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We prove that the Hitchin parametrization provides geodesic coordinates at the Fuchsian locus for the pressure metric in the Hitchin component $\mathcal{H}_3(S)$ of surface group representations into $\mathrm{PSL}(3,\mathbb{R})$.

The proof consists of the following elements: We compute first derivatives of the pressure metric using the thermodynamic formalism. We invoke a gauge-theoretic formula to compute the first and second variations of the reparametrization functions by studying flat connections from Hitchin's equations and their parallel transports. We then extend these expressions of integrals over closed geodesics to integrals over the two-dimensional surface. Symmetries of the Liouville measure then provide cancellations, which show that the first derivatives of the pressure metric tensors vanish at the Fuchsian locus.

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1 Introduction

The Weil–Petersson metric on Teichmüller space is a central object in classical Teichmüller theory. Quite a bit is known about it: it is a negatively curved real analytic

Kähler metric with isometry group induced from the extended mapping class group (see Ahlfors [1], Tromba [36] and Masur and Wolf [25]). Although it is not complete (see Wolpert [38] and Chu [11]), it resembles a complete negative curved metric and shares many similar nice properties (see Wolpert [38; 39]).

In recent years, considerable attention has focused on higher-rank Teichmüller spaces; see Goldman [13], Hitchin [15] and Labourie [19]. It is natural to seek metric structures on these spaces with the hope that such structure will reflect important properties of the spaces. To that end, Bridgeman, Canary, Labourie and Sambarino [8] have extended the Weil–Petersson metric from Teichmüller space to an analytic Riemannian metric by techniques from thermodynamic formalism, called the pressure metric on Hitchin components. The Hitchin component $\mathcal{H}_n(S)$, defined by Hitchin in [15], is a special component of the representation space of the fundamental group of a closed surface S of genus $g \geq 2$ into $\mathrm{PSL}(n, \mathbb{R})$. In particular, the Teichmüller space $\mathcal{T}(S)$, identified as representations into $\mathrm{PSL}(2, \mathbb{R})$, embeds in this component and is called the Fuchsian locus. To define the pressure metric, we associate a geodesic flow to each Hitchin representation and describe these reparametrized geodesic flows by some Hölder functions, called reparametrization functions. Our pressure metric is defined on the tangent space of a Hitchin component by taking the variance of the first variations of the reparametrization functions that record the infinitesimal change of the representations.

Bridgeman, Canary, Labourie and Sambarino have proved that the pressure metric in fact restricts to a multiple of the Weil–Petersson metric on the Fuchsian locus and is invariant under the action of the mapping class group. Despite this nice coincidence, very little is presently known about the pressure metric. Some C^0 properties of the pressure metric have recently been identified by Labourie and Wentworth [20]. In particular, they show that, when restricted to the Fuchsian locus, the pressure metric is proportional to a Petersson-type pairing for variation given by holomorphic differentials. Building upon their work, our goal in this paper is to investigate some variational C^1 properties of the pressure metric using tools from thermodynamic formalism.

One may be curious to what extent the pressure metric in Hitchin components resembles Weil–Petersson geometry. Inspired by Ahlfors' work [1] showing the Bers coordinates are geodesic for Weil–Petersson metric, we will show that, for one particular case of the Hitchin component, similar coordinates are geodesic for the pressure metric near the Fuchsian locus. The Hitchin component we consider is $\mathcal{H}_3(S)$, which coincides with the space of convex real projective structures; see Choi and Goldman [10]. It is a prototypical example of higher-rank Teichmüller spaces. We expect similar results will hold for general cases of Hitchin components $\mathcal{H}_n(S)$.

Inspired by the methods of Labourie and Wentworth [20] for the C^0 properties of the pressure metric, we will find and evaluate expressions for the derivatives of the pressure metric at the Fuchsian locus for the case of $\mathrm{PSL}(3,\mathbb{R})$ and its Hitchin component $\mathcal{H}_3(S)$.

The coordinates we choose are very natural in the setting of Hitchin components from a Higgs bundle perspective. Picking $(q_1,\dots,q_{6g-6})$ to be a basis for $H^0(X,K^2)$ over $\mathbb{R}$ and $(q_{6g-5},\dots,q_{16g-16})$ to be a basis for $H^0(X,K^3)$ over $\mathbb{R}$, every element of $\mathcal{H}_3(S)$ corresponds to some

$$m(\xi)=\xi_1 q_1+\cdots+\xi_l q_l$$

with $\xi=(\xi_1,\dots,\xi_l)\in\mathbb{R}^l$ and $l=16g-16$.

The ξ_i are coordinate functions and the coordinate system is realized by the Hitchin parametrization $\mathcal{H}_3(S)\cong H^0(X,K^2)\oplus H^0(X,K^3)$. The Hitchin parametrization is given by the Hitchin section of the Hitchin fibration, which was defined by Hitchin in [15] and will be explained in the next section.

We will show:

Theorem 1.1 *Let S be a closed oriented surface with genus $g\geq 2$. For any point $\sigma\in\mathcal{T}(S)\subset\mathcal{H}_3(S)$, let X be the Riemann surface corresponding to σ. Then the Hitchin parametrization $H^0(X,K^2)\oplus H^0(X,K^3)$ provides geodesic coordinates for the pressure metric at σ.*

More explicitly, if we denote components of the pressure metric at σ by $g_{ij}(\sigma)$ with respect to the coordinates given by Hitchin parametrization, then $\partial_k g_{ij}(\sigma)=0$ for all possible i, j and k ranging from 1 to $16g-16$.

The proof will be a combination of techniques from the theory of thermodynamic formalism and the theory of Higgs bundles. On the one hand, we will use thermodynamic formalism to study the pressure metric and investigate its C^1 properties. On the other hand, reparametrization functions and their variations need to be understood via their Higgs bundle invariants. We now outline some important ingredients of our computations and proofs.

Since there are two types of tangential directions in $\mathcal{H}_3(S)$ — directions given by quadratic differentials and directions given by cubic differentials (corresponding to directions along the Fuchsian locus and transverse to it, respectively) — the derivatives of the metric tensor will be divided into different cases according to this distinction:

- The vanishing of a few types of first derivative of the metric tensor follows easily from the geometric facts that the Fuchsian locus is a totally geodesic embedding into the Hitchin component and that the Bers coordinates on Teichmüller space are geodesic.

- On the other hand, to compute the bulk of the components, we need to invoke thermodynamic formalism to obtain an explicit formula for first derivatives of the pressure metric. We find a formula for the first variations of the pressure metric by computing third derivatives of pressure functions using the theory of the Ruelle operator. This expression involves the first and second variations of the reparametrization functions.

- We start from studying the first and second variations of the reparametrization functions on closed geodesics. Because vectors tangent to periodic geodesics are dense in tangent bundles of hyperbolic surfaces, the computation of the first and second variations of the reparametrization functions on closed geodesics can be extended to the unit tangent bundle after an argument that the natural extensions are Hölder functions.

- To study the first variations of the reparametrization functions on closed geodesics, we recall a gauge-theoretic formula from [20]. We then interpret the resulting formula as defining a system of homogeneous ordinary differential equations, which we proceed to solve.

- Finding the second variations of the reparametrization functions is equivalent to understanding the first variations of our gauge-theoretic formula from the previous paragraph. The difficulty here is in describing how projections onto the eigenvectors for the holonomy map vary when we have a family of representations in the Hitchin component. Indeed, it turns out that we need to understand the variations of all of the eigenvectors of our holonomy map. We interpret this problem in terms of solving a system of nonhomogeneous ordinary differential equations with suitable boundary conditions, which we then proceed to solve.

- For some types of metric tensors that involve both the tangential directions and transverse directions to the Fuchsian locus, analyzing flat connections associated to these directions require understanding the corresponding harmonic metrics that are solutions of Hitchin's equations. The harmonic metrics are no longer diagonalizable when leaving the Fuchsian locus along these mixed directions. We break up the infinitesimal version of Hitchin's equation system and obtain nine scalar equations. We analyze them by maximum principles and Bochner techniques to compute the second variations of the reparametrization functions.

- The evaluation of first derivatives of the pressure metric can be lifted to the Poincaré disk following an idea from [20]. Here is where it becomes important that we are taking first derivatives of the pressure metric rather than zero derivatives of the pressure metric. In particular, we find formulas involving iterated integrals of these holomorphic differentials. Specifying a point on the unit tangent bundle, we can identify the Poincaré disk as our coordinate chart and write down the analytic expansions of our holomorphic differentials on this chart. Using geodesic flow invariance and rotational invariance of the Liouville measure, we find that no nonzero coefficients of our analytic expansions remain after integration.

There are more cases of tangential directions along Fuchsian locus in $\mathcal{H}_n(S)$ for $n \geq 4$, where the harmonic metrics are not known to be diagonalizable. Despite the fact that this makes the analysis difficult, the $n = 3$ case suggests the following conjecture:

Conjecture 1.2 *Let S be a closed oriented surface with genus $g \geq 2$ and $n \geq 4$. For any point $\sigma \in \mathcal{T}(S) \subset \mathcal{H}_n(S)$, let X be the Riemann surface corresponding to σ; the Hitchin parametrization $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ provides geodesic coordinates for the pressure metric at σ.*

Recently, a Riemannian metric in $\mathcal{H}_n(S)$ associated to periods given by the first simple root length, $L_{\alpha_1}(\rho(\gamma)) = \log\bigl(\lambda_1(\rho(\gamma))/\lambda_2(\rho(\gamma))\bigr)$, has been defined by Bridgeman, Canary, Labourie and Sambarino [9], where $\lambda_1(\rho(\gamma))$ and $\lambda_2(\rho(\gamma))$ are the largest and second largest moduli of eigenvalues of $\rho(\gamma)$. This Riemannian metric is called the Liouville pressure quadratic form in [9]. Our methods of computing first derivatives of metric tensors can be applied to the Liouville pressure quadratic form. We expect similar geodesic coordinate results to hold in that setting as well.

Structure of the article In Section 2, we recall some fundamental results from the theory of thermodynamic formalism and reparametrizations of geodesic flows. We define the pressure metric. We also introduce Higgs bundles and Hitchin deformation for defining our coordinates in Hitchin components. Section 3 is devoted to preliminary proofs by thermodynamic formalism machinery. We compute the formula for third derivatives of the pressure function. In Section 4, we start the proof of the main theorem and divide the components of first derivatives of metric tensors into several types. We also include a gauge-theoretic formula given by Labourie and Wentworth [20] here. Then, in Section 5, we derive the second variations of the reparametrization functions by studying infinitesimal variation of parallel transport equations. In Section 6, we evaluate the first derivatives of the pressure metric and show they are zero following the steps explained above. We finally generalize the arguments to all types of metric tensors in Section 7.

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2 Background and notation

In this section, we develop the notation and background material that we will need. We begin in Section 2.1 with a discussion of reparametrization of geodesic flows. Then, in Section 2.2, we recall the elements of thermodynamic formalism that we will need, and finally, in Section 2.3, we conclude with some notation from the theory of Higgs bundles which arises in our arguments.

Let S be a closed oriented surface with genus $g \geq 2$. We will define all the concepts for introducing the pressure metric in the context of Hitchin components $\mathcal{H}_n(S)$. The reader can find a more general version in [8]. The Hitchin components $\mathcal{H}_n(S)$ will be briefly introduced in Section 2.3.

Equip S with a complex structure J such that $X = (S, J)$ is a Riemann surface and thus a point in Teichmüller space. Let σ be the hyperbolic metric in the conformal class of X. We denote the unit tangent bundle of X with respect to σ by UX and the geodesic flow on (X, σ) by Φ.

2.1 Reparametrization function

We now introduce how we reparametrize the geodesic flow Φ by reparametrization functions. In particular, we introduce Livšic's theorem and geodesic flows for Hitchin representations.

Suppose $f\colon UX \to \mathbb{R}$ is a positive Hölder function and a a closed orbit. We will reparametrize the flow Φ by the function f so that, for the new flow Φ^f, the flow's direction remains the same everywhere but the speed of the flow changes. In particular, for a Φ–periodic orbit a, denoting its period with respect to Φ by $l(a)$, we want the period of a for the new flow Φ^f to be

$$l_f(a) = \int_0^{l(a)} f(\Phi_s(x))\, ds,$$

where x is any point on a.

This leads to the following definition of reparametrization:

Definition 2.1 Let $f\colon UX \to \mathbb{R}$ be a positive Hölder continuous function. We define the reparametrization of Φ by f to be the flow Φ^f on UX such that, for any $(x,t) \in UX \times \mathbb{R}$,

$$\Phi_t^f(x) = \Phi_{\alpha_f(x,t)}(x),$$

where $\kappa_f(x,t) = \int_0^t f(\Phi_s(x))\, ds$ and $\alpha_f\colon UX \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\alpha_f(x, \kappa_f(x,t)) = t.$$

Remark 2.2 Suppose O is the set of periodic orbits of Φ. If $a \in O$, then its period as a Φ_t^f–periodic orbit is $l_f(a)$ because

$$\Phi^f_{l_f(a)}(x) = \Phi_{\alpha_f(x, l_f(a))}(x) = \Phi_{l(a)}(x) = x.$$

We introduce Livšic cohomology classes [22]. Livšic-cohomologous Hölder functions turn out to reparametrize a flow in "equivalent" ways.

Let $C^h(UX)$ denote the set of real-valued Hölder functions on UX.

Definition 2.3 For $f, g \in C^h(UX)$, we say they are Livšic cohomologous if there exists a Hölder continuous function $V\colon UX \to \mathbb{R}$ that is differentiable in the flow's direction such that

$$f(x) - g(x) = \left.\frac{\partial V(\Phi_T(x))}{\partial t}\right|_{t=0}.$$

If f is Livšic cohomologous to g, then we will denote it by $f \sim g$.

We have the following important properties of Livšic-cohomologous functions:

(1) (Livšic's theorem [23]) Two Hölder continuous function f and g are Livšic cohomologous if and only if $l_f(a) = l_g(a)$ for every $a \in O$.

(2) If f and g are Livšic cohomologous, then they have the same integral over any Φ–invariant measure. This is because $\int_{UX} V(\Phi_t(x))\,dm = \text{const}$ for any Φ–invariant measure m and any $t \in \mathbb{R}$.

(3) [17, Proposition.19.2.8] If f and g are positive and Livšic cohomologous, then the reparametrized flows Φ^f and Φ^g are Hölder conjugate, ie there exists a Hölder homeomorphism $h\colon UX \to UX$ such that, for all $x \in UX$ and $t \in \mathbb{R}$,
$$h(\Phi_t^f(x)) = \Phi_t^g(h(x)).$$

The procedure of reparametrizing geodesic flows can be applied to Hitchin components $\mathcal{H}_n(S)$ and provides reparametrization functions as codings for representations. This idea was first introduced by Sambarino to study counting problems associated to Anosov representations [33]. It has also been elaborated later in [34; 31] and other work of Sambarino. In the setting we are working in, similar ideas lead to a construction of a geodesic flow Φ^ρ associated to each (conjugacy class of a) Hitchin representation $\rho \in \mathcal{H}_n(S)$. We refer the reader to [8] for the explicit construction. In particular, this flow relates $\mathcal{H}_n(S)$ to thermodynamic formalism. We will describe here some of the important properties of Φ^ρ:

- Φ^ρ is an Anosov flow.
- There exists a Hölder function $f_\rho\colon UX \to \mathbb{R}^+$, called the reparametrization function of ρ, such that the reparametrized flow Φ^{f_ρ} of Φ is Hölder conjugate to Φ^ρ [33].
- The period of the orbit associated to $[\gamma] \in \pi_1(S)$ is $\log \Lambda_\gamma(\rho)$, where $\Lambda_\gamma(\rho)$ is the spectral radius of $\rho(\gamma)$, ie the largest modulus of the eigenvalues of $\rho(\gamma)$.

Remark 2.4 One can also reparametrize the geodesic flow by a Hölder function with periods given by simple root lengths $L_{\alpha_1}(\rho(\gamma)) = \log\bigl(\lambda_1(\rho(\gamma))/\lambda_2(\rho(\gamma))\bigr)$, where $\lambda_1(\rho(\gamma))$ and $\lambda_2(\rho(\gamma))$ are the largest and second largest moduli of eigenvalues of $\rho(\gamma)$. This will lead to the Liouville pressure quadratic form, which also gives rise to a Riemannian metric in $\mathcal{H}_n(S)$ (see [9, Theorem 1.6]). However we will mainly focus on the spectrum radius length $\Lambda_\gamma(\rho)$ and its associated pressure metric in this paper.

2.2 Thermodynamic formalism

Next we will introduce some concepts arising from the thermodynamic formalism needed for our proofs. The introduction of most of the material here can also be found in [8]. After the introduction, we will define the pressure metric on Hitchin components.

As usual, we let Φ denote the geodesic flow on a hyperbolic surface (X, σ). We denote by $\mathcal{M}^{\Phi}$ the set of Φ–invariant probability measures on UX. Recall $l(a)$ denotes the period of the periodic point a with respect to Φ. Let

$$R_T = \{a \text{ closed orbit of } \Phi \mid l(a) \leq T\}.$$

Definition 2.5 The topological entropy of Φ is defined as

$$h(\Phi) = \limsup_{T \to \infty} \frac{\log \# R_T}{T}.$$

Recall, for a Hölder function $f : UX \to \mathbb{R}$, we write

$$l_f(a) = \int_0^{l(a)} f(\Phi_s(x))\, ds.$$

Definition 2.6 The topological pressure (or simply pressure) of a continuous function $f : UX \to \mathbb{R}$ with respect to Φ is defined by

$$\boldsymbol{P}(\Phi, f) = \limsup_{T \to \infty} \frac{1}{T} \log\biggl(\sum_{a \in R_T} e^{l_f(a)} \biggr).$$

Remark 2.7 From this definition, we see the pressure of a function f only depends on the periods of f, ie the collection of numbers $\{l_f(a)\}$ for any $a \in O$. From Livšic's theorem, we conclude the pressure of a function only depends on its Livšic cohomology class.

In statistical mechanics, suppose we are given a physical system with different possible states $i = 1, \dots, n$ and the energies of these states are $E_1, E_2, \dots, E_n$ with probability p_i that state i occurs. When energy is fixed, the principle "nature maximizes entropy h" says that the entropy $h(p_1, \dots, p_n) = \sum_{i=1}^{n} -p_i \log p_i$ of the distribution will be maximized with right choices of p_i. However, when the physical system is put in contact with a much larger "heat source" which is at a fixed temperature T and energy is allowed to pass between the original system and the heat source, "nature minimizes the free energy" will instead apply by reaching the "Gibbs distribution". The free energy is $E - kTh$, where k is a physical constant and $E = \sum_{i=1}^{n} p_i E_i$ is the average energy. In the thermodynamic formalism, energy potentials E_i of different states are encoded by continuous functions and "Gibbs distributions" for discrete probability spaces are generalized to equilibrium states. The principle "nature minimizes free energy" motivates the following:

Proposition 2.8 (variational principle) *Denoting the measure-theoretic entropy of Φ with respect to a measure $m \in \mathcal{M}^\Phi$ as $h(\Phi, m)$, the (topological) pressure of a continuous function $f : UX \to \mathbb{R}$ satisfies*

$$\boldsymbol{P}(\Phi, f) = \sup_{m \in \mathcal{M}^\Phi} \left(h(\Phi, m) + \int_{UX} f \, dm \right).$$

In particular, the topological entropy is the supremum of all measure-theoretic entropies,

$$\boldsymbol{P}(\Phi, 0) = \sup_{m \in \mathcal{M}^\Phi} (h(\Phi, m)) = h(\Phi).$$

Remark 2.9 One can also take Proposition 2.8 as definitions of pressure and topological entropies.

We shall omit the background geodesic flow Φ in the notation of pressure and simply write

$$\boldsymbol{P}(\cdot) = \boldsymbol{P}(\Phi, \cdot).$$

Definition 2.10 A measure $m \in \mathcal{M}^\Phi$ on UX such that

$$\boldsymbol{P}(f) = h(\Phi, m) + \int_{UX} f \, dm$$

is called an equilibrium state of f.

Proposition 2.11 (Bowen and Ruelle [6]) *For any Hölder function $f : UX \to \mathbb{R}$, with respect to the geodesic flow Φ, there exists a unique equilibrium state for f, denoted by m_f. Moreover, m_f is ergodic.*

Remark 2.12 By the definition of equilibrium states, if $f - g$ is Livšic cohomologous to a constant, then f and g have the same equilibrium states.

Definition 2.13 The equilibrium state m_0 for $f = 0$ is called a probability measure of maximal entropy. It is also called the Bowen–Margulis measure of Φ. We also denote it by m_Φ. It satisfies

$$\boldsymbol{P}(0) = \boldsymbol{P}(\Phi, 0) = h(\Phi, m_\Phi) = h(\Phi).$$

Remark 2.14 The Liouville measure m_L, the normalized Riemannian measure on UX, is a probability measure of maximal entropy for geodesic flows of closed hyperbolic manifolds (see [16, Section 2]). Thus, when considering the geodesic flow Φ of a hyperbolic surface (X, σ), we have $m_L = m_\Phi$.

Given f a positive Hölder continuous function on UX, denoting $h(f) = h(\Phi^f)$ to be the topological entropy of the reparametrized flow Φ^f, we have the following lemma, which allows us to "normalize" a Hölder function to have pressure zero:

Lemma 2.15 (Sambarino [33]; Bowen and Ruelle [6]) *The pressure satisfies*

$$\boldsymbol{P}(-hf) = 0$$

if and only if $h = h(f) = h(\Phi^f)$.

Potrie and Sambarino show, in the Hitchin component $\mathcal{H}_n(S)$, the topological entropy is maximized only along the Fuchsian locus. In particular, it is a constant on the Fuchsian locus.

Theorem 2.16 (Potrie and Sambarino [31]) *If* $\rho \in \mathcal{H}_n(S)$, *then* $h(\rho) \leq 2/(n-1)$. *Moreover, if* $h(\rho) = 2/(n-1)$, *then* ρ *lies in the Fuchsian locus.*

We start to define variance and covariance which will be important. The convergence of them for mean zero functions is classical.

Definition 2.17 For g a Hölder continuous function on UX with mean zero with respect to m_f (ie $\int_{UX} g \, dm_f = 0$), the variance of g with respect to f is defined as

$$\mathrm{Var}(g, m_f) = \lim_{T\to\infty} \frac{1}{T} \int_{UX} \left(\int_0^T g(\Phi_s(x)) \, ds \right)^2 dm_f(x). \tag{2-1}$$

Definition 2.18 For g_1 and g_2 Hölder continuous functions on UX with mean zero with respect to m_f (ie $\int_{UX} g_1 \, dm_f = \int_{UX} g_2 \, dm_f = 0$), the covariance of g_1, g_2 with respect to f is defined as

$$\begin{aligned} \mathrm{Cov}(g_1, g_2, m_f) \\ = \lim_{T\to\infty} \frac{1}{T} \int_{UX} \left(\int_0^T g_1(\Phi_s(x)) \, ds \right) \left(\int_0^T g_2(\Phi_s(x)) \, ds \right) dm_f(x). \end{aligned} \tag{2-2}$$

Note these expressions are finite:

Proposition 2.19 *For* g_1 *and* g_2 *Hölder continuous function on* UX *with mean zero with respect to* m_f, *the covariance of* g_1 *and* g_2 *is finite*:

$$\mathrm{Cov}(g_1, g_2, m_f) < \infty.$$

The convergence is guaranteed by decay of correlations (see [26]).

Definition 2.20 We define an operator $P_m\colon C^h(UX) \to C^h(UX)$ associated to a probability measure m on UX to be

$$P_m(g)(x) = g(x) - m(g),$$

where we use the notation $m(g) = \int_{UX} g\,dm$ for a probability measure m.

The following corollary will be useful:

Corollary 2.21 *It suffices to have $m_f(g_1) = 0$ and $m_f(g_2) < \infty$ to guarantee the convergence of covariance and*

$$\operatorname{Cov}(g_1, g_2, m_f) = \operatorname{Cov}(g_1, P_{m_f}(g_2), m_f) < \infty. \tag{2-3}$$

The same applies to the case $m_f(g_2) = 0$ and $m_f(g_1) < \infty$.

Proof We have

$$\begin{aligned}
&\frac{1}{T}\int_{UX}\left(\int_0^T g_1(\Phi_s(x))\,ds\right)\left(\int_0^T g_2(\Phi_s(x)) - P_{m_f}\big(g_2(\Phi_s(x))\big)\,ds\right)dm_f(x)\\
&\quad= \frac{1}{T}\int_{UX}\left(\int_0^T g_1(\Phi_s(x))\,ds\right)\left(\int_0^T m_f(g_2)\,ds\right)dm_f(x)\\
&\quad= m_f(g_2)\int_{UX}\int_0^T g_1(\Phi_s(x))\,ds\,dm_f(x) && \text{(as } m_f(g_2) \text{ is a constant)}\\
&\quad= m_f(g_2)\int_0^T\int_{UX} g_1(\Phi_s(x))\,dm_f(x)\,ds && \text{(by Fubini's theorem)}\\
&\quad= m_f(g_2)\int_0^T\int_{UX} g_1(x)\,dm_f(x)\,ds && \text{(as } m_f \text{ is } \Phi\text{–invariant)}\\
&\quad= 0.
\end{aligned}$$

Letting $T \to \infty$, we obtain the desired result. □

We will also need the following characterization of covariance for later use:

Proposition 2.22 (Pollicott [29]) *For g_1 and g_2 Hölder continuous functions with mean zero with respect to m_f (ie $\int_{UX} g_1\,dm_f = \int_{UX} g_2\,dm_f = 0$), the covariance of g_1 and g_2 may also be written as*

$$\operatorname{Cov}(g_1, g_2, m_f) = \lim_{T\to\infty}\int_{UX} g_2(x)\left(\int_{-T/2}^{T/2} g_1(\Phi_s(x))\,ds\right)dm_f(x).$$

Proof We have

$$
\begin{aligned}
&\operatorname{Cov}(g_1, g_2, m_f)\\
&\quad = \lim_{T\to\infty} \frac{1}{T} \int_{UX} \left(\int_0^T g_1(\Phi_s(x))\,ds\right)\left(\int_0^T g_2(\Phi_s(x))\,ds\right) dm_f(x)\\
&\quad = \lim_{T\to\infty} \frac{1}{T} \int_{UX} \left(\int_{-T/2}^{T/2} g_1(\Phi_s(x))\,ds\right)\left(\int_{-T/2}^{T/2} g_2(\Phi_s(x))\,ds\right) dm_f(x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{as } m_f \text{ is } \Phi\text{–invariant})\\
&\quad = \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) \frac{1}{T}\left(\int_{-T/2}^{T/2} g_2(\Phi_s(x))\,ds\right) dm_f(x)\,dt.
\end{aligned}
$$

Because $m \in \mathcal{M}^{\Phi}$, the following does not vary with s:

$$
\begin{aligned}
\text{const} &= \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) g_2(\Phi_s(x))\,dm_f(x)\,dt \qquad (\text{for all } s \in \mathbb{R})\\
&= \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) \frac{1}{S}\left(\int_{-S/2}^{S/2} g_2(\Phi_s(x))\,ds\right) dm_f(x)\,dt\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{average over } s \in \left[-\tfrac{1}{2}S, \tfrac{1}{2}S\right]\right)\\
&= \lim_{S\to\infty} \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) \frac{1}{S}\left(\int_{-S/2}^{S/2} g_2(\Phi_s(x))\,ds\right) dm_f(x)\,dt\\
&= \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) \frac{1}{T}\left(\int_{-T/2}^{T/2} g_2(\Phi_s(x))\,ds\right) dm_f(x)\,dt\\
&= \operatorname{Cov}(g_1, g_2, m_f).
\end{aligned}
$$

In particular, setting $s = 0$ gives

$$
\operatorname{Cov}(g_1, g_2, m_f) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \int_{UX} g_1(\Phi_t(x)) g_2((x))\,dm_f(x)\,dt.
$$

Rearranging the integrals gives the desired result. □

Higher correlation and higher covariance are introduced for Anosov diffeomorphism in [18]. For geodesic flows, we define:

Definition 2.23 For g_1, g_2 and g_3 Hölder continuous functions with mean zero with respect to m_f, we define the higher covariance by

$$
\begin{aligned}
&\operatorname{Cov}(g_1, g_2, g_3, m_f)\\
&\quad = \lim_{T\to\infty} \frac{1}{T} \int_{UX} \int_0^T g_1(\Phi_t(x))\,dt \int_0^T g_2(\Phi_t(x))\,dt \int_0^T g_3(\Phi_t(x))\,dt\,dm_f(x).
\end{aligned}
$$

Equivalently,

$$\operatorname{Cov}(g_1, g_2, g_3, m_f) = \lim_{T\to\infty} \int_{UX} g_1(x)\left(\int_{-T/2}^{T/2} g_2(\Phi_s(x))\,ds\right)\left(\int_{-T/2}^{T/2} g_3(\Phi_s(x))\,ds\right) dm_f(x).$$

This equivalence is clear from the proof of Proposition 2.22. The convergence of $\operatorname{Cov}(h_1, h_2, h_3, m)$ is guaranteed by "exponential multiple mixing" for geodesic flow on negatively curved compact surfaces (see Pollicott's note [30]). These definitions will be used later when we introduce first derivatives of the pressure metric.

We use the general notation in the sequel

$$\partial_s f(0) = \frac{df(s)}{ds}\bigg|_{s=0}, \quad \partial_s^2 f(0) = \frac{d^2 f(s)}{ds^2}\bigg|_{s=0}. \tag{2-4}$$

If there is more than one parameter, for example $f(s_1, s_2, \dots, s_k)$ and $k \geq 2$, then we specify the indexes that we are taking derivatives of, such as

$$\partial_{s_{i_1}\dots s_{i_j}} f(0) = \frac{\partial^j f(s_1, s_2, \dots, s_k)}{\partial s_{i_1} \cdots \partial s_{i_j}}\bigg|_{s_1=s_2=\dots=0}. \tag{2-5}$$

Theorem 2.24 (Parry and Pollicott [27]; McMullen [26]) *Let f_s be a smooth family of functions in $C^h(UX)$. Then*:

(1) *The first derivative of $\boldsymbol{P}(f_s)$ at $s=0$ is given by*

$$\frac{d\boldsymbol{P}(f_s)}{ds}\bigg|_{s=0} = \int_{UX} \partial_s f_0 \, dm_{f_0}. \tag{2-6}$$

(2) *If the first derivative is zero*, *then*

$$\frac{d^2\boldsymbol{P}(f_s)}{ds^2}\bigg|_{s=0} = \operatorname{Var}(\partial_s f_0, m_{f_0}) + \int_{UX} \partial_s^2 f_0 \, dm_{f_0}. \tag{2-7}$$

(3) *If the first derivative is zero*, *then* $\operatorname{Var}(\partial_s f_0, m_{f_0}) = 0$ *if and only if* $\partial_s f_0$ *is Livšic cohomologous to zero.*

Remark 2.25 If $f(s,t)$ is a smooth two-parameter family in $C^h(UX)$, then

$$\begin{aligned} &\frac{\partial \boldsymbol{P}(f(s,t))}{\partial t\,\partial s}\bigg|_{s=t=0} \\ &\qquad = \operatorname{Cov}\big(P_{m_{f(0)}}(\partial_s f(0)), P_{m_{f(0)}}(\partial_t f(0)), m_{f(0)}\big) + \int_{UX} \partial_{st} f(0)\, dm_{f(0)}. \end{aligned} \tag{2-8}$$

Define $\mathcal{P}(UX)$ to be the set of pressure zero Hölder functions on UX, ie

$$\mathcal{P}(UX) = \{f \in C^h(UX) : \boldsymbol{P}(f) = 0\}.$$

The tangent space of $\mathcal{P}(UX)$ at f is the set

$$T_f\mathcal{P}(UX) = \ker d_f \boldsymbol{P} = \left\{h \in C^h(UX) \;\middle|\; \int_{UX} h \, dm_f = 0\right\}.$$

We define a pressure seminorm on the tangent space of $\mathcal{P}(UX)$ at f, by letting:

Definition 2.26 The pressure seminorm of $g \in T_f\mathcal{P}(UX)$ is defined as

$$\langle g, g\rangle_P = -\frac{\operatorname{Var}(g, m_f)}{\int_{UX} f \, dm_f}.$$

One notices, for $g \in T_f\mathcal{P}(UX)$, the variance $\operatorname{Var}(g, m_f) = 0$ if and only if g is Livšic cohomologous to 0, ie $g \sim 0$.

2.3 Higgs bundles and Hitchin deformation

We next introduce all the notation from the theory of Higgs bundles that will arise in our arguments. We also introduce a coordinate system on the Hitchin component at the end of the section.

Recall S is a closed oriented surface with genus $g \geq 2$ and $X = (S, J)$ is a Riemann surface.

Definition 2.27 A rank n Higgs bundle over X is a pair (E, Φ), where E is a holomorphic vector bundle of rank n and $\Phi \in H^0(X, \operatorname{End}(E) \otimes K)$ is called a Higgs field. An $\mathrm{SL}(n, \mathbb{C})$–Higgs bundle is a Higgs bundle (E, Φ) satisfying $\det E = \mathcal{O}$ and $\operatorname{Tr} \Phi = 0$.

Definition 2.28 (1) A Higgs bundle (E, Φ) is semistable if every proper Φ–invariant holomorphic subbundle F of E satisfies

$$\frac{\deg(F)}{\operatorname{rank}(F)} \leq \frac{\deg(E)}{\operatorname{rank}(E)}$$

and stable if this inequality is strict.

(2) A semistable Higgs bundle (E, Φ) is polystable if it decomposes as a direct sum of stable Higgs bundles.

Theorem 2.29 *It is classical that, for a holomorphic vector bundle E with holomorphic structure $\overline{\partial}_E$ and a Hermitian metric H, there exists a unique connection $\nabla_{\overline{\partial}_E,H}$, called the Chern connection, such that*:

(1) $\nabla^{0,1}_{\overline{\partial}_E,H} = \overline{\partial}_E$.
(2) $\nabla_{\overline{\partial}_E,H}$ *is unitary.*

We will from now on restrict our interest to degree zero Higgs bundles.

Theorem 2.30 (Hitchin [14]; Simpson [35]) *Let (E,Φ) be a rank n, degree zero Higgs bundle on X. Then E admits a Hermitian metric H satisfying Hitchin's equation if and only if (E,Φ) is polystable. Here Hitchin's equation is*

$$F_{\overline{\partial},H} + [\Phi,\Phi^{*_H}] = 0, \tag{2-9}$$

where $F_{\overline{\partial},H}$ is the curvature of the Chern connection $\nabla_{\overline{\partial}_E,H}$ and Φ^{*_H} is the Hermitian adjoint of Φ.

Remark 2.31 Define a connection D_H on (E,Φ,H) as

$$D_H = \nabla_{\overline{\partial}_E,H} + \Phi + \Phi^{*_H}. \tag{2-10}$$

D_H is flat if and only if Hitchin's equation is satisfied.

We define the Higgs bundles moduli space and de Rham moduli space as:

Definition 2.32
- The space of gauge equivalence classes of polystable $\mathrm{SL}(n,\mathbb{C})$–Higgs bundles is called the moduli space of $\mathrm{SL}(n,\mathbb{C})$–Higgs bundles and is denoted by $\mathcal{M}_{\mathrm{Higgs}}(\mathrm{SL}(n,\mathbb{C}))$.
- The space of gauge equivalence classes of reductive flat $\mathrm{SL}(n,\mathbb{C})$ connections is called the de Rham moduli space and is denoted by $\mathcal{M}_{\mathrm{de\,Rham}}(\mathrm{SL}(n,\mathbb{C}))$.

Remark 2.33 The Hitchin–Simpson theorem gives a one-to-one correspondence between $\mathcal{M}_{\mathrm{Higgs}}(\mathrm{SL}(n,\mathbb{C}))$ and $\mathcal{M}_{\mathrm{de\,Rham}}(\mathrm{SL}(n,\mathbb{C}))$ from the above remark. It is also called the Hitchin–Kobayashi correspondence.

We will introduce the Hitchin fibration and Hitchin section following Baraglia's work [2]. We refer the reader to [2, Section 2] for a more comprehensive exposition.

Given a principal 3–dimensional subalgebra $\mathfrak{s} = \mathrm{span}\{x,e,\tilde{e}\}$ of $\mathfrak{sl}(n,\mathbb{C})$ consisting of a semisimple element x and regular nilpotent elements e and $\tilde{e}$ with commutation relations

$$[x,e]=e, \quad [x,\tilde{e}]=-\tilde{e}, \quad [e,\tilde{e}]=x,$$

the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ decomposes into a direct sum of irreducible subspaces under the adjoint representation of $\mathfrak{s}$,

$$\mathfrak{sl}(n,\mathbb{C}) = \bigoplus_{i=1}^{n-1} V_i.$$

We take $e_1,\dots,e_{n-1}$ as highest-weight elements of $V_1,\dots,V_{n-1}$, where $e_1 = e$. With these defined, there exists a basis of $\mathrm{SL}(n,\mathbb{C})$–invariant homogeneous polynomials p_i of degree i on $\mathfrak{sl}(n,\mathbb{C})$, where $2 \le i \le n$, such that, for all elements $f \in \mathfrak{sl}(n,\mathbb{C})$ of the form

$$f = \tilde{e} + \alpha_2 e_1 + \cdots + \alpha_n e_{n-1},$$

we have $p_i(f) = \alpha_i$.

Definition 2.34 The Hitchin fibration is a map from the moduli space of $\mathrm{SL}(n,\mathbb{C})$–Higgs bundles over X to the direct sum of holomorphic differentials given by

$$p\colon \mathcal{M}_{\mathrm{Higgs}}(\mathrm{SL}(n,\mathbb{C})) \to \bigoplus_{i=2}^{i=n} H^0(X,K^i), \quad (E,\Phi) \mapsto (p_2(\Phi),\dots,p_n(\Phi)),$$

where p_i are the homogeneous invariant polynomials defined above.

Definition 2.35 A Hitchin section s of the Hitchin fibration is a map back from $\bigoplus_{i=2}^{i=n} H^0(X,K^i)$ to $\mathcal{M}_{\mathrm{Higgs}}(\mathrm{SL}(n,\mathbb{C}))$. For $q = (q_2,q_3,\dots,q_n) \in \bigoplus_{i=2}^{i=n} H^0(X,K^i)$, we define $s(q)$ to be a Higgs bundle $E = K^{(n-1)/2} \oplus K^{(n-3)/2} \oplus \cdots \oplus K^{(1-n)/2}$ with its Higgs field given by

$$\Phi(q) = \tilde{e} + q_2 e_1 + q_3 e_2 + \cdots + q_n e_{n-1}.$$

More explicitly, we have

$$\Phi(q) = \begin{bmatrix} 0 & r_1q_2 & r_1r_2q_3 & r_1r_2r_3q_4 & \cdots & \prod_{i=1}^{n-2} r_iq_{n-1} & \prod_{i=1}^{n-1} r_iq_n \\ 1 & 0 & r_2q_2 & r_2r_3q_3 & \cdots & \cdots & \prod_{i=2}^{n-1} r_iq_{n-1} \\ 0 & 1 & 0 & r_3q_2 & r_3r_4q_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & r_{n-1}q_2 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix} : E \to E \otimes K,$$

where $r_i = \frac{1}{2} i(n-i)$ and $K^{1/2}$ is a holomorphic line bundle with its square to be the canonical line bundle K. The notation for e_i we use here can be found in [2; 21].

Remark 2.36 There exists an involutive automorphism σ on $\mathfrak{sl}(n,\mathbb{C})$ such that

$$\sigma(e_i) = -e_i, \quad \sigma(\tilde{e}) = -\tilde{e}.$$

Composing with the compact real form ρ on $\mathfrak{sl}(n,\mathbb{C})$ given by $\rho(X) = -X^*$, we can obtain the split real involution given by $\lambda = \rho \circ \sigma$. The fixed-point set of λ is the real split form $\mathfrak{sl}(n,\mathbb{R})$. A detailed exposition for this can be found in [2].

From the fact that $\lambda(\Phi(q)) = \Phi(q)^*$, one can see the flat connection (2-10) has holonomy in the split real form of $\mathfrak{sl}(n,\mathbb{C})$. Hitchin therefore shows that the Higgs bundles in the image of the Hitchin section have holonomy in $\mathrm{SL}(n,\mathbb{R})$ (see [15]). The representation space of these Higgs bundles up to conjugacy equivalence forms a connected component of the representation variety $\mathrm{Rep}(\pi_1(S), \mathrm{SL}(n,\mathbb{R}))$, called the Hitchin component $\mathcal{H}_n(S)$. Here we recall that the representation variety $\mathrm{Rep}(\pi_1(S), \mathrm{SL}(n,\mathbb{R}))$ is the space of conjugacy classes of reductive representations from $\pi_1(S)$ to $\mathrm{SL}(n,\mathbb{R})$.

Remark 2.37 The isomorphism between $\mathcal{H}_n(S)$ and $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ yields a parametrization of the Hitchin component $\mathcal{H}_n(S)$. We call $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ the Hitchin base. In particular, the tangent space at the Fuchsian point X is identified with the Hitchin base.

Fixing $E = K^{(n-1)/2} \oplus K^{(n-3)/2} \cdots \oplus K^{(1-n)/2}$, we consider the following map as an infinitesimal change of a family of Higgs fields Φ_ϵ associated to q:

$$\chi\colon \bigoplus_{i=2}^{i=n} H^0(X, K^i) \to \Omega^{1,0}(X, \mathfrak{sl}(\mathfrak{n},\mathbb{R})), \quad \chi(q) = \sum_{i=2}^{n} q_i \otimes e_{i-1}.$$

In particular, the infinitesimal change of a family of flat connections (2-10) in the space $\mathcal{M}_{\mathrm{de\,Rham}}(\mathrm{SL}(n,\mathbb{C}))$ associated to q defines an isomorphism of $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ with the tangent space of the Hitchin component $T_X\mathcal{H}_n(S)$. Associated to $\chi(q)$, the deformation of flat connections which is the infinitesimal version of (2-10) is:

Definition 2.38 At the Fuchsian point X, we define our Hitchin deformation associated to q to be

$$\varphi(q) := \chi(q) + \lambda(\chi(q)),$$

where λ is the antilinear involution for the split real form of $\mathfrak{sl}(\mathfrak{n},\mathbb{C})$ defined above.

This type of deformation will be the tangential objects we consider for the pressure metric.

Remark 2.39 The Hitchin parametrization in Remark 2.37 gives a coordinate system for $\mathcal{H}_n(S)$ based at X. More explicitly, given a basis $\{q_i\}_{i=1}^{i=l}$ of $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ with $l = 2(n^2-1)(g-1)$, the coordinate system is given by

$$m(\xi) = \xi_1 q_1 + \cdots + \xi_l q_l,$$

where $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{R}^l$. Because of the isomorphism between $\mathcal{H}_n(S)$ and $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$, the vector $\xi = (\xi_1, \dots, \xi_l)$ provides local parameters on $\mathcal{H}_n(S)$ and $\xi_i \colon \mathcal{H}_n(S) \to \mathbb{R}$ is a coordinate function for $1 \le i \le l$.

2.4 The pressure metric on Hitchin components

We define the pressure metric for Hitchin components $\mathcal{H}_n(S)$ in this subsection and state some known results about it.

Recall $\mathcal{H}(UX)$ is the space of pressure zero Hölder functions modulo Livšic coboundaries. We relate $\mathcal{H}(UX)$ to the Hitchin component $\mathcal{H}_n(S)$ by the following thermodynamic mapping:

Definition 2.40 The thermodynamic mapping $\Psi \colon \mathcal{H}_n(S) \to \mathcal{H}(UX)$ from a Hitchin component $\mathcal{H}_n(S)$ to the space $\mathcal{H}(UX)$ of Livšic cohomology classes of pressure zero Hölder functions on UX is defined as

$$\Psi(\rho) = [-h(\rho) f_\rho],$$

where $h(\rho) = h(f_\rho) = h(\Phi^{f_\rho})$ is the topological entropy of the reparametrized flow Φ^{f_ρ}.

The mapping Ψ admits local analytic lifts to the space $\mathcal{P}(UX)$ of pressure zero Hölder functions. In particular, the map $\widetilde{\Psi} \colon \mathcal{H}_n(S) \to \mathcal{P}(UX)$ given by $\widetilde{\Psi}(\rho) = -h(\rho) f_\rho$ is an analytic local lift of Ψ. This enables us to pull back the pressure form on $\mathcal{P}(UX)$ to obtain a pressure form on $\mathcal{H}_n(S)$.

We will from now on write $f_\rho^N = -h(\rho) f_\rho$ for the normalized reparametrization function.

Given an analytic family $\{\rho_s\}_{s \in (-1,1)}$ of (conjugacy classes of) representations in the Hitchin component $\mathcal{H}_n(S)$, we define $\dot{\rho}_0 = \partial_s \rho_0 = \partial_s \rho_s(0)$. Let $\{f_{\rho_s}\}_{s \in (-1,1)}$ be associated reparametrization functions, we pull back the pressure form on $\mathcal{P}(UX)$ to obtain

$$\begin{aligned}\langle\dot\rho_0,\dot\rho_0\rangle_P &= \langle d\widetilde\Psi(\dot\rho_0), d\widetilde\Psi(\dot\rho_0)\rangle_P\\ &= \left\langle \frac{\partial(-h(\rho_s)f_{\rho_s})}{\partial s}\bigg|_{s=0}, \frac{\partial(-h(\rho_s)f_{\rho_s})}{\partial s}\bigg|_{s=0}\right\rangle_P\\ &= \langle\partial_s(f_{\rho_s}^N)(0), \partial_s(f_{\rho_s}^N)(0)\rangle_P\\ &= -\frac{\operatorname{Var}(\partial_s(f_{\rho_s}^N)(0), m_{f_{\rho_0}^N})}{\int_{UX} f_{\rho_0}^N\,dm_{f_{\rho_0}^N}}.\end{aligned}$$

It is proved in [8] that the pullback pressure form is nondegenerate and thus defines a Riemannian metric on $\mathcal{H}_n(S)$:

Definition 2.41 If $\{\rho_s\}_{s\in(-1,1)}$ and $\{\eta_t\}_{t\in(-1,1)}$ are two analytic families of (conjugacy classes of) representations in the Hitchin component $\mathcal{H}_n(S)$ such that $\rho_0=\eta_0$, the pressure metric for $\dot{\rho_0},\dot{\eta_0}\in T_{\rho_0}\mathcal{H}_n(S)$ is defined as

$$\langle\dot{\rho_0},\dot{\eta_0}\rangle_P = -\frac{\operatorname{Cov}\big(\partial_s(f_{\rho_s}^N)(0), \partial_s(f_{\eta_s}^N)(0), m_{f_{\rho_0}^N}\big)}{\int_{UX} f_{\rho_0}^N\,dm_{f_{\rho_0}^N}}.$$

For simplicity, later we will also write $\partial_s(f_{\rho_s}^N)(0)=\partial_s f_{\rho_0}^N$ and $\partial_s(f_{\eta_s}^N)(0)=\partial_s f_{\eta_0}^N$. The principle is that we always first normalize a family of reparametrization functions to be pressure zero and then take derivatives.

Because of the identification of $\bigoplus_{i=2}^{i=n} H^0(X,K^i)$ with the tangent space of the Hitchin component $T_X\mathcal{H}_n(S)$, our Hitchin deformation $\varphi(q)$ introduced in Definition 2.38 can be thought of as tangent vectors in $T_X\mathcal{H}_n(S)$. With this understood, we introduce the following important results of Labourie and Wentworth [20]:

Let q_i be a holomorphic differential of degree k on X and let $\varphi(q_i)$ be the associated Hitchin deformation. Labourie and Wentworth [20] show the pressure metric satisfies

$$\langle\varphi(q_i),\varphi(q_i)\rangle_P = C(n,k)\langle q_i,q_i\rangle_X,$$

where $C(n,k)>0$ is a constant that does not depend on σ and $\langle q_i,q_i\rangle_X$ is the Petersson pairing

$$\langle q_i,q_i\rangle_X = \int_X q_i\bar q_i\sigma^{-k}(z)\,dA_\sigma$$

with $dA_\sigma=\sigma(z)\,dx\wedge dy$ denoting the area form for the hyperbolic metric σ.

If q_i and q_j are holomorphic differentials of the same degree, then

$$\begin{aligned}\langle \varphi(q_i), \varphi(q_j)\rangle_P &= \tfrac{1}{4}[\langle \varphi(q_i+q_j), \varphi(q_i+q_j)\rangle_P - \langle \varphi(q_i-q_j), \varphi(q_i-q_j)\rangle_P] \\ &= \tfrac{1}{4}C(n,k)\langle q_i+q_j, q_i+q_j\rangle_X - \tfrac{1}{4}C(n,k)\langle q_i-q_j, q_i-q_j\rangle_X \\ &= C(n,k)\langle q_i, q_j\rangle_X.\end{aligned}$$

If q_i and q_j are holomorphic differentials of different degrees on X, Labourie and Wentworth [20] show that

$$\langle \varphi(q_i), \varphi(q_j)\rangle_P = 0. \tag{2-11}$$

We denote the pressure metric components with respect to the coordinates introduced in Remark 2.39 by g_{ij}. Equivalently, the metric tensor $g_{ij}(\xi)$ means that the pressure metric $\langle \cdot, \cdot \rangle_P$ is evaluated at ξ with tangential vectors parallel to the q_i–axis and q_j–axis. In particular, at the point X, we have $g_{ij}(0) = g_{ij}(\sigma) = \langle \varphi(q_i), \varphi(q_j)\rangle_P$. It is always possible to choose an orthonormal basis $\{q_i\}$ with respect to our pressure metric from the vector space $\bigoplus_{i=2}^{i=n} H^0(X, K^i)$ so that $g_{ij}(\delta) = \delta_{ij}$.

3 More thermodynamic formalism

Bowen and Ruelle's work [3; 4; 6] guarantees that many of the results in the thermodynamic formalism proved for subshifts of finite type by the Ruelle operator still hold for Axiom A diffeomorphisms and Axiom A flows. We adopt this idea of simplifying the rather complicated object "flow" by discretizing it and studying a relative simple object "shift" given by symbolic coding. We will compute the formula for the third derivatives of pressure functions using subshifts of finite type. The reader can find an introduction for modeling hyperbolic diffeomorphisms by subshifts of finite type and modeling hyperbolic flows by suspension flows through Markov partition and symbolic dynamics in [5, Sections 3 and 4; 27, Appendix III].

Section 3.1 is devoted to the Ruelle operator and Ruelle–Perron–Frobenius theorem. These are important tools for studying subshifts of finite types. Then, in Section 3.2, we will compute the third derivatives of pressure functions in Lemma 3.8. These will be important for the proof of the main theorem in the next section.

3.1 Ruelle operator and others

We start with a cursory introduction to the elements of thermodynamic formalism for subshifts of finite types. A complete description is in [26; 27].

Definition 3.1 Let A be a $k \times k$ matrix of zeros and ones; we define the associated two-sided shift of finite type (Σ, σ_A), where Σ is the set of sequences

$$\Sigma = \{x = (x_n)_{n=-\infty}^{\infty} : x_n \in \{1, \dots, k\}, n \in \mathbb{Z}, A(x_n, x_{n+1}) = 1\}$$

and $\sigma_A \colon \Sigma \to \Sigma$ is defined by $\sigma_A(x) = y$, where $y_n = x_{n+1}$.

If instead we consider $x = (x_n)_{n=0}^{\infty}$ with the same restriction given by the matrix A and $\sigma(x) = y$, ie $y_n = x_{n+1}$ for $n \geq 0$, then we obtain a one-sided shift of finite type.

The set $\{1, \dots, k\}$ is equipped with the discrete topology and the two-sided (or one-sided) shift space Σ_A is equipped with the associated product topology.

Given $\alpha \in (0, 1)$, we can metrize the topology on the two-sided shift space Σ by defining a metric $d_\alpha(x, y) = \alpha^N$, where N is the largest nonnegative integer such that $x_i = y_i$ for $|i| < N$. Similarly, we have a metric d_α defined for one-sided shift space.

We let $C(\Sigma)$ be the space of real-valued continuous functions on Σ and $C^\alpha(\Sigma)$ be the space of real-valued Hölder functions on Σ with Hölder exponent α with respect to d_α.

The two-sided (one-sided) shift of finite type (Σ, σ_A) is called a subshift of finite type if σ_A is topologically transitive.

We define the pullback operator on $C^\alpha(\Sigma)$ by $(\sigma_A^* f)(y) = f(\sigma_A(y))$. Similarly to Definition 2.3, we define:

Definition 3.2 f_1 and f_2 in $C^\alpha(\Sigma)$ are (Livšic) cohomologous if

$$f_1 - f_2 = f_3 - \sigma_A^* f_3$$

for some $f_3 \in C^\alpha(\Sigma)$.

From now on, we assume our subshift of finite type (Σ, σ_A) to be one-sided unless otherwise specified.

Definition 3.3 Given $w \in C^\alpha(\Sigma)$, the Ruelle operator (or transfer operator) on $f \in C^\alpha(\Sigma)$ is defined by

$$\mathcal{L}_w(f)(x) = \sum_{\sigma_A(y)=x} e^{w(y)} f(y).$$

Theorem 3.4 (Ruelle, Perron and Frobenius) *Suppose (Σ, σ_A) is topologically mixing (ie $A_{i,j}^M > 0$ for all i and j for some $M > 0$, also called irreducible and aperiodic) and $w \in C^\alpha(\Sigma)$. Then:*

(1) *There is a simple maximal positive eigenvalue* $\rho(\mathcal{L}_w)$ *of* $\mathcal{L}_w : C^\alpha(\Sigma) \to C^\alpha(\Sigma)$ *with a corresponding strictly positive eigenfunction* e^ψ:

$$\mathcal{L}_w(e^\psi) = \rho(\mathcal{L}_w) e^\psi.$$

(2) *The remainder of the spectrum of* $\mathcal{L}_w$ *(excluding* $\rho(\mathcal{L}_w)$*) is contained in a disk of radius strictly smaller than* $\rho(w)$.

(3) *There is a unique probability measure* μ_w *on* Σ *such that*

$$\mathcal{L}_w{}^* \mu_w = e^\psi \mu_w.$$

The pressure $\boldsymbol{P}(w)$ of w, which can be defined in an analogous way as the pressure of functions on UX by the variational principle Proposition 2.8, turns out to be related to the spectral radius of the Ruelle operator: $\boldsymbol{P}(w) = \log \rho(\mathcal{L}_w)$ (see [5, Theorem 1.22]).

Associated to μ_w is another measure $m_w = e^\psi \mu_w$. It is called the equilibrium measure of w. It is a σ_A–invariant and ergodic probability measure and satisfies $\mathcal{L}_w{}^* m_w = m_w$.

We will from now on assume $\boldsymbol{P}(w) = 0$. As pressure functions and equilibrium measures depend only on cohomology class, we can modify w by a coboundary so that $\mathcal{L}_w(1) = 1$ and $\mu_w = m_w$. One notices this implies $\mathcal{L}_w(\sigma_A^* f) = f$.

Fixing m_w, we define an inner product $\langle f_1, f_2 \rangle := \int_\Sigma f_1 f_2 \, dm_w$ on the Banach space $C^\alpha(\Sigma)$.

For convenience, we also write $S_n(f, x) = \sum_{i=0}^{n-1} f(\sigma_A^i x)$.

The following two lemmas are applications of Ruelle operators and will be useful in the next subsection:

Lemma 3.5 (McMullen [26, Theorems 3.2 and 3.3]) *For any* $g \in C(\Sigma)$ *and* $f \in C^\alpha(\Sigma)$ *with* $\int_\Sigma f \, dm_w = 0$,

$$\lim_{n\to\infty} \left\langle g, \frac{S_n(f)^2}{n} \right\rangle = \mathrm{Var}(f, m_w) \int_\Sigma g \, dm_w = 0,$$

where $\mathrm{Var}(f, m_w) = \lim_{n\to\infty} (1/n) \langle S_n(f), S_n(f) \rangle$.

Lemma 3.6 *For any* $f \in C^\alpha(\Sigma)$ *with* $\int_\Sigma f \, dm_w = 0$,

$$\lim_{n\to\infty} \frac{1}{n} \int_\Sigma (S_n(f))^3 \, dm_w < \infty.$$

Proof This proof is similar to Theorem.3.3 of [26]. We have

$$\frac{1}{n} \int_\Sigma (S_n(f))^3 \, dm = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \langle f \circ \sigma_A^i \cdot f \circ \sigma_A^j, f \circ \sigma_A^k \rangle.$$

When $k > j > i$,

$$\begin{aligned}
\langle f \circ \sigma_A^i \cdot f \circ \sigma_A^j, f \circ \sigma_A^k \rangle &= \langle \sigma_A^{*i}(f \cdot f \circ \sigma_A^{j-i}), \sigma_A^{*i}(f \circ \sigma_A^{k-i}) \rangle \\
&= \langle f \cdot f \circ \sigma_A^{j-i}, f \circ \sigma_A^{k-i} \rangle \qquad \text{(by } \sigma_A\text{–invariance of } m_w\text{)} \\
&= \langle f, f \circ \sigma_A^{j-i} \cdot f \circ \sigma_A^{k-i} \rangle \\
&= \langle f, \sigma_A^{*(j-i)}(f \cdot f \circ \sigma_A^{k-j}) \rangle \\
&= \langle \mathcal{L}_w^{j-i}(f), f \cdot f \circ {\sigma_A}^{k-j} \rangle \\
&\qquad \text{(as } \mathcal{L}_w(\sigma_A^* f) = f \text{ and } {\mathcal{L}_w}^* m_w = m_w\text{)} \\
&= \langle f \cdot \mathcal{L}_w^{j-i}(f), f \circ {\sigma_A}^{k-j} \rangle.
\end{aligned}$$

We define a projection operator on $C^\alpha(\Sigma)$ by $P_{m_w}(h)(x) = h(x) - \int_\Sigma h\, dm_w$. Because $P_{m_w}(h)$ has mean zero with respect to m_w, the spectrum of the operator $T_w = \mathcal{L}_w \circ P_{m_w}$ lies in a disk of radius $r < 1$ by the Ruelle–Perron–Frobenius theorem.

One has

$$\langle h_1, h_2 \circ \sigma \rangle = \langle T_w(h_1), h_2 \rangle \tag{3-1}$$

whenever h_1 or h_2 has mean zero.

Because f is mean zero with respect to m_w, $T_w(f) = \mathcal{L}_w(f)$. Moreover,

$$\begin{aligned}
\langle f \cdot \mathcal{L}_w^{j-i}(f), f \circ {\sigma_A}^{k-j} \rangle &= \langle f \cdot T_w^{j-i}(f), f \circ {\sigma_A}^{k-j} \rangle \\
&= \langle T_w^{k-j}(f \cdot T_w^{j-i}(f)), f \rangle \qquad \text{(by (3-1))} \\
&\le \|T^{k-j}\| \|T^{j-i}\| \|f\|^3 \\
&\le C r^{k-i} \qquad \text{(for some } C > 0\text{)},
\end{aligned}$$

where the norm for T is the operator norm.

Thus,

$$\begin{aligned}
&\frac{1}{n} \sum_{0 \le i < j < k \le n-1} \langle f \circ \sigma_A^i \cdot f \circ \sigma_A^j, f \circ \sigma_A^k \rangle \\
&\qquad \le \frac{C}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{k} (k-i) r^{k-i} \qquad \text{(by the estimate above)} \\
&\qquad = \frac{C}{n} \sum_{k=0}^{n-1} \sum_{s=0}^{k} s r^s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Cr}{1-r}\left(1-\frac{1}{n}\sum_{k=1}^{n} r^k\right) \\
&< \infty \qquad\qquad\qquad\qquad \text{(when } n \to \infty\text{)}.
\end{aligned}
$$

This shows $\lim_{n\to\infty}(1/n)\int_\Sigma (S_n(f))^3\, dm_w < \infty$. □

3.2 Third derivatives of pressure functions

Our goal in this subsection is to compute the third derivatives of pressure functions in Lemma 3.8. For this, we first need to compute the third derivatives of pressure functions for subshifts of finite type by the method of the Ruelle operator and generalize it to our setting of suspension flows.

We start from introducing suspension flows. We will also recall Bowen's celebrated results, applied to our setting, that suspension flows efficiently model the geodesic flow on UX.

Definition 3.7 Suppose (Σ, σ_A) is a two-sided shift of finite type. Given a roof function $r\colon \Sigma \to \mathbb{R}^+$, the suspension flow of (Σ, σ_A) under r is the quotient space

$$\Sigma_r = \{(x,t) \in \Sigma \times \mathbb{R} : 0 \le t \le r(x),\ x \in \Sigma\}/(x, r(x)) \sim (\sigma_A(x), 0)$$

equipped with the natural flow $\sigma^r_{A,s}(x,t) = (x, t+s)$

Any σ_A–invariant probability measure m on Σ induces a natural $\sigma^r_{A,s}$–invariant probability measure on Σ_r

$$dm_r = \frac{dm\, dt}{\int_\Sigma r\, dm}. \tag{3-2}$$

This correspondence gives a bijection between σ_A–invariant probability measures and $\sigma^r_{A,s}$–invariant probability measures.

Bowen [3] shows the construction of Markov partitions for Axiom A diffeomorphisms. He then shows how to model Axiom A flows via the Markov partition and symbolic dynamics in [4]. We illustrate the version of this celebrated result in our context (see also [32]): the geodesic flow Φ admits a Markov coding (Σ_A, π, r), where (Σ_A, σ_A) is a topologically mixing two-sided shift of finite type, the roof function $r\colon \Sigma_A \to \mathbb{R}^+$ is Hölder continuous, and the map $\pi\colon \Sigma_A \to UX$ is also Hölder continuous. The suspension flow $\sigma^r_{A,t}$ models Φ_t effectively in the following sense:

- π is surjective.
- π is one-to-one on a set of full measure (for any ergodic measure of full support) and on a residual set.
- π is finite-to-one.
- $\pi\sigma^r_{A,t} = \Phi_t\pi$ for all $t \in \mathbb{R}$.

Now we are able to state and prove the major result in this subsection:

Lemma 3.8 *Let F_s be a smooth family in $C^h(UX)$ such that $\boldsymbol{P}(F_0) = 0$ and $\partial_s\boldsymbol{P}(F_s)(0)$. Then*

$$\begin{aligned}\text{(3-3)}\quad \frac{d^3\boldsymbol{P}(F_s)}{ds^3}\bigg|_{s=0} &= \\ &\int_{UX}\partial_s^3F_0(x)\,dm_{F_0}(x) \\ &\quad + \lim_{r\to\infty}\frac{1}{r}\biggl(3\int_{UX}\int_0^r\partial_sF_0(\Phi_t(x))\,dt\int_0^r\partial_s^2F_0(\Phi_t(x))\,dt\,dm_{F_0}(x) \\ &\qquad + \int_{UX}\biggl(\int_0^r\partial_sF_0(\Phi_t(x))\,dt\biggr)^3dm_{F_0}(x)\biggr).\end{aligned}$$

In particular, if $F(u,v,w)$ is a smooth three-parameter family of Hölder functions on UX such that $\boldsymbol{P}(F(0,0,0)) = 0$ and all of the first variations of $\boldsymbol{P}(F(u,v,w))$ are zero, then

$$\begin{aligned}\text{(3-4)}\quad \frac{\partial^3\boldsymbol{P}(F(u,v,w))}{\partial u\,\partial v\,\partial w}\bigg|_{u=v=w=0} &= \\ &\int_{UX}\partial_u\partial_v\partial_wF(0)(x)\,dm_{F(0)}(x) \\ &+ \lim_{r\to\infty}\frac{1}{r}\biggl(\int_{UX}\biggl(\int_0^r\partial_uF(0)(\Phi_t(x))\,dt\biggr)\biggl(\int_0^r\partial_vF(0)(\Phi_t(x))\,dt\biggr) \\ &\qquad\cdot\biggl(\int_0^r\partial_wF(0)(\Phi_t(x))\,dt\biggr)dm_{F(0)}(x) \\ &\quad + \int_{UX}\biggl(\int_0^r\partial_uF(0)(\Phi_t(x))\,dt\biggr)\biggl(\int_0^r\partial_{vw}F(0)(\Phi_t(x))\,dt\biggr)dm_{F(0)}(x) \\ &\quad + \int_{UX}\biggl(\int_0^r\partial_vF(0)(\Phi_t(x))\,dt\biggr)\biggl(\int_0^r\partial_{uw}F(0)(\Phi_t(x))\,dt\biggr)dm_{F(0)}(x) \\ &\quad + \int_{UX}\biggl(\int_0^r\partial_wF(0)(\Phi_t(x))\,dt\biggr)\biggl(\int_0^r\partial_{uv}F(0)(\Phi_t(x))\,dt\biggr)dm_{F(0)}(x)\biggr).\end{aligned}$$

Proof The proof proceeds in two steps. In the first step, we find a formula for the third derivatives of pressure functions for topologically mixing shifts of finite type. In the second step, we show how the computation can be carried to geodesic flows through symbolic coding and suspension flows.

Step 1 The computation of the first and second derivatives of pressure functions for aperiodic shifts of finite type is shown by Parry and Pollicott [27] using the Ruelle operator. We will give a computation of the third derivative by the same method and then generalize it to our flow case.

Let (Σ_A,σ_A) be a (one-sided or two-sided) shift of finite type that is topologically mixing. We assume f_s is a smooth family of functions on $C^\alpha(\Sigma_A)$ such that $\boldsymbol{P}(f_0)=0$ and $\partial_s\boldsymbol{P}(f_s)(0)$. We will prove

$$\partial_s^3\boldsymbol{P}(f_s)(0)=\lim_{n\to\infty}\frac{1}{n}\int_X(S_n(\partial_s f_0))^3\,dm_{f_0} + \lim_{n\to\infty}\frac{3}{n}\int_X S_n(\partial_s f_0)S_n(\partial_s^2 f_0)\,dm_{f_0}+\int_X\partial_s^3 f_0\,dm_{f_0}. \tag{3-5}$$

Any Hölder function on a two-sided shift space is cohomologous to a Hölder function depending only on the corresponding one-sided shift space (see [27, Proposition 1.2]). It suffices to prove (3-5) for one-sided shifts of finite type. We assume (Σ_A,σ_A) is one-sided and f_s is a smooth family of Hölder functions (with possibly a different Hölder exponent from α) on Σ_A.

We change f_0 in its cohomology class so that $\mathcal{L}_{f_0}(1)=1$.

Following the method in [27], let $Q(s)$ be a projection-valued function which is analytic in s and satisfies

$$\mathcal{L}_{f_s}Q(s)=Q(s)\mathcal{L}_{f_s}.$$

Let $w(s)\colon\Sigma_A\to\mathbb{R}$ be $w(s)(x):=Q(s)\cdot 1$. So

$$\mathcal{L}_{f_s}w(s)=e^{\boldsymbol{P}(f_s)}w(s) \tag{3-6}$$

and $w(0)(x)=Q(0)\cdot 1=1$.

Iterate (3-6) n times and take third s–derivatives of both sides at $s=0$:

$$\partial_s^3\Bigl(\sum_{\sigma_A y=x}e^{S_n(f_s)(y)}w(s)(y)\Bigr)(0)=\partial_s^3(e^{n\boldsymbol{P}(f_s)}w(s))(0). \tag{3-7}$$

Notice $\boldsymbol{P}(f_0)=0$, $\partial_s \boldsymbol{P}(f_s)(0)=0$ and $\int_{UX} \partial_s f_0 \, dm_{f_0}=0$. Integrating both sides of (3-7) with respect to m_{f_0} yields

$$\begin{aligned}
&3n\partial_s^2 \boldsymbol{P}(f_s)(0)\int_X \partial_s w(0)\, dm_{f_0} + n\partial_s^3 \boldsymbol{P}(f_s)(0)\\
&= \int_X S_n(\partial_s^3 f_0)\, dm_{f_0} + 3\int_X (S_n(\partial_s f_0)^2 + S_n(\partial_s^2 f_0))\partial_s w(0)\, dm_{f_0}\\
&\qquad + 3\int_X S_n(\partial_s f_0)\partial_s^2 w(0)\, dm_{f_0} + 3\int_X S_n(\partial_s f_0) S_n(\partial_s^2 f_0)\, dm_{f_0}\\
&\qquad\qquad + \int_X S_n(\partial_s f_0)^3\, dm_{f_0}.
\end{aligned}$$

Divide by n and take $n\to\infty$. From ergodicity of m_{f_0}, we may evaluate two of the resulting terms:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\int_X S_n(\partial_s f_0)\partial_s^2 w(0)\, dm_{f_0} &= \int_X \partial_s f_0\, dm_{f_0}\int_X \partial_s^2 w(0)\, dm_{f_0} = 0,\\
\lim_{n\to\infty}\frac{1}{n}\int_X S_n(\partial_s^2 f_0)\partial_s w(0)\, dm_{f_0} &= \int_X \partial_s^2 f_0\, dm_{f_0}\int_X \partial_s w(0)\, dm_{f_0}.
\end{aligned}$$

We also notice that, by applying Lemma 3.5 and the formula for second derivatives of pressure functions,

$$\begin{aligned}
&\partial_s^2 \boldsymbol{P}(f_s)(0)\int_X \partial_s w(0)\, dm_{f_0}\\
&\qquad = \lim_{n\to\infty}\frac{1}{n}\int_X S_n(\partial_s f_0)^2 \partial_s w(0)\, dm_{f_0} + \lim_{n\to\infty}\frac{1}{n}\int_X S_n(\partial_s^2 f_0)\partial_s w(0)\, dm_{f_0}.
\end{aligned}$$

Therefore, we obtain a formal expression

$$\begin{aligned}
\partial_s^3 \boldsymbol{P}(f_s)(0) = \lim_{n\to\infty}\frac{1}{n}\int_X (S_n(\partial_s f_0))^3\, dm_{f_0} + \lim_{n\to\infty}\frac{3}{n}\int_X S_n(\partial_s f_0) S_n(\partial_s^2 f_0)\, dm_{f_0}\\
+ \int_X \partial_s^3 f_0\, dm_{f_0}.
\end{aligned}$$

We observe each term of the right-hand side converges: finiteness of the first limit has been shown in Lemma 3.6 and that of the second is guaranteed by Corollary 2.21.

Step 2 We now explain how we obtain the flow version of the above formula.

Suppose F_s is a smooth family of functions in $C^h(UX)$ such that $\boldsymbol{P}(F_s)=0$. We have a topologically mixing Markov coding (Σ_A,π,r) for UX. Because of the conjugacy $\pi\sigma_{A,t}^r = \Phi_t\pi$ between geodesic flow and the suspension flow of (Σ_A,π,r), it suffices to prove (3-3) for $F_s\circ\pi\colon \Sigma_{A,r}\to\mathbb{R}$ on suspension space with pullback measure $\pi^* m_{F_0}$. For simplicity, we still write $F_s\circ\pi$ as F_s and $\pi^* m_{F_0}$ as m_{F_0}.

We then want to reduce the problem of proving (3-3) for suspension flows to proving it for subshifts of finite type. We construct a function $\widehat{F}_s\colon \Sigma_A \to \mathbb{R}$ from the function F_s on the suspension space as

$$\widehat{F}_s(x) = \int_0^{r(x)} F_s(x,t)\,dt. \tag{3-8}$$

As F_s and r are Hölder on $\Sigma_{A,r}$ and Σ_A, respectively, the function $\widehat{F}_s$ is clearly Hölder. Denoting the set of σ^r_A–invariant probability measures by $\mathcal{M}^{\sigma^r_A}$ and the set of σ_A–invariant probability measures by $\mathcal{M}^{\sigma_A}$, we have

$$\begin{aligned} \boldsymbol{P}(\sigma^r_{A,t}, F_s) &= \sup_{m_r \in \mathcal{M}^{\sigma^r_A}} \left(h(\sigma^r_{A,1}, m_r) + \int_{\Sigma_{A,r}} F_s\,dm_r \right) \\ &= \sup_{m \in \mathcal{M}^{\sigma_A}} \frac{h(\sigma_A, m) + \int_{\Sigma_A} F_s\,dm}{\int_{\Sigma_A} r\,dm}. \end{aligned}$$

Let $c_s = \boldsymbol{P}(\sigma^r_{A,t}, F_s)$, we have the relation between the pressure function of F_s and the pressure function of $\widehat{F}_s$ (also see [6])

$$\boldsymbol{P}(\sigma_A, \widehat{F}_s - c_s r) = 0. \tag{3-9}$$

Let $\partial_s c_0 = \partial_s(c_s)(0)$ and $\partial_{ss} c_0 = \partial_s^2(c_s)(0)$.

We have the assumption $\partial_s c_0 = 0$. Without loss of generality, we can also assume $\partial_s^2 c_0 = 0$. Otherwise, we consider the family of functions $\widetilde{F}_s := F_s - \frac{1}{2} s^2 \partial_s^2 c_0$. Clearly $\partial_s \boldsymbol{P}(\widetilde{F}_s)(0) = \partial_s^2 \boldsymbol{P}(\widetilde{F}_s)(0) = 0$ and $\partial_s^3 \boldsymbol{P}(\widetilde{F}_s)(0) = \partial_s^3 \boldsymbol{P}(F_s)(0)$.

Now let's take the third s–derivative of (3-9) with the assumptions $\partial_s c_0 = \partial_s^2 c_0 = 0$. By (3-5),

$$\begin{aligned} 0 &= \partial_s^3 \boldsymbol{P}(\widehat{F}_s - c_s r)(0) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} (S_n(\partial_s \widehat{F}_0))^3\,dm_{\widehat{F}_0} + \lim_{n\to\infty} \frac{3}{n} \int_{\Sigma_A} S_n(\partial_s \widehat{F}_0) S_n(\partial_s^2 \widehat{F}_0)\,dm_{\widehat{F}_0} \\ &\qquad + \int_{\Sigma_A} (\partial_s^3 \widehat{F}_0 - \partial_s^3 c_0 r)\,dm_{\widehat{F}_0}. \end{aligned}$$

This yields

$$\partial_s^3 c_0 = \partial_s^3 \boldsymbol{P}(\sigma^r_{A,t}, F_s)(0) = \left(\int_{\Sigma_A} r\,dm_{\widehat{F}_0} \right)^{-1} \partial_s^3 \boldsymbol{P}(\sigma_A, \widehat{F}_s)(0).$$

Therefore, proving (3-3) for F_s is equivalent to proving

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r}\int_{\Sigma_{A,r}}\left(\int_0^r \partial_s F_0\,dt\right)^3 dm_{F_0}+\lim_{r\to\infty}\frac{3}{r}\int_{\Sigma_{A,r}}\int_0^r \partial_s F_0\,dt\int_0^r \partial_{ss}F_0\,dt\,dm_{F_0}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\int_{\Sigma_{A,r}}\partial_s^3 F_0\,dm_{F_0}\\
&=\left(\int_{\Sigma_A} r\,dm_{\widehat{F}_0}\right)^{-1}\\
&\qquad\cdot\left(\lim_{n\to\infty}\frac{1}{n}\int_{\Sigma_A}(S_n(\partial_s\widehat{F}_0))^3\,dm_{\widehat{F}_0}+\lim_{n\to\infty}\frac{3}{n}\int_{\Sigma_A}S_n(\partial_s\widehat{F}_0)S_n(\partial_s^2\widehat{F}_0)\,dm_{\widehat{F}_0}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\left(\int_{\Sigma_A} r\,dm_{\widehat{F}_0}\right)^{-1}\int_{\Sigma_A}\partial_s^3\widehat{F}_0\,dm_{\widehat{F}_0}.
\end{aligned}$$

Each term on the left is actually equal to the corresponding term on the right. We show here how to obtain

$$\begin{aligned}
\text{(3-10)}\quad &\lim_{r\to\infty}\frac{1}{r}\int_{\Sigma_{A,r}}\left(\int_0^r \partial_s F_0(\sigma_t^r(y))\,dt\right)^3 dm_{F_0}(y)\\
&\qquad=\left(\int_{\Sigma_A} r\,dm_{\widehat{F}_0}\right)^{-1}\lim_{n\to\infty}\frac{1}{n}\int_{\Sigma_A}(S_n(\partial_s\widehat{F}_0)(x))^3\,dm_{\widehat{F}_0}(x).
\end{aligned}$$

The other two terms follow a similar analysis.

To see (3-10), we begin by noting the identity [28], where $y=(x,u)$,

$$\partial_s F_0(\sigma_{A,t}^r(x,u))=\sum_{n\in\mathbb{Z}}\left(\int_0^{r(\sigma_A^n x)}\partial_s F_0(\sigma_A^n x,v)\delta(u+t-v-r^n(x))\,dv\right),$$

where $r^n(x)=r(x)+r(\sigma_A x)+\cdots+r(\sigma_A^{n-1}x)$ for $n>0$ and $r^0(x)=0$ and $r^{-n}(x)=-(r(\sigma_A{}^{-1}x)+\cdots+r(\sigma_A^{-n}x))$ for $n\ge 1$.

One has from Proposition 2.22, the measure correspondence (3-2) and (3-8) that

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r}\int_{\Sigma_{A,r}}\left(\int_0^r \partial_s F_0(\sigma_{A,t}^r(y))\,dt\right)^3 dm_{F_0}(y)\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\Sigma_{A,r}}\partial_s F_0(y)\partial_s F_0(\sigma_{A,t}^r(y))\partial_s F_0(\sigma_{A,v}^r(y))\,dm_{F_0}(y)\,dt\,dv\\
&=\left(\int_{\Sigma_A} r\,dm_{\widehat{F}_0}\right)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\Sigma_A}\int_0^{r(x)}\partial_s F_0(x,u)\partial_s F_0(\sigma_{A,t}^r(x,u))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\partial_s F_0(\sigma_{A,v}^r(x,u))\,du\,dm_{\widehat{F}_0}(x)\,dt\,dv
\end{aligned}$$

$$= \left(\int_{\Sigma_A} r \, dm_{\widehat{F}_0} \right)^{-1} \sum_{m,n \in \mathbb{Z}} \int_{\Sigma_A} dm_{\widehat{F}_0}(x) \int_0^{r(x)} \partial_s F_0(x,u) \, du \cdot \int_0^{r(\sigma_A^n x)} \partial_s F_0(\sigma_A^n x, v) \, dv \int_0^{r(\sigma_A^m x)} \partial_s F_0(\sigma_A^m x, v) \, dv$$

$$= \left(\int_{\Sigma_A} r \, dm_{\widehat{F}_0} \right)^{-1} \sum_{m,n \in \mathbb{Z}} \int_{\Sigma_A} \partial_s \widehat{F}_0(x) \partial_s \widehat{F}_0(\sigma_A^n x) \partial_s \widehat{F}_0(\sigma_A^m x) \, dm_{\widehat{F}_0}(x)$$

$$= \left(\int_{\Sigma_A} r \, dm_{\widehat{F}_0} \right)^{-1} \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_A} (S_n(\partial_s \widehat{F}_0)(x))^3 \, dm_{\widehat{F}_0}(x).$$

We therefore obtain a suspension flow version of (3-5) for F_s.

The arguments for three-parameter families are the same as the one-parameter case. In fact, since the operator $\partial_u \partial_v \partial_w$ is a symmetric multilinear map in u, v and w that is completely characterized by its values on the diagonal, one can deduce (3-4) for multivariable cases directly from (3-3) for one-parameter families. □

Next we introduce a formula for taking derivatives of integrals over varying measures by tools of thermodynamic formalism. This formula will be very useful in later proofs.

Lemma 3.9 *Suppose $\{f_s\}_{s \in (-1,1)}$ is a smooth family of pressure zero Hölder functions over UX and suppose $\{m_{f_s}\}_{s \in (-1,1)}$ is the associated family of equilibrium states. Suppose furthermore that $\{w_s\}_{s \in (-1,1)}$ is another smooth family of Hölder functions over UX. Then*

$$\partial_s \left(\int_{UX} w_s \, dm_{f_s} \right)(0) = \operatorname{Cov}(w_0, \partial_s f_0, m_{f_0}) + \int_{UX} \partial_s w_0 \, dm_{f_0}. \tag{3-11}$$

Proof We have

$$\begin{aligned}
&\partial_s \left(\int_{UX} w_s \, dm_{f_s} \right)(0) \\
&\quad = \partial_s \left(\frac{\partial \boldsymbol{P}(f_s + t w_s)}{\partial t} \bigg|_{t=0} \right)(0) && \text{(by (2-6))} \\
&\quad = \frac{\partial^2 \boldsymbol{P}(f_s + t w_s)}{\partial s \, \partial t} \bigg|_{s=t=0} \\
&\quad = \operatorname{Cov}(P_{m_{f_0}}(w_0), P_{m_{f_0}}(\partial_s f_0), m_{f_0}) + \int_{UX} \partial_s w_0 \, dm_{f_0} && \text{(by (2-8))} \\
&\quad = \operatorname{Cov}(P_{m_{f_0}}(w_0), \partial_s f_0, m_{f_0}) + \int_{UX} \partial_s w_0 \, dm_{f_0} \\
&\quad = \operatorname{Cov}(w_0, \partial_s f_0, m_{f_0}) + \int_{UX} \partial_s w_0 \, dm_{f_0} && \text{(by Corollary 2.21).} \quad \square
\end{aligned}$$

4 Proof of the main theorem: initial steps

We first restate our main theorem:

Theorem 1.1 *Let S be a closed oriented surface with genus $g \geq 2$. For any point $\sigma \in \mathcal{T}(S) \subset \mathcal{H}_3(S)$, let X be the Riemann surface corresponding to σ. Then the Hitchin parametrization $H^0(X, K^2) \oplus H^0(X, K^3)$ provides geodesic coordinates for the pressure metric at σ.*

We want to show $\partial_k g_{ij}(\sigma) = 0$ for the pressure metric components g_{ij} with respect to the coordinates introduced in Remark 2.39 for all possible i, j and k.

4.1 Some geometrical observation

In this subsection, we conclude some derivatives of metric tensors vanish by a geometric observation. Starting from the next section, we will develop a general method to compute first derivatives of the pressure metric via the thermodynamic formalism.

From now on, we restrict ourselves to the Hitchin component $\mathcal{H}_3(S)$. Suppose $\{q_i\}$ is a basis of holomorphic differentials in $H^0(X, K^2) \oplus H^0(X, K^3)$ and suppose $\{\varphi(q_i)\}$ is the associated Hitchin deformation given in Definition 2.38. Recall we use the notation $g_{ij}(\sigma) = \langle \varphi(q_i), \varphi(q_j) \rangle_P$ to emphasize the metric tensor is evaluated at $\sigma \in \mathcal{T}(S)$. We also assume $g_{ij}(\delta) = \delta_{ij}$.

Furthermore, instead of using the Latin letters i, j and k to denote arbitrary holomorphic differentials of degree 2 and 3, we let the Latin letters i, j and k only refer to quadratic differentials $q_i, q_j, q_k \in H^0(X, K^2)$ from now on. Therefore, the corresponding Hitchin deformations $\varphi(q_i)$, $\varphi(q_j)$ and $\varphi(q_k)$ are tangential directions to the Fuchsian locus in $T_X\mathcal{H}_3(S)$. We use the Greek letters α, β and γ to refer to cubic differentials $q_\alpha, q_\beta, q_\gamma \in H^0(X, K^3)$. Then the corresponding Hitchin deformations $\varphi(q_\alpha)$, $\varphi(q_\beta)$ and $\varphi(q_\gamma)$ are normal directions to the Fuchsian locus in $T_X\mathcal{H}_3(S)$ with respect to the pressure metric.

With the above notation understood, we have in total six types of first derivative of metric tensors that need to be considered: $\partial_k g_{ij}$, $\partial_j g_{i\alpha}$, $\partial_\alpha g_{ij}$, $\partial_i g_{\alpha\beta}$, $\partial_\beta g_{i\alpha}$ and $\partial_\gamma g_{\alpha\beta}$. Our goal is to prove they all vanish.

We first notice the following facts:

(1) $\partial_k g_{ij}(\sigma) = 0$.

To see this, note that the pressure metric is a constant multiple of the Weil–Petersson metric on Teichmüller space $\mathcal{T}(S)$. Because the coordinates system in terms of quadratic differentials from the Hitchin reparametrization agrees with Bers coordinates through second order in the case of $\mathcal{T}(S)$ [37, Corollaries 5.2 and 5.4]. That the Bers coordinates are geodesic [1] for the Weil–Petersson metric implies that, for the pressure metric, $\partial_k g_{ij}(\sigma) = 0$.

(2) $\partial_j g_{\alpha i}(\sigma) = 0$ implies $\partial_\alpha g_{ij}(\sigma) = 0$.

The contragredient involution $\kappa\colon \mathrm{PSL}(3,\mathbb{R}) \to \mathrm{PSL}(3,\mathbb{R})$ given by $\kappa(g) = (g^{-1})^t$ induces an involution $\hat{\kappa}$ on $\mathcal{H}_3(S)$ by $\hat{\kappa}(\rho)(\gamma) = \kappa(\rho(\gamma))$. Because $\hat{\kappa}$ is an isometry of $\mathcal{H}_3(S)$ with respect to the pressure metric and the fixed-point set of $\hat{\kappa}$ is $\mathcal{T}(S)$, the Fuchsian locus is in fact totally geodesic in $\mathcal{H}_3(S)$ (see [7]). So, for $\tilde{\nabla}$ the Levi-Civita connection of the pressure metric and any $X, Y \in T_\sigma \mathcal{T}(S)$, we have

$$\Pi(X,Y) = (\tilde{\nabla}_X Y)^\perp = 0. \tag{4-1}$$

Thus, the Christoffel symbols for the connection $\tilde{\nabla}$ satisfy $\Gamma^\alpha_{ij}(\sigma) = 0$ and, because

$$\begin{aligned} \Gamma^\alpha_{ij} &= \tfrac{1}{2} g^{\beta\alpha}(\partial_j g_{i\beta} + \partial_i g_{j\beta} - \partial_\beta g_{ji}) \quad (\text{since } g_{k\alpha}(\sigma) = 0 \text{ and } g^{k\alpha}(\sigma) = 0) \\ &= \tfrac{1}{2} g^{\alpha\alpha}(\partial_j g_{i\alpha} + \partial_i g_{j\alpha} - \partial_\alpha g_{ji}) \quad (\text{since } g_{\alpha\beta} = \sigma_{\alpha\beta}), \end{aligned}$$

it suffices to know $\partial_j g_{i\alpha}(\sigma) = 0$ and $\partial_i g_{j\alpha}(\sigma) = 0$ to conclude $\partial_\alpha g_{ij}(\sigma) = 0$.

(3) $\partial_\beta g_{\alpha\alpha}(\sigma) = 0$ implies $\partial_\gamma g_{\alpha\beta}(\sigma) = 0$, and $\partial_i g_{\alpha\alpha}(\sigma) = 0$ implies $\partial_i g_{\alpha\beta}(\sigma) = 0$.

This is because

$$\begin{aligned} \partial_\gamma g_{\alpha\beta} &= \tfrac{1}{2}(\partial_\gamma g_{\alpha+\beta,\alpha+\beta} - \partial_\gamma g_{\alpha\alpha} - \partial_\gamma g_{\beta\beta}), \\ \partial_i g_{\alpha\beta} &= \tfrac{1}{2}(\partial_i g_{\alpha+\beta,\alpha+\beta} - \partial_i g_{\alpha\alpha} - \partial_i g_{\beta\beta}). \end{aligned}$$

The remaining four cases left to prove are as follows:

(i) $\partial_\beta g_{\alpha\alpha}(\sigma) = 0$.

(ii) $\partial_i g_{\alpha\alpha}(\sigma) = 0$.

(iii) $\partial_j g_{\alpha i}(\sigma) = 0$.

(iv) $\partial_\beta g_{\alpha i}(\sigma) = 0$.

We will have a general method to prove them. We first give a general formula for first derivatives of the pressure metric in the next subsection. The computation for the model case $\partial_\beta g_{\alpha\alpha}(\sigma)$ will be shown in Sections 5 and 6. The other three cases will be discussed in Section 7.

4.2 First derivatives of the pressure metric

This subsection is devoted to a formula for first derivatives of the pressure metric. We also prove we have some freedom to choose representatives for the variations of the reparametrization functions from the Livšic cohomology classes.

Suppose $\{\rho(u,v,w)\}_{(u,v,w)\in\{(-1,1)\}^3}$ is an analytic three-parameter family of representations in the Hitchin component $\mathcal{H}_n(S)$ with basepoint $\rho(0,0,0)\in\mathcal{T}(S)$ corresponding to X. Suppose $\{f_{\rho(u,v,w)}\}_{(u,v,w)\in\{(-1,1)\}^3}$ are associated reparametrization functions. For simplicity of notation, we denote the renormalized reparametrization functions by

$$F(u,v,w)=f^N_{\rho(u,v,w)}=-h(\rho(u,v,w))f_{\rho(u,v,w)}.$$

We also write $F(0)=F(0,0,0)$ and $\rho(0)=\rho(0,0,0)$.

In the case of the Fuchsian representation, the topological entropy and the reparametrization function are simple. We have $h(\rho(0))=1$ (see Theorem 2.16). Since $\Phi_{\rho(0)}=\Phi$, the reparametrization function $f_{\rho(0)}$ can be chosen to be 1 in the Livšic cohomology class. Therefore, one can choose $F(0)=-1$.

The following characterization of the equilibrium measure for $F(0)$ is important:

Lemma 4.1 *The equilibrium state $m_{F(0)}$ for $F(0)$ is the Liouville measure m_L.*

Proof Since the Liouville measure m_L coincides with the Bowen–Margulis measure (Remark 2.14), this follows easily from the variational principle (Proposition 2.8). $\square$

The Liouville measure m_L is both Φ_t–invariant and rotationally invariant on UX, ie $(e^{i\theta})^*m_L=m_L$, where $e^{i\theta}$ acts on UX by usual multiplication. We will repeatedly use these important properties of the Liouville measure for our proofs later.

Proposition 4.2 *The first derivatives of the pressure metric at $\rho(0)$ satisfy*

$$\begin{aligned}&\partial_w\big(\langle\partial_u\rho(0,0,w),\partial_v\rho(0,0,w)\rangle_P\big)(0)\\&=\lim_{r\to\infty}\frac{1}{r}\bigg(\int_{UX}\int_0^r\partial_uF(0)\,dt\int_0^r\partial_vF(0)\,dt\int_0^r\partial_wF(0)\,dt\,dm_0\\&\qquad+\int_{UX}\int_0^r\partial_uF(0)\,dt\int_0^r\partial_{wv}F(0)\,dt\,dm_0\\&\qquad+\int_{UX}\int_0^r\partial_vF(0)\,dt\int_0^r\partial_{wu}F(0)\,dt\,dm_0\bigg),\end{aligned}$$

where the flow $\Phi_t(x)$ is omitted for simplicity.

Proof Starting from the Fuchsian point $\rho(0)$, along the ray with parametrization $\{(0,0,w)\}_{w\in(-1,1)}$, the pressure metric $\langle\cdot,\cdot\rangle_P : T_{(0,0,w)}\mathcal{H}_n(S)\times T_{(0,0,w)}\mathcal{H}_n(S)\to\mathbb{R}$ satisfies

$$
\begin{aligned}
&\langle \partial_u\rho(0,0,w),\partial_v\rho(0,0,w)\rangle_P\\
&\qquad= -\frac{\mathrm{Cov}(\partial_u F(0,0,w),\partial_v F(0,0,w),m_{F(0,0,w)})}{\int_{UX} F(0,0,w)\,dm_{F(0,0,w)}}\\
&\qquad= -\frac{\partial_v\partial_u \boldsymbol{P}(F(0,0,w)) - \int_{UX}\partial_{uv}F(0,0,w)\,dm_{F(0,0,w)}}{\int_{UX} F(0,0,w)\,dm_{F(0,0,w)}} \quad \text{(by (2-8))}.
\end{aligned}
$$

We first notice $\int_{UX} F(0)\,dm_0 = -1$ and, from (3-11),

$$
\begin{aligned}
\partial_w\biggl(\int_{UX} F(0,0,w)\,dm_{F(0,0,w)}\biggr)(0) &= \mathrm{Cov}(F(0),\partial_w F(0),m_0) + \int_{UX}\partial_w F(0)\,dm_0\\
&= 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\partial_w\bigl(\langle \partial_u\rho(0,0,w),\partial_v\rho(0,0,w)\rangle_P\bigr)(0)\\
&= \partial_w\partial_v\partial_u \boldsymbol{P}(F(0)) - \partial_w\biggl(\int_{UX}\partial_{uv}F(0)\,dm_{F(0,0,w)}\biggr)(0)\\
&= \partial_w\partial_v\partial_u \boldsymbol{P}(F(0)) - \mathrm{Cov}(\partial_{uv}F(0),\partial_w F(0),m_0) - \int_{UX}\partial_{uvw}F(0)\,dm_0 \quad \text{(by (3-11))}\\
&= \lim_{r\to\infty}\frac{1}{r}\biggl(\int_{UX}\int_0^r \partial_u F(0)\,dt \int_0^r \partial_v F(0)\,dt \int_0^r \partial_w F(0)\,dt\,dm_0\\
&\qquad\qquad + \int_{UX}\int_0^r \partial_u F(0)\,dt\int_0^r \partial_{vw}F(0)\,dt\,dm_0\\
&\qquad\qquad\quad + \int_{UX}\int_0^r \partial_v F(0)\,dt\int_0^r \partial_{wu}F(0)\,dt\,dm_0\biggr) \quad \text{(by (3-4))}. \qquad \square
\end{aligned}
$$

Proposition 4.3 *The formula in Proposition 4.2 for the first derivatives of the pressure metric only depends on the Livšic class of each component function $\partial_u F(0)$, $\partial_v F(0)$, $\partial_w F(0)$, $\partial_{wv}F(0)$ and $\partial_{wu}F(0)$.*

Proof We know from the proof of Proposition 4.2 that

$$
\begin{aligned}
&\partial_w\bigl(\langle \partial_u\rho(0,0,w),\partial_v\rho(0,0,w)\rangle_P\bigr)(0)\\
&\qquad= \partial_w\partial_v\partial_u \boldsymbol{P}(F(0)) - \int_{UX}\partial_{uvw}F(0)\,dm_0 - \mathrm{Cov}(\partial_{uv}F(0),\partial_w F(0),m_0).
\end{aligned}
$$

By (2-8), in general, if we take two mean-zero Hölder functions h_1 and h_2 with respect to m_0, then

$$\mathrm{Cov}(h_1, h_2, m_0) = \partial_u \partial_v \boldsymbol{P}(F(0) + uh_1 + vh_2)(0).$$

As the value of the pressure function $\boldsymbol{P}$ only depends on the Livšic class, we see changing h_1 and h_2 in its cohomology class does not change $\mathrm{Cov}(h_1, h_2, m_0)$. In particular, this holds for $\mathrm{Cov}(\partial_{uv} F(0), \partial_w F(0), m_0)$.

Similarly, from (3-3), it is clear that

$$\begin{aligned}\partial_w \partial_v \partial_u \boldsymbol{P}(F(0)) - \int_{UX} \partial_{uvw} F(0)\, dm_0 \\ = \partial_w \partial_v \partial_u \boldsymbol{P}\big(F(0) + u\partial_u F(0) + v\partial_v F(0) + w\partial_w F(0) \\ + uv\partial_{uv} F(0) + uw\partial_{uw} F(0) + vw\partial_{vw} F(0)\big)(0).\end{aligned}$$

Again the above pressure function $\boldsymbol{P}$ does not change value if we change each component function. So, altogether, we know the first derivatives of the pressure metric only depend on the Livšic class of each component function $\partial_u F(0)$, $\partial_v F(0)$, $\partial_w F(0)$, $\partial_{wv} F(0)$ and $\partial_{wu} F(0)$. □

4.3 A gauge-theoretical formula

In [20], Labourie and Wentworth show the variations of the reparametrization functions can be expressed by a gauge-theoretical formula. This formula will be crucial for our computation in the next section. We include the formula and its proof here for completeness. We add some assumptions which are natural for our case of Hitchin components $\mathcal{H}_n(S)$.

We consider (E, H) a rank n Hermitian bundle over the surface S equipped with a Riemannian metric g. We let γ be a closed curve on S with arc-length parametrization $\gamma(t)$. Suppose D_{A^0} is a flat connection on E whose holonomy has distinct eigenvalues along γ. Suppose λ_γ is one eigenvalue with a corresponding eigenline $\mathcal{L}_\gamma$ and $\mathcal{H}_\gamma$ is the complementary hyperplane stabilized by the holonomy. We denote by $\mathcal{L}_\gamma(t)$ the line generated by the parallel transports of $\mathcal{L}_\gamma$ along γ at time t, by $\mathcal{H}_\gamma(t)$ the hyperplane generated by complementary eigenvectors, and by $\pi(t)$ the projection on $\mathcal{L}_\gamma(t)$ along $\mathcal{H}_\gamma(t)$. Then we have:

Proposition 4.4 (Labourie and Wentworth [20]) *For D_{A^s} a smooth one-parameter family of flat connections, we have a unique smooth function $\lambda_\gamma(s)$ such that, for*

s small enough, $\lambda_\gamma(s)$ is the eigenvalue of the holonomy of D_{A^s} with $\lambda_\gamma(0) = \lambda_\gamma$. Moreover,

$$\left.\frac{d \log \lambda_\gamma(s)}{ds}\right|_{s=0} = -\int_0^{l_\gamma} \mathrm{Tr}(\partial_s D_{A^0}(t) \cdot \pi(t))\, dt. \tag{4-2}$$

Here the notation is $\partial_s D_{A^0}(t) := \partial_s D_{A^s}(\dot\gamma(t)\,\partial/\partial t)(0)$, where $\partial_s D_{A^0}$ is an $\mathrm{End}(E)$–valued 1–form and $\dot\gamma(t)\,\partial/\partial t$ is the tangent vector field along $\gamma(t)$.

Proof We prove (4-2) here.

Let $\{g_s\}$ be a family of gauge transformations acting on $\{D_{A^s}\}$ with $g_0 = \mathrm{id}$. Define the new connection 1–forms $\widetilde{A}^s := g_s^* A^s$. We first prove

$$\int_0^{l_\gamma} \mathrm{Tr}(\partial_s D_{A^0}(t) \cdot \pi(t))\, dt = \int_0^{l_\gamma} \mathrm{Tr}(\partial_s D_{\widetilde{A}^0}(t) \cdot \pi(t))\, dt.$$

Note here $\partial_s D_{A^0}(t)$ is a 0–form since we have contracted the 1–form $\partial_s D_{A^s}(0)$ with the tangential vector field. Therefore, $\mathrm{Tr}(\partial_s D_{A^0}(t) \cdot \pi(t))$ is a function in t or in $\dot\gamma(t)$.

Taking the derivative of $\widetilde{A}^s := g_s^* A^s$ at $s = 0$ yields

$$\partial_s D_{\widetilde{A}^0} = \partial_s D_{A^0} + D_{A^0}\dot g,$$

where $\dot g$, denoting $\partial g_s/\partial_s|_{s=0}$, is a section of $\mathrm{End}(E)$ and the connection D_{A^0} acts on $\dot g$ as $D_{A^0}\dot g = d\dot g + [A^0, \dot g]$.

We want to show

$$\int_0^{l_\gamma} \mathrm{Tr}((D_{A^0}\dot g)\pi)\, dt = 0.$$

To simplify the notation, we will always omit the variable t when writing our formulas. For example, here $(D_{A^0}\dot g)\pi := (D_{A^0}(t)\dot g(t))\pi(t)$.

We start by proving that π is a D_{A^0}–parallel section in $\mathrm{End}(E)$. Given any section $v \in \Gamma(E)$, we can write it as a linear combination of eigenvectors of holonomy. Set $v(t) = \sum_{i=1}^n a_i(t) e_i(t)$, where $e_i(t)$ satisfies the parallel transport equation $D_{A_0} e_i = 0$ with boundary conditions $e_i(l_\gamma) = \lambda_\gamma^i e_i(0)$ and $\|e_i(0)\| = 1$. In particular, we assume $\lambda_\gamma^1 = \lambda_\gamma$ and $\mathcal{L}_\gamma(t)$ is generated by $e_1(t)$. Then

$$\begin{aligned}
(D_{A^0}\pi)(v) &= [D_{A^0}, \pi]v = D_{A^0}(\pi v) - \pi(D_{A^0} v) \\
&= D_{A^0}(a_1(t)e_1(t)) - \pi\left(\sum_{i=1}^n (da_i(t)e_i(t) + a_i(t) D_{A^0} e_i(t))\right) \\
&= da_1(t)e_1(t) - da_1(t)e_1(t) \\
&= 0.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\int_0^{l_\gamma} & \frac{d}{dt}(\mathrm{Tr}(\dot g\cdot\pi))\,dt \\
&= \int_0^{l_\gamma} \mathrm{Tr}\Big(\frac{\partial}{\partial t}(\dot g\pi)\Big)\,dt \\
&= \int_0^{l_\gamma} \mathrm{Tr}(D_{A^0}(\dot g\pi))\,dt && (\text{since } \mathrm{Tr}([A^0,\dot g\pi])=0) \\
&= \int_0^{l_\gamma} \mathrm{Tr}([D_{A^0},\dot g\pi])\,dt && (\text{notice } \dot g\pi\in\Gamma(\mathrm{End}(E))) \\
&= \int_0^{l_\gamma} \mathrm{Tr}([D_{A^0},\dot g]\pi+\dot g[D_{A^0},\pi])\,dt \\
&= \int_0^{l_\gamma} \mathrm{Tr}((D_{A^0}\dot g)\pi+\dot g(D_{A^0}\pi))\,dt && (\text{action of a connection on } \Gamma(\mathrm{End}(E))) \\
&= \int_0^{l_\gamma} \mathrm{Tr}((D_{A^0}\dot g)\pi)\,dt && (\text{since } D_{A^0}\pi=0).
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^{l_\gamma} \mathrm{Tr}((D_{A^0}\dot g)\pi)\,dt &= \int_0^{l_\gamma} \frac{d}{dt}(\mathrm{Tr}(\dot g\cdot\pi))\,dt \\
&= \mathrm{Tr}(\dot g(l_\gamma)\pi(l_\gamma)) - \mathrm{Tr}(\dot g(0)\pi(0)) \\
&= 0.
\end{aligned}
$$

As s varies, the eigenline $\mathcal{L}_\gamma^s(t)$ corresponding to $\lambda_\gamma(s)$ varies according to s and so does the complementary hyperplane $\mathcal{H}_\gamma^s(t)$. By picking suitable gauges $\{g_s\}$, we can assume, for $\widetilde{A}^s := g_s^* A^s$, the eigenlines $\widetilde{\mathcal{L}}_\gamma^s(t)$ and complementary hyperplanes $\widetilde{\mathcal{H}}_\gamma^s(t)$ satisfy $\widetilde{\mathcal{L}}_\gamma^s(t)=\mathcal{L}_\gamma(t)$ and $\widetilde{\mathcal{H}}_\gamma^s(t)=\mathcal{H}_\gamma(t)$.

Without loss of generality, we assume D_{A^s} is itself the connection for a suitable gauge and $\{e_i^s\}$ are eigenvectors for A^s with e_1^s corresponding to $\mathcal{L}_\gamma^s$. Thus,

$$D_{A^s} e_i^s(t)=0,\quad e_i^s(l_\gamma)=\lambda_\gamma^i(s)e_i^s(0).$$

In particular, we can assume

$$D_{A^s} e_1^s(t)=0,\quad e_1^s(t)=c_s(t)e_1^0(t),\quad e_1^s(l_\gamma)=\lambda_\gamma^1(s)e_1^s(0),\quad e_1^s(0)=e_1^0(0).$$

So

$$e_1^s(l_\gamma)=c_s(l_\gamma)e_1^0(l_\gamma)=c_s(l_\gamma)\lambda_\gamma^1(0)e_1^0(0)=\lambda_\gamma^1(s)e_1^s(0)=\lambda_\gamma^1(s)e_1^0(0)$$

and thus $c_s(l_\gamma) = \lambda^1_\gamma(s)/\lambda^1_\gamma(0)$ and $c_0(l_\gamma) = 1$. Notice

$$\frac{H(e_1^0(t), D_{A^s} e_1^0(t))}{H(e_1^0(t), e_1^0(t))} = \frac{H\big(e_1^0(t), D_{A^s}(e_1^s(t)/c_s(t))\big)}{H(e_1^0(t), e_1^s(t)/c_s(t))} = \frac{\partial_t(1/c_s(t))}{1/c_s(t)} = -\frac{\partial(\log c_s(t))}{\partial t}.$$

So

$$\begin{aligned}\int_0^{l_\gamma} \mathrm{Tr}(\partial_s D_{A^0}\pi)\,dt &= \int_0^{l_\gamma} \frac{H(e_1^0(t), \partial_s D_{A^0} e_1^0(t))}{H(e_1^0(t), e_1^0(t))}\,dt \\ &= -\int_0^{l_\gamma} \frac{\partial}{\partial s}\left(\frac{\partial(\log c_s(t))}{\partial t}\right)\bigg|_{s=0} dt \\ &= -\frac{d \log \lambda^1_\gamma(s)}{ds}\bigg|_{s=0}. \end{aligned}$$

□

5 Computation of the variations of the reparametrization functions for a model case

In this section and the next, we consider the model case $\partial_\beta g_{\alpha\alpha}(\sigma)$. Note the treatment of this case will involve all the steps needed for the other cases. This justifies the expositional decision that we consider it here first and in isolation.

In this case, we are given parameters $(u, v) \in \{(-1, 1)\}^2$ with (conjugacies classes of) representations $\{\rho(u, v)\}$ in $\mathcal{H}_3(S)$ corresponding to

$$\{(0, uq_\alpha + vq_\beta)\} \subset H^0(X, K^2) \oplus H^0(X, K^3)$$

by Hitchin parametrization (see Remark 2.37). In particular, at the Fuchsian point $\rho(0) = X$, we identify $\partial_u \rho(0, 0)$ with $\varphi(q_\alpha)$ and $\partial_v \rho(0, 0)$ with $\varphi(q_\beta)$, where φ is the Hitchin deformation given in Definition 2.38. We suppose $\{f_{\rho(u,v)}\}$ is an associated two-parameter family of reparametrization functions. By Proposition 4.2, the formula for $\partial_\beta g_{\alpha\alpha}(\sigma)$ is

$$\begin{aligned}\partial_\beta g_{\alpha\alpha}(\sigma) &= \partial_v\big(\langle \partial_u \rho(0, v), \partial_u \rho(0, v)\rangle_P\big)(0) \\ &= \lim_{r\to\infty} \frac{1}{r}\bigg[\int_{UX}\bigg(\int_0^r \partial_u f^N_{\rho(0)}\,dt\bigg)^2 \int_0^r \partial_v f^N_{\rho(0)}\,dt\,dm_0 \\ &\qquad\qquad + 2\int_{UX}\int_0^r \partial_u f^N_{\rho(0)}\,dt \int_0^r \partial_{uv} f^N_{\rho(0)}\,dt\,dm_0\bigg].\end{aligned}$$

Because $\partial_u h(\rho(u, 0)) = \partial_v h(\rho(0, v)) = 0$ on Fuchsian locus $\mathcal{T}(S)$. By Theorem 2.16, the variations of the reparametrization functions that need to be computed are the following:

(i) $\partial_u f^N_{\rho(0)} = -\partial_u f_{\rho(0)}$.

(ii) $\partial_v f^N_{\rho(0)} = -\partial_v f_{\rho(0)}$.

(iii) $\partial_{uv} f^N_{\rho(0)} = -\partial_{uv} h(\rho(0)) - \partial_{uv} f_{\rho(0)}$.

Before proceeding to compute (i), (ii) and (iii), we explain our general strategy to compute the variations of the reparametrization functions. Our computation will be based on Proposition 4.4 and tools from Higgs bundles theory. Let us first set up our Higgs bundles.

In the component $\mathcal{H}_3(S)$ we are considering, the rank-3 holomorphic vector bundle is fixed as $E = K \oplus \mathcal{O} \oplus K^{-1}$. Associated to a representation ρ in $\mathcal{H}_3(S)$ is a Hermitian metric H on E that solves Hitchin's equation (2-9) and a flat connection $D_H = \nabla_{\bar\partial_E, H} + \Phi + \Phi^{*H}$, where $\nabla_{\bar\partial_E, H}$ is the Chern connection (see Theorem 2.29).

Given a parameter $s \in (-1, 1)$, suppose we are considering a family of conjugacy classes of representations $\{\rho_s\}$ in $\mathcal{H}_3(S)$. On the one hand, there is a family of flat connections $\{D_{H(s)}\}$ given by (2-10) associated to $\{\rho_s\}$. On the other hand, there is a family of reparametrization functions $\{f_{\rho_s}\}_{s\in(-1,1)}$ associated to $\{\rho_s\}$ from the thermodynamical point of view. Recall our notation (2-4)–(2-5). For a family of flat connections $\{D_{H(s)}\}$, we write

$$\partial_s D_{H(0)} = \frac{\partial D_{H(s)}}{\partial s}\bigg|_{s=0}$$

and, for a family of reparametrization functions $\{f_{\rho_s}\}$,

$$\partial_s f_{\rho_0} = \frac{\partial f_{\rho_s}}{\partial s}\bigg|_{s=0}.$$

By Proposition 4.4 and Livšic's theorem, the Hölder function $-\mathrm{Tr}(\partial_s D_{H(0)}\pi)(x)$ and $\partial_s f_{\rho_0}(x)$ are in the same Livšic cohomology class. Recalling our notation in Definition 2.3,

$$\partial_s f_{\rho_0}(x) \sim -\mathrm{Tr}(\partial_s D_{H(0)}\pi)(x). \tag{5-1}$$

Here we define $\mathrm{Tr}(\partial_s D_{H(0)}\pi)(\Phi_t(x)) := \mathrm{Tr}(\partial_s D_{H(0)}(t)\pi(t))$, following Proposition 4.4. The curve $\gamma(t)$ in Proposition 4.4 from now on will be a unit-speed geodesic starting from x. Therefore, $x = \dot\gamma(0)\,\partial/\partial t$ and $\Phi_t(x) = \dot\gamma(t)\,\partial/\partial t$.

Proposition 4.3 allows us to consider the first and second variations of the reparametrization functions in terms of Livšic cohomology classes instead of individual functions.

From now on, for the first and second variations of the reparametrization functions, we will no longer distinguish cohomologous elements.

Because X is a hyperbolic surface and the geodesic flow is Anosov, the vectors tangent to periodic geodesics are dense in TX. To recover the information of $\partial_s f_{\rho_0}$, it suffices to compute $\operatorname{Tr}(\partial_s D_{H(0)}\pi)$ on each closed geodesic. Similarly, to compute the second variations of the reparametrization functions, it suffices to compute them on each closed geodesic.

Now we start to give a complete computation of the first and second variations of the reparametrization functions for the case $\partial_\beta g_{\alpha\alpha}(\sigma)$. The steps of our argument are divided into different subsections:

(1) We set up coordinates adapted to the closed geodesics we study and conclude special properties of affine metrics with respect to chosen coordinates on these geodesics.

(2) We first construct a homogeneous ODE arising from the parallel transport equation for the base flat connection at $\rho(0) = \sigma \in \mathcal{T}(S)$. This leads to formulas for the first variations of the reparametrization functions proved in [20].

(3) We consider a family of parallel transport equations associated to a family of flat connections by solving Hitchin's equations based at $\rho(0) = \sigma \in \mathcal{T}(S)$. The variation of this family of parallel transport equations at σ gives rise to some nonhomogeneous ODEs and yields solutions for the second variations of the reparametrization functions on the closed geodesics we consider.

(4) We extend our computation from the closed geodesics to the surface.

5.1 Setting up coordinates on surfaces

In this subsection, we set up coordinates adapted to the closed geodesics we study. We will obtain some important properties for the affine metric after setting up the coordinates. They can be used in the computation of the first and second variations of the reparametrization functions in the following sections. The first variations have been computed in [20] by advanced Lie-theoretic methods.

The convention we use for a Hermitian metric H on E is it is $\mathbb{C}$–linear in the second variable and conjugate-linear in the first variable. Suppose on a coordinate chart (U, z), the bundle $E = K \oplus \mathcal{O} \oplus K^{-1}$ is trivialized as $E|_U \cong U \times \mathbb{C}^3$. Locally we have a holomorphic frame (s_1, s_2, s_3) on U. With respect to the local holomorphic frame and our convention of the Hermitian metric, the $(1,0)$–part of the Chern connection $\nabla_{\bar\partial_E,H}$ is $H^{-1}\,\partial H$. The Hermitian conjugate is $\Phi^{*H} = H^{-1}\bar\Phi^t H$. The connection 1–form A of the flat connection D_H is thus

$$A = H^{-1}\,\partial H + \Phi + \Phi^{*H}.$$

Associated to representations $\{\rho(u,v)\}$ are a two-parameter family of flat connections $\{D_{H(u,v)}\}$. We will study their connection 1–forms in holomorphic frames with respect to some carefully chosen coordinates on the surface X.

When the Higgs field is

$$\Phi(u,v) = \begin{bmatrix} 0 & 0 & uq_\alpha + vq_\beta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

Baraglia proves the Hermitian metric $H(u,v)$ that solves Hitchin's equation (2-9) is diagonal (see [2]). Following Baraglia's notation [2], we denote the Hermitian metric by $H(u,v) = e^{2\Omega(u,v)}$. We have

$$H(u,v) = \begin{bmatrix} h(u,v)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h(u,v) \end{bmatrix},$$

where $h = h(u,v)$ is a section of $\bar K \otimes K$ and

$$\Omega(u,v) = \begin{bmatrix} -\omega(u,v) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega(u,v) \end{bmatrix}$$

with $\omega(u,v) = \frac{1}{2}\log h(u,v)$.

We denote the corresponding flat connection by

$$D_{H(u,v)} = \nabla_{\bar\partial_E,H(u,v)} + \Phi(u,v) + \Phi(u,v)^{*H(u,v)}.$$

The connection 1–form $A(u,v) \in \Gamma(T^*X \otimes \operatorname{End} E)$ is thus

$$A(u,v) = \begin{bmatrix} -2\,\partial\omega(u,v) & h(u,v) & uq_\alpha + vq_\beta \\ 1 & 0 & h(u,v) \\ h^{-2}(u\bar q_\alpha + v\bar q_\beta) & 1 & 2\,\partial\omega(u,v) \end{bmatrix}. \tag{5-2}$$

In fact, $2h(u,v)$ is an affine metric for some hyperbolic affine sphere in the conformal class of σ (see [2]).

We let $\phi = \log(2h/\sigma)$. Note $\phi = \phi(u,v,z)$ is actually a globally well-defined function on X that does not depend on coordinate systems. Hitchin's equation (2-9), using the integrability condition for affine sphere (see [24]), can be written as an equation of ϕ as

$$\Delta_\sigma \phi + 16\|uq_\alpha + vq_\beta\|_\sigma^2 e^{-2\phi} - 2e^{\phi} + 2 = 0, \tag{5-3}$$

where $\|\cdot\|_\sigma$ is the induced norm on cubic differentials. It satisfies $\|q\|_\sigma^2 = |q|^2/\sigma^3$. The notation we adopt for Laplacian is $\Delta_\sigma = 4\partial_{\bar{z}}\partial_z/\sigma$.

For simplicity of notation, we sometimes omit variables and write ϕ as $\phi(u,v)$ or $\phi(z)$ depending on our needs.

We have the following observation from (5-3):

- When $(u,v) = (0,0)$, the only solution of (5-3) is $\phi = \phi(0,0) = 0$. The affine metric $2h = \sigma$ is indeed the hyperbolic metric of constant curvature -1.
- Taking the u–derivative or v–derivative of (5-3) at $(u,v) = (0,0)$ yields

$$\Delta_\sigma \phi_u - 2e^{\phi}\phi_u = 0, \tag{5-4}$$

$$\Delta_\sigma \phi_v - 2e^{\phi}\phi_v = 0. \tag{5-5}$$

  Therefore, the fact that $\phi = \phi(0,0) = 0$ implies $\phi_u = \phi_u(0,0) = 0$ and $\phi_v = \phi_v(0,0) = 0$.

We now choose a special coordinate system that facilitates the study of holonomy problems on a closed geodesic. Let z be a local holomorphic coordinate on X. Suppose the affine metric in this coordinate is $e^{\psi(u,v,z)}|dz|^2$ and the hyperbolic metric in this coordinate is $\sigma = e^{\delta(z)}|dz|^2$. Suppose $\gamma(t)$ is any closed geodesic with respect to the hyperbolic metric σ on the Riemann surface X. Then, written in the z–coordinate, it is

$$\gamma(t) = z(t) = \operatorname{Re}\gamma(t) + i\operatorname{Im}\gamma(t)$$

and

$$\dot\gamma(t)\frac{\partial}{\partial t} = (\operatorname{Re}\dot\gamma(t) + i\operatorname{Im}\dot\gamma(t))\frac{\partial}{\partial z} + (\operatorname{Re}\dot\gamma(t) - i\operatorname{Im}\dot\gamma(t))\frac{\partial}{\partial \bar z}.$$

In particular, we can model $\gamma(t)$ on a strip $S = \left\{x + iy : |y| < \frac{\pi}{2}\right\}$ with the hyperbolic metric $ds = |dz|/\cos y$ and $\gamma(t) = (t,0)$. This coordinate around γ is called a Fermi coordinate and satisfies $\operatorname{Re}\dot\gamma(t) = 1$ and $\operatorname{Im}\dot\gamma(t) = 0$. Thus, it's easy to check that, on γ, one has $\gamma^* \, ds = |dz|$ and $\delta(z) = 0$.

The variable t is then the arc-length parameter for our choice of coordinates. Therefore, if one writes $\dot\gamma(0)\,\partial/\partial t = x \in UX$, then $\dot\gamma(t)\,\partial/\partial t = \Phi_t(x)$. We will always assume $\dot\gamma(0)\,\partial/\partial t = x$ in our discussion.

With the Fermi coordinate understood, from the fact that the only solution of (5-3) is $\phi = 0$, we conclude

$$\psi(z) = \phi(z) + \delta(z) = \delta(z) = 0.$$

From (5-4) together with (5-5) and their solutions $\phi_u = \phi_v = 0$, we obtain

$$\psi_u(z) = \phi_u(z) = 0, \quad \psi_v(z) = \phi_v(z) = 0.$$

Also $\psi(z) = 0$ implies

$$\psi_z(z) = \delta_z(z) = 0.$$

All this information about the affine metric ψ with respect to the Fermi coordinate will be important in computation in later sections.

5.2 Homogeneous ODEs for holonomy and first variations of the reparametrization functions

In this subsection, we show a formula for the first variations of the reparametrization functions from [20]. We also construct homogeneous ODEs arising from the parallel transport equations for the base flat connection at $\sigma \in \mathcal{T}(S)$. These serve as the first step for the computation of the second variations in later subsections.

We first explain our notation. For $q_i = q_i(z)\,dz^2$ any quadratic differential and $q_\alpha = q_\alpha(z)\,dz^3$ any cubic differential, we also use q_i and q_α to denote Hölder functions on the unit tangent bundle UX as follows. We let $q_i \colon UX \to \mathbb{C}$ and $q_\alpha \colon UX \to \mathbb{C}$ be

$$q_i(x) := q_i(z)\,dz^2(x,x) = q_i(z)(dz(x))^2, \tag{5-6}$$

$$q_\alpha(x) := q_\alpha(z)\,dz^3(x,x,x) = q_\alpha(z)(dz(x))^3. \tag{5-7}$$

The first variations of the reparametrization functions for our cases have been computed in [20] as follows:

Proposition 5.1 [20, Theorem 4.0.2] *The first variations of the reparametrization functions $\partial_u f_{\rho(0)} \colon UX \to \mathbb{R}$ and $\partial_v f_{\rho(0)} \colon UX \to \mathbb{R}$ for our model case $\partial_\beta g_{\alpha\alpha}(\sigma)$ satisfy*

$$-\partial_u f_{\rho(0)}(x) \sim \operatorname{Re} q_\alpha(x), \quad -\partial_v f_{\rho(0)}(x) \sim \operatorname{Re} q_\beta(x),$$

where the notation $\sim$ is Livšic equivalence (Definition 2.3).

Proposition 5.1 is proved in [20] as a consequence of (5-1).

We then study parallel transport equations for the connection $D_{H(0)}$ arising from holonomy problems based at $\rho(0) \in \mathcal{T}(S)$. With the coordinates introduced in the last section, they become homogenous ODE systems that are easy to solve. We list some important computations involved here. These will be important for the second variations of the reparametrization functions.

The parallel transport equation for the connection $D_{H(0)}$ on the closed geodesic γ is

$$D_{H(0),\dot{\gamma}} V = 0, \tag{5-8}$$

where $V \in \Gamma(E)$ is a parallel section with boundary conditions

$$V(l_\gamma) = \lambda_i(\gamma, \rho(0)) V(0).$$

Here $\lambda_i(\gamma, \rho(0))$ is one of the eigenvalues for holonomy of $D_{H(0)}$ on γ for $i = 1, 2, 3$. We want to write (5-8) on a specific holomorphic frame, which can be constructed as follows.

We cover γ by m charts $\{(U_i, z_i)\}_{i=1}^m$ such that $z_i : U_i \to z_i(U_i) \subset \mathbb{C}$ is a diffeomorphism for $1 \le i \le m$. We assume our holomorphic bundle E is trivialized on each U_i. Furthermore, we assume the transition map on every overlap is either the identity or a hyperbolic translation viewed on the universal cover $\mathbb{D}$. Since dz_i is a local holomorphic section of K on U_i and $\partial/\partial z_i$ is a local holomorphic section of K^{-1} on U_i, we can define a local holomorphic frame $s^i = (s^i_1, s^i_2, s^i_3)$ for $E = K \oplus \mathcal{O} \oplus K^{-1}$ on U_i, where $s^i_1 = dz_i$ and $s^i_2 = 1$ and $s^i_3 = \partial/\partial z_i$. Setting $(U_{m+1}, z_{m+1}) = (U_1, z_1)$ and $s^{m+1}_j = s^1_j$, this yields a well-defined holomorphic frame for γ because, on each overlap and for $j = 1, 2, 3$, we have $s^i_j = s^{i+1}_j$ on $\gamma|_{U_i} \cap \gamma|_{U_i+1}$ with $1 \le i \le m$.

We will simply write the holomorphic frame on γ as s_j for $j = 1, 2, 3$. With respect to this frame, the parallel transport equation for $V(t) = \sum_{i=1}^3 V_i(t) s_i(t)$ becomes

$$\partial_t \begin{bmatrix} V^1(t) \\ V^2(t) \\ V^3(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V^1(t) \\ V^2(t) \\ V^3(t) \end{bmatrix} = 0.$$

There are three eigenvalues for this ODE system: $\lambda_1(\gamma, \rho(0)) = e^{l_\gamma}$, $\lambda_2(\gamma, \rho(0)) = 1$ and $\lambda_3(\gamma, \rho(0)) = e^{-l_\gamma}$. The solutions for V (assuming norm 1 at the starting point with respect to the Hermitian metric $H(0)$), denoted by e_i corresponding to $\lambda_i(\gamma)$ for $i = 1, 2, 3$, are

$$e_1 = \frac{\sqrt{2}}{2} e^t \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}, \quad e_2 = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad e_3 = \frac{\sqrt{2}}{2} e^{-t} \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}.$$

We note at the Fuchsian point $\rho(0) \in \mathcal{T}(S)$, the eigenvectors e_1, e_2 and e_3 are orthogonal. In our holomorphic frame, the projection $\pi(0) = \pi(\rho(0))$ can be computed as

$$\pi(0) = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ -1 & 1 & -\frac{1}{2} \\ 1 & -1 & \frac{1}{2} \end{bmatrix}.$$

The eigenvectors e_i and projection π will play important roles in later sections.

5.3 Inhomogeneous ODEs and the second variations of the reparametrization functions

We will compute the second variation of the reparametrization functions $\partial_{uv} f_{\rho(0)}$ in this and the next subsection. With our formula (5-1), we have

$$\begin{aligned}\partial_{uv} f_{\rho(0)} &\sim -\partial_v\big(\mathrm{Tr}(\partial_u D_{H(0,v)}\pi(0,v))\big)(0) \\ &= -\mathrm{Tr}\bigg(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}\pi(0)\bigg) - \mathrm{Tr}(\partial_u D_{H(0)}\partial_v \pi(0)) \\ &=: -\mathrm{I} - \mathrm{II}.\end{aligned} \tag{5-9}$$

In this subsection, we compute $\partial_{uv} f_{\rho(0)}$ along a closed geodesic by computing I and II. We study variation of holonomy problems along a closed geodesic and construct associated inhomogeneous ODEs. In the next subsection, we extend the computation of $\partial_{uv} f_{\rho(0)}$ to the whole surface.

Compute I With the holomorphic frames and Fermi coordinates setup as before, one obtains, on γ,

$$\partial_{uv} D_{H(0)}(x) = \begin{bmatrix} -(\psi_z)_{uv}(z) & \frac{1}{2}\psi_{uv}(z) & 0 \\ 0 & 0 & \frac{1}{2}\psi_{uv}(z) \\ 0 & 0 & (\psi_z)_{uv}(z) \end{bmatrix}.$$

Thus,

$$\mathrm{Tr}\bigg(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}(x)\pi(0)\bigg) = -\tfrac{1}{2}\psi_{uv}(z).$$

More explicitly, $\mathrm{Tr}((\partial^2 D_{H(0)}/\partial u\,\partial v)\pi(0))\colon UX \to \mathbb{R}$ satisfies

$$\mathrm{Tr}\bigg(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}\pi(0)\bigg)(x) = -\tfrac{1}{2}\psi_{uv}\big(z(p(x))\big) = -\tfrac{1}{2}\phi_{uv}(p(x)),$$

where $p\colon UX \to X$ is the projection from the unit tangent bundle to our surface and z is the Fermi coordinate we choose evaluating at the point $p(x) \in X$. Note that the affine metric ψ is always real and $\phi = \psi - \sigma$ does not depend on the coordinates we choose.

Compute II To study $\partial_v \pi(0)$ takes some effort. We set $u=0$ and take a family of flat connections $\{D_{H(v)}\}$ with connection 1–forms $A(0,v)$ (recall (5-2)). Associated to each of them is a parallel transport equation along the closed geodesic γ on (S,σ),

$$D_{H(v),\dot{\gamma}} V(v,t) = 0, \tag{5-10}$$

with the assumption $\|V(v,0)\|_{H(0)} = 1$.

In [19], Labourie proves the images of every Hitchin representation are purely loxodromic. For $\rho(0,v)$ in $\mathcal{H}_3(S)$, we know $\rho(0,v)(\gamma)$ has distinct eigenvalues $\lambda_1(\gamma,v) > \lambda_2(\gamma,v) > \lambda_3(\gamma,v)$. The holonomy problem for $\rho(0,v)$ has three distinct eigenvectors which are parallel sections $\{e_i(v,t)\}_{i=1}^3$ along $\gamma(t)$. Each section $V(v,t) = e_i(v,t)$ satisfies (5-10). In addition to the norm 1 condition at the starting point, $\|V(v,0)\|_{H(0)} = 1$, we also impose another boundary condition in order to guarantee these are eigenvectors. The boundary conditions are, for $i = 1,2,3$,

(i) $\|e_i(v,0)\|_{H(0)} = 1$;

(ii) $e_i(v,l_\gamma) = \lambda_i(\gamma,v) e_i(v,0)$.

The reader may notice that, up to now, there are two frames for E along γ mentioned, the holomorphic frame (s_1,s_2,s_3) and the frame spanned by eigenvectors (e_1,e_2,e_3). On the one hand, we can write our holomorphic frames as linear combinations of eigenvectors $s_i(t) = \sum_{j=1}^3 a_{ij}(v,t) e_j(v,t)$ for $i = 1,2,3$. On the other hand, we can write the eigenvectors as linear combinations of our holomorphic frames $e_j(v,t) = \sum_{k=1}^3 e_{jk}(v,t) s_k(t)$ for $j = 1,2,3$. We have the following observation:

With respect to the holomorphic frame (s_1,s_2,s_3), the projection onto e_1 along the hyperplane spanned by (e_2,e_3) in matrix form is

$$\begin{aligned}
\pi(v,t) &= \big[\pi(v,t)s_1(t) \;\; \pi(v,t)s_2(t) \;\; \pi(v,t)s_3(t)\big] \\
&= \big[a_{11}(v,t)e_1(v,t) \;\; a_{21}(v,t)e_1(v,t) \;\; a_{31}(v,t)e_1(v,t)\big] \\
&= \begin{bmatrix} a_{11}(v,t)e_{11}(v,t) & a_{21}(v,t)e_{11}(v,t) & a_{31}(v,t)e_{11}(v,t) \\ a_{11}(v,t)e_{12}(v,t) & a_{21}(v,t)e_{12}(v,t) & a_{31}(v,t)e_{12}(v,t) \\ a_{11}(v,t)e_{13}(v,t) & a_{21}(v,t)e_{13}(v,t) & a_{31}(v,t)e_{13}(v,t) \end{bmatrix}.
\end{aligned}$$

To understand $\partial_v \pi(0)$, we need to know $\partial_v e_1(0)$ and $\partial_v \alpha_{i1}(0)$ for $i = 1,2,3$. One can check, in the holomorphic frame,

$$\begin{aligned}
\operatorname{Tr}(\partial_u D_{A(0)} \partial_v \pi(0)) &= q_\alpha(\partial_v a_{11}(0) e_{13}(0) + a_{11}(0) \partial_v e_{13}(0)) \\
&\quad + 4\bar{q}_\alpha(\partial_v a_{31}(0) e_{11}(0) + a_{31}(0) \partial_v e_{11}(0)),
\end{aligned} \tag{5-11}$$

where $e_{11}(0)$ and $e_{13}(0)$ are known. Thus, we need to compute $\partial_v e_1(0)$ and $\partial_v a_{11}(0)$ and $\partial_v a_{31}(0)$.

We first show how to obtain $\partial_v e_1(0,t)$ as the solution of an inhomogeneous ODE system arising from taking the v–derivative for a family of parallel transport equations (5-10) at $v=0$,

$$\partial_t \begin{bmatrix} \partial_v e_{11}(0,t) \\ \partial_v e_{12}(0,t) \\ \partial_v e_{13}(0,t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_v e_{11}(0,t) \\ \partial_v e_{12}(0,t) \\ \partial_v e_{13}(0,t) \end{bmatrix} = -\frac{\sqrt{2}}{2} e^t \begin{bmatrix} q_\beta(\Phi_t(x)) \\ 0 \\ 2\bar{q}_\beta(\Phi_t(x)) \end{bmatrix},$$

with boundary conditions

$$\begin{aligned} &H(\partial_v e_1(0,0), e_1(0,0)) = 0, \\ &\qquad \partial_v e_1(0,l_\gamma) = -e^{l_\gamma}\left(\int_0^{l_\gamma} \operatorname{Re} q_\beta(\Phi_s(x))\,ds\right) e_1(0,0) + e^{l_\gamma}\partial_v e_1(0,0). \end{aligned}$$

The boundary conditions arise from taking the v–derivative for boundary conditions (i) and (ii) of the parallel transport equation (5-10) that the maximum eigenvector e_1 satisfies.

With these boundary conditions, we solve

$$\begin{aligned} \begin{bmatrix} \partial_v e_{11}(t) \\ \partial_v e_{12}(t) \\ \partial_v e_{13}(t) \end{bmatrix} &= \begin{bmatrix} -\frac{\sqrt{2}}{2}\int_0^t e^s(\cosh(t-s)\operatorname{Re} q_\beta + i \operatorname{Im} q_\beta)\,ds \\ \sqrt{2}\int_0^t e^s \sinh(t-s)\operatorname{Re} q_\beta\,ds \\ -\sqrt{2}\int_0^t e^s(\cosh(t-s)\operatorname{Re} q_\beta - i\operatorname{Im} q_\beta)\,ds \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{\sqrt{2}}{4}(e^{2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{2s-t}\operatorname{Re} q_\beta\,ds - \frac{\sqrt{2}}{2} i (e^{l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^s \operatorname{Im} q_\beta\,ds \\ -\frac{\sqrt{2}}{2}(e^{2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{2s-t}\operatorname{Re} q_\beta\,ds \\ -\frac{\sqrt{2}}{2}(e^{2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{2s-t}\operatorname{Re} q_\beta\,ds + \sqrt{2} i (e^{l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^s \operatorname{Im} q_\beta\,ds \end{bmatrix}. \end{aligned}$$

Here q_β refers to $q_\beta(\Phi_s(x))$ defined in (5-7).

We continue to compute $\partial_v \alpha_{11}(0)$ and $\partial_v \alpha_{31}(0)$. Combining

$$e_j(v,t) = \sum_{k=1}^{3} e_{jk}(v,t) s_k(t) \quad \text{and} \quad s_i(t) = \sum_{j=1}^{3} a_{ij}(v,t) e_j(v,t)$$

gives

$$\sum_{j=1}^{j=3} a_{ij}(v,t) e_{jk}(v,t) = \sigma_{ik}. \tag{5-12}$$

Recall the $e_{jk}(0,t)$ are known:

$$e_1(0,t)=\begin{bmatrix}e_{11}(0,t)\\ e_{12}(0,t)\\ e_{13}(0,t)\end{bmatrix}=\tfrac{\sqrt{2}}{2}e^t\begin{bmatrix}\tfrac12\\ -1\\ 1\end{bmatrix},$$

$$e_2(0,t)=\begin{bmatrix}e_{21}(0,t)\\ e_{22}(0,t)\\ e_{23}(0,t)\end{bmatrix}=\tfrac12\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix},$$

$$e_3(0,t)=\begin{bmatrix}e_{31}(0,t)\\ e_{32}(0,t)\\ e_{33}(0,t)\end{bmatrix}=\tfrac{\sqrt{2}}{2}e^{-t}\begin{bmatrix}\tfrac12\\ 1\\ 1\end{bmatrix}.$$

Then one obtains

$$a(0,t)=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\ a_{21}&a_{22}&a_{23}\\ a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}\tfrac{\sqrt{2}}{2}e^{-t}&-1&\tfrac{\sqrt{2}}{2}e^t\\ -\tfrac{\sqrt{2}}{2}e^{-t}&0&\tfrac{\sqrt{2}}{2}e^t\\ \tfrac{\sqrt{2}}{4}e^{-t}&\tfrac12&\tfrac{\sqrt{2}}{4}e^t\end{bmatrix}.$$

Taking the v–derivative of (5-12) at $v=0$,

$$\sum_{j=1}^{j=3}\partial_v a_{ij}(0,t)e_{jk}(0,t)+\sum_{j=1}^{j=3}a_{ij}(0,t)\partial_v e_{jk}(0,t)=0.$$

Solutions of $\partial_v a_{ij}(0,t)$ can be expressed in terms of $\partial_v e_1(0,t)$, $\partial_v e_2(0,t)$ and $\partial_v e_3(0,t)$. We have just solved $\partial_v e_1$. Similarly,$\partial_v e_2(0)$ and $\partial_v e_3(0)$ are solutions of another two systems of nonhomogeneous ODEs deduced from (5-10). We now proceed to solve $\partial_v e_2(0,t)$ and $\partial_v e_3(0,t)$.

(1) For $\partial_v e_2(0,t)$, we have

$$\partial_t\begin{bmatrix}\partial_v e_{21}(0,t)\\ \partial_v e_{22}(0,t)\\ \partial_v e_{23}(0,t)\end{bmatrix}+\begin{bmatrix}0&\tfrac12&0\\ 1&0&\tfrac12\\ 0&1&0\end{bmatrix}\begin{bmatrix}\partial_v e_{21}(0,t)\\ \partial_v e_{32}(0,t)\\ \partial_v e_{23}(0,t)\end{bmatrix}=\begin{bmatrix}-q_\beta(\Phi_t(x))\\ 0\\ 2\bar{q}_\beta(\Phi_t(x))\end{bmatrix}$$

with boundary conditions

$$H(\partial_v e_2(0,0),e_2(0,0))=0,$$

$$\partial_v e_2(0,l_\gamma)=2\left(\int_0^{l_\gamma}\operatorname{Re}q_\beta(\Phi_s(x))\,ds\right)e_2(0,0)+\partial_v e_2(0,0).$$

(2) For $\partial_v e_3(0,t)$, we get

$$\partial_t\begin{bmatrix}\partial_v e_{31}(0,t)\\ \partial_v e_{32}(0,t)\\ \partial_v e_{33}(0,t)\end{bmatrix}+\begin{bmatrix}0&\tfrac12&0\\ 1&0&\tfrac12\\ 0&1&0\end{bmatrix}\begin{bmatrix}\partial_v e_{31}(0,t)\\ \partial_v e_{32}(0,t)\\ \partial_v e_{33}(0,t)\end{bmatrix}=-\tfrac{\sqrt{2}}{2}e^{-t}\begin{bmatrix}q_\beta(\Phi_t(x))\\ 0\\ 2\bar{q}_\beta(\Phi_t(x))\end{bmatrix}$$

with boundary conditions

$$H(\partial_v e_3(0,0), e_3(0,0)) = 0,$$
$$\partial_v e_3(0,l_\gamma) = -e^{-l_\gamma}\left(\int_0^{l_\gamma} \operatorname{Re} q_\beta(\Phi_s(x))\,ds\right)e_3(0,0) + e^{-l_\gamma}\partial_v e_3(0,0).$$

We obtain respective solutions from

$$\begin{bmatrix}\partial_v e_{21}(t)\\ \partial_v e_{22}(t)\\ \partial_v e_{23}(t)\end{bmatrix} = \begin{bmatrix}-\int_0^t \operatorname{Re} q_\beta + i\cosh(t-s)\operatorname{Im} q_\beta\,ds\\ 2\int_0^t i\sinh(t-s)\operatorname{Im} q_\beta\,ds\\ 2\int_0^t \operatorname{Re} q_\beta - i\cosh(t-s)\operatorname{Im} q_\beta\,ds\end{bmatrix}$$
$$+\begin{bmatrix}-\frac{i}{2}\int_0^{l_\gamma}\operatorname{Im} q_\beta((1-e^{l_\gamma})^{-1}e^{l_\gamma+t-s} + (1-e^{-l_\gamma})^{-1}e^{-l_\gamma-t+s})\,ds\\ i\int_0^{l_\gamma}\operatorname{Im} q_\beta((1-e^{l_\gamma})^{-1}e^{l_\gamma+t-s} - (1-e^{-l_\gamma})^{-1}e^{-l_\gamma-t+s})\,ds\\ -i\int_0^{l_\gamma}\operatorname{Im} q_\beta((1-e^{l_\gamma})^{-1}e^{l_\gamma+t-s} + (1-e^{-l_\gamma})^{-1}e^{-l_\gamma-t+s})\,ds\end{bmatrix}$$

and

$$\begin{bmatrix}\partial_v e_{31}(t)\\ \partial_v e_{32}(t)\\ \partial_v e_{33}(t)\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2}\int_0^t e^{-s}(\cosh(t-s)\operatorname{Re} q_\beta + i\operatorname{Im} q_\beta)\,ds\\ \sqrt{2}\int_0^t e^{-s}\sinh(t-s)\operatorname{Re} q_\beta\,ds\\ -\sqrt{2}\int_0^t e^{-s}(\cosh(t-s)\operatorname{Re} q_\beta - i\operatorname{Im} q_\beta)\,ds\end{bmatrix}$$
$$+\begin{bmatrix}-\frac{\sqrt{2}}{4}(e^{-2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{t-2s}\operatorname{Re} q_\beta\,ds - \frac{\sqrt{2}}{2}i(e^{-l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{-s}\operatorname{Im} q_\beta\,ds\\ \frac{\sqrt{2}}{2}(e^{-2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{t-2s}\operatorname{Re} q_\beta\,ds\\ -\frac{\sqrt{2}}{2}(e^{-2l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{t-2s}\operatorname{Re} q_\beta\,ds + \sqrt{2}i(e^{-l_\gamma}-1)^{-1}\int_0^{l_\gamma} e^{-s}\operatorname{Im} q_\beta\,ds\end{bmatrix},$$

where q_β in the solutions again refers to $q_\beta(\Phi_s(x))$ defined in (5-7).

We are therefore able to solve $\partial_v a_{ij}(0,t)$ from $\partial_v e_1(0,t)$, $\partial_v e_2(0,t)$ and $\partial_v e_3(0,t)$. For a closed geodesic γ of length l_γ starting from $\dot\gamma(0) = x$, we compute, from (5-11),

$$\begin{aligned}\text{(5-13)}\quad &\operatorname{Tr}(\partial_u D_{A(0)}\partial_v\pi(0))(\Phi_t(x))\\ &= \operatorname{Re} q_\alpha(\Phi_t(x))\int_0^t (e^{2(t-s)} - e^{2(s-t)})\operatorname{Re} q_\beta(\Phi_s(x))\,ds\\ &\quad + 2\operatorname{Im} q_\alpha(\Phi_t(x))\int_0^t (e^{t-s} - e^{s-t})\operatorname{Im} q_\beta(\Phi_s(x))\,ds\\ &\quad + \operatorname{Re} q_\alpha(\Phi_t(x))\int_0^{l_\gamma}\left(\frac{e^{2(t-s)}}{e^{-2l_\gamma}-1} - \frac{e^{2(s-t)}}{e^{2l_\gamma}-1}\right)\operatorname{Re} q_\beta(\Phi_s(x))\,ds\\ &\quad + 2\operatorname{Im} q_\alpha(\Phi_t(x))\int_0^{l_\gamma}\left(\frac{e^{t-s}}{e^{-l_\gamma}-1} - \frac{e^{s-t}}{e^{l_\gamma}-1}\right)\operatorname{Im} q_\beta(\Phi_s(x))\,ds.\end{aligned}$$

In particular, at $t=0$,

$$\begin{aligned}(5\text{-}14)\quad & \operatorname{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x) \\ &= \operatorname{Re} q_\alpha(x) \int_0^{l_\gamma} \left(\frac{e^{-2s}}{e^{-2l_\gamma}-1} - \frac{e^{2s}}{e^{2l_\gamma}-1} \right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ &\qquad + 2 \operatorname{Im} q_\alpha(x) \int_0^{l_\gamma} \left(\frac{e^{-s}}{e^{-l_\gamma}-1} - \frac{e^{s}}{e^{l_\gamma}-1} \right) \operatorname{Im} q_\beta(\Phi_s(x))\, ds.\end{aligned}$$

Remark 5.2 Every point on the closed geodesic γ plays an equivalent role. We can always let $y=\Phi_t(x)$ be the initial point of our γ and set up boundary conditions for our ODEs based at y instead of x. The solution of this new ODE system is (5-14), treating $y=\Phi_t(x)$ as the initial point. It is in fact the same as starting from x and obtaining $\operatorname{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(\Phi_t(x))$ from (5-13).

5.4 Hölder extension to the surface

The holonomy problems only yield solutions on closed geodesics as they can be simplified as linear ODEs with boundary conditions. However, it is still possible to extend the computation for the second variations of the reparametrization functions from closed geodesics to the Riemann surface X. This will be our goal in this subsection. In particular, We will prove in the end of this subsection the main proposition about second variations of the reparametrization functions.

Proposition 5.3 *The second variation of the reparametrization functions*

$$\partial_{uv} f_{\rho(0)} \colon UX \to \mathbb{R}$$

for our model case $\partial_\beta g_{\alpha\alpha}(\sigma)$ satisfies

$$\partial_{uv} f_{\rho(0)}(x) \sim \tfrac{1}{2}\phi_{uv}(p(x)) - \eta(x),$$

where we recall that ϕ is defined in (5-3) *and $p\colon UX \to X$ is the projection from the unit tangent bundle UX to our Riemann surface X, and $\eta\colon UX \to \mathbb{R}$ is given by*

$$\begin{aligned}\eta(x) = &-\operatorname{Re} q_\alpha(x) \int_0^\infty e^{-2s} \operatorname{Re} q_\beta(\Phi_s(x))\, ds - \operatorname{Re} q_\alpha(x) \int_{-\infty}^0 e^{2s} \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ &-2 \operatorname{Im} q_\alpha(x) \int_0^\infty e^{-s} \operatorname{Im} q_\beta(\Phi_s(x))\, ds - 2 \operatorname{Im} q_\alpha(x) \int_{-\infty}^0 e^{s} \operatorname{Im} q_\beta(\Phi_s(x))\, ds.\end{aligned}$$

We will prove that $\eta(x)$ coincides with $\mathrm{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x)$ on periodic orbits and that $\eta(x)$ is a Hölder function. Denoting the subset of UX that consists of all unit tangent vectors to closed geodesics by W, we first show:

Proposition 5.4 *For any $x \in W$, $\eta(x) = \mathrm{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x)$.*

To prove Proposition 5.4, from the computation of $\mathrm{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x)$ in (5-14), we introduce an intermediate function $\psi\colon W \times \mathbb{R}^+ \to \mathbb{R}$, given by

$$\psi(x,r) = \operatorname{Re} q_\alpha(x) \int_0^r \left(\frac{e^{-2s}}{e^{-2r}-1} - \frac{e^{2s}}{e^{2r}-1} \right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ + 2 \operatorname{Im} q_\alpha(x) \int_0^r \left(\frac{e^{-s}}{e^{-r}-1} - \frac{e^{s}}{e^{r}-1} \right) \operatorname{Im} q_\beta(\Phi_s(x))\, ds.$$

Given $x \in W$, if we denote the closed geodesic that x is tangential to by γ_x, with length l_{γ_x}, then clearly $\mathrm{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x) = \psi(x, l_{\gamma_x})$. To prove Proposition 5.4 for the set W, we need the following lemma, which states that $\psi(x,r)$ attains the same value when r is any positive integer multiple of l_{γ_x}:

Lemma 5.5 *$\psi(x, k l_{\gamma_x}) = \psi(x, l_{\gamma_x})$ for all $x \in W$ and $k \in \mathbb{Z}^+$.*

Proof For any $k \in \mathbb{Z}^+$, we have

$$\int_0^{k l_{\gamma_x}} \left(\frac{e^{-2s}}{e^{-2k l_{\gamma_x}}-1} - \frac{e^{2s}}{e^{2k l_{\gamma_x}}-1} \right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ = \sum_{i=1}^{k} \int_{(i-1) l_{\gamma_x}}^{i l_{\gamma_x}} \left(\frac{e^{-2s}}{e^{-2k l_{\gamma_x}}-1} - \frac{e^{2s}}{e^{2k l_{\gamma_x}}-1} \right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ = \frac{1}{e^{-2k l_{\gamma_x}}-1} \sum_{i=1}^{k} \int_{(i-1) l_{\gamma_x}}^{i l_{\gamma_x}} e^{-2s} \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ - \frac{1}{e^{2k l_{\gamma_x}}-1} \sum_{i=1}^{k} \int_{(i-1) l_{\gamma_x}}^{i l_{\gamma_x}} e^{2s} \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ = \int_0^{l_{\gamma_x}} \left(\frac{e^{-2s}}{e^{-2 l_{\gamma_x}}-1} - \frac{e^{2s}}{e^{2 l_{\gamma_x}}-1} \right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds.$$

Similar arguments hold for $\int_0^{l_{\gamma_x}} (e^{-s}/(e^{-l_{\gamma_x}}-1) - e^s/(e^{l_{\gamma_x}}-1)) \operatorname{Im} q_\beta(\Phi_s(x))\, ds$. Thus, we obtain $\psi(x, k l_{\gamma_x}) = \psi(x, l_{\gamma_x})$. □

Remark 5.6 This equality is clear if one understands that $\psi(x, k l_\gamma)$ is the solution of the holonomy problem that goes around our closed geodesic γ k times with the same boundary conditions.

Proof of Proposition 5.4 Instead of flowing from x to $\Phi_{l_{\gamma_x}}(x)$, we view x as our midpoint and consider our flow from $\Phi_{-l_{\gamma_x}/2}(x)$ to x and then from x to $\Phi_{l_{\gamma_x}/2}(x)$. From this point of view, we can write $\psi(x, l_{\gamma_x})$ as

$$\begin{aligned}\psi(x,l_{\gamma_x}) = \operatorname{Re} q_\alpha(x) &\int_0^{l_{\gamma_x}/2} \left(\frac{e^{-2s}}{e^{-2l_{\gamma_x}}-1} - \frac{e^{2s}}{e^{2l_{\gamma_x}}-1}\right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ &+ \operatorname{Re} q_\alpha(x) \int_{-l_{\gamma_x}/2}^{0} \left(\frac{e^{2s}}{e^{-2l_{\gamma_x}}-1} - \frac{e^{-2s}}{e^{2l_{\gamma_x}}-1}\right) \operatorname{Re} q_\beta(\Phi_s(x))\, ds \\ &+ 2\operatorname{Im} q_\alpha(x) \int_0^{l_{\gamma_x}/2} \left(\frac{e^{-s}}{e^{-l_{\gamma_x}}-1} - \frac{e^{s}}{e^{l_{\gamma_x}}-1}\right) \operatorname{Im} q_\beta(\Phi_s(x))\, ds \\ &+ 2\operatorname{Im} q_\alpha(x) \int_{-l_{\gamma_x}/2}^{0} \left(\frac{e^{s}}{e^{-l_{\gamma_x}}-1} - \frac{e^{-s}}{e^{l_{\gamma_x}}-1}\right) \operatorname{Im} q_\beta(\Phi_s(x))\, ds.\end{aligned}$$

The above also holds if we replace l_{γ_x} by kl_{γ_x}. We will now conclude by taking $k \to \infty$ in the above formula.

Suppose $\max_{x\in UX}\{|\operatorname{Re} q_\alpha(x)|, |\operatorname{Im} q_\alpha(x)|, |\operatorname{Re} q_\beta(x)|, |\operatorname{Im} q_\beta(x)|\} = M$. Then notice

$$\begin{aligned}|\psi(x,kl_{\gamma_x}) - \eta(x)| \le 2M^2 \int_{kl_{\gamma_x}/2}^{\infty} e^{-2s}\, ds &+ 4M^2 \int_{kl_{\gamma_x}/2}^{\infty} e^{-s}\, ds \\ &+ 2M^2 \int_0^{kl_{\gamma_x}/2} \left| \frac{e^{-2s}}{e^{-2kl_{\gamma_x}}-1} - \frac{e^{2s}}{e^{2kl_{\gamma_x}}-1} + e^{-2s} \right| ds \\ &+ 4M^2 \int_0^{kl_{\gamma_x}/2} \left| \frac{e^{-s}}{e^{-kl_{\gamma_x}}-1} - \frac{e^{s}}{e^{kl_{\gamma_x}}-1} + e^{-s} \right| s \\ \to 0 \quad \text{when } k \to \infty.&\end{aligned}$$

Thus, by Lemma 5.5, we obtain, for any $x \in W$,

$$\operatorname{Tr}(\partial_u D_{A(0)} \partial_v \pi(0))(x) = \psi(x, l_{\gamma_x}) = \lim_{k\to\infty} \psi(x, kl_{\gamma_x}) = \eta(x). \qquad \square$$

We also need the following proposition about regularity of the function η:

Proposition 5.7 $\eta(x)\colon UX \to \mathbb{R}$ *is a Hölder function.*

Proof We start by showing $\int_0^\infty e^{-s} \operatorname{Im} q_\beta(\Phi_s(x))\, ds$ is Hölder. Let x and y be close, with $d(x,y) = \epsilon \ll 1$. It is classical for a hyperbolic surface (S,σ) that we have standard ODE estimates on the geodesic flow

$$d(\Phi_s(x), \Phi_s(y)) \le Ne^s d(x,y) = \epsilon N e^s,$$

where $N > 0$ is some constant and the distance function d on UX is induced from the canonical (Sasaki) metric $\langle\cdot,\cdot\rangle$ on UX.

Consider $T = -\log(\epsilon)$. Then, dividing the integral into two parts, from 0 to T and from T to ∞, yields

$$\begin{aligned}
&\left|\int_0^\infty e^{-s}\operatorname{Im} q_\beta(\Phi_s(x))\,ds - \int_0^\infty e^{-s}\operatorname{Im} q_\beta(\Phi_s(y))\,ds\right| \\
&\quad= \left|\int_0^T e^{-s}\big(\operatorname{Im} q_\beta(\Phi_s(x)) - \operatorname{Im} q_\beta(\Phi_s(y))\big)\,ds\right| \\
&\qquad\qquad\qquad + \left|\int_T^\infty e^{-s}\big(\operatorname{Im} q_\beta(\Phi_s(x)) - \operatorname{Im} q_\beta(\Phi_s(y))\big)\,ds\right| \\
&\quad\le \int_0^T e^{-s} N_1 N\epsilon e^s\,ds + 2N_2 e^{-T} \\
&\quad\le -N_1 N\epsilon\log(\epsilon) + 2N_2\epsilon \\
&\quad\le (N_1 N + 2N_2)\,d(x,y)^{1/2}.
\end{aligned}$$

Here we use the fact that $\operatorname{Im} q_\beta$ is smooth, so we can assume its Lipschitz constant to be N_1. We also use that UX is compact and we assume $\sup_{x\in UX}\operatorname{Im} q_\beta(x) = N_2$.

It then follows easily that $\operatorname{Im} q_\alpha(x)\int_0^\infty e^{-2s}\operatorname{Im} q_\beta(\Phi_s(x))\,ds$ is also a Hölder function. The arguments to show that the other three terms in $\eta(x)$ are Hölder are the same. We therefore conclude that $\eta(x)$ is a Hölder function. □

Finally, with Propositions 5.4 and 5.7, we are able to prove Proposition 5.3 about the second variations of the reparametrization functions on the Riemann surface X.

Proof of Proposition 5.3 We have most of the necessary elements for this proof in previous estimates. We assemble everything here. Because $\operatorname{Tr}(\partial_u D_{A(0)}\partial_v\pi(0))(\Phi_t(x))$ is a Hölder function and it equals the Hölder function $\eta(x)$ on a dense subset of UX, we conclude it coincides with $\eta(x)$ everywhere on UX. We obtain

$$\begin{aligned}
\partial_{uv} f_{\rho(0)} &\sim -\partial_u\big(\operatorname{Tr}(\partial_v D_{H(0)}\pi(0))\big) \\
&= -\operatorname{Tr}\left(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}\pi(0)\right) - \operatorname{Tr}(\partial_v D_{H(0)}\partial_u\pi(0)) \\
&= \tfrac{1}{2}\phi_{uv}(p(x)) - \eta(x),
\end{aligned}$$

where we recall here $\phi = \log(2h/\sigma)$ is a globally well-defined function defined in (5-3) evaluating at the point $p(x)\in X$ and $p\colon UX\to X$ is the projection from the unit tangent bundle to our surface. □

6 Evaluation on the Poincaré disk for the model case

After the computation of the first and second variations of the reparametrization functions on UX in the last two sections, we are able to evaluate $\partial_\beta g_{\alpha\alpha}(\sigma)$. Our goal in this section is to show the following:

Proposition 6.1 *For $\sigma \in \mathcal{T}(S)$, $\partial_\beta g_{\alpha\alpha}(\sigma) = 0$.*

Let's first write down the expression for $\partial_\beta g_{\alpha\alpha}(\sigma)$,

$$\begin{aligned}
&\partial_\beta g_{\alpha\alpha}(\sigma)\\
&= \partial_v\big(\langle \partial_u \rho(0,v), \partial_u \rho(0,v)\rangle_P\big)(0)\\
&= \lim_{r\to\infty} \frac{1}{r}\biggl[\int_{UX}\biggl(\int_0^r \partial_u f^N_{\rho(0)}\,dt\biggr)^2 \int_0^r \partial_v f^N_{\rho(0)}\,dt\,dm_0\\
&\qquad\qquad\qquad + 2\int_{UX}\int_0^r \partial_u f^N_{\rho(0)}\,dt \int_0^r \partial_{uv} f^N_{\rho(0)}\,dt\,dm_0\biggr]\\
&= \lim_{r\to\infty}\frac{1}{r}\int_{UX}\biggl(\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\biggr)^2 \int_0^r \operatorname{Re} q_\beta(\Phi_t(x))\,dt\,dm_0\\
&\quad + \lim_{r\to\infty}\frac{1}{r}\int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \int_0^r -\partial_{uv} h(\rho(0)) - \partial_{uv} f_{\rho(0)}(\Phi_t(x))\,dt\,dm_0\\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The formula for $\partial_{uv} f_{\rho(0)}$ is given in Proposition 5.3.

We aim to prove both I and II are zero. The following lemma will be crucial:

Lemma 6.2 *For any $t, s \in \mathbb{R}$, we have*

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\,dm_0(x) = 0, \tag{6-1}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Im} q_\alpha(\Phi_t(x)) \operatorname{Im} q_\beta(\Phi_s(x))\,dm_0(x) = 0. \tag{6-2}$$

We use the methods in [20] to show the integrals are zero. Similarly to the proof of Theorem 6.3.1 in [20], the key is to use the symmetry properties of the Liouville measure $m_0 = m_L$ and homogeneity of holomorphic differentials viewed as functions on UX. We transfer the problem of evaluating the integrals in (6-1) and (6-2) to analyzing the Fourier coefficients of holomorphic differentials.

Before we start our proof, we first explain the coordinates we will use to do the computation following [20]. We take the Poincaré disk as our charts. Pick a point $x \in UX$. We identify the universal cover of (X, σ) with $\mathbb{D}$ by the unique isometry that takes $\pi(x) \in X$ to $0 \in \mathbb{D}$ and identify the vector $x \in UX$ with the vector $(1,0) \in T_0\mathbb{D}$.

We express our holomorphic differentials in these coordinates. The holomorphic cubic differential q_α has the analytic expansion in the coordinate based on x,

$$q_{\alpha,x}(z) = \sum_{n=1}^{\infty} a_n(x) z^n \, dz^3.$$

Recall the hyperbolic distance d_H in the Poincaré disk model satisfies

$$d_H(0, Re^{i\theta}) = r(R) = \tfrac{1}{2}\log\Big(\frac{1+R}{1-R}\Big).$$

Thus, $\partial/\partial r = (1-R^2)\,\partial/\partial R$ and

$$dz\Big(\frac{\partial}{\partial r}\Big)\Big|_{Re^{i\theta}} = (1-R^2)e^{i\theta}.$$

Denoting $\tilde{q}_{\alpha,x}(z) := \mathrm{Re}(q_{\alpha,x}(z)(\partial/\partial r, \partial/\partial r, \partial/\partial r))$, one has

$$\mathrm{Re}\, q_\alpha(\Phi_r(e^{i\theta}x)) = \tilde{q}_{\alpha,x}(Re^{i\theta}) = \mathrm{Re}\bigg(\sum_{n=0}^{\infty} a_n(x) R^n (1-R^2)^3 e^{i(n+3)\theta}\bigg). \tag{6-3}$$

In particular, when $r = 0$,

$$\lim_{R\to 0} dz\Big(\frac{\partial}{\partial r}\Big)\Big|_{Re^{i\theta}} = e^{i\theta}.$$

Therefore,

$$\begin{aligned} \mathrm{Re}\, q_\alpha(e^{i\theta}x) &= \tilde{q}_{\alpha,x}(0\cdot e^{i\theta}) \\ &= \lim_{R\to 0} \mathrm{Re}\Big(q_{\alpha,x}(Re^{i\theta})\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big)\Big) \\ &= \mathrm{Re}(a_0(x)e^{i3\theta}). \end{aligned} \tag{6-4}$$

Suppose the coefficients of the analytic expansion for q_β are b_n; then

$$\mathrm{Re}\, q_\beta(\Phi_r(e^{i\theta}x)) = \tilde{q}_{\beta,x}(Re^{i\theta}) = \mathrm{Re}\bigg(\sum_{n=0}^{\infty} b_n(x) R^n (1-R^2)^3 e^{i(n+3)\theta}\bigg). \tag{6-5}$$

For the convenience of computation later for other cases, we also write down here two analytic expansions for holomorphic quadratic differentials q_i and q_j, with coefficients

c_n and d_n, respectively,

$$\operatorname{Re} q_i(\Phi_r(e^{i\theta}x)) = \tilde{q}_{i,x}(Re^{i\theta}) = \operatorname{Re}\biggl(\sum_{n=0}^{\infty} c_n(x) R^n (1-R^2)^2 e^{i(n+2)\theta}\biggr). \tag{6-6}$$

$$\operatorname{Re} q_j(\Phi_r(e^{i\theta}x)) = \tilde{q}_{j,x}(Re^{i\theta}) = \operatorname{Re}\biggl(\sum_{n=0}^{\infty} d_n(x) R^n (1-R^2)^2 e^{i(n+2)\theta}\biggr). \tag{6-7}$$

Proof of Lemma 6.2 We begin with showing (6-1).

The proof of it will be divided into two cases:

(1) $t \geq 0$ and $s \geq 0$.

(2) $t < 0$ or $s < 0$.

In the first case, we work with the analytic expansions (6-3) and (6-5). We choose two special situations: $s = t$ and $s = \frac{1}{2}t$. We observe some symmetries in these two situations and argue from these symmetries that (6-1) holds for the first case. We then apply the results for the first case to the second case by flow-invariance properties of m_L. Equation (6-2) then follows easily from (6-1) once we find the relation between them.

Since $m_0 = m_L$ is rotationally invariant, ie $(e^{i\theta})^* m_L = m_L$, we have

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{UX} \operatorname{Re} q_\alpha(e^{i\theta}x) \operatorname{Re} q_\alpha(\Phi_t(e^{i\theta}x)) \operatorname{Re} q_\beta(\Phi_s(e^{i\theta}x))\, dm_0(x)\, d\theta.\end{aligned}$$

(1) We restrict ourselves to the case $t, s \geq 0$ of (6-1) so that we can work with the analytic expansions (6-3) and (6-5).

We let $t(T) = \frac{1}{2}\log((1+T)/(1-T))$ and $s(S) = \frac{1}{2}\log((1+S)/(1-S))$. We first consider $t > 0$ and $s > 0$. Then, if we first integrate over the θ–variable, in terms of the analytic expansion, we get

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\biggl(\int_{UX} \operatorname{Re}(a_0 a_n \bar{b}_{n+3})\, dm_0\, T^n(1-T^2)^3 S^{n+3}(1-S^2)^3 \\ &\qquad + \int_{UX} \operatorname{Re}(a_0 \bar{a}_{n+3} b_n)\, dm_0\, T^{n+3}(1-T^2)^3 S^n (1-S^2)^3\biggr).\end{aligned} \tag{6-8}$$

We let $A_n = \int_{UX} \operatorname{Re}(a_0 a_n \bar{b}_{n+3})\, dm_0$ and $B_n = \int_{UX} \operatorname{Re}(a_0 \bar{a}_{n+3} b_n)\, dm$. To show (6-1) holds for $t, s \geq 0$, it suffices to prove, for $n \geq 0$,

$$A_n = B_n = 0. \tag{6-9}$$

If $t = 0$ or $s = 0$, equation (6-1) is equivalent to

$$A_0 = B_0 = 0,$$

which are included in (6-9). To prove (6-9), we consider two special cases of (6-1): flow times $s = t$ and $s = \frac{1}{2}t$.

By the Φ_t–invariance of m_0, flow time $s = t$ satisfies

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_t(x))\, dm_0(x) = \int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(x)\, dm_0(x).$$

A convenient observation is that flowing from x backwards for time t is the opposite of flowing forwards for time t from $-x$, ie $\Phi_{-t}(x) = -\Phi_t(-x)$. Let $y = -x$ and notice $(e^{i\pi})^* m_0 = m_0$, so we have

$$\int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(x)\, dm_0(x) = -\int_{UX} \operatorname{Re} q_\alpha(\Phi_t(y)) \operatorname{Re} q_\alpha(y) \operatorname{Re} q_\beta(y)\, dm_0(y).$$

Therefore,

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_t(x))\, dm_0(x) = -\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(x)\, dm_0(x).$$

This implies

$$\sum_{n=0}^{\infty} (A_n + B_n) T^{2n+3} (1 - T^2)^6 = -B_0 T^3 (1 - T^2)^3.$$

The coefficient of T^0 yields

$$A_0 + 2B_0 = 0. \tag{6-10}$$

Similarly, for flow time $s = \frac{1}{2}t$, we let $y = -x$ and again use the fact $(e^{i\pi})^* m_0 = m_0$:

$$
\begin{aligned}
\int_{UX} &\operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_{t/2}(x))\, dm_0(x) \\
&= \int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_{-t/2}(x))\, dm_0(x) \\
&= -\int_{UX} \operatorname{Re} q_\alpha(\Phi_t(-x)) \operatorname{Re} q_\alpha(-x) \operatorname{Re} q_\beta(\Phi_{t/2}(-x))\, dm_0(x) \\
&= -\int_{UX} \operatorname{Re} q_\alpha(\Phi_t(y)) \operatorname{Re} q_\alpha(y) \operatorname{Re} q_\beta(\Phi_{t/2}(y))\, dm_0(y).
\end{aligned}
$$

Thus, $\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_{t/2}(x))\, dm_0(x) = 0$.

Recall $t(T) = \frac{1}{2}\log((1+T)/(1-T))$ and $s = \frac{1}{2}\log((1+S)/(1-S))$. In the case $s = \frac{1}{2}t$, we have $T = 2S/(S^2+1)$. The analytic expansion for

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_{t/2}(x))\, dm_0(x) = 0$$

with condition $T = 2S/(S^2+1)$ simplifies to

$$\sum_{n=0}^{\infty} (A_n(S^2+1)^3 + 8B_n)\left(\frac{2S^2}{S^2+1}\right)^n = 0.$$

Let $W = S^2/(S^2+1)$ with $0 < W < \frac{1}{2}$. Then the above is equivalent to

$$\sum_{n=0}^{\infty}\left(A_n \sum_{k=0}^{\infty} \tfrac{1}{2}(k+1)(k+2)W^k + 8B_n\right)2^n W^n = 0.$$

This give relations

$$2^{n+3}B_n + \sum_{k=0}^{n}(n-k+1)(n-k+2)2^{k-1}A_k = 0, \quad n \geq 0.$$

When $n = 0$, combining with (6-10), we obtain $A_0 = B_0 = 0$. Then (6-10) yields $A_n + B_n = 0$ for all $n \in \mathbb{N}$. This fact, combined with the above formula, gives $A_n = B_n = 0$ and (6-1) holds for $t, s \geq 0$.

(2) For $t < 0$ or $s < 0$, there are three cases we need to discuss.

- If $t \leq s$ and $t < 0$, then, as m_0 is Φ_t–invariant,

$$
\begin{aligned}
\int_{UX} &\operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\
&= \int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_{s-t}(x))\, dm_0(x).
\end{aligned}
$$

This is the same as the $s, t \geq 0$ case.

- If $s < t \le 0$, then

$$\begin{aligned}\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\ &= \int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_{s-t}(x))\, dm_0(x) \\ &= -\int_{UX} \operatorname{Re} q_\alpha(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_{t-s}(-x))\, dm_0(x) \\ &= 0.\end{aligned}$$

This is from the observation that the analytic expansion of $\operatorname{Re} q_\beta(\Phi_r(-e^{i\theta}x))$ based at x for $r > 0$ is

$$\begin{aligned}\operatorname{Re} q_\beta(\Phi_r(-e^{i\theta}x)) = \operatorname{Re} q_\beta(\Phi_r(e^{i(\theta+\pi)}x)) &= \tilde{q}_{\beta,x}(Re^{i(\theta+\pi)}) \\ &= \operatorname{Re}\biggl(\sum_{n=0}^{\infty} b_n(x) R^n (1-R^2)^3 e^{i(n+3)(\theta+\pi)}\biggr)\end{aligned}$$

and that, for $n \ge 0$,

$$e^{-i(n+6)\pi} \int_{UX} \operatorname{Re}(a_0 a_n \bar{b}_{n+3})\, dm_0 = 0, \quad e^{i(n+3)\pi} \int_{UX} \operatorname{Re}(a_0 \bar{a}_{n+3} b_n)\, dm_0 = 0.$$

- If $s < 0 \le t$, then we consider

$$\begin{aligned}\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\ &= \int_{UX} \operatorname{Re} q_\alpha(\Phi_t(-x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_{t-s}(-x))\, dm_0(x) \\ &= 0.\end{aligned}$$

The argument is essentially the same as the other cases. This finishes the proof of (6-1).

Equation (6-2) follows easily from (6-1) since, for all $t, s \in \mathbb{R}$,

$$\begin{aligned}&\operatorname{Re}\biggl(\int_{UX} \operatorname{Re} q_\alpha(x) q_\alpha(\Phi_t(x)) q_\beta(\Phi_s(x))\, dm_0(x)\biggr) \\ &\quad= \int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\beta(\Phi_s(x))\, dm_0(x) \\ &\qquad\qquad - \int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Im} q_\alpha(\Phi_t(x)) \operatorname{Im} q_\beta(\Phi_s(x))\, dm_0(x)\end{aligned}$$

and

$$\int_{UX} \operatorname{Re} q_\alpha(x) q_\alpha(\Phi_t(x)) q_\beta(\Phi_s(x))\, dm_0(x) = 0. \tag{6-11}$$

This is easy to see from the fact that $\int_0^{2\pi} \mathrm{Re}(a_0(x)e^{i3\theta})e^{i(n+3)\theta}e^{i(m+3)\theta}\,d\theta = 0$ for all $n, m \geq 0$ and thus, for $t, s > 0$,

$$\begin{aligned}
&\int_{UX} \mathrm{Re}\, q_\alpha(x) q_\alpha(\Phi_t(x)) q_\beta(\Phi_s(x))\,dm_0(x) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{UX} \mathrm{Re}\, q_\alpha(e^{i\theta}x) q_\alpha(\Phi_t(e^{i\theta}x)) q_\beta(\Phi_s(e^{i\theta}x))\,dm_0(x)\,d\theta \\
&= \frac{1}{2\pi}\sum_{m,n\geq 0}\int_{UX}\int_0^{2\pi} \mathrm{Re}(a_0(x)e^{i3\theta})(a_n(x)T^n(1-T^2)^3 e^{i(n+3)\theta}) \\
&\qquad\qquad \cdot (b_m(x)S^{m+3}(1-S^2)^3 e^{i(m+3)\theta})\,d\theta\,dm_0(x) \\
&= 0.
\end{aligned}$$

The argument for $t \leq 0$ or $s \leq 0$ can be transferred back to the $t > 0$ and $s > 0$ cases. One needs the observation that $-\Phi_{-t}(-x) = \Phi_t(x)$ and $-e^{i\theta}x = e^{i(\theta+\pi)}x$. We conclude (6-11) holds for all $t, s \in \mathbb{R}$ and thus (6-2) holds. □

Proof of Proposition 6.1 We start to show $\mathrm{I} = \mathrm{II} = 0$.

$\mathrm{I} = 0$ reduces to (6-1) of Lemma 6.2 if we take $r \to \infty$ in

$$\begin{aligned}
&\frac{1}{r}\int_{UX}\left(\int_0^r \mathrm{Re}\, q_\alpha(\Phi_t(x))\,dt\right)^2 \int_0^r \mathrm{Re}\, q_\beta(\Phi_t(x))\,dt\,dm_0 \\
&= \frac{1}{r}\int_0^r\int_0^r\int_0^r\int_{UX} \mathrm{Re}\, q_\alpha(\Phi_t(x))\,\mathrm{Re}\, q_\alpha(\Phi_s(x))\,\mathrm{Re}\, q_\beta(\Phi_\mu(x))\,dm_0\,d\mu\,dt\,ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&= \frac{1}{r}\int_0^r\int_0^r\int_0^r\int_{UX} \mathrm{Re}\, q_\alpha(\Phi_{t-s}(x))\,\mathrm{Re}\, q_\alpha(x)\,\mathrm{Re}\, q_\beta(\Phi_{\mu-s}(x))\,dm_0\,d\mu\,dt\,ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } m_0 \text{ is } \Phi_t\text{–invariant)} \\
&= 0.
\end{aligned}$$

We next look into II:

$$\begin{aligned}
\mathrm{II} &= \lim_{r\to\infty}\frac{1}{r}\int_{UX} 2\int_0^r \mathrm{Re}\, q_\alpha(\Phi_t(x))\,dt \int_0^r -\partial_{uv}h(\rho(0)) - \partial_{uv} f_{\rho(0)}(\Phi_t(x))\,dt\,dm_0 \\
&= -\lim_{r\to\infty}\frac{1}{r}\int_{UX} 2\int_0^r \mathrm{Re}\, q_\alpha(\Phi_t(x))\,dt \int_0^r \partial_{uv}h(\rho(0))\,dt\,dm_0 \\
&\quad - \lim_{r\to\infty}\frac{1}{r}\int_{UX} 2\int_0^r \mathrm{Re}\, q_\alpha(\Phi_t(x))\,dt \int_0^r \mathrm{Tr}\left(\frac{\partial^2 D_{A(0)}}{\partial u\,\partial v}\pi(0)\right)(\Phi_t(x))\,dt\,dm_0 \\
&\quad + \lim_{r\to\infty}\frac{1}{r}\int_{UX} 2\int_0^r \mathrm{Re}\, q_\alpha(\Phi_t(x))\,\mathrm{Tr}(\partial_v D_{A(0)}\partial_u \pi(0))(\Phi_t(x))\,dt\,dm_0.
\end{aligned}$$

There are three terms here. Since $\partial_{uv}h(\rho(0))$ is a constant, the first term is

$$\lim_{r\to\infty}\frac{1}{r}\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\int_0^r \partial_{uv}h(\rho(0))\,dt\,dm_0 = \lim_{r\to\infty}2\partial_{uv}h(\rho(0))\int_{UX}\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\,dm_0.$$

Recall our expressions given by (6-3) and (6-6). Then

$$\begin{aligned}
\int_{UX}\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\,dm_0
&= \int_0^r\int_{UX}\operatorname{Re} q_\alpha(\Phi_t(x))\,dm_0\,dt\\
&= \int_0^r\int_{UX}\operatorname{Re} q_\alpha(x)\,dm_0\,dt && \text{(since } m_0 \text{ is } \Phi_t\text{–invariant)}\\
&= \frac{1}{2\pi}\int_0^r\int_{UX}\int_0^{2\pi}\operatorname{Re} q_\alpha(e^{i\theta}x)\,d\theta\,dm_0\,dt && \text{(since } m_0 \text{ is rotationally invariant)}\\
&= \frac{r}{2\pi}\int_{UX}\int_0^{2\pi}\operatorname{Re}(a_0(x)e^{i3\theta})\,d\theta\,dm_0\\
&= 0.
\end{aligned}$$

The second term in II is

$$-\lim_{r\to\infty}\frac{1}{r}\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\int_0^r \operatorname{Tr}\Bigl(\frac{\partial^2 D_{A(0)}}{\partial u\,\partial v}\pi(0)\Bigr)(\Phi_t(x))\,dt\,dm_0 = \lim_{r\to\infty}\frac{1}{r}\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\int_0^r \tfrac{1}{2}\phi_{uv}(\Phi_t(x))\,dt\,dm_0,$$

recalling that ϕ is a globally well-defined function on X (see formula (5-3)), and

$$\tfrac{1}{2}\phi_{uv}\bigl(p(\Phi_t(x))\bigr) = \tfrac{1}{2}\phi_{uv}\bigl(p(\Phi_t(e^{i\theta}x))\bigr).$$

So

$$\begin{aligned}
\frac{1}{r}\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\int_0^r \tfrac{1}{2}\phi_{uv}(\Phi_t(x))\,dt\,dm_0
&= \frac{1}{r}\int_{UX}\int_0^{2\pi}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(e^{i\theta}x))\,dt\int_0^r \tfrac{1}{2}\phi_{uv}\bigl(p(\Phi_t(e^{i\theta}x))\bigr)\,dt\,d\theta\,dm_0\\
&= \frac{1}{r}\int_{UX}\int_0^{2\pi}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(e^{i\theta}x))\,dt\int_0^r \tfrac{1}{2}\phi_{uv}\bigl(p(\Phi_t(x))\bigr)\,dt\,d\theta\,dm_0\\
&= \frac{1}{r}\int_0^r\int_0^r\int_{UX}\phi_{uv}\bigl(p(\Phi_{t-s}(x))\bigr)\int_0^{2\pi}\operatorname{Re} q_\alpha(e^{i\theta}x)\,d\theta\,dm_0\,ds\,dt.
\end{aligned}$$

Again by the fact $\int_0^{2\pi} \operatorname{Re} q_\alpha(e^{i\theta}x)\,d\theta = \int_0^{2\pi} \operatorname{Re}(a_0(x)e^{i3\theta})\,d\theta = 0$, we conclude

$$\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha\,dt \int_0^r \operatorname{Tr}\left(\frac{\partial^2 D_{A(0)}}{\partial u\,\partial v}\pi(0)\right) dt\,dm_0 = 0.$$

It remains to show

$$\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha\,dt \int_0^r \operatorname{Tr}(\partial_v D_{A(0)}\partial_u \pi(0))\,dt\,dm_0 = 0.$$

This is

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \int_0^r \eta(\Phi_t(x))\,dt\,dm_0 \\
&= -\lim_{r\to\infty} \frac{1}{r}\Biggl(\int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \\
&\qquad\qquad \cdot \int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x)) \int_0^\infty e^{-2s} \operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0 \\
&\qquad + \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \\
&\qquad\qquad \cdot \int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x)) \int_{-\infty}^0 e^{2s} \operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0 \\
&\qquad + \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \\
&\qquad\qquad \cdot \int_0^r 2\operatorname{Im} q_\alpha(\Phi_\mu(x)) \int_0^\infty e^{-s} \operatorname{Im} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0 \\
&\qquad + \int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \\
&\qquad\qquad \cdot \int_0^r 2\operatorname{Im} q_\alpha(\Phi_\mu(x)) \int_{-\infty}^0 e^{s} \operatorname{Im} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0\Biggr).
\end{aligned}$$

We have estimates for these tail terms

$$\begin{aligned}
&\frac{1}{r}\int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x)) \int_r^\infty e^{-2s} \operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0 \\
&\quad + \frac{1}{r}\int_{UX} 2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt \int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x)) \int_{-\infty}^{-r} e^{2s} \operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0 \\
&\qquad\qquad \le \frac{4M^3}{r} r^2 \int_r^\infty e^{-2s}\,ds \\
&\qquad\qquad = 2M^3 r e^{-2r} \xrightarrow{r\to\infty} 0.
\end{aligned}$$

The other two tail terms with integrals involving $\operatorname{Im} q_\alpha$ and $\operatorname{Im} q_\beta$ also go to zero for the same reason. So, in fact,

$$
\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r}\int_{UX}2\int_0^r \operatorname{Re} q_\alpha\,dt\int_0^r \operatorname{Tr}(\partial_v D_{A(0)}\partial_u\pi(0))\,dt\,dm_0\\
&=-\lim_{r\to\infty}\frac{1}{r}\biggl(\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\\
&\qquad\qquad\cdot\int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x))\int_0^r e^{-2s}\operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0\\
&\qquad+\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\\
&\qquad\qquad\cdot\int_0^r \operatorname{Re} q_\alpha(\Phi_\mu(x))\int_{-r}^0 e^{2s}\operatorname{Re} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0\\
&\qquad+\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\\
&\qquad\qquad\cdot\int_0^r 2\operatorname{Im} q_\alpha(\Phi_\mu(x))\int_0^r e^{-s}\operatorname{Im} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0\\
&\qquad+\int_{UX}2\int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\,dt\\
&\qquad\qquad\cdot\int_0^r 2\operatorname{Im} q_\alpha(\Phi_\mu(x))\int_{-r}^0 e^{s}\operatorname{Im} q_\beta(\Phi_{\mu+s}(x))\,ds\,d\mu\,dm_0\biggr).
\end{aligned}
$$

Similar to I, the above equaling 0 reduces to (6-2). This finishes our proof of Proposition 6.1 and so concludes the discussion of the model case $\partial_\beta g_{\alpha\alpha}(\sigma)$. □

7 The remaining cases

We will show in this section the proofs of the remaining three cases, ie $\partial_i g_{\alpha\alpha}(\sigma)=0$, $\partial_j g_{\alpha i}(\sigma)=0$ and $\partial_\beta g_{\alpha i}(\sigma)=0$. They provide a complete proof of Theorem 1.1.

7.1 The case of $\partial_i g_{\alpha\alpha}(\sigma)$

In this case, given parameters $(u,v)\in\{(-1,1)\}^2$, we obtain a family of (conjugacy classes of) representations $\{\rho(u,v)\}$ in $\mathcal{H}_3(S)$ corresponding to $\{(vq_i,uq_\alpha)\}\subset H^0(X,K^2)\oplus H^0(X,K^3)$ by the Hitchin parametrization. In particular, $\partial_u\rho(0,0)$ is identified with $\varphi(q_\alpha)$ and $\partial_v\rho(0,0)$ is identified with $\varphi(q_i)$. The formula for $\partial_i g_{\alpha\alpha}(\sigma)$

is

$$\begin{aligned}\partial_i g_{\alpha\alpha}(\sigma) &= \partial_v\big(\langle \partial_u\rho(0,v), \partial_u\rho(0,v)\rangle_P\big)(0)\\ &= \lim_{r\to\infty}\frac{1}{r}\bigg[\int_{UX}\bigg(\int_0^r \partial_u f^N_{\rho(0)}\,dt\bigg)^2\int_0^r \partial_v f^N_{\rho(0)}\,dt\,dm_0\\ &\qquad\qquad + 2\int_{UX}\int_0^r \partial_u f^N_{\rho(0)}\,dt\int_0^r \partial_{uv} f^N_{\rho(0)}\,dt\,dm_0\bigg],\end{aligned}$$

where the first and second variations are

(i) $\partial_u f^N_{\rho(0)} = -\partial_u f_{\rho(0)}$;

(ii) $\partial_v f^N_{\rho(0)} = -\partial_v f_{\rho(0)}$;

(iii) $\partial_{uv} f^N_{\rho(0)} = -\partial_{uv} h(\rho(0)) - \partial_{vu} f_{\rho(0)}$.

7.1.1 First and second variations of the reparametrization functions We compute the first and second variations for the case of $\partial_i g_{\alpha\alpha}(\sigma)$ in this subsection.

We have Higgs field

$$\Phi(u,v) = \begin{bmatrix} 0 & vq_i & uq_\alpha\\ 1 & 0 & vq_i\\ 0 & 1 & 0\end{bmatrix}.$$

Following the steps and methods for our model case $\partial_\beta g_{\alpha\alpha}(\sigma)$ in Section 5, we show in this subsection:

Proposition 7.1 *The first variations of the reparametrization functions $\partial_u f_{\rho(0)}\colon UX \to \mathbb{R}$ and $\partial_v f_{\rho(0)}\colon UX \to \mathbb{R}$ for the case $\partial_i g_{\alpha\alpha}(\sigma)$ satisfy*

$$\partial_u f_{\rho(0)}(x) \sim -\operatorname{Re} q_\alpha(x), \quad \partial_v f_{\rho(0)}(x) \sim 2\operatorname{Re} q_i(x)$$

and the second variation of the reparametrization functions $\partial_{vu} f_{\rho(0)}\colon UX \to \mathbb{R}$ for the case $\partial_i g_{\alpha\alpha}(\sigma)$ satisfies

$$\begin{aligned}\partial_{uv} f_{\rho(0)}(x) &\sim \tfrac{1}{2}\operatorname{Re} y_{21}(x)\\ &\qquad - 2\operatorname{Im} q_\alpha(x)\bigg(\int_0^\infty \operatorname{Im} q_i(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_i(\Phi_s(x))e^{s}\,ds\bigg),\end{aligned}$$

where $p\colon UX \to X$ is the projection from the unit tangent bundle UX to our Riemann surface X. Understanding a section of $\operatorname{End}(E)$ as a linear map on each fiber of $E = K \oplus \mathcal{O} \oplus K^{-1}$ over a point of X, the element y_{21} is the component of the section $Y = H^{-1}\partial_{uv}H$ that takes K to $\mathcal{O}$. As a function on UX, y_{21} transforms as $y_{21}(e^{i\theta}x) = e^{-i\theta}y_{21}(x)$.

Proof The first variations are found in [20]. The computation of the second variation of the reparametrization functions $\partial_{uv} f_{\rho(0)}$ of (5-9) is again divided into computations of I and II.

Compute I The major difference between the case $\partial_i g_{\alpha\alpha}(\sigma)$ and $\partial_\beta g_{\alpha\alpha}(\sigma)$ is the computation of this term. As before, our flat connection is

$$D_{H(u,v)} = \nabla_{\bar{\partial}_E, H(u,v)} + \Phi(u,v) + \Phi(u,v)^{*H(u,v)}.$$

For the computation of $\partial_u f_{\rho(0)}$ and $\partial_v f_{\rho(0)}$, when $u = 0$ or $v = 0$, the harmonic metric $H(u,v)$ is diagonal and one obtains

$$\partial_u D_{H(0)} = \begin{bmatrix} 0 & 0 & q_\alpha \\ 0 & 0 & 0 \\ 4\bar{q}_\alpha & 0 & 0 \end{bmatrix}, \quad \partial_v D_{H(0)} = \begin{bmatrix} 0 & q_i & 0 \\ 2\bar{q}_i & 0 & q_i \\ 0 & 2\bar{q}_i & 0 \end{bmatrix}.$$

However, when $u \neq 0$ and $v \neq 0$ both hold, the harmonic metric $H(u,v)$ corresponding to our Higgs field $\Phi(u,v)$ is not diagonal. The computation of $\partial^2 D_{H(0)}/\partial u\, \partial v$ requires an analysis of Hitchin's equations.

We start from the family of Hitchin's equations

$$F_{D_{H(u,v)}} + [\Phi(u,v), \Phi(u,v)^{*H(u,v)}] = 0. \tag{7-1}$$

We take u– and v–derivatives of Hitchin's equations (7-1) at $u, v = 0$:

$$\partial_u \partial_v \big(F_{D_{H(u,v)}} + [\Phi(u,v), \Phi(u,v)^{*H(u,v)}]\big)(0,0)(0) = 0. \tag{7-2}$$

We consider taking $H^{-1}\partial_{vu} H$ as a variable. We define

$$Y = H^{-1}\partial_{vu} H = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}.$$

$Y = H^{-1}\partial_{uv} H$ is a section of $\mathrm{End}(E)$.

We now work with local coordinates and local trivialization. When varying the real parameters u and v, the holomorphic structure of our bundle E does not change. Thus, fixing a local holomorphic frame for all u and v, the Chern connection 1–form under this frame compatible with the Hermitian metric $H(u,v)$ is $A(u,v) = H(u,v)^{-1}\,\partial H(u,v)$. The curvature term in our holomorphic frame is

$$F_{D_{H(u,v)}} = dA(u,v) + A(u,v) \wedge A(u,v) = \bar{\partial}(H(u,v)^{-1}\,\partial H(u,v)).$$

The section $Y \in \Gamma(\mathrm{End}(E))$ in a local holomorphic frame has the following properties:

(i) $\mathrm{Tr}(Y) = 0$.

(ii) $H(u,v)^* = H(u,v)$. Also, because u and v are real parameters, we have $\partial_{uv} H = \partial_{uv}(H^*) = (\partial_{uv} H)^*$ and $Y^* = HYH^{-1}$.

We can express $\partial^2 D_{H(0)}/\partial u\,\partial v$ in terms of Y on γ. With respect to the local holomorphic frame introduced in the model case adapted to the Fermi coordinate, we have $\partial H = 0$ on γ. So

$$\begin{aligned}(7\text{-}3)\quad \partial_{uv}(D_{H(0)})|_\gamma &\\ &= \partial_{uv}(H(u,v)^{-1}\,\partial H(u,v) + \Phi(u,v) + \Phi(u,v)^{*H(u,v)})(0,0)(0)\\ &= -YH^{-1}\,\partial H + H^{-1}\,\partial H Y + \partial Y + \Phi^{*H} Y - Y\Phi^{*H}\\ &= \partial Y + \Phi^{*H} Y - Y\Phi^{*H}.\end{aligned}$$

We want to simplify (7-2) as an equation about Y and then solve Y from (7-2).

Before we continue, we first fix some notation. We will write

$$H = H(0,0), \quad \partial_u H = \frac{\partial H(u,v)}{\partial u}\bigg|_{u,v=0}, \quad \Phi = \Phi(0,0), \quad \partial_v H = \frac{\partial H(u,v)}{\partial v}\bigg|_{u,v=0}, \quad \partial_{uv} H = \frac{\partial H(u,v)}{\partial u\,\partial v}\bigg|_{u,v=0}.$$

As a generalization of the classic result of Ahlfors, the first variations of the harmonic metric vanish at the Fuchsian point (see [20, Theorem 3.5.1]). In particular,

$$\partial_u H = \partial_v H = 0.$$

Taking $H^{-1}\partial_{uv} H$ as a variable, one can verify from (7-2) that

$$\begin{aligned}(7\text{-}4)\quad 0 = \bar\partial\partial(H^{-1}\partial_{uv}H) - H^{-1}\partial H \wedge \bar\partial(H^{-1}\partial_{uv}H) - \bar\partial(H^{-1}\partial_{uv}H) \wedge H^{-1}\,\partial H\\ + \bar\partial(H^{-1}\,\partial H)H^{-1}\partial_{uv}H - H^{-1}\partial_{uv}H\,\bar\partial(H^{-1}\,\partial H)\\ + [\partial_u\Phi, (\partial_v\Phi)^{*H}] + [\partial_v\Phi, (\partial_u\Phi)^{*H}] + [\Phi, [-H^{-1}\partial_{uv}H, \Phi^{*H}]].\end{aligned}$$

Equation (7-4) can be simplified by the observation

$$\begin{aligned}\bar\partial(H^{-1}\,\partial H)H^{-1}\partial_{uv}H - H^{-1}\partial_{uv}H\,\bar\partial(H^{-1}\,\partial H) + [\Phi, [-H^{-1}\partial_{uv}H, \Phi^{*H}]]&\\ = [H^{-1}\partial_{uv}H, [\Phi, \Phi^{*H}]] - [\Phi, [H^{-1}\partial_{uv}H, \Phi^{*H}]] &\quad \text{(Hitchin's equation)}\\ = [[H^{-1}\partial_{uv}H, \Phi], \Phi^{*H}] &\quad \text{(Jacobi identity).}\end{aligned}$$

As $Y = H^{-1}\partial_{uv}H$, this yields

$$\begin{aligned}(7\text{-}5)\quad \bar\partial\partial Y + [\Phi^{*H}, [Y,\Phi]] - H^{-1}\,\partial H \wedge \bar\partial Y - \bar\partial Y \wedge H^{-1}\,\partial H&\\ = -[\partial_u\Phi, (\partial_v\Phi)^{*H}] - [\partial_v\Phi, (\partial_u\Phi)^{*H}].&\end{aligned}$$

The PDE system (7-5) in local holomorphic frames is equivalent to the following nine scalar equations about y_{ij}:

(1) $\bar\partial\partial y_{11}+h(y_{22}-y_{11})=0$.
(2) $\bar\partial\partial y_{22}+h(y_{33}-2y_{22}+y_{11})=0$.
(3) $\bar\partial\partial y_{33}+h(y_{22}-y_{33})=0$.
(4) $\bar\partial\partial y_{21}+h(y_{32}-y_{21})+h^{-1}\,\partial h\,\bar\partial y_{21}=h^{-2}q_i\bar q_\alpha$.
(5) $\bar\partial\partial y_{32}+h(y_{21}-y_{32})+h^{-1}\,\partial h\,\bar\partial y_{32}=-h^{-2}q_i\bar q_\alpha$.
(6) $\bar\partial\partial y_{12}+h(y_{23}-2y_{12})-h^{-1}\,\partial h\,\bar\partial y_{12}=h^{-1}q_\alpha\bar q_i$.
(7) $\bar\partial\partial y_{23}+h(y_{12}-2y_{23})-h^{-1}\,\partial h\,\bar\partial y_{23}=-h^{-1}q_\alpha\bar q_i$.
(8) $\bar\partial\partial y_{31}+2h^{-1}\,\partial h\,\bar\partial y_{31}=0$.
(9) $\bar\partial\partial y_{13}+2hy_{13}-2h^{-1}\,\partial h\,\bar\partial y_{13}=0$.

From property (ii) of Y, one can thus verify (4) is equivalent to (6), (5) is equivalent to (7), and (8) is equivalent to (9). Thus, it suffices to consider the following six equations:

- $\bar\partial\partial y_{11}+h(y_{22}-y_{11})=0$.
- $\bar\partial\partial y_{22}+h(y_{33}-2y_{22}+y_{11})=0$.
- $\bar\partial\partial y_{33}+h(y_{22}-y_{33})=0$.
- $\bar\partial\partial y_{21}+h(y_{32}-y_{21})+h^{-1}\,\partial h\,\bar\partial y_{21}=h^{-2}q_i\bar q_\alpha$.
- $\bar\partial\partial y_{32}+h(y_{21}-y_{32})+h^{-1}\,\partial h\,\bar\partial y_{32}=-h^{-2}q_i\bar q_\alpha$.
- $\bar\partial\partial y_{31}+2h^{-1}\,\partial h\,\bar\partial y_{31}=0$.

We first take a look at the first three equations. We deduce from them

$$\begin{aligned}\bar\partial\partial(y_{11}+y_{22}+y_{33})&=0,\\ \bar\partial\partial(y_{11}-y_{33})-h(y_{11}-y_{33})&=0,\\ \bar\partial\partial(y_{11}+y_{33})+h(2y_{22}-(y_{11}+y_{33}))&=0.\end{aligned}$$

As $Y=H^{-1}\partial_{uv}H$ is a section of $\mathrm{End}(E)$, the components $y_{ii}\in\Gamma(\mathcal{O})$ are actually just functions on the surface X for $i=1,2,3$. Recall our notation $\Delta_\sigma=4\partial_z\partial_{\bar z}/\sigma$ and the fact $h=h(0,0)=\frac12\sigma$, so the above equations can be written independent of coordinate charts on our surface as

$$\begin{aligned}\Delta_\sigma(y_{11}+y_{22}+y_{33})&=0,\\ \Delta_\sigma(y_{11}-y_{33})-2(y_{11}-y_{33})&=0,\\ \Delta_\sigma(y_{11}+y_{33})+2(2y_{22}-(y_{11}+y_{33}))&=0.\end{aligned}$$

We have the following observations:

- From the first equation, we obtain $y_{11}+y_{22}+y_{33}=C$, where C is a constant.
- Since all eigenvalues of Δ_σ should be nonpositive, the second equation can hold only when $y_{11}-y_{33}=0$.
- The third equation is $\Delta_\sigma(y_{11}+y_{33})-6(y_{11}+y_{33})=-4C$. By a maximum principle argument, one gets $y_{11}+y_{33}=\frac{2}{3}C$.

Thus, property (i) of Y gives $y_{11}=y_{22}=y_{33}=0$.

We then continue on the other three equations. From them, we deduce

$$\begin{aligned}\bar\partial\partial(y_{21}+y_{32})+h^{-1}\,\partial h\,\bar\partial(y_{21}+y_{32})&=0,\\ \bar\partial\partial(y_{21}-y_{32})-2h(y_{21}-y_{32})+h^{-1}\,\partial h\,\bar\partial(y_{21}-y_{32})&=2h^{-2}q_i\bar q_\alpha,\\ \bar\partial\partial y_{31}+2h^{-1}\,\partial h\,\bar\partial y_{31}&=0.\end{aligned}$$

Let $w=y_{21}+y_{32}$. We want to compute $\Delta_h\|w\|_h^2$, where the h–norm $\|\cdot\|_h$ is defined as

$$\|s\|_h^2=h^{-i}s\bar s$$

for a section $s\in\Gamma(K^i)$ and $i\in\mathbb{Z}$.

Because $h=h(0,0)=\frac{1}{2}\sigma$ and $\sigma=e^{\delta(z)}|dz|^2$ is a hyperbolic metric with curvature $K(\sigma)=-\Delta_\sigma(\log\sigma)=-1$, we have that h satisfies

$$\bar\partial\partial h=\frac{\partial h\,\bar\partial h}{h}+\tfrac{1}{2}h^2. \tag{7-6}$$

Note $w\in\Gamma(K^{-1})$. The metric h induces a Chern connection ∇^h on K^{-1} and, in our local holomorphic frames, one has

$$\nabla^{h,(1,0)}w=\partial w+h^{-1}\,\partial h\,w.$$

One recognizes $\nabla^{h,(1,0)}w$ is a section of $\Omega^{(1,0)}(K^{-1})=\Gamma(\mathcal{O})$. Therefore,

$$\|\nabla^{h,(1,0)}w\|_h^2=(\partial w+h^{-1}\,\partial h\,w)\overline{(\partial w+h^{-1}\,\partial h\,w)}. \tag{7-7}$$

Combining (7-6) and (7-7) gives

$$\Delta_h\|w\|_h^2=\frac{4\bar\partial\partial(hw\bar w)}{h}=2\|w\|_h^2+4\|\partial\bar w\|_h+4\|\nabla^{h,(1,0)}w\|_h^2\geq 0.$$

This is an inequality independent of coordinates valid on the Riemann surface. By a maximum principle argument, $\|w\|_h^2$ must be a constant M. If $M\neq 0$, then $0=\Delta_h(M)\geq 2M>0$, leading to a contradiction. Thus, $M=0$ and $y_{21}+y_{32}=0$.

We have similar arguments for $\overline{\partial}\partial y_{31} + 2h^{-1}\,\partial h\,\overline{\partial} y_{31} = 0$. We begin with computing $\Delta_h \|y_{31}\|_h^2$.

Since y_{31} is a section of $\Gamma(K^{-2})$, in local holomorphic frames, the Chern connection ∇^h induced from h in this case acts as $\nabla^{h,(1,0)} y_{31} = \partial y_{31} + h^{-2}\,\partial(h^2) y_{31}$.

We obtain

$$\Delta_h \|y_{31}\|_h^2 = \frac{4\overline{\partial}\partial(h^2 y_{31}\bar{y}_{31})}{h} = \|y_{31}\|_h^2 + 4\|\overline{\partial} y_{31}\|_h + 4\|\nabla^{h,(1,0)} y_{31}\|_h^2 \ge 0.$$

Similar to the argument for w, this leads to $y_{31} = 0$.

We conclude up to this point that $Y = H^{-1}\partial_{vu} H \in \Gamma(\mathrm{End}(E))$ in our local frame is of the form

$$Y = H^{-1}\partial_{uv} H = \begin{bmatrix} 0 & h\bar{y}_{21} & 0 \\ y_{21} & 0 & -h\bar{y}_{21} \\ 0 & -y_{21} & 0 \end{bmatrix}$$

with $\overline{\partial}\partial y_{21} - 2h y_{21} + h^{-1}\,\partial h\,\overline{\partial} y_{21} = h^{-2} q_i \bar{q}_\alpha$.

With respect to the Fermi coordinate, we have $h(z) = \frac{1}{2}$ and $\partial_z h = 0$ on γ. Also, we know $Y^* = HYH^{-1}$, so we finally obtain on γ, from (7-3),

$$\mathrm{Tr}\biggl(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}\pi(0)\biggr)(x) = \mathrm{Tr}\biggl(\frac{\partial^2 D_{H(0)}}{\partial u\,\partial v}(x)\pi(0)\biggr) = -\tfrac{1}{2}\,\mathrm{Re}\, y_{21}(x).$$

Remark 7.2 We have $y_{21}(x) = y_{21}(z)$, where $x = \dot{\gamma}(0)$ is the starting point of γ. Recall y_{21} is the component of $Y \in \Gamma(\mathrm{End}(E))$ taking $\mathcal{K}$ to $\mathcal{O}$ and $y_{21}(z)$ is y_{21} evaluating at $p(x)$ in the trivialization given by the holomorphic frame adapted to the Fermi coordinate z for γ.

In particular, if we consider another closed geodesic γ_2 starting from $\gamma_2'(0) = e^{i\theta}x$ with its Fermi coordinate around γ_2 to be w, then $y_{21}(e^{i\theta}x) = y_{21}(w)$. We have $y_{21}(w) = y_{21}(z)\,dw/dz = y_{21}(z)e^{i\theta}$.

Because the vectors tangent to periodic orbits are dense in TX, we can extend y_{21} to be everywhere defined on UX. We conclude that, as a function on UX, y_{21} transfers as

$$y_{21}(e^{i\theta}x) = e^{-i\theta} y_{21}(x).$$

This finishes the computation of I on UX. We now move to II; together, these provide an expression for the second variations of the reparametrization functions.

Compute II We have

$$
\begin{aligned}
&\operatorname{Tr}(\partial_u D_{H(0)} \partial_v \pi(0)) \\
&\quad = q_\alpha(\partial_v a_{11}(0) e_{13}(0) + a_{11}(0)\partial_v e_{13}(0)) + 4\bar{q}_\alpha(\partial_v a_{31}(0) e_{11}(0) + a_{31}(0)\partial_v e_{11}(0)).
\end{aligned}
$$

Similar to the model case $\partial_\beta g_{\alpha\alpha}$, here $\partial_v e_1(0) = y$ is the solution of a nonhomogeneous ODE system which arises from taking a v–derivative on the system of the parallel transport equation (5-10) at $v = 0$,

$$
\partial_t \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \frac{\sqrt{2}}{2} e^t \begin{bmatrix} q_i(\Phi_t(x)) \\ -2 \operatorname{Re} q_i(\Phi_t(x)) \\ 2\bar{q}_i(\Phi_t(x)) \end{bmatrix},
$$

with boundary conditions

$$
H(y(0), e_1(0,0)) = 0, \quad y(l_\gamma) = e^{l_\gamma} \left(\int_0^{l_\gamma} 2 \operatorname{Re} q_i(\Phi_s(x))\, ds \right) e_1(0,0) + e^{l_\gamma} y(0).
$$

The boundary conditions are set up based on the same consideration as the case of $\partial_\beta g_{\alpha\alpha}(\sigma)$. The solution is

$$
\begin{aligned}
&\begin{bmatrix} \partial_v e_{11}(t) \\ \partial_v e_{12}(t) \\ \partial_v e_{13}(t) \end{bmatrix} \\
&\quad = \begin{bmatrix} \frac{\sqrt{2}}{2} \int_0^t (e^t \operatorname{Re} q_i + i e^s \operatorname{Im} q_i)\, ds \\ -\sqrt{2} \int_0^t e^t \operatorname{Re} q_i \, ds \\ \sqrt{2} \int_0^t (e^t \operatorname{Re} q_i - i e^s \operatorname{Im} q_i)\, ds \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{2}}{2} (e^{l_\gamma} - 1)^{-1} \int_0^{l_\gamma} i e^s \operatorname{Im} q_i \, ds \\ 0 \\ -\sqrt{2} (e^{l_\gamma} - 1)^{-1} \int_0^{l_\gamma} i e^s \operatorname{Im} q_i \, ds \end{bmatrix}.
\end{aligned}
$$

Similarly, one can compute $\partial_v e_2(0)$ and $\partial_v e_3(0)$ by this method. It turns out that

$$
\begin{aligned}
&\operatorname{Tr}(\partial_u D_{H(0)} \partial_v \pi(0))(\Phi_t(x)) \\
&\quad = 2 \operatorname{Im} q_\alpha(\Phi_t(x)) \int_0^t (e^{s-t} - e^{t-s}) \operatorname{Im} q_i(\Phi_s(x))\, ds \\
&\qquad\qquad + 2 \operatorname{Im} q_\alpha(\Phi_t(x)) \int_0^{l_\gamma} \left(\frac{e^{s-t}}{e^{l_\gamma} - 1} - \frac{e^{t-s}}{e^{-l_\gamma} - 1} \right) \operatorname{Im} q_i(\Phi_s(x))\, ds.
\end{aligned}
$$

We therefore obtain, for a closed geodesic γ of length l_γ starting from $\dot{\gamma(0)} = x$,

$$
\operatorname{Tr}(\partial_u D_H(0) \partial_v \pi(0))(x) = 2 \operatorname{Im} q_\alpha(x) \int_0^{l_\gamma} \left(\frac{e^s}{e^{l_\gamma} - 1} - \frac{e^{-s}}{e^{-l_\gamma} - 1} \right) \operatorname{Im} q_i(\Phi_s(x))\, ds.
$$

Similar to our model case of $g_{\alpha\alpha,\beta}(\sigma)$, one can define a function $\eta\colon W \to \mathbb{R}$,

$$
\eta(x) = 2 \operatorname{Im} q_\alpha(x) \left(\int_0^\infty e^{-s} \operatorname{Im} q_i(\Phi_s(x))\, ds + \int_{-\infty}^0 e^s \operatorname{Im} q_i(\Phi_s(x))\, ds \right),
$$

and we verify that $\eta(x)$ is Hölder, so that $\operatorname{Tr}(\partial_u D_{H(0)} \partial_v \pi(0))(x) \equiv \eta(x)$ on UX.

We conclude

$$\begin{aligned}
&\partial_{uv} f_{\rho(0)}(x) \\
&\sim -\partial_v\big(\mathrm{Tr}(\partial_u D_{H(0)}\pi(0))\big)(x) \\
&= \tfrac{1}{2}\operatorname{Re} y_{21}(x) - 2\operatorname{Im} q_\alpha(x)\left(\int_0^\infty e^{-s}\operatorname{Im} q_i(\Phi_s(x))\,ds + \int_{-\infty}^0 e^{s}\operatorname{Im} q_i(\Phi_s(x))\,ds\right).
\end{aligned}$$

This finishes the proof of Proposition 7.1. □

Remark 7.3 Instead of starting from the first variation of the reparametrization functions $\partial_u f_{\rho(0)}(x) \sim -\mathrm{Tr}(\partial_u D_{H(0)}\pi(0))(x)$, we can take the first variation of the reparametrization functions to be $\partial_v f_{\rho(0)}(x) \sim -\mathrm{Tr}(\partial_v D_{H(0)}\pi(0))(x)$ by (5-1) and consider

$$\begin{aligned}
\partial_{vu} f_{\rho(0)}(x) &\sim -\partial_u\big(\mathrm{Tr}(\partial_v D_{H(0)}\pi(0))\big)(x) \\
&= -\mathrm{Tr}\left(\frac{\partial^2 D_{H(0)}}{\partial v\,\partial u}\pi(0)\right)(x) - \mathrm{Tr}(\partial_v D_{H(0)}\partial_u\pi(0))(x).
\end{aligned}$$

By the same method, we get

$$\begin{aligned}
&\mathrm{Tr}(\partial_v D_{H(0)}\partial_u\pi(0))(\Phi_t(x)) \\
&\quad = 2\operatorname{Im} q_i(\Phi_t(x))\int_0^t (e^{s-t} - e^{t-s})\operatorname{Im} q_\alpha(\Phi_s(x))\,ds \\
&\qquad\qquad + 2\operatorname{Im} q_i(\Phi_t(x))\int_0^{l_\gamma}\left(\frac{e^{s-t}}{e^{l_\gamma}-1} - \frac{e^{t-s}}{e^{-l_\gamma}-1}\right)\operatorname{Im} q_\alpha(\Phi_s(x))\,ds.
\end{aligned}$$

One can verify, by Fubini's theorem,

$$\int_0^{l_\gamma}\mathrm{Tr}(\partial_u D_{H(0)}\partial_v\pi(0))(\Phi_t(x))\,dt = \int_0^{l_\gamma}\mathrm{Tr}(\partial_v D_{H(0)}\partial_u\pi(0))(\Phi_t(x))\,dt.$$

This agrees with the fact that $\partial_v\big(\mathrm{Tr}(\partial_u D_{H(0)}\pi(0))\big)(x)$ and $\partial_u\big(\mathrm{Tr}(\partial_v D_{H(0)}\pi(0))\big)(x)$ should be in the same Livšic class by Livšic's theorem.

7.1.2 Evaluation on the Poincaré disk With the computation in the last section, we have

$$\begin{aligned}
\partial_i g_{\alpha\alpha}(\sigma) &= \partial_v\big(\langle\partial_u\rho(0,v), \partial_u\rho(0,v)\rangle_P\big)(0) \\
&= \lim_{r\to\infty}\frac{1}{r}\Bigg[\int_{UX}\left(\int_0^r \operatorname{Re} q_\alpha\,dt\right)^2\int_0^r -2\operatorname{Re} q_i\,dt\,dm_0 \\
&\qquad\qquad - 2\int_{UX}\int_0^r \operatorname{Re} q_\alpha\,dt\int_0^r \partial_{uv} f^N_{\rho(0)}\,dt\,dm_0\Bigg],
\end{aligned}$$

where $\partial_{uv} f^N_{\rho(0)} = -\partial_{uv} h(\rho(0)) - \partial_{vu} f_{\rho(0)}$ and

$$\begin{aligned}&\partial_{uv} f_{\rho(0)}(x)\\ &\sim \tfrac{1}{2}\operatorname{Re} y_{21}(x) - 2\operatorname{Im} q_\alpha(x)\Big(\int_0^\infty \operatorname{Im} q_i(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_i(\Phi_s(x))e^{s}\,ds\Big).\end{aligned}$$

We show in this subsection:

Proposition 7.4 *For $\sigma \in \mathcal{T}(S)$, $\partial_i g_{\alpha\alpha}(\sigma) = 0$.*

The argument for this proposition boils down to the following lemma:

Lemma 7.5 *For any $t, s \in \mathbb{R}$,*

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0(x) = 0, \tag{7-8}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0(x) = 0, \tag{7-9}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Im} q_\alpha(\Phi_t(x)) \operatorname{Im} q_i(\Phi_s(x))\, dm_0(x) = 0. \tag{7-10}$$

Proof The proof of this lemma is basically the same as the proof of Lemma 6.2 except that flow time $s = \frac{1}{2}t$ tells us nothing in this case. We instead choose the flow times to be the three special cases $s = t$, $s = 2t$ and $s = 3t$. We recall our analytic expansions for q_i and q_α are

$$\begin{aligned} q_i(\Phi_r(e^{i\theta}x)) &= \Big(\sum_{n=0}^\infty c_n(x) R^n (1-R^2)^2 e^{i(n+2)\theta}\Big),\\ q_\alpha(\Phi_r(e^{i\theta}x)) &= \Big(\sum_{n=0}^\infty a_n(x) R^n (1-R^2)^3 e^{i(n+3)\theta}\Big).\end{aligned}$$

We have, when $t, s > 0$,

$$\begin{aligned}&\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0(x)\\ &\qquad = \frac{1}{2\pi}\int_0^{2\pi}\int_{UX} \operatorname{Re} q_i(e^{i\theta}x) \operatorname{Re} q_\alpha(\Phi_t(e^{i\theta}x)) \operatorname{Re} q_\alpha(\Phi_s(e^{i\theta}x))\, dm_0(x)\, d\theta\\ &\qquad = \frac{1}{4}\sum_{n=0}^\infty \int_{UX} \operatorname{Re}(c_0 a_n \bar{a}_{n+2})\, dm_0 T^n S^n (1-T^2)^3 (1-S^2)^3 (S^2+T^2).\end{aligned}$$

Consider $t = s > 0$. Then

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_i(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_t(x))\, dm_0(x)\\ &= \int_{UX} \operatorname{Re} q_i(\Phi_{-t}(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(x)\, dm_0(x)\\ &= \int_{UX} \operatorname{Re} q_i(-\Phi_t(-x)) \operatorname{Re} q_\alpha(-x) \operatorname{Re} q_\alpha(-x)\, dm_0(x)\\ &= \int_{UX} \operatorname{Re} q_i(\Phi_t(y)) \operatorname{Re} q_\alpha(y) \operatorname{Re} q_\alpha(y)\, dm_0(y) \qquad \text{(with } y = -x).\end{aligned}$$

The analytic expansions of the left- and right-hand sides of the above equation give

$$\tag{7-11} \frac{1}{2}\sum_{n=0}^{\infty} \int_{UX} \operatorname{Re}(c_0 a_n \bar{a}_{n+2})\, dm_0 T^{2n}(1-T^2)^6 T^2 = \frac{1}{4}\int_{UX} \operatorname{Re}(a_0 a_0 \bar{c}_4)\, dm_0 (1-T^2)^2 T^4.$$

We let $C_n = \int_{UX} \operatorname{Re}(c_0 a_n \bar{a}_{n+2})\, dm_0$ and $D_n = \int_{UX} \operatorname{Re}(a_0 a_n \bar{c}_{n+4})\, dm_0$ for $n \ge 0$. We proceed to prove $C_n = 0$ for $n \ge 0$.

The coefficients of T^0 and T^2 yield

$$C_0 = 0, \quad 2C_1 - 8C_0 = D_0.$$

On the other hand, if we consider $s = 2t$ and $s = 3t$, they lead to

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_i(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{2t}(x))\, dm_0(x)\\ &= -\int_{UX} \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(-x))\, dm_0(y)\end{aligned}$$

and

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_i(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{3t}(x))\, dm_0(x)\\ &= -\int_{UX} \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_{2t}(-x))\, dm_0(y).\end{aligned}$$

When $s = 2t$, we have $S = 2T/(T^2+1)$ and $S = 2T + O(T^3)$. When $s = 3t$, we have $S = (3T + T^3)/(3T^2+1)$ and $S = 3T + O(T^3)$.

Compare coefficients of T^4 of the analytic expansions of the above two equations and use the relations $S = 2T + O(T^3)$ and $S = 3T + O(T^3)$ to obtain $D_0 = 0$. Therefore, from (7-11), we conclude $C_n = 0$ for $n \ge 0$ and (7-8) holds for $t, s > 0$. For $s \le 0$ or

$t \le 0$, the argument for (7-8) to hold is an analogy of the $\partial_\beta g_{\alpha\alpha}(\sigma)$ case. We omit it here.

Equation (7-9) then follows from (7-8) by a Φ_t–invariance argument for m_0. To prove (7-10), we just need

$$\int_{UX} \operatorname{Re} q_\alpha(x) q_i(\Phi_t(x)) q_i(\Phi_s(x))\, dm_0(x) = 0 \quad \text{for all } t, s \in \mathbb{R}.$$

The argument is the same as the argument for Lemma 6.2. This finishes the proof of Lemma 7.5. □

Proof of Proposition 7.4 We begin by showing

$$\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2 \int_0^r \partial_u f^N_{\rho(0)}\, dt \int_0^r \operatorname{Tr}\left(\frac{\partial^2 D_{H(0)}}{\partial u\, \partial v} \pi(0) \right) dt\, dm_0 = 0$$

by evaluating the integral on the Poincaré disk.

Recall from the last subsection that y_{21} is the solution of $\bar\partial \partial y_{21} - 2h y_{21} + h^{-1}\, \partial h\, \bar\partial y_{21} = h^{-2} q_i \bar q_\alpha$. Because q_i and $\bar q_\alpha$ are real analytic and because $h = h(0,0) = \frac{1}{2}\sigma$ is also real analytic, we know y_{21} is real analytic by analytic elliptic regularity theory [12].

As discussed before, the function y_{21} on UX transfers as $y_{21}(e^{i\theta}x) = e^{-i\theta} y_{21}(x)$. Similarly to the model case of $g_{\alpha\alpha,\beta}$, we write the real analytic expansion for y_{21} in the coordinates given by the Poincaré disk model based on x,

$$y_{21,x}(z) = \sum_{n,m\ge 0} b_{n,m}(x) z^n \bar z^m \frac{\partial}{\partial z}.$$

Define $\tilde y_{21,x}(z) := \operatorname{Re}(y_{21,x}(z)(dr))$. Recall $r(R) = \frac{1}{2}\log((1-R)/(1+R))$. One has

$$y_{21}(\Phi_r(e^{i\theta}x)) = \tilde y_{21,x}(Re^{i\theta}) = \operatorname{Re}\left(\sum_{n,m\ge 0} b_{n,m}(x) R^{n+m} (1-R^2)^{-1} e^{i(n-m-1)\theta} \right).$$

Thus,

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2 \int_0^r \partial_u f^N_{\rho(0)}\, dt \int_0^r \operatorname{Tr}\left(\frac{\partial^2 D_{H(0)}}{\partial u\, \partial v} \pi(0) \right) dt\, dm_0 \\
&= \lim_{r\to\infty} \frac{1}{r} \int_{UX} \int_0^r \operatorname{Re} q_\alpha(\Phi_t(x))\, dt \int_0^r \operatorname{Re} y_{21}(\Phi_t(x))\, dt\, dm_0 \\
&= \lim_{r\to\infty} \frac{1}{r} \int_0^r \int_0^r \int_{UX} \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} y_{21}(\Phi_s(x))\, dm_0\, dt\, ds \\
&= \lim_{r\to\infty} \frac{1}{r} \int_0^r \int_0^r \int_{UX} \operatorname{Re} q_\alpha(\Phi_{t-s}(x)) \operatorname{Re} y_{21}(x)\, dm_0\, dt\, ds.
\end{aligned}$$

When $\mu = t - s \geq 0$,

$$\int_{UX} \operatorname{Re} q_\alpha(\Phi_\mu(x)) \operatorname{Re} y_{21}(x)\, dm_0 \\ = \frac{1}{2\pi} \int_{UX} \int_0^{2\pi} \operatorname{Re} q_\alpha(\Phi_\mu(e^{i\theta} x)) \operatorname{Re} y_{21}(e^{i\theta} x)\, d\theta\, dm_0.$$

However, $\int_0^{2\pi} \operatorname{Re}(e^{-i\theta} b_{0,0}) \operatorname{Re}(a_n e^{i(n+3)\theta})\, d\theta = 0$ for all $n \geq 0$, which implies the above is zero.

It also holds for $\mu \leq 0$ by simply observing that $\operatorname{Re} q_\alpha(\Phi_{-\mu}(-x)) = -\operatorname{Re} q_\alpha(\Phi_\mu(x))$. Therefore, we conclude

$$\lim_{r\to\infty} \frac{1}{r} \int_{UX} 2 \int_0^r \partial_u f^N_{\rho(0)}\, dt \int_0^r \operatorname{Tr}\biggl(\frac{\partial^2 D_{H(0)}}{\partial u\, \partial v} \pi(0)\biggr)\, dt\, dm_0 = 0.$$

Arguments for the other terms in $\partial_i g_{\alpha\alpha}(\sigma)$ to be equal to zero are analogous to the model case of $\partial_\beta g_{\alpha\alpha}(\sigma)$. They all reduce to Lemma 7.5. We thus finish the proof of Proposition 7.4. □

7.2 The case of $\partial_j g_{\alpha i}(\sigma)$

The proofs for the case of $\partial_j g_{\alpha i}(\sigma)$ in this subsection and the case of $\partial_\beta g_{\alpha i}(\sigma)$ in the next subsection are basically the same as the cases for $\partial_\beta g_{\alpha\alpha}(\sigma)$ and $\partial_i g_{\alpha\alpha}(\sigma)$. Although there are no new ingredients in the proofs, we include them here for completeness.

For $\partial_j g_{\alpha i}(\sigma)$, we have three parameters $\{(u, v, w)\} \in \{(-1, 1)\}^3$. The representations $\{\rho(u, v, w)\}$ in $\mathcal{H}_3(S)$ correspond to $\{(vq_i + wq_j, uq_\alpha)\} \subset H^0(X, K^2) \oplus H^0(X, K^3)$ by Hitchin parametrization. In particular, we have $\partial_u \rho(0, 0, 0)$ is identified with $\varphi(q_\alpha)$ and $\partial_v \rho(0, 0, 0)$ is identified with $\varphi(q_i)$. Also $\partial_w \rho(0, 0, 0)$ is identified with $\varphi(q_j)$. The formula for $\partial_j g_{\alpha i}(\sigma)$ is

$$\begin{aligned} \partial_j g_{\alpha i}(\sigma) &= \partial_w\bigl(\langle \partial_u \rho(0, 0, w), \partial_v \rho(0, 0, w)\rangle_P\bigr)(0) \\ &= \lim_{r\to\infty} \frac{1}{r} \biggl(\int_{UX} \int_0^r \partial_u f^N_{\rho(0)}\, dt \int_0^r \partial_v f^N_{\rho(0)}\, dt \int_0^r \partial_w f^N_{\rho(0)}\, dt\, dm_0 \\ &\qquad + \int_{UX} \int_0^r \partial_u f^N_{\rho(0)}\, dt \int_0^r \partial_{vw} f^N_{\rho(0)}\, dt\, dm_0 \\ &\qquad + \int_{UX} \int_0^r \partial_v f^N_{\rho(0)}\, dt \int_0^r \partial_{uw} f^N_{\rho(0)}\, dt\, dm_0 \biggr), \end{aligned}$$

where the first and second variations are:

(i) $\partial_u f^N_{\rho(0)} = -\partial_u f_{\rho(0)}$.

(ii) $\partial_v f^N_{\rho(0)} = -\partial_v f_{\rho(0)}$.

(iii) $\partial_{uw} f^N_{\rho(0)} = -\partial_{uw} h(\rho(0)) - \partial_{uw} f_{\rho(0)}$.

(iv) $\partial_{vw} f^N_{\rho(0)} = -\partial_{vw} h(\rho(0)) - \partial_{vw} f_{\rho(0)}$.

7.2.1 First and second variations of the reparametrization functions Our Higgs field in this case is

$$\Phi(u,v,w) = \begin{bmatrix} 0 & vq_i + wq_j & uq_\alpha \\ 1 & 0 & vq_i + wq_j \\ 0 & 1 & 0 \end{bmatrix}.$$

Following the steps and methods from the cases $\partial_\beta g_{\alpha\alpha}(\sigma)$ and $\partial_\beta g_{\alpha\alpha}(\sigma)$, we have:

Proposition 7.6 *The first variations of the reparametrization functions $\partial_u f_{\rho(0)} \colon UX \to \mathbb{R}$, $\partial_v f_{\rho(0)} \colon UX \to \mathbb{R}$ and $\partial_w f_{\rho(0)} \colon UX \to \mathbb{R}$ for the case $\partial_j g_{\alpha i}(\sigma)$ satisfy*

$$\partial_u f_{\rho(0)}(x) \sim -\operatorname{Re} q_\alpha(x), \quad \partial_v f_{\rho(0)}(x) \sim 2\operatorname{Re} q_i(x), \quad \partial_w f_{\rho(0)}(x) \sim 2\operatorname{Re} q_j(x),$$

and the second variations of the reparametrization functions $\partial_{uw} f_{\rho(0)} \colon UX \to \mathbb{R}$ and $\partial_{vw} f_{\rho(0)} \colon UX \to \mathbb{R}$ satisfy

$$\begin{aligned} \partial_{uw} f_{\rho(0)} &\sim \tfrac{1}{2}\operatorname{Re} y_{21}(x) \\ &\qquad -2\operatorname{Im} q_\alpha(x)\biggl(\int_0^\infty \operatorname{Im} q_i(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_i(\Phi_s(x))e^{s}\,ds\biggr), \\ \partial_{vw} f_{\rho(0)}(x) &\sim \tfrac{1}{2}\phi_{vw}(p(x)) \\ &\qquad +2\operatorname{Im} q_i(x)\biggl(\int_0^\infty \operatorname{Im} q_j(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_j(\Phi_s(x))e^{s}\,ds\biggr), \end{aligned}$$

where $p \colon UX \to X$ and y_{21} are defined as before.

Proof For the second variations of the reparametrization functions, we have computed $\partial_{uw} f_{\rho(0)}$ in the $\partial_i g_{\alpha\alpha}(\sigma)$ case:

$$\begin{aligned} \partial_{uw} f_{\rho(0)} &\sim \tfrac{1}{2}\operatorname{Re} y_{21}(x) \\ &\qquad -2\operatorname{Im} q_\alpha(x)\biggl(\int_0^\infty \operatorname{Im} q_i(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_i(\Phi_s(x))e^{s}\,ds\biggr). \end{aligned}$$

The computation of

$$\partial_{vw} f_{\rho(0)} \sim -\operatorname{Tr}\biggl(\frac{\partial^2 D_{A(0)}}{\partial w\,\partial v}\pi(0)\biggr) - \operatorname{Tr}(\partial_v D_{H(0)} \partial_w \pi(0)) =: -\mathrm{I} - \mathrm{II}$$

is divided into computations of I and II.

Compute I We set $u=0$; the Higgs field is

$$\Phi(v,w)=\begin{bmatrix}0 & vq_i+wq_j & 0\\ 1 & 0 & vq_i+wq_j\\ 0 & 1 & 0\end{bmatrix}.$$

The harmonic metric $H(v,w)$ is diagonalizable and the computation of $\partial_{vw}D_{H(0)}$ is the same as in the model case of $\partial_\beta g_{\alpha\alpha}(\sigma)$.

With respect to the notation defined in the model case of $\partial_\beta g_{\alpha\alpha}(\sigma)$, one obtains

$$\operatorname{Tr}\biggl(\frac{\partial^2 D_{H(0)}}{\partial v\,\partial w}\pi(0)\biggr)(x)=-\tfrac{1}{2}\psi_{vw}\bigl(z(p(x))\bigr)=-\tfrac{1}{2}\phi_{vw}(p(x)),$$

where $p\colon UX\to X$ is the projection from the unit tangent bundle to our surface and z is the Fermi coordinate we choose evaluating at the point $p(x)\in X$.

Compute II Both $\partial_v D_{H(0)}$ and $\partial_w\pi(0)$ have been computed in the $\partial_i g_{\alpha\alpha}(\sigma)$ case. One can check

$$\begin{aligned}
&\operatorname{Tr}(\partial_v D_{H(0)}\partial_w\pi(0))(\Phi_t(x))\\
&=q_i(\partial_w a_{11}(0)e_{12}(0)+a_{11}(0)\partial_w e_{12}(0)+\partial_w a_{21}(0)e_{13}(0)+a_{21}(0)\partial_w e_{13}(0))\\
&\qquad+2\bar{q}_i(\partial_w a_{21}(0)e_{11}(0)+a_{21}(0)\partial_w e_{11}(0)+\partial_w a_{31}(0)e_{12}(0)+a_{31}(0)\partial_w e_{12}(0))\\
&=2\operatorname{Im}q_i(\Phi_t(x))\int_0^t\operatorname{Im}q_j(\Phi_s(x))(e^{t-s}-e^{s-t})\,ds\\
&\qquad\qquad+2\operatorname{Im}q_i(\Phi_t(x))\int_0^{l_\gamma}\operatorname{Im}q_j(\Phi_s(x))\biggl(\frac{e^{t-s}}{e^{-l_\gamma}-1}-\frac{e^{s-t}}{e^{l_\gamma}-1}\biggr)\,ds.
\end{aligned}$$

In particular,

$$\operatorname{Tr}(\partial_v D_{H(0)}\partial_w\pi(0))(x)=2\operatorname{Im}q_i(x)\int_0^{l_\gamma}\operatorname{Im}q_j(\Phi_s(x))\biggl(\frac{e^{-s}}{e^{-l_\gamma}-1}-\frac{e^{s}}{e^{l_\gamma}-1}\biggr)\,ds.$$

Similarly to the cases of $\partial_\beta g_{\alpha\alpha}(\sigma)$ and $\partial_i g_{\alpha\alpha}(\sigma)$, one can then define a function $\eta\colon UX\to\mathbb{R}$,

$$\eta(x)=-2\operatorname{Im}q_i(x)\biggl(\int_0^\infty\operatorname{Im}q_j(\Phi_s(x))e^{-s}\,ds+\int_{-\infty}^0\operatorname{Im}q_j(\Phi_s(x))e^{s}\,ds\biggr),$$

and verify that $\eta(x)$ is Hölder and such that $\operatorname{Tr}(\partial_v D_{H(0)}\partial_w\pi(0))(x)\equiv\eta(x)$ on UX.

We finally obtain

$$\begin{aligned}
&\partial_{vw}f_{\rho(0)}(x)\\
&\quad\sim-\operatorname{Tr}\biggl(\frac{\partial^2 D_{H(0)}}{\partial v\,\partial w}\pi(0)\biggr)(x)-\operatorname{Tr}(\partial_v D_{H(0)}\partial_w\pi(0))(x)
\end{aligned}$$

$$= \tfrac{1}{2}\phi_{vw}(p(x)) \\ + 2\operatorname{Im} q_i(x)\left(\int_0^{\infty} \operatorname{Im} q_j(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^{0} \operatorname{Im} q_j(\Phi_s(x))e^{s}\,ds\right). \quad \square$$

7.2.2 Evaluation on the Poincaré disk We show in this subsection:

Proposition 7.7 *For $\sigma \in \mathcal{T}(S)$, $\partial_j g_{\alpha i}(\sigma) = 0$.*

For the same reasoning as before, the proof of the above proposition reduces to the following lemma:

Lemma 7.8 *For any $t, s \in \mathbb{R}$,*

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\,dm_0(x) = 0, \tag{7-12}$$

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Im} q_j(\Phi_t(x)) \operatorname{Im} q_\alpha(\Phi_s(x))\,dm_0(x) = 0, \tag{7-13}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_j(\Phi_s(x))\,dm_0(x) = 0, \tag{7-14}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Im} q_i(\Phi_t(x)) \operatorname{Im} q_j(\Phi_s(x))\,dm_0(x) = 0. \tag{7-15}$$

Proof We just need to show (7-12). Equations (7-13), (7-14) and (7-15) follow easily using the methods we developed in the former cases.

We start from a special case of (7-12), with $q_i = q_j$:

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\,dm_0(x) = 0 \quad \text{for all } t, s \in \mathbb{R}. \tag{7-16}$$

The proof of this case is an analogy of the case $\partial_\beta g_{\alpha\alpha}(\sigma)$ since, for flow times $s = t$ and $s = \frac{1}{2}t$,

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_t(x))\,dm_0(x) \\ = -\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(x)\,dm_0(x)$$

and

$$\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{t/2}(x))\,dm_0(x) = 0.$$

For $t,s>0$, recall our analytic expansions given in (6-3) and (6-6) lead to

$$\begin{aligned}
&\int_{UX}\operatorname{Re} q_i(x)\operatorname{Re} q_i(\Phi_t(x))\operatorname{Re} q_\alpha(\Phi_s(x))\,dm_0(x)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\int_{UX}\operatorname{Re} q_i(e^{i\theta}x)\operatorname{Re} q_i(\Phi_t(e^{i\theta}x))\operatorname{Re} q_\alpha(\Phi_s(e^{i\theta}x))\,dm_0(x)\,d\theta\\
&=\frac{1}{4}\sum_{n=0}^{\infty}\biggl(\int_{UX}\operatorname{Re}(c_0c_n\bar{a}_{n+1})\,dm_0T^n(1-T^2)^2S^{n+1}(1-S^2)^3\\
&\qquad\qquad+\int_{UX}\operatorname{Re}(c_0\bar{c}_{n+3}a_n)\,dm_0T^{n+3}(1-T^2)^2S^n(1-S^2)^3\biggr).
\end{aligned}$$

Let $E_n=\int_{UX}\operatorname{Re}(c_0c_n\bar{a}_{n+1})\,dm_0$ and $F_n=\int_{UX}\operatorname{Re}(c_0\bar{c}_{n+3}a_n)\,dm_0$. We argue, for $n\ge 0$,

$$E_n=F_n=0. \tag{7-17}$$

The case $t=0$ or $s=0$ of (7-16) is included in the $n=0$ case of (7-17).

For flow time $s=t$, we have

$$\sum_{n=0}^{\infty}(E_nT^{2n+1}(1-T^2)^5+F_nT^{2n+3}(1-T^2)^5)=-F_0T^3(1-T^2)^2. \tag{7-18}$$

This implies

$$E_0=0,\quad E_1=-2F_0.$$

For flow time $s=\frac{1}{2}t$, we obtain

$$\sum_{n=0}^{\infty}(E_nT^n(1-T^2)^2S^{n+1}(1-S^2)^3+F_nT^{n+3}(1-T^2)^2S^n(1-S^2)^3)=0,$$

where $T=2S/(1+S^2)$.

It simplifies to

$$\sum_{n=0}^{\infty}\biggl(E_n+8F_n\frac{S^2}{(S^2+1)^3}\biggr)\biggl(\frac{2S^2}{S^2+1}\biggr)^n=0.$$

Let $W=S^2/(1+S^2)$; we have

$$\sum_{n=0}^{\infty}\biggl(E_n\sum_{k=0}^{\infty}(k+1)W^k+8F_nW\biggr)(2W)^n=0.$$

This gives relations

$$E_0 = 0, \qquad \sum_{k=0}^{n} 2^k (n-k+1) E_k + 2^{n+2} F_{n-1} = 0, \quad n \geq 1. \tag{7-19}$$

Combining with (7-18), we get $E_1 = F_0 = 0$. Therefore, the right-hand side of (7-18) is zero and we obtain from it $E_{n+1} + F_n = 0$ for $n \geq 0$. Combining this with (7-19) and by an induction argument, one concludes $E_n = F_n = 0$. This proves (7-17) for $s, t \geq 0$. The case $s, t < 0$ is similar to before.

Now we proceed to prove (7-12). The above case implies, for $q_i \neq q_j$,

$$\begin{aligned}
\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0 &= 0,\\
\int_{UX} \operatorname{Re} q_j(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0 &= 0,\\
\int_{UX} \operatorname{Re}(q_i + q_j)(x) \operatorname{Re}(q_i + q_j)(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0 &= 0.
\end{aligned}$$

Therefore, for all $t, s \in \mathbb{R}$,

$$\begin{aligned}
\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0 \qquad&\\
+ \int_{UX} \operatorname{Re} q_j(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0 &= 0.
\end{aligned}\tag{7-20}$$

Recall the analytic expansion for q_j is given in (6-7). Consider $t, s > 0$:

$$\begin{aligned}
&\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_s(x))\, dm_0(x)\\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{UX} \operatorname{Re} q_i(e^{i\theta} x) \operatorname{Re} q_j(\Phi_t(e^{i\theta} x)) \operatorname{Re} q_\alpha(\Phi_s(e^{i\theta} x))\, dm_0(x)\, d\theta\\
&= \frac{1}{4} \sum_{n=0}^{\infty} \biggl(\int_{UX} \operatorname{Re}(c_0 d_n \bar{a}_{n+1})\, dm_0 T^n (1-T^2)^2 S^{n+1} (1-S^2)^3\\
&\qquad\qquad + \int_{UX} \operatorname{Re}(c_0 \bar{d}_{n+3} a_n)\, dm_0 T^{n+3} (1-T^2)^2 S^n (1-S^2)^3 \biggr).
\end{aligned}$$

Let $G_n = \int_{UX} \operatorname{Re}(c_0 d_n \bar{a}_{n+1})\, dm_0$ and $H_n = \int_{UX} \operatorname{Re}(c_0 \bar{d}_{n+3} a_n)\, dm_0$. We want to show $G_n = H_n = 0$ for $n \geq 0$.

Let m be an integer and $m \geq 2$. Consider the flow time $s = mt$. Observe

$$\begin{aligned}
&\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{mt}(x))\, dm_0(x)\\
&= \int_{UX} \operatorname{Re} q_i(\Phi_{-t}(x)) \operatorname{Re} q_j(x) \operatorname{Re} q_\alpha(\Phi_{(m-1)t}(x))\, dm_0(x)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{UX} \operatorname{Re} q_j(y) \operatorname{Re} q_i(\Phi_t(y)) \operatorname{Re} q_\alpha(\Phi_{-(m-1)t}(y))\, dm_0(y) \\
&\hspace{20em} (\text{let } y = -x \text{ and } \Phi_t(-y) = -\Phi_{-t}(y)) \\
&= \int_{UX} \operatorname{Re} q_i(y) \operatorname{Re} q_j(\Phi_t(y)) \operatorname{Re} q_\alpha(\Phi_{-(m-1)t}(y))\, dm_0(y) \qquad (\text{by (7-20)}) \\
&= -\int_{UX} \operatorname{Re} q_j(x) \operatorname{Re} q_i(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{mt}(x))\, dm_0(x) \\
&\hspace{20em} (\text{exchange the roles of } q_i \text{ and } q_j) \\
&= -\int_{UX} \operatorname{Re} q_i(x) \operatorname{Re} q_j(\Phi_t(x)) \operatorname{Re} q_\alpha(\Phi_{(m-1)t}(-x))\, dm_0(x)
\end{aligned}
$$

When $s = mt$, we have $S = S(m) = ((1+T)^m - (1-T)^m)/((1+T)^m + (1-T)^m) = mT + O(T^3)$. From the analytic expansion

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \bigl(G_n T^n S(m)^{n+1} (1 - S(m)^2)^3 + G_n e^{in\pi} T^n S(m-1)^{n+1} (1 - S(m-1)^2)^3 \bigr) \\
&= -\sum_{n=0}^{\infty} \bigl(-H_n e^{in\pi} T^{n+3} S(m-1)^n (1 - S(m-1)^2)^3 + H_n T^{n+3} S(m)^n (1 - S(m)^2)^3 \bigr),
\end{aligned}
$$

the coefficients of T^1 and T^3 and T^5 yield, respectively,

$$
\begin{aligned}
G_0 &= 0, \\
(m^2 - (m-1)^2) G_1 &= 0, \\
(m^3 + (m-1)^3) G_2 &= -(2m-1) H_1 + (6m-3) H_0.
\end{aligned}
$$

The cases $m = 2$, $m = 3$ and $m = 4$ together give $H_0 = H_1 = G_2 = 0$. By induction, assuming $G_k = H_{k-1} = 0$ for $1 \le k < n$, the coefficient of T^{2n+1} gives

$$
(m^{n+1} + e^{in\pi} (m-1)^{n+1}) G_n = (e^{i(n-1)\pi} (m-1)^{n-1} - m^{n-1}) H_{n-1}.
$$

We conclude $G_n = H_n = 0$ for $n \ge 0$ by choosing two different m. This finishes the proof of (7-12) for $t, s > 0$. Equation (7-12) for $t \le 0$ and $s \le 0$ can be proved similarly to the former cases. □

7.3 The case of $\partial_\beta g_{\alpha i}(\sigma)$

This is the last case. In this case, the representations $\{\rho(u, v, w)\}$ in $\mathcal{H}_3(S)$ correspond to $\{(vq_i, uq_\alpha + wq_\beta)\} \subset H^0(X, K^2) \oplus H^0(X, K^3)$ by Hitchin parametrization. Our metric tensor is

$$
\begin{aligned}
&\partial_\beta g_{\alpha i}(\sigma)\\
&= \partial_w\big(\langle \partial_u \rho(0,0,w), \partial_v \rho(0,0,w)\rangle_P\big)(0)\\
&= \lim_{r\to\infty}\frac{1}{r}\bigg(\int_{UX}\int_0^r \partial_u f^N_{\rho(0)}\,dt \int_0^r \partial_v f^N_{\rho(0)}\,dt \int_0^r \partial_w f^N_{\rho(0)}\,dt\,dm_0\\
&\qquad\qquad + \int_{UX}\int_0^r \partial_u f^N_{\rho(0)}\,dt \int_0^r \partial_{vw} f^N_{\rho(0)}\,dt\,dm_0\\
&\qquad\qquad\qquad + \int_{UX}\int_0^r \partial_v f^N_{\rho(0)}\,dt \int_0^r \partial_{uw} f^N_{\rho(0)}\,dt\,dm_0\bigg),
\end{aligned}
$$

where the first and second variations are

(i) $\partial_u f^N_{\rho(0)} = -\partial_u f_{\rho(0)}$;

(ii) $\partial_v f^N_{\rho(0)} = -\partial_v f_{\rho(0)}$;

(iii) $\partial_{uw} f^N_{\rho(0)} = -\partial_{uw} h(\rho(0)) - \partial_{uw} f_{\rho(0)}$;

(iv) $\partial_{vw} f^N_{\rho(0)} = -\partial_{vw} h(\rho(0)) - \partial_{vw} f_{\rho(0)}$.

7.3.1 First and second variations of the reparametrization functions Our Higgs field in this case is

$$
\Phi(u,v,w) = \begin{bmatrix} 0 & vq_i & uq_\alpha + wq_\beta \\ 1 & 0 & vq_i \\ 0 & 1 & 0 \end{bmatrix}.
$$

Proposition 7.9 *The first variations of the reparametrization functions* $\partial_u f_{\rho(0)}\colon UX \to \mathbb{R}$ *and* $\partial_v f_{\rho(0)}\colon UX \to \mathbb{R}$ *for the case* $\partial_\beta g_{\alpha i}(\sigma)$ *satisfy*

$$
\partial_u f_{\rho(0)}(x) \sim -\operatorname{Re} q_\alpha(x), \quad \partial_v f_{\rho(0)}(x) \sim 2\operatorname{Re} q_i(x), \quad \partial_w f_{\rho(0)}(x) \sim -\operatorname{Re} q_\beta(x),
$$

and the second variations of the reparametrization functions $\partial_{uw} f_{\rho(0)}\colon UX \to \mathbb{R}$ *and* $\partial_{vw} f_{\rho(0)}\colon UX \to \mathbb{R}$ *satisfy*

$$
\begin{aligned}
&\partial_{uw} f_{\rho(0)}(x)\\
&\sim \tfrac{1}{2}\phi_{uw}(p(x)) + \operatorname{Re} q_\alpha(x)\int_0^\infty e^{-2s}\operatorname{Re} q_\beta(\Phi_s(x))\,ds\\
&\quad + \operatorname{Re} q_\alpha(x)\int_{-\infty}^0 e^{2s}\operatorname{Re} q_\beta(\Phi_s(x))\,ds + 2\operatorname{Im} q_\alpha(x)\int_0^\infty e^{-s}\operatorname{Im} q_\beta(\Phi_s(x))\,ds\\
&\qquad\qquad + 2\operatorname{Im} q_\alpha(x)\int_{-\infty}^0 e^{s}\operatorname{Im} q_\beta(\Phi_s(x))\,ds
\end{aligned}
$$

and

$$\begin{aligned}&\partial_{vw} f_{\rho(0)}(x)\\&\quad= \partial_{wv} f_{\rho(0)}(x)\\&\quad\sim \tfrac{1}{2}\operatorname{Re} y_{21}(x) - 2\operatorname{Im} q_\beta(x)\biggl(\int_0^\infty \operatorname{Im} q_i(\Phi_s(x))e^{-s}\,ds + \int_{-\infty}^0 \operatorname{Im} q_i(\Phi_s(x))e^{s}\,ds\biggr),\end{aligned}$$

where $p\colon UX \to X$ and y_{21} are defined as before.

Proof All of the computations have been done in the former cases. □

7.3.2 Evaluation on the Poincaré disk We show in this subsection:

Proposition 7.10 *For $\sigma \in \mathcal{T}(S)$, $\partial_\beta g_{\alpha i}(\sigma) = 0$.*

For the same reasoning as before, the proof of the above proposition reduces to the following lemma:

Lemma 7.11 *For any $t, s \in \mathbb{R}$,*

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0(x) = 0, \tag{7-21}$$

$$\int_{UX} \operatorname{Im} q_\alpha(x) \operatorname{Im} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0(x) = 0, \tag{7-22}$$

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Im} q_\beta(\Phi_t(x)) \operatorname{Im} q_i(\Phi_s(x))\, dm_0(x) = 0. \tag{7-23}$$

Proof We just need to show (7-21). Equations (7-22) and (7-23) follow easily, similar to the former cases.

From the computation of $\partial_i g_{\alpha\alpha}(\sigma)$, we know

$$\begin{aligned}\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0 &= 0,\\ \int_{UX} \operatorname{Re} q_\beta(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0 &= 0,\\ \int_{UX} \operatorname{Re}(q_\alpha + q_\beta)(x) \operatorname{Re}(q_\alpha + q_\beta)(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0 &= 0.\end{aligned}$$

We deduce

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0 \\ + \int_{UX} \operatorname{Re} q_\beta(x) \operatorname{Re} q_\alpha(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0 = 0.$$

Similar to $\partial_j g_{\alpha i}(\sigma)$, we consider $s = mt$ for $m \in \mathbb{N}$ and $m \geq 2$. We observe

$$\int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_{mt}(x))\, dm_0 \\ = \int_{UX} \operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_{(m-1)t}(-x))\, dm_0.$$

We recall the Poincaré disk model and our analytic expansion for q_α, q_β and q_i in (6-3), (6-5) and (6-7). For $t, s \geq 0$, the analytic expansion

$$\begin{aligned}\int_{UX} &\operatorname{Re} q_\alpha(x) \operatorname{Re} q_\beta(\Phi_t(x)) \operatorname{Re} q_i(\Phi_s(x))\, dm_0(x) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{UX} \operatorname{Re} q_\alpha(e^{i\theta} x) \operatorname{Re} q_\beta(\Phi_t(e^{i\theta} x)) \operatorname{Re} q_i(\Phi_s(e^{i\theta} x))\, dm_0(x)\, d\theta \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \biggl(\int_{UX} \operatorname{Re}(a_0 b_n \bar{c}_{n+4})\, dm_0 T^n (1-T^2)^3 S^{n+4} (1-S^2)^2 \\ &\qquad + \int_{UX} \operatorname{Re}(a_0 \bar{b}_{n+2} c_n)\, dm_0 S^n (1-S^2)^2 T^{n+2} (1-T^2)^3 \biggr).\end{aligned}$$

Denoting $I_n = \int_{UX} \operatorname{Re}(a_0 b_n \bar{c}_{n+4})\, dm_0$ and $J_n = \int_{UX} \operatorname{Re}(a_0 \bar{b}_{n+2} c_n)\, dm_0$ for $n \geq 0$, we argue

$$I_n = J_n = 0.$$

When $s = mt$, we have $S = S(m) = ((1+T)^m - (1-T)^m)/((1+T)^m + (1-T)^m) = mT + O(T^3)$. The analytic expansions give

$$\sum_{n=0}^{\infty} \bigl(I_n T^n S(m)^{n+4} (1-S(m)^2)^2 - I_n e^{in\pi} T^n S(m-1)^{n+4} (1-S(m-1)^2)^2 \bigr) \\ = \sum_{n=0}^{\infty} \bigl(-J_n T^{n+2} S(m)^n (1-S(m)^2)^2 + J_n e^{in\pi} T^{n+2} S(m-1)^n (1-S(m-1)^2)^2 \bigr).$$

The coefficients of T^4 yield

$$(m^4 - (m-1)^4) I_0 = -(2m-1) J_1 + (4m-2) J_0.$$

The cases $m=2$, $m=3$ and $m=4$ give $I_0=J_0=J_1=0$. By induction, assuming $I_k=J_{k+1}=0$ for $1\le k<n$, the coefficient of T^{2n+4} gives

$$(m^{n+4}-e^{in\pi}(m-1)^{n+4})I_n=(e^{i(n+1)\pi}(m-1)^{n+1}-m^{n+1})J_{n+1}.$$

We conclude $I_n=J_n=0$ for $n\ge 0$ by choosing two different m. This finishes the proof of (7-21) for $t,s>0$. Equation (7-21) for $t\le 0$ and $s\le 0$ can be proved similarly to the former cases. Lemma 7.11 and also Proposition 7.6 therefore hold. □

We have shown

(i) $\partial_\beta g_{\alpha\alpha}(\sigma)=0$,

(ii) $\partial_i g_{\alpha\alpha}(\sigma)=0$,

(iii) $\partial_j g_{\alpha i}(\sigma)=0$, and

(iv) $\partial_\beta g_{\alpha i}(\sigma)=0$.

This finishes the proof of our Theorem 1.1.

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