# ERRATUM TO "ALGEBRAIC COBORDISM AND ÉTALE COHOMOLOGY" 

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Let $\pi: A \rightarrow B$ be a finite, étale, degree $d$ morphism of commutative rings and let $\mathcal{E}$ be an $\mathrm{MGL}_{A}$-module. We let $p: \operatorname{Spec} B \rightarrow \operatorname{Spec} A$ denote the map induced by $\pi$. In [2, Proposition D.1(ii)], it was claimed that the map $p_{*} p^{*}: \mathcal{E}^{* *}(A) \rightarrow \mathcal{E}^{* *}(A)$ is multiplication by $d$, which is in general false. As counter-example, we can take $X$ to be a regular noetherian scheme of finite Krull dimension, with $1 / 2 \in \mathcal{O}_{X}(X)$, and admitting a non-trivial line bundle $L \rightarrow X$ that is 2 -torsion in $\operatorname{Pic}(X)$, i.e. $L^{\otimes 2}$ is the trivial line bundle. Let $p: Y \rightarrow X$ be the inverse image of the 1-section of $L^{\otimes 2} \cong \mathbb{A}_{X}^{1}$ via the squaring map $s q: L \rightarrow L^{\otimes 2}$, and take $\mathcal{E}=\operatorname{KGL} \in \operatorname{SH}(X)$, representing Quillen algebraic $K$-theory.

We claim that $p_{*}\left(\mathcal{O}_{Y}\right) \cong \mathcal{O}_{X} \oplus \mathcal{L}^{\vee}$, where $\mathcal{L}$ is the invertible sheaf of sections of $L$ and $\mathcal{L}^{\vee}$ is the dual; taking determinants then shows that the class of $p_{*}\left(\mathcal{O}_{Y}\right)$ in $K_{0}(X)$ is not $2=2\left[\mathcal{O}_{X}\right]$. Indeed, if we take an open cover $\left\{U_{i}\right\}$ of $X$ trivializing $L$, with local coordinate functions $t_{i}: L_{\mid U_{i}} \xrightarrow{\sim} \mathbb{A}_{U_{i}}^{1}$, then $L$ is given by the cocycle $\left\{\xi_{i j}=t_{i} \circ t_{j}^{-1} \in \mathcal{O}_{X}^{\times}\left(U_{i} \cap U_{j}\right)\right\}$, and $Y \cap L_{\mid U_{i}}$ is defined by an equation $t_{i}^{2}=u_{i}$, with $u_{i} \in \mathcal{O}_{X}^{\times}\left(U_{i}\right)$ and $\xi_{i j}^{2}=u_{i} \cdot u_{j}^{-1}$ on $U_{i} \cap U_{j}$. Thus, for $U \subset X$ open, each section $s$ of $p_{*}\left(\mathcal{O}_{Y}\right)(U)$ has restriction $s_{i} \in p_{*}\left(\mathcal{O}_{Y}\right)\left(U \cap U_{i}\right)$ uniquely of form $s_{i}=s_{i}^{0}+s_{i}^{1} \cdot t_{i}$, for $s_{i}^{0}, s_{i}^{1} \in \mathcal{O}_{X}\left(U \cap U_{i}\right)$. Since $s_{i}=s_{j}$ on $U \cap U_{i} \cap U_{j}$, we see that the $s_{i}^{0}$ define a section $s^{0}$ of $\mathcal{O}_{X}(U)$ and the $s_{i}^{1}$ define a section $s^{1}$ of $L^{-1}(U)$, giving the decomposition $p_{*}\left(\mathcal{O}_{Y}\right) \cong \mathcal{O}_{X} \oplus \mathcal{L}^{\vee}$ as claimed.

For an example with $X$ (and hence $Y$ ) affine, one can consider a smooth projective curve $C$ over $\mathbb{C}$, of genus $g \geq 1$, choose a point $p \in C$ and take $X$ to be the affine curve $C \backslash\{p\}$. Then $\operatorname{Pic}(X)$ is the Jacobian of $C$, so has $2 g-1$ non-trivial 2-torsion elements.

As a correction, we will show here that, in case $A$ is noetherian, there is a nilpotent element $x \in \mathrm{MGL}^{0,0}(A)$ such that $p_{*} p^{*}: \mathcal{E}^{* *}(A) \rightarrow \mathcal{E}^{* *}(A)$ is multiplication by $d+x$.

In [2], Proposition D. 1 was used to show, for a finite, étale, degree $d$, Galois extension $A \rightarrow B$ with group $G$, that $p^{*}: \mathcal{E}^{* *}(A)[1 / d] \rightarrow \mathcal{E}^{* *}(B)[1 / d]$ identifies $\mathcal{E}^{* *}(A)[1 / d]$ with the $G$-invariants in $\mathcal{E}^{* *}(B)[1 / d]$, which in turn allowed us to define a transfer map $\mathcal{E}^{* *}(B)[1 / d] \rightarrow \mathcal{E}^{* *}(A)[1 / d]$, by sending $x \in \mathcal{E}^{* *}(B)[1 / d]$ to $\sum_{g \in G} g^{*}(x)$. As we only used loc. cit. in case $A$ is noetherian, and inverting $d$ makes $d+x$ a unit, the construction of the transfer map goes through with the modified version of Proposition D.1(ii) proven here.

The result does not require the schemes in question to be affine, so we replace Spec $B \rightarrow \operatorname{Spec} A$ with a finite, étale, degree $d$ morphism $p: Y \rightarrow X$ of noetherian schemes.

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Lemma 1. Let $x \in X$ be a closed point.
(1) Suppose the residue field $k(x)$ is infinite. Then there is an affine open neighborhood $U$ of $x$ in $X$, and a closed immersion $i_{x}: p^{-1}(U) \rightarrow \mathbb{A}_{U}^{1}$ over $U$.
(2) Suppose the residue field $k(x)$ is a finite field. Let $\ell$ be a prime. Then there is an affine open neighborhood $U$ of $x$ in $X$ an integer $n \geq 1$, a finite étale morphism $a: U^{\prime} \rightarrow U$ of degree $\ell^{n}$, and a closed immersion $i_{x}: Y \times_{X} U^{\prime} \rightarrow \mathbb{A}_{U^{\prime}}^{1}$ over $U^{\prime}$. In addition, $U^{\prime}$ admits a closed immersion over $U, U^{\prime} \hookrightarrow \mathbb{A}_{U}^{1}$.

Proof. For (1), we take $V=\operatorname{Spec} A \subset X$ to be an affine open neighborhood of $x \in X$; since $p$ is an affine map, we have $p^{-1}(V)=\operatorname{Spec} B$ for $\pi: A \rightarrow B$ a finite, étale, degree $d$ extension of commutative, noetherian rings.

Let $\mathfrak{m}_{x} \subset A$ be the maximal ideal corresponding to $x$ and let $\bar{B}=B / m_{x} B$. Then $\bar{B}$ is a finite étale $k(x)$-algebra, and we can thus write $\bar{B}=\prod_{i=1}^{r} F_{i}$, where each $F_{i}$ is a finite separable field extension of $k(x)$. In particular, each $F_{i}$ can be written as $F_{i}=k(x)[T] / g_{i}(T)$, for some irreducible polynomial $g_{i}(T) \in k(x)[T]$, and as such, we have for each $i$ a closed immersion $\alpha_{i}: \operatorname{Spec} F_{i} \rightarrow \mathbb{A}_{k(x)}^{1}$. Clearly the set of elements $\lambda \in k(x)$ such that translation by $\lambda$ maps $\alpha_{i}\left(\operatorname{Spec} F_{i}\right)$ to $\alpha_{j}\left(\operatorname{Spec} F_{j}\right)$ is finite for each pair $i, j$, and since $k(x)$ is infinite, translating $\alpha_{i}$ by a suitable $\lambda_{i} \in k(x)$ and changing notation, we may assume that $\alpha_{i}\left(\operatorname{Spec} F_{i}\right)$ and $\alpha_{j}\left(\operatorname{Spec} F_{j}\right)$ are disjoint for each $i \neq j$. This gives us the closed immersion $\alpha: \operatorname{Spec} \bar{B} \rightarrow \mathbb{A}_{k(x)}^{1}$. In other words, we have an isomorphism $\bar{B} \cong k(x)[T] / g(T)$ for some $g(T) \in k(x)[T]$. Since $\bar{B}$ is finite and étale over $k(x)$, we may choose $g$ to be monic, and the ideal $\left(g(T), g^{\prime}(T)\right)$ is the unit ideal in $k(x)[T]$.

In particular, if $\bar{b} \in \bar{B}$ maps to the residue of $T$ under this isomorphism, it follows that $\bar{B}$ is the free $k(x)$-module on elements $1, \bar{b}, \ldots, \bar{b}^{d-1}$, where $d$ is the degree of $f$. Now lift $\bar{b}$ to an element $b \in B$. By Nakayama's lemma, there is an $f \in A$, $f(x) \neq 0$, such that $B[1 / f]$ is generated by $1, b, \ldots, b^{d-1}$ as $A[1 / f]$-module. Thus, we can write $b^{d}=\tilde{g}(b)$, for some $\tilde{g}(T) \in A[1 / f][T]$ lifting $g(T)$. This gives us the finite $A[1 / f]$-algebra $A[1 / f][T] /(\tilde{g}(T))$, with its canonical embedding in $\mathbb{A}_{A[1 / f]}^{1}$, as well as a surjective $A[1 / f]$-algebra homomorphism $\phi: A[1 / f][T] /(\tilde{g}(T)) \rightarrow B[1 / f]$. Since $B[1 / f]$ is flat and finite over $A[1 / f]$, we may assume, after localizing further, that $B[1 / f]$ is the free $A[1 / f]$-module on $1, b, \ldots, b^{d-1}$, from which it follows that $\phi$ is an isomorphism. Since $\left(g(T), g^{\prime}(T)\right)=k(x)[T]$ and $A[1 / f][T] /(\tilde{g}(T))$ is finite over $A[1 / f]$, we may localize $A[1 / f]$ further to achieve that $\left(\tilde{g}(T), \tilde{g}^{\prime}(T)\right)=A[1 / f][T]$, so $A[1 / f] \rightarrow B[1 / f]$ is étale. This proves (1).

For (2), let $A, B$ and $\bar{B}$ be as in (1); by assumption $k(x)=\mathbb{F}_{q}$ for some prime power $q$. The argument in (1) shows that, if we pass to $\mathbb{F}_{q^{e \infty}}:=\cup_{n \geq 0} \mathbb{F}_{q^{e^{n}}}$, we have a closed immersion $\operatorname{Spec} \bar{B} \otimes_{k(x)} \mathbb{F}_{q^{\ell \infty}} \hookrightarrow \mathbb{A}_{\mathbb{F}_{q^{\ell \infty}}}^{1}$, in other words, there is a $g(T) \in \mathbb{F}_{q^{e \infty}}[T]$ and an isomorphism $\psi: \bar{B} \otimes_{k(x)} \mathbb{F}_{q^{\ell \infty}} \xrightarrow{\sim} \mathbb{F}_{q^{\ell \infty}}[T] /(g(T))$. Thus there is an integer $n$ such that $g$ is an element of $\mathbb{F}_{q^{n}}[T]$ and that $\psi$ is the baseextension of an isomorphism $\bar{B} \otimes_{k(x)} \mathbb{F}_{q^{e n}} \cong \mathbb{F}_{q^{e n}}[T] /(g(T))$. Writing $\mathbb{F}_{q^{e n}}=k(x)[\zeta]$ for some $\zeta \in \mathbb{F}_{q^{e^{n}}}$, let $e(X) \in \mathbb{F}_{q}[X]$ be the monic irreducible polynomial of $\zeta$, and lift $e(X)$ to a $\tilde{e}(X) \in A[X]$. After inverting some $f \in A, f(x) \neq 0$, we have that $A^{\prime}:=A[1 / f][X] /(\tilde{e}(X))$ is a finite étale $A[1 / f]$-algebra of degree $\ell^{n}$ over $A[1 / f]$, giving a closed immersion $\operatorname{Spec} A^{\prime} \hookrightarrow \mathbb{A}_{A[1 / f]}^{1}$ over $A[1 / f]$. Let $x^{\prime} \in \operatorname{Spec} A^{\prime}$ be the closed point lying over $x$, so $k\left(x^{\prime}\right)=\mathbb{F}_{q^{\ell n}}$. Using the closed immersion $\bar{i}: \operatorname{Spec}\left(\bar{B} \otimes_{k(x)} k\left(x^{\prime}\right)\right) \hookrightarrow \mathbb{A}_{k\left(x^{\prime}\right)}^{1}$ defined by $g(T)$, and arguing as in the proof of (1), we may extend $\bar{i}$ to a closed embedding $i_{x}: \operatorname{Spec}\left(B \otimes_{A} A^{\prime}\right) \hookrightarrow \mathbb{A}_{A^{\prime}}^{1}$ over $A^{\prime}$, after a further localization on $A$ if needed.

Lemma 2. Let $\mathcal{E} \in \mathrm{SH}(X)$ be an $\mathrm{MGL}_{X}$-module, and let $x \in X$ be a closed point. Then there is an open neighborhood $U \subset X$ of $x$ such that, letting $p_{U}: V \rightarrow U$ be the map induced by $p, V:=p^{-1}(U)$, we have $p_{U *} \circ p_{U}^{*}: \mathcal{E}^{* *}(U) \rightarrow \mathcal{E}^{* *}(U)$ is multiplication by $d:=\operatorname{deg} p$.

Proof. Suppose we have an open neighborhood $U \subset X$ of $x$, and a closed immersion $i_{0}: V:=p^{-1}(U) \hookrightarrow \mathbb{A}_{U}^{1}$ over $U$. Since $Y$ is finite and étale over $X$, it follows that $i_{0}$ extends to a regular codimension one embedding $i: V \hookrightarrow \mathbb{P}_{U}^{1}$. Thus $i(V)$ is an effective Cartier divisor $D$ on $\mathbb{P}_{U}^{1}$, so its ideal sheaf is the invertible sheaf $\mathcal{L}:=\mathcal{O}_{\mathbb{P}_{U}^{1}}(-D)$. After shrinking $U$ if necessary, we may assume that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}_{U}^{1}}(-d)$. By [1, Proposition 3.3.10], it follows that $p_{U *} \circ p_{U}^{*}$ is multiplication by $d$.

If the residue field $k(x)$ is infinite, we apply Lemma $1(1)$ to give an affine open neighborhood $U$ of $x$, and a closed immersion $V \hookrightarrow \mathbb{A}_{U}^{1}$ over $U$, where $V:=p^{-1}(U)$, which finishes the proof. If $k(x)$ is a finite field, we choose a prime $\ell$. By Lemma 1(2), we have an affine open neighborhood $U$ of $x$, a degree $\ell^{n}$ finite étale morphism $q: U^{\prime} \rightarrow U$, giving the Cartesian diagram

where $V:=p^{-1}(U)$, and we have closed immersions $U^{\prime} \hookrightarrow \mathbb{A}_{U}^{1}, V^{\prime} \hookrightarrow \mathbb{A}_{U^{\prime}}^{1}$, over $U$ and $U^{\prime}$, respectively. Applying the argument above to $p_{U}^{\prime}$ and $q$, we have

$$
p_{U *}^{\prime} p_{U}^{\prime *}=\times d, q_{*} q^{*}=\times \ell^{n}
$$

shrinking $U$ if necessary. Thus the map $q^{*}: \mathcal{E}^{* *}(U)[1 / \ell] \rightarrow \mathcal{E}^{* *}\left(U^{\prime}\right)[1 / \ell]$ is injective. Since

$$
q^{*}\left(p_{U *} p_{U}^{*}(x)\right)=p_{U *}^{\prime} q^{\prime *} p_{U}^{*}(x)=p_{U *}^{\prime} p_{U}^{* *}\left(q^{*}(x)\right)=d \cdot q^{*}(x)=q^{*}(d \cdot x)
$$

it follows that $p_{U *} \circ p_{U}^{*}: \mathcal{E}^{* *}(U)[1 / \ell] \rightarrow \mathcal{E}^{* *}(U)[1 / \ell]$ is multiplication by $d$.
Taking a second prime $\ell^{\prime} \neq \ell$ and repeating the above argument, we find that $p_{U *} \circ p_{U}^{*}: \mathcal{E}^{* *}(U)\left[1 / \ell^{\prime}\right] \rightarrow \mathcal{E}^{* *}(U)\left[1 / \ell^{\prime}\right]$ is again multiplication by $d$ (after shrinking $U$ again, if necessary). As the diagonal map

$$
\mathcal{E}^{* *}(U) \rightarrow \mathcal{E}^{* *}(U)[1 / \ell] \times \mathcal{E}^{* *}(U)\left[1 / \ell^{\prime}\right]
$$

is injective, it follows that $p_{U *} \circ p_{U}^{*}: \mathcal{E}^{* *}(U) \rightarrow \mathcal{E}^{* *}(U)$ is multiplication by $d$.
Proposition 3. Let $p: Y \rightarrow X$ be a finite, étale, degree $d$ map of noetherian schemes, and let $\mathcal{E} \in \mathrm{SH}(X)$ be an $\mathrm{MGL}_{X}$-module. Then there is a nilpotent element $x \in \operatorname{MGL}^{0,0}(X)$ such that $p_{*} \circ p^{*}: \mathcal{E}^{* *}(X) \rightarrow \mathcal{E}^{* *}(X)$ is multiplication by $d+x$.

Proof. It suffices to handle the case $\mathcal{E}=\mathrm{MGL}_{A}$. Define $x \in \operatorname{MGL}^{0,0}(X)$ by

$$
x=p_{*} p^{*}\left(1_{\mathrm{MGL}(X)}\right)-d ;
$$

by the projection formula, $p_{*} p^{*}$ is multiplication by $d+x$, so we need only show that $x$ is nilpotent.

By Lemma 2, $X$ admits a finite open cover $\mathcal{U}=\left\{U_{i} \mid i=1, \ldots m\right\}$ such that, letting $V_{i}:=p^{-1}\left(U_{i}\right)$, and letting $p_{i}: V_{i} \rightarrow U_{i}$ be the map induced by $p$, we have $p_{i *} \circ p_{i}^{*}\left(1_{\mathrm{MGL}^{0,0}\left(U_{i}\right)}\right)=d$. Thus, we see that $x$ restricts to zero in $\operatorname{MGL}^{0,0}\left(U_{i}\right)$.

Let $W_{r}=\cup_{i=1}^{r} U_{i}$, and let $j_{r}: W_{r} \rightarrow X$ be the inclusion. We show by induction on $r$ that $j_{r}^{*}(x)^{r}=0$, the case $r=1$ having just been settled and the case $r=m$ the result we wish to prove.

So fix $r, 1 \leq r<m$, and suppose that $j_{r}^{*}(x)^{r}=0$. Let $j: W_{r} \rightarrow W_{r+1}$ be the open immersion, and let $i: Z:=W_{r+1} \backslash W_{r} \rightarrow W_{r+1}$ be the closed complement. We have the long exact sequence in $\mathcal{E}$-cohomology

$$
\ldots \rightarrow \mathcal{E}_{Z}^{0,0}\left(W_{r+1}\right) \xrightarrow{i_{*}} \mathcal{E}^{0,0}\left(W_{r+1}\right) \xrightarrow{j^{*}} \mathcal{E}^{0,0}\left(W_{r}\right) \rightarrow \ldots
$$

which is a sequence of $\mathcal{E}^{0,0}\left(W_{r+1}\right)$-modules.
By assumption $j^{*}\left(j_{r+1}^{*}(x)^{r}\right)=0$, so there is an $\alpha \in \mathcal{E}_{Z}^{0,0}\left(W_{r+1}\right)$ with $j_{r+1}^{*}(x)^{r}=$ $i_{*}(\alpha)$. On the other hand, letting $j^{\prime}: U_{r+1} \rightarrow X$ be the inclusion, we have already seen that $j^{\prime *}(x)=0$. As $i: Z \rightarrow W_{r+1}$ factors through $i^{\prime}: Z \rightarrow U_{r+1}$, we see that $j_{r+1}^{*}(x) \cdot \alpha=0$. Thus

$$
j_{r+1}^{*}(x)^{r+1}=j_{r+1}^{*}(x) \cdot i_{*}(\alpha)=i_{*}\left(j_{r+1}^{*}(x) \cdot \alpha\right)=0
$$

and the induction continues.

## References

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