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Top-dimensional quasiflats in CAT(0) cube complexes

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We show that every n–quasiflat in an n–dimensional CAT(0) cube complex is at finite Hausdorff distance from a finite union of n–dimensional orthants. Then we introduce a class of cube complexes, called *weakly special* cube complexes, and show that quasi-isometries between their universal covers preserve top-dimensional flats. This is the foundational result towards the quasi-isometric classification of right-angled Artin groups with finite outer automorphism group.

Some of our arguments also extend to CAT(0) spaces of finite geometric dimension. In particular, we give a short proof of the fact that a top-dimensional quasiflat in a Euclidean building is Hausdorff close to a finite union of Weyl cones, which was previously established by Kleiner and Leeb (1997), Eskin and Farb (1997) and Wortman (2006) by different methods.

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1 Introduction

1.1 Summary of results

A quasiflat of dimension d in a metric space X is a quasi-isometric embedding $\phi\colon \mathbb{E}^d \to X$, ie there exist positive constants L, A such that for all $x, y \in \mathbb{E}^d$,

$$L^{-1}d(x,y) - A \le d(\phi(x),\phi(y)) \le Ld(x,y) + A.$$

Top-dimensional (or maximal) flats and quasiflats in spaces of higher rank are analogues of geodesics and quasigeodesics in Gromov hyperbolic spaces, which play a key role in understanding the large scale geometry of these spaces. In particular, several quasi-isometric rigidity results were established on the study of such flats or quasiflats. Here is a list of examples:

- Euclidean buildings and symmetric spaces of noncompact type; see Mostow [34], Kleiner and Leeb [30], Eskin and Farb [17], Kramer and Weiss [32].
- Universal covers of certain Haken manifolds (see Kapovich and Leeb [27]); higher-dimensional graph manifolds (see Frigerio, Lafont and Sisto [18]); two-dimensional tree groups and their higher dimensional analogues (see Behrstock, Januszkiewicz and Neumann [7; 5].

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- $\mathrm{CAT}(0)$ 2–complexes; see Bestvina, Kleiner and Sageev [9], with applications to the quasi-isometric rigidity of atomic right-angled Artin groups in their paper [8].
- Flats generated by Dehn twists in mapping class groups; see Behrstock, Kleiner, Minsky and Mosher [6].

In this paper, we will mainly focus on top-dimensional quasiflats and flats in $\mathrm{CAT}(0)$ cube complexes. All cube complexes in this paper will be finite-dimensional. Our first main result shows how the cubical structure interacts with quasiflats.

Theorem 1-1 *If X is a $\mathrm{CAT}(0)$ cube complex of dimension n, then for every n–quasiflat Q in X, there is a finite collection $O_1, \dots, O_k$ of n–dimensional orthant subcomplexes in X such that*

$$d_H\left(Q, \bigcup_{i=1}^{k} O_k\right) < \infty,$$

where d_H denotes the Hausdorff distance.

An *orthant* O of X is a convex subset which is isometric to the Cartesian product of finitely many half-lines $\mathbb{R}_{\geq 0}$. If O is both a subcomplex and an orthant, then O is called an *orthant subcomplex*. We caution the reader that the definition of orthant subcomplex here is slightly different from other places, ie we require an orthant subcomplex to be convex with respect to the $\mathrm{CAT}(0)$ metric.

The 2–dimensional case of Theorem 1-1 was proved in [9]. We will use this theorem as one of the main ingredients to study the coarse geometry of right-angled Artin groups (see Corollary 1-4 below and the remarks after). Also note that recently Behrstock, Hagen, and Sisto have obtained a quasiflat theorem of quite a different flavor in [4]. Their result does not imply our result and vice versa.

Based on Theorem 1-1, we study how the top-dimensional flats behave under quasi-isometries. In general, quasi-isometries between $\mathrm{CAT}(0)$ complexes of the same dimension do not necessarily preserve top dimension flats up to finite Hausdorff distance, even if the underlying spaces are cocompact. However, motivated by Haglund and Wise [20], we can define a large class of cube complexes such that top-dimensional flats behave nicely with respect to quasi-isometries between universal covers of these complexes. Our class contains all compact nonpositively curved special cube complexes up to finite cover [20, Proposition 3.10].

Definition 1-2 A cube complex W is *weakly special* if and only if it has the following properties:

(1) W is nonpositively curved.

(2) No hyperplane *self-osculates* or *self-intersects*.

The notions of self-osculation and self-intersection were introduced in [20, Definition 3.1].

Theorem 1-3 *Let W_1' and W_2' be two compact weakly special cube complexes with $\dim(W_1') = \dim(W_2') = n$, and let W_1 and W_2 be the universal covers of W_1' and W_2', respectively. If $f\colon W_1 \to W_2$ is an (L, A)–quasi-isometry, then there exists a constant $C = C(L, A)$ such that for any top-dimensional flat $F \subset W_1$, there exists a top-dimensional flat $F' \subset W_2$ with $d_H(f(F), F') < C$.*

We now apply this result to *right-angled Artin groups* (RAAGs). Recall that for every finite simplicial graph Γ with its vertex set denoted by $\{v_i\}_{i\in I}$, one can define a group using the following presentation:

$$\langle \{v_i\}_{i\in I} \mid [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are adjacent}\rangle.$$

This is called the *right-angled Artin group with defining graph* Γ, and we denote it by $G(\Gamma)$. Each $G(\Gamma)$ can be realized as the fundamental group of a nonpositively curved cube complex $\bar{X}(\Gamma)$, which is called the Salvetti complex (see Charney [13] for a precise definition). The 2–skeleton of the Salvetti complex is the usual presentation complex for $G(\Gamma)$. The universal cover of $\bar{X}(\Gamma)$ is a CAT(0) cube complex, which we denote by $X(\Gamma)$.

Corollary 1-4 *Let Γ_1, Γ_2 be finite simplicial graphs, and let $\phi\colon X(\Gamma_1) \to X(\Gamma_2)$ be an (L, A)–quasi-isometry. Then:*

(1) $\dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$.

(2) *There is a constant $D = D(L, A)$ such that for any top-dimensional flat F_1 in $X(\Gamma_1)$, we can find a flat F_2 in $X(\Gamma_2)$ such that $d_H(\phi(F_1), F_2) < D$.*

This is the foundation for a series of work on quasi-isometric classification and rigidity of RAAGs by the author and Kleiner [21; 22; 24; 23].

Remark We could also use Theorem 1-3 to obtain an analogous statement for quasi-isometries between the Davis complexes of certain right-angled Coxeter groups, but in general the dimensions of maximal flats in a Davis complex are strictly smaller than the dimension of complex itself, so we need extra condition on the right-angled Coxeter groups; see Corollary 5-18 for a precise statement.

Corollary 1-4 implies that ϕ maps chains of top-dimensional flats to chains of top-dimensional flats, and this gives rise to several quasi-isometry invariants for RAAGs. More precisely, we consider a graph $\mathcal{G}_d(\Gamma)$ where the vertices are in 1–1 correspondence to top-dimensional flats in $X(\Gamma)$ and two vertices are connected by an edge if and only if the coarse intersection of the corresponding flats has dimension $\geq d$. The connectedness of $\mathcal{G}_d(\Gamma)$ can be read off from Γ, which gives us the desired invariants.

Definition 1-5 Let $d \geq 1$ be an integer. Let Γ be a finite simplicial graph and let $F(\Gamma)$ be the flag complex that has Γ as its 1–skeleton. Γ has *property* (P_d) if and only if:

(1) Any two top-dimensional simplices Δ_1 and Δ_2 in $F(\Gamma)$ are connected by a $(d-1)$*–gallery*.

(2) For any vertex $v \in F(\Gamma)$, there is a top-dimensional simplex $\Delta \subset F(\Gamma)$ such that Δ contains at least d vertices that are adjacent to v.

A sequence of n–dimensional simplices $\{\Delta_i\}_{i=1}^{p}$ in $F(\Gamma)$ is a k*–gallery* if $\Delta_i \cap \Delta_{i+1}$ contains a k–dimensional simplex for $1 \leq i \leq p-1$.

Theorem 1-6 *$\mathcal{G}_d(\Gamma)$ is connected if and only if Γ has property (P_d). In particular, for any $d \geq 1$, property (P_d) is a quasi-isometry invariant for RAAGs.*

Remark Another interesting fact in the case $d = 1$ is that one can tell whether Γ admits a nontrivial join decomposition by looking at the diameter of $\mathcal{G}_1(\Gamma)$. This basically follows from the argument in Dani and Thomas [14]. See Theorem 5-30 for a precise statement. Thus in the case of $X(\Gamma)$, one can determine whether the space splits as a product by looking at the intersection pattern of top-dimensional flats. We ask whether this is true in general: if Z is a cocompact geodesically complete CAT(0) space that has n–flats but not $(n+1)$–flats, can one determine whether Z splits as a product of two unbounded CAT(0) spaces by looking at the intersection pattern of n–flats in Z?

Actually, a large portion of our discussion generalizes to n–dimensional quasiflats in CAT(0) spaces of geometric dimension $= n$ (the notion of geometric dimension and its relation to other notions of dimension are discussed in Kleiner [28]). This will be discussed in the appendix and see Theorem A-18 and Theorem A-19 for a summary.

In particular, this leads to a short proof of the following result, which was previously established in Kleiner and Leeb [30], Eskin and Farb [17], and Wortman [37] by different methods, and it is one of the main ingredients in proving quasi-isometric rigidity for Euclidean buildings.

Theorem 1-7 *Let Y be a Euclidean building of rank n, and let $Q \subset Y$ be an n–quasiflat. Then there exist finitely many Weyl cones $\{W_i\}_{i=1}^h$ such that*

$$d_H\left(Q, \bigcup_{i=1}^h W_i\right) < \infty.$$

On the way to Theorem 1-7, we also give a more accessible proof of the following weaker version of one of the main results in Kleiner and Lang [29].

Theorem 1-8 *Let $q\colon Y \to Y'$ be a quasi-isometric embedding, where Y and Y' are $\mathrm{CAT}(0)$ spaces of geometric dimension $\le n$. Then q induces a monomorphism $q_*\colon H_{n-1}(\partial_T Y) \to H_{n-1}(\partial_T Y')$. If q is a quasi-isometry, then q_* is an isomorphism.*

Here $\partial_T Y$ and $\partial_T Y'$ denote the Tits boundary of Y and Y' respectively.

1.2 Sketch of proofs

1.2.1 Proof of Theorems 1-1 and 1-7 The proof of Theorem 1-1 has five steps, as below. The first one follows Bestvina, Kleiner and Sageev [9] closely, but the others are different, since part of the argument in [9] depends heavily on special features of dimension 2, and does not generalize to the n–dimensional case.

Let X be a $\mathrm{CAT}(0)$ piecewise Euclidean polyhedral complex with $\dim(X) = n$, and let $Q\colon \mathbb{E}^n \to X$ be a top-dimensional quasiflat in X.

Step 1 Following [9], one can replace the top-dimensional quasiflat, which usually contains local wiggles, by a minimizing object which is more rigid.

More precisely, let us assume without of loss of generality that Q is a continuous quasi-isometric embedding. Let $[\mathbb{E}_n]$ be the fundamental class in the n^{th} locally finite homology group of $\mathbb{E}^n$ and let $[\sigma] = Q_*([\mathbb{E}_n])$. Let S be the support set (Definition 3-1) of $[\sigma]$. It turns out that S has nice local property (it is a subcomplex with geodesic extension property) and asymptotic property (it looks like a cone from far away). Moreover, $d_H(S, Q) < \infty$.

In the next few steps, we study the structure of S by looking at its "boundary".

Recall that X has a Tits boundary $\partial_T X$, whose points are asymptotic classes of geodesic rays in X, and the asymptotic angle between two geodesic rays induces a metric on $\partial_T X$. See Section 2.2 for a precise definition. We define the boundary of S, denoted $\partial_T S$, to be the subset of $\partial_T X$ corresponding to geodesic rays inside S.

Step 2 We produce a collection of orthants in X from S. More precisely, we find an embedded simplicial complex $K \subset \partial_T X$ such that $\partial_T S \subset K$. Moreover, K is made of right-angled spherical simplices, each of which is the boundary of an isometrically embedded orthant in X. This step depends on the cubical structure of X, and is discussed in Section 4.1.

Step 3 We show $\partial_T S$ is actually a cycle. Namely, it is the "boundary cycle at infinity" of the homology class $[\sigma]$. This step does not depend on the cubical structure of X and is actually true in greater generality by the much earlier, but still unpublished work of Kleiner and Lang [29]. However, their paper was based on metric current theory. Under the assumption of Theorem 1-1, we are able to give a self-contained account which only requires homology theory; see Section 4.2.

Step 4 We deduce from the previous two steps that $\partial_T S$ is a cycle made of $(n-1)$–dimensional all-right spherical simplices. Moreover, each simplex is the boundary of an orthant in X.

Step 5 We finish the proof by showing S is Hausdorff close to the union of these orthants. See Section 4.3 for the last two steps.

If X is a Euclidean building, then it is already clear that the cycle at infinity can be represented by a cellular cycle, since the Tits boundary is a polyhedral complex (a spherical building). The problem is that X itself may not be a polyhedral complex. There are several ways to get around this point. Here we deal with it by generalizing several results of [9] to CAT(0) spaces of finite geometric dimension, which is of independent interest.

1.2.2 Proof of Theorem 1-3 Let W_1 and W_2 be the universal covers of two weakly special cube complexes. We also assume $\dim(W_1) = \dim(W_2) = n$. Our starting point is similar to the treatment in Kleiner and Leeb [30] and Bestvina, Kleiner and Sageev [8]. Let $\mathcal{KQ}(W_i)$ be the lattice generated by finite unions, coarse intersections and coarse subtractions of top-dimensional quasiflats in W_i, modulo finite Hausdorff distance. Any quasi-isometry $q\colon W_1 \to W_2$ will induce a bijection $q_\sharp\colon \mathcal{KQ}(W_1) \to \mathcal{KQ}(W_2)$.

It suffices to study the combinatorial structure of $\mathcal{KQ}(W_i)$. By Theorem 1-1, each element $[A] \in \mathcal{KQ}(W_i)$ is made of a union of top-dimensional orthants, together with several lower dimensional objects. We denote the number of top-dimensional orthants in $[A]$ by $\|[A]\|$. $[A]$ is *essential* if $\|[A]\| > 0$, and $[A]$ is *minimal essential* if for any $[B] \in \mathcal{KQ}(W_i)$ with $[B] \subset [A]$ (ie B is coarsely contained in A) and $[B] \neq [A]$, we have $\|[B]\| = 0$.

It suffices to study the minimal essential elements of $\mathcal{KQ}(W_i)$, since every element in $\mathcal{KQ}(W_i)$ can be decomposed into minimal essential elements together with several lower dimensional objects. In the case of universal covers of special cube complexes, these elements have nice characterizations and behave nicely with respect to quasi-isometries:

Theorem 1-9 *If $[A] \in \mathcal{KQ}(W_i)$ is minimal essential, then there exists a convex subcomplex $K \subset W_i$ which is isometric to $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$ such that $[K] = [A]$.*

Theorem 1-10 $|q_\sharp([A])| = \|[A]\|$ *for any minimal essential element $[A] \in \mathcal{KQ}(W_1)$.*

Theorem 1-3 essentially follows from the above two results.

1.3 Organization of the paper

In Section 2 we will recall several basic facts about CAT(κ) spaces and CAT(0) cube complexes. We also collect several technical lemmas in this section, which will be used later.

In Section 3 we will review the discussion in [9] which will enable us to replace the top-dimensional quasiflat by the support set of the corresponding homology class. In Section 4 we will study the geometry of this support set and prove Theorem 1-1. In Sections 5.1 and 5.2, we look at the behavior of top-dimensional flats in the universal covers of weakly special cube complexes and prove Theorem 1-3 and Corollary 1-4. In Section 5.3, we use Corollary 1-4 to establish several quasi-isometric invariants for RAAGs.

In the appendix, we generalize some results of Sections 3 and 4 to CAT(0) spaces of finite geometric dimension and prove Theorem 1-8 and Theorem 1-7.

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2 Preliminaries

We start with some basic notation. The open balls and closed balls of radius r in a metric space will be denoted by $B(p,r)$ and $\overline{B}(p,r)$ respectively. The sphere of radius r centered at p is denoted by $S(p,r)$. The open r–neighborhood of a set A in a metric space is denoted by $N_r(A)$. The diameter of A is denoted by $\operatorname{diam}(A)$.

For a metric space K, $C_\kappa K$ denotes the κ–cone over K; see [11, Definition I.5.6]. When $\kappa = 0$, we call it the Euclidean cone over K and denote it by CK for simplicity. All products of metric spaces in this paper will be l^2–products.

The closed and open stars of a vertex v in a polyhedral complex are denoted by $\overline{st}(v)$ and $st(v)$ respectively. We use "$*$" for the join of two polyhedral complexes and "$\circ$" for the join of two graphs.

2.1 M_k–polyhedral complexes with finite shapes

In this section, we summarize some results about M_k–polyhedral complexes with finitely many isometry types of cells from [11, Chapter I.7], see also [10].

An M_k–polyhedral complex is obtained by taking a disjoint union of a collection of convex polyhedra from the complete simply connected n–dimensional Riemannian manifolds with constant curvature equal to k (n is not fixed) and gluing them along isometric faces. The complex is endowed with the quotient metric (see [11, Definition I.7.37]). Note that the topology induced by the quotient metric may be different from the topology as a cell complex.

An M_1–polyhedral complex is also called a piecewise spherical complex. If the complex is made of right-angled spherical simplices, then it is also called an all-right spherical complex. A M_0–polyhedral complex is also called a piecewise Euclidean complex.

We are mainly interested in the case where the collection of convex polyhedra we use to build the complex has only finitely many isometry types. Following [11], we denote the isometry classes of cells in K by $\mathrm{Shape}(K)$. Note that we can barycentrically subdivide any M_k–polyhedral complex twice to get an M_k–simplicial complex.

For an M_k–polyhedral complex K and a point $x \in K$, we denote the unique open cell of K which contains x by $\mathrm{supp}(x)$ and the closure of $\mathrm{supp}(x)$ by $\mathrm{Supp}(x)$. We also denote the geometric link of x in K by $\mathrm{Lk}(x, K)$; see [11, Section I.7.38]. In this paper, we always truncate the usual length metric on $\mathrm{Lk}(x, K)$ by π. If an ϵ–ball $B(x, \epsilon)$ around x has the properties that

- $B(x, \epsilon)$ is contained in the open star of x in K,
- $B(x, \epsilon)$ is isometric to the ϵ–ball centered at the cone point in $C_k(\mathrm{Lk}(x, K))$,

then we call $B(x, \epsilon)$ a *cone neighborhood of* x.

Theorem 2-1 [11, Theorem I.7.39] *Suppose K is an M_k–polyhedral complex with* $\mathrm{Shape}(K)$ *finite. Then for every $x \in K$, there exists a positive number ϵ (depending on x) such that $B(x, \epsilon)$ is a cone neighborhood of x.*

Theorem 2-2 [11, Theorem I.7.50] *Suppose K is an M_k–polyhedral complex with* Shape(K) *finite. Then K is a complete geodesic metric space.*

Lemma 2-3 *If K is an M_k–polyhedral complex with* Shape(K) *finite, then there exist positive constants c_1 and c_2, which depend on* Shape(K), *such that every geodesic segment in K of length L is contained in a subcomplex which is a union of at most $c_1 L + c_2$ closed cells.*

This lemma follows from [11, Corollaries I.7.29 and I.7.30].

2.2 CAT(κ) spaces

Please see [11] for an introduction to CAT(κ) spaces.

Let X be a CAT(0) space and pick $x, y \in X$. We denote by $\overline{xy}$ the unique geodesic segment joining x and y. For any $y, z \in X \setminus \{x\}$, we denote the comparison angle between $\overline{xy}$ and $\overline{xz}$ at x by $\overline{\angle}_x(y,z)$ and the Alexandrov angle by $\angle_x(y,z)$.

The Alexandrov angle induces a distance on the space of germs of geodesics emanating from x. The completion of this metric space is called the *space of directions* at x and is denoted by $\Sigma_x X$. The tangent cone at x, denoted $T_x X$, is the Euclidean cone over $\Sigma_x X$. Following [9], we define the logarithmic map $\log_p\colon X \setminus \{x\} \to \Sigma_x X$ by sending $y \in X \setminus \{x\}$ to the point in $\Sigma_x X$ represented by $\overline{xy}$. Similarly, one can define $\log_x\colon X \to T_x X$. For a constant speed geodesic $l\colon [a,b] \to X$, we denote by $l^-(t)$ and $l^+(t)$ respectively the incoming and outgoing directions in $\Sigma_{l(t)} X$ for $t \in [a,b]$. Note that if X is a CAT(0) M_k–polyhedral complex with finitely many isometry types of cells, then $\Sigma_x X$ is naturally isometric to Lk(x, X), so we will identify these two objects.

Let us denote the Tits boundary of X by $\partial_T X$. We also have a well-defined map $\log_x\colon \partial_T X \to \Sigma_x X$. For $\xi_1, \xi_2 \in \partial_T X$, recall that the Tits angle $\angle_T(\xi_1, \xi_2)$ between them is defined as

$$\angle_T(\xi_1,\xi_2) = \sup_{x \in X} \angle_x(\xi_1,\xi_2).$$

This induces a metric on $\partial_T X$, which is called the *angular metric*. There are several different ways to define $\angle_T(\xi_1,\xi_2)$ (see [30, Section 2.3] or [11, Chapter II.9]):

Lemma 2-4 *Let X be a complete* CAT(0) *space and let ξ_1, ξ_2 be as above. Pick a base point $p \in X$, and let l_1 and l_2 be two unit speed geodesic rays emanating from p such that $l_i(\infty) = \xi_i$ for $i = 1, 2$. Then:*

(1) $\angle_T(\xi_1,\xi_2) = \lim_{t,t'\to\infty} \overline{\angle}_p(l_1(t), l_2(t'))$

(2) $\angle_T(\xi_1,\xi_2) = \lim_{t\to\infty} \angle_{l_1(t)}(\xi_1,\xi_2)$

(3) $2\sin(\angle_T(\xi_1,\xi_2)/2) = \lim_{t\to\infty} d(l_1(t), l_2(t))/t$

The space $(\partial_T X, \angle_T)$ is $\mathrm{CAT}(1)$; see [11, Chapter II.9]. We denote the Tits cone, which is the Euclidean cone over $\partial_T X$, by $C_T X$. Note that $C_T X$ is $\mathrm{CAT}(0)$. Denote the cone point of $C_T X$ by o. Then for each $p \in X$, there is a well-defined 1–Lipschitz logarithmic map $\log_p\colon C_T X \to X$ sending a geodesic ray $\overline{o\xi} \subset C_T X$ ($\xi \in \partial_T X$) to the geodesic ray $\overline{p\xi} \subset X$. This also gives rise to two other 1–Lipschitz logarithmic maps,

$$\log_p\colon C_T X \to T_p X, \quad \log_p\colon \partial_T X \to \Sigma_p X.$$

We always have $\angle_p(\xi_1,\xi_2) \le \angle_T(\xi_1,\xi_2)$, and the following flat sector lemma (see [30, Section 2.3] or [11, Chapter II.9]) describes when the equality holds.

Lemma 2-5 *Let X, ξ_1, ξ_2, l_1, l_2 and p be as above. If $\angle_T(\xi_1,\xi_2) = \angle_p(\xi_1,\xi_2) < \pi$, then the convex hull of l_1 and l_2 in X is isometric to a sector of angle $\angle_p(\xi_1,\xi_2)$ in the Euclidean plane.*

Any convex subset $C \subset X$ is also a $\mathrm{CAT}(0)$ space (with the induced metric) and there is an isometric embedding $i\colon \partial_T C \to \partial_T X$. There is a well-defined nearest point projection $\pi_C\colon X \to C$, which has the following properties.

Lemma 2-6 *Let X, C and π_C be as above. Then:*

(1) *π_C is 1–Lipschitz.*

(2) *For $x \notin C$ and $y \in C$ such that $y \neq \pi_C(x)$, we have $\angle_{\pi_C(x)}(x,y) \ge \frac{\pi}{2}$.*

See [11, Chapter II.2] for a proof of the above lemma.

Two convex subset C_1 and C_2 are *parallel* if $d(\cdot, C_1)|_{C_2}$ and $d(\cdot, C_2)|_{C_1}$ are constant. In this case, the convex hull of C_1 and C_2 is isometric to $C_1 \times [0, d(C_1, C_2)]$. Moreover, $\pi_{C_1}|_{C_2}$ and $\pi_{C_2}|_{C_1}$ are isometric inverse to each other; see [30, Section 2.3.3] or [11, Chapter II.2].

Let $Y \subset X$ be a closed convex subset. We define the *parallel set* of Y, denoted by P_Y, to be the union of all convex subsets which are parallel to Y. P_Y is not a convex set in general, but when Y has the geodesic extension property, P_Y is closed and convex.

Now we turn to $\mathrm{CAT}(1)$ spaces. In this paper, $\mathrm{CAT}(1)$ spaces are assumed to have diameter $\le \pi$ (we truncate the length metric on the space by π). We say a subset of a $\mathrm{CAT}(1)$ space is *convex* if it is π–convex.

For a CAT(1) space Y and $p \in Y$, $K \subset Y$, we define the *antipodal set of z in* K to be $\mathrm{Ant}(p, K) := \{v \in K \mid d(v, p) = \pi\}$.

Let Y and Z be two metric spaces. Their *spherical join*, denoted by $Y * Z$, is the quotient space of $Y \times Z \times \left[0, \frac{\pi}{2}\right]$ under the identifications $(y, z_1, 0) \sim (y, z_2, 0)$ and $\left(y_1, z, \frac{\pi}{2}\right) \sim \left(y_2, z, \frac{\pi}{2}\right)$. One can write the elements in $Y * Z$ as formal sums $(\cos\alpha)y + (\sin\alpha)z$, where $\alpha \in \left[0, \frac{\pi}{2}\right]$, $y \in Y$ and $z \in Z$. Let

$$w_1 = (\cos\alpha_1)y_1 + (\sin\alpha_1)z_1, \quad w_2 = (\cos\alpha_2)y_2 + (\sin\alpha_2)z_2.$$

Their distance in $Y * Z$ is defined to be

$$d_{Y*Z}(w_1, w_2) = \cos\alpha_1 \cos\alpha_2 \cos\bigl(d_Y^{\pi}(y_1, y_2)\bigr) + \sin\alpha_1 \sin\alpha_2 \sin\bigl(d_Z^{\pi}(z_1, z_2)\bigr),$$

where d_Y^{π} is the metric on Y truncated by π, similarly for d_Z^{π}.

When Y is only one point, $Y * Z$ is the spherical cone over Z. When Y consists of two points a distance π from each other, $Y * Z$ is the spherical suspension of Z. The spherical join of two CAT(1) spaces is still CAT(1).

Definition 2-7 (cell structure on the link) Let X be an M_κ–polyhedral complex and pick a point $x \in X$. Suppose σ_x is the unique closed cell which contains x as its interior point. Then $\mathrm{Lk}(x, X)$ is isometric to $\mathrm{Lk}(x, \sigma_x) * \mathrm{Lk}(\sigma_x, X) = \mathbb{S}^{k-1} * \mathrm{Lk}(\sigma_x, X)$, where k is the dimension of σ_x. Note that $\mathrm{Lk}(\sigma_x, X)$ has a natural M_1–polyhedral complex structure which is induced from the ambient space X.

When X is made of Euclidean rectangles, $\mathrm{Lk}(\sigma_x, X)$ is an all-right spherical complex. Moreover, there is a canonical way to triangulate $\mathrm{Lk}(x, \sigma_x)$ into an all-right spherical complex which is isomorphic to an octahedron as simplicial complexes. The vertices of $\mathrm{Lk}(x, \sigma_x)$ come from segments passing through x which are parallel to edges of σ_x. Thus $\mathrm{Lk}(\sigma_x, X)$ has a natural all-right spherical complex structure. In general, there is no canonical way to triangulate $\mathrm{Lk}(x, \sigma_x)$. However, there are still cases when we want to treat $\mathrm{Lk}(x, X)$ as a piecewise spherical complex. In such cases, one can pick an arbitrary all-right spherical complex structure on $\mathrm{Lk}(x, \sigma_x)$.

If X is CAT(0), then we can identify $\Sigma_x X$ with $\mathrm{Lk}(x, X)$. In this case, $\Sigma_x X$ is understood to be equipped with the above polyhedral complex structure.

Any two points of distance less than π from each other in a CAT(1) space are joined by a unique geodesic. A generalization of this fact would be the following.

Lemma 2-8 *Let Y be* CAT(1) *and let $\Delta \subset Y$ be an isometrically embedded spherical k–simplex with its vertices denoted by $\{v_i\}_{i=0}^k$. Pick $v \in \Delta$ and $v' \in Y$. If $d(v', v_i) \le d(v, v_i)$ for all i, then $v = v'$.*

By spherical simplices, we always means those which are not too large, ie those contained in an open hemisphere.

Proof We proceed by induction. When $k=1$, it follows from the uniqueness of geodesics. In general, since $\triangle = \triangle_1 * \triangle_2$, where $\triangle_1$ is spanned by vertices $\{v_i\}_{i=0}^{k-2}$ and $\triangle_2$ is spanned by v_{k-1} and v_k, there exists $w \in \triangle_2$ such that $v \in \triangle_1 * \{w\}$. Triangle comparison implies $d(v', w) \le d(v, w)$, so we can apply the induction assumption to the $(k-1)$–simplex $\triangle_1 * \{w\}$, which implies $v = v'$. □

Lemma 2-9 *Let Y be a* CAT(1) *piecewise spherical complex with finitely many isometry types of cells, and let $K \subset Y$ be a subcomplex which is a spherical suspension (in the induced metric). Pick a suspension point $v \in K$. Then all points in* Supp(v) *are suspension points of K and we have a splitting $K = \mathbb{S}^k * K'$, where $k = \dim(\mathrm{Supp}(v))$ and $\mathbb{S}^k$ is the standard sphere of dimension k.*

Proof By Theorem 2-1, v has a small neighborhood isometric to the ϵ–ball centered at the cone point in the spherical cone over $\Sigma_v K$. Since v is a suspension point, $K = \mathbb{S}^0 * \mathrm{Lk}(v, K) = \mathbb{S}^0 * \Sigma_v K$. However, $\Sigma_v K = \Sigma_v \,\mathrm{Supp}(v) * K' = \mathbb{S}^{k-1} * K'$ for some K', thus $K = \mathbb{S}^k * K'$. Also every point in Supp(v) belongs to the $\mathbb{S}^k$–factor, hence is a suspension point. □

2.3 CAT(0) cube complexes

All cube complexes in this paper are assumed to be finite-dimensional.

Every cube complex X (a polyhedral complex whose building blocks are cubes) has a canonical cubical metric: endow each n–cube with the standard metric of the unit cube in Euclidean n–space $\mathbb{E}^n$, then glue these cubes together to obtain a piecewise Euclidean metric on X. This metric is complete and geodesic if X is of finite dimension, and is CAT(0) if the link of each vertex is a flag complex [11; 19].

Now we come to the notion of hyperplane, which is the cubical analogue of "track" introduced in [16]. A *hyperplane* h in a cube complex X is a subset such that:

(1) h is connected.

(2) For each cube $C \subset X$, $h \cap C$ is either empty or a union of mid-cubes of C.

(3) h is minimal, ie if there exists $h' \subset h$ satisfying (1) and (2), then $h = h'$.

Recall that a *mid-cube* of $C = [0,1]^n$ is a subset of form $f_i^{-1}\left(\frac{1}{2}\right)$, where f_i is one of the coordinate functions.

For each edge $e \in X$, there exists a unique hyperplane which intersects e in one point. This is called the hyperplane *dual* to the edge e. Following [20], we say a hyperplane h *self-intersects* if there exists a cube C such that $C \cap h$ contains at least two different mid-cubes. A hyperplane h *self-osculates* if there exist two different edges e_1 and e_2 such that (1) $e_1 \cap e_2 \neq \varnothing$; (2) e_1 and e_2 are not consecutive edges in a 2–cube; (3) $e_i \cap h \neq \varnothing$ for $i = 1, 2$.

Let X be a CAT(0) cube complex, and let $e \subset X$ be an edge. Denote the hyperplane dual to e by h_e. Suppose $\pi_e \colon X \to e \cong [0,1]$ is the CAT(0) projection. It is known that:

1. h_e is embedded, ie the intersection of h_e with every cube in X is either a mid-cube, or an empty set.
2. h_e is a convex subset of X, and h_e with the induced cell structure from X is also a CAT(0) cube complex.
3. $h_e = \pi_e^{-1}\left(\frac{1}{2}\right)$.
4. $X \setminus h_e$ has exactly two connected components; they are called *halfspaces*.
5. If N_h is a union of closed cells in X which has nontrivial intersection with h_e, then N_h is a convex subcomplex of X and N_h is isometric to $h_e \times [0,1]$. We call N_h the *carrier* of h_e. Note that $N_h = P_e$, where P_e is the parallel set of e.

We refer to [36] for more information about hyperplanes.

Now we investigate the coarse intersection of convex subcomplexes. The following lemma adjusts [8, Lemma 2.3] to our cubical setting.

Lemma 2-10 *Let X be a* CAT(0) *cube complex of dimension n, and let C_1, C_2 be convex subcomplexes. Suppose $\triangle = d(C_1, C_2)$, $Y_1 = \{y \in C_1 \mid d(y, C_2) = \triangle\}$ and $Y_2 = \{y \in C_2 \mid d(y, C_1) = \triangle\}$. Then:*

1. *Y_1 and Y_2 are not empty.*
2. *Y_1 and Y_2 are convex; π_{C_1} maps Y_2 isometrically onto Y_1 and π_{C_2} maps Y_1 isometrically onto Y_2; the convex hull of $Y_1 \cup Y_2$ is isometric to $Y_1 \times [0, \triangle]$.*
3. *Y_1 and Y_2 are subcomplexes.*
4. *There exists $A = A(\Delta, n, \epsilon)$ such that if $p_1 \in C_1$, $p_2 \in C_2$ and $d(p_1, Y_1) \geq \epsilon > 0$, $d(p_2, Y_2) \geq \epsilon > 0$, then*

$$d(p_1, C_2) \geq \triangle + A d(p_1, Y_1), \quad d(p_2, C_1) \geq \triangle + A d(p_2, Y_2). \tag{2-11}$$

Proof For assertion (1), since X has finite dimension, X has only finitely many isometry types of cells; we use the "quasicompact" argument of Bridson [10]. Suppose we have sequences of points $\{x_n\}_{n=1}^{\infty}$ in C_1 and $\{y_n\}_{n=1}^{\infty}$ in C_2 such that

$$d(x_n, y_n) < \Delta + \tfrac{1}{n}. \tag{2-12}$$

Then by Lemma 2-3, there exists an integer N such that for every n, the geodesic joining x_n and y_n is contained in a subcomplex K_n which is a union of at most N closed cells. Write $C_{1n} = C_1 \cap K_n$ and $C_{2n} = C_2 \cap K_n$, which are also subcomplexes. Since there are only finitely many isomorphisms types among $\{K_n\}_{n=1}^{\infty}$, we can assume, up to a subsequence, that there exist a finite complex K_∞ and subcomplexes $C_{1\infty}$, $C_{2\infty}$ of K_∞ such that for any n, there is a simplicial isomorphism $\varphi_n\colon K_n \to K$ with $\varphi_n(C_{1n}) = C_{1\infty}$ and $\varphi_n(C_{2n}) = C_{2\infty}$. By (2-12), $d_{K_\infty}(C_{1\infty}, C_{2\infty}) \le \Delta$ in the intrinsic metric of K_∞, so there exist $x_\infty \in C_{1\infty}$ and $y_\infty \in C_{2\infty}$ such that $d_{K_\infty}(x_\infty, y_\infty) \le \Delta$ by compactness of K_∞. It follows that $d_X(\varphi_n^{-1}(x_\infty), \varphi_n^{-1}(y_\infty)) \le d_{K_n}(\varphi_n^{-1}(x_\infty), \varphi_n^{-1}(y_\infty)) \le \Delta$.

We prove (4) with $\epsilon = 1$; the other cases are similar. A similar argument as above implies that there is a constant $A > 0$, such that if $x \in C_1$ and $d(x, Y_1) = 1$, then $d(x, C_2) > A + \Delta$. Note that the combinatorial complexity depends on Δ and n, so A also depends on Δ and n. Now for any $p_1 \in C_1$ and $d(p_1, Y_1) \ge 1$, let $p_0 = \pi_{Y_1}(p_1)$ and let $l\colon [0, d(p_0, p_1)] \to X$ be the unit speed geodesic from p_0 to p_1. We have $l(1) \in \{x \in C_1 \mid d(x, Y_1) = 1\}$, so $d(l(1), C_2)) > A + \Delta$ while $d(l(0), C_2)) = \Delta$. Then (2-11) follows from the convexity of the function $d(\cdot, C_2)$.

The assertion (2) is a standard fact in [11, Chapter II.2].

To prove (3), it suffices to prove that for every $y_1 \in Y_1$, we have $\mathrm{Supp}(y_1) \in Y_1$. Denote $y_2 = \pi_{C_2}(y_1) \in Y_2$ (hence $y_1 = \pi_{C_1}(y_2)$ by (2)) and $l\colon [0, \Delta] \to X$ the unit speed geodesic from y_1 to y_2. Recall that we use $l^-(t)$ and $l^+(t)$ to denote the incoming and outgoing directions of l in $\Sigma_{l(t)} X$ for $t \in [0, \Delta]$. Our goal is to construct a "parallel transport" of $\mathrm{Supp}(y_1)$ (which is a k–cube) along l.

Since X has only finitely many isometry types of cells, l is contained in a finite union of closed cells, and we can find a sequence of closed cubes $\{B_i\}_{i=1}^{N}$ and $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \Delta$ such that each B_i contains $\{l(t) \mid t_{i-1} < t < t_i\}$ as interior points. We denote $\mathrm{Supp}(y_1)$ by $\square_{t_0}$ from now on.

Starting At $l(0) = y_1$, we have a splitting $\Sigma_{y_1} X = \Sigma_{y_1} \square_{t_0} * K_1$ for some convex subset $K_1 \subset \Sigma_{y_1} X$. Since $y_1 = \pi_{C_1}(y_2)$ and $\square_{t_0} \subset C_1$, by Lemma 2-6 we know $d_{\Sigma_{y_1} X}(l^+(t_0), \Sigma_{y_1} \square_{t_0}) \ge \frac{\pi}{2}$. Thus $l^+(t_0) \in K_1$ and $d_{\Sigma_{y_1} X}(v, l^+(t_0)) = \frac{\pi}{2}$ for any $v \in \Sigma_{y_1} \square_{t_0}$. It follows that the segment $B_1 \cap l$ is orthogonal to $\square_{t_0}$ in B_1. Since

$\square_{t_0}$ is a subcube of B_1, by geometry of cubes, there is an isometric embedding $e_1\colon \square_{t_0} \times [0, t_1] \to B_1$ with $e_1(y_1, t) = l(t)$. Denote $\square_{t_1} = e_1(\square_{t_0} \times \{t_1\})$; then $l(t_1) \in \square_{t_1} \subseteq \mathrm{Supp}(l(t_1)) \subseteq B_1 \cap B_2$. Note that $\square_{t_1}$ is not necessarily a subcomplex of B_1 (or B_2), but it is always parallel to some subcube of B_1 (or B_2).

Continuing By construction we know $d_{\Sigma_{l(t_1)}}(l^-(t_1), v) = \frac{\pi}{2}$ for $v \in \Sigma_{l(t_1)}\square_{t_1}$, so $d_{\Sigma_{l(t_1)}}(l^+(t_1), \Sigma_{l(t_1)}\square_{t_1}) \geq \frac{\pi}{2}$, since $d_{\Sigma_{l(t_1)}}(l^-(t_1), l^+(t_1)) = \pi$. However, there is a splitting $\Sigma_{l(t_1)}X = \Sigma_{l(t_1)}\square_{t_1} * K_2$ for some convex subset $K_2 \subset \Sigma_{l(t_1)}X$. Thus $l^+(t_1) \in K_2$ and $d_{\Sigma_{l(t_1)}X}(v, l^+(t_1)) = \frac{\pi}{2}$ for any $v \in \Sigma_{l(t_1)}\square_{t_1}$. It follows that inside B_2, the segment $B_2 \cap l$ is orthogonal to $\square_{t_1}$. Recall that $\square_{t_1}$ is parallel to a subcube of B_2, hence by geometry of cubes, we have an isometric embedding

$$e_2\colon \square_{t_1} \times [t_1, t_2] \to B_2$$

with $e_2(y, t) = l(t)$ for some $y \in \square_{t_1}$. Write $\square_{t_2} = e_2(\square_{t_1} \times \{t_2\})$; we know $\square_{t_2}$ is parallel to some subcube of B_3, so one can proceed to construct an isometric embedding e_3 as before. More generally, we can build $e_i\colon \square_{t_{i-1}} \times [t_{i-1}, t_i] \to B_i$ with $e_i(y, t) = l(t)$ for some $y \in \square_{t_{i-1}}$ and $\square_{t_i} = e_i(\square_{t_{i-1}} \times \{t_i\})$ inductively. Note that $l(t_i) \in \square_{t_i} \subseteq \mathrm{Supp}(l(t_i)) \subseteq B_i \cap B_{i+1}$ by construction.

Arriving Since $y_2 = l(t_N) \in B_N \cap C_2$, where B_N and C_2 are subcomplexes, we have $l(t_N) \in \square_{t_N} \subseteq \mathrm{Supp}(l(t_N)) \subseteq B_N \cap C_2$ by construction. Moreover, we can concatenate the embeddings $\{e_i\}_{i=1}^N$ constructed in the previous step to obtain a map $e\colon \square_{t_0} \times [0, \triangle] \to X$ such that

- $e(y, t) = l(t)$ for some $y \in \square_{t_0}$;
- $e(\square_{t_0} \times \{0\}) \subseteq C_1$;
- $e(\square_{t_0} \times \{\triangle\}) \subseteq C_2$;
- e is 1–Lipschitz (e is actually an isometric embedding, since e is a local isometric embedding by construction).

Therefore $d(y, C_2) \leq \triangle$ for any $y \in \square_{t_0}$ (recall that $\mathrm{Supp}(y_1) = \square_{t_0}$), which implies assertion (3). □

Remark 2-13 (1) By the same proof, items (1), (2) and (4) in the above lemma are true for piecewise Euclidean CAT(0) complexes with finitely many isometry types of cells. However, (3) might not be true in such generality.

(2) If C_1 and C_2 are orthant subcomplexes, then by items (2) and (3), Y_1 (or Y_2) is isometric to $O \times \prod_{i=1}^k I_i$, where O is an orthant and each I_i is a finite interval. In other words, there exists an orthant subcomplex $O \subset X$ such that $d_H(Y_1, O) < \infty$.

(3) Equation (2-11) implies that for any $R_1, R_2 > 0$, we have
$$N_{R_1}(C_1) \cap N_{R_2}(C_2) \subset N_{R'_1}(Y_1), \quad N_{R_1}(C_1) \cap N_{R_2}(C_2) \subset N_{R'_2}(Y_2),$$
where
$$R'_1 = \min\Big(1, \frac{R_1+R_2-\Delta}{A} + R_2\Big), \quad R'_2 = \min\Big(1, \frac{R_1+R_2-\Delta}{A} + R_1\Big),$$
with $A = A(\Delta, n, 1)$. Moreover, $\partial_T C_1 \cap \partial_T C_2 = \partial_T Y_1 = \partial_T Y_2$.

The last remark implies that Y_1 and Y_2 capture the information about how C_1 and C_2 intersect coarsely. We use the notation $\mathcal{I}(C_1, C_2) = (Y_1, Y_2)$ to describe this situation, where $\mathcal{I}$ stands for the word "intersect". The next lemma gives a combinatorial description of Y_1 and Y_2.

Lemma 2-14 *Let X, C_1, C_2, Y_1 and Y_2 be as in Lemma 2-10. Pick an edge e in Y_1 (or Y_2), and let h be the hyperplane dual to e. Then $h \cap C_i \neq \varnothing$ for $i = 1, 2$. Conversely, if a hyperplane h' satisfies $h' \cap C_i \neq \varnothing$ for $i = 1, 2$, then*
$$\mathcal{I}(h' \cap C_1, h' \cap C_2) = (h' \cap Y_1, h' \cap Y_2)$$
and h' comes from the dual hyperplane of some edge e' in Y_1 (or Y_2).

Proof The first part of the lemma follows from the proof of Lemma 2-10. Let $\mathcal{I}(h' \cap C_1, h' \cap C_2) = (Y'_1, Y'_2)$. Pick $x \in Y'_1$ and set $x' = \pi_{h' \cap C_2}(x) \in Y'_2$. Then $\pi_{h' \cap C_1}(x') = x$. We identify the carrier of h' with $h' \times [0, 1]$. Since C_i is a subcomplex, $(h' \cap C_i) \times \big(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\big) = C_i \cap \big(h' \times \big(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\big)\big)$ for $i = 1, 2$ and $\epsilon < \frac{1}{2}$. Thus for any $y \in C_2$, one has $\angle_{x'}(x, y) \geq \frac{\pi}{2}$, which implies that $x' = \pi_{C_2}(x)$. Similarly, $x = \pi_{C_1}(x') = \pi_{C_1} \circ \pi_{C_2}(x)$, hence $x \in Y_1$ and $Y'_1 \subset Y_1$. By the same argument, $Y'_2 \subset Y_2$, thus $Y'_i = Y_i \cap h'$ for $i = 1, 2$ and the lemma follows. □

Definition 2-15 We call an isometrically embedded orthant O *straight* if for any $x \in O$, the space $\Sigma_x O$ is a subcomplex of $\Sigma_x X$ (see Definition 2-7 for the cell structure on $\Sigma_x X$). In particular, if the orthant is 1–dimensional, we will call it a *straight geodesic ray*. Note that O itself may not be a subcomplex.

Remark 2-16 Any k–dimensional straight orthant $O \subset X$ is Hausdorff close to an orthant subcomplex of X.

To see this, let $k' = \max_{x \in O}\{\dim(\mathrm{Supp}(x))\}$; we proceed by induction on $k' - k$. The case $k' - k = 0$ is clear. Assume $k' - k = m \geq 1$ and pick $x_0 \in O$ such that $\dim(\mathrm{Supp}(x_0)) = k'$. Then there exists $B \subset \mathrm{Supp}(x_0)$ such that $B \cong [0, 1]^k$, B is parallel to a k–dimensional subcube of $\mathrm{Supp}(x_0)$ and $O \cap \mathrm{Supp}(x_0) \subset B$. Choose a line segment $e \cong [0,1]$ in $\mathrm{Supp}(x_0)$ such that $x_0 \in e$, e is orthogonal to B and e is parallel to some edge e' of $\mathrm{Supp}(x_0)$.

Suppose h is the hyperplane dual to e and suppose $N_h \cong e \times h$ is the carrier of h. For any other point $x \in O$, the segment $\overline{x_0 x}$ is orthogonal to e by our construction, thus there exists a point $y \in e$ such that $O \subset \{y\} \times h \subset N_h$. Now we can endow $\{y\} \times h$ with the induced cubical structure and use our induction hypothesis to find an orthant complex O_1 in the k–skeleton of $\{y\} \times h$ such that $d_H(O, O_1) < \infty$. Since $N_h \cong e \times h$, we can slide O_1 along e in N_h to get an orthant subcomplex in the k–skeleton of X.

Lemma 2-17 *Let X be a* CAT(0) *cube complex. If l_1 and l_2 are two straight geodesic rays in X, then either $\angle_T(l_1, l_2) = 0$, or $\angle_T(l_1, l_2) \geq \frac{\pi}{2}$.*

Proof We can assume without loss of generality that l_1 and l_2 are in the 1–skeleton and $l_1(0)$ is a vertex of X. We parametrize these two geodesic rays by unit speed. Let $\{b_m\}_{m=1}^{\infty}$ be the collection of hyperplanes in X such that $b_m \cap l_1 = l_1\left(\frac{1}{2} + m\right)$, and let h_m be the halfspace bounded by b_m which contains l_1 up to a finite segment. Suppose N_m is the carrier of b_m.

Suppose $l_2 \cap b_m \neq \varnothing$ for infinitely many m. Since each b_m separates X, there exists an m_0 such that $l_2 \cap b_m \neq \varnothing$ for all $m \geq m_0$. Recall that l_2 is in the 1–skeleton, so for each $m \geq m_0$, there exists an edge e_m such that $e_m \subset l_2$, $e_m \subset N_m$ and $e_m \cap b_m$ is a point. Consider the function $f(t) = d(l_2(t), l_1)$ for $t \geq 0$. Then f is convex and there exist infinitely many intervals of unit length (they come from e_m for $m \geq m_0$) such that f restricted to each interval is constant, so there exists a t_0 such that $f|_{[t_0,\infty)}$ is constant, which implies $\angle_T(l_1, l_2) = 0$.

If $l_2 \cap b_m \neq \varnothing$ for finitely many m, then there exists an m_0 such that $h_{m_0} \cap l_2 = \varnothing$, which implies the CAT(0) projection of l_2 to l_1 is a finite segment. If $\angle_T(l_1, l_2) < \frac{\pi}{2}$, then $\pi_{l_1}(l_2)$ is an infinite segment by Lemma 2-4, which is a contradiction, so $\angle_T(l_1, l_2) \geq \frac{\pi}{2}$. □

We will see later on that the subset of $\partial_T X$ which is responsible for the behavior of top-dimensional quasiflats is spanned by those points represented by straight geodesic rays. The following lemma makes the word "span" precise.

Lemma 2-18 *Let X be a* CAT(0) *cube complex, and let $\{l_i\}_{i=1}^{k}$ be a collection of straight geodesic rays in X, emanating from the same base point p, such that $\angle_T(l_i, l_j) = \angle_p(l_i, l_j) = \frac{\pi}{2}$ for $i \neq j$. Then the convex hull of $\{l_i\}_{i=1}^{k}$ is a k–dimensional straight orthant.*

One may compare this lemma with [3, Propositions 2.10 and 2.11].

Proof By Lemma 2-5, l_i and l_j together bound an isometrically embedded quarter plane for $i \neq j$. We prove the lemma by induction and assume the claim is true for $\{l_i\}_{i=1}^{k-1}$. We parametrize l_k by arc length and denote by O_0 the straight orthant spanned by $\{l_i\}_{i=1}^{k-1}$. Note that $O_0 \cap l_k = p$.

For $s > 0$ and $1 \le i \le k-1$, let c_i be the geodesic ray such that (1) c_i is in the quarter plane spanned by l_k and l_i; (2) c_i starts at $l_k(s)$; (3) c_i is parallel to l_i. Thus $\angle_T(c_i, c_j) = \frac{\pi}{2}$ and $\angle_{l_k(s)}(c_i, c_j) \le \frac{\pi}{2}$ for $i \neq j$. Note that $\{c_i\}_{i=1}^{k-1}$ are also straight geodesic rays, and $\{\log_{l_k(s)} c_i\}_{i=1}^{k-1}$ are distinct points in the 0–skeleton of $\Sigma_{l_k(s)} X$. It follows that actually $\angle_{l_k(s)}(c_i, c_j) = \frac{\pi}{2}$ for $i \neq j$. Hence by the induction assumption, there is a straight orthant O_s spanned by $\{c_i\}_{i=1}^{k-1}$.

By Lemma 2-8, $\partial_T O_0 = \partial_T O_s$. Let $l \subset O_s$ be a unit-speed geodesic ray emanating from $l_k(s)$. Then $d(l(t), O_0)$ is a bounded convex function. Since $\Sigma_{l_k(s)} O_s$ is an all-right spherical simplex in $\Sigma_{l_k(s)} X$ spanned by $\{\log_{l_k(s)} c_i\}_{i=1}^{k-1}$, we have $\angle_{l_k(s)}(l(t), l_k(0)) = \frac{\pi}{2}$ for any $t > 0$. Similarly, we have $\angle_{l_k(0)}(y, l_k(s)) = \frac{\pi}{2}$ for any $y \in O_0 \setminus \{l_s(0)\}$. Hence by triangle comparison, $d(l(t), O_0)$ attains its minimum at $t = 0$. Thus $d(l(t), O_0)$ has to be a constant function. Thus $d(x, O_0) \equiv s$ for any $x \in O_s$, and similarly $d(x, O_s) \equiv s$ for any $x \in O_0$, which implies the convex hull of O_0 and O_s is isometric to $O_0 \times [0, s]$; see eg [11, Chapter II.2]. Moreover, the convex hull of O_0 and O_s is contained in the convex hull of O_0 and $O_{s'}$ for $s \le s'$. So the convex hull of $\{l_i\}_{i=1}^{k}$ is a straight orthant O. □

3 Proper homology classes of bounded growth

In this section we summarize some results from [9] about locally finite homology classes of certain polynomial growth and make some generalizations to adjust the results to our situation.

3.1 Proper homology and supports of homology classes

For an arbitrary metric space Z, we define the proper (singular) homology of Z with coefficients in an abelian group G, denoted $H_*^{\mathrm{p}}(Z; G)$, as follows. Elements in the proper n–chain group $C_n^{\mathrm{p}}(Z; G)$ are of the form $\Sigma_{\lambda \in \Lambda} g_\lambda \sigma_\lambda$ (here Λ may be infinite, $g_\lambda \in G$ and the σ_λ are singular n–simplices) such that for every bounded set $K \subset Z$, the set $\{\lambda \in \Lambda \mid g_\lambda \neq \mathrm{Id} \text{ and } \sigma_\lambda(\Delta^n) \cap K \neq \varnothing\}$ is finite. The usual boundary map gives rise to a group homomorphism $\partial\colon C_n^{\mathrm{p}}(Z; G) \to C_{n-1}^{\mathrm{p}}(Z; G)$, yielding a chain complex $C_*^{\mathrm{p}}(Z; G)$, and $H_*^{\mathrm{p}}(Z; G)$ is defined to be the homology of this chain complex.

We will use Greek letters $\alpha, \beta, \dots$ to denote (proper) singular chains. We denote the union of images of singular simplices in a (proper) singular chain α by $\operatorname{Im}\alpha$. If α is a (proper) cycle, we denote the corresponding (proper) homology class by $[\alpha]$.

We also define the relative version of proper homology $H^{\mathrm{p}}_*(Z,Y)$ for $Y \subset Z$ in a similar way (Y is endowed with the induced metric). Then there is a long exact sequence

$$\cdots \to H^{\mathrm{p}}_n(Y) \to H^{\mathrm{p}}_n(Z) \to H^{\mathrm{p}}_n(Z,Y) \to H^{\mathrm{p}}_{n-1}(Y) \to H^{\mathrm{p}}_{n-1}(Z) \to \cdots.$$

Moreover, by the usual procedure of subdividing the chains, we know excision holds. Namely, for a subspace W such that the closure of W is in the interior of Y, the map $H^{\mathrm{p}}_*(Z-W, Y-W) \to H^{\mathrm{p}}_*(Z,Y)$ induced by inclusion is an isomorphism. As a corollary, if $B \subset Z$ is bounded, then there is a natural isomorphism $H^{\mathrm{p}}_*(Z, Z-A) \cong H_*(Z, Z-A)$, since we can replace the pair $(Z, Z-A)$ by $(O, O-B)$ by excision, where O is a bounded open neighborhood of B. Pick a point $z \in Z \setminus Y$; then there is a homomorphism $i\colon H^{\mathrm{p}}_k(Z,Y) \to H^{\mathrm{p}}_k(Z, Z\setminus\{z\}) \cong H_k(Z, Z\setminus\{z\})$ induced by the inclusion of pairs $(Z,Y) \to (Z, Z-\{z\})$. The map i is called the *inclusion homomorphism*.

If Z is also a simplicial complex or polyhedral complex, we can similarly define the proper simplicial (or cellular) homology by considering the former sum of simplices or cells such that for every bounded subset $K \subset Z$, we have only finitely many terms which intersect K nontrivially. The simplicial version (or the cellular version) of the homology theory is isomorphic to the singular version in a simplicial complex (or polyhedral complex) by the usual proof in algebraic topology.

The proper homology depends on the metric of the space, so it is not a topological invariant. By definition, every proper chain is locally finite and we have a group homomorphism $H^{\mathrm{p}}_*(Z,Y) \to H^{\mathrm{lf}}_*(Z,Y)$, where $H^{\mathrm{lf}}_*(Z,Y)$ is the locally finite homology defined in [9]. If Z is a proper metric space, then these two homology theories are the same.

A continuous map $f\colon Z_1 \to Z_2$ is *(metrically) proper* if the inverse image of every bounded subset is bounded. In this case, we have an induced map on proper homology $f_*\colon H^{\mathrm{p}}_k(Z_1, G) \to H^{\mathrm{p}}_k(Z_2, G)$.

In the rest of this paper, we will always take $G = \mathbb{Z}/2\mathbb{Z}$ and omit G when we write the homology.

Definition 3-1 For $z \in Z \setminus Y$, let $i\colon H^{\mathrm{p}}_k(Z,Y) \to H_k(Z, Z\setminus\{z\})$ be the inclusion homomorphism defined as above. For $[\sigma] \in H^{\mathrm{p}}_k(Z,Y)$, we define the *support set* of $[\sigma]$, denoted $S_{[\sigma],Z,Y}$, to be $\{z \in Z \setminus Y \mid i_*[\sigma] \neq \mathrm{Id}\}$. We will write $S_{[\sigma],Z}$ if Y is empty, and use $S_{[\sigma]}$ to denote the support set if the underlying spaces Z and Y are clear.

It follows that $S_{[\sigma]} = \big(\bigcap_{[\sigma']=[\sigma]\in H_k^{\mathrm{p}}(Z,Y)} \operatorname{Im}\sigma'\big) \setminus Y$.

If $Z \subset Z_1$, then $S_{[\sigma],Z,Y} \supseteq S_{[\sigma],Z_1,Y}$. These two sets are equal if Z is open in Z_1. If Z is a polyhedral complex and $Y = \varnothing$, then the support set is always a subcomplex. In particular, if $[\sigma] \in H_n^{\mathrm{p}}(Z)$ is a nontrivial top-dimensional class, then $[\sigma]$ can be represented by a top-dimensional polyhedral cycle, which implies the support set $S_{[\sigma]} \neq \varnothing$. But the support of a nontrivial class can be empty if it is not top-dimensional.

The support sets of (proper) homology classes behave like the support sets of currents in the following situation.

Lemma 3-2 *Let Z_1 be a metric space of homological dimension $\leq n$, and let $Y_1 \subset Z_1$ be a subspace. Pick $[\sigma] \in H_n^{\mathrm{p}}(Z_1, Y_1)$. If $f\colon (Z_1, Y_1) \to (Z_2, Y_2)$ is a proper map, then $S_{f_*[\sigma]} \subset f(S_{[\sigma]})$.*

Recall that Z_1 has $(\mathbb{Z}/2\mathbb{Z})$–homological dimension $\leq n$ if $H_r(U, V) = 0$ for any $r > n$ and U, V open in Z_1.

Proof Pick $y \in S_{f_*[\sigma]}$. Since $f^{-1}(y)$ is bounded, we have the following commutative diagram:

$$\begin{array}{ccc} H_n^{\mathrm{p}}(Z_1, Y_1) & \xrightarrow{f_*} & H_n^{\mathrm{p}}(Z_2, Y_2) \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} \\ H_n(Z_1, Z_1 \setminus f^{-1}(y)) & \xrightarrow{f_*} & H_n(Z_2, Z_2 \setminus \{y\}) \end{array}$$

Thus if $S_{f_*[\sigma]} \neq \varnothing$, then $[\sigma'] = i_*[\sigma] \in H_n(Z_1, Z_1 \setminus f^{-1}(y))$ is nontrivial. It suffices to show there exists $x \in f^{-1}(y)$ such that $[\sigma']$ is nontrivial when viewed as an element in $H_n(Z_1, Z_1 \setminus \{x\})$.

Fix a singular chain $\sigma' \in C_n(Z_1, Z_1 \setminus f^{-1}(y))$ which represents $[\sigma']$. We argue by contradiction and assume that $[\sigma']$ is trivial in $H_n(Z_1, Z_1 \setminus \{x\})$ for any $x \in f^{-1}(y)$. Let $K = f^{-1}(y) \cap \operatorname{Im}\sigma'$. For each $x \in K$, there exists an $\epsilon(x) > 0$ such that $\bar{B}(x, 2\epsilon(x)) \cap \operatorname{Im}\partial\sigma' = \varnothing$ and $[\sigma']$ is trivial in $H_n\big(Z_1, Z_1 \setminus \bar{B}(x, 2\epsilon(x))\big)$. Since $f^{-1}(y)$ is bounded and closed, K is compact and we can find finitely many points $\{x_i\}_{i=1}^N$ in K such that $K \subset \bigcup_{i=1}^N B(x_i, \epsilon(x_i))$. Let $U = \bigcup_{i=1}^N B(x_i, \epsilon(x_i))$ and let $U' = (Z_1 \setminus f^{-1}(y)) \cup U$. Then $\operatorname{Im}\sigma' \subset U'$ and $[\sigma']$ is trivial in $H_n(Z_1, U')$. We put $U'' = U' \setminus \big(\bigcup_{i=1}^N \bar{B}(x_i, 2\epsilon(x_i))\big)$. Then $\operatorname{Im}\partial\sigma' \subset U''$ and $U'' \subset Z_1 \setminus f^{-1}(y)$. So if we can show that $[\sigma']$ is trivial in $H_n(Z_1, U'')$, then $[\sigma']$ must be trivial in $H_n(Z_1, Z_1 \setminus f^{-1}(y))$, which yields a contradiction.

Let us assume $N = 1$. Then there is a Mayer–Vietoris sequence

$$H_{n+1}\big(Z_1, U' \cup (Z_1 \setminus \bar{B}(x_1, 2\epsilon(x_1)))\big) \to H_n(Z_1, U'') \\ \to H_n(Z_1, U') \oplus H_n\big(Z_1, Z_1 \setminus \bar{B}(x_1, 2\epsilon(x_1))\big).$$

The first term is trivial since Z_1 has homological dimension $\le n$ and $[\sigma']$ is trivial in the last term by construction, so $[\sigma']$ has to be trivial in the second term. Using an induction argument, we can obtain the contradiction similarly for $N \ge 2$. □

Remark 3-3 (1) The assumption on Z_1 is satisfied if Z_1 is a CAT(κ) space of geometric dimension $\le n$, see [28, Theorem A].

(2) The assumption on Z_1 is satisfied if Z_1 is a locally finite n–dimensional polyhedral complex (with the topology of a cell complex) or an M_k–polyhedral complex of finite shape, since such a space supports a CAT(1) metric which induces the same topology as its original metric.

3.2 The growth condition

In Sections 3.2–3.3, Y will be a piecewise Euclidean CAT(0) complex of dimension n. The following result shows every top-dimensional quasiflat in Y is Hausdorff close to the support set of some proper homology class. Therefore to understand quasiflats, we can focus on the support sets, which have nice local and asymptotic properties.

Lemma 3-4 [9, Lemma 4.3] *If $Q \subset Y$ is an (L, A)–quasiflat of dimension n, then there exists $[\sigma] \in H_n^{\mathrm{p}}(Y)$ satisfying the following conditions:*

(1) *There exists a constant $D = D(L, A)$ such that $d_H(S, Q) \le D$, where S is the support set of $[\sigma]$.*

(2) *There exists $a = a(L, A)$ such that for every $p \in Y$,*

$$\mathcal{H}^n(B(p, r) \cap S) \le a(1 + r)^n. \tag{3-5}$$

Here $\mathcal{H}^n$ denotes the n–dimensional Hausdorff measure and d_H denotes the Hausdorff distance.

Since Y is uniformly contractible, we can approximate the (L, A)–quasi-isometric embedding $q\colon \mathbb{R}^n \to Y$ by a Lipschitz (L, A)–quasi-isometric embedding $\tilde{q}$, which is proper. Then $[\sigma]$ is chosen to be the pushforward of the fundamental class of $\mathbb{R}^n$ under $\tilde{q}$.

The support set of a top-dimensional homology class enjoys the following geodesic extension property.

Lemma 3-6 [9, Lemma 3.1] *Let S be the support set of some top-dimensional proper homology class in Y. Pick arbitrary $p \in Y$ and $y \in S$. Then there is a geodesic ray $\overline{y\xi} \subset S$, which fits together with the geodesic segment $\overline{py}$ to form a geodesic ray $\overline{p\xi}$.*

Note that this lemma does not imply S is convex; see [9, Remark 3.2]. However, we still can define the Tits boundary of S.

Definition 3-7 Let Z be a CAT(0) space and let $A \subset Z$ be any subset. We define the *Tits boundary* of A, denoted $\partial_T A$, to be the set of points $\xi \in \partial_T Z$ such that there exists a geodesic ray $\overline{x\xi}$ such that $\overline{x\xi} \subset A$. The Tits boundary $\partial_T A$ is endowed with the usual Tits metric. We define the *Tits cone* of A, denoted $C_T A$, to be the Euclidean cone over $\partial_T A$.

Let S be as in Lemma 3-6. Then $\partial_T S$ is nonempty if S is nonempty.

There is a similar version of the geodesic extension property for the link $\Sigma_y S \subset \Sigma_y Y$, where $y \in S$.

Lemma 3-8 *Let S be as in Lemma 3-6. Then for any point $y \in Y$, $\Sigma_y S$ is the support set of some top-dimensional homology class in $\Sigma_y Y$.*

Proof By subdividing Y in an appropriate way, we may assume y is a vertex of Y. Suppose $S = S_{[\sigma]}$. We can represent $[\sigma]$ as a cellular cycle $\sigma = \Sigma_{\lambda \in \Lambda} \eta_\lambda$, where the η_λ are closed top-dimensional cells in Y (recall that we are using $\mathbb{Z}/2\mathbb{Z}$ coefficients, so all the coefficients are either 0 or 1). Then $S = \bigcup_{\lambda \in \Lambda} \eta_\lambda$. Let $\Lambda_y = \{\lambda \in \Lambda \mid y \in \eta_\lambda\}$. Since η is a cycle, $\eta_y = \Sigma_{\lambda \in \Lambda_y} \mathrm{Lk}(y, \eta_\lambda)$ is a top-dimensional cycle in the link $\mathrm{Lk}(y, Y) \cong \Sigma_y Y$. Moreover, $S_{[\eta_y]} = \bigcup_{\lambda \in \Lambda_y} \mathrm{Lk}(y, \eta_\lambda) = \mathrm{Lk}(y, S)$. □

Lemma 3-9 *Let K be a k–dimensional* CAT(1) *piecewise spherical complex, and let $K' \subset K$ be the support set of a top-dimensional homology class. Pick arbitrary $w \in K$, $v \in K'$, and suppose $\overline{wv}$ is a local geodesic joining v and w. Then there is a (nontrivial) local geodesic segment $\overline{vv'} \subset K'$ which fits together with $\overline{wv}$ to form a local geodesic segment $\overline{wv'}$. Moreover,* length$(\overline{vv'})$ *can be as large as we want.*

Now we turn to the global properties of S. Since we are in a CAT(0) space, for any $p \in Y$ and $0 < r \le R$, we have a map $\Phi_{r,R}\colon B(p, R) \to B(p, r)$ obtained by contracting points toward p by a factor of r/R. This contracting map together with Lemma 3-6 implies $B(p, r) \cap S \subset \Phi_{r,R}(B(p, R) \cap S)$ [9, Corollary 3.3, item 1].

Since $\Phi_{r,R}$ is (r/R)–Lipschitz, we have the following result.

Theorem 3-10 [9, Corollary 3.3] *Let S be the support set of some top-dimensional proper homology class in Y, and let $n = \dim(Y)$. Then:*

(1) **Monotonicity of density** *For all $0 \le r \le R$,*

$$\frac{\mathcal{H}^n(B(p,r)\cap S)}{r^n} \le \frac{\mathcal{H}^n(B(p,R)\cap S)}{R^n}. \tag{3-11}$$

(2) **Lower density bound** *For all $p \in S$ and $r > 0$,*

$$\mathcal{H}^n(B(p,r)\cap S) \ge \omega_n r^n, \tag{3-12}$$

with equality only if $B(p,r)\cap S$ is isometric to an r–ball in $\mathbb{E}^n$, where ω_n is the volume of an n–dimensional Euclidean ball of radius 1.

From (3-11) we know the quantity

$$\frac{\mathcal{H}^n(B(p,r)\cap S)}{r^n} \tag{3-13}$$

is monotone increasing with respect to r, and (3-5) tells us that if S comes from a top-dimensional quasiflat, then (3-13) is bounded above by some constant. Thus the limit exists and is finite as $r \to \infty$. More generally, we will consider those top-dimensional proper homology classes whose support sets S satisfy

$$\lim_{r\to+\infty} \frac{\mathcal{H}^n(B(p,r)\cap S)}{r^n} < \infty, \tag{3-14}$$

where $n = \dim(Y)$. We call them *proper homology classes of Cr^n growth*. These classes form a subgroup of $H^{\mathrm{p}}_n(Y)$, which will be denoted by $H^{\mathrm{p}}_{n,n}(Y)$.

The following lemma can be proved by a packing argument.

Lemma 3-15 [9, Lemma 3.12] *Pick $[\sigma] \in H^{\mathrm{p}}_{n,n}(Y)$ and let $S = S_{[\sigma]}$. Then given a base point $p \in Y$, for all $\epsilon > 0$ there is an N such that for all $r \ge 0$, the set $B(p,r)\cap S$ does not contain an ϵr–separated subset of cardinality greater than N.*

Lemma 3-16 *Let S and p be as in Lemma 3-15. Denote the cone point in $C_T S$ by o. Then*

$$\lim_{r\to+\infty} d_{\mathrm{GH}}\big(\tfrac{1}{r}(B(p,r)\cap S), B(o,1)\big) = 0. \tag{3-17}$$

Here d_{GH} denotes the Gromov–Hausdorff distance, $B(o,1)$ is the ball of radius 1 in $C_T S$ centered at o and $\frac{1}{r}(S\cap B(p,r))$ means we rescale the metric on $S\cap B(x,r)$ by a factor $\frac{1}{r}$.

Proof We follow the argument in [9]. It suffices to prove that for any $\epsilon > 0$, there exists $R > 0$ such that for any $r > R$, we can find an ϵ–isometry between $\frac{1}{r}(B(p,r) \cap S)$ and $B(o,1)$.

For $r > 0$, we denote the maximal cardinality of an ϵr–separated net in $B(p,r) \cap S$ by m_r. By Lemma 3-15, there exists N_0 such that $m_r \le N_0$ for all r. Pick R_1 such that $m_r \le m_{R_1}$ for all $r \neq R_1$ and denote the corresponding ϵR_1–net in $B(p, R_1) \cap S$ by $\{x_i\}_{i=1}^N$. By Lemma 3-6, for each i, we can extend the geodesic $\overline{px_i}$ to obtain a geodesic ray $\overline{p\xi_i}$ such that $\overline{x_i\xi_i} \subset S$. Let l_i: $[0,\infty) \to Y$ be a constant-speed geodesic ray joining p and ξ_i such that $l_i(R_1) = x_i$ and $l_i(0) = p$.

Since the quantity $d(l_i(t), l_j(t))/t$ is monotone increasing, $\{l_i(t)\}_{i=1}^N$ is a maximal ϵt–separated net in $B(p, R_1) \cap S$ for $t \ge R_1$. We pick $R > R_1$ such that for all $t > R$ and $1 \le i, j \le N$, we have

$$\left| \frac{d(l_i(t), l_j(t))}{t} - \lim_{t \to +\infty} \frac{d(l_i(t), l_j(t))}{t} \right| < \epsilon. \tag{3-18}$$

Now we fix $t > R$ and define a map such that for each i, $l_i(t) \in B(p,t) \cap S$ is mapped to the point $y_i \in B(o,1) \subset C_T S$ satisfying $y_i \in \overline{o\xi_i}$ and $d(y_i, o) = d(l_i(t), p)/t$. It follows from (3-18) that

$$\left| \frac{d(l_i(t), l_j(t))}{t} - d(y_i, y_j) \right| < \epsilon. \tag{3-19}$$

We claim $\{y_i\}_{i=1}^N$ is an ϵ–net in $B(o,1)$.

Pick an arbitrary $y \in B(o,1)$ and suppose $y \in \overline{o\xi}$ for $\xi \in \partial_T S$. We parametrize the geodesic ray $\overline{p\xi}$ by constant speed $= d(y,o)$ and denote this ray by l. Since there exists a geodesic $\overline{p'\xi} \subset S$ such that $d_H(\overline{p\xi}, \overline{p'\xi}) = C < \infty$, we can find $x \in \overline{p'\xi} \subset S$ with $d(x, l(t)) < C$ for every t. Thus $x \in B(p, td(y,o) + C) \cap S \subset B(p, t+C) \cap S$, which implies there exists some i such that $d(l_i(t+C), x) \le \epsilon(t+C)$. These estimates together with $d(l_i(t+C), l_i(t)) \le C$ (the ray l_i has speed ≤ 1) imply

$$d(l(t), l_i(t)) \le \epsilon t + (2+\epsilon)C. \tag{3-20}$$

Here i might depend on t, but we can choose a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to +\infty$ and

$$d(l(t_k), l_{i_0}(t_k)) \le \epsilon t + (2+\epsilon)C \tag{3-21}$$

for every k with i_0 fixed, thus

$$d(y, y_{i_0}) = \lim_{k \to +\infty} \frac{d(l(t_k), l_{i_0}(t_k))}{t_k} \le \epsilon. \tag{3-22}$$

So $\{y_i\}_{i=1}^N$ is an ϵ–net in $B(o,1)$; this fact together with (3-19) give us the ϵ–isometry as required. □

Remark 3-23 (1) Define $\partial_{p,r}S=\{\xi\in\partial_T S \mid \overline{p\xi}\subset B(p,r)\cup S\}$. Then the above proof shows

$$\lim_{r\to+\infty} d_H(\partial_{p,r}S,\partial_T S)=0. \tag{3-24}$$

(2) $\partial_T S$ has similar behavior to the Tits boundary of a convex subset in the following aspect. Let $l\colon[0,\infty)\to Y$ be a constant-speed geodesic ray. If there exist a constant $C<\infty$ and a sequence $t_i\to+\infty$ such that $d(l(t_i),S)<C$, then $\partial_T l$ is an accumulation point of $\partial_T S$. The proof is similar to the above argument.

3.3 ϵ–splittings

As we have seen from Lemma 3-16, the growth bound (3-14) implies that S looks more and more like a cone if one observes S from a farther and farther away point (this is called asymptotic conicality in [9]). So one would expect some regularity of S near infinity. The following key lemma from [9] will be our starting point.

Lemma 3-25 [9, Lemma 3.13] *Let S and p be as in Lemma 3-15. Then for all $\beta>0$ there is an $r<\infty$ such that if $x\in S\setminus B(p,r)$, then the antipodal set satisfies*

$$\operatorname{diam}(\operatorname{Ant}(\log_x p,\Sigma_x S))<\beta. \tag{3-26}$$

The proof of this lemma in [9] actually shows something more. Given a base point $p\in Y$ and $x\in S$, we define the *antipode at* ∞ of $\log_x p$ in S, denoted $\operatorname{Ant}_\infty(\log_x p,S)$, to be $\{\xi\in\partial_T S\mid \overline{x\xi}\subset S \text{ and } x\in\overline{p\xi}\}$. Then we have:

Lemma 3-27 *Let S and p be as in Lemma 3-15. Then for all $\beta>0$ there is an $r<\infty$ such that if $x\in S\setminus B(p,r)$, then*

$$\operatorname{diam}(\operatorname{Ant}_\infty(\log_x p,S))<\beta. \tag{3-28}$$

The diameter here is with respect to the angular metric on $\partial_T X$.

Lemma 3-25 tells us that $\Sigma_y S$ looks more and more like a suspension as $d(y,p)\to\infty$ (for $y\in S$). If we also assume $\operatorname{Shape}(Y)$ is finite, then for all $y\in S$, $\Sigma_y S$ is built from cells of finitely many isometry types. Moreover, by Theorem 3-10, there is a positive constant N such that $\Sigma_y S$ has at most N cells for any $y\in S$. Thus the family $\{\Sigma_y S\}_{y\in S}$ has only finite possible combinatorics. As $\beta\to 0$, one may expect $\Sigma_y S$ to actually be a suspension (this is called isolated suspension in [9]).

Now we restrict ourselves to the case of finite-dimensional CAT(0) cube complexes of finite dimension. Then the spaces of directions are finite-dimensional all-right spherical CAT(1) complexes (see Definition 2-7 for the definition of cell structure on the spaces of directions).

Lemma 3-29 *Suppose $\mathcal{F}$ is a family of all-right spherical* CAT(1) *complexes with dimension at most d. Then for every $\alpha > 0$ and $N > 0$, there is a constant $\beta = \beta(d, N, \alpha) > 0$ such that if K' satisfies the following conditions:*

1. *$K' \subset K$ is a subcomplex of some $K \in \mathcal{F}$,*
2. *the number of cells in K' is bounded above by N,*
3. *K' has the geodesic extension property in the sense of Lemma 3-9,*
4. *there exists $v \in K$ such that* $\mathrm{diam}(\mathrm{Ant}(v, K')) < \beta$,

then K' is a metric suspension (in the metric induced from K) and v lies at a distance $< \alpha$ from a suspension point of K'.

Proof We prove the lemma by contradiction. Suppose there exist $\alpha > 0$, $N > 0$ and a sequence $\{K'_i\}_{i=1}^{\infty}$ such that for each i, K'_i satisfies conditions (2) and (3), K'_i is a subcomplex of some $K_i \in \mathcal{F}$, and there exists a $v_i \in K_i$ such that

$$\mathrm{diam}(\mathrm{Ant}(v_i, K'_i)) < \tfrac{1}{i}, \tag{3-30}$$

but no point in the α–neighborhood of v_i is suspension point of K'_i.

Let w_i be the point in K'_i which realizes the minimal distance to v_i in the length metric of K_i (note that the original metric on K_i is the length metric truncated by π). If l_i is the geodesic segment (in the length metric) joining v_i and w_i, then by (3-30) and the geodesic extension property of K'_i, there exists a C such that $\mathrm{length}(l_i) < C$ for all i. So for any i, there exists a subcomplex $L_i \subset K_i$ such that $l_i \subset L_i$ and the number of cells in L_i are uniformly bounded by constant N_1 (by Lemma 2-3).

Let M_i be the full subcomplex spanned by $K'_i \cup L_i$, ie M_i is the union of simplices in K_i whose vertex sets are in $K'_i \cup L_i$. Then M_i is a π–convex, hence CAT(1), subcomplex of K_i, and the number of cells in M_i is uniformly bounded above by some constant N_2. Without loss of generality, we can replace K_i by M_i. Since M_i has only finitely many possible isometry types, after passing to a subsequence, we can assume there exist a finite CAT(1) complex M and a subcomplex $K' \subset M$ such that for every i, there is a simplicial isomorphism $\phi_i\colon M_i \to M$ mapping K'_i onto K' (here ϕ_i is also an isometry).

Since M is compact, there is a subsequence of $\{\phi_i(v_i)\}_{i=1}^{\infty}$ converging to a point $v \in M$. We claim $\mathrm{Ant}(v, K')$ is exactly one point. First $\mathrm{Ant}(v, K') \neq \varnothing$ by the geodesic extension property of K'. If there were two distinct points $u, u' \in \mathrm{Ant}(v, K')$, then we could extend the geodesic segment $\overline{v_i u}$, $\overline{v_i u'}$ into K', yielding a contradiction with (3-30) for large i.

Suppose $\mathrm{Ant}(v, K') = \{v'\}$. Then $\mathrm{Ant}(v', K') = v$. In fact, if this were not true, then we would have some point $w \in \mathrm{Ant}(v', K')$ such that $0 < d(v, w) < \pi$. Then we could extend the geodesic segment $\overline{vw}$ into K' to get a local geodesic $\overline{vw'}$ with $w' \in K'$ and $\mathrm{length}(\overline{vw'}) = \pi$. This would actually be a geodesic since we are in a CAT(1) space, thus $w' \in \mathrm{Ant}(v, K')$ and $w' \neq v'$, contradicting $\mathrm{Ant}(v, K') = v'$.

Now pick a point $k \in K'$, with $k \neq v$ and $k \neq v'$. Then $d(k, v') < \pi$ and $d(k, v) < \pi$. We extend the geodesic segment $\overline{v'k}$ into K' to get a geodesic of length π, then the other end must hit v since $\mathrm{Ant}(v', K') = v$. Thus $\overline{kv} \subset K'$ by the uniqueness of geodesic joining k and v. Similarly we know $\overline{kv'} \subset K'$, thus there is a geodesic segment in K' passing through k and joining v and v'. By CAT(1) geometry, K' (with the induced metric from M) splits as a metric suspension and v, v' are suspension points. However, by the assumption at the beginning of the proof, $\{\phi_i(v_i)\}_{i=1}^{\infty}$ should have distance at least α from a suspension point for every i, so v should also be α–away from a suspension point; this contradiction finishes the proof. □

Remark 3-31 (1) The above proof also shows the following result. Let K be a piecewise spherical CAT(1) complex, and let $K' \subset K$ be a subcomplex with geodesic extension property in the sense of Lemma 3-9. Pick $v \in K$. If $\mathrm{Ant}(v, K')$ is exactly one point, then $v \in K'$ and v is a suspension point of K'.

(2) By the same proof, it is not hard to see Lemma 3-29 is also true when $\mathcal{F}$ is a finite family of finite piecewise spherical CAT(0) complexes (not necessarily all-right).

From Lemmas 3-4, 3-25 and 3-29, we have the following analogue of [9, Theorem 3.11].

Theorem 3-32 *Let X be an n–dimensional* CAT(0) *cube complex, and let $S = S_{[\sigma]}$, where $\sigma \in H^p_{n,n}(X)$. Then for every $p \in X$ and every $\epsilon > 0$, there is an $r < \infty$ such that if $x \in S \setminus B(p, r)$, then $\Sigma_x S$ is a suspension and $\log_x p$ is ϵ–close to a suspension point of $\Sigma_x S$.*

Remark 3-33 By the same proof, the conclusion of Theorem 3-32 is also true if X is a proper n–dimensional CAT(0) complex with a cocompact (cellular) isometry group.

4 The structure of the top-dimensional support sets

Throughout this section, X is an n–dimensional CAT(0) cube complex. Pick a homology class $[\sigma] \in H^{\mathrm{p}}_{n,n}(X)$ and let $S = S_{[\sigma]}$. Also recall that $\Sigma_x X$ is an all-right spherical CAT(1) complex for each $x \in X$; see Definition 2-7.

Let Δ^k be the k–dimensional all-right spherical simplex, and let Δ^k_{mod} be the quotient of Δ^k by the action of its isometry group (Δ^k_{mod} is endowed with the quotient metric). Define the function $\chi\colon \Delta^k \to (0, +\infty)$ by

$$\chi(v) = \inf\{d(v, v') \mid v' \in \Delta^k \text{ and } \mathrm{Supp}(v') \cap v = \varnothing\}. \tag{4-1}$$

Recall that $\mathrm{Supp}(v')$ denotes the unique closed face of Δ^k which contains v' as an interior point. By symmetry of Δ^k, χ descends to a function $\chi\colon \Delta^k_{\mathrm{mod}} \to (0, +\infty)$.

For any $k' \geq k$, we have a canonical isometric embedding $i\colon \Delta^k_{\mathrm{mod}} \hookrightarrow \Delta^{k'}_{\mathrm{mod}}$ with $\chi = \chi \circ i$. Let $\Delta_{\mathrm{mod}} = \varinjlim \Delta^k_{\mathrm{mod}}$ be the corresponding direct limit of metric spaces.

Let Y be an all-right spherical CAT(1) complex. Then there is a well-defined 1–Lipschitz map

$$\theta\colon Y \to \Delta_{\mathrm{mod}}$$

such that θ restricted to any k–face $\Delta^k \subset Y$ is the map $\Delta^k \to \Delta^k_{\mathrm{mod}} \hookrightarrow \Delta_{\mathrm{mod}}$. Moreover, for $v \in Y$,

(1) $v \in \mathrm{Supp}(v')$ if $d(v, v') < \chi(\theta(v))$,

(2) $\chi \circ \theta$ is continuous on the interior of each face of Y.

When $Y = \Sigma_x X$ for some $x \in X$ and $v \in \Sigma_x X$, we also call $\theta(v)$ the Δ_{mod} *direction of* v.

4.1 Producing orthants

In this section, we study geodesic rays with constant Δ_{mod} direction, ie unit-speed geodesic rays $l\colon [0, \infty) \to S$ with $\theta(l^-(t)) = \theta(l^+(t)) = \theta(l^-(t')) = \theta(l^+(t'))$ for any $t \neq t'$. Here are two examples.

(1) If a geodesic ray l stays inside an orthant subcomplex of $O \subset Y$ (or more generally a straight orthant), then it has constant Δ_{mod} direction. Moreover, the Δ_{mod} direction of $\partial_T l$ in $\partial_T O$ is equal to $\theta(l^{\pm}(t))$.

(2) If Y is a product of trees, then each geodesic ray in $l \in Y$ has constant Δ_{mod} direction. Again, the Δ_{mod} direction of $\partial_T l$ in $\partial_T Y$ (in this case $\partial_T Y$ is an all-right spherical complex) is equal to $\theta(l^{\pm}(t))$.

Later, geodesic rays with constant Δ_{mod} direction will play an important role in the construction of orthants; see Lemma 4-9. First we show such geodesics exist in the support set of a top-dimensional proper cycle and there are plenty of them.

Lemma 4-2 *If Y is a k–dimensional all-right spherical* $\mathrm{CAT}(1)$ *complex and if $K \subset Y$ is the support set of some top-dimensional homology class, then for any $v \in K$, there exists a $v' \in K$ such that $d(v, v') = \pi$ and $\theta(v) = \theta(v')$.*

Recall that the metric on Y is the length metric on Y truncated by π.

Proof The lemma is clear when $k = 1$ by Lemma 3-9. We assume it is true for $i \le k-1$. Denote $k' = \dim(\mathrm{Supp}(v))$. We endow $\mathbb{S}^{k'}$ with the structure of an all-right spherical complex and pick $w \in \mathbb{S}^{k'}$ such that $\theta(v) = \theta(w)$. Suppose $w' = \mathrm{Ant}(w, \mathbb{S}^{k'})$ and suppose $\gamma' \colon [0, \pi] \to \mathbb{S}^{k'}$ is a unit-speed geodesic joining w and w'. It is clear that $\theta(w) = \theta(w')$. Our goal is to construct a unit-speed local geodesic $\gamma \colon [0, \pi] \to K$ such that $\gamma(0) = v$ and $\theta(\gamma(s)) = \theta(\gamma'(s))$ for all $s \in [0, \pi]$, as in the following diagram:

$$\begin{array}{ccc} [0,\pi] & \xrightarrow{\ \gamma'\ } & \mathbb{S}^{k'} \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ K & \xrightarrow{\ \theta\ } & \Delta_{\mathrm{mod}} \end{array}$$

Then γ is actually a geodesic and we can take $v' = \gamma(\pi)$ to finish the proof.

There exists a sequence of faces $\{\Delta'_j\}_{j=1}^N$ in $\mathbb{S}^{k'}$ and $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \pi$ such that each Δ'_j contains $\{\gamma'(t) \mid t_{j-1} < t < t_j\}$ as interior points. Let $\Delta_1 = \mathrm{Supp}(v)$. Then since $\theta(v) = \theta(w)$, we can find $v_1 \in \Delta_1$ such that there exists an isometry $\Phi \colon \Delta_1 \to \Delta'_1$ with $\Phi(v) = w$ and $\Phi(v_1) = \gamma'(t_1)$, in particular $\theta(v_1) = \theta(\gamma'(t_1))$. Define $\gamma \colon [0, t_1] \to K$ to be the geodesic segment $\overline{vv_1}$.

Recall that we have identified $\Sigma_{v_1} Y$ with $\mathrm{Lk}(v_1, Y)$; see Definition 2-7. Now let $\sigma_1 = \mathrm{Supp}(v_1)$ and $k_1 = \dim(\sigma) - 1$. Then $\Sigma_{v_1} Y = \mathrm{Lk}(v_1, \sigma_1) * \mathrm{Lk}(\sigma_1, Y) = \mathbb{S}^{k_1} * \mathrm{Lk}(\sigma_1, Y)$. Similarly, $\Sigma_{v_1} K = \mathrm{Lk}(v_1, \sigma_1) * \mathrm{Lk}(\sigma_1, K) = \mathbb{S}^{k_1} * \mathrm{Lk}(\sigma_1, K)$. Let $K_1 = \mathrm{Lk}(\sigma_1, K)$ and $Y_1 = \mathrm{Lk}(\sigma_1, Y)$. Then they are all-right spherical complexes, K_1 is a subcomplex of Y_1, and Y_1 is $\mathrm{CAT}(1)$. Moreover, since $\Sigma_{v_1} K$ is the support set of some top-dimensional homology class in $\Sigma_{v_1} Y$ (Lemma 3-8), so is K_1 in Y_1. As $\gamma^-(t_1) \in \Sigma_{v_1} K = \mathbb{S}^{k_1} * K_1$, we write

$$\gamma^-(t_1) = (\cos\alpha_1) x_1 + (\sin\alpha_1) y_1 \tag{4-3}$$

for $x_1 \in \mathbb{S}^{k_1}$ and $y_1 \in K_1$. By the induction assumption, we can find $y'_1 \in \mathrm{Ant}(y_1, K_1)$ such that $\theta(y'_1) = \theta(y_1)$. Let $x'_1 = \mathrm{Ant}(x_1, \mathbb{S}^{k_1})$. Suppose $\Delta_2 \subset K$ is the unique face ($v_1 \in \Delta_2$) such that $\mathrm{Supp}((\cos\alpha_1)x_1' + (\sin\alpha_1)y_1') = \Sigma_{v_1}\Delta_2$. Let $\overline{v_1v_2} \subset \Delta_2$ be the geodesic segment which starts at v_1, and goes along the direction $(\cos\alpha_1)x_1' + (\sin\alpha_1)y_1'$ until it hits some boundary point v_2 of Δ_2. Note that $\overline{vv_1}$ and $\overline{v_1v_2}$ fit together to form a local geodesic in K.

On the other hand, at $\gamma'(t_1) \in \mathbb{S}^{k'}$, we have $\Sigma_{\gamma'(t_1)}\mathbb{S}^{k'} = \mathrm{Lk}(\gamma'(t_1), \sigma_1') * \mathrm{Lk}(\sigma_1', \mathbb{S}^{k'}) = \mathbb{S}^{k_1} * \mathrm{Lk}(\sigma_1', \mathbb{S}^{k'})$, where $\sigma_1' = \mathrm{Supp}(\gamma'(t_1))$ and k_1 is the same as the previous paragraph. Write $\gamma'^{-}(t_1) = (\cos\alpha_1)u_1 + (\sin\alpha_1)v_1$ for $u_1 \in \mathbb{S}^{k_1}$ and $v_1 \in \mathrm{Lk}(\sigma_1', \mathbb{S}^{k'})$, where we have the same α_1 as (4-3) since Φ is an isometry. Then $\gamma'^{+}(t_1) = (\cos\alpha_1)u_1' + (\sin\alpha_1)v_1'$ for $u_1' = \mathrm{Ant}(u_1, \mathbb{S}^{k_1})$ and $v_1' = \mathrm{Ant}(v_1, \mathrm{Lk}(\sigma_1', \mathbb{S}^{k'}))$. Note that $\theta(v_1') = \theta(v_1) = \theta(y_1) = \theta(y_1')$, so we can extend the isometry Φ to get a map $\Phi'\colon \Delta_1 \cup \Delta_2 \to \Delta_1' \cup \Delta_2'$ such that Φ' is an isometry with respect to the length metric on both sides and $\Phi'(\overline{v_1v_2}) = \gamma([t_1, t_2])$. Thus $d(v_1, v_2) = t_2 - t_1$ and we can define $\gamma\colon [t_1, t_2] \to K$ to be the geodesic segment $\overline{v_1v_2}$. It is clear that $\theta(\gamma(s)) = \theta(\gamma'(s))$ for all $s \in [0, t_2]$. We can repeat this process to define the required local geodesic $\gamma\colon [0, \pi] \to K$. □

Corollary 4-4 *For any $x \in S$ and $v \in \Sigma_x S$, there exists a geodesic ray $\overline{x\xi} \subset S$ which has constant Δ_{mod} direction and $\log_x \xi = v$.*

Proof Since x has a cone neighborhood in S, we can find a short geodesic segment $\overline{xx'}$ in the cone neighborhood such that $\log_x x' = v$. There is a unique closed cube $C_1 \subset S$ such that $\overline{xx'} \subset C_1$ and v is an interior point of $\Sigma_x C_1$. We extend $\overline{xx'}$ in C_1 until it hits the boundary of C_1 at x_1. By Lemma 4-2, there exists a $v_1 \in \mathrm{Ant}(\log_{x_1} x, \Sigma_{x_1} S)$ with $\theta(v_1) = \theta(\log_{x_1} x) = \theta(v)$. Now we choose cube $C_2 \subset S$ and segment $\overline{x_1x_2} \subset C_2$ with $\log_{x_1} x_2 = v_1$ as before. Note that $\overline{xx_1}$ and $\overline{x_1x_2}$ together form a local geodesic segment (hence a geodesic segment). We repeat the previous process to extend the geodesic. Since S is a closed set, the extension can not terminate, which will give us the geodesic ray $\overline{x\xi}$ as required. □

Corollary 4-5 *The set of points in $\partial_T S$ which can be represented by a geodesic ray in S with constant Δ_{mod} direction is dense.*

Proof Fix a base point p, pick some $\xi \in \partial_T S$. For any $\epsilon > 0$, by (3-24), we can find an r_1 such that

$$d_H(\partial_{p,r} S, \partial_T S) < \tfrac{1}{2}\epsilon \tag{4-6}$$

for all $r > r_1$. By Lemma 3-27, we can find r_2 such that for $r > r_2$,

$$\mathrm{diam}(\mathrm{Ant}_\infty(\log_x p, S)) < \tfrac{1}{2}\epsilon \tag{4-7}$$

for any $x \in S \setminus B(p, r)$.

If $r_0 = \max\{r_1, r_2\}$, then we can find $\overline{p\xi_1} \subset B(p, r_0) \cup S$ such that $\angle_T(\xi_1, \xi) \le \frac{\epsilon}{2}$ by (4-6). Pick $x \in \overline{p\xi_1}$ such that $d(p, x) > r_0$. By Corollary 4-4, we can find a geodesic ray $\overline{x\xi_2} \subset S$ of constant Δ_{mod} direction such that $\overline{x\xi_2}$ fits together with $\overline{px}$ to form a geodesic ray $\overline{p\xi_2}$, thus $\angle_T(\xi_1, \xi_2) < \frac{\epsilon}{2}$ by (4-7). Then $\angle_T(\xi, \xi_2) < \epsilon$, which finishes the proof since ϵ and ξ are arbitrary. □

Let l be a geodesic ray of constant Δ_{mod} direction. Then we define $\theta(l)$ to be $\theta(l^{\pm}(t))$. The definition does not depend on the choice of sign $\pm$ and t.

Lemma 4-8 *If $l \subset S$ is a unit-speed geodesic ray of constant Δ_{mod} direction, then there exists a $t_0 < \infty$, which depends on the position of $l(0)$ and $\theta(l)$, such that for any $t > t_0$, $\Sigma_{l(t)} S = \Sigma_{l(t)} l * Y_t$ for some $Y_t \subset \Sigma_{l(t)} S$.*

Proof We apply Theorem 3-32 with $p = l(0)$ and $\epsilon = \chi(\theta(l))$ (see (4-1) for the definition of χ) to get $t_0 < \infty$ such that $\Sigma_{l(t)} S$ is a metric suspension and the suspension point is $\chi(\theta(l))$–close to $l^+(t)$ (or $l^-(t)$) for all $t > t_0$. By Lemma 2-9, $l^+(t)$ and $l^-(t)$ are suspension points, thus $\Sigma_{l(t)} S = \Sigma_{l(t)} l * Y_t$. □

Based on Lemma 4-8, we define a parallel transport of $\Sigma_{l(t)} S$ along l as follows. Let t_0 be as in Lemma 4-8. For any $t > t_0$, $l(t)$ has a product neighborhood in S of form $X_t \times (t - \epsilon, t + \epsilon)$, where X_t is some subset of X with the induced metric. So for any $|t' - t| < \epsilon$, we can identify $\Sigma_{l(t)} S$ and $\Sigma_{l(t')} S$. Moreover, for any $t_1 > t_0$ and $t_2 > t_0$, we can cover the geodesic segment $\overline{l(t_1) l(t_2)}$ by finitely many product neighborhoods, which will induce an identification of $\Sigma_{l(t_1)} S$ and $\Sigma_{l(t_2)} S$. This identification does not depend on the covering we choose.

To see this identification more concretely, take $t > t_0$, a product neighborhood $X_t \times (t - \epsilon, t + \epsilon)$ of $l(t)$ in S, and $v \in \Sigma_{l(t)} S$, we construct a short geodesic $\overline{l(t) x_t} \subset S$ in the cone neighborhood of $l(t)$ going along the direction v. If $|t' - t| < \epsilon$, we can find an isometrically embedded parallelogram in the product neighborhood such that $\overline{l(t) x_t}$ and $\overline{l(t') x_{t'}}$ are opposite sides of the parallelogram and $\overline{l(t) l(t')}$ is one of the remaining sides (we might have to shorten $\overline{l(t) x_t}$ a little).

In general, for any $t' > t_0$, we can cover the geodesic segment $\overline{l(t) l(t')}$ by finitely many product neighborhoods as before and construct a local isometric embedding ϕ from a parallelogram to X such that two opposite sides of the parallelogram are mapped to some geodesic segments $\overline{l(t) x_t}$ and $\overline{l(t') x_{t'}}$ and one of the remaining sides is mapped to $\overline{l(t) l(t')}$. Since X is CAT(0), ϕ is actually an isometric embedding. So we have a well-defined *parallel transport* of $\Sigma_{l(t)} S$ along $l(t)$ for $t > t_0$.

In the construction of the above parallelogram, the length of $\overline{l(t) x_t}$ (or $\overline{l(t') x_{t'}}$) may go to 0 as $|t' - t| \to \infty$. However, in the special case where there exists $s_0 > 0$ such that $\Sigma_{l(t)} l$ is contained in the 0–skeleton of $\Sigma_{l(t)} S$ for all $t > s_0$ (or equivalently $l([s_0, \infty))$ is parallel to some geodesic ray in the 1–skeleton of X), $l([t_0, \infty))$ has a product neighborhood of the form $X' \times [t_0, \infty)$ in S, where X' is a subcomplex of X with the induced metric. Therefore for any $t > t_0$ and a segment $\overline{l(t)x_t} \subset S$ short enough, we can parallel transport $\overline{l(t)x_t}$ along l to infinity, ie there is an isometrically embedded "infinite parallelogram" with one side $\overline{l(t)x_t}$ and one side $l([t, \infty))$.

Lemma 4-9 *If $l\colon [0, \infty) \to S$ is a unit-speed geodesic ray of constant Δ_{mod} direction, then there exists an orthant subcomplex $O \subset X$ satisfying the following conditions:*

(1) $\partial_T l \in \partial_T O$.

(2) *If $\dim(O) = k$ and if $\{l_i\}_{i=1}^k$ are the geodesic rays emanating from the tip of the orthant such that O is the convex hull of $\{l_i\}_{i=1}^k$, then $\partial_T l_i \in \partial_T S$ for all i.*

Proof By the previous lemma, we can choose some $r > 0$ such that for $t > r$, the $l^{\pm}(t)$ are suspension points in $\Sigma_{l(t)} S$. Pick some $t > r$ and let k be the dimension of $\mathrm{Supp}(l^+(t))$. Let $\{v_i^t\}_{i=1}^k$ be vertices of $\mathrm{Supp}(l^+(t))$. Suppose that $\alpha_i = d_{\Sigma_{l(t)} S}(v_i^t, l^+(t))$, where the values of k and α_i are the same for all $t > r$ by the splitting in Lemmas 4-8 and 2-9. Moreover, we would like the labels v_i^t to be consistent under parallel transportation, ie for $t' \neq t$ (with $t' > r$), $v_i^{t'}$ is the parallel transport of v_i^t along l. By Theorem 3-32, we can choose $r' \geq r$ such that if $x \in S \setminus B(l(0), r')$, then $\log_x l(0)$ is ϵ–close to some suspension point in $\Sigma_x S$ for $\epsilon = \min_{1 \leq i \leq k} \bigl\{\frac{1}{2}\bigl(\frac{\pi}{2} - \alpha_i\bigr)\bigr\}$.

Now we pick some $t > r'$, and construct a short geodesic segment $\overline{l(t)x_i} \subset S$ going along the direction of v_i^t in the cone neighborhood of $l(t)$. We choose an arbitrary extension of $\overline{l(t)x_i}$ into S and call the geodesic ray l_i^t for $1 \leq i \leq k$. We claim that for any $y \in l_i^t$ (with $y \neq l(t)$),

$$\Sigma_y S = \Sigma_y l_i^t * Y \tag{4-10}$$

for some $Y \subset \Sigma_y S$, hence the extension is unique and l_i^t is a straight geodesic.

Suppose the claim were not true. Pick the first point $y_i \in l_i^t$ such that (4-10) is not satisfied. Since $\angle_{l(t)}(y_i, l(0)) \geq \pi - \alpha_i > \frac{\pi}{2}$, hence $d(y_i, l(0)) > d(l(0), l(t)) > r'$. By our choice of r', there is a suspension point in $\Sigma_{y_i} S$ which has distance less than $\frac{1}{2}(\frac{\pi}{2} - \alpha_i)$ from $\log_{y_i} l(0)$. Since $\angle_{y_i}(l(0), l(t)) < \alpha_i$, $\log_{y_i} l(t)$ has distance less than $\alpha_i + \frac{1}{2}\bigl(\frac{\pi}{2} - \alpha_i\bigr) < \frac{\pi}{2}$ from a suspension point. Since all points in l_i^t between $l(t)$ and y_i satisfy (4-10), $\log_{y_i} l(t)$ is a vertex in the all-right spherical complex $\Sigma_{y_i} S$. Thus $\log_{y_i} l(t)$ is also a suspension point and (4-10) must hold at $y = y_i$, which is a contradiction.

We claim next that l_i^t is parallel to $l_i^{t'}$ for any $t > r'$ and $t' > r'$. In fact, fixing t, by the discussion before Lemma 4-9 and the uniqueness of $l_i^{t'}$, we know the claim is true for $|t' - t| < \epsilon$, where ϵ depends on t. For the general case, we can apply a covering argument as before.

Fix $t_0 > r'$. By Lemma 2-4, for all i,

$$\angle_T(l_i^{t_0}, l) = \lim_{t \to +\infty} \angle_{l(t)}(l_i^t, l) = \alpha_i < \tfrac{\pi}{2}. \tag{4-11}$$

Thus $\angle_T(l_i^{t_0}, l) = \angle_{l(t_0)}(l_i^{t_0}, l) = \alpha_i$ for all i. It follows that l and $l_i^{t_0}$ bound a flat sector by Lemma 2-5.

We fix a pair i, j with $i \neq j$, and parametrize $l_i^{t_0}$ by arc length. We can assume without loss of generality that $l(t_0)$ is in the 0–skeleton. Let $\{h_m\}_{m=1}^{\infty}$ be the collection of hyperplanes such that $h_m \cap l_i^{t_0} = l_i^{t_0}\big(m + \frac{1}{2}\big)$. Then (4-11) and Lemma 2-4 imply that the CAT(0) projection of l onto $l_i^{t_0}$ is surjective, thus there exists a sequence $\{t_m\}_{m=1}^{\infty}$ such that $l(t_m) \in h_m$. Recalling that $j \neq i$, note that $l_j^{t_m}$ starts at $l(t_m)$, $\angle_{l(t_m)}(l_i^{t_m}, l_j^{t_m}) = \frac{\pi}{2}$ and $l_i^{t_m}$ is orthogonal to h_m, so $l_j^{t_m} \subset h_m$.

By convexity of h_m, we can find a geodesic ray c_m which starts at $l_i^{t_0}\big(m + \frac{1}{2}\big)$, stays inside h_m and is asymptotic to $l_j^{t_m}$ for every m, thus by Lemma 2-4,

$$\angle_T(l_i^{t_0}, l_j^{t_0}) = \lim_{m \to +\infty} \angle_{l_i^{t_0}(m+\frac{1}{2})}(l_i^{t_0}, c_m) = \tfrac{\pi}{2}. \tag{4-12}$$

By (4-12), $\angle_T(l_i^{t_0}, l_j^{t_0}) = \angle_{l(t_0)}(l_i^{t_0}, l_j^{t_0}) = \frac{\pi}{2}$ for $i \neq j$. By Lemma 2-18, we know that the geodesic rays $\{l_i^{t_0}\}_{i=1}^{k}$ span a straight orthant O. Moreover, Lemma 2-8 together with (4-11) imply $\partial_T l \in \partial_T O$. By Remark 2-16, we can replace O by an orthant subcomplex which is Hausdorff close to O. □

4.2 Cycle at infinity

By Lemma 4-9 and Corollary 4-5, there exists a dense subset G of $\partial_T S$ such that for any $v \in G$, there exists an orthant subcomplex $O_v \in X$ such that $v \in \Delta_v = \partial_T O_v$. Denote the vertices of Δ_v by $\mathrm{Fr}(v)$, then $\mathrm{Fr}(v) \subset \partial_T S$ by Lemma 4-9.

It is clear that $G \subset \bigcup_{v \in G} \Delta_v$. We claim $\bigcup_{v \in G} \Delta_v$ is a finite union of all-right spherical simplices. In fact, it suffices to show $\bigcup_{v \in G} \mathrm{Fr}(v)$ is a finite set, which follows from Lemmas 2-17, 3-15 and 4-9 (note that each point in $\bigcup_{v \in G} \mathrm{Fr}(v)$ is represented by a straight geodesic contained in S).

Moreover, $\bigcup_{v \in G} \Delta_v$ has the structure of a finite simplicial complex. Take two simplices Δ_{v_1} and Δ_{v_2}. We know $\Delta_{v_i} = \partial_T O_{v_i}$ for orthant subcomplex O_{v_i}, and Remark 2-13 implies $\Delta_{v_1} \cap \Delta_{v_2}$ is a face of Δ_1 (or Δ_2).

We endow $K=\bigcup_{v\in G}\Delta_v\subset\partial_T Y$ with the angular metric and denote the Euclidean cone over K by CK, which is a subset of $C_T X$.

Lemma 4-13 (1) *K is a topologically embedded finite simplicial complex in $\partial_T X$.*

(2) *CK is linearly contractible.*

Recall that linearly contractible means there exists a constant C such that for any $d>0$, every cycle of diameter $\leq d$ can be filled in by a chain of diameter $\leq Cd$.

Proof Let $\angle_T$ be the angular metric on K and d_l be the length metric on K as an all-right spherical complex. Our goal is to show that Id: $(K,\angle_T)\to(K,d_l)$ is a bi-Lipschitz homeomorphism. Let $\{\Delta_i\}$ be the collection of faces of K (each Δ_i is an all-right spherical simplex). Suppose $\{O_i\}_{i=1}^N$ are orthant subcomplexes of X such that $\partial_T O_i=\Delta_i$. If points x and y are in the same Δ_i for some i, then

$$d_l(x,y)=\angle_T(x,y). \tag{4-14}$$

If x and y are not in the same simplex, then we put $\Delta_i=\mathrm{Supp}(x)$, $\Delta_j=\mathrm{Supp}(y)$ and $\Delta_k=\Delta_i\cap\Delta_j$. Assume without loss of generality that $d_l(x,\Delta_k)\geq\frac{1}{2}d_l(x,y)$. Let $(Y_1,Y_2)=\mathcal{I}(O_i,O_j)$. Then $\partial_T Y_1=\partial_T Y_2=\Delta_k$. Moreover, it follows from (2-11) and Lemma 2-4 that

$$\angle_T(x,y)\geq 2\arcsin\bigl(\tfrac{1}{2}A\sin(d_l(x,\Delta_k))\bigr)\geq 2\arcsin\bigl(\tfrac{1}{2}A\sin\bigl(\tfrac{1}{2}d_l(x,y)\bigr)\bigr), \tag{4-15}$$

where A can be chosen to be independent of i and j since $\{O_i\}_{i=1}^N$ is a finite collection. Equations (4-14) and (4-15) imply Id: $(K,\angle_T)\to(K,d_l)$ is a bi-Lipschitz homeomorphism, thus (1) is true.

To see (2), it suffices to prove $(K,\angle_T)$ is linearly locally contractible, ie there exist $C<\infty$ and $R>0$ such that for any $d<R$, every cycle of diameter $\leq d$ can be filled in by a chain of diameter $\leq Cd$. By the above discussion, we only need to prove (K,d_l) is locally linearly contractible.

Since (K,d_l) is compact and can be covered by finitely many cone neighborhoods (see Theorem 2-1), it suffices to show each cone neighborhood is linearly contractible; but any cone neighborhood is isometric to a metric ball in the spherical cone of some lower dimensional finite piecewise spherical complex, thus we can finish the proof by induction on dimension. □

Since G is a dense subset of $\partial_T S$ and K is compact, it follows that $\partial_T S\subset K$ and $C_T S\subset CK\subset C_T X$. We denote the base point of $C_T X$ by o.

Lemma 4-16 *K has the structure of an $(n-1)$–simplicial cycle.*

Proof In the following proof, we will use d to denote the metric on X, and use $\bar{d}$ to denote the metric on $C_T X$.

Pick a base point $p \in X$. By the proof of Lemma 3-16, we know that for any $\epsilon > 0$, there exist a finite collection of constant speed geodesic rays $\{l_i\}_{i=1}^N$ and an $R_\epsilon < \infty$ such that $l_i(t) \in S$ and $\{l_i(t)\}_{i=1}^N$ is a ϵt–net in $B(p,t) \cap S$ for $t \geq R_\epsilon$. Write $\xi_i = \partial_T l_i$ and define $f_\epsilon \colon S \to C_T S \subset CK$ by sending $l_i(t)$ to the point in $\overline{o\xi_i} \subset C_T S$ which has distance $d(l_i(t), p)$ from o ($t \geq R_\epsilon$). For $x \notin \bigcup_{i=1}^N l_i[R_\epsilon, \infty)$, we pick a point $y \in \bigcup_{i=1}^N l_i[R_\epsilon, \infty)$ which is nearest to x and define $f_\epsilon(x) = f_\epsilon(y)$.

It is clear that

$$|d(x,p) - \bar{d}(f_\epsilon(x), o)| \leq \epsilon \max\{d(x,p), R_\epsilon\} \tag{4-17}$$

for any $x \in S$, and

$$|d(x,y) - \bar{d}(f_\epsilon(x), f_\epsilon(y))| \leq \epsilon \max\{d(p,x), d(p,y), R_\epsilon\} \tag{4-18}$$

for any $x \in S$ and $y \in S$. Moreover,

$$\bar{d}_H(f_\epsilon(B(p,r) \cap S), B(o,r) \cap C_T S) \leq \epsilon \max\{r, R_\epsilon\}. \tag{4-19}$$

We might need to pick a larger R_ϵ for (4-19).

We want to approximate f_ϵ by a continuous map. Let us cover S by a collection of open sets $\{B(x, r_x) \cap S\}_{x \in S}$, where $r_x = \epsilon \max\{d(x,p), R_\epsilon\}$. Since S has topological dimension $\leq n$, this covering has a refinement $\{U_i\}_{i=1}^\infty$ of order $\leq n$; see [25, Chapter V]. Note that $\operatorname{diam}(U_i) \leq 2\epsilon \max\{d(p, U_i), R_\epsilon\}$. Denote the nerve of $\{U_i\}_{i=1}^\infty$ by N, which is a simplicial complex of dimension $\leq n$.

Now we define a map $b' \colon N \to CK$ as follows. For any vertex $v_i \in L$, pick $x_i \in U_i$ where U_i is the open set associated with vertex v_i, then set $b'(v_i) = f_\epsilon(x_i)$. Then use the linear contractibility of CK to extend the map skeleton by skeleton to get b'. By choosing a partition of unity subordinate to the covering $\{U_i\}_{i=1}^\infty$, we obtain a barycentric map b from S to the nerve N (see [25, Chapter V]), then the continuous map $b' \circ b \colon S \to CK$ also satisfies (4-17)–(4-19) with ϵ replaced by $L'\epsilon$, where L' is some constant which only depends on the linear contractibility constant of CK. So we can assume without loss of generality that $f_\epsilon \colon S \to CK$ is continuous and (4-17)–(4-19) still hold for f_ϵ.

Recall that S is the support set of some top-dimensional proper homology class $[\sigma]$. We can also view $[\sigma]$ as the fundamental class of S and assume σ is the proper singular cycle representing this class. If $\alpha = f_\epsilon(\sigma)$, then $[\alpha] \in H_n^{\mathrm{p}}(CK)$ since f_ϵ is a proper map by (4-17). Our next goal is to show

$$S_{[\alpha]} = CK \tag{4-20}$$

for ϵ small enough. Since K is a simplicial complex, (4-20) would imply K also has a fundamental class whose support set is exactly K itself, proving Lemma 4-16.

Recall that we have a 1–Lipschitz logarithmic map $\log_p\colon C_T X \to X$ sending base point o to p. By (4-17) and (4-18), there exists a constant $L < \infty$ such that

$$\bar{d}(z,o) = d(\log_p(z), p) \tag{4-21}$$

for all $z \in \operatorname{Im} f_\epsilon$, and

$$|\bar{d}(z,w) - d(\log_p(z), \log_p(w))| \le L\epsilon \max\{\bar{d}(o,z), \bar{d}(o,w), R_\epsilon\} \tag{4-22}$$

for all $z, w \in \operatorname{Im} f_\epsilon$. Moreover,

$$d(\log_p \circ f_\epsilon(x), x) \le L\epsilon \max\{d(x,p), R_\epsilon\} \tag{4-23}$$

for all $x \in S$.

By (4-21), $\log_p$ is proper. Let $\beta = \log_p(\alpha) = \log_p \circ f_\epsilon(\sigma)$. By (4-23), the geodesic homotopy between $\log_p \circ f_\epsilon\colon S \to X$ and the inclusion map $i\colon S \to X$ is proper, thus $[\beta] = [\sigma]$ and $S_{[\beta]} = S_{[\sigma]} = S$. By Lemma 3-2,

$$\log_p(S_{[\alpha]}) \supset S_{[\beta]} = S. \tag{4-24}$$

Equations (4-24), (4-22) and (4-23) imply there exists $L < \infty$ such that

$$\bar{d}_H(B(o,r) \cap S_{[\alpha]}, B(o,r) \cap \operatorname{Im} f_\epsilon) \le L\epsilon \max\{r, R_\epsilon\}. \tag{4-25}$$

This together with (4-19) imply

$$\bar{d}_H(B(o,r) \cap S_{[\alpha]}, B(o,r) \cap C_T S) \le L\epsilon \max\{r, R_\epsilon\}. \tag{4-26}$$

Since K is a simplicial complex, $S_{[\alpha]} = CK'$, where K' is some subcomplex of K. Recall that by the construction of K, the only subcomplex of K that contains $\partial_T S$ is K itself. Now (4-26) implies the Hausdorff distance between $\partial_T S$ and K' is bounded above by $L\epsilon$, thus for ϵ small enough, $K' = K$ and (4-20) holds. We also know $\partial_T S$ is dense in K from this. □

We actually defined a boundary map

$$\partial\colon H^{\mathrm{p}}_{n,n}(X) \to H_{n-1}(\partial_T X) \tag{4-27}$$

in the proof of the above lemma; namely, for ϵ small enough, we send $[\sigma] \in H^{\mathrm{p}}_{n,n}(X)$ to $f_{\epsilon *}[\sigma] \in H^{\mathrm{p}}_n(C_T X)$, which passes to an element in $H_{n-1}(\partial_T X)$ via the map $H^{\mathrm{p}}_n(C_T X) \to H_n(C_T X, C_T X \setminus \{o\}) \cong H_{n-1}(\partial_T X)$.

In the construction of f_ϵ, we have to choose a base point, the geodesic rays $\{l_i(t)\}_{i=1}^N$, the covering $\{U_i\}_{i=1}^\infty$ and the maps b and b'. However, different choices give maps in the same proper homotopy class if the corresponding ϵ is small enough. Also the geodesic homotopy from f_{ϵ_1} to f_{ϵ_2} is proper if ϵ_1 and ϵ_2 are small enough, so the above boundary map is well-defined.

Next we construct a map in the opposite direction as follows. Let η' be a Lipschitz $(n-1)$–cycle in $\partial_T X$. Let α' be the cone over η'. Note that one can cone off elements in $C_{n-1}(\partial_T X)$ to obtain elements in $C_n^{\mathrm{p}}(C_T X)$, which induces a homomorphism $H_{n-1}(\partial_T X) \to H_n^{\mathrm{p}}(C_T X)$. Actually $[\alpha'] \in H_{n,n}^{\mathrm{p}}(C_T X)$ since the cone over a Lipschitz cycle would satisfy the required growth condition. If $\sigma' = \log(\alpha')$, then $[\sigma'] \in H_{n,n}^{\mathrm{p}}(X)$ since log is 1–Lipschitz. Now we define the "coning off" map

$$c\colon H_{n-1}(\partial_T X) \to H_{n,n}^{\mathrm{p}}(X) \tag{4-28}$$

by sending $[\eta']$ to $[\sigma']$. The base point in the definition of log does not matter because different base points give maps which are of bounded distance from each other. It is easy to see that c is a group homomorphism.

For $\epsilon > 0$, pick a finite ϵ–net of $\operatorname{Im}\eta'$ and denote it by $\{\xi_i\}_{i=1}^N$. Suppose $p = \log(o)$ and suppose $\{l_i\}_{i=1}^N$ are the unit-speed geodesic rays emanating from p with $\partial_T l_i = \xi_i$. Pick $R_\epsilon > 0$ such that

$$\left| \frac{d(l_i(t), l_j(t))}{t} - \lim_{t\to+\infty} \frac{d(l_i(t), l_j(t))}{t} \right| < \epsilon \tag{4-29}$$

for $t > R_\epsilon$. Let $I_{\sigma'}$ be the smallest subcomplex of X which contains $\operatorname{Im}\sigma'$. By using the rays $\{l_i\}_{i=1}^N$ as in the proof of Lemma 4-16, we can construct a continuous proper map $g_\epsilon\colon I_{\sigma'} \to C_T X$ skeleton by skeleton so that

$$d(g_\epsilon \circ \log(x), x) \le L\epsilon \max\{d(x,o), R_\epsilon\} \tag{4-30}$$

for $x \in \operatorname{Im}\alpha'$, which implies $g_{\epsilon *}[\sigma'] = [\alpha']$ for ϵ small.

Let $[\sigma'']$ be the fundamental class of $S_{[\sigma']}$ and let $f_{\epsilon *}\colon S_{[\sigma']} \to C_T X$ be the map in Lemma 4-16. We claim that $g_{\epsilon *}[\sigma'] = f_{\epsilon *}[\sigma'']$ for ϵ small, which would imply

$$\partial \circ c = \mathrm{Id}. \tag{4-31}$$

To see the claim, note that $[\sigma'] = [\sigma'']$ in $H_n^{\mathrm{p}}(I_{\sigma'})$. For ϵ small, there is a proper geodesic homotopy between $g_\epsilon|_{S_{[\sigma']}}$ and f_ϵ by (4-23) and (4-30), thus $g_{\epsilon *}[\sigma''] = f_{\epsilon *}[\sigma'']$. Moreover, $g_{\epsilon *}[\sigma''] = g_{\epsilon *}[\sigma']$, so $f_{\epsilon *}[\sigma''] = g_{\epsilon *}[\sigma'] = [\alpha']$.

From (4-23) and the discussion after it we know

$$c \circ \partial = \mathrm{Id}. \tag{4-32}$$

Thus ∂ is also a group homomorphism and we have the following result.

Corollary 4-33 *If X is an n–dimensional* CAT(0) *cube complex, then:*

(1) *$\partial\colon H^{\mathrm{p}}_{n,n}(X)\to H_{n-1}(\partial_T X)$ is a group isomorphism, and the inverse is given by $c\colon H_{n-1}(\partial_T X)\to H^{\mathrm{p}}_{n,n}(X)$.*

(2) *If $q\colon X\to X'$ is a quasi-isometric embedding from X to another n–dimensional* CAT(0) *cube complex X', then q induces a monomorphism $q_*\colon H_{n-1}(\partial_T X)\to H_{n-1}(\partial_T X')$. If q is a quasi-isometry, then q_* is an isomorphism.*

Proof We only need to prove (2). Let us approximate q by a Lipschitz quasi-isometric embedding and denote the smallest subcomplex of X' that contains $\operatorname{Im} q$ by I_q. Now we have a homomorphism

$$q_*\colon H^{\mathrm{p}}_{n,n}(X)\to H^{\mathrm{p}}_{n,n}(I_q)\hookrightarrow H^{\mathrm{p}}_{n,n}(X'). \tag{4-34}$$

We can define a continuous map $p\colon I_q\to X$ skeleton by skeleton in such a way that $d(x,p\circ q(x))<D$ for all $x\in X$ (here D is some positive constant), which induces $p_*\colon H^{\mathrm{p}}_{n,n}(I_q)\to H^{\mathrm{p}}_{n,n}(X)$. It is easy to see $p_*\circ q_*=\mathrm{Id}$ and $q_*\circ p_*=\mathrm{Id}$, so the first map in (4-34) is an isomorphism. Note that the second map in (4-34) is a monomorphism, thus q_* is injective and (2) follows from (1). □

We refer to Theorem A-19 and the remarks after it for generalizations of the above corollary.

Remark 4-35 Though we are working with $\mathbb{Z}/2$ coefficients, it is easy to check that the analogue of Corollary 4-33 for arbitrary coefficients is also true (the same proof goes through).

Remark 4-36 By the above proof and the argument in Lemma 4-16, there exists a positive D', which depends on the quasi-isometry constant of q, such that

$$d_H(q(S_{[\tilde\sigma]}),S_{q_*[\tilde\sigma]})<D' \tag{4-37}$$

for any $[\tilde\sigma]\in H^{\mathrm{p}}_{n,n}(X)$.

4.3 Cubical coning

Note that the above coning map c does not give us much information about the combinatorial structure of the support set. Now we introduce an alternative coning procedure based on the cubical structure. We can assume, by Lemma 4-16, that $K=\bigcup_{i=1}^N\Delta_i$, where each Δ_i is an all-right spherical $(n-1)$–simplex. Let $\{O_i\}_{i=1}^N$ be the collection of top-dimensional orthant subcomplexes in X such that $\partial_T O_i=\Delta_i$. By (2-11), we can pass to suborthants and assume $\{O_i\}_{i=1}^N$ is a disjoint collection.

The natural quotient map $\bigsqcup_{i=1}^N \Delta_i \to K$ induces a quotient map $Q\colon \bigsqcup_{i=1}^N O_i \to CK$ sending the tip of each O_i to the cone point of CK. We define an inverse map $F\colon CK \to \bigsqcup_{i=1}^N O_i \subset X$ by sending each $x \in CK$ to some point in $Q^{-1}(x)$.

Lemma 4-38 *$F\colon CK \to X$ is a quasi-isometric embedding.*

Recall that CK is endowed with the induced metric from $C_T X$.

Proof Let o_i be the tip of O_i and $L = \max_{i \neq j} d(o_i, o_j)$. For $x \in O_i$ and $y \in O_j$, let $c_{i,x} \subset O_i$ be the constant-speed geodesic ray with $c_{i,x}(0) = o_i$ and $c_{i,x}(1) = x$. We can define $c_{j,y} \subset O_j$ similarly. Let c'_j be the geodesic ray which (1) is asymptotic to $c_{j,y}$; (2) has the same speed as $c_{j,y}$; (3) satisfies $c'_j(0) = o_i$. Then by Lemma 2-4 and convexity of $d(c_i(t), c'_j(t))$,

$$
\begin{aligned}
\text{(4-39)}\qquad d(Q(x), Q(y)) &= \lim_{t\to\infty} \frac{d(c_{i,x}(t), c_{j,y}(t))}{t} \\
&= \lim_{t\to\infty} \frac{d(c_{i,x}(t), c'_j(t))}{t} \\
&\geq d(c_{i,x}(1), c'_j(1)) \\
&\geq d(c_{i,x}(1), c_{j,y}(1)) - d(c'_j(1), c_{j,y}(1)) \\
&\geq d(x, y) - d(o_i, o_j) \\
&\geq d(x, y) - L.
\end{aligned}
$$

It follows that

$$
\text{(4-40)}\qquad d(F(x), F(y)) \leq d(x, y) + L
$$

for any $x, y \in CK$.

For the other direction, pick $x \in O_i$ and $y \in O_j$, and let us assume without loss of generality that $i \neq j$ and x, y are interior points of O_i and O_j. We extend $\overline{o_i x}$ (or $\overline{o_j y}$) to get a ray $\overline{o_i \xi_1} \subset O_i$ (or $\overline{o_j \xi_2} \subset O_j$). Let $(Y_1, Y_2) = \mathcal{I}(O_i, O_j)$. Since $d(x, y) \leq d(x, Y_1) + d(Y_1, Y_2) + d(y, Y_2) \leq d(x, Y_1) + d(y, Y_2) + L$, we can assume without loss of generality that

$$
\text{(4-41)}\qquad d(x, Y_1) \geq \tfrac{1}{2}(d(x, y) - L).
$$

From (4-15), we have

$$
\begin{aligned}
\text{(4-42)}\quad d(F(x), F(y)) \geq d(x, o_i) \sin(\angle_T(\xi_1, \xi_2)) &\geq \tfrac{1}{2} A d(x, o_i) \sin(\angle_T(\xi_1, \partial_T Y_1)) \\
&\geq \tfrac{1}{2} A d(x, Y_1) - L' \\
&\geq \tfrac{1}{4} A d(x, y) - L' - \tfrac{1}{2} L
\end{aligned}
$$

if $\angle_T(\xi_1,\xi_2)<\frac{\pi}{2}$, and

$$d(F(x),F(y))\ge d(x,o_i)\ge d(x,Y_1)-L'\ge d(x,y)-L'-\tfrac{1}{2}L \tag{4-43}$$

if $\angle_T(\xi_1,\xi_2)\ge\frac{\pi}{2}$. Here A and L' depend on O_i and O_j, but there are finitely many orthants, so we can make A and L' uniform. □

Since X is linearly contractible, we can approximate F by a continuous quasi-isometric embedding F' such that $d(F(x),F'(x))\le L$ for some constant L and any $x\in CK$. Let $K^{(n-2)}$ be the $(n-2)$–skeleton of K and define $\rho\colon CK\to[0,1]$ to be

$$\rho(x)=\begin{cases}1 & \text{if } d(x,CK^{(n-2)})\le 1,\\ 2-d(x,CK^{(n-2)}) & \text{if } 1<d(x,CK^{(n-2)})<2,\\ 0 & \text{if } d(x,CK^{(n-2)})\ge 2.\end{cases}$$

Let

$$F_1(x)=\rho(x)F'(x)+(1-\rho(x))F(x)$$

for $x\in CK$, where $\rho(x)F'(x)+(1-\rho(x))F(x)$ denotes the point in the geodesic segment $\overline{F'(x)F(x)}$ which has distance $\rho(x)d(F'(x),F(x))$ from $F(x)$. Though F may not be continuous, F_1 is continuous, since the only discontinuity points of F are in the 1–neighborhood of $CK^{(n-2)}$, however inside such a neighborhood we have $F_1=F'$ by definition. Also note that $d(F(x),F_1(x))\le L'$ for all $x\in CK$.

Since $F_1=F$ outside the 2–neighborhood of $CK^{(n-2)}$, there exists an orthant subcomplex $O'_i\subset O_i$ such that $F_1^{-1}(O'_i)$ is an orthant in CK for $1\le i\le N$ and

$$d_H\Big(\operatorname{Im}F_1,\bigcup_{i=1}^{N}O'_i\Big)<\infty. \tag{4-44}$$

Let $[CK]\in H_n^{\mathrm{p}}(CK)$ be the fundamental class. If $[\tau]=(F_1)_*[CK]\in H_{n,n}^{\mathrm{p}}(X)$, then

$$\bigcup_{i=1}^{N}O'_i\subset S_{[\tau]}\subset\operatorname{Im}F_1. \tag{4-45}$$

The first inclusion follows from the construction of O'_i and the second follows from Lemma 3-2. Equations (4-44) and (4-45) immediately imply:

Lemma 4-46 $$d_H\Big(S_{[\tau]},\bigcup_{i=1}^{N}O'_i\Big)<\infty.$$

Now we are ready to prove the main result.

Theorem 4-47 *Let X be a* CAT(0) *cube complex of dimension n. Pick $[\sigma] \in H^{\mathrm{p}}_{n,n}(X)$ and suppose $S = S_{[\sigma]}$. Then there is a finite collection $O_1, \dots, O_k$ of n–dimensional orthant subcomplexes of S such that*

$$d_H\left(S, \bigcup_{i=1}^{k} O_k\right) < \infty.$$

Proof By Lemma 4-46, it suffices to show $[\sigma] = [\tau]$ in $H^{\mathrm{p}}_n(X)$. Note that (4-45) implies $\partial_T S_{[\tau]} = K$, so $\partial([\tau]) = [K] = \partial([\sigma])$, where $[K]$ is the fundamental class of K and ∂ is the map in (4-27). Thus $[\sigma] = [\tau]$ by Corollary 4-33. □

In particular, by Lemma 3-4 and Theorem 4-47, we have:

Theorem 4-48 *If X is a* CAT(0) *cube complex of dimension n, then for every n–quasiflat Q in X, there is a finite collection $O_1, \dots, O_k$ of n–dimensional orthant subcomplexes in X such that*

$$d_H\left(Q, \bigcup_{i=1}^{k} O_k\right) < \infty.$$

5 Preservation of top-dimensional flats

5.1 The lattice generated by top-dimensional quasiflats

We investigate the coarse intersection of the top-dimensional quasiflats in this section.

Let X be a finite-dimensional CAT(0) cube complex. For two subsets A and B, we say they are *coarsely equivalent* (denoted $A \sim B$) if $d_H(A, B) < \infty$. We assume the empty subset is coarsely equivalent to any bounded subset. Denote by $[A]$ the coarse equivalence class which contains A. We say $[A] \subset [B]$ if there exists an $r < \infty$ such that $A \subset N_r(B)$. If $[A] \subset [B]$ and $[A] \neq [B]$, we will write $[A] \subsetneq [B]$. Also we define the union $[A] \cup [B]$ to be $[A \cup B]$, but intersection is not well-defined in general.

The class $[A]$ is *admissible* if it can be represented by a subset which is a finite union of (not necessarily top-dimensional) orthant subcomplexes in X (here A is allowed to be empty). Let $\mathcal{A}(X)$ be the collection of admissible classes of subsets in X. Pick $[A_1], [A_2] \in \mathcal{A}(X)$. We define another two operations between $[A_1]$ and $[A_2]$ as follows.

(1) By Lemma 2-10(4), there exists an $r < \infty$ such that

$$[N_{r_1}(A_1) \cap N_{r_1}(A_2)] = [N_{r_2}(A_1) \cap N_{r_2}(A_2)]$$

for any $r_1 \geq r$ and $r_2 \geq r$. We define the intersection $[A_1] \cap [A_2]$ to be $[N_r(A_1) \cap N_r(A_2)]$, which is also admissible.

(2) By Lemma 2-10(4), there exists an $r < \infty$ such that

$$[A_1 \setminus N_{r_1}(A_2)] = [A_1 \setminus N_{r_2}(A_2)]$$

for any $r_1 \geq r$ and $r_2 \geq r$. Define the subtraction $[A_1] - [A_2]$ to be $[A_1 \setminus N_r(A_2)]$, which is also admissible.

If Y is another CAT(0) cube complex with $\dim(Y) = \dim(X)$ and there is a quasi-isometry $f\colon X \to Y$, then we define $f_\sharp([A])$ to be $[f(A)]$. This is well-defined since $A \sim B$ implies $f(A) \sim f(B)$. Note that:

(1) $f_\sharp([A]) \cup f_\sharp([B]) = f_\sharp([A] \cup [B])$.

(2) If $[A], [B], [f(A)]$ and $[f(B)]$ are all admissible, then

$$f_\sharp([A]) \cap f_\sharp([B]) = f_\sharp([A] \cap [B]) \quad \text{and} \quad f_\sharp([A]) - f_\sharp([B]) = f_\sharp([A] - [B]).$$

We only verify the last equality. Since f is a quasi-isometry, there exist constants $a > 1$, $b > 0$ such that for r large enough, we have

$$f(A) \setminus N_{ar+b}(f(B)) \subset f(A \setminus N_r(B)) \subset f(A) \setminus N_{(r/a)-b}(f(B)).$$

Since $[f(A)]$ and $[f(B)]$ are admissible, the first term and the last term of the above inequality are in the same coarse class for r large enough. This finishes the proof.

Let $\mathcal{Q}(X)$ be the collection of top-dimensional quasiflats in X, modulo the above equivalence relation. Theorem 4-48 implies $\mathcal{Q}(X) \subset \mathcal{A}(X)$. Let $\mathcal{KQ}(X)$ be the smallest subset of $\mathcal{A}(X)$ which contains $\mathcal{Q}(X)$ and is closed under union, intersection and subtraction as defined above. More precisely, each element $\mathcal{KQ}(X)$ can be written as a finite string of elements of $\mathcal{Q}(X)$ with union, intersection or subtraction between adjacent terms and braces which indicate the order of these operations. Let $f\colon X \to Y$ be a quasi-isometry. Then by induction on the length of the string, one can show $[f(A)]$ is admissible and $[f(A)] \in \mathcal{KQ}(Y)$ for each $[A] \in \mathcal{KQ}(X)$. By considering the quasi-isometry inverse of f, we have the following theorem.

Theorem 5-1 *Let X and Y be n–dimensional* CAT(0) *cube complexes. If $f\colon X \to Y$ is a quasi-isometry, then f induces a bijection $f_\sharp\colon \mathcal{KQ}(X) \to \mathcal{KQ}(Y)$. Moreover, for*

$[A], [B] \in \mathcal{KQ}(X)$, *we have:*

$$f_\sharp([A]) \cup f_\sharp([B]) = f_\sharp([A] \cup [B]),$$
$$f_\sharp([A]) \cap f_\sharp([B]) = f_\sharp([A] \cap [B]),$$
$$f_\sharp([A]) - f_\sharp([B]) = f_\sharp([A] - [B]).$$

For $[A]$ admissible, pick a representative in $[A]$ which is a finite union of orthant complexes. Define the *order* of $[A]$, denoted $\|[A]\|$, to be the number of top-dimensional orthant complexes in the representative. By Lemma 2-10, this definition does not depend on the choice of representative. Since every element in $\mathcal{KQ}(X)$ is admissible, we have a map $\mathcal{KQ}(X) \to \{0\} \cup \mathbb{Z}^+$ with the following properties:

1. $\|[Q]\| \geq 2^{\dim X}$ for $[Q] \in \mathcal{Q}(X)$.
2. $\|[A] \cup [B]\| = \|[A]\| + \|[B]\| - \|[A] \cap [B]\|$ for $[A], [B] \in \mathcal{KQ}(X)$.
3. Let f be as in Theorem 5-1. Then $\|[A]\| = 0$ if and only if $|f_\sharp([A])| = 0$ for $[A] \in \mathcal{KQ}(X)$.

The first assertion follows from (3-12).

We say an element $[A] \in \mathcal{KQ}(X)$ is *essential* if $\|[A]\| > 0$. We call $[A]$ a *minimal essential element* if for any $[B] \in \mathcal{KQ}(X)$ with $[B] \subsetneq [A]$, we have $\|[B]\| = 0$. Minimal essential elements have the following properties:

1. For any $[A] \in \mathcal{KQ}(X)$, there is a decomposition $[A] = \left(\bigcup_{i=1}^N [A_i]\right) \cup [B]$ such that each $[A_i]$ is a minimal essential element and $\|[B]\| = 0$. We also require $[B]$ and each $[A_i]$ to be in $\mathcal{KQ}(X)$.
2. For two different minimal essential elements $[A_1], [A_2] \in \mathcal{KQ}(X)$, we have $\|[A_1] \cap [A_2]\| = 0$, thus $\|[A_1] \cup [A_2]\| = \|[A_1]\| + \|[A_2]\|$.
3. Let f be as above. If $[A]$ is a minimal essential element in $\mathcal{KQ}(X)$, then $f_\sharp([A])$ is also a minimal essential element.

We only prove (1). For each top-dimensional orthant subcomplex $[O_i]$ such that $[O_i] \subset [A]$, let $[A_i]$ be the minimal element in $\mathcal{KQ}(X)$ which contains $[O_i]$. We claim that $[A_i]$ is minimal essential. Suppose the contrary true, ie there exists $[A'_i] \in \mathcal{KQ}(X)$ such that $\|[A'_i]\| \neq 0$ and $[A'_i] \subsetneq [A_i]$. The minimality of $[A_i]$ implies $[O_i] \subset [A'_i]$ does not hold. However, in such a case $[O_i] \subset [A_i] - [A'_i] \subsetneq [A_i]$, which contradicts the minimality of $[A_i]$. We choose $[B] = [A] - \left[\bigcup_{i=1}^N A_i\right]$.

Lemma 5-2 *Let X, Y be n–dimensional* CAT(0) *cube complexes and let $f\colon X \to Y$ be an (L', A')–quasi-isometry. If $|f_\sharp([A])| = \|[A]\|$ for any minimal essential element $[A]$ in $\mathcal{KQ}(X)$, then there exists a constant $C = C(L', A')$ such that for any top-dimensional flat $F \subset X$, there exists a top-dimensional flat $F' \subset Y$ such that $d_H(f(F), F') < C$.*

Proof By Theorem 5-1 and the above discussion, we know $|f_\sharp([A])| = \|[A]\|$ for any $[A] \in \mathcal{KQ}(X)$, in particular $\|[f(F)]\| = \|[F]\| = 2^n$ (here $n = \dim(X) = \dim(Y)$). By Lemma 3-4, let $[\sigma] \in H_n^{\mathrm{p}}(Y)$ be the class such that $d_H(S_{[\sigma]}, f(F)) < \infty$. By Theorem 4-47, $S_{[\sigma]}$ is Hausdorff close to a union of 2^n orthant subcomplexes. Thus $\partial_T S_{[\sigma]}$ is contained in 2^n right-angled spherical simplices of dimension $n-1$. Then $\mathcal{H}^{n-1}(\partial_T S_{[\sigma]}) \le \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. We pick a base point $p \in S_{[\sigma]}$ and consider the logarithmic map $\log_p\colon C_T Y \to Y$. Lemma 3-6 implies $S_{[\sigma]} \subset \log_p(C_T S_{[\sigma]})$. Thus

$$\frac{\mathcal{H}^n(B(p,r)\cap S_{[\sigma]})}{r^n} \le \frac{\mathcal{H}^n(B(p,r)\cap \log_p(C_T S_{[\sigma]}))}{r^n} \le \frac{\mathcal{H}^n(B(o,r)\cap C_T S_{[\sigma]})}{r^n} \le \omega_n.$$

Here o is the cone point in $C_T Y$ and ω_n is the volume of unit ball in $\mathbb{E}^n$. The second inequality follows from the fact that $\log_p$ is 1–Lipschitz and the third inequality follows from $\mathcal{H}^{n-1}(\partial_T S_{[\sigma]}) \le \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. By Theorem 3-10(2), $S_{[\sigma]}$ is isometric to $\mathbb{E}^n$. □

5.2 The weakly special cube complexes

It is shown in [8] that the assumption of Lemma 5-2 is satisfied for 2–dimensional RAAGs. Our goal in this section is to find an appropriate class of cube complexes which shares some key properties of the canonical CAT(0) cube complexes of RAAGs such that the assumption of Lemma 5-2 is satisfied. In [20], Haglund and Wise introduced a class of RAAG-like cube complexes, which are called special cube complexes. We adjust their definition for our purposes in the following way.

Definition 5-3 A cube complex K is *weakly special* if:

(1) K is nonpositively curved.

(2) No hyperplane *self-osculates* or *self-intersects*.

The second condition means that for any vertex v and two distinct edges e_1 and e_2 such that $v \in e_1 \cap e_2$, the hyperplanes dual to e_1 and e_2 are different.

If K is compact, then there exists a finite sheet, weakly special cover $\bar{K}$ of K such that every hyperplane in $\bar{K}$ is *two-sided*, ie there exists a small neighborhood of the hyperplane which is a trivial interval bundle over the hyperplane. This follows from the argument in [20, Proposition 3.10].

In the rest of this section, we will denote by W' a compact weakly special cube complex, and W the universal cover of W'. Since we mainly care about W, there is no loss of generality in assuming every hyperplane in W' is two-sided. The goal of this section is to prove the following theorem.

Theorem 5-4 *Let W_1' and W_2' be two compact weakly special cube complexes with $\dim(W_1') = \dim(W_2') = n$. Suppose W_1, W_2 are the universal covers of W_1', W_2' respectively. If $f\colon W_1 \to W_2$ is a (L, A)–quasi-isometry, then there exists a constant $C = C(L, A)$ such that for any top-dimensional flat $F \subset W_1$, there exists a top dimensional flat $F' \subset W_2$ with $d_H(f(F), F') < C$.*

This theorem follows from Lemma 5-2 and the following lemma.

Lemma 5-5 *Let W_1, W_2 and f be as in Theorem 5-4. If $f_\sharp\colon \mathcal{KQ}(W_1) \to \mathcal{KQ}(W_2)$ is the induced bijection in Theorem 5-1, then $|f_\sharp([A])| = |[A]|$ for any minimal essential element $[A] \in \mathcal{KQ}(W_1)$.*

In the rest of this section, we will prove Lemma 5-5.

We label the vertices and edges of W' by $\{\overline{v}_i\}_{i=1}^{N_v}$ and $\{\overline{e}_i\}_{i=1}^{N_e}$ such that: (1) different vertices have different labels; (2) two edges have the same label if and only if they are dual to the same hyperplane. We also choose an orientation for each edge such that if two edges are dual to the same hyperplane, then their orientations are consistent with parallelism (this is possible since each hyperplane is two-sided). All the labelings and orientations lift to the universal cover W. The edges in W dual to the same hyperplane also share the same label.

For every edge-path ω in W' or W, define $L(\omega)$ to be the word $\overline{v}_i \overline{e}_{i_1}^{\epsilon_{i_1}} \overline{e}_{i_2}^{\epsilon_{i_2}} \overline{e}_{i_3}^{\epsilon_{i_3}} \cdots$, where $\overline{v}_i$ is the label of the initial vertex of ω, $\overline{e}_{i_j}$ is the label of the j^{th} edge and $\epsilon_{i_j} = \pm 1$ records the orientation of the j^{th} edge.

Definition 5-3 and the way we label W' imply:

(1) For two edges e_1' and e_2' in W' dual to the same hyperplane, e_1' is embedded if and only if e_2' is embedded, ie its end points are distinct.

(2) Pick any vertex $v_i' \in W'$. Then two distinct edges e_1' and e_2' with $v_i' \in e_1' \cap e_2'$ have different labels.

(3) If ω_1' and ω_2' are two edge paths in W' such that $L(\omega_1') = L(\omega_2')$, then $\omega_1' = \omega_2'$. If ω_1 and ω_2 are two edge paths in W such that $L(\omega_1) = L(\omega_2)$, then there exists a unique deck transformation γ such that $\gamma(\omega_1) = \omega_2$.

We will be using the following simple observation repeatedly.

Lemma 5-6 *Pick vertices v_1 and v_2 in W which have the same label. For $i = 1, 2$, let $\{l_{ij}\}_{j=1}^{k}$ be a collection such that each l_{ij} is a geodesic ray, a geodesic segment or a complete geodesic that contains v_i. Suppose that:*

(1) *Each l_{ij} is a subcomplex of W.*

(2) *For each j, there is a graph isomorphism $\phi_j\colon l_{1j} \to l_{2j}$ which preserves the labels of vertices and edges and the orientations of edges, moreover $\phi_j(v_1) = v_2$.*

(3) *The convex hull of $\{l_{1j}\}_{j=1}^k$, which we denote by K_1, is a subcomplex isometric to $\prod_{j=1}^k l_{1j}$.*

Then the convex hull of $\{l_{2j}\}_{j=1}^k$, which we denote by K_2, is a subcomplex isometric to $\prod_{j=1}^k l_{2j}$. Moreover, let γ be the deck transformation such that $\gamma(v_1) = v_2$. Then $\gamma(K_1) = K_2$.

Let $\dim(W) = n$ and let O be a top-dimensional orthant subcomplex in W. We now construct a suitable doubling of O which will serve as a basic move to analyze the minimal essential elements in $\mathcal{KQ}(W)$.

Let $\{r_j\}_{i=1}^n$ be the geodesic rays emanating from the tip of O such that O is the convex hull of $\{r_j\}_{j=1}^n$. We parametrize r_1 by arc length. Since the labeling of W is finite, we can find a sequence of nonnegative integers $\{n_j\}_{j=1}^{\infty}$ with $n_j \to \infty$ such that the label and orientation of the incoming edge at $r_1(n_j)$, the label and orientation of the outgoing edge at $r_1(n_j)$ and the label of $r_1(n_j)$ do not depend on j.

We identify O with $[0,\infty)\times O'$, where O' is an $(n-1)$–dimensional orthant orthogonal to r_1. By our choice of $r_1(n_1)$ and $r_1(n_2)$, we can extend $\overline{r_1(n_2)r_1(n_1)}$ over $r_1(n_1)$ to reach a vertex v such that $L(\overline{r_1(n_1)v}) = L(\overline{r_1(n_2)r_1(n_1)})$. Here v does not need to lie on r_1; see Figure 1.

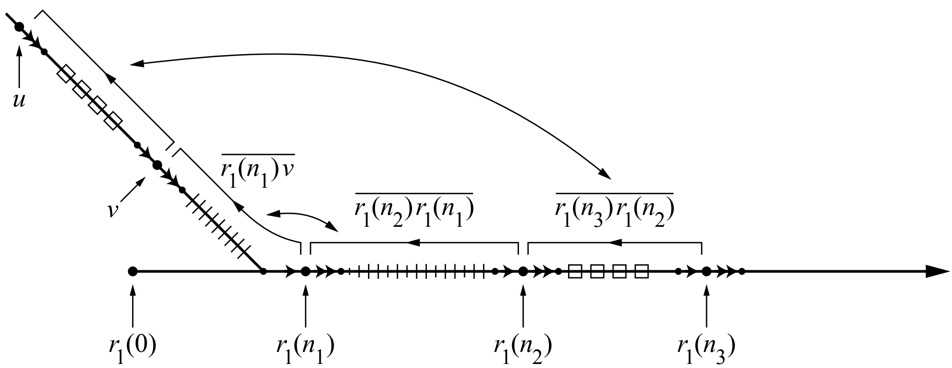

Figure 1

Let K_1 be the convex hull of $\{n_2\}\times O'$ and $\overline{r_1(n_2)r_1(n_1)}$. Then K_1 is of form $K_1 = [n_1,n_2]\times O'$. Note that the parallelism map between $\{n_1\}\times O'$ and $\{n_2\}\times O'$ preserves labeling and orientation of edges. Then it follows from Lemma 5-6 that the convex hull of $\{n_1\} \times O'$ and $\overline{r_1(n_1)v}$ is a subcomplex isometric to $\overline{r_1(n_1)v} \times O'$. (Actually if $\gamma \in \pi_1(W')$ is the deck transformation satisfying $\gamma(r_1(n_2)) = r_1(n_1)$, then $\gamma(K_1)$ is the convex hull of $\{n_1\} \times O'$ and $\overline{r_1(n_1)v}$.) We call this subcomplex the *mirror* of K_1 and denote it by K'_1. Since $K_1 \cap K'_1 = \{n_1\} \times O'$, it follows that $K'_1 \cup ([n_1, \infty) \times O')$ is again an orthant; see Figure 2.

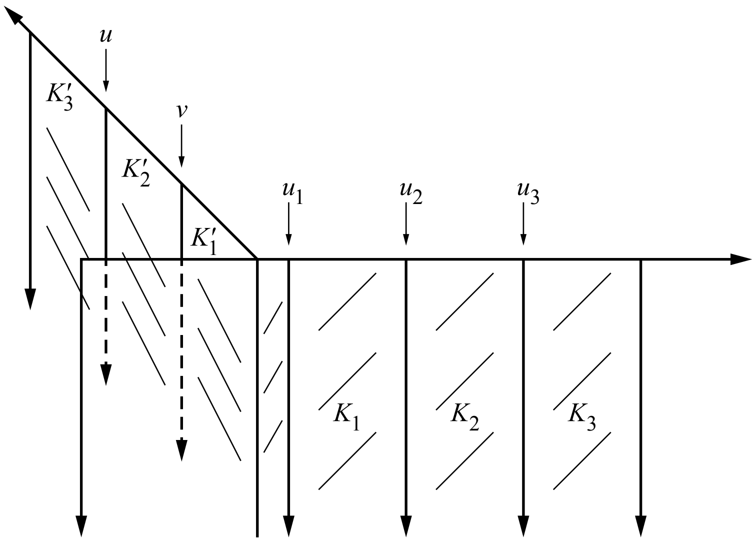

Figure 2

Let $K_2 = [n_2, n_3] \times O'$. We extend $\overline{r_1(n_1)v}$ over v to reach a vertex u such that $L(\overline{vu}) = L(\overline{r_1(n_3)r_1(n_2)})$. Note that the parallelism map between $\{v\} \times O'$ and $\{n_3\} \times O'$ preserves labeling and orientation of edges. Then it follows from Lemma 5-6 that the convex hull of $\{v\} \times O'$ and $\overline{vu}$ is a subcomplex isometric to K_2. (Actually if $\gamma \in \pi_1(W')$ is the deck transformation satisfying $\gamma(r_1(n_3)) = v$, then $\gamma(K_2)$ is the convex hull of $\{v\} \times O'$ and $\overline{vu}$.) This convex hull is called the mirror of K_2, and is denoted by K'_2. Since $\overline{vu}$ and $\overline{r_1(n_1)v}$ fit together to form a geodesic segment, $K'_1 \cap K'_2 = \{v\} \times O'$. Thus $K'_2 \cup K'_1 \cup ([n_1, \infty) \times O')$ is again an orthant. We can continue this process, and consecutively construct the mirror of $K_i = [n_i, n_{i+1}] \times O'$ in W (denoted K'_i) arranged in the pattern indicated in the above picture. Similarly one can verify that K'_i is isometric to K_i, and $K'_i \cap K'_{i+1}$ is isometric to O'.

Now we obtain a subcomplex $K = \left(\bigcup_{i=1}^{\infty} K_i\right) \cup \left(\bigcup_{i=1}^{\infty} K'_i\right)$. It is clear that $[O] \subset [K]$. The discussion in the previous paragraph implies that $\bigcup_{i=1}^{\infty} K'_i$ is also a top-dimensional orthant. We will call it the *mirror* of O. Moreover, K is isometric to $\mathbb{R} \times (\mathbb{R}_{\geq 0})^{n-1}$. More generally, by the same argument as above and Lemma 5-6, we have the following result.

Lemma 5-7 *If $K \subset W$ is a convex subcomplex isometric to $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$, then there exists a convex subcomplex K' isometric to $(\mathbb{R}_{\geq 0})^{k-1} \times \mathbb{R}^{n-k+1}$ such that $[K] \subset [K']$.*

Pick a minimal essential element $[A] \in \mathcal{KQ}(W)$. Then there exists a top-dimensional orthant O with $[O] \subset [A]$, where $[O]$ may not be an element in $\mathcal{KQ}(W)$. Using Lemma 5-7, we can double the orthant n times to get a top-dimensional flat F with $[O] \subset [F]$. Since $[A]$ is minimal, $[A] \cap [F] = [A]$, which implies the following result.

Corollary 5-8 *If $[A] \in \mathcal{KQ}(W)$ is a minimal essential element, then there exists a top-dimensional flat $[F]$ such that $[A] \subset [F]$. In particular, $\|[A]\| \leq 2^{\dim(W)} = 2^n$.*

Pick a top-dimensional orthant subcomplex O and denote the $(n-1)$–faces of O by $\{O_i\}_{i=1}^n$. We say that $[O_i]$ is *branched* if there exist top-dimensional orthant subcomplexes O' and O'' such that $[O]$, $[O']$ and $[O'']$ are distinct elements and $[O] \cap [O'] = [O] \cap [O''] = [O_i]$; otherwise $[O_i]$ is called *unbranched.*

Lemma 5-9 *If O and O_i are as above, then $[O_i]$ is branched if and only if there exists a suborthant $O_i' \subset O_i$ and geodesic rays l_1, l_2 and l_3 emanating from the tip of O_i' such that:*

1. *$[O_i'] = [O_i]$.*
2. *$[l_1]$, $[l_2]$ and $[l_3]$ are distinct.*
3. *The convex hull of l_j and O_i' is a top-dimensional orthant for $1 \leq j \leq 3$.*

Proof If O_i is branched, let O' and O'' be the orthant subcomplexes as above. We can assume $O' \cap O = O'' \cap O = \varnothing$. Let $(Y_1, Y_2) = \mathcal{I}(O, O')$. Since Y_1 and Y_2 bound a copy of $Y_1 \times [0, d(O, O')]$ inside W, we have $\dim(Y_1) = \dim(Y_2) \leq n-1$. However, (2-11) implies $[Y_1] = [O] \cap [O'] = [O_i]$, so Y_1 and Y_2 are $(n-1)$–dimensional orthant subcomplexes. We can find a copy of $Y_2 \times [0, \infty)$ inside O' and we claim $Y_1 \times [0, d(O, O')] \cup Y_2 \times [0, \infty)$ is also a top-dimensional orthant subcomplex.

To see this, note that $(Y_1 \times [0, d(O, O')]) \cap (Y_2 \times [0, \infty)) = Y_2$. Pick $y \in Y_2$, and let $\{v_i\}_{i=1}^{n-1}$ be mutually orthogonal directions in $\Sigma_y Y_2$. Moreover, we can assume each v_i is in the 0–skeleton of $\Sigma_y Y_2 \subset \Sigma_y W$. Let $v \in \Sigma_y(Y_1 \times [0, d(O, O')])$ be the direction corresponding to the $[0, d(O, O')]$ factor and let $v' \in \Sigma_y(Y_2 \times [0, \infty))$ be the direction corresponding to the $[0, \infty)$ factor. It is clear that v and v' are distinct points in the 0–skeleton of $\Sigma_y W$. If $d(v, v') = \frac{\pi}{2}$, then v, v' and $\{v_i\}_{i=1}^{n-1}$ would be mutually orthogonal directions, which yields a contradiction with the fact that $\dim(W) = n$.

Thus $d(v,v')=\pi$ and $Y_1\times[0,d(O,O')]\cup Y_2\times[0,\infty)$ is indeed a top-dimensional orthant subcomplex.

Note that the orthant constructed above is the convex hull of Y_1 and some geodesic ray l emanating from the tip of Y_1. We can repeat this argument for O'' to obtain the required suborthants and geodesic rays in the lemma. The other direction of the lemma is trivial. □

Let O, $\{r_j\}_{j=1}^n$, $\{n_i\}_{i=1}^\infty$, K_i and K_i' be as in the discussion before Lemma 5-7. Let $a_j=n_{j+1}-n_1$ for $j\ge 0$. We identify $\big(\bigcup_{i=1}^\infty K_i\big)\cup\big(\bigcup_{i=1}^\infty K_i'\big)$ with $\mathbb{R}\times\prod_{j=2}^n r_j$ such that $K_i=[a_{i-1},a_i]\times\prod_{j=2}^n r_j$. Thus $K_i'=[-a_i,-a_{i-1}]\times\prod_{j=2}^n r_j$. Let l be the unit-speed complete geodesic line in W such that $l(0)=r_1(n_1)$ and it is parallel to the $\mathbb{R}$ factor. For $x\in\mathbb{R}$, we denote the geodesic ray in $\big(\bigcup_{i=1}^\infty K_i\big)\cup\big(\bigcup_{i=1}^\infty K_i'\big)$ that starts at $l(x)$ and goes along the r_j factor by $\{x\}\times r_j$.

Let $\gamma_i\in\pi_1(W')$ be the deck transformation satisfying $\gamma_i(l(a_i))=l(-a_{i-1})$. Then by our construction, $\gamma_i(K_i)=K_i'$. Moreover, under the product decomposition $K_i=[a_{i-1},a_i]\times\prod_{j=2}^n r_j$ and $K_i'=[-a_i,-a_{i-1}]\times\prod_{j=2}^n r_j$, we have that γ_i maps $[a_{i-1},a_i]$ to $[-a_i,-a_{i-1}]$ and fixes the factor $\prod_{j=2}^n r_j$ pointwise.

Let $\widetilde{O}=\bigcup_{i=1}^\infty K_i'$ be the mirror of O. There is an isometry ρ acting on

$$\Big(\bigcup_{i=1}^{\infty}K_i\Big)\cup\Big(\bigcup_{i=1}^{\infty}K_i'\Big)=\mathbb{R}\times\prod_{j=2}^{n}r_j$$

by flipping the $\mathbb{R}$ factor (the other factors are fixed). For $1\le j\le n$, let O_j be the $(n-1)$–face of O which is orthogonal to r_j and let $\widetilde{O}_j$ be the $(n-1)$–face of $\widetilde{O}$ such that $[\rho(\widetilde{O}_j)]=[O_j]$. (Recall that $[O]=\big[\bigcup_{i=1}^\infty K_i\big]$.)

Lemma 5-10 *$[O_j]$ is branched if and only if $[\widetilde{O}_j]$ is branched.*

Proof If $j=1$, then $[O_1]=[\widetilde{O}_1]=[O]\cap[\widetilde{O}]$ and the lemma is trivial, so let us assume $j\ne 1$. If $[O_j]$ is branched, then by Lemma 5-9, we can assume without loss of generality (one might need to modify K_i and K_i' by cutting off suitable pieces and replace l by a geodesic in $\mathbb{R}\times\prod_{j=2}^n r_j$ which is parallel to l) that there exist $i_0\ge 0$ and geodesic rays c_1, c_2, c_3 emanating from $l(a_{i_0})$ such that $[c_1]$, $[c_2]$, $[c_3]$ are distinct elements and the convex hull of c_m, $l([a_{i_0},\infty))$ and $\{a_{i_0}\}\times r_k$ (for $k\ne 1,j$), which we denote by H_m, is a top-dimensional orthant subcomplex for $1\le m\le 3$.

Let γ be the deck transformation satisfying $\gamma(l(a_{i_0}))=l(-a_{i_0})$. Such a γ exists by the construction of l (in the previous paragraph, we might possibly replace the original l by a geodesic parallel to l, however the same γ works). Let $\widetilde{c}_m=\gamma(c_m)$

for $1 \le m \le 3$. Then $[\widetilde{c}_1]$, $[\widetilde{c}_2]$, $[\widetilde{c}_3]$ are distinct since γ is an isometry. Since γ is label and orientation preserving, c_m and $\widetilde{c}_m$ correspond to the same word for $1 \le m \le 3$, moreover $\gamma(\{a_i\} \times r_k) = \{-a_i\} \times r_k$ for $k \ne 1$. To prove $[\widetilde{O}_j]$ is branched, it suffices to show the convex hull of $\widetilde{c}_m$, $l((-\infty, -a_{i_0}])$ and $\{-a_{i_0}\} \times r_k$ (for $k \ne 1, j$) is a top-dimensional orthant subcomplex.

For $m = 1$, we chop H_1 into pieces so that

$$H_1 = \bigcup_{i=i_0+1}^{\infty} L_i, \quad \text{where} \quad L_i = c_1 \times l([a_{i-1}, a_i]) \times \prod_{k \ne 1, j} r_k.$$

Let γ_i be the deck transformation defined before Lemma 5-10 and let $L'_i = \gamma(L_i)$. We claim that

$$\gamma_i\Bigl(c_1 \times \{a_{i-1}\} \times \prod_{k \ne 1, j} r_k\Bigr) = \gamma_{i+1}\Bigl(c_1 \times \{a_{i+1}\} \times \prod_{k \ne 1, j} r_k\Bigr)$$

for $i \ge i_0 + 1$. This claim follows from the following two observations: (1) both sides of the equality contain $l(-a_i)$; (2) γ_i, γ_{i+1} and the parallelism between $c_1 \times \{a_{i-1}\} \times \prod_{k \ne 1, j} r_k$ and $c_1 \times \{a_{i+1}\} \times \prod_{k \ne 1, j} r_k$ preserve labeling and orientation of edges. It follows from the claim that $H'_1 = \bigcup_{i=i_0+1}^{\infty} L'_i$ is a top-dimensional orthant subcomplex. By a similar argument as before, we know

$$\gamma\Bigl(c_1 \times \{a_i\} \times \prod_{k \ne 1, j} r_k\Bigr) = \gamma_{i_0+1}\Bigl(c_1 \times \{a_{i_0+1}\} \times \prod_{k \ne 1, j} r_k\Bigr),$$

thus H'_1 is the convex hull of $\widetilde{c}_1$, $l((-\infty, -a_i])$ and $\{-a_i\} \times r_k$, for $k \ne 1, j$. Moreover, $[H'_1] \cap [\widetilde{O}] = [\widetilde{O}_j]$. We can repeat this construction for $\widetilde{c}_2$ and $\widetilde{c}_3$, which implies $[\widetilde{O}_j]$ is branched. By the same argument, if $[\widetilde{O}_j]$ is branched, we can prove $[O_j]$ is branched. □

Remark 5-11 It is important that we keep track of information from the labels of O while constructing the mirror of O; in other words, if we construct $\widetilde{O}$ by the pattern indicated in Figure 3, we will not be able to conclude that $[O_j]$ is branched from the fact that $[\widetilde{O}_j]$ is branched.

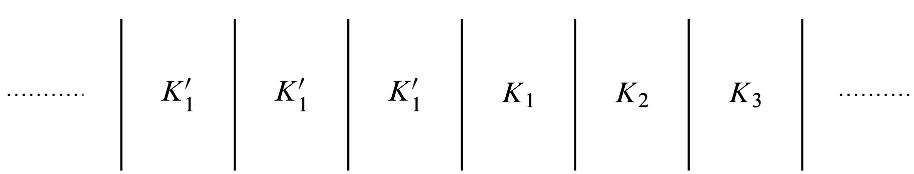

Figure 3

Lemma 5-12 *If $[A] \in \mathcal{KQ}(W)$ is a minimal essential element, then:*

(1) $\|[A]\| = 2^i$ *for some integer i with $1 \le i \le n$.*

(2) *There exists a top-dimensional flat F and another $2^{n-i}-1$ minimal essential elements $\{A_j\}_{j=1}^{2^{n-i}-1}$ with $\|[A_j]\| = \|[A]\|$ such that $[F] = [A] \cup \big(\bigcup_{j=1}^{2^{n-i}-1}[A_j]\big)$.*

Proof We find a top-dimensional orthant subcomplex O such that $[O] \subset [A]$. By the argument before Lemma 5-7, we can double this orthant n times to a get top-dimensional flat F such that $[O] \subset [F]$. Assume without loss of generality that $O \subset F$. Denote by $\{O_i\}_{i=1}^n$ the $(n-1)$–faces of O and let $\rho_i\colon F \to F$ be the isometry that fixes O_i pointwise and flips the direction orthogonal to O_i.

Let G be the group generated by $\{\rho_i\}_{i=1}^n$. Then $G \cong (\mathbb{Z}/2)^n$. We define

$$\Lambda_b = \{1 \le i \le n \mid [O_i] \text{ is branched}\}, \quad \Lambda_u = \{1 \le i \le n \mid [O_i] \text{ is unbranched}\}.$$

Let G_b be the subgroup generated by $\{\rho_i\}_{i\in\Lambda_b}$ and let G_u be the subgroup generated by $\{\rho_i\}_{i\in\Lambda_u}$. We denote by G_i the subgroup generated by $\{\rho_1 \cdots \rho_{i-1}, \rho_{i+1} \cdots \rho_n\}$.

Claim 1 For any $\gamma \in G$, $[O_i]$ is branched if and only if $[\gamma(O_i)]$ is branched.

Proof Writing $\gamma = \rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}$, we prove it by induction on k. The case $k = 0$ is trivial. In general, suppose $[O_i]$ is branched if and only if $[\rho_{i_2}\cdots\rho_{i_k}(O_i)]$ is branched. It follows from the way we construct F that $[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}(O)]$ is the mirror of $[\rho_{i_2}\cdots\rho_{i_k}(O)]$. So by Lemma 5-10, $[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}(O_i)]$ is branched if and only if $[\rho_{i_2}\cdots\rho_{i_k}(O_i)]$ is branched, thus the claim is true. □

Claim 2
$$[A] \subset \Big[\bigcup_{\gamma\in G_u} \gamma(O)\Big].$$

Proof If $[O_i]$ is branched, by Lemma 5-9 there exists a subcomplex M_i isometric to $(\mathbb{R}_{\ge 0})^{n-1} \times \mathbb{R}$ such that $[M_i] \cap [F] = [O]$. By Lemma 5-7 we can find a top-dimensional flat F_i such that $[M_i] \subset [F_i]$. Since $F_i \cap F \ne \varnothing$, by Lemma 2-10 $[F \cap F_i] = [F] \cap [F_i]$. Note that $F \cap F_i$ is a convex subcomplex of F with $\|[F \cap F_i] \cap [\rho_i(O)]\| = 0$, so $[F \cap F_i] \subset \big[\bigcup_{\gamma\in G_i}\gamma(O)\big]$. Recall that $[A]$ is a minimal essential element, so

$$[A] \subset [F] \cap \Big(\bigcap_{i\in\Lambda_b}[F_i]\Big) = \bigcap_{i\in\Lambda_b}([F_i]\cap[F]) \subset \bigcap_{i\in\Lambda_b}\Big[\bigcup_{\gamma\in G_i}\gamma(O)\Big] = \Big[\bigcup_{\gamma\in G_u}\gamma(O)\Big]. \quad \square$$

Claim 3
$$\Big[\bigcup_{\gamma\in G_u}\gamma(O)\Big] \subset [A].$$

Proof First we need the following observation. Let $[P_1]$ and $[P_2]$ be two different top-dimensional orthant complexes. Suppose each $[Q] \in \mathcal{Q}(X)$ satisfies the property that either $[P_1] \subset [Q]$ and $[P_2] \subset [Q]$, or $[P_1] \not\subseteq [Q]$ and $[P_2] \not\subseteq [Q]$. Then this property is also true for each element in $\mathcal{KQ}(X)$. To see this, let $\mathcal{A}_{P_1,P_2}(X)$ be the collection of elements in $\mathcal{A}(X)$ which satisfy this property. Then one readily verifies that $\mathcal{A}_{P_1,P_2}(X)$ is closed under union, intersection and subtraction. Moreover, $\mathcal{Q}(X) \subset \mathcal{A}_{P_1,P_2}(X)$. Thus $\mathcal{KQ}(X) \subset \mathcal{A}_{P_1,P_2}(X)$.

Pick an unbranched face $[O_i]$. By Lemma 4-16, Equation (4-34) and Remark 4-36, for every top-dimensional quasiflat Q with $[O] \subset [Q]$, there exists another top-dimensional orthant complex O' such that $[O'] \subset [Q]$ and $\partial_T O' \cap \partial_T O = \partial_T O_i$. This together with Lemma 2-10 (see also Remark 2-13) imply $[O] \cap [O'] = [O_i]$, thus $[O'] = [\rho_i(O)]$ and $[\rho_i(O)] \subset [Q]$ (recall that $[O_i]$ is unbranched). Similarly, one can prove if $[\rho_i(O)] \subset [Q]$ for a top-dimensional quasiflat Q, then $[O] \subset [Q]$. It follows from the above observation that $[\rho_i(O)] \subset [A]$ for $i \in \Lambda_u$.

Let $\gamma \in G_u$. Write $\gamma = \rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}$ with $i_j \in \Lambda_u$ for $1 \le j \le n$. We will prove Claim 3 by induction on k. The case $k = 1$ is already done by the previous paragraph. In general, we assume $[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{k-1}}(O)] \subset [A]$. Note that $[O] \cap [\rho_{i_k}(O)] = [O_{i_k}]$, where $[O_{i_k}]$ is unbranched, so

$$[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{k-1}}(O)] \cap [\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}(O)] = [\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{k-1}}(O_{i_k})].$$

Claim 1 implies $[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_{k-1}}(O_{i_k})]$ is also unbranched, so $[\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}(O)] \subset [A]$ by the same argument as in the previous paragraph. □

Claim 2 and Claim 3 imply $\left[\bigcup_{\gamma\in G_u}\gamma(O)\right] = [A]$. So $\|[A]\| = |G_u|$, where $|G_u|$ is the order of G_u. Now the first assertion of the lemma follows. Moreover, for any $\gamma \in G$, let $[A_\gamma] \in \mathcal{KQ}(W)$ be the unique minimal essential element such that $[\gamma(O)] \subset [A_\gamma]$. Claim 1 implies $\{[\gamma(O_i)]\}_{i\in\Lambda_b}$ and $\{[\gamma(O_i)]\}_{i\in\Lambda_u}$ are the branched faces and unbranched faces of $[\gamma(O)]$ respectively. By the same argument as in Claim 2 and Claim 3, we can show $[A_\gamma] = \left[\bigcup_{\gamma'\in\gamma G_u}\gamma'(O)\right]$, where γG_u denotes the corresponding coset of G_u. Since there are $|G|/|G_u|$ cosets of G_u, the second assertion of the lemma also follows. □

Proof of Lemma 5-5 If $\|[A]\| = 2^n$, by Corollary 5-8 we know there exists a top-dimensional flat F such that $[A] \subset [F]$, so actually $[A] = [F]$. Then $f(A)$ is a top-dimensional quasiflat, thus $|f_\sharp([A])| \ge 2^n$. However, $f_\sharp([A])$ is also minimal essential, so by Corollary 5-8 we actually have $|f_\sharp([A])| = 2^n = \|[A]\|$. Let g be a quasi-isometry inverse of f. If $[A'] \in \mathcal{KQ}(W_2)$ is a minimal essential element, then by the same argument, we know that $\|[A']\| = 2^n$ implies $|g_\sharp([A'])| = 2^n = \|[A']\|$. So $\|[A]\| = 2^n$ if and only if $|f_\sharp([A])| = 2^n$ for minimal essential element $[A] \in \mathcal{KQ}(W_1)$.

In general, we assume inductively that $\|[A]\| = k$ if and only if $|f_\sharp([A])| = k$ for any $k \geq 2^{n-i+1}$ and any minimal essential element $[A] \in \mathcal{KQ}(W_1)$ (we are doing induction on i). If $[B_1] \in \mathcal{KQ}(W_1)$ is a minimal essential element with $\|[B_1]\| = 2^{n-i}$, then by Lemma 5-12, we can find a top-dimensional flat F and another $2^i - 1$ minimal essential elements $\{[B_j]\}_{j=2}^{2^i}$ such that $\|[B_j]\| = \|[B_1]\|$ and

(5-13) $$[F] = \bigcup_{j=1}^{2^i} [B_j].$$

Since $f(F)$ is a top-dimensional flat, we have

(5-14) $$|f_\sharp(F)| = \left| f_\sharp\left(\bigcup_{j=1}^{2^i} [B_j] \right) \right| = \left| \bigcup_{j=1}^{2^i} f_\sharp([B_j]) \right| = \sum_{j=1}^{2^i} |f_\sharp([B_j])| \geq 2^n.$$

But our induction assumption implies

(5-15) $$|f_\sharp([B_j])| < 2^{n-i+1}.$$

Since $f_\sharp([B_j])$ is minimal essential element for each j, Equation (5-15) together with assertion (1) of Lemma 5-12 imply

(5-16) $$|f_\sharp([B_j])| \leq 2^{n-i}.$$

Now (5-14) and (5-16) imply

(5-17) $$\|[B_j]\| = |f_\sharp([B_j])| = 2^{n-i}$$

for all j. By considering the quasi-isometry inverse, we know $\|[B]\| = 2^{n-i}$ if and only if $|f_\sharp([B])| = 2^{n-i}$ for minimal essential element $[B] \in \mathcal{KQ}(W_1)$. By Lemma 5-12(1) and our induction assumption, we have actually proved that $\|[B]\| = k$ if and only if $|f_\sharp([B])| = k$ for any $k \geq 2^{n-i}$ and any minimal essential element $[B] \in \mathcal{KQ}(W_1)$. □

5.3 Application to right-angled Coxeter groups and Artin groups

5.3.1 The right-angled Coxeter group case For a finite simplicial graph Γ with vertex set $\{v_i\}_{i \in I}$, there is an associated right-angled Coxeter group (RACG), denoted by $C(\Gamma)$, with the following presentation:

$$\langle \{v_i\}_{i \in I} \mid v_i^2 = 1 \text{ for all } i;\ [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are joined by an edge} \rangle.$$

The group $C(\Gamma)$ has a nice geometric model $D(\Gamma)$, called the *Davis complex*. The 1–skeleton of $D(\Gamma)$ is the Cayley graph of $C(\Gamma)$ with edges corresponding to v_i, v_i^{-1} identified. For $n \geq 2$, the n–skeleton $D^{(n)}(\Gamma)$ of $D(\Gamma)$ is obtained from $D^{(n-1)}(\Gamma)$ by attaching an n–cube whenever one finds a copy of the $(n-1)$–skeleton of an n–cube inside $D^{(n-1)}(\Gamma)$. This process will terminate after finitely many steps and one obtains a CAT(0) cube complex where $C(\Gamma)$ acts properly and cocompactly.

The action of $C(\Gamma)$ on $D(\Gamma)$ is not free; however, $D(\Gamma)$ can be realized as the universal cover of a compact cube complex. The following construction is from [15]. Let $\{e_i\}_{i\in I}$ be the standard basis of $\mathbb{R}^I$ and let $\square^I = [0,1]^I \subset \mathbb{R}^I$ be the unit cube with the standard cubical structure. Let $F(\Gamma)$ be the flag complex of Γ. For each simplex $\Delta \subset F(\Gamma)$, let $\mathbb{R}^\Delta$ be the linear subspace spanned by $\{e_i\}_{v_i\in\Delta}$. Define

$$K(\Gamma) = \bigcup_{\Delta} \{\text{faces of } \square^I \text{ parallel to } \mathbb{R}^\Delta\},$$

where Δ varies among all simplices in $F(\Gamma)$. Then the Davis complex $D(\Gamma)$ is exactly the universal cover of $K(\Gamma)$; see [15, Proposition 3.2.3].

One can verify that $K(\Gamma)$ is weakly special. In order to apply Theorem 5-4 in a nontrivial way, we need the following extra condition:

$(*)$ There is an embedded top-dimensional hyperoctahedron in $F(\Gamma)$.

One can check there exists a top-dimensional flat in $D(\Gamma)$ if and only if $(*)$ is true.

Corollary 5-18 *Let Γ_1 and Γ_2 be two finite simplicial graphs satisfying $(*)$. If $\phi\colon D(\Gamma_1) \to D(\Gamma_2)$ is an (L,A)–quasi-isometry, then $\dim(D(\Gamma_1)) = \dim(D(\Gamma_2))$. Moreover there is a constant $D = D(L,A)$ such that for any top-dimensional flat F_1 in $D(\Gamma_1)$, we can find a flat F_2 in $D(\Gamma_2)$ such that*

$$d_H(\phi(F_1), F_2) < D.$$

Proof It suffices to show that $\dim(D(\Gamma_1)) = \dim(D(\Gamma_2))$. The rest follows from Theorem 5-4 and the above discussion.

We can assume the quasi-isometry ϕ is defined on the 0–skeleton of $D(\Gamma_1)$. Since $D(\Gamma_2)$ is CAT(0), we can extend ϕ skeleton by skeleton to obtain a continuous quasi-isometry. Similarly, we assume the quasi-isometry inverse ϕ' is also continuous. Since ϕ and ϕ' are proper, there are induced homomorphisms for the proper homology $\phi_*\colon H_*^{\mathrm{p}}(D(\Gamma_1)) \to H_*^{\mathrm{p}}(D(\Gamma_2))$ and $\phi'_*\colon H_*^{\mathrm{p}}(D(\Gamma_2)) \to H_*^{\mathrm{p}}(D(\Gamma_1))$; see Section 3.1. Note that the geodesic homotopy between $\phi' \circ \phi$ (or $\phi \circ \phi'$) and the identity map is proper, so $\phi_* \circ \phi'_* = \mathrm{Id}$ and $\phi'_* \circ \phi_* = \mathrm{Id}$. Hence ϕ_* is an isomorphism.

By symmetry, it suffices to show $\dim(D(\Gamma_1)) \geq \dim(D(\Gamma_2))$. If $\dim(D(\Gamma_1)) < \dim(D(\Gamma_2))$, then $H_n^{\mathrm{p}}(D(\Gamma_1))$ is trivial ($n = \dim(D(\Gamma_2))$) since there are no n–dimensional cells in $D(\Gamma_1)$. On the other hand, $(*)$ implies there is a top-dimensional flat in $D(\Gamma_2)$, thus $H_n^{\mathrm{p}}(D(\Gamma_2))$ is nontrivial, which yields a contradiction. □

5.3.2 The right-angled Artin group case Recall that for every simplicial graph Γ, there is a corresponding RAAG $G(\Gamma)$. Suppose $\bar{X}(\Gamma)$ is the Salvetti complex of $G(\Gamma)$. Then the 1–cells and 2–cells of $\bar{X}(\Gamma)$ are in 1–1 correspondence with the vertices and edges in Γ receptively. The closure of each k–cell in $\bar{X}(\Gamma)$ is a k–torus, which we call a *standard k–torus*. One can verify that the Salvetti complex $\bar{X}(\Gamma)$ is a weakly special cube complex.

We label the vertices of Γ by distinct letters (they correspond to the generators of $G(\Gamma)$), which induces a labeling of the edges of the Salvetti complex. We choose an orientation for each edge in the Salvetti complex and this gives us a directed labeling of the edges in $X(\Gamma)$. If we specify some base point $v \in X(\Gamma)$ (here v is a vertex), then there is a 1–1 correspondence between words in $G(\Gamma)$ and edge paths in $X(\Gamma)$ which start at v.

A subgraph $\Gamma' \subset \Gamma$ is a *full subgraph* if there does not exist an edge $e \subset \Gamma$ such that the two endpoints of e belong to Γ' but $e \not\subseteq \Gamma'$. In this case, there is an embedding $\bar{X}(\Gamma') \hookrightarrow \bar{X}(\Gamma)$ which is locally isometric. If $p\colon X(\Gamma) \to \bar{X}(\Gamma)$ is the universal cover, then each connected component of $p^{-1}(\bar{X}(\Gamma'))$ is a convex subcomplex isometric to $X(\Gamma')$. Following [8], we call these components *standard subcomplexes associated with* Γ'. Note that there is a 1–1 correspondence between standard subcomplexes associated with Γ' and left cosets of $G(\Gamma')$ in $G(\Gamma)$. A *standard k–flat* is the standard complex associated with a complete subgraph of k vertices. When $k = 1$, we also call it a *standard geodesic*.

Given a subcomplex $K \subset X(\Gamma)$, we denote the collection of labels of edges in K by $\mathrm{label}(K)$ and the corresponding collection of vertices in Γ by $V(K)$.

Let $V \subset \Gamma$ be a set of vertices. We define the *orthogonal complement* of V, denoted by $V^{\perp}$, to be the set $\{w \in \Gamma \mid d(w, v) = 1 \text{ for any } v \in V\}$.

The following theorem follows from Theorem 5-4.

Theorem 5-19 *Let Γ_1, Γ_2 be finite simplicial graphs, and let $\phi\colon X(\Gamma_1) \to X(\Gamma_2)$ be an (L, A)–quasi-isometry. Then $\dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$. Moreover there is a constant $D = D(L, A)$ such that for any top-dimensional flat F_1 in $X(\Gamma_1)$, we can find a flat F_2 in $X(\Gamma_2)$ such that*

$$d_H(\phi(F_1), F_2) < D.$$

One can argue as in Corollary 5-18 or using the invariance of cohomological dimension to show $\dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$.

Using Theorem 5-19, we can set up some immediate quasi-isometry invariants for RAAGs. Let $F(\Gamma)$ be the flag complex of Γ. We will assume $n = \dim(F(\Gamma))$ in the following discussion, then $\dim(X(\Gamma)) = n + 1$.

We construct a family of new graphs $\{\mathcal{G}_d(\Gamma)\}_{d=1}^n$, where the vertices of $\mathcal{G}_d(\Gamma)$ are in 1–1 correspondence with the top-dimensional flats in $X(\Gamma)$, and two vertices v_1 and v_2 are joined by an edge if and only if the associated flats F_1 and F_2 satisfy the condition that there exists an $r > 0$ such that $N_r(F_1) \cap N_r(F_2)$ contains a flat of dimension d. Let $\mathcal{G}_d^s(\Gamma)$ be the full subgraph of $\mathcal{G}_d(\Gamma)$ spanned by those vertices representing standard flats of top dimension.

Lemma 2-10 and Theorem 5-19 yield the following result.

Corollary 5-20 *Given a pair of finite simplicial graphs Γ_1, Γ_2 and a quasi-isometry $q\colon X(\Gamma_1) \to X(\Gamma_2)$, there is an induced graph isomorphism $q_*\colon \mathcal{G}_d(\Gamma_1) \to \mathcal{G}_d(\Gamma_2)$ for $1 \le d \le \dim(F(\Gamma_1)) = \dim(F(\Gamma_2))$.*

The relation between $\mathcal{G}_d(\Gamma)$ and Γ is complicated, but several basic properties of $\mathcal{G}_d(\Gamma)$ can be directly read from Γ. We first investigate the connectivity of $\mathcal{G}_d(\Gamma)$.

Lemma 5-21 *Suppose $1 \le d \le n$. Then $\mathcal{G}_d(\Gamma)$ is connected if and only if $\mathcal{G}_d^s(\Gamma)$ is connected.*

Proof For the $\Leftarrow$ direction, it suffices to show every point $v \in \mathcal{G}_d(\Gamma)$ is connected to some point in $\mathcal{G}_d^s(\Gamma)$. Let F_v be the associated top-dimensional flats. Pick a vertex $x \in F_v$ and suppose $\{e_i\}_{i=1}^{n+1}$ are mutually orthogonal edges in F_v emanating from x. Let $e_1^\perp$ be the subspace of F_v orthogonal to e_1 and let l_i be the unique standard geodesic such that $e_i \subset l_i$. Then by Lemma 5-6, the convex hull of l_1 and $e_1^\perp$ is a top-dimensional flat $F_{v,1}$. By construction, $F_{v,1}$ is adjacent to F_v in $\mathcal{G}_d(\Gamma)$. Now we can replace F_v by $F_{v,1}$, and run the same argument with respect to l_2. After finitely many steps, we will arrive at a standard flat F which is the convex hull of $\{l_i\}_{i=1}^{n+1}$, moreover F is connected to F_v in $\mathcal{G}_d(\Gamma)$. Note that F only depends on the choice of base vertex x and the frame $\{e_i\}_{i=1}^{n+1}$ at x. So we also denote F by $F = F_v(x, e_1, \dots, e_{n+1})$.

Now we prove the other direction. Pick a different base point $x' \in F_v$ and frame $\{e_i'\}_{i=1}^{n+1}$ at x'. We claim $F_v(x, e_1, \dots, e_{n+1})$ and $F_v(x', e_1', \dots, e_{n+1}')$ are connected in $\mathcal{G}_d^s(\Gamma)$. Note that:

(1) If $x' = x$, $e'_i = e_i$ for $2 \le i \le n+1$ and $e'_1 = -e_1$, then F and F' are adjacent inside $\mathcal{G}^s_d(\Gamma)$.

(2) If x' is the other end point of e_1, $e'_1 = \overrightarrow{x'x}$ and e'_i is parallel to e_i for $2 \le i \le n+1$, then $F = F'$.

In general, we can connect x and x' by an edge path $\omega \subset F_v$ and use the previous two properties to induct on the combinatorial length of ω.

Let $\{F_i\}_{i=1}^m$ be a chain of top-dimensional flats representing an edge path in $\mathcal{G}_d(\Gamma)$ such that F_1 and F_m are standard flats. Pick i, let $(Y, Y') = \mathcal{I}(F_i, F_{i+1})$ and let $\phi\colon Y \to Y'$ be the isometry induced by CAT(0) projection as in Lemma 2-10(2). Since Y contains a d–dimensional flat, for vertex $x \in Y$ there are d mutually orthogonal edges $\{e_i\}_{i=1}^d$ such that $x \in e_i \subset Y$. Let $x' = \phi(x)$ and let $e'_i = \phi(e_i)$. We add more edges such that $\{e_i\}_{i=1}^{n+1}$ and $\{e'_i\}_{i=1}^{n+1}$ become bases for F_i and F_{i+1} respectively. Let $F_{i,i} = F_i(x, e_1, \dots, e_{n+1})$ and $F_{i+1,i} = F_{i+1}(x', e'_1, \dots, e'_{n+1})$. By Lemma 2-14, $F_{i,i}$ and $F_{i+1,i}$ are adjacent in $\mathcal{G}^s_d(\Gamma)$ for $1 \le i \le m-1$. Moreover, for $2 \le i \le m-1$, $F_{i,i}$ and $F_{i,i-1}$ are connected by a path inside $\mathcal{G}^s_d(\Gamma)$ by the previous claim. Thus F_1 and F_m are connected inside $\mathcal{G}^s_d(\Gamma)$. □

Recall that the notion of k–gallery is defined in Definition 1-5.

Lemma 5-22 *$\mathcal{G}^s_d(\Gamma)$ is connected if and only if Γ satisfies the following conditions:*

(1) *For any vertex $v \in F(\Gamma)$, there is a top-dimensional simplex $\Delta \subset F(\Gamma)$ such that $\Delta \cap v^\perp$ contains at least d vertices.*

(2) *Any two top-dimensional simplices Δ_1 and Δ_2 in $F(\Gamma)$ are connected by a $(d-1)$–gallery.*

Proof For the only if part, pick vertex $x \in X(\Gamma)$ and let $\Gamma_{d,x}$ be the full subgraph of $\mathcal{G}^s_d(\Gamma)$ generated by those vertices representing top-dimensional standard flats containing x. Then there is a canonical surjective simplicial map $\phi\colon \mathcal{G}^s_d(\Gamma) \to \Gamma_{d,x}$ by sending any top-dimensional standard flat F to the unique standard flat F' with $x \in F'$ and $\mathrm{label}(F) = \mathrm{label}(F')$. Since $\mathcal{G}^s_d(\Gamma)$ is connected, $\Gamma_{d,x}$ is also connected and (2) is true.

To see (1), suppose there exists a vertex $v \in F(\Gamma)$ such that for any top-dimensional simplex $\Delta \subset F(\Gamma)$, $\Delta \cap v^\perp$ contains less than d vertices. Pick a vertex $x_1 \in X(\Gamma)$. If $e \subset X(\Gamma)$ is the unique edge such that $V(e) = v$ and $x_1 \in e$, then by our assumption, e is not contained in any top-dimensional standard flat. This is also true for any edge parallel to e. Let h be the hyperplane dual to e. Suppose x_2 is the other endpoint of e.

For $i=1,2$, let F_i be the top-dimensional standard flat such that $x_i \in F_i$. Then F_1 and F_2 are separated by h. Since F_1 and F_2 are joined by a chain of top-dimensional standard flats such that each flat in the chain has trivial intersection with h (otherwise some edge parallel to e would be contained in a top-dimensional standard flat), we can find F_1' and F_2' in this chain which are adjacent in $\mathcal{G}^s_d(\Gamma)$ and separated by h. Let $(Y_1, Y_2) = \mathcal{I}(F_1', F_2')$. Then for a vertex $y \in Y_1$, there are d mutually orthogonal edges $\{e_i\}_{i=1}^d$ such that $y \in e_i \subset Y_1$. Let h_i be the hyperplane dual to e_i. Then $h_i \cap h \neq \varnothing$ for $1 \le i \le d$ by Lemma 2-14, hence in Γ we have $d(V(e_i), V(e)) = d(V(e_i), v) = 1$ for $1 \le i \le d$, which yields a contradiction.

For the other direction, note that (2) implies that $\Gamma_{d,x}$ is connected for any vertex $x \in X(\Gamma)$ and (1) implies that for adjacent vertices $x_1, x_2 \in X(\Gamma)$, there exist $v_i \in \Gamma_{d,x_i}$ for $i=1,2$ such that v_1 and v_2 are either adjacent or identical in $\mathcal{G}^s_d(\Gamma)$, thus $\mathcal{G}^s_d(\Gamma)$ is connected. □

The next result follows from Corollary 5-20, Lemma 5-21 and Lemma 5-22.

Theorem 5-23 *Given $G(\Gamma_1)$ and $G(\Gamma_2)$ which are quasi-isometric to each other, for $1 \le d \le \dim(F(\Gamma_1))$, the graph Γ_1 satisfies conditions (1) and (2) in Lemma 5-22 if and only if Γ_2 also satisfies these conditions.*

Now we turn to the diameter of $\mathcal{G}_1(\Gamma)$.

If Γ admits a nontrivial join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2$, then $\operatorname{diam}(\mathcal{G}_1(\Gamma)) \le 2$. To see this, take two arbitrary top-dimensional flats F_1 and F_2 in $X(\Gamma)$, then $F_i = A_i \times B_i$, where A_i and B_i are top-dimensional flats in $X(\Gamma_1)$ and $X(\Gamma_2)$ respectively for $i=1,2$; see [30, Lemma 2.3.8]. Let $F = A_1 \times B_2$. Then $\operatorname{diam}(N_r(F_i) \cap N_r(F)) = \infty$ for some $r>0$ and $i=1,2$, thus $\operatorname{diam}(\mathcal{G}_1(\Gamma)) \le 2$. Our next goal is to prove the following converse.

Lemma 5-24 *If $\operatorname{diam}(\mathcal{G}_1(\Gamma)) \le 2$ and if Γ is not one point, then Γ admits a nontrivial join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2$.*

In the first part of the following proof, we will use the argument in [14, Section 4.1].

Proof Following [14, Section 4.2], let Γ^c be the complement graph of Γ. So Γ_c and Γ have the same vertex set, and two vertices are adjacent in Γ_c if and only if they are not adjacent in Γ. It suffices to show Γ_c is disconnected.

We argue by contradiction and suppose Γ_c is connected. Pick a top-dimensional simplex Δ in the flag complex $F(\Gamma)$ of Γ, where we identify Γ with the 1–skeleton of $F(\Gamma)$. Note that Δ corresponds to a top-dimensional standard torus T_Δ in the Salvetti complex. For any vertex $x \in X(\Gamma)$, we denote the unique standard flat in $X(\Gamma)$ which contains x and covers T_Δ by $F_{\Delta,x}$.

If Γ does not contain vertices other than those in Δ, then we are done; otherwise we can find a vertex $\widetilde{v} \in \Gamma$ with

$$\widetilde{v} \notin \Delta. \tag{5-25}$$

Let ω be an edge path of Γ^c which starts at $\widetilde{v}$, ends at $\widetilde{v}$ and travels through every vertex in Γ^c. By recording the labels of consecutive vertices in ω, we obtain a word W. Let W' be the concatenation of eight copies of W.

Pick a vertex $x_1 \in X(\Gamma)$ and let l be the edge path which starts at x_1 and corresponds to the word W'. Let x_2 be the other endpoint of l. Note that l is actually a geodesic segment by our construction of W'. For $i = 1, 2$, let $F_i = F_{\Delta,x_i}$ and let w_i be the vertex in $\mathcal{G}_1(\Gamma)$ corresponding to F_i. We claim $d(w_1, w_2) > 2$.

If $d(w_1, w_2) \le 2$, then there exists a top-dimensional flat F such that

$$\operatorname{diam}(N_r(F_i) \cap N_r(F)) = \infty \tag{5-26}$$

for some $r > 0$ and $i = 1, 2$. Let $(Y_1, Y) = \mathcal{I}(F_1, F)$. By (5-26) and Lemma 2-10, Y_1 is not a point (and neither is Y) and we can identify the convex hull of $Y \cup Y_1$ with $Y_1 \times [0, d(F_1, F)]$. Pick an edge $e_a \subset Y_1$ and let K_1 be the strip $e_a \times [0, d(F_1, F)]$ inside $Y_1 \times [0, d(F_1, F)]$. By considering the pair F and F_2, we can similarly find an edge $e_b \subset F_2$ and a strip K_2 isometric to $e_b \times [0, d(F_2, F)]$ which joins F and F_2. See Figure 4.

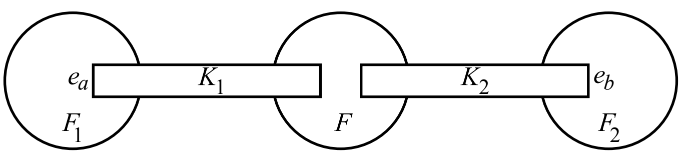

Figure 4

We parametrize the geodesic segment $l = \overline{x_1 x_2}$ by arc length such that $l(0) = v_1$. Assume $l(N) = x_2$. Let h_i be the hyperplane dual to the edge $\overline{l(i-1)l(i)}$ for $1 \le i \le N$. Note that

$$h_j \text{ separates } h_i \text{ and } h_k \text{ for } i < j < k. \tag{5-27}$$

Moreover, each h_i separates F_1 and F_2 by (5-25), hence also separates e_a and e_b. Consider the set $K_1 \cup F \cup K_2$, which is connected and contains e_a and e_b, so each h_i must have nontrivial intersection with at least one of K_1, F and K_2.

We claim each of K_1, F and K_2 could intersect at most $2M$ hyperplanes from the collection $\{h_i\}_{i=1}^N$, where M is the length of word W (and $M > 1$ since Γ contains more than one vertex). This will yield a contradiction since $N = 8M$. We prove the claim for K_1; the case of K_2 is similar.

Let h_a be the hyperplane dual to e_a and let $\Lambda = \{1 \le i \le N \mid h_a \cap h_i \neq \varnothing\}$. Then

$$\{1 \le i \le N \mid K_1 \cap h_i \neq \varnothing\} \subset \Lambda. \tag{5-28}$$

If $h_a = h_{i_0}$ for some i_0, then by (5-27), h_{i_0} is the only hyperplane in $\{h_i\}_{i=1}^N$ intersecting K_1. Hence we are done in this case. Now we assume $h_a \notin \{h_i\}_{i=1}^N$. Let e_i be an edge dual to h_i. Then it follows from $h_a \cap h_i \neq \varnothing$ that for any $i \in \Lambda$,

$$d(V(e_i), V(e_a)) = 1 \tag{5-29}$$

in Γ. If the claim for K_1 is not true, then (5-28) implies Λ has cardinality bigger than $2M$; moreover, it follows from (5-27) that if $i \in \Lambda$ and $j \in \Lambda$, then $k \in \Lambda$ for any $i \le k \le j$. By the construction of the word W, we know every vertex of Γ is contained in the collection $\{V(e_i)\}_{i \in \Lambda}$, which contradicts (5-29).

Now we prove the claim for F. Suppose $F \cap h_i \neq \varnothing$. Then $V(e_i) \in \Delta$. By the construction of W', we know there exist positive integers $a, a' < M$ such that $d(V(e_{i+a}), V(e_i)) \ge 2$ and $d(V(e_{i-a'}), V(e_i)) \ge 2$ in Γ. Then $F \cap h_j = \varnothing$ for $j = i + a$ and $j = i - a'$. By (5-27), $F \cap h_j = \varnothing$ for $j > i + a$ and $j < i - a'$. Thus the claim is true for F. □

Theorem 5-30 *The following are equivalent:*

(1) $\operatorname{diam}(\mathcal{G}_1(\Gamma)) < \infty$.

(2) $\operatorname{diam}(\mathcal{G}_1(\Gamma)) \le 2$.

(3) Γ *admits a nontrivial join decomposition or* Γ *is one point.*

Moreover, these properties are quasi-isometry invariants for right-angled Artin groups.

Note that (1) $\Rightarrow$ (3) follows by considering the concatenation of arbitrarily many copies of W in Lemma 5-24 and applying the same argument.

Remark 5-31 It is shown in [2] and [1] that $G(\Gamma)$ has linear divergence if and only if Γ is a nontrivial join, which also implies that Γ being a nontrivial join is quasi-isometry invariant. Moreover, their results together with [26, Theorem B and Proposition 4.7] implies the following stronger statement.

Given $X = X(\Gamma)$ and $X' = X(\Gamma')$, let $\Gamma = \Delta \circ \Gamma_1 \circ \cdots \circ \Gamma_k$ be the join decomposition such that Δ is the maximal clique factor, and each Γ_i does not allow nontrivial further join decomposition. Similarly, let $\Gamma = \Delta' \circ \Gamma'_1 \circ \cdots \circ \Gamma'_{k'}$. Let $X = \mathbb{R}^n \times \prod_{i=1}^k X(\Gamma_i)$ and let $X' = \mathbb{R}^{n'} \times \prod_{j=1}^{k'} X(\Gamma'_j)$ be the corresponding product decomposition. If $\phi\colon X \to X'$ is an (L, A) quasi-isometry, then $n = n'$, $k = k'$ and there exist constants $L' = L'(L, A)$, $A' = A'(L, A)$, $D = D(L, A)$ such that after re-indexing the factors in X', we have an (L', A') quasi-isometry $\phi_i\colon X(\Gamma_i) \to X(\Gamma'_i)$ so that

$$d\left(p' \circ \phi, \prod_{i=1}^{k} \phi_i \circ p\right) < D,$$

where $p\colon X \to \prod_{i=1}^k X(\Gamma_i)$ and $p'\colon X' \to \prod_{i=1}^k X(\Gamma'_i)$ are the projections.

More generally, let X and X' be locally compact CAT(0) cube complexes which admit a cocompact and essential action. Let $X = \prod_{j=1}^n Z_j \times \prod_{i=1}^k X_i$ be the finest product decomposition of X, where the Z_j are exactly the factors which are quasi-isometric to $\mathbb{R}$. Suppose $Z = \prod_{j=1}^n Z_j$. Then $X = Z \times \prod_{i=1}^k X_i$. Similarly, we decompose X' as $X' = Z' \times \prod_{i=1}^{k'} X'_i$. Then any quasi-isometry between X and X' respects such product decompositions in the sense of the previous paragraph. This is a consequence of [26, Theorem B], [26, Proposition 4.7] and [12, Theorem 6.3].

Appendix: Top-dimensional support sets in spaces of finite geometric dimension, and application to Euclidean buildings

In this section we adjust previous arguments to study the structure of top-dimensional quasiflats in Euclidean buildings and prove the following result.

Theorem A-1 *If Y is a Euclidean building of rank n and $[\sigma] \in H^{\mathrm{p}}_{n,n}(Y)$, then there exist finitely many Weyl cones $\{W_i\}_{i=1}^h$ such that*

$$d_H\left(S_{[\sigma]}, \bigcup_{i=1}^{h} W_i\right) < \infty.$$

Moreover, we can assume $W_i \subset S_{[\sigma]}$ for all i.

For the case of discrete Euclidean buildings, our previous method goes through without much modification. One way to handle the nondiscrete case is to use [30, Lemma 6.2.2], which says the support set of a top-dimensional class locally looks like a polyhedral complex, to reduce to the discrete case. But this lemma relies on the local structure of Euclidean buildings. We introduce another way, based on the generalization of results in Section 3.2 to CAT(0) spaces of finite geometric dimension (or homological dimension), which is of independent interest.

Lemma A-2 *Lemma 3-4 is true under the assumption that Y is a* CAT(0) *space of homological dimension $\le n$.*

Proof Let $\phi\colon \mathbb{E}^n \to Y$ be a top-dimensional (L, A)–quasiflat. We can assume ϕ is Lipschitz as before since Y is CAT(0). Let $[\mathbb{E}^n]$ be the fundamental class of $\mathbb{E}^n$. Pick $\sigma = \phi_*([\mathbb{E}^n])$ and let $S = S_{[\sigma]}$ be the support set. Pick $\epsilon > 0$, suppose U is the 1–neighborhood of $\mathrm{Im}\,\phi$ and suppose $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of U, where each U_λ is an open subset of U with diameter < 1. Since every metric space is paracompact, we can assume this covering is locally finite and define a continuous map $\varphi\colon U \to \mathbb{E}^n$ via the nerve complex of this covering as in Lemma 4-16, such that there exists a constant C such that

$$d(\varphi \circ \phi(x), x) < C \tag{A-3}$$

for any $x \in \mathbb{E}^n$, thus $\varphi_*([\sigma]) = [\mathbb{E}^n]$. Then we have $S_{[\mathbb{E}^n]} = \mathbb{E}^n \subset \varphi(S_{[\sigma]})$ by Lemma 3-2. It follows that $d_H(S, \mathrm{Im}\,\phi) < D = D(L, A)$. □

Remark A-4 In the above proof, we need to define φ in an open neighborhood of $\mathrm{Im}\,\phi$ since $S_{[\sigma],\mathrm{Im}\,\phi}$ might be strictly smaller than $S_{[\sigma],Y}$. Also we do not need to bound the dimension of the nerve complex of $\{U_\lambda\}_{\lambda\in\Lambda}$ as in Lemma 4-16 since Y is CAT(0), while in Lemma 4-16, the target space CK is linearly contractible with the contractibility constant possibly greater than 1.

Recall that in a polyhedral complex, every top-dimensional homology class can be represented by a cycle with image inside the support of the homology class. However, we do not know if this is still true in the case of an arbitrary metric space of homological dimension n. The following result helps us to get around this point.

Lemma A-5 *Let Y be a metric space of homological dimension $\le n$ and $[\sigma] \in H_n^{\mathrm{p}}(Y)$. If O is an open neighborhood of $S_{[\sigma]}$, then there exists a proper cycle σ' such that $[\sigma] = [\sigma']$ and $\mathrm{Im}\,\sigma' \subset O$.*

Proof We first prove a relative version of the above lemma for the usual homology theory. Let $V \subset U$ be open sets in Y. Pick $[\alpha] \in H_n(U, V)$ and let $K = S_{[\alpha]}$. We claim for any open neighborhood $O \supseteq K$, there exist chains β and γ such that $\mathrm{Im}\,\beta \subset U$, $\mathrm{Im}\,\gamma \subset V \cup O$ and $\alpha = \partial\beta + \gamma$.

Let $K' = \mathrm{Im}\,\alpha \setminus (V \cup O)$. For every point $x \in K'$, there exists $\epsilon(x) > 0$ such that $\bar{B}(x, 2\epsilon(x)) \subset U \setminus \mathrm{Im}\ \partial\alpha$ and $[\alpha]$ is trivial in $H_n(U, U \setminus \bar{B}(x, 2\epsilon(x)))$. Since K' is compact, we can find finitely many points $\{x_i\}_{i=1}^N$ in K' such that $K' \subset \bigcup_{i=1}^N B(x_i, \epsilon(x_i))$. Suppose $U_{K'} = \bigcup_{i=1}^N B(x_i, \epsilon(x_i))$ and $W = V \cup O \cup U_{K'}$.

Then $\operatorname{Im}\alpha \subset W$ and $[\alpha]$ is trivial in $H_n(U,W)$. Let $W' = W \setminus \big(\bigcup_{i=1}^{N} \overline{B}(x_i, 2\epsilon(x_i))\big)$. Then $\operatorname{Im}\partial\alpha \subset W' \subset V \cup O$, so it suffices to show $[\alpha]$ is trivial in $H_n(U,W')$, but this follows from the Mayer–Vietoris argument in Lemma 3-2.

Now we turn to the case of $[\sigma] \in H_n^{\mathrm{p}}(Y)$. Pick a base point $p \in Y$, put

$$\begin{aligned} U_1 &:= B(p,4), & U_i &:= B(p,3^i+1) \setminus \overline{B}(p,3^{i-1}-1) \quad \text{for } i>1;\\ U_1' &:= Y \setminus \overline{B}(p,3), & U_i' &:= B(p,3^{i-1}) \cup (Y \setminus \overline{B}(p,3^i)) \quad \text{for } i>1. \end{aligned}$$

By barycentric subdivision, we can assume every singular simplex in σ has image of diameter $\le \frac{1}{3}$.

Set $\sigma_0 = \sigma$, $V_0 = Y$ and $V_i = Y \setminus \overline{B}(p,3^i)$ for $i \ge 1$. Given σ_i with $\operatorname{Im}\sigma_i \subset (O \cup V_i)$ (this is trivially true for $i=0$), we define σ_{i+1} inductively as follows. First subdivide σ_i to get a proper cycle σ'_{i+1} such that

- $\operatorname{Im}\sigma_i = \operatorname{Im}\sigma'_{i+1}$,
- $\sigma'_{i+1} = \sigma_i + \partial\beta_{i+1}$ with $\operatorname{Im}\beta_{i+1} \subset U_{i+1}$, and
- $\sigma'_{i+1} = \sigma'_{i+1,1} + \sigma'_{i+1,2}$ for $\operatorname{Im}\,\sigma'_{i+1,1} \subset U_{i+1}$ and $\operatorname{Im}\sigma'_{i+1,2} \subset U'_{i+1}$.

It follows that $\operatorname{Im}\partial\sigma'_{i+1,1} \subset U_{i+1} \cap U'_{i+1}$ and $\operatorname{Im}\partial\sigma'_{i+1,1} \subset \operatorname{Im}\sigma_i \subset (O \cup V_i)$. So we can view $[\sigma'_{i+1,1}]$ as an element in $H_n(U_{i+1}, U_{i+1} \cap U'_{i+1} \cap (O \cup V_i))$. Then by the previous claim, there exists a chain β'_{i+1} with $\operatorname{Im}\beta'_{i+1} \subset U_{i+1}$ such that $\operatorname{Im}(\sigma'_{i+1,1} + \partial\beta'_{i+1}) \subset F$, where

$$\begin{aligned} F = (U_{i+1} \cap U'_{i+1} \cap (O \cup V_i)) \cup (O \cap U_{i+1}) &= (U_{i+1} \cap U'_{i+1} \cap V_i) \cup (O \cap U_{i+1})\\ &= (U_{i+1} \cap V_{i+1}) \cup (O \cap U_{i+1})\\ &= (O \cup V_{i+1}) \cap U_{i+1}. \end{aligned}$$

Let $\sigma_{i+1} = \sigma_i + \partial(\beta_{i+1} + \beta'_{i+1})$. Then

$$\operatorname{Im}\sigma_{i+1} \subset (F \cup \operatorname{Im}\,\sigma'_{i+1,2}) \subset (F \cup (O \cup V_i)) \subset (O \cup V_{i+1})$$

and the induction goes through.

Let $\sigma' = \sigma + \sum_{i=1}^{\infty} \partial(\beta_i + \beta'_i)$. Since $\operatorname{Im}(\beta_i + \beta'_i) \subset U_i$, the infinite summation makes sense and σ' is a proper cycle. Also $\operatorname{Im}\sigma' \subset O$ by construction. □

Remark A-6 The above proof also shows that $[\sigma] \in H_n^{\mathrm{p}}(Y)$ is nontrivial if and only if $S_{[\sigma]} \neq \varnothing$. This is not true for lower-dimensional cycles.

Corollary A-7 *Let Z be a* CAT(1) *space of homological dimension n. If $[\sigma] \in H_n(Z)$ is a nontrivial element, then for every point $x \in Z$, there exists a point $y \in S_{[\sigma]}$ such that $d(x,y) = \pi$.*

Proof We argue by contradiction and assume there exists a point $x \in Z$ such that $S_{[\sigma]} \subset B(x,\pi)$. Then by Lemma A-5, there exists a cycle σ' such that $\mathrm{Im}\,\sigma' \subset B(x,\pi)$ and $[\sigma'] = [\sigma]$. However, $B(x,\pi)$ is contractible and $[\sigma']$ must be trivial, which yields a contradiction. □

Let Y be a CAT(0) space. Pick $p \in Y$ and let T_pY be the tangent cone at p. Denote the base point of T_pY by o. Recall that there are logarithmic maps $\log_p\colon Y \to T_pY$ and $\log_p\colon Y\setminus\{p\} \to \Sigma_pY$. By [31, Theorem 3.5] (see also [33]), $\log_p\colon Y\setminus\{p\} \to \Sigma_pY$ is a homotopy equivalence. Thus we get:

Lemma A-8 *The map $(\log_p)_*\colon H_*(Y, Y\setminus\{p\}) \to H_*(T_pY, T_pY\setminus\{o\})$ is an isomorphism.*

We need a simple observation about support sets in cones before we proceed. Let Z be a metric space and let CZ be the Euclidean cone over Z with base point o. Pick a $[\sigma] \in H_i(CZ, CZ\setminus B(o,r))$. Recall that there is an isomorphism

$$\partial\colon H_i(CZ, CZ\setminus B(o,r)) \to H_{i-1}(Z)$$

induced by the boundary map.

Lemma A-9 *Suppose $S = S_{\partial[\sigma],Z}$ and suppose CS is the cone over S inside CZ. Then $S_{[\sigma],CZ,CZ\setminus B(o,r)} = CS \cap B(o,r)$.*

The next lemma is an immediate consequence of [28, Theorem A].

Lemma A-10 *If Z is a CAT(κ) space of homological dimension $\le n$, then for any $p \in Z$, Σ_pZ is of homological dimension $\le n-1$.*

Now we are ready to prove the geodesic extension property for top-dimensional support sets. The argument is similar to [9, Lemma 3.1].

Lemma A-11 *Let Y be a CAT(0) space of homological dimension n. Pick an element $[\sigma] \in H^{\mathrm{p}}_n(Y)$ and let $S = S_{[\sigma]}$. Then for a geodesic segment $\overline{pq} \subset Y$ with $q \in S$, there exists a geodesic ray $\overline{q\xi} \subset S$ such that $\overline{pq}$ and $\overline{q\xi}$ fit together to form a geodesic ray.*

Proof First we claim that for any $\epsilon > 0$, there exists a point $z \in S \cap S(p,\epsilon)$ such that the concatenation of $\overline{pq}$ and $\overline{qz}$ is a geodesic. Let

$$\log_q\colon (Y, Y\setminus B(q,2\epsilon)) \to (T_qY, T_qY\setminus B(o,2\epsilon)),$$

and let $\alpha = \log_q(\sigma)$. By Lemma A-10, the homological dimension of T_pY is bounded above by n. Then by Lemma 3-2,

$$S_{[\alpha],T_qY,T_qY\setminus B(o,2\epsilon)} \subset f(S_{[\sigma],Y,Y\setminus B(q,2\epsilon)}) = f(S\cap B(q,2\epsilon)). \tag{A-12}$$

Let $\partial\colon H_n(T_qY, T_qY\setminus B(o,2\epsilon)) \to H_{n-1}(\Sigma_qY)$ be the isomorphism and let $[\beta]=\partial[\alpha]$. Then $[\beta]$ is nontrivial in $H_{n-1}(\Sigma_qY)$ by Lemma A-8 and this commuting diagram:

$$\begin{array}{ccc} H_n(Y, Y\setminus B(q,2\epsilon)) & \xrightarrow{(\log_q)_*} & H_n(T_qY, T_qY\setminus B(o,2\epsilon)) \\ \downarrow & & \downarrow \\ H_n(Y, Y\setminus\{q\}) & \xrightarrow{(\log_q)_*} & H_n(T_qY, T_qY\setminus\{o\}) \end{array}$$

Let $CS_{[\beta]}$ be the Euclidean cone over $S_{[\beta]}\subset\Sigma_qY$ inside T_qY. Then

$$CS_{[\beta]}\cap B(o,2\epsilon) = S_{[\alpha],T_qY,T_qY\setminus B(o,2\epsilon)} \subset f(S\cap B(q,2\epsilon)) \tag{A-13}$$

by (A-12) and Lemma A-9. Moreover, by Corollary A-7 and Lemma A-10, there exists an $x\in S_{[\beta]}$ such that

$$d(x,\log_q(p)) = \pi, \tag{A-14}$$

where $\log_q\colon Y\setminus\{q\}\to\Sigma_qY$. So the claim follows from (A-13).

By repeatedly applying the above claim, for each positive integer n we can obtain a unit-speed geodesic $c_n\colon [0,\epsilon]\to Y$ such that $c(0)=q$, $c(m\epsilon/2^n)\in S$ for any integer $0\le m\le 2^n$ and $\log_q(c(\epsilon))=x$, where x is the point in (A-14). Note that $S\cap\overline{B}(q,\epsilon)$ is compact, so we assume without loss of generality that $r=\lim_{n\to\infty} c_n(\epsilon)$. If $c\colon [0,\epsilon]\to Y$ is the unit-speed geodesic joining q and r, then c_n converges uniformly to c, which implies $c([0,\epsilon])\subset S$. Moreover, $\log_q(c(\epsilon))=x$. Thus the concatenation of $\overline{pq}$ and $\overline{qr}$ is a geodesic by (A-14). Now we can repeatedly apply this ϵ–extension procedure to obtain the geodesic ray as required. □

In general, the above set $S\cap\overline{B}(q,\epsilon)$ is not equal to the geodesic cone $C_q(S\cap S(q,\epsilon))$ based at q over $S\cap S(q,\epsilon)$ no matter how small ϵ is. However, we have

$$\lim_{\epsilon\to 0} d_{\mathrm{GH}}\big(\tfrac{1}{\epsilon}(C_q(S\cap S(q,\epsilon))), CS_{[\beta]}\cap\overline{B}(o,1)\big) = \lim_{\epsilon\to 0} d_{\mathrm{GH}}\big(\tfrac{1}{\epsilon}(S\cap\overline{B}(q,\epsilon)), CS_{[\beta]}\cap\overline{B}(o,1)\big) = 0.$$

Thus the tangent cone of S exists for every point in S.

Remark A-15 By the same proof, we know Lemma A-11 is still true if Y is an Alexandrov space which has curvature bounded above and homological dimension $=n$. In this case, $\overline{p\xi}$ is locally geodesic.

Lemma A-16 *Let Z be a* CAT(1) *space of homological dimension $\le n$ and let $[\sigma] \in \widetilde{H}_n(Z)$ be a nontrivial class. Then the following assertions hold.*

(1) $\mathcal{H}^n(S_{[\sigma]}) \ge \mathcal{H}^n(\mathbb{S}^n)$.

(2) *Let $V(n,r)$ be the volume of r–ball in $\mathbb{S}^n$. Then for any $0 \le r \le R < \pi$ and any $p \in S_{[\sigma]}$,*

$$1 \le \frac{\mathcal{H}^n(B(p,r) \cap S_{[\sigma]})}{V(n,r)} \le \frac{\mathcal{H}^n(B(p,R) \cap S_{[\sigma]})}{V(n,R)}.$$

(3) *If $\mathcal{H}^n(S_{[\sigma]}) = \mathcal{H}^n(\mathbb{S}^n)$, then $S_{[\sigma]}$ is an isometrically embedded copy of $\mathbb{S}^n$.*

Here $\mathbb{S}^n$ denotes the n–dimensional standard sphere with constant curvature 1.

Proof We claim there exists a 1–Lipschitz map from a subset of $S_{[\sigma]}$ to a full-measure subset of $\mathbb{S}^n$. Let us assume this is true for $i = n-1$. Pick $p \in S_{[\sigma]}$, let $\mathbb{S}^0 * \Sigma_p Z$ be the spherical suspension of $\Sigma_p Z$ and let o be one of the suspension points. Then there is a well-defined 1–Lipschitz map $\log_p\colon B(p,\pi) \to \mathbb{S}^0 * \Sigma_p Z$ sending p to o. Let $[\beta]$ be the image of $[\sigma]$ under the map

$$\widetilde{H}_n(Z) \to H_n(Z, Z \setminus \{p\}) \to H_n(B(p,\pi), B(p,\pi) \setminus \{p\}) \xrightarrow{(\log_p)_*} H_n(B(o,\pi), B(o,\pi) \setminus \{o\}) \to \widetilde{H}_{n-1}(\Sigma_p Z).$$

We can slightly adjust the proof of Lemma A-11 to show that

$$\log_p(S_{[\sigma]} \cap B(p,\pi)) \supseteq (\mathbb{S}^0 * S_{[\beta]}) \cap B(o,\pi). \tag{A-17}$$

The induction assumption implies that there are a subset $K \in S_{[\beta]}$ and a 1–Lipschitz map $f\colon K \to \mathbb{S}^{n-1}$ such that $\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus f(K)) = 0$. Note that f induces a 1–Lipschitz map $\tilde{f}\colon \mathbb{S}^0 * K \to \mathbb{S}^0 * \mathbb{S}^{n-1} = \mathbb{S}^n$ whose image also has full measure, thus by (A-17), there exists $K' \subset S_{[\sigma]}$ such that the image of $\tilde{f} \circ \log_p\colon K' \to \mathbb{S}^n$ has full measure. It follows that $\mathcal{H}^n(S_{[\sigma]}) \ge \mathcal{H}^n(\mathbb{S}^n)$.

The first inequality of (2) follows from (1) and (A-17). The second inequality follows from Remark A-15 and the proof of [9, Corollary 3.3].

Now we prove (3). By Remark A-15, for every point $x \in S_{[\beta]}$, there exists a geodesic segment $l_x \subset S_{[\sigma]}$ emanating from p along the direction x, which has length $= \pi$. Let A be the closure of $\bigcup_{x \in S_{[\beta]}} l_x$. Then $A \subset S_{[\sigma]}$ and $\mathcal{H}^n(A) \ge \mathcal{H}^n(\mathbb{S}^n)$. Then (2) implies that actually $A = S_{[\sigma]}$. Pick arbitrary $q \in S_{[\sigma]} \cap B(p,\pi)$. Then there exists a sequence $\{q_n\}_{n=1}^{\infty} \subset \bigcup_{x \in S_{[\beta]}} l_x$ such that $\lim_{n\to\infty} q_n = q$. Since $\overline{q_n p} \subset S_{[\sigma]}$ by construction, $\overline{qp} \subset S_{[\sigma]}$. It follows that $S_{[\sigma]}$ is π–convex in Y. Then $S_{[\sigma]}$ can be viewed as a compact and geodesically complete CAT(1) space. By [35, Proposition 7.1], $S_{[\sigma]}$ is isometric to $\mathbb{S}^n$. □

We can recover the monotonicity (3-11) and the lower density bound (3-12) from Lemma A-11 and Lemma A-16, then we can define the group $H^{\mathrm{p}}_{n,n}(Y)$ as before when Y is a CAT(0) space of homological dimension $\le n$ and the rest of the discussion in Section 3.2 goes through without any change. Recall that the homological dimension of a CAT(0) space is equal to its geometric dimension [28, Theorem A], so the following result holds.

Theorem A-18 *Let Y be a* CAT(0) *space of geometric dimension n. Pick an element $[\sigma] \in H^{\mathrm{p}}_{n,n}(Y)$ and let $S = S_{[\sigma]}$. Then:*

(1) **Local property I** *Each point $p \in Y$ has a well-defined tangent cone $T_p Y$.*

(2) **Local property II** *S has the geodesic extension property in the sense of Lemma 3-6.*

(3) **Monotonicity and lower density bound** *For all $0 \le r \le R$ and $p \in Y$,*

$$\frac{\mathcal{H}^n(B(p,r) \cap S)}{r^n} \le \frac{\mathcal{H}^n(B(p,R) \cap S)}{R^n}.$$

If $p \in S$, then

$$\mathcal{H}^n(B(p,r) \cap S) \ge \omega_n r^n,$$

with equality only if $B(p,r) \cap S$ is isometric to an r–ball in $\mathbb{E}^n$. Here ω_n is the volume of an n–dimensional Euclidean ball of radius 1.

(4) **Asymptotic conicality I** *Let $B(o,1)$ be the unit ball in $C_T S$ centered at the cone point o. For any $p \in Y$,*

$$\lim_{r \to +\infty} d_{\mathrm{GH}}\big(\tfrac{1}{r}(B(p,r) \cap S), B(o,1)\big) = 0.$$

Moreover, putting $\partial_{p,r} S := \{\xi \in \partial_T S \mid \overline{p\xi} \subset B(p,r) \cup S\}$, then

$$\lim_{r \to +\infty} d_H(\partial_{p,r} S, \partial_T S) = 0.$$

(5) **Asymptotic conicality II** *For all $\beta > 0$ there is an $r < \infty$ such that if $x \in S \setminus B(p,r)$, then*

$$\mathrm{diam}(\mathrm{Ant}_\infty(\log_x p, S)) < \beta,$$

where the diameter is with respect to the angular metric on $\partial_T Y$.

Now we reinterpret the group $H^{\mathrm{p}}_{n,n}(Y)$. Recall that there is another logarithmic map $\log_p \colon C_T Y \to Y$ sending the base point o of $C_T Y$ to $p \in Y$. Since $\log_p$ is proper and 1–Lipschitz, it induces a map $H^{\mathrm{p}}_{n,n}(C_T Y) \to H^{\mathrm{p}}_{n,n}(Y)$.

Next we define a map in the other direction. Pick $[\sigma]\in H^{\mathrm{p}}_{n,n}(Y)$, let $S=S_{[\sigma]}$ be the support set and let U_S be the 1–neighborhood of S. By Lemma A-5, we can assume $\operatorname{Im}\sigma\subset U_S$. For $\epsilon>0$, we define the map $f_\epsilon\colon U_S\to C_T S$ as in Lemma 4-16. To approximate f_ϵ by a continuous map, we choose a locally finite covering of U_S by its open subsets which satisfies the diameter condition in Lemma 4-16, then proceed as before to obtain a continuous map $f_\epsilon\colon U_S\to C_T X$. Here the image may not stay inside $C_T S$, however it is sublinearly close to $C_T S$. Now we define $\exp_*([\sigma])=\lim_{\epsilon\to 0} f_{\epsilon *}([\sigma])$; note that $f_{\epsilon *}([\sigma])$ does not depend on ϵ when it is small. Since (4-23) is still true, $(\log_p)_*\circ\exp_*=\mathrm{Id}$.

To see that $\exp_*\circ(\log_p)_*=\mathrm{Id}$, we follow the proof of (4-30); the only difference is that we need to replace $I_{\sigma'}$ there by the 1–neighborhood of $\operatorname{Im}\sigma'$, then use the nerve complex of a suitable covering to approximate g_ϵ as we did for f_ϵ. So

$$(\log_p)_*\colon H^{\mathrm{p}}_{n,n}(C_T Y)\to H^{\mathrm{p}}_{n,n}(Y)$$

is an isomorphism, with the inverse map $\exp_*$ defined as above.

Let $h_\lambda\colon C_T Y\to C_T Y$ be the homothety map with respect to the base point o and a factor λ. Then h_λ is properly homotopic to h_1 for any $0<\lambda<\infty$, so for any $[\beta]\in H^{\mathrm{p}}_i(C_T Y)$, we have $h_{\lambda *}([\beta])=[\beta]$ and $h_\lambda(S_{[\beta]})=S_{[\beta]}$. It follows that every cycle in $H^{\mathrm{p}}_i(C_T Y)$ is conical. Thus the map

$$j\colon H^{\mathrm{p}}_i(C_T Y)\to H_i(C_T Y, C_T Y\setminus\{o\})\to H_{i-1}(\partial_T Y)$$

is an isomorphism with inverse given by the "coning off" procedure. It follows that the map defined in (4-27) and (4-28) are isomorphisms, and the analogues of Corollary 4-33 and Remark 4-36 in the case of CAT(0) spaces with finite homological dimension are still true (again, for our argument to go through, we need to replace the set I_q in the proof of Corollary 4-33 by some r–neighborhood of the image of q). This discussion can be summarized as follows.

Theorem A-19 *Let $q\colon Y\to Y'$ be a quasi-isometric embedding, where Y and Y' are* CAT(0) *spaces of geometric dimension $\le n$. Then:*

(1) *The map $\partial := j\circ(\exp_*)\colon H^{\mathrm{p}}_{n,n}(Y)\to H_{n-1}(\partial_T Y)$ is a group isomorphism, with inverse given by the coning off map $c\colon H_{n-1}(\partial_T Y)\to H^{\mathrm{p}}_{n,n}(Y)$; see (4-28).*

(2) *The map q induces a monomorphism $q_*\colon H_{n-1}(\partial_T Y)\to H_{n-1}(\partial_T Y')$. If q is a quasi-isometry, then q_* is an isomorphism.*

(3) *There exists a $D'>0$, depending on the quasi-isometry constants of q, such that*

$$d_H(q(S_{[\tilde\sigma]}), S_{q_*[\tilde\sigma]})<D'$$

for any $[\tilde\sigma]\in H^{\mathrm{p}}_{n,n}(Y)$.

We refer to the work of Kleiner and Lang [29] for a more general version of the above theorem.

Remark A-20 Pick $[\tau] \in H_{n-1}(\partial_T Y)$. Then by Lemma A-8 and Theorem A-19,

$$S_{c([\tau])} = \{y \in Y \mid [\tau] \text{ is nontrivial under } (\log_y)_*\colon H_{n-1}(\partial_T Y) \to H_{n-1}(\Sigma_y Y)\},$$

where c is the coning off map in (4-28).

Now we are ready to prove Theorem A-1. To avoid repetition, we will only sketch the main steps.

Proof of Theorem A-1 If Y is a Euclidean building of rank n, it follows from [30, Corollary 6.1.1] that the homological dimension of Y is less than or equal to n. This also follows from [28, Theorem A] by noticing that $\Sigma_p Y$ is a spherical building of dimension $n-1$ for any $p \in Y$. Let $[\sigma] \in H^{\mathrm{p}}_{n,n}(Z)$.

Step 1 Let $[\alpha] = \exp_*([\sigma]) \in H^{\mathrm{p}}_{n,n}(C_T Y)$. Since $\partial_T Y$ is a spherical building, $S_{[\alpha]}$ is a cone over K, where $K = \bigcup_{i=1}^h C_i$ and each C_i is a chamber in $\partial_T Y$.

Step 2 Let $W_i \subset Y$ be a Weyl cone such that $\partial_T W_i = C_i$. Note that for any $i \neq j$, there is an apartment of $\partial_T Y$ which contains C_i and C_j. Thus we can assume W_i and W_j are contained in a common apartment of Y. So W_i and W_j satisfy inequalities similar to (2-11). The quotient map $\bigsqcup_{i=1}^h C_i \to K$ induces a map $\varphi\colon CK \to Y$ which is a quasi-isometric embedding as in Lemma 4-38. We can assume φ is continuous. Put $[\tau] = \varphi_*([CK])$, where $[CK]$ is the fundamental class of CK. Then it follows from the proof of Lemma 4-46 that $d_H\big(S_{[\tau]}, \bigcup_{i=1}^h W_i\big) < \infty$. Moreover, we can assume that $W_i \subset S_{[\tau]}$.

Step 3 It suffices to show that $[\sigma] = [\tau]$. Pick $p \in Y$. Note that there exists a $D > 0$ such that $d(\log_p(x), \varphi(x)) < D$ for any $x \in CK$. Then

$$[\tau] = \varphi_*([CK]) = (\log_p)_*([\alpha]) = ((\log_p)_* \circ \exp_*)([\sigma]) = [\sigma]. \qquad \square$$

The following result is an immediate consequence of Lemma A-2 and Theorem A-1.

Corollary A-21 *If Y is a Euclidean building of rank n and $Q \subset Y$ is an n–quasiflat, then there exist finitely many Weyl cones $\{W_i\}_{i=1}^h$ such that*

$$d_H\Big(Q, \bigcup_{i=1}^h W_i\Big) < \infty.$$

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