\ell = \widehat{\gamma}_0 \cdots \widehat{\gamma}_{q-1}$ are homotopic relative to their endpoints. Thus the terminal point of $\widehat{\gamma}$ (ie $\widehat{x}_\ell$) is sent to the terminal point of $\widehat{\gamma}_{q-1}$, ie $\widehat{x}_0$. Hence $\widehat{f}_\ell(\widehat{x}_\ell) = \widehat{f}_\ell(\widehat{x}_0) = \widehat{x}_0$. □

We can now complete the proof of Proposition 5.7. Lemma 5.9 provides the map we are looking for. The second point becomes a consequence of Lemma 5.8. □

Lemma 5.10 *The map φ induces an automorphism of $\boldsymbol{H}$.*

Proof It follows from Proposition 5.7 that $\varphi(\boldsymbol{H})$ is a subgroup of $\boldsymbol{H}$. It follows from Lemma 6.0.6 in [4] that the restriction of φ to $\boldsymbol{H}$ is an automorphism. $\square$

The abelianization of $\boldsymbol{H}$ Now we complete the proof of Proposition 5.2. Let d be the rank of the free group $\boldsymbol{H}$. We consider the abelianization morphism $\boldsymbol{H} \to \mathbb{Z}^d$. In particular, φ induces an automorphism φ_{ab} of $\mathbb{Z}^d$. We denote by $\mathcal{C}$ the image in $\mathbb{Z}^d$ of the free basis $\mathcal{B}$ of $\boldsymbol{H}$ given by Lemma 5.4. The first $(d-1)$ elements of $\mathcal{B}$ (the ones corresponding to oriented edges in $\mathcal{F}$) are conjugates of yellow loops of Γ. However, f, and thus $\widehat{f}$, maps yellow edges to yellow edges. Hence the subgroup $\mathbb{Z}^{d-1}$ generated by the first $(d-1)$ elements of $\mathcal{C}$ is invariant under φ_{ab}. By Proposition 5.7, $\widehat{f}(\widehat{\gamma})$ is homotopic relative to $\{\widehat{x}_0\}$ to $\widehat{\gamma}_0\widehat{\gamma}_1\cdots\widehat{\gamma}_{q-1}$. It follows from Lemma 5.5 that the matrix R of φ_{ab} in the basis $\mathcal{C}$ has the following shape:

$$R = \left(\begin{array}{ccc|c} \star & \cdots & \star & \star \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & \star & \star \\ \hline 0 & \cdots & 0 & q \end{array}\right).$$

Since $q \geqslant 2$, the determinant of R cannot be invertible in $\mathbb{Z}$, which contradicts the fact that φ_{ab} is an automorphism. We have thus proved that $\sigma^\infty(e_\bullet)$ is not shift-periodic. $\square$

Proposition 5.11 *There exists an integer $m \geqslant 2$ such that for every $p \in \mathbb{N}$, as a word over $\mathcal{E}_\bullet$, $\mathrm{Red}(f_\#^p(e_\bullet))$ does not contains an m^{th} power.*

Proof According to Lemma 5.1, for every $p \in \mathbb{N}$, the red word associated to $f_\#^p(e_\bullet)$ is exactly $\sigma^p(e_\bullet)$. However, the substitution σ is primitive and the infinite word $\sigma^\infty(e_\bullet)$ is not shift-periodic. Hence the result follows from Proposition 2.1. $\square$

6 The automorphism of $B_r(n)$ induced by φ

6.1 A criterion of nontriviality in $B_r(n)$

Let us have a pause in order to introduce a key ingredient for the sequel of the proof of Proposition 4.11. As explained in the introduction, we need a tool to decide whether two elements in a free Burnside group are distinct. The main theorem of [12] will play that role. In [12], Coulon considers a more general situation than the one we are interested in. Given a nonelementary torsion-free hyperbolic group G, he studies the natural projection $G \to G/G^n$, where G^n stands for the (normal) subgroup of G

generated by the n^{th} power of all elements of G. He provides a criterion to decide whether two elements $g, g' \in G$ have the same image in the quotient G/G^n. For our purpose, we focus on the case where G is a free group. This is the situation that we describe below.

Let (X, x_0) be a pointed simplicial tree. Given two points x and x' of X, we denote by $|x-x'|$ the distance between them, whereas $[x, x']$ stands for the geodesic joining x and x'. Let g be an isometry of X. Its *translation length*, denoted by $\|g\|$, is the quantity $\|g\| = \inf_{x\in X} |gx-x|$. If X is the Cayley graph of $\boldsymbol{F}_r$, then $\|g\|$ is exactly the length of the conjugacy class of $g \in \boldsymbol{F}_r$. The set of points $A_g = \{x \in X \mid |gx-x| = \|g\|\}$ is called the *axis* of g. It is a subtree of X. It is known that either $\|g\| = 0$ and A_g is the set of fixed points of g, or $\|g\| > 0$ and A_g is a bi-infinite geodesic on which g acts by translation of length $\|g\|$. In the first case, g is said to be *elliptic*, in the second one *hyperbolic*. For more details, we refer the reader to [16]. We now assume that $\boldsymbol{F}_r$ acts by isometries on X.

Definition 6.1 Let $n \in \mathbb{N}$ and $\xi \in \mathbb{R}_+$. Let y and z be two points of X. We say that z is the image of y by an (n,ξ)*–elementary move* (or simply *elementary move*) if there is a hyperbolic element $u \in \boldsymbol{F}_r$ such that

(1) $\operatorname{diam}\big([x_0, y] \cap A_u\big) \geqslant \big(\frac{n}{2} - \xi\big)\|u\|$,

(2) $z = u^{-n} y$.

The point z is the image of y by a *sequence of* (n,ξ)*–elementary moves* if there is a finite sequence $y = y_0, y_1, \dots, y_\ell = z$ such that for all $i \in \{0, \dots, \ell-1\}$, the point y_{i+1} is the image of y_i by an (n,ξ)–elementary move.

Knowing that the hyperbolicity constant of a tree is zero, this notion of (n,ξ)–elementary move is exactly the one defined in [12]. The next statement is a particular case of the main theorem of [12] when the group G is free and the underlying space X is a tree.

Proposition 6.2 (Coulon [12]) *Assume that $\boldsymbol{F}_r$ acts properly cocompactly by isometries on (X, x_0). There exist $n_1 \in \mathbb{N}$ and $\xi \in \mathbb{R}_+$ such that for every odd exponent $n \geqslant n_1$ the following holds: two isometries $g, g' \in \boldsymbol{F}_r$ have the same image in $\boldsymbol{B}_r(n)$ if and only if there exist two finite sequences of (n,ξ)–elementary moves which respectively send gx_0 and $g'x_0$ to the same point of X.*

Remark 6.3 Roughly speaking, an elementary move allows us to replace a subword of the form v^m by v^{m-n} provided m is sufficiently large. Assume indeed that (X, x_0) is the Cayley graph of $\boldsymbol{F}_r$ pointed at the identity element of $\boldsymbol{F}_r$. There is a natural one-to-one correspondence between reduced words and geodesics of X starting at x_0. More precisely, given an element $g \in \boldsymbol{F}_r$, the reduced word w which represents g labels the geodesic between x_0 and gx_0. Let us suppose now that w can be written (as a reduced word) $w = pv^m s$ with $m \geqslant \frac{n}{2} - \xi$. It follows that

$$\operatorname{diam}\bigl([x_0, gx_0] \cap A_u\bigr) \geqslant \|u^m\| \geqslant \bigl(\tfrac{n}{2} - \xi\bigr)\|u\|,$$

where u is the element of $\boldsymbol{F}_r$ represented by pvp^{-1}. Thus $u^{-n}g$, which is represented by $pv^{m-n}s$, is the image of g by an elementary move. With this dictionary in mind, Theorem 1.5 becomes a direct application of Proposition 6.2, where (X, x_0) is the Cayley graph of $\boldsymbol{F}_r$ based at 1.

Later in the proof, the tree X will be the universal cover of an RTT. Therefore, this formulation, which extends the idea of substituting subwords, is more appropriate for our purpose.

Proposition 6.2 provides in particular a criterion for detecting trivial elements in $\boldsymbol{B}_r(n)$.

Corollary 6.4 *Assume that $\boldsymbol{F}_r$ acts properly cocompactly by isometries on (X, x_0). There exist $n_1 \in \mathbb{N}$ and $\xi \in \mathbb{R}_+$ such that for every odd exponent $n \geqslant n_1$, the following holds: an element $g \in \boldsymbol{F}_r$ is trivial in $\boldsymbol{B}_r(n)$ if and only if there exists a finite sequence of (n, ξ)–elementary moves which sends gx_0 to x_0.*

However, despite the similarity with the word problem in a group, Corollary 6.4 is not equivalent to Proposition 6.2. This comes from the fact that (n, ξ)–elementary moves are not symmetric. One first has to see a large power along the geodesic $[x_0, gx_0]$ before performing an elementary move. For instance, if a and b are two distinct primitive elements of $\boldsymbol{F}_r$, there is no sequence of elementary moves that sends a^n to b^n. Corollary 6.4 only implies a weaker form of Proposition 6.2 in the sense that we need to allow a larger class of elementary moves: those of the form $pv^m s \to pv^{m-n}s$ with $m \geqslant \frac{n}{4} - \frac{\xi}{2}$.

In our situation, we will apply this criterion with two elements of the form g and $\varphi^p(g)$, where $g \in \boldsymbol{F}_r$ and φ is the automorphism we want to study. The theory of train-track provides much information about the path $[x_0, \varphi^p(g)x_0]$. Therefore, it is also more natural to have a criterion that uses conditions on $[x_0, gx_0]$ and $[x_0, \varphi^p(g)x_0]$ rather than $[gx_0, \varphi^p(g)x_0]$.

Remark Proposition 6.2 is "well known" to the experts of Burnside's groups. To our knowledge, it has never been formulated in such a level of simplicity, though. The reader can, for instance, compare our definition of elementary moves with the one of *simple r–reversal of rank α* used by Adian; see [1, Section 4.18, pages 8–16] for the prerequisites.

After consulting Adian and Ol'shanskiĭ, it seems that the closest published statements to Proposition 6.2 are [1, Chapter VI, Lemma 2.8] and [32, Lemma 5.5]. They should lead to a similar result, but with a weaker requirement to perform elementary moves. Adian's approach would provide an analogue of Theorem 1.5 with a sharper critical exponent ($n_1 = 667$) but where an elementary move is allowed as soon as $m \geqslant 90$ (instead of $m \geqslant \frac{n}{2} - \xi$). This is unfortunately not enough for our purpose. In the Appendix, we explain how our results on $\mathrm{Out}(\boldsymbol{B}_r(n))$ can be proved using Ol'shanskiĭ's work instead of Proposition 6.2. In particular, we prove an analogue of Theorem 1.5 where elementary moves are allowed as soon as $m \geqslant \frac{n}{3}$; see Proposition A.2.

6.2 Performing elementary moves in $\widetilde{G}$

We get back to the proof of Proposition 4.11. The notation is the same as in Section 5.

Metrics on $\widetilde{G}$ For our purpose, the pointed tree (X, x_0) that appears in Proposition 6.2 will be the universal cover $(\widetilde{G}, \tilde{x}_0)$ of G where $\tilde{x}_0$ is preimage of x_0. By declaring that any edge of $\widetilde{G}$ is isometric to the unit real segment $[0,1]$, we obtain an $\boldsymbol{F}_r$–invariant length metric on $\widetilde{G}$: the *combinatorial metric*. We denote by $|\alpha|$ the resulting *combinatorial length* of a path α in $\widetilde{G}$.

We also define a pseudolength metric on $\widetilde{G}$ in the following way. We first consider that any yellow edge has length zero. Recall that $\mathcal{E}$ is the set of all the oriented red edges of G. We chose a preferred set of oriented edges $\vec{\mathcal{E}}$. Recall that the transition matrix M of the red stratum of f is aperiodic. We denote by $\lambda > 1$ the Perron–Frobenius dominant eigenvalue of M, and we consider a positive right eigenvector $l = (l_e)_{e \in \vec{\mathcal{E}}}$ associated to λ. We declare the lifts of e isometric to the real segment $[0, l_e]$. The resulting pseudometric is called the *PF-pseudometric*. We denote by $|\alpha|_{\mathrm{PF}}$ the resulting length of the path α in $\widetilde{G}$: this is called the *PF-length* of α. This length only depends on the red word $\mathrm{Red}(\alpha) \in \mathcal{E}^*$. If α is a red-legal path, we thus get that for all $p \in \mathbb{N}$,

$$|f_\#^p(\alpha)|_{\mathrm{PF}} = \lambda^p |\alpha|_{\mathrm{PF}}.$$

Unless stated otherwise we will work with $\widetilde{G}$ endowed with the combinatorial metric.

The element g and its orbit Recall that $e_\bullet$ is the red edge fixed at the beginning of Section 5. For all $p \in \mathbb{N}$, we have that $f_\#^p(e_\bullet)$ is a path starting by $e_\bullet$. Its yellow-red decomposition is a splitting. The red stratum H is aperiodic. Thus if p is a sufficiently large integer, one can find another occurrence of $e_\bullet$ in $f_\#^p(e_\bullet)$: namely $f_\#^p(e_\bullet) = e_\bullet \nu_0 e_\bullet \nu_1$. The path $\nu = e_\bullet \nu_0$ is a red-legal circuit, and the yellow-red decomposition of ν is a splitting. We denote by g the element of $\pi_1(G, x_0)$ represented by ν. By construction, the geodesic $[\tilde{x}_0, g\tilde{x}_0]$ is the lift in $\widetilde{G}$ of ν starting at $\tilde{x}_0$.

Lemma 6.5 *There exists an integer n_2 with the following property. Let $p \in \mathbb{N}$. Let β be a path of $\widetilde{G}$ such that the red words respectively associated to β and $f_\#^p(\nu)$ are the same. For all $u \in \boldsymbol{F}_r \setminus \{1\}$, if*

$$\operatorname{diam}(\beta \cap A_u) > n_2 \|u\|,$$

then the axis of u only contains yellow edges.

Proof By construction, there exists $p_0 \in \mathbb{N}$ such that ν is a prefix of $f_\#^{p_0}(e_\bullet)$. More generally, $f_\#^p(\nu)$ is a prefix of $f_\#^{p+p_0}(e_\bullet)$ for every $p \in \mathbb{N}$. According to Proposition 5.11, there exists $m \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, the red word associated to $f_\#^p(\nu)$ does not contain an m^{th} power. Put $n_2 = m + 2$. Note that n_2 does not depend on the path β. Let u be a nontrivial element of $\boldsymbol{F}_r$ such that

$$\operatorname{diam}(\beta \cap A_u) > n_2 \|u\|.$$

In particular, there is a vertex $x \in A_u$ such that for every $j \in \{0, \dots, m\}$, the point $u^j x$ belongs to β. Assume now that A_u contains a red edge e. Since A_u is a u–invariant bi-infinite geodesic, the geodesic $[x, ux]$ contains some red edges. In particular, if α stands for the path $[x, u^m x]$, then $\operatorname{Red}(\alpha)$ contains an m^{th} power. However, β is a path of $\widetilde{G}$. Consequently, $[x, u^m x]$ is a subset, hence a subpath, of β. Therefore, the red word associated to β, and thus to $f_\#^p(\nu)$, contains an m^{th} power. This is a contradiction. □

We finish this section with the proof of Proposition 4.11.

Proof of Proposition 4.11 Recall that g is the element of $\pi_1(G, x_0)$ represented by the red legal circuit $\nu = e_\bullet \nu_0$. Our goal is to prove that for sufficiently large odd integers n, the sequence $(\varphi^p(g))_{p \in \mathbb{N}}$ of elements of $\boldsymbol{F}_r$ is embedded in $\boldsymbol{B}_r(n)$. Since φ is an automorphism, it is sufficient to check that $\varphi^p(g) \not\equiv g$ in $\boldsymbol{B}_r(n)$ for all $p \in \mathbb{N}^*$. We are going to use the criterion of Section 6.1. Recall that the geodesic $[\tilde{x}_0, g\tilde{x}_0]$ is a lift in $\widetilde{G}$ of ν. We denote by n_1, ξ and n_2 the constants given, accordingly, by Proposition 6.2 and Lemma 6.5. For the rest of the proof, we fix an odd integer n larger than

$$n_0 = \max\bigl\{n_1,\ 2n_2 + 2\xi + 1,\ 2|\tilde{x}_0 - g\tilde{x}_0| + 2\xi + 1\bigr\}.$$

Note that this lower bound only depends on the outer automorphism Φ and the RTT f.

Let $p \in \mathbb{N}^*$. By construction, the path $\beta = [\tilde{x}_0, \varphi^p(g)\tilde{x}_0]$ is a lift of $f_\#^p(\nu)$. Assume now that $\varphi^p(g) \equiv g$ in $\boldsymbol{B}_r(n)$. By Proposition 6.2, there exist two sequences of (n, ξ)–elementary moves which respectively send $g\tilde{x}_0$ and $\varphi^p(g)\tilde{x}_0$ to the same point of $\widetilde{G}$. However, we fixed $n > 2|\tilde{x}_0 - g\tilde{x}_0| + 2\xi$. Therefore, no (n, ξ)–elementary move can be performed on $[\tilde{x}_0, g\tilde{x}_0]$. It follows that there exists a sequence of (n, ξ)–elementary moves which sends $\varphi^p(g)\tilde{x}_0$ to $g\tilde{x}_0$. We denote by β_i the reduced path obtained from β after the i^{th} (n,ξ)–elementary move. In particular, $\beta_0 = \beta$. Note that the initial point of β_i is always $\tilde{x}_0$. Recall that $\boldsymbol{F}_r^n$ is the normal subgroup of $\boldsymbol{F}_r$ generated by the n^{th} power of every element. We are going to show, by induction on i, that

(H1) the endpoints of a maximal yellow subpath of β_i are not in the same $\boldsymbol{F}_r^n$–orbit,

(H2) $\mathrm{Red}(\beta_i) = \mathrm{Red}(\beta)$.

The base of induction Let α be a maximal yellow subpath of β_0. Recall that β_0 is a lift of $f_\#^p(\nu)$. On the other hand, $f_\#^p(\nu)$ is a subpath of $f_\#^q(e_\bullet)$ for some $q \in \mathbb{N}$. It follows from our assumption that α is not mapped by $\tilde{G} \twoheadrightarrow G$ to a loop. In particular, its endpoints are not in the same $\boldsymbol{F}_r$–orbit, which provides (H1). Assertion (H2) is obvious.

The inductive step Assume that these two conditions hold for i. For simplicity, we denote by $\tilde{z}_i$ the terminal point of β_i; hence $\beta_i = [\tilde{x}_0, \tilde{z}_i]$. We focus on the $(i+1)^{\text{st}}$ elementary move. Let us denote by A_u the axis of the elementary move performed on β_i. In particular, $\operatorname{diam}(\beta_i \cap A_u) \geqslant \left(\frac{n}{2} - \xi\right)\|u\|$ in $\tilde{G}$. By hypothesis (H2), the red words associated to β_i, β and $f_\#^p(\nu)$ are the same. By Lemma 6.5, the axis A_u only contains yellow edges. In particular, A_u crosses β_i along (a part of) a maximal yellow subpath of β_i that we denote by α; see Figure 5.

Let $\tilde{y}$ and $\tilde{y}'$ be the respective initial and terminal points of α. By (H1), $\tilde{y} \neq u^{-n}\tilde{y}'$. Recall that the action of $\boldsymbol{F}_r$ on $\tilde{G}$ respects the yellow-red decomposition. Consequently, the path β_{i+1} is exactly

$$\beta_{i+1} = [\tilde{x}_0, \tilde{y}] \cup [\tilde{y}, u^{-n}\tilde{y}'] \cup [u^{-n}\tilde{y}', u^{-n}\tilde{z}_i].$$

In particular, $\mathrm{Red}(\beta_{i+1}) = \mathrm{Red}(\beta_i)$. Combined with (H2), we get $\mathrm{Red}(\beta_{i+1}) = \mathrm{Red}(\beta)$, which corresponds to (H2) at step $i+1$. The maximal yellow subpaths of β_{i+1} are of three kinds:

- the ones of $[\tilde{x}_0, \tilde{y}]$ which are actually maximal yellow subpaths of β_i,
- the ones of $[u^{-n}\tilde{y}', u^{-n}\tilde{z}_i]$ which are translates of maximal yellow subpaths of β_i,
- the geodesic $[\tilde{y}, u^{-n}\tilde{y}']$.

By (H1), the endpoints of any maximal yellow path of the first two kinds are not in the same $\boldsymbol{F}_r^n$–orbit. Thus $\tilde{y}$ and $\tilde{y}'$ are not in the same $\boldsymbol{F}_r^n$–orbit, being the endpoints of α. Hence neither are $\tilde{y}$ and $u^{-n}\tilde{y}'$. This gives (H2) at step $i+1$, which completes the induction.

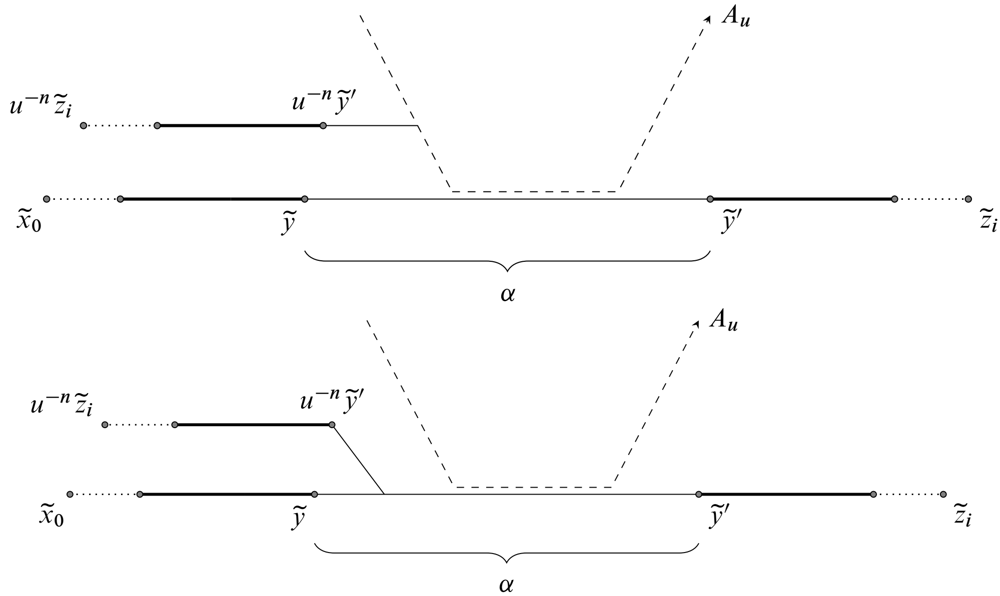

Figure 5: Performing a move on β_i, the two possible configurations. The thin lines refer to yellow paths, the thick ones to red paths. Top: α does not contain the full n^{th} power of u. Bottom: α contains the full n^{th} power of u, but cannot be totally removed.

Recall that (β_i) is the collection of paths obtained by the sequence of elementary moves which sends $\varphi^p(g)$ to g. It follows from the previous discussion that at each step i, $|\beta_i|_{\mathrm{PF}} = |\beta|_{\mathrm{PF}}$. In particular, $|f_\#^p(\nu)|_{\mathrm{PF}} = |\beta|_{\mathrm{PF}} = |\nu|_{\mathrm{PF}}$. However, we build ν in such a way that $|f_\#^p(\nu)|_{\mathrm{PF}} = \lambda^p|\nu|_{\mathrm{PF}}$. This contradicts our original assumption. Therefore, $\varphi^p(g) \not\equiv g$ in $\boldsymbol{B}_r(n)$ for every $p \in \mathbb{N}$. In particular, φ (respectively Φ) induces an automorphism (respectively outer automorphism) of $\boldsymbol{B}_r(n)$ of infinite order. $\square$

7 Comments and questions

7.1 About other possible strategies of proof

In the introduction, we recalled the argument given by Cherepanov. It is easy to elaborate a generalization to a wider class of automorphisms which does not require the criterion stated in Proposition 6.2.

An outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ is *irreducible with irreducible powers* (or simply *iwip*) if there is no (conjugacy class of a) proper free factor of $\boldsymbol{F}_r$ which is invariant by some positive power of Φ. An iwip outer automorphism can be represented by an (absolute) train-track map $f\colon G \to G$ with a primitive transition matrix [6]. Roughly speaking, it implies that there are no "yellow strata" which were the ones responsible for having large powers in our words. As a particular case of Proposition 5.11, there exists a loop γ in G and an integer n_2 with the following property. For every $p \in \mathbb{N}$, the word labeling the loop $f_\#^p(\gamma)$ does not contain an n_2^{th} power (as a complete word, not just its red part). Consequently, Proposition 1.4 is sufficient to conclude. Note also that, in this context, Proposition 5.11 can be proved in a much easier way by using either the action of $\boldsymbol{F}_r$ on the stable tree associated to Φ [18, Theorem 2.1] or the fact that the attracting laminations of Φ cannot be carried by a subgroup of rank 1 [3, Proposition 2.4].

However, as we explained in the introduction, there exist automorphisms for which one cannot use the same strategy. Consider, for instance, the automorphism ψ of $\boldsymbol{F}_4 = \boldsymbol{F}(a,b,c,d)$ defined in the introduction by $\psi(a) = a$, $\psi(b) = ba$, $\psi(c) = cbcd$ and $\psi(d) = c$. One can view ψ as a relative train-track map on the rose: there is only one exponential stratum (the "red stratum" which corresponds to the free factor $\langle c,d\rangle$) and the restriction of ψ to $\langle a,b\rangle$ has polynomial growth (and $\langle a,b\rangle$ gives rise to a "yellow stratum"). We saw that a^{p-1} occurs as subword of $\psi^p(d)$. Nevertheless, we still do not need Proposition 6.2 to conclude here that the automorphism ψ satisfies the statement of Theorem 1.3. It is sufficient to pass to the quotients of $\boldsymbol{F}_r$ and $\boldsymbol{B}_r(n)$ by the normal subgroup generated by a and b, and then to argue as previously.

Nevertheless, given an arbitrary automorphism, this trick (passing to a well chosen quotient) seems to be less easy to run. Look at the automorphism ψ of $\boldsymbol{F}_4 = \boldsymbol{F}(a,b,c,d)$ defined by

$$\psi\colon a \mapsto a, \quad b \mapsto ba, \quad c \mapsto cd^{-1}bd, \quad d \mapsto dcd^{-1}bd.$$

This automorphism grows exponentially. However, if one considers the quotient of $\boldsymbol{F}_4$ by the normal subgroup generated by a and b, it induces the Dehn twist $c \mapsto c$, $d \mapsto dc$, which has finite order as an automorphism of $\boldsymbol{B}_2(n)$.

Let φ be an automorphism of $\boldsymbol{F}_r$. The geometry of the suspension $\boldsymbol{F}_r \rtimes_\varphi \mathbb{Z}$ might provide an alternative proof of Theorem 1.3. In [13], the first author solved indeed the case where $\boldsymbol{F}_r \rtimes_\varphi \mathbb{Z}$ is a hyperbolic group. Generalizing the Delzant–Gromov approach of the Burnside Problem, he constructed a sequence of groups $\boldsymbol{H}_j$ with $\varinjlim \boldsymbol{H}_j = \boldsymbol{B}_r(n)$ such that for every j,

- φ induces an automorphism of infinite order of $\boldsymbol{H}_j$,
- $\boldsymbol{H}_j \rtimes_\varphi \mathbb{Z}$ is a hyperbolic group obtained from $\boldsymbol{H}_{j-1} \rtimes_\varphi \mathbb{Z}$ by small cancellation.

It follows from the hyperbolicity that φ induces an automorphism of infinite order of $\boldsymbol{B}_r(n)$.

If φ is an arbitrary exponentially growing automorphism, then $\boldsymbol{F}_r \rtimes_\varphi \mathbb{Z}$ is no more hyperbolic. However, F Gautero and M Lustig proved that $\boldsymbol{F}_r \rtimes_\varphi \mathbb{Z}$ is hyperbolic relatively to a family of subgroups which consists of conjugacy classes that grow polynomially under iteration by φ [19; 20]. Therefore, one could use a generalization of the iterated small cancellation theory to relative hyperbolic groups. We refer the reader to [14] for a detailed presentation of the Delzant–Gromov approach to the Burnside problem and to [15] for a generalization. See also [34] for a theory of small cancellation in relatively hyperbolic groups.

7.2 Quotients of $\mathrm{Out}(\boldsymbol{F}_r)$

The following remark is due to M Sapir. Proposition 4.1 says that for every integer $n \geqslant 1$, polynomially growing automorphisms of $\boldsymbol{F}_r$ induce automorphisms of finite order of $\boldsymbol{B}_r(n)$. More precisely, their orders divide

$$p(r,n) = n^{2(2^{r-1}-1)}.$$

Let us denote by $\mathcal{Q}_{r,n}$ the quotient of $\mathrm{Out}(\boldsymbol{F}_r)$ by the (normal) subgroup generated by

$$\{\Phi^{p(r,n)} \mid \Phi \in \mathrm{Out}(\boldsymbol{F}_r) \text{ polynomially growing}\}.$$

In particular, the $p(r,n)^{\text{th}}$ power of the Nielsen transformations which generate $\mathrm{Out}(\boldsymbol{F}_r)$ are trivial in $\mathcal{Q}_{r,n}$. It follows from Proposition 4.1 that the map $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$ induces a natural map $\mathcal{Q}_{r,n} \to \mathrm{Out}(\boldsymbol{B}_r(n))$. Therefore, we have the following results:

Theorem 7.1 *Let $r \geqslant 3$. There exists n_0 such that for all odd integers $n \geqslant n_0$, the group $\mathcal{Q}_{r,n}$ contains copies of $\boldsymbol{F}_2$ and $\mathbb{Z}^{\lfloor r/2 \rfloor}$.*

Proof This is a consequence of [13] Theorems 1.8 and 1.10. □

Theorem 7.2 *Let Φ be an outer automorphism of $\boldsymbol{F}_r$. The following assertions are equivalent:*

(1) *Φ has exponential growth;*

(2) *there exists $n \in \mathbb{N}$ such that the image of Φ in $\mathcal{Q}_{r,n}$ has infinite order;*

(3) *there exist $\kappa, n_0 \in \mathbb{N}$ such that for all odd integers $n \geqslant n_0$, the image of Φ in $\mathcal{Q}_{r,\kappa n}$ has infinite order.*

Proof This is a consequence of our main theorem. □

7.3 Exponentially growing automorphisms of the free group can have finite order in a free Burnside group

The constant n_0 in Theorem 1.3 does depend on the outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$. Indeed, we give in this section explicit examples of automorphisms in the kernel of the natural map $\mathrm{Aut}(\boldsymbol{F}_r) \to \mathrm{Aut}(\boldsymbol{B}_r(n))$ which have exponential growth. In particular, there are iwip automorphisms in this kernel.

7.3.1 A first family of examples An outer automorphism $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ induces, by abelianization, an automorphism of $\mathbb{Z}^r$. This defines a homomorphism $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{GL}(r,\mathbb{Z})$, $\Phi \mapsto M_\Phi$. Nielsen proved that for $r = 2$, this morphism is an isomorphism [28]. Moreover, Φ has exponential growth if and only if the absolute value of the trace of $M_\Phi^2 \in \mathrm{GL}(2,\mathbb{Z})$ is larger than 2.

Examples Let $\{a,b\}$ be a basis of the free group $\boldsymbol{F}_2$. For $n \in \mathbb{N}^*$, we define $\varphi_n \in \mathrm{Aut}(\boldsymbol{F}_2)$ by $\varphi_n(a) = a(ba^n)^n$, $\varphi_n(b) = ba^n$. We denote by Φ_n the corresponding outer class in $\mathrm{Out}(\boldsymbol{F}_2)$. The outer class Φ_n has exponential growth since the trace of $M_{\Phi_n}^2$ equals n^4+4n^2+2. However, the outer automorphism of $\boldsymbol{B}_2(n)$ induced by Φ_n is the identity.

For $r > 2$, we consider a splitting of $\boldsymbol{F}_r$ as a free product $\boldsymbol{F}_r = \boldsymbol{F}_2 * \boldsymbol{F}_{r-2}$. For $n \in \mathbb{N}^*$, we consider the automorphism $\psi_n = \varphi_n * \mathrm{Id}$ which is equal to φ_n (defined in the previous paragraph) when restricted to the first factor of the splitting and to the identity when restricted to the second factor. Again, the outer class Ψ_n of ψ_n has exponential growth (since Φ_n has), but the outer automorphism of $\boldsymbol{B}_r(n)$ induced by Ψ_n is the identity.

These examples show that the constant n_0 in Theorem 1.3 is not uniform: it does depend on the outer class $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$. The automorphisms Φ_n are iwip automorphisms. But this is not the case of the automorphisms Ψ_n. We fix this point in the next subsection.

7.3.2 Iwip automorphisms of $\boldsymbol{F}_r$ trivial in $\mathrm{Out}(\boldsymbol{B}_r(n))$ To produce iwip automorphisms in the kernel of the canonical map $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$, one can follow the idea of W Thurston to generically produce pseudo-Anosov homeomorphisms of a surface by composing well chosen Dehn twist homeomorphisms [37].

In the context of automorphisms of free groups, there is a notion of a Dehn twist (outer) automorphism (see for instance [10]) which generalizes the notion of a Dehn twist homeomorphism of a surface: Example 4.3 provides such a Dehn twist automorphism. In [9], M Clay and A Pettet explain how to generate iwip automorphisms of $\boldsymbol{F}_r$ by composing two Dehn twist automorphisms associated to a filling pair of cyclic splittings of $\boldsymbol{F}_r$. We will not explicitly state these definitions here. For our purpose, we only need to know that

- Dehn twist automorphisms have polynomial growth (in fact linear growth), and
- there exist Dehn twist automorphisms $\Delta_1, \Delta_2 \in \mathrm{Out}(\boldsymbol{F}_r)$ satisfying the hypothesis of the following theorem.

Theorem 7.3 (Clay and Pettet [9, Theorem 5.3]) *Let $\Delta_1, \Delta_2 \in \mathrm{Out}(\boldsymbol{F}_r)$ be the Dehn twist outer automorphisms for a filling pair of cyclic splittings of $\boldsymbol{F}_r$. There exists $N \in \mathbb{N}$ such that for every $p, q > N$,*

- *the subgroup of $\mathrm{Out}(\boldsymbol{F}_r)$ generated by Δ_1^p and Δ_2^q is a free group of rank 2,*
- *if $\Phi \in \langle \Delta_1^p, \Delta_2^q \rangle$ is not conjugate to a power of either Δ_1^p or Δ_2^q, then Φ is an iwip outer automorphism.*

We fix an exponent $n \in \mathbb{N}$. We consider two such Dehn twist outer automorphisms Δ_1 and Δ_2, and the integer $N \in \mathbb{N}$ given by Theorem 7.3. Since Δ_1 and Δ_2 have polynomial growth, they induce an automorphism of finite order of $\boldsymbol{B}_r(n)$. In particular, there exists $p > N$ such that $\Phi = \Delta_1^p \Delta_2^p$ is in the kernel of the map $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$. However, Theorem 7.3 ensures that Φ is an iwip outer automorphism of $\boldsymbol{F}_r$.

7.4 Growth rates in $\mathrm{Out}(\boldsymbol{F}_r)$ and $\mathrm{Out}(\boldsymbol{B}_r(n))$

Let Φ be an exponentially growing automorphism of $\boldsymbol{F}_r$. Our study in Section 6 seems to indicate that for odd exponents n large enough, some structure of Φ is preserved in $\boldsymbol{B}_r(n)$. Therefore, we wonder how much information could be carried through the map $\mathrm{Out}(\boldsymbol{F}_r) \to \mathrm{Out}(\boldsymbol{B}_r(n))$. In particular, what can we say about the growth rate of Φ?

Let $\boldsymbol{G}$ be a group generated by a finite set S. We endow $\boldsymbol{G}$ with the word-metric with respect to S. The length of the conjugacy class of $g \in \boldsymbol{G}$, denoted by $\|g\|$, is the length of the shortest element conjugated to g. An outer automorphism Φ of $\boldsymbol{G}$ naturally acts on the set of conjugacy classes of $\boldsymbol{G}$. Consequently, as in the free group, one can define the *(exponential) growth rate* of Φ by

$$\mathrm{EGR}(\Phi) = \sup_{g \in \boldsymbol{G}} \limsup_{p \to +\infty} \sqrt[p]{\|\Phi^p(g)\|}.$$

Since the word-metrics for two distinct finite generating sets of $\boldsymbol{G}$ are bi-Lipschitz equivalent, this rate does not depend on S. The automorphism Φ is said to have *exponential growth* if $\mathrm{EGR}(\Phi) > 1$.

By our knowledge, it is not known if there exist outer automorphisms of Burnside groups with exponential growth. We would like to ask the following questions:

- Are there automorphisms of $\boldsymbol{B}_r(n)$ with exponential growth?
- Let $\Phi \in \mathrm{Out}(\boldsymbol{F}_r)$ with exponential growth. Is there an integer n_0 such that for all (odd) exponents $n \geqslant n_0$, the automorphism $\widehat{\Phi}_n$ of $\boldsymbol{B}_r(n)$ induced by Φ has exponential growth? Such that $\mathrm{EGR}(\widehat{\Phi}_n) = \mathrm{EGR}(\Phi)$?
- Are there automorphisms of $\boldsymbol{B}_r(n)$ of infinite order which do not have exponential growth?

On the other hand, it could be very interesting to understand to what extent the structure of the attracting laminations associated to an outer automorphism of $\boldsymbol{F}_r$ is preserved in $\boldsymbol{B}_r(n)$. Recall that theses laminations are the fundamental tool used by Bestvina, Feighn and Handel to prove that $\mathrm{Out}(\boldsymbol{F}_r)$ satisfies the Tits alternative [4; 5].

Appendix

Proposition 6.2 can be seen as a weak form of a Dehn algorithm associated to the following presentation of the free Burnside group:

$$\boldsymbol{B}_r(n) = \langle a_1, \dots, a_r \mid x^n = 1 \text{ for all } x \rangle. \tag{4}$$

Let w be a reduced word over the alphabet $\{a_1, \dots a_r\}$. If w contains a subword v corresponding to almost half a relation from (4), we allow v to be replaced, in w, by its complement. Proposition 6.2 states that if w represents the trivial element, then after finitely many steps we will get the trivial word.

For our purpose, we actually do not need such a strong statement. The aim of this appendix is to explain how Theorem 1.3 can be proved using only Ol'shanskiĭ's work on free Burnside groups [32]. It might be possible to proceed in the same way using the Novikov–Adian approach [1]. The exposition would, however, be more technical. We first recall some results of Ol'shanskiĭ, and then list the modifications that need to be made to our original proof of Proposition 4.11.

Let (X, x_0) be a pointed simplicial tree endowed with an action by isometries of $\boldsymbol{F}_r$.

Definition A.1 Let $n \in \mathbb{N}^*$ and $c \in (0,1)$. Let y and z be two points of X. We say that z is the image of y by an (c,n)*–weak elementary move* if there is a hyperbolic element $u \in \boldsymbol{F}_r$ such that

(1) $\mathrm{diam}\big([x_0, y] \cap A_u\big) \geqslant cn\|u\|$,

(2) $z = u^{-n} y$.

The point z is the image of y by a *sequence of* (c,n)*–weak elementary moves* if there is a finite sequence $y = y_0, y_1, \dots, y_\ell = z$ such that for all $i \in \{0, \dots, \ell-1\}$, the point y_{i+1} is the image of y_i by a (c,n)–weak elementary move.

Remark As we recalled previously, in our original framework, we allowed a regular move to be performed if $[x_0, y]$ contained almost half of a relation. Here we relax this condition: one can perform a weak move even if $[x_0, y]$ contains a (much) smaller ratio of a relation. The allowed ratio is given by c. In practice we will always have $c \leqslant \frac{1}{3}$.

Let us focus first on the case where X is the Cayley graph of $\boldsymbol{F}_r$ with respect to the free basis $\{a_1, \dots, a_r\}$. To avoid any ambiguity, we denote it by T. Let t_0 be the vertex of T corresponding to the identity. The following result is a consequence of Ol'shanskiĭ's work [32].

Proposition A.2 *Let $n > 10^{10}$ be an odd integer. An element $g \in \boldsymbol{F}_r$ is trivial in $\boldsymbol{B}_r(n)$ if and only if there exists a finite sequence of $\left(\frac{1}{3}, n\right)$–weak elementary moves which sends gt_0 to t_0.*

Remark The proof below relies on Ol'shanskiĭ's diagrammatical approach of the Burnside problem. To keep the appendix short, we do not recall all the necessary background on diagrams. In particular, we use the vocabulary and notations of [32] without any further explanation. For an extensive introduction to diagrams, we refer the reader to [33].

Proof The "if" part directly follows from the definition of weak elementary moves. Let us focus on the "only if" part. Let (C_j) be the system of independent relations of $\boldsymbol{B}_r(n)$ defined in [32, page 203]:

$$\boldsymbol{B}_r(n) = \langle a_1, \dots, a_r \mid C_1^n, C_2^n, \dots, C_i^n, \dots \rangle.$$

Let $g \in \boldsymbol{F}_r \setminus \{1\}$ whose image in $\boldsymbol{B}_r(n)$ is trivial. Let w be the noncontractible word over the group alphabet $\{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$ representing g. As g is trivial in $\boldsymbol{B}_r(n)$, there exists $i \geqslant 1$ such that g is trivial in $\langle a_1, \dots, a_r \mid C_1^n, C_2^n, \dots, C_i^n \rangle$. In other words, w labels the contour of a diagram of rank i that we denote by Δ. Without loss of generality, we can assume that Δ is minimal [32, page 205].

The next arguments are a variation of [32, Lemma 5.5]. This lemma covers indeed two cases, where Δ is a *disc* diagram or an *annular* diagram and requires therefore the label of Δ to be cyclically noncontractible. In our case, we only need to consider disc diagrams, hence the assumption that w is noncontractible will be sufficient. Since w is not trivial, Δ contains a least one cell. By [32, Lemma 3.6], Δ admits a θ–cell. Recall that $\theta = 0.985$, while $\gamma = 10^{-6}/7$. Consequently, its degree of contiguity to a section p of the contour of Δ is at least $\frac{1}{3} + 400\gamma$. Let Π be a θ–cell whose corresponding contiguity subdiagram Γ has minimal type $\tau(\Gamma)$. Its contour is decomposed as $p_1 q_1 p_2 q_2$, where $q_1 = \Gamma \wedge p$ and $q_2 = \Gamma \wedge \Pi$. Applying [32, Lemma 2.1] one observes that $|p_i| \leqslant 2\gamma n \min\{|C_k|, |C_l|\}$, where $k = r(\Pi)$ and $l = r(p)$. It follows that $|p_1| + |p_2| \leqslant 400\gamma|\partial\Pi|$; thus $|q_2| \geqslant \frac{1}{3}|\partial\Pi| + 100(|p_1| + |p_2|)$. Applying [32, Lemma 5.4], one gets that q_1 and q_2 have a common subpath q such that $|q| \geqslant |\partial\Pi|/3$. In other words, the contour of Π can be decomposed as $q\bar{q}$ where q^{-1} is a section of the contour of Δ and $|q| \geqslant |\partial\Pi|/3$. As Δ is a diagram of rank i, there exists $j \leqslant i$ and a cyclic permutation D of $C_j^{\pm 1}$ such that the label of $\bar{q}q$ is D^n. On the other hand, the contour of Δ can be written $rq^{-1}s$.

We now rephrase this observation using our geometric point of view. Let v and h be the element of $\boldsymbol{F}_r$ represented by the respective labels of $\bar{q}q$ (ie D) and r. Let $u = hvh^{-1}$. Let γ be the path of T starting at t_0 and labeled as q^{-1}. Recall that the contour $rq^{-1}s$ of Δ is labeled by the noncontractible word w; hence $h\gamma$ is a subpath of the geodesic $[t_0, gt_0]$. On the other hand, the collection of words (C_i) has been chosen in a minimal way. In particular, C_i and thus D are cyclically reduced. As a consequence, γ is contained in $[t_0, v^{-n}t_0]$ which lies in the axis of v. It follows that $h\gamma$ is a path contained in hA_v, ie the axis of u. Hence

$$\operatorname{diam}\big([t_0, gt_0] \cap A_u\big) \geqslant |\gamma| \geqslant \tfrac{1}{3}|\partial\Pi| = \tfrac{1}{3}|D| = \tfrac{n}{3}\|u\|.$$

In other words, $u^n g$ is obtained from g by performing a $\left(\frac{1}{3}, n\right)$–weak elementary move.

Let Δ' be the diagram obtained from Δ by removing the cell Π. Its contour is exactly $r\bar{q}s$. By our choice of v and h, its label represents $u^n g$. In other words, removing Π is equivalent to performing a $\left(\frac{1}{3}, n\right)$–weak elementary move. By the very definition of diagrams, Δ only contains finitely many cells. An induction on the number of cells in Δ shows that after finitely many weak elementary moves gt_0 is sent to t_0. □

The proof of Proposition 4.11 does not take place in the Cayley graph of $\boldsymbol{F}_r$ but in the universal cover of the underlying graph of an RTT map. Therefore, we need an analogue of Proposition A.2 in an arbitrary tree. From now on, (X, x_0) is a pointed simplicial tree. We assume that $\boldsymbol{F}_r$ acts properly cocompactly by isometries on X. There exists a natural $\boldsymbol{F}_r$–equivariant map $F\colon T \to X$ sending t_0 to x_0. Since $\boldsymbol{F}_r$ acts properly cocompactly on X, there exist $k \geqslant 1$ and $l \geqslant 0$ such that F is a (k, l)–quasiisometry, meaning that for every $t, t' \in T$,

$$k^{-1}|t - t'| - l \leqslant |F(t) - F(t')| \leqslant k|t - t'| + l.$$

We denote by ∂X the boundary at infinity of X.

Lemma A.3 *There exists $B \geqslant 0$ with the following property. Let $t, t' \in T$. Let $u \in \boldsymbol{F}_r \setminus \{1\}$ and $m \in \mathbb{N}$. Assume that we have*

$$\operatorname{diam}\bigl([t, t'] \cap A_u\bigr) \geqslant m\|u\|$$

in T. Then the following holds in X:

$$\operatorname{diam}\bigl([F(t), F(t')] \cap A_u\bigr) \geqslant m\|u\| - B.$$

Remark By abuse of notation, A_u stands for the axis of u in both T and X. Similarly with $\|u\|$.

Proof Recall first a well known statement of hyperbolic geometry: the stability of quasigeodesics. There exists $d \geqslant 0$ with the following property. The Hausdorff distance between a (k, l)–quasigeodesic of X and any geodesic with the same endpoints (possibly in ∂X) is bounded above by d [11, Chapitre 3, Théorème 1.3].

By assumption, there exists a point s in $[t, t'] \cap A_u$ such that $u^m s$ still belongs to $[t, t'] \cap A_u$. Recall that the axis of u in T is an u–invariant geodesic. Hence its image under F is a u–invariant (k, l)–quasigeodesic of X. It follows from the stability of quasigeodesics that $F(s)$ and $u^m F(s)$ lie in the d–neighborhood of the axis of u in X. In the same way, we see that $F(s)$ and $u^m F(s)$ lie in the d neighborhood of $[F(t), F(t')]$. Consequently, the following holds in X:

$$m\|u\| \leqslant |u^m F(s) - F(s)| \leqslant \operatorname{diam}\bigl([F(t), F(t')]^{+d} \cap A_u^{+d}\bigr). \tag{5}$$

Here the notation Y^{+d} stands for the d–neighborhood of $Y \subset X$. However, we observe that

$$\operatorname{diam}\bigl([F(t), F(t')]^{+d} \cap A_u^{+d}\bigr) \leqslant \operatorname{diam}\bigl([F(t), F(t')] \cap A_u\bigr) + 2d. \tag{6}$$

The result follows from (5) and (6) with $B = 2d$. □

Proposition A.4 *There exists $n_1 \in \mathbb{N}$ such that for every odd integer $n \geqslant n_0$, the following holds: an element $g \in \boldsymbol{F}_r$ is trivial in $\boldsymbol{B}_r(n)$ if and only if there exists a finite sequence of $\left(\frac{1}{4}, n\right)$–weak elementary moves which sends gx_0 to x_0.*

Proof Let B be the parameter given by Lemma A.3. Let $n_1 = \max\{10^{10}, 12B\}$. Let $n \geqslant n_0$. Let g be an element of $\boldsymbol{F}_r$. The tree X being simplicial, the translation length in X of any nontrivial element of $\boldsymbol{F}_r$ is at least 1. It follows from our choice of n_0 that for every $u \in \boldsymbol{F}_r \setminus \{1\}$, we have the following property. If $\operatorname{diam}\bigl([t_0, gt_0] \cap A_u\bigr) \geqslant \frac{n}{3}\|u\|$ in T, then $\operatorname{diam}\bigl([x_0, gx_0] \cap A_u\bigr) \geqslant \frac{n}{4}\|u\|$ in X. In other words, to any $\left(\frac{1}{3}, n\right)$–weak elementary move in T corresponds a $\left(\frac{1}{4}, n\right)$–weak elementary move in X. Hence the result follows from Proposition A.2. □

Corollary A.5 *There exists $n_1 \in \mathbb{N}$ such that for every odd integer $n \geqslant n_0$, the following holds: two isometries $g, g' \in \boldsymbol{F}_r$ have the same image in $\boldsymbol{B}_r(n)$ if and only if there exist two finite sequences of $\left(\frac{1}{8}, n\right)$–weak elementary moves which respectively send gx_0 and $g'x_0$ to the same point of X.*

Proof We apply Proposition A.4 with $g^{-1}g'$. □

Let us come back to Proposition 4.11. The main idea of the proof was the following. The criterion (Proposition 6.2) gave us a sequence of moves performed in $\widetilde{G}$ to send $\varphi^p(g)\widetilde{x}_0$ to $\widetilde{x}_0$. However, performing a move required us first to see a large part of a relation along $[\widetilde{x}_0, \varphi^p(g)\widetilde{x}_0]$. Because of Lemma 6.5, the support of the moves only contained yellow letters. Therefore, the red part was preserved, which led to a contradiction. Note that it does not matter whether the requirement to perform a move is to see one half, one fourth or one tenth of the relation. Therefore, the proof of Proposition 4.11 works in exactly the same way with the following modifications:

(1) Replace Proposition 6.2 by Corollary A.5.

(2) Define the critical exponent n_0 as
$$n_0 = \max\{n_1,\ 8n_2 + 1,\ 8|g\widetilde{x}_0 - \widetilde{x}_0| + 1\}.$$

(3) Replace every (n, ξ)–elementary move by a $\left(\frac{1}{8}, n\right)$–weak elementary move.

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