Geometry & Topology

Volume 27 (2023)

Birational geometry of the intermediate Jacobian fibration of a cubic fourfold

Giulia Saccà
Appendix by Claire Voisin

msp

Geometry & Topology **27**:4 (2023) 1479–1538
DOI: 10.2140/gt.2023.27.1479
Published: 15 June 2023

Birational geometry of the intermediate Jacobian fibration of a cubic fourfold

GIULIA SACCÀ
APPENDIX BY CLAIRE VOISIN

We show that the intermediate Jacobian fibration associated to *any* smooth cubic fourfold X admits a hyper-Kähler compactification $J(X)$ with a regular Lagrangian fibration $\pi\colon J \to \mathbb{P}^5$. This builds upon work of Laza, Saccà and Voisin (2017), where the result is proved for general X, as well as on the degeneration techniques introduced in the work of Kollár, Laza, Saccà and Voisin, and the minimal model program. We then study some aspects of the birational geometry of $J(X)$: for very general X we compute the movable and nef cones of $J(X)$, showing that $J(X)$ is not birational to the twisted version of the intermediate Jacobian fibration, nor to an OG10–type moduli space of objects in the Kuznetsov component of X; for any smooth X we show, using normal functions, that the Mordell–Weil group $\mathrm{MW}(\pi)$ of the fibration is isomorphic to the integral degree-4 primitive algebraic cohomology of X, ie $\mathrm{MW}(\pi) \cong H^{2,2}(X,\mathbb{Z})_0$.

14D06, 14J42

Introduction

The geometry of smooth cubic fourfolds has ties to that of K3 surfaces and, more generally, to that of higher-dimensional hyper-Kähler manifolds. For example, with certain special cubic fourfolds one can associate a K3 surface via Hodge-theoretic (Hassett [34]) or derived categorical (Kuznetsov [43]) methods. From a more geometric perspective, given a smooth cubic fourfold X, hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$–type are constructed geometrically, via parameter spaces of rational curves of certain degrees on X (Beauville and Donagi [12] and Lehn, Lehn, Sorger and van Straten [49]), or as moduli spaces of objects in the Kuznetsov component of X (Bayer, Lahoz, Macrì, Nuer, Perry and Stellari [6] and Lahoz, Lehn, Macrì and Stellari [44]). These constructions give rise to 20–dimensional families of polarized hyper-Kähler manifolds, the maximal possible dimension of families of polarized hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$–type. As the cubic fourfold becomes special, for example when it acquires more algebraic classes, the geometry of these hyper-Kähler manifolds also becomes more interesting. For example, when X has an associated K3 surface in the sense of Addington and Thomas [2], Hassett [34], Huybrechts [37] and Kuznetsov [43], these hyper-Kähler manifolds become isomorphic, or birational, to moduli spaces of objects in the derived category of the corresponding K3 surface; see Addington [1] and Bayer, Lahoz, Macrì, Nuer, Perry and Stellari [6].

Laza, Saccà and Voisin [47] constructed a Lagrangian fibered hyper-Kähler manifold starting from a general cubic fourfold. This hyper-Kähler manifold is a deformation of O'Grady's 10–dimensional exceptional example. More precisely, let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold and let $\pi_U : J_U \to U \subset (\mathbb{P}^5)^\vee$ be the family of intermediate Jacobians of the smooth hyperplane sections of X. This fibration was considered by Donagi and Markman in [25], where they showed that the total space has a holomorphic symplectic form. The main result of [47] was to construct, for general X, a smooth projective hyper-Kähler compactification J of J_U, with a flat morphism $J \to (\mathbb{P}^5)^\vee$ extending π_U, and to show that this hyper-Kähler 10–fold is deformation equivalent to O'Grady's 10–dimensional example. In [80], Voisin constructed a hyper-Kähler compactification J^T of a natural J_U–torsor J^T_U, which is nontrivial for very general X. The two hyper-Kähler manifolds J and J^T are birational over countably many hypersurfaces in the moduli space of cubic fourfolds. These two constructions give rise to two 20–dimensional families of hyper-Kähler manifolds of OG10–type, each of which forms an open subset of a codimension-two locus inside the moduli space of hyper-Kähler manifolds in this deformation class.

If one wishes to study the geometry of these hyper-Kähler manifolds as the cubic fourfold becomes special, a first step is to check if a hyper-Kähler compactification of the fibration $J_U \to U$ can be constructed for an arbitrary smooth cubic fourfold. The starting result of this paper is that this can indeed be done.

Theorem 1 (Theorem 1.6) *Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, and let*

$$\pi_U \colon J_U \to U \subset (\mathbb{P}^5)^\vee$$

be the Donagi–Markman fibration. There exists a smooth projective hyper-Kähler compactification J of J_U with a morphism $\pi \colon J \to (\mathbb{P}^5)^\vee$ extending π_U.

The same techniques also give the existence of a Lagrangian fibered hyper-Kähler compactification for the nontrivial J_U–torsor $J_U^T \to U$ of [80] for any smooth X; see Remark 1.14. Moreover, with little extra work, the theorem is proved also for mildly singular cubic fourfolds such as, for example, cubic fourfolds with a simple node; see Proposition 1.17. For a general cubic fourfolds with one node, the existence of such a Lagrangian fibered hyper-Kähler manifold provides a positive answer to a question of Beauville [11]; see Remark 1.18.

We should point out that as a consequence of the "finite monodromy implies smooth filling" results of Kollár, Laza, Saccà and Voisin [41], we prove in Proposition 1.5 that J_U admits projective birational model that is hyper-Kähler. Theorem 1 shows that there exists a hyper-Kähler model with a Lagrangian fibration extending π_U.

There are several ingredients in the construction of the hyper-Kähler compactification of [47]: a cycle-theoretic construction of the holomorphic symplectic form, the problem of the existence of so-called very good lines for any hyperplane section of X, a smoothness criterion for relative compactified Prym varieties, the independence of the compactification from the choice of a very good line. Here we have pursued a different direction, and instead rely on the existence of a hyper-Kähler compactification for general X, use the degeneration techniques introduced in [41], and implement some results from birational geometry and the minimal model program, following Kollár [40] and Lai [45]. One advantage of our method is that it opens the door to using birational geometry to compactify Lagrangian fibrations.

The second result of this paper is concerned with the hyper-Kähler birational geometry of J. We show that the relative theta divisor Θ of the fibration is a prime exceptional divisor and that for general X it can be contracted after a Mukai flop.

Theorem 2 (Theorem 4.1) *Let q be the Beauville–Bogomolov form on $H^2(J,\mathbb{Z})$. The relative theta divisor $\Theta \subset J$ is a prime exceptional divisor with $q(\Theta) = -2$. For very general X, there is a unique other hyper-Kähler birational model of J, denoted by N, which is the Mukai flop $p\colon J \dashrightarrow N$ of J along the image of the zero section. N admits a divisorial contraction $h\colon N \to \bar{N}$, which contracts the proper transform of Θ onto an 8–dimensional variety which is birational to the LLSv 8–fold $Z(X)$.*

Thus, for very general X, J is the unique hyper-Kähler birational model with a Lagrangian fibration, it is not birational to J^T (Corollary 3.10), and its movable cone is the union of its nef cone and the nef cone of N. This answers a question by Voisin [80]. As a consequence of this theorem we show that for very general X, J is not birational to a moduli spaces of objects in the Kuznetsov component $\mathcal{K}u(X)$ of X; see Corollary 4.2. In the opposite direction, it was recently proved by Li, Pertusi and Zhao [51] that the twisted hyper-Kähler manifold J^T is birational to a moduli space of objects of OG10–type in $\mathcal{K}u(X)$. By objects of OG10–type, we mean objects whose Mukai vector is of the form $2w$, with $w^2 = 2$. As a consequence, the family of intermediate Jacobian fibrations is the only known family of hyper-Kähler manifolds associated with cubic fourfolds whose very general point cannot be described as a moduli space of objects in the Kuznetsov component of X.

Given $J = J(X)$, a hyper-Kähler compactification of the intermediate Jacobian fibration for any smooth cubic fourfold X, a natural question to ask is how the geometry of J changes as X becomes less general. One way to answer this question is the following theorem, describing the Mordell–Weil group of π in terms of the primitive algebraic cohomology of X. In Section 5 we prove:

Theorem 3 (Theorem 5.1) *Let $\mathrm{MW}(\pi)$ be the Mordell–Weil group of $\pi : J \to \mathbb{P}^5$, ie the group of rational sections of π, and let $H^{2,2}(X,\mathbb{Z})_0$ be the primitive degree-4 integral cohomology of X. The natural group homomorphism*

$$\phi_X : H^{2,2}(X,\mathbb{Z})_0 \to \mathrm{MW}(\pi)$$

induced by the Abel–Jacobi map is an isomorphism.

The proof of this result uses the theory of normal functions, as developed by Griffiths and Zucker, as well as the techniques used by Voisin to prove the integral Hodge conjecture for cubic fourfolds. A consequence of this is a geometric description of the Lagrangian fibered hyper-Kähler manifolds with maximal Mordell–Weil rank, whose existence was proved by Oguiso in [64]: indeed, Oguiso's examples are (birationally) given by $J = J(X) \to \mathbb{P}^5$, where X is a smooth cubic fourfold with $H^{2,2}(X, \mathbb{Z})$ of maximal rank.

Plan of the paper

In Section 1 we prove the existence of a hyper-Kähler compactification for J_U and for J_U^T, in the case of any smooth, or mildly singular, X. This uses some results from the minimal model program, which are briefly recalled. In Section 2 we review some basic results about moduli spaces of OG10–type and we compute, using the Bayer–Macrì techniques adapted to these singular moduli spaces by Meachan and Zhang [58], the nef and movable cones of certain moduli spaces of OG10–type that appear as limits of the intermediate Jacobian fibration. The main result of Section 3 is the computation that $q(\Theta) = -2$. Section 4 is devoted to the proof of Theorem 2 and its preparation: Given a family of cubic fourfolds degenerating to the chordal cubic, we construct a certain degeneration of the intermediate Jacobian fibration and identify the limit of the corresponding degeneration of the relative Theta divisor. By the results of Section 2, the limiting theta divisor can be contracted after a Mukai flop of the zero section and we deduce the analogous result for Θ. The computation of the Mordell–Weil group occupies Section 5.

Finally, in the appendix by C Voisin, some applications to the Beauville conjecture on the polynomial relations in the Chow group of a projective hyper-Kähler manifold are given for $J = J(X)$, in the case of very general J of Picard number 2 or 3. This is obtained as an application of the computation of $q(\Theta) = -2$ from Theorem 2.

Acknowledgements

I would like to thank J Kollár for pointing my attention to the techniques of [40; 45], which are used in this paper. It is my pleasure to thank E Arbarello, C Camere, G Di Cerbo, K Hulek, R Laza, E Macrì, C Onorati, G Pearlstein, A Rapagnetta, L Tasin and C Voisin for useful and interesting discussions related to the topic of this paper. I also thank the anonymous referee for having read this paper very carefully and for many useful comments. Finally, I would like to warmly thank Collège de France and École Normal Supérieure for the hospitality and the great working conditions while the final version of this manuscript was being prepared. This work is partially supported by NSF grant DMS-1801818.

1 A hyper-Kähler compactification of the intermediate Jacobian fibration for any smooth cubic fourfold

We denote by $X \subset \mathbb{P}^5$ a smooth cubic fourfold, by $(\mathbb{P}^5)^\vee$ the dual projective space parametrizing hyperplane sections $Y = X \cap H \subset X$, and by $U \subset (\mathbb{P}^5)^\vee$ the open subset parametrizing smooth hyperplane sections. The dual hypersurface of X, parametrizing singular hyperplane sections, is denoted by $X^\vee \subset (\mathbb{P}^5)^\vee$. Its smooth locus

$$U_1 := (\mathbb{P}^5)^\vee \setminus \mathrm{Sing}(X^\vee) \subset (\mathbb{P}^5)^\vee$$

parametrizes hyperplane sections of X that are smooth or have one simple node and no other singularities. In what follows, we freely drop the $^\vee$ from $(\mathbb{P}^5)^\vee$ and write simply $\mathbb{P}^5$. From the context it will be clear if we are referring to the projective space parametrizing hyperplane sections of X or the projective space containing X. For a smooth cubic threefold Y, the Griffiths intermediate Jacobian of Y will be denoted by

$$\mathrm{Jac}(Y) \cong H^1(Y, \Omega^2_Y)^\vee / H_3(Y, \mathbb{Z}).$$

It is a principally polarized abelian fivefold which parametrizes rational equivalence classes of homologically trivial 1–cycles on Y [79, Theorem 6.24].

Over U consider the Donagi–Markman fibration

$$\pi_U \colon J_U = J_U(X) \to U, \tag{1-1}$$

whose fiber over a smooth hyperplane section $Y = X \cap H$ is the intermediate Jacobian $\mathrm{Jac}(Y)$. By [25], J_U is quasiprojective and admits a holomorphic symplectic form σ_{J_U}, with respect to which π_U is Lagrangian. The main result of [47] is the following theorem.

Theorem 1.1 [47] *Let X be a general cubic fourfold. Then there exists a smooth projective compactification $J = J(X)$ of J_U, with a flat morphism $\pi \colon J \to (\mathbb{P}^5)^\vee$ extending π_U, which has irreducible fibers and which admits a rational zero section $s \colon (\mathbb{P}^5)^\vee \dashrightarrow J$. Moreover, J is an irreducible holomorphic symplectic manifold, deformation equivalent to O'Grady's 10–dimensional exceptional example.*

We will say that X is general *in the sense of LSV* if the construction of [47] works for $J_U(X)$, and we refer to $J = J(X)$ as in Theorem 1.1 as the *LSV fibration*. A necessary condition for this to happen is that the hyperplane sections of X are palindromic; see [17]. For example, a cubic fourfold containing a plane is not general in the sense of LSV.

To extend the theorem above for any X, we use the existence of a hyper-Kähler compactification for general X, the cycle-theoretic description of the holomorphic symplectic form that was given in [47], the degeneration results from [41], and techniques from the minimal model program, following [40; 45]. We start by recalling the construction of a natural partial compactification of J_U, which already appeared in [25; 47].

Lemma 1.2 [25; 47] *For any smooth X, there is a canonical partial compactification $J_{U_1} = J_{U_1}(X)$ of J_U, with a projective morphism $\pi_{U_1}\colon J_{U_1} \to U_1$ with irreducible fibers extending π_U. This J_{U_1} is smooth and has a holomorphic symplectic form $\sigma_{J_{U_1}}$ extending σ_{J_U}.*

Proof This is already proved in [25, Section 8.5.2 and Theorem 8.18]. Alternatively, one can use [23, Corollary 2.38], and [47, Definitions 2.2 and 2.9, Proposition 1.4 and Lemma 5.2]. □

Before giving an application of the cycle-theoretic construction of the holomorphic symplectic form [47, Section 1], we recall the definition of symplectic variety.

Definition 1.3 A normal projective variety M is called symplectic if its smooth locus carries a holomorphic symplectic form which extends to a regular (ie holomorphic) form on any resolution of singularities of M.

Lemma 1.4 *Let $\overline{J}$ be a normal projective compactification of J_U. Then*:

(1) *The smooth locus of $\overline{J}$ admits a holomorphic two-form extending σ_{J_U}. In particular, the canonical class $K_{\overline{J}}$ of $\overline{J}$ is effective and is trivial if and only if $\overline{J}$ is a symplectic variety.*

(2) *$\overline{J}$ is not uniruled.*

Proof (1) The first statement is [47, Theorem 1.2(iii)], while the second follows from the fact that the canonical class of $\overline{J}$ is the (closure of the) codimension-one locus where the generically nondegenerate holomorphic two-form is degenerate.

(2) Let $\widetilde{J} \to \overline{J}$ be a resolution of singularities. By (1), $\widetilde{J}$ has effective canonical class and thus by [59] it is not uniruled. □

The following is an application of the degeneration techniques of [41].

Proposition 1.5 *Let X be a smooth cubic fourfold and let $J_U = J_U(X)$ be as above. Then there exists a smooth projective hyper-Kähler manifold M birational to J_U and of $OG10$–type.*

Proof Let $\mathcal{X} \to \Delta$ be a family of smooth cubic fourfolds with $\mathcal{X}_0 = X$. Here Δ is an open affine subset of a smooth projective curve, or a small disk. We will use the notation $t = 0$ to denote a chosen special point in Δ, and $t \neq 0$ to denote any other point. Up to restricting Δ if necessary, assume that for $t \neq 0$, $\mathcal{X}_t$ is general in the sense of LSV. By [47, Proposition 2.10], we can assume that for any $t \neq 0$ all the hyperplane sections of $\mathcal{X}_t$ admit a very good line; see [47, Definition 2.9]. Consider the open set $\mathcal{V} = (\mathbb{P}^5)^\vee \times \Delta \setminus \mathrm{Sing}(\mathcal{X}_0^\vee) \times \{0\}$, so that $\mathcal{V}_t = (\mathbb{P}^5)^\vee$ for $t \neq 0$ and $\mathcal{V}_0 = U_1 \times \{0\}$ parametrizes the hyperplane sections of $\mathcal{X}_0 = X$ that have at most one nodal point and no other singularities. The construction of [47, Section 5] can be carried out in families, yielding a projective morphism

$$\mathcal{J}_\mathcal{V} \to \mathcal{V},$$

which is fibered in compactified Prym varieties and is such that, denoting by $\mathcal{J}_t$ the fiber of the induced smooth quasiprojective morphism $\mathcal{J}_\mathcal{V} \to \Delta$ for $t \neq 0$, $\mathcal{J}_t$ is the LSV fibration $J(\mathcal{X}_t)$, and $\mathcal{J}_0 = J_{U_1}(X)$. Let $\widetilde{\mathcal{J}} \to \Delta$ be a projective morphism extending $\mathcal{J}_\mathcal{V} \to \Delta$. The central fiber $\mathcal{J}_0$ has a multiplicity-one component which contains J_{U_1} as dense open subset. By Lemma 1.4, this component is not uniruled. By [41, Corollary 5.2] there is a birational model M of $J_{U_1}(X)$ that is a hyper-Kähler manifold, deformation equivalent to the smooth fibers $\mathcal{J}_t = J(\mathcal{X}_t)$, for $t \neq 0$. □

By [57], given a hyper-Kähler manifold M with a Lagrangian fibration $\pi\colon M \to \mathbb{P}^n$, the locus inside $\mathrm{Def}(M)$ where the Lagrangian fibration deforms is an open subset of the hypersurface where the class $\pi^*\mathcal{O}(1)$ stays of type $(1,1)$. However, this fact alone is not enough to imply the existence of a hyper-Kähler compactification of J_{U_1} for *any* smooth X.

This is what we prove in the following theorem, whose proof uses the mmp following Kollár [40, Section 8] and Lai [45]. In Section 1.1 we will recall some basic facts about the mmp that are needed in the proof of Theorems 1.6 and 1.19. We refer to [42] and to [32] for the basic definitions and fundamental results.

Theorem 1.6 *For any smooth cubic fourfold X, there exists a smooth projective hyper-Kähler compactification $J = J(X)$ of $J_U(X)$, with a projective flat morphism $\pi\colon J \to \mathbb{P}^5$ extending π_U.*

Proof Let $\bar{J} \to \mathbb{P}^5$ be any normal projective compactification of J_{U_1} with a regular morphism $\bar{\pi}\colon \bar{J} \to \mathbb{P}^5$. By Lemma 1.4, there is a holomorphic two-form $\bar{\sigma}$ on the smooth locus of $\bar{J}$ extending $\sigma_{J_{U_1}}$, the canonical class $K_{\bar{J}} \geq 0$ is effective, and $K_{\bar{J}} = 0$ if and only if $\bar{J}$ is a symplectic variety. Since $K_{\bar{J}}$ is supported on the complement of J_{U_1}, $\operatorname{codim} \bar{\pi}(\operatorname{Supp}(K_{\bar{J}})) \geq 2$. By definition [40, Definition 7], this means that $K_{\bar{J}}$ is $\bar{\pi}$–exceptional, if it is nontrivial. If this is the case, then by [61, III 5.1] (see also [45, Lemma 2.10]), $K_{\bar{J}}$ is not $\bar{\pi}$–nef. More precisely, there is a component of $K_{\bar{J}}$ that is covered by curves that are contracted by $\bar{\pi}$ and that intersect $K_{\bar{J}}$ negatively.

Let $\widetilde{J} \to \mathbb{P}^5$ be a smooth projective compactification of J_{U_1} admitting a regular morphism $\widetilde{\pi}\colon \widetilde{J} \to \mathbb{P}^5$, and let $K_{\widetilde{J}}$ be its canonical class. If the effective divisor $K_{\widetilde{J}}$ is not trivial, we use the mmp to contract $\operatorname{Supp}(K_{\widetilde{J}})$ relatively to $\mathbb{P}^5$. Let H be a $\widetilde{\pi}$–ample $\mathbb{Q}$–divisor such that the pair $(\widetilde{J}, H)$ is klt and $K_{\widetilde{J}} + H$ is relatively big and nef. The mmp with scaling over $\mathbb{P}^5$ (see Section 1.1 below) produces a sequence of birational maps

$$\widetilde{J} = J_0 \overset{\psi_0}{\dashrightarrow} J_1 \overset{\psi_1}{\dashrightarrow} \cdots \dashrightarrow J_i \overset{\psi_i}{\dashrightarrow} \cdots \tag{1-2}$$

over $\mathbb{P}^5$ — ie there are projective morphisms $\pi_i\colon J \to \mathbb{P}^5$ such that $\pi_0 = \widetilde{\pi}$ and $\pi_i := \pi_{i-1} \circ \psi_i^{-1}$ — and a nonincreasing sequence of nonnegative rational numbers $t_0 = 1 \geq t_1 \geq \dots t_i \geq \cdots \geq 0$, with the following properties:

(1) For every $i \geq 0$, $K_{J_i} + t_i H_i$ is π_i–big and π_i–nef.

(2) For every $i \geq 0$, J_i is a $\mathbb{Q}$–factorial terminal compactification of J_{U_1}. The fact that the birational morphisms ψ_i are isomorphisms away from J_{U_1} follows from the fact that the K_{J_i}–negative rays of the mmp correspond to rational curves that are contained in the support of K_{J_i}. Thus, by Lemma 1.4, the smooth locus of J_i carries a holomorphic two-form σ_i extending $\sigma_{J_{U_1}}$.

(3) K_{J_i} is effective and, if not trivial, it has a component covered by K_{J_i}–negative curves which are contracted by π_i.

(4) The process stops if and only if there exists an i such that K_{J_i} is π_i–nef. This holds if and only if $K_{J_i} = 0$.

The number of irreducible components of the support of K_{J_i} is nonincreasing, since the birational maps of the mmp extract no divisors. In fact, we claim that this number is eventually strictly decreasing. By (4) above, this happens if and only if the process eventually stops. Suppose that this is not the case. Then by Lemma 1.13, $\lim t_i = 0$. Recall, as already observed, that if $K_{J_k} \neq 0$, then there exists a component that is covered by K_{J_k}–negative curves that are contracted by π_k. Since we are assuming that $\lim t_i = 0$, this implies that for $i \gg 0$, t_i is small enough that this component is contained in the relative stable base locus $\mathbb{B}((K_{J_k} + t_i H_k)/\mathbb{P}^5)$. Since by Lemma 1.12, the divisorial components of $\mathbb{B}((K_{J_k} + t_i H_k)/\mathbb{P}^5)$ are contracted by $J_k \dashrightarrow J_i$, it follows that for $i \gg 0$, the number of irreducible components of the effective divisor K_{J_i} is strictly less than the number of components of K_{J_k}. Thus, the claim is proved and for some $i \gg 0$, the process gives a model with $K_{J_i} = 0$. By Lemma 1.4, $\bar{J} := J_i$ is a $\mathbb{Q}$–factorial terminal symplectic compactification of J_{U_1}. Finally, by Proposition 1.7 below, $\bar{J}$ is smooth and the theorem is proved. □

Proposition 1.7 (Greb–Lehn–Rollenske) *Let $\bar{M}$ be a $\mathbb{Q}$–factorial terminal symplectic variety. Suppose that $\bar{M}$ is birational to a smooth hyper-Kähler manifold M. Then $\bar{M}$ is smooth.*

Proof This is [29, Proposition 6.5]. □

Remark 1.8 The techniques used to prove the theorem above can be applied to similar contexts to give $\mathbb{Q}$–factorial terminal symplectic compactifications of other quasiprojective Lagrangian fibrations. We plan to come back to this in upcoming work.

As a consequence of Theorem 1.19 below, we will give a slightly stronger version of the theorem just proved (see Remark 1.20) showing that, given a family of smooth cubic fourfolds whose general fiber is general in the sense of [47], then up to a base change and birational transformations, the corresponding family of LSV intermediate Jacobian fibrations can be filled with a Lagrangian fibered smooth projective hyper-Kähler compactification of the Donagi–Markman fibration of the limiting cubic fourfold.

Another approach to Theorem 1 would be to show that the rational map $M \dashrightarrow \mathbb{P}^5$ induced by the birational map $\phi\colon M \dashrightarrow J_{U_1}$ of Proposition 1.5 is almost holomorphic [56, Definition 1]. By [56] this would imply the existence of a birational hyper-Kähler model of M with a regular morphism to $\mathbb{P}^5$. It seems, however, that controlling the mmp of Proposition 1.5 to ensure that $M \dashrightarrow \mathbb{P}^5$ is almost holomorphic is not too far from running the relative mmp as in the proof of Theorem 1.6.

Given a smooth cubic fourfold X, we will refer to both the Donagi–Markman fibration J_U and to any hyper-Kähler compactification J of J_U as in Theorem 1.6, as the intermediate Jacobian fibration. Hopefully, it will be clear from the context which one we are referring to.

Remark 1.9 Unlike the compactification of [47], the proof of Theorem 1.6 is not constructive and, for a given X, the hyper-Kähler compactification that we show to exist may not be unique. We will return to this question in Section 4.

1.1 The mmp with scaling

In this subsection we recall some basic tools and known results from the minimal model program (mmp) that are used to prove Theorems 1.6 and 1.19. For the basic notions and the fundamental results we refer to [42] and [32]. In this section, by divisor we will mean a $\mathbb{Q}$–divisor.

Let M be a normal $\mathbb{Q}$–factorial variety with a projective morphism $\pi\colon M \to B$ to a normal quasiprojective variety B. Let Δ be an effective divisor on M and let H be a general divisor on M that is ample (or big) over B. We assume that the pair $(M, \Delta + H)$ is klt and that $K_M + \Delta + H$ is nef over B.

The mmp with scaling of H [32, Section 5.E] produces a sequence of birational maps $\psi_i\colon M_i \dashrightarrow M_{i+1}$ over B, such that $M_0 = M$, $\Delta_{i+1} = (\psi_i)_*\Delta_i$, $H_{i+1} = (\psi_i)_* Hi$ and ψ_i is the flip or the divisorial contraction for a $(K_{M_i} + \Delta_i)$–negative relative extremal ray R_i over B. We let π_i be the induced regular morphism $M_i \to B$. The sequence is defined inductively in the following way. Let

$$t_i = \inf\{t \geq 0 \mid K_{M_i} + \Delta_i + tH_i \text{ is nef over } B\}.$$

If $t_i = 0$, then $K_{M_i} + \Delta_i$ is nef over B and the process stops. Otherwise, there is a $0 < t' \leq t_i$ such that $K_{M_i} + \Delta_i + t'H_i$ is not nef over B. By the cone theorem (see [42, Chapter 3] or [32, Theorem 5.4]) $K_{M_i} + \Delta_i + t_i H_i$ is nef over B and there exists a $(K_{M_i} + \Delta_i)$–negative extremal ray R_i over B such that $(K_{M_i} + \Delta_i + t_i H_i) \cdot R_i = 0$.

Let $c_i\colon M_i \to Z_i$ be the extremal contraction over B associated to R_i, which exists by the “contraction” part of the cone theorem [32, (5.4.3)–(5.4.4)]. If $\dim Z_i < \dim M_i$, then c_i is a Mori fiber space and we stop. If c_i is not a Mori fiber space then it is either a divisorial or flipping contraction. In the first case, we let $M_{i+1} = Z_i$ and $\psi_i = c_i$. In second case, we let $\psi_i\colon M_i \dashrightarrow M_{i+1}$ be the $(K_{M_i} + \Delta_i + t'H_i)$–flip (which exists by [32, Corollary 5.73]). By construction, ψ_i extracts no divisors, meaning that ψ_i^{-1} contracts no divisors.

By the contraction part of the cone theorem, the divisor $K_{M_{i+1}} + \Delta_{i+1} + t_i H_{i+1}$ is nef over B. The pair $(M_i, \Delta_{i+1} + t_i H_{i+1})$ is klt (see [42, Corollaries 3.42–3.44]) and M_i is $\mathbb{Q}$–factorial (see [42, Corollary 3.18]). If $\Delta = 0$ and M is terminal, then so is M_i. As long as $K_{M_i} + \Delta_i$ is not π_i–nef, t_{i+1} is nonzero and $\Delta_{i+1} + t_i H_{i+1}$ is big over B. Thus we can keep going, producing a nonincreasing sequence $t_i \geq t_{i+1} \geq \cdots$ of nonnegative rational numbers and a sequence of birational maps $\psi_i \colon M_i \dashrightarrow M_{i+1}$ over B. The process stops if there exists an N such that $c_N \colon M_N \to Z_N$ is a Mori fiber space over B or such that $K_{M_N} + \Delta_N$ is nef over B. Otherwise, the sequence is infinite.

The pair $(M_i, \Delta_i + t_i H_i)$ is a log terminal model (ltm) for $(M, \Delta + t_i H)$ over B; see Definition 5.29 and Lemma 5.31 of [32]. We will need the following lemmas:

Lemma 1.10 *For any $i > j$, let $\psi_{ij} \colon M_j \dashrightarrow M_i$ be the induced birational morphism over B. Then ψ_{ij} is not an isomorphism.*

Proof This is [32, Lemma 5.62]. □

Lemma 1.11 [32, Exercise 5.10] *Let (M, Δ) be a klt pair as above and suppose that Δ is big over B and that $K_M + \Delta$ is nef over B. Then $K_M + \Delta$ is semiample over B, ie there exists a projective morphism $f \colon M \to Z$ over B and an ample divisor L on B such that $K_M + \Delta \sim_{\mathbb{Q},B} f^*L$.*

Proof Since Δ is big over B, we can write $\Delta \sim_{\mathbb{Q},B} A + C$, where A is ample over B and $C \geq 0$. Choose an $0 < \epsilon \ll 1$ such that (M, Δ') is klt, where $\Delta' = (1-\epsilon)\Delta + \epsilon C$. Then

$$(K_M + \Delta) - (K_M + \Delta') = \epsilon A$$

is ample over B. By the basepoint-free theorem (see eg [32, Theorem 5.1]), $K_M + \Delta$ is semiample over B. □

Lemma 1.12 *Let the notation be as above and for any $i > 0$, let $\phi_i \colon M \dashrightarrow M_i$ be the induced birational map over B. Then the divisors contracted by ϕ_i are the divisorial components of $\mathbb{B}((K_{M_i} + \Delta_i + t_i H_i)/B)$, the stable base locus over B; cf* [32, Section 2.E]. *Similarly, $\psi_{ij} \colon M_j \dashrightarrow M_i$ contracts the divisorial components of $\mathbb{B}((K_{M_j} + \Delta_j + t_i H_j)/B)$.*

Proof Since $(M_i, \Delta_i + t_i H_i)$ is klt, $\Delta_i + t_i H_i$ is big over B, and $K_{M_i} + \Delta_i + t_i H_i$ is nef over B, by the lemma above, $K_{M_i} + \Delta_i + t_i H_i$ is semiample over B.

Let W be a smooth birational model resolving ϕ_i, and let p and q be the induced birational morphisms to M and M_i. By [32, Lemma 5.31] the pair $(M_i, \Delta_i + t_i H_i)$ is a log terminal model for $(M, \Delta + t_i H)$ over B; see [32, Definition 5.29]. Thus,

$$p^*(K_M + \Delta + t_i H) = q^*(K_{M_i} + \Delta_i + t_i H_i) + E, \tag{1-3}$$

where

$$E = \sum_F (a(F; M, \Delta + t_i H) - a(F; M_i, \Delta_i + t_i H))F$$

is an effective q–exceptional divisor whose support contains the divisors contracted by ϕ_i. Since

$$\begin{aligned} p^{-1}\mathbb{B}((K_M + \Delta + t_i H)/B) &= \mathbb{B}(p^*(K_M + \Delta + t_i H)/B) \\ &= \mathbb{B}(q^*(K_{M_i} + \Delta_i + t_i H_i) + E/B) \\ &= \operatorname{Supp}(E), \end{aligned}$$

the first statement follows. The second statement is proved in the same way, since by [32, Lemma 5.31], the pair $(M_i, \Delta_i + t_i H_i)$ is a log terminal model for $(M_j, \Delta_j + t_i H_j)$ over B and hence the equivalent of (1-3) holds. □

Lemma 1.13 *Let the notation be as above. If the mmp with scaling does not terminate, then*

$$\lim_{i \to \infty} t_i = 0.$$

Proof This is [26, Proposition 3.2]. The only difference is the relative setting, but the proof is the same: Suppose the mmp does not terminate and that $\lim t_i = t_\infty > 0$. By [13, Theorem E] there are finitely many log terminal models of $(M, \Delta + (t_\infty + t)H)$, with $t \in [0, 1 - t_\infty]$. We have already observed that $(M_i, \Delta_i + t_i H_i)$ is an ltm for $(M, \Delta + t_i H) = (M, \Delta + t_\infty H + (t_i - t_\infty)H))$ over B. Thus, if the sequence is infinite there are integers $i > j$ such that the birational map $M_j \dashrightarrow M_i$ is an isomorphism. This gives a contradiction with Lemma 1.10 above. □

1.2 Variants

In this section we give some variants of the results of the previous section. First we notice that the compactification result of Theorem 1.6 holds also for the twisted intermediate Jacobian fibration; see Remark 1.14. Then we consider the case of the intermediate Jacobian fibration associated to a mildly singular cubic fourfold; see Proposition 1.15 and Remark 1.16. We then give a slightly stronger version of Theorem 1.6, in that we show that the Lagrangian fibered hyper-Kähler compactification works in families; see Proposition 1.17 and Theorem 1.19. As an application, we give a positive answer to a question of Beauville; see Remark 1.18.

Remark 1.14 (the twisted case) In [80], Voisin constructed a nontrivial J_U–torsor $J_U^T \to U$ defined from a class in $H^1(U, \mathcal{J}_U[3])$, where $\mathcal{J}_U$ is the sheaf of holomorphic sections of $J_U \to U$ and where $\mathcal{J}_U[3] \subset \mathcal{J}_U$ is the sheaf of 3–torsion points. The nontriviality (for very general X) of this class corresponds to the nonexistence, for the universal family of hyperplanes sections of X, of a relative one-cycle of degree one. The main result of the paper is to produce, for general X, a hyper-Kähler compactification $J^T = J^T(X)$ with Lagrangian fibration to $\mathbb{P}^5$ extending $J_U^T \to U$. This builds on the compactification of [47]. We will refer to this hyper-Kähler manifold as the twisted intermediate Jacobian fibration. This hyper-Kähler manifold is deformation equivalent to the nontwisted version $J(X)$, as they agree as soon as X has a two-cycle which restricts to a one-cycle of degree one or two on its hyperplane sections. Lemma 1.4, Proposition 1.5 and Theorem 1.6 work the same for the nontrivial torsor $J_U^T \to U$, giving a Lagrangian fibered hyper-Kähler $J^T = J^T(X)$ for every smooth X. In Section 4.1 we will return to the twisted intermediate Jacobian fibration and in Corollary 3.10 we prove that for very general X these two fibrations are not birational and that on J there is a unique isotropic class in the movable cone of J. This fact will be used in the appendix.

Finally, we show that the Lagrangian fibered hyper-Kähler compactification exists generically also over $\mathcal{C}_6$, the divisor in the moduli space of cubic fourfolds whose general point parametrizes cubics with one A_1 singularity. The following proposition is an adaptation of [47, Section 2] to the case of a cubic fourfold with mild singularities.

Proposition 1.15 *Let $X_0 \subset \mathbb{P}^5$ be a cubic fourfold with one simple node $o \in X_0$ and no other singularities. Let $U \subset \mathbb{P}^5$ be the open locus parametrizing smooth hyperplane sections, and let $\pi_U \colon J_U = J_U(X_0) \to U$ be the Donagi–Markman fibration. Then there exists a holomorphic symplectic form σ_U on J_U, which extends to a holomorphic two-form on any smooth projective compactification. As a consequence, Lemma 1.4 holds for J_U, namely any projective compactification of J_U has smooth locus admitting a generically nondegenerate holomorphic two-form extending σ_U, and is not uniruled. Similarly, for the twisted intermediate Jacobian, $J_U^T = J_U^T(X_0)$.*

Proof Let $\widetilde{X}_0$ (resp. $\widetilde{\mathbb{P}}^5$) be the blowup of X_0 (resp. $\mathbb{P}^5$) at the point o. Let $E \subset \widetilde{\mathbb{P}}^5$ be the exceptional divisor. Projection from o determines an isomorphism $\widetilde{X}_0 \cong BL_S\mathbb{P}^4$, where S is the $(2,3)$ complete intersection in $\mathbb{P}^3$ parametrizing lines in X_0 by o. The surface S is a smooth K3 surface and thus $H^1(\widetilde{X}_0, \Omega^3_{\widetilde{X}_0})$ is one-dimensional; let η be a generator. The same argument as in [47, Theorem 1.2] shows that η induces a holomorphic two-form σ on J_U, with respect to which the fibers of $J_U \to U$ are isotropic. To show that σ is nondegenerate, it suffices to show that for any smooth hyperplane section Y (which in particular does not pass by the point o), the map

$$T_{[Y]}U = H^0(Y, \mathcal{O}_Y(1)) \to H^1(Y, \Omega^2_Y) = H^0(J_U, \Omega^1_{J_U}) \tag{1-4}$$

induced by σ, via the fact that the fibers of $J_U \to U$ are isotropic, is an isomorphism. By [47, Theorem 1.2(ii)], this map is given by the cup product with a class $\eta_Y \in H^1(Y, \Omega^2_Y(-1))$ defined in the following way: let $\eta_{|Y} \in H^1(Y, (\Omega^3_{\widetilde{X}_0})_{|Y})$ be the restriction of η to Y. Since $H^1(Y, \Omega^3_Y) = 0$, the exact sequence

$$0 \to \Omega^2_Y(-1) \to (\Omega^3_{\widetilde{X}_0})_{|Y} \to \Omega^3_Y \to 0$$

implies that $\eta_{|Y}$ lifts to a class $\eta_Y \in H^1(\Omega^2_Y(-1))$. By Griffiths residue theory [47, Lemma 1.7], $H^1(\Omega^2_Y(-1))$ is one-dimensional and cup product with any nonzero element induces an isomorphism $H^0(Y, \mathcal{O}_Y(1)) \to H^1(Y, \Omega^2_Y)$; more precisely, using the canonical isomorphism $\Omega^2_Y(-1) = T_Y(3)$, this space is spanned by the class of the nontrivial extension $0 \to T_Y \to (T_{\mathbb{P}^4})_{|Y} \to \mathcal{O}_Y(-3) \to 0$. It follows that to show that (1-4) is an isomorphism, we only need to show that $\eta_Y \neq 0$, which amounts to showing that $\eta_{|Y} \neq 0$. Under the isomorphism $\Omega^3_{\widetilde{X}_0} = T_{\widetilde{X}_0}(-3)(2E)$, the class of a generator of $H^1(\widetilde{X}_0, \Omega^3_{\widetilde{X}_0})$ corresponds to the class of the extension $0 \to T_{\widetilde{X}} \to (T_{\widetilde{\mathbb{P}}^5})_{|\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(3)(-2E) \to 0$. Restricting to Y and considering the tangent bundle sequence for Y in $\mathbb{P}^4$, we get the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (T_{\widetilde{X}})_{|Y} \arrow[r] & (T_{\widetilde{\mathbb{P}}^5})_{|Y} \arrow[r] & \mathcal{O}_Y(3) \arrow[r] & 0 \\
0 \arrow[r] & T_Y \arrow[r] \arrow[u] & (T_{\mathbb{P}^4})_{|Y} \arrow[r] \arrow[u, "\alpha"] & \mathcal{O}_Y(3) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the first two vertical arrows are injective. The extension class of the first row is $\eta_{|Y}$ and the second row is nonsplit, as we already observed. Since $\operatorname{coker}(\alpha) = \mathcal{O}_Y(1)$, we have $\operatorname{Hom}(\mathcal{O}_Y(3), \operatorname{coker}(\alpha)) = 0$. Thus any splitting of the first row would induce a splitting of the second row, giving a contraction. $\square$

Remark 1.16 Proposition 1.15 holds, more generally, for any cubic fourfold with isolated singularities, as long as a general one-parameter smoothing of it has finite monodromy. This corresponds to the K3 surface S of lines through one of the singular points having canonical singularities. The case of the degeneration to the chordal cubic [34], which has finite monodromy but central fiber with 2–dimensional singular locus, will be discussed at length in Section 4.2.

Proposition 1.17 *Let $X_0 \subset \mathbb{P}^5$ be as in Proposition 1.15 (or as in Remark 1.16) and let $\pi_U \colon J_U \to U$ be the corresponding intermediate Jacobian fibration. Then there exists a hyper-Kähler compactification $J = J(X_0)$ of J_U, with a regular flat morphism to $(\mathbb{P}^5)^\vee$ extending π_U. Moreover, if $\mathcal{X} \to \Delta$ is a general family of smooth cubic fourfolds degenerating to X_0, then up to a base change, there exists a family of Lagrangian fibered hyper-Kähler manifolds*

$$\mathcal{J} \to \mathbb{P}^5_\Delta \to \Delta$$

such that for $t \neq 0$, $\mathcal{J}_t = J(\mathcal{X}_t)$ is the LSV compactification and, for $t = 0$, $\mathcal{J}_0$ is a hyper-Kähler compactification of $J_U = J_U(X_0)$. Similarly, the analogous statement holds for the twisted intermediate Jacobian.

Proof By Proposition 1.15 above, J_U has a holomorphic symplectic form that extends to a regular form on any smooth projective compactification. As in Lemma 1.4, it follows that J_U is not uniruled. Let $\mathcal{X} \to \Delta$ be a family of smooth cubic fourfolds degenerating to $\mathcal{X}_0 = X_0$ with the property that for $t \neq 0$, $\mathcal{X}_t$ is general in the sense of LSV. As in the beginning of Theorem 1.6, let $\mathcal{J}_\mathcal{V} \to \mathcal{V}$ be such that the fiber over $t \neq 0$ of $\mathcal{J}_\mathcal{V} \to \Delta$ is the LSV compactification $J(\mathcal{X}_t)$ and, over $t = 0$, is $J_U \to U$. We are thus in the position of applying Theorem 1.19 below, which proves the proposition. □

A consequence of this proposition is a positive answer to a question of Beauville [11], as explained in the following remark.

Remark 1.18 Given a smooth cubic threefold Y, let $\ell \subset Y$ be a line. In [9; 26] it is shown that the moduli space of Ulrich bundles on Y with rank 2, $c_1 = 0$ and $c_2 = 2\ell$ is birational to the intermediate Jacobian of Y; more precisely, it can be identified with the blowup of the intermediate Jacobian fibration along the Fano surface. Now let X_0 be cubic fourfold with one simple node and let $S \subset \mathbb{P}^4$ be the $(2,3)$ complete intersection K3 surface parametrizing lines through the singular point of X_0. Consider the Mukai vector $v = 2v_0 = 2(1,0,-1) \in H^*(S,\mathbb{Z})$ and let $\widetilde{M}_{2v_0}(S)$ be the symplectic resolution of the singular moduli space of OG10–type; cf Section 2.

By considering the relative moduli spaces of Ulrich bundles supported on the five-dimensional family of cubic threefolds containing S and by restricting the bundles to S, Beauville [11, Section 5, Example $d = 3$] shows that there is a birational map $J_U \dashrightarrow M_v(S)$. This induces a rational map $M_{2v_0}(S) \dashrightarrow \mathbb{P}^5$, and Beauville asks whether there exists a hyper-Kähler manifold birational to $M_v(S)$ which admits a *regular* morphism to $\mathbb{P}^5$. Proposition 1.17 thus gives a positive answer to this question.

The proof of the proposition above relies on the following theorem, which is the Lagrangian fibration analogue of results from [41, Theorem 2.1 and Corollary 5.2]. Theorem 1.19 will be used also in Section 4 for the proof of Proposition 4.5 (and thus also of Theorem 4.1). As usual, Δ is an open affine subset of a smooth curve, or a small analytic disk. In both cases, we keep the notation $t = 0$ to denote a chosen special point in Δ, and $t \neq 0$ to denote any other point.

Theorem 1.19 *Let $\tilde{f}\colon \tilde{\mathcal{J}} \to \Delta$ be a projective degeneration of hyper-Kähler manifolds of dimension $2n$. Suppose that there is a commutative diagram*

$$\begin{tikzcd} \tilde{\mathcal{J}} \arrow[r, "\tilde{\pi}"] \arrow[dr, "\tilde{f}"'] & \mathbb{P}^n_\Delta \arrow[d, "p"] \\ & \Delta \end{tikzcd}$$

where $\tilde{\mathcal{J}} \to \mathbb{P}^n_\Delta$ is a projective fibration such that for $t \neq 0$, $\mathcal{J}_t \to \mathbb{P}^n_t$ is a Lagrangian fibration. Assume that the central fiber $\tilde{\mathcal{J}}_0 = Y_0 + \sum_{i\in I} m_i Y_i$ has a reduced component Y_0 which is not uniruled. Suppose, furthermore, that there is an open subset of $Y_0 \setminus \bigcup_{i\geq 1}(Y_i \cap Y_0)$ such that the morphism to $\mathbb{P}^n_0$ is a fibration $J_{U_0} \to U_0 \subset \mathbb{P}^n_0$ in abelian varieties. Then:

1. *There exists a projective degeneration $\bar{f}\colon \bar{\mathcal{J}} \to \Delta$ of hyper-Kähler manifolds such that*
   (a) *$\bar{\mathcal{J}}$ is $\mathbb{Q}$–factorial, terminal and isomorphic to $\tilde{\mathcal{J}}$ over Δ^*,*
   (b) *the central fiber $\bar{\mathcal{J}}_0$ is a reduced, irreducible, and a normal symplectic variety with canonical singularities and admitting a symplectic resolution, and*
   (c) *there is a relative Lagrangian fibration $\bar{\pi}\colon \bar{\mathcal{J}} \to \mathbb{P}^n_\Delta$ compatible, via the birational map $\bar{\mathcal{J}} \dashrightarrow \tilde{\mathcal{J}}$, with $\tilde{\pi}$ and such that, up to restricting the open set $U_0 \subset \mathbb{P}^n_0$, the morphism $\bar{\mathcal{J}}_0 \to \mathbb{P}^n_0$ extends the abelian fibration $J_{U_0} \to U_0$.*
2. *Up to a base change $\Delta' \to \Delta$, there exists a (not necessarily projective) family $\mathcal{J} \to \Delta'$ of hyper-Kähler manifolds, with a birational morphism $\mathcal{J} \to \bar{\mathcal{J}}' := \bar{\mathcal{J}} \times_{\Delta'} \Delta$ over Δ', which is an isomorphism away from the central fiber and in the central fiber is a symplectic resolution of $\bar{\mathcal{J}}_0$. Moreover, $\mathcal{J}$ has a family of Lagrangian fibrations $\pi'\colon \mathcal{J} \to \mathbb{P}^n_{\Delta'}$, compatible with the base change of $\bar{\pi}$.*

Proof The proof follows ideas from [75; 40; 41]. Up to passing to a log resolution of the pair $(\tilde{\mathcal{J}}, \tilde{\mathcal{J}}_0)$, we can assume that $\tilde{\mathcal{J}}_0 = Y_0 + \sum_{i=1}^k m_i Y_i$ is a normal crossing divisor. By [41, Theorem 2.1 and Corollary 5.2], running the mmp over Δ contracts the components Y_i for $i \geq 1$, and yields a birational model of $\widetilde{\mathcal{J}}$ with an irreducible central fiber which is a symplectic variety. In particular, Y_0 is the unique component of $\widetilde{\mathcal{J}}_0$ that is not uniruled; cf [41, Remark 2.2]. To prove the theorem we only need to show that the birational maps required to contract the other components can be preformed relatively to $\mathbb{P}^n_\Delta$ and, furthermore, that they induce isomorphism away from $\bigcup_{i\geq 1} Y_i$. This is to ensure that the central fiber has a Lagrangian fibration extending $J_{U_0} \to U_0$ (maybe up to restricting the open subset $U_0 \subset \mathbb{P}^n_0$).

The canonical class $K_{\widetilde{\mathcal{J}}}$ is trivial over Δ^*, so it is $\tilde{f}$–equivalent to a divisor of the form $\sum_{i=0}^k a_i' Y_i$. Following [75, Section 2.3, point (1)] we set $r = \min a_i'/m_i$, so[1]

$$K_{\widetilde{\mathcal{J}}} =_{\mathbb{Q},\Delta} \sum_{i=0}^{k} a_i Y_i,$$

where $a_i = a_i' - r m_i \geq 0$ are nonnegative rational numbers and $a_i = 0$ for at least one i. Let $J \subsetneq \{0, 1, \dots, k\}$ be the set of indices such that $a_i > 0$ and let J^c be its complement. By [75, Proposition 5.1]:

(1) For every $j \in J$, the irreducible component Y_j is uniruled.

(2) If $|J^c| \geq 2$, then for every $j \in J^c$, the irreducible component Y_j is uniruled.

Since Y_0 is not uniruled, it follows that $J = \{1, \dots, k\}$ and thus

$$K_{\widetilde{\mathcal{J}}} =_{\mathbb{Q},\tilde{\pi}} \sum_{i=1}^{k} a_i Y_i, \quad \text{with } a_i > 0.$$

By assumption, for every $i \geq 1$, the closed subset $Y_0 \cap Y_i$ is in the complement of J_{U_0} and, since the fibers of $\widetilde{\mathcal{J}}_0 \to \mathbb{P}^n_\Delta$ are connected, it follows that the induced map $Y_i \to \mathbb{P}^n_0$ is not dominant. Thus, the codimension of $\tilde{\pi}(Y_i)$ in $\mathbb{P}^n_\Delta$ is greater or equal to two. In other words, Y_i is $\tilde{\pi}$–exceptional.

We are in the same setting of Theorem 1.6, namely a projective morphism from a smooth quasiprojective variety with a canonical class that is relatively $\mathbb{Q}$–linearly equivalent to an effective divisor all of whose components are relatively exceptional. We can thus argue as in the proof of Theorem 1.6, running the mmp over $\mathbb{P}^n_\Delta$ with scaling of an ample divisor in order to contract each of the Y_i, for $i \geq 1$. This yields a birational map $\widetilde{\mathcal{J}} \dashrightarrow \overline{\mathcal{J}}$ over $\mathbb{P}^n_\Delta$, where $\overline{\mathcal{J}} \to \Delta$ has irreducible fibers and the fibration

[1] For a projective morphism $f: A \to B$ and two $\mathbb{Q}$–Cartier divisors D and D' on A, we write $D =_{\mathbb{Q},B} D'$ or $D \sim_{\mathbb{Q},f} D'$ if and only if D and D' are $\mathbb{Q}$–linearly equivalent up to the pullback of a $\mathbb{Q}$–Cartier divisor from B.

$\bar{\mathcal{J}} \to \mathbb{P}^n_\Delta$ has $\mathbb{Q}$–factorial terminal total space and is such that $K_{\bar{\mathcal{J}}} = \tilde{\pi}^* B$ for some $\mathbb{Q}$–divisor B on $\mathbb{P}^n_\Delta$. Since at each step the K–negative rays of the mmp are contained in uniruled components of the central fiber, it follows that the birational map $\tilde{\mathcal{J}} \dashrightarrow \bar{\mathcal{J}}$ is an isomorphism away from $\bigcup_{i\geq 1} Y_i$. In particular, the central fiber $\bar{\mathcal{J}}_0$, which is irreducible, has an open subset which is isomorphic to J_{U_0}. Since $(K_{\bar{\mathcal{J}}})_{|\mathcal{J}_t} = 0$ for $t \neq 0$, we get that $B_{|\mathbb{P}^n_t} = 0$ for $t \neq 0$. In particular, B is p–trivial, where $p\colon \mathbb{P}^n_\Delta \to \Delta$ is the projection, and thus $K_{\bar{\mathcal{J}}}$ is $\tilde{f}$–trivial. We can now argue as in the last part of the proof of [47, Theorem 1.1] to show that $\bar{\mathcal{J}}_0$ is normal with canonical singularities. As in [47, Corollary 4.2] it follows that $\bar{\mathcal{J}}_0$ is a symplectic variety and that, up to a base change $\Delta' \to \Delta$, there exists a smooth family $\mathcal{J} \to \Delta'$ with a birational morphism $\mathcal{J} \to \bar{\mathcal{J}}' := \bar{\mathcal{J}} \times_{\Delta'} \Delta$ with the desired properties. □

Remark 1.20 Theorem 1.19 gives another proof of Theorem 1.6, as well as the stronger statement of the existence of a relative intermediate Jacobian fibration $\mathcal{J} \to \mathbb{P}^5_\Delta$ associated to any family $\mathcal{X} \to \Delta$ of smooth cubic fourfolds for which the general fiber is general in the sense of LSV.

2 Moduli spaces of OG10–type

By [47, Corollary 6.3] (see also [41, Section 6.3]) any hyper-Kähler compactification J of J_U is deformation equivalent to O'Grady's 10–dimensional example. We start this section by recalling the basic definitions and first properties of those singular moduli spaces of sheaves on a K3 surface whose symplectic resolutions are hyper-Kähler manifolds in this deformation class. Then we use the methods of Bayer and Macrì, as adapted by Meachan and Zhang to this class of singular moduli spaces, to study the movable cone of certain moduli spaces that appear naturally as limits of the intermediate Jacobian fibration, when the underlying cubic fourfold degenerates to the chordal cubic; see Section 4.2.

We start by recalling the following fundamental theorem.

Theorem 2.1 [60; 82; 63; 50; 38; 66] *Let (S, H) be a general polarized K3 surface and let $v_0 \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ be a primitive Mukai vector which we suppose to be positive in the sense of* [8, Definition 5.1], *see also* [18, Remark 3.1.1]. *Let $m \geq 2$ be an integer. The moduli space $M_{mv_0,H}(S)$ of H–semistable sheaves on S with Mukai vector mv_0 is an irreducible normal projective symplectic variety of dimension $m^2v_0^2 + 2$, which* *admits a symplectic resolution if and only if $m = 2$ and $v_0^2 = 2$. When this is the case, the symplectic resolution $\widetilde{M}_{2v_0,H}(S) \to M_{2v_0,H}(S)$ is the blow up of the singular locus $\mathrm{Sym}^2 M_{v_0,H}(S) \subset M_{2v_0,H}(S)$, with its reduced induced structure. Moreover, $\widetilde{M}_{2v_0,H}(S)$ is an irreducible holomorphic symplectic manifold and its deformation class is independent of (S,H) and of v_0; in particular, $\widetilde{M}_{2v_0,H}(S)$ is deformation equivalent to O'Grady's original 10–dimensional exceptional example.*

We will refer to a Mukai vector of the form $2v_0$ with $v_0^2 = 2$ as a Mukai vector of OG10–type and to a hyper-Kähler manifold in this deformation class as a hyper-Kähler of OG10–type.

2.1 Contracting the relative theta divisor on the relative Jacobian of curves

It is known [4; 7; 8; 5] that the birational geometry of moduli spaces of pure dimension one sheaves on a K3 surface is related to Brill–Noether loci. For example, on the degree $g-1$ Beauville–Mukai system of a genus g linear system on a K3 surface, the relative theta divisor can be contracted, possibly after performing a finite sequence of birational transformations. This is the content of the following example.

Example 2.2 [4; 5] Let (S,C) be a general polarized K3 surface of genus g, with $\mathrm{NS}(S) = \mathbb{Z}C$. Set $v = (0,C,0) \in H^*(S,\mathbb{Z})$ and let M_v be the moduli space of C–stable sheaves on S with Mukai vector[2] v. Since we are assuming (S,C) to be general in moduli, we are suppressing the polarization from the notation — thus M_v will denote the moduli space of C–semistable sheaves on S with Mukai vector v; when we consider instead a Bridgeland stability condition σ, the corresponding moduli space will be denoted by $M_{v,\sigma}$. This moduli space is smooth and $M_v \to \mathbb{P}^g = |C|$ is the degree $g-1$ relative compactified Jacobian of the genus g linear system $|C|$ on S. There is a naturally defined effective, irreducible, relatively ample theta divisor $\theta \subset M_v$ which parametrizes sheaves with a nontrivial global section and which can be realized as the zero locus of a canonical section of the determinant line bundle; see [48, Section 2.3] or [3, Theorem 5.3]. Recall that there is a Hodge isometry $\mathrm{NS}(M_v) \cong v^\perp = \langle (0,0,1),(1,0,0)\rangle$; see for example [7, Theorem 3.6].

[2]This Mukai vector is not positive in the sense defined above, since both the first and last entry are zero. However, since for general (S,C), tensoring by C induces an isomorphism with $M_{v'}$, where $v' = (0,C,g-1)$, the results of [7] still hold. See also [66] for other considerations about the last entry of the Mukai vector.

The class $\ell := (0,0,1)$ is the class of the isotropic line bundle inducing the Lagrangian fibration $M_v \to \mathbb{P}^g$ while the theta divisor θ corresponds to the class $-(1,0,1) = -v(\mathcal{O}_S)$; see [48, page 643] or also [7, Proposition 7.1 and Theorem 12.3].[3] Since $\theta^2 = -2$, the irreducible effective divisor θ is prime exceptional. By [27], it can be contracted on a hyper-Kähler birational model of M_v. Since the rays corresponding to divisorial contractions and to Lagrangian fibrations must be in the boundary of the movable cone [36], it follows that

$$\overline{\mathrm{Mov}}(M_v) = \mathbb{R}_{\geq 0}\ell + \mathbb{R}_{\geq 0}h,$$

where $h = (-1,0,1) \in \theta^\perp \cap v^\perp$ is a big line which is nef on some birational model of M_v; this also follows from [7, Theorem 12.3]. Using [7], the walls of the nef cones of the various birational models can be computed. Since we don't need this, we omit the computation.

2.2 Movable cones of certain moduli spaces of OG10–type

If we consider a nonprimitive genus g linear system $|mC|$, with $m \geq 2$, then the relative compactified Jacobian of degree $g-1$ is singular. For singular moduli spaces of OG10–type, ie when $v = 2w$ with $w^2 = 2$, Meachan and Zhang [58] adapted the techniques of Bayer and Macrì [8; 7] to compute the nef and movable cones of these moduli spaces. We refer to [16; 4; 8; 7] for the relevant definitions and main results on Bridgeland stability conditions on K3 surfaces, and to [58] for the results on moduli spaces of OG10 type.

By [58, Theorem 7.6(3)], all birational models of $M_{2w} = M_{2w,C}$ which are isomorphic to M_{2w} in codimension one are isomorphic to a Bridgeland moduli space $M_{2w,\sigma}$ for some Bridgeland stability condition σ on S. Moreover, by [58, Corollary 2.8],

$$\mathrm{NS}(M_{2w,\sigma}) \cong w^\perp. \tag{2-1}$$

We now apply the results of [58] to describe the nef and movable cones of certain singular models of OG10 appearing as limits of the intermediate Jacobian fibration. By [67], the factoriality properties of a singular moduli space M_{2w} of OG10–type depend on the divisibility of the primitive Mukai vector $w \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$. More precisely, by [67, Theorem 1.1], M_{2w} is factorial if and only if $w \cdot u \in 2\mathbb{Z}$ for every $u \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$. Otherwise, M_{2w} is 2–factorial. Since there can be different birational models with

[3] Compared to [7], there is a difference in a choice of sign in the isomorphism $\mathrm{NS}(M_v) \cong v^\perp$.

different factoriality properties (cf Remark 2.3), it is important to choose the correct model to work with.

Now let (S,C) be a general K3 surface of degree 2 and set

$$v_k := (0, C, k-2). \tag{2-2}$$

The Le Potier morphism $\pi\colon M_{2v_k} \to \mathbb{P}^5$ realizes the singular moduli space M_{2v_k} as a compactification of the degree $2k$ relative Jacobian of the genus-five hyperelliptic linear system $|2C|$. Composing π with the symplectic resolution $m\colon \widetilde{M}_{2v_k} \to M_{2v_k}$, we get a natural Lagrangian fibration

$$\tilde{\pi}\colon \widetilde{M}_{2v_k} \to \mathbb{P}^5. \tag{2-3}$$

By the result of Perego and Rapagnetta mentioned above, M_{2v_k} is factorial if and only if k is even. It turns out that the birational class of these moduli spaces is independent of k, but the isomorphism class depends on the parity of k [18, Proposition 3.2.7]: indeed, tensoring a pure dimension one sheaf by $\mathcal{O}_S(C)$ determines an isomorphism

$$M_{2v_k} \xrightarrow{\sim} M_{2v_{k+2}}. \tag{2-4}$$

Remark 2.3 Tensoring a line bundle supported on a smooth hyperelliptic curve of genus 5 by the unique g^1_2 on the curve defines a birational morphism $M_{2v_k} \dashrightarrow M_{2v_{k+1}}$. (I thank A Rapagnetta for pointing out this to me.) As a side remark, notice that the map thus defined is *not* an isomorphism in codimension one. Indeed, it can be checked that when passing to the birational morphism $\widetilde{M}_{2v_k} \dashrightarrow \widetilde{M}_{2v_{k+1}}$ between the two resolutions, which is an isomorphism in codimension two, the exceptional divisor of one model is exchanged with the proper transform of the locus parametrizing sheaves on reducible curves on the other model.

In view of Lemma 4.4 below and the isomorphism (2-4), we will focus on the case $k=0$.

Remark 2.4 For general (S,C) it is not hard to check that the structure sheaf of every curve in $|2C|$ satisfies the numerical criterion for C–stability and hence that the fibration $M_{2v_0} \to \mathbb{P}^5$ admits a regular zero section. Notice also that the image of this section is not contained in the singular locus of M_{2v_0}.

By [58, Corollary 2.8], $\mathrm{NS}(M_{2v_0}) \cong v_0^{\perp} = U = \langle (0,0,1), (-1,C,0)\rangle$, where, as above,

$$\ell = (0,0,1) \tag{2-5}$$

is the line bundle inducing the Lagrangian fibration $\pi\colon M_{2v_0} \to \mathbb{P}^5 = |2C|$. Under the isomorphism $M_{2v_2} \cong M_{2v_0}$ induced by tensoring with $\mathcal{O}_S(-C)$, the relative theta divisor is mapped isomorphically to the prime exceptional divisor

$$\theta := -(1, -C, 2) \tag{2-6}$$

parametrizing sheaves which receive a nontrivial morphism from the spherical object $\mathcal{O}_S(-C)$; see also Lemma 2.6. Indeed, the relative theta divisor in M_{2v_2} parametrizes sheaves with a nontrivial morphism from $\mathcal{O}_S$ and thus its image in M_{2v_0} is exactly the divisor θ. Notice that

$$\theta^2 = -2. \tag{2-7}$$

For later use we highlight the following remark.

Remark 2.5 The effective divisor $\theta \subset M_{2v_0}$ with cohomology class (2-6) does not contain the singular locus of M_{2v_0}: using the description of θ as the zero locus of a section of the determinant line bundle [3, Theorem 5.3], which is compatible with S–equivalence classes, it is enough to show that the section defining θ is not identically zero on the singular locus of M_{2v_0}. It is therefore sufficient to show that there are S–equivalence classes of polystable sheaves all of whose members have a zero space of global sections. This is clear, since the generic semistable sheaf with Mukai vector $2v_0$ is an extension of two degree-one line bundles each supported on two distinct curves of genus two.

The following lemma is an application of [58, Theorems 5.1–5.3] to M_{2v_0}. (Note that Example 8.6 of loc. cit. is for odd k, so in view of Remark 2.3 it is concerned with a birational model of M_{2v_0} which is not isomorphic in codimension one, and hence we cannot immediately apply it here.)

Lemma 2.6 *Let the notation be as above. Then*

$$\operatorname{Nef}(M_{2v_0}) = \mathbb{R}_{\geq 0}\ell + \mathbb{R}_{\geq 0}h_0, \quad \operatorname{Mov}(M_{2v_0,C}) = \mathbb{R}_{\geq 0}\ell + \mathbb{R}_{\geq 0}h,$$

where

$$\ell = (0,0,1), \quad h_0 = (-1, C, 1), \quad h = (-1, C, 0).$$

Moreover, the wall spanned by $h_0 = (-1, C, 1)$ contracts the zero section of $M_{2v_0} \to \mathbb{P}^5$ and the class corresponding to $h = (-1, C, 0)$ is big and nef on the Mukai flop of M_{2v_0} along the zero section and contracts the proper transform of θ.

Proof Since $\ell = \pi^*\mathcal{O}_{\mathbb{P}^5}(1)$ is nef and isotropic, it is one of the two rays of both the Nef and the movable cones of M_{2v_0}. By [58, Theorem 5.3] there is a divisorial contraction of BNU–type (notation as in loc. cit.), determined by the spherical class $s = (1, -C, 2)$, which is orthogonal to v_0. The second ray of the movable cone is thus determined by $s^\perp \cap v_0^\perp$. We pick $h = (-1, C, 0)$ as a generator of this ray, since $h \cdot \ell > 0$. By the same theorem in [58], the flopping walls are determined by $w^\perp \cap v_0^\perp$ for w spherical and such that $w \cdot v_0 = 2$. There is a unique ray in $\mathrm{Mov}(M_{2v_0})$ that is of this form. It is determined by $w = (1, 0, 1) = v(\mathcal{O}_S)$ or, equivalently, by $w' = (-1, 2C, -5) = 2v_0 - w$. We can choose $h_0 = (-1, C, 1)$ as generator of this ray. As in [58, Remark 8.5], we can see that this wall corresponds to the flop of the $\mathbb{P}^5$ corresponding to the sheaves with a morphism from $\mathcal{O}_S$, ie of the image of the zero section. □

Remark 2.7 It can be shown that the birational model on the other side of the wall can be identified with the Gieseker moduli space M_{2w_0}, where $w_0 = (2, C, 0)$. Since we don't need this in the rest of the paper, we omit the proof.

Remark 2.8 The theta divisor θ is Cartier, since by [66] M_{2v_0} is factorial; see also Section 2.2. Moreover, it is relatively ample over $\mathbb{P}^5$, since by the description of the Nef cone of Lemma 2.6 we can write θ as a sum of an ample line bundle and a multiple of $\ell = \pi^*\mathcal{O}_{\mathbb{P}^5}(1)$.

3 The relative theta divisor on the intermediate Jacobian fibration

For any smooth cubic threefold Y, there is a canonically defined theta divisor in $\mathrm{Jac}(Y)$ which is (-1)–invariant and whose unique singular point lies at the origin. For the hyper-Kähler compactification $J = J(X) \to (\mathbb{P}^5)^\vee$ of the intermediate Jacobian fibration associated to a smooth cubic 4–fold X, there is an effective relative theta divisor $\Theta \subset J$, which is defined as the closure of the union of the canonical theta divisor in the smooth fibers. More precisely, by [20; 21], see also [47, Lemma 5.4], Θ can be defined as the closure of the image of the Abel–Jacobi difference mapping

$$\mathcal{F} \times_{(\mathbb{P}^5)^\vee} \mathcal{F} \dashrightarrow J, \quad (\ell, \ell', Y) \mapsto \phi_Y(\ell - \ell'). \tag{3-1}$$

The relative theta divisor Θ played an important role in [47], where it was shown that for general X the divisor Θ is π–ample and J is identified with the relative Proj of the sheaf of $\mathcal{O}_{\mathbb{P}^5}$–algebras associated with this divisor. Another useful way of realizing the theta divisor is using twisted cubics [21]. Let $Z = Z(X)$ be the Lehn–Lehn–Sorger–van Straten 8–fold [49]. Then Z is the blowdown $g\colon Z' \to Z$ of a smooth 10–fold Z' whose points parametrize nets of (generalized) twisted cubics. The exceptional locus of g parametrizes non-ACM cubics and its image in Z is isomorphic to the cubic itself. Let

$$r\colon \mathbb{P}_{Z'} \to Z'$$

be the $\mathbb{P}^1$–bundle over Z' whose fiber over a twisted cubic $[C] \in Z'$ is the pencil $\mathbb{P}^1_C$ of hyperplane sections of X containing $\Sigma_C := X \cap \langle C\rangle$. Here $\langle C\rangle = \mathbb{P}^3$ is the linear span of the curve. By [47, Sublemma 5.5], see also [21, Section 4] or [41, Proposition 6.10], the Abel–Jacobi map

$$\varphi\colon \mathbb{P}_{Z'} \dashrightarrow J, \quad (C, Y) \mapsto \phi_Y(C - h^2), \tag{3-2}$$

is birational onto its image, which is precisely Θ. Here h^2 is the class of the intersection of two hyperplanes in Y.

Remark 3.1 For later use, we note the following two facts. First of all, the restriction of $\mathbb{P}_{Z'}$ to the locus of nonCM cubics is mapped to the zero section of $J \to \mathbb{P}^5$ (which lies in Θ). Second, using the Gauss map, see [20, Section 12] or also [33, Section 3], one can see that if C is a twisted cubic in a smooth cubic threefold Y with the property that $\phi_Y(C - h^2) = 0$ in $\mathrm{Jac}(Y)$, then the cubic surface $\Sigma_C = Y \cap \langle C\rangle$ is singular.

For every X, the Néron–Severi group of $J = J(X)$ has at least rank two, since

$$\mathrm{NS}(J(X)) \supset \langle L, \Theta\rangle.$$

Here $L = \pi^*\mathcal{O}_{\mathbb{P}^5}(1)$ and Θ is, as above, the relative theta divisor obtained as the closure of the image of (3-1).

Lemma 3.2 *For any smooth X, there is an isomorphism of rational Hodge structures $H^2(J, \mathbb{Q})_{\mathrm{tr}} \cong H^4(X, \mathbb{Q})_{\mathrm{tr}}$. In particular, $\rho(J) = \mathrm{rk}\, H^{2,2}(X, \mathbb{Q}) + 1$.*

Proof The first statement was already noted in [47], while the second follows from the first and the fact that $b_2(J) = 24$ and $b_4(X) = 23$. $\square$

Remark 3.3 The locus, inside $\mathrm{Def}(J)$, parametrizing intermediate Jacobian fibrations is of codimension two and corresponds to the locus where the classes L and Θ stay of type $(1,1)$. By [74, Theorem 6], a Lagrangian fibration with a section deforms, as Lagrangian fibration with a section, over a smooth codimension-two locus of the deformation space of the underlying hyper-Kähler manifold. Since by Theorem 1.1 for general X the LSV compactification $J(X)$ has a section, it follows that the codimension-two locus where L and Θ stay algebraic is exactly the locus where the section deforms.

We highlight the following corollary for future reference.

Corollary 3.4 *For very general X, we have that $\rho(J)=2$. Thus*:

(1) *J is the only projective hyper-Kähler birational model of J_U where L is nef. In particular, any hyper-Kähler compactification of J_U with a Lagrangian fibration extending $J_U \to U$ is isomorphic to the compactification of [47].*

(2) *There is at most one prime exceptional divisor on J.*

Proof (1) Since $\rho(J)=2$, the boundary of the movable cone of J has two rays, of which L is one.

(2) If there is a prime exceptional divisor, its class has to be orthogonal to the second extremal ray of the movable cone [52, Theorem 1.5]. Since two prime exceptional divisors with proportional classes have to be isomorphic [53, Corollary 3.6(3)], there is at most one prime exceptional divisor. □

The following lemma was communicated to me by K Hulek and R Laza. I thank them for sharing this observation with me and for raising the question of computing $q(\Theta)$.

Lemma 3.5 *We have that $q(L,\Theta)=1$. In particular, $\langle L,\Theta\rangle$ is a primitive sublattice of $\mathrm{NS}(J)$, isomorphic to the standard hyperbolic lattice U of rank two. For very general X, $\mathrm{NS}(J)=U$.*

Proof The computation of $q(L,\Theta)$ goes as in [73, Lemma 1]: one expands in t the Fujiki equality $q(L+t\Theta)^5 = c(L+t\Theta)^{10}$, where $c=945$ is the Fujiki constant [71], and uses the fact that $\Theta^5L^5 = (\Theta_{|J_{[H]}})^5 = 5!$. The final statement follows from Lemma 3.2. □

I thank C Onorati for many discussions around Θ and for his interest in the following computation.

Proposition 3.6 *The irreducible divisor $\Theta \subset J$ is prime exceptional. In particular, it can be contracted on some projective birational hyper-Kähler model of J. Moreover, $q(\Theta)=-2$.*

Proof Let $\varphi\colon \mathbb{P}_{Z'} \dashrightarrow \Theta$ be the Abel–Jacobi map as in (3-2) and let $V \subset Z'$ be a nonempty open subset such that the restriction of φ to $r^{-1}(V) =: \mathbb{P}_V$ is regular. By restricting V if necessary, we can assume that all twisted cubics parametrized by V are such that Σ_C is smooth and, in particular, that C is ACM.

Recall that for any twisted cubic $[C] \in V$, we have set $\mathbb{P}^1_C = r^{-1}(C)$. The rational curve $\varphi(\mathbb{P}^1_C) \subset J$ is smooth because it maps to $\pi(\varphi(\mathbb{P}^1_C)) \subset \mathbb{P}^5$, which is the pencil of hyperplane sections of X that contain the curve C. Moreover, $(\pi \circ \varphi)(\mathbb{P}_V)$ intersects the dual variety $X^\vee$ in a dense open subset and, similarly, $\varphi(\mathbb{P}_V)$ intersects $\Theta_{X^\vee}$, the restriction of Θ to $X^\vee$, in a dense open subset. (This statement follows from [21] and the fact, proven there, that for a cubic threefold with one A_1 singularity the Abel–Jacobi mapping is birational onto its image.)

We start by showing that for a general $[C] \in V$, the smooth rational curve $\varphi(\mathbb{P}^1_C)$ is contained in the smooth locus of Θ. The singular locus Θ_{sing} of Θ has an irreducible component that is equal to the closure of the zero section of $J_U \to U$, while any other irreducible component of the singular locus is properly contained in $\Theta_{X^\vee}$, the restriction of Θ to $X^\vee$. Since V parametrizes ACM curves, by Remark 3.1 it follows that the intersection of $\varphi(\mathbb{P}_V)$ with the image of the zero section of $J \to \mathbb{P}^5$ is contained in $J_{X^\vee}$, the restriction of J to $X^\vee \subset \mathbb{P}^5$. Let $B := \varphi^{-1}(\Theta_{\mathrm{sing}} \cap \varphi(\mathbb{P}_V))$ be the locus in $\mathbb{P}_V$ parametrizing points mapped to the singular locus of Θ and let $W_V := (\pi \circ \varphi)^{-1}(X^\vee \cap (\pi \circ \varphi)(\mathbb{P}_V))$ be the locus in $\mathbb{P}_V$ of pairs (C, Y) such that Y is singular. By what we have observed, it follows that $B \subseteq W_V$. Notice that W_V is irreducible of dimension 8, because it maps to an open subset of $X^\vee$, with fibers parametrizing equivalence classes of twisted cubics contained in a cubic threefold with one A_1 singularity; these form an irreducible subset of $Z(X)$, as follows from [21, Section 3]. We have already observed that the general point of $\Theta_{X^\vee}$ is contained in the image $\varphi(\mathbb{P}_V)$. Thus, if Y_0 corresponds to a general point in $X^\vee$, there is a twisted cubic $C \subset Y_0$, with $[C] \in V$ and such that $\varphi(C, Y_0) = \phi_{Y_0}(C)$ lies in the smooth locus of Θ. It follows that B is strictly contained in W_V. Since W_V is irreducible, $\dim B = 7$. Thus B does not dominate V and hence the image of the open subset $\mathbb{P}_{V'} := r^{-1}(V')$, where $V' := V \setminus r(B)$ is contained in the smooth locus of Θ.

For the general point in $(C, Y) \in \mathbb{P}_V$, let $R := \varphi(\mathbb{P}^1_C) \subset \Theta$ be the corresponding element of the ruling. By generic smoothness, the differential of φ is of maximal rank at a general point $x \in R$, so by [39, Chapter II, Proposition 3.4], the vector bundle $(T_\Theta)_{|R}$ is globally generated at $x \in R$. It follows that $(T_\Theta)_{|R} = \bigoplus \mathcal{O}_R(a_i)$, with $a_i \geq 0$. By Lemma 3.8 below, the restriction of the tangent bundle of J to the smooth rational curve R is of the form $\mathcal{O}_R^{\oplus 8} \oplus \mathcal{O}_R(2) \oplus \mathcal{O}_R(-2)$. Using this and the fact that R is contained in the smooth locus of Θ, we find that $(T_\Theta)_{|R} = \mathcal{O}_R(2) \oplus \mathcal{O}_R^{\oplus 8}$ and hence that $N_{R|\Theta} = \oplus \mathcal{O}_R^{\oplus 8}$. In particular, $\Theta \cdot R = -2$.

Consider the lattice embedding $H^2(J,\mathbb{Z}) \subset H^2(J,\mathbb{Z})^\vee = H_2(J,\mathbb{Z})$ induced by the Beauville–Bogomolov form. We claim that under this embedding, the classes of R and of Θ are equal, ie that $R \cdot x = q(\Theta, x)$ for every $x \in H^2(J,\mathbb{Z})$. This immediately proves the proposition, as it implies that $q(\Theta) = \Theta \cdot R = -2$. By [53, Corollary 3.6(1)] and [27, Proposition 4.5], the class of the ruling of a prime exceptional divisor is proportional, via a positive constant, to the class of the exceptional divisor. Thus, to prove the claim it suffices to show that Θ is prime exceptional, since the constant would have to be equal to 1, as both $R \cdot L$ and $q(\Theta, L)$ are equal to 1.

To prove that Θ is prime exceptional we use standard techniques on deformations of maps from rational curves to hyper-Kähler manifolds, following [53, Section 5.1] or also [19, Section 3]. We include a proof because the setting of Markman is different and because the proof in [19, Section 3] is for projective families of hyper-Kähler manifolds. Choose $R \subset \Theta$ a general element in the ruling and let $\mathrm{Def}(J)_R \subset \mathrm{Def}(J)$ be the smooth hypersurface in the deformation space of J where the class of R stays of Hodge type. Let $\mathcal{H}ilb \to \mathrm{Def}(J)_R$ be the component of the relative Douady space containing the point $[R]$. Since $N_{R|J} = \mathcal{O}_R(-2) \oplus \mathcal{O}_R^{\oplus 8}$, then by [70, Theorem 1] it follows that the morphism $\rho\colon \mathcal{H}ilb \to \mathrm{Def}(J)_R$ is smooth at R and of relative dimension 8. Let $T \subset \mathrm{Def}(J)_R$ be a general curve containing 0 (in particular we can assume that for very general $t \in T$, the Néron–Severi of the corresponding deformation $\mathcal{J}_t$ of J is one-dimensional and spanned by a line bundle whose class is proportional to R_t, the parallel transport of the class of R to $\mathcal{J}_t$) and let $\rho_T : \mathcal{H}ilb_T \to T$ be the component of the base change to T of $\mathcal{H}ilb \to \mathrm{Def}(J)_R$ that contains $[R]$. Since ρ is smooth at $[R]$, ρ_T is dominant of relative dimension 8. Up to a base change and to restricting T, we can assume that $\mathcal{H}ilb_T \to T$ has irreducible fibers for $t \neq 0$. Let $\mathcal{J}_T \to T$ be the base change of the universal family to $T \to \mathrm{Def}(J)$ and let $\mathcal{D} \subset \mathcal{J}_T$ be the image of the universal family over $\mathcal{H}ilb_T$ under the evaluation map. Then $\mathcal{D}$ is irreducible of relative codimension one. Moreover, $\mathcal{D}_t$ is irreducible for $t \neq 0$, and $D := \mathcal{D}_0$ is a union of effective uniruled divisors containing Θ as an irreducible component (with a given multiplicity $m \geq 1$). By the choice of T, for very general t, $\rho(\mathcal{J}_t) = 1$. It follows that the class of $\mathcal{D}_t$ is proportional to the class of R_t and hence that the class of $D = \mathcal{D}_0$ is proportional to that of R. Moreover, the proportionality constant is positive, as both D and R intersect positively with a Kähler class. Hence, since $\Theta \cdot R$ is negative, so is $q(\Theta, D)$. Moreover, since the product of two distinct irreducible uniruled divisors is nonnegative, it follows that $q(\Theta, D) \geq m\, q(\Theta, \Theta)$. Thus $q(\Theta, \Theta) < 0$, ie Θ is prime exceptional. Thus, as already observed, the classes of Θ and of R have to be the same and hence $q(\Theta, \Theta) = -2$. □

Remark 3.7 A posteriori, once we know that Θ is prime exceptional, we can use [53, Lemma 5.1] to show that $\mathcal{D}_0 = \Theta$.

Lemma 3.8 *Let M be a hyper-Kähler manifold of dimension $2n$ and $R \subset M$ be a smooth rational curve. Suppose R is a general ruling of a uniruled divisor. Then*

$$(T_M)_{|R} = \mathcal{O}_R^{\oplus 2n-2} \oplus \mathcal{O}_R(2) \oplus \mathcal{O}_R(-2),$$

and thus

$$N_{R|M} = \mathcal{O}_R(-2) \oplus \mathcal{O}_R^{\oplus 2n-2}.$$

Proof Since T_M is self dual, $(T_M)_{|R} = \bigoplus_i \mathcal{O}_R(a_i) \oplus \bigoplus_i \mathcal{O}_R(-a_i)$, where $a_i \geq 0$. Since R is general and its deformations sweep out a divisor, by [39, Chapter II, Proposition 3.4], the rank of the evaluation map $\mathrm{rk}[H^0(R, (T_M)_{|R}) \otimes \mathcal{O}_R \to (T_M)_{|R}]$ at a general point of R is equal to $2n-1$. Hence $a_2 = \cdots = a_n = 0$ and $a_1 \geq 2$; cf [27, Proposition 4.5]. Since the normal sheaf of R in M is torsion-free and contains the quotient $\mathcal{O}_R(a_1)/T_R = \mathcal{O}_R(a_1)/\mathcal{O}_R(2)$, it follows that $a_1 = 2$. □

Notice that the same argument as the last part of the proof of Proposition 3.6 shows the following.

Proposition 3.9 *Let M be a hyper-Kähler manifold of dimension $2n$ and let $E \subset M$ be an irreducible uniruled divisor. Suppose that a general curve R in the ruling is smooth and that $E \cdot R < 0$, eg if R is contained in the smooth locus of E. Then E is prime exceptional and hence, under the lattice embedding $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{Z})^\vee = H_2(M, \mathbb{Z})$ induced by the Beauville–Bogomolov form, the classes of E and R are proportional by a positive constant.*

Corollary 3.10 *For very general X, the movable cone of $J(X)$ is spanned by L and H, where H is a generator of $\Theta^\perp \subset \mathrm{NS}(J)$ with $q(H, L) > 0$ and $q(H) > 0$; ie*

$$\mathrm{Mov}(J) = \mathbb{R}_{\geq 0} L + \mathbb{R}_{\geq 0} H.$$

In particular, there is a unique hyper-Kähler model of J with a Lagrangian fibration, and J is not birational to the twisted intermediate Jacobian fibration J^T.

Proof We already know that one of the rays of the movable cone of J is spanned by L. By [53, Theorem 1.5] the closure of the movable cone is spanned by classes that intersect nonnegatively with all prime exceptional divisors. Since by Proposition 3.6, Θ is prime exceptional, the second ray of the movable cone is determined by $\Theta^{\perp}$, which is spanned by a class H which is big and nef on some birational hyper-Kähler model of J. Thus, $q(H) > 0$ and $q(H, L) > 0$. In particular, the movable cone is strictly contained in the positive cone, implying that the only isotropic class that is movable is L. □

In terms of the other projective hyper-Kähler birational models of J, we can actually prove something more precise. The main result of Section 4 describes, for general X, which birational model of J the proper transform of Θ can be contracted on.

3.1 Induced automorphisms

For hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$–type, a considerable amount of literature has been devoted to the study and classification of automorphism groups. This includes studying the automorphisms induced from a K3 surface to the moduli spaces of sheaves on it. In view of Theorem 1.6, a natural question is to study the induced action on J of the automorphism group of X in relation to the Lagrangian fibration structures. I thank G Pearlstein for asking questions that led me to the following observations.

Let X be a smooth cubic fourfold and let τ be an automorphism of X. Then τ acts on the universal family of hyperplane sections of X and thus also on the Donagi–Markman fibration $J_U \to U$, which is identified with the relative Pic^0 of the family of Fano surfaces of the hyperplane sections of X. By abuse of notation we denote by

$$\tau\colon J \dashrightarrow J$$

the induced birational morphism. Notice that τ preserves Θ and L, so the induced action of τ^* is the identity on $U = \langle L, \Theta\rangle \subset \mathrm{NS}(J)$.

Proposition 3.11 *Let X be a smooth cubic fourfold and suppose that the fibers of $\pi\colon J \to \mathbb{P}^5$ are irreducible.* (By [47] this happens for general X.) *Then*:

(1) *Θ is π–ample and so is any $B \in \mathrm{NS}(J)$ with $q(L, B) > 0$.*

(2) *Any birational automorphism $\tau\colon J \dashrightarrow J$ which fixes $L = \pi^*\mathcal{O}(1)$ extends to a regular automorphism.*

(3) *L is nef on a unique hyper-Kähler birational model of J. In other words, if J' is a birational hyper-Kähler model of J, with birational map $f\colon J' \dashrightarrow J$, and the induced map $\pi'\colon J' \dashrightarrow J \to \mathbb{P}^5$ is regular, then f is an isomorphism.*

Proof (1) Let H be an ample line bundle on J and let J_t be a smooth fiber of $J \to \mathbb{P}^5$. Then $[H_{|J_t}] = m[\Theta_{|J_t}]$ for a positive integer m, so the restrictions of H and $m\Theta$ are topologically equivalent for any smooth fiber. Since the fibers of π are irreducible, it follows that the restrictions of H and $m\Theta$ to any fiber are numerically equivalent; see [80, Lemma 4.4]. By Nakai–Moishezon, $m\Theta$ is π–ample. Similarly, if $q(B,L) > 0$, then there exists positive integers a and b such that aB and $b\Theta$ are numerically equivalent on every fiber.

(2) By assumption, $\tau^*L = L$ so $q(\tau^*\Theta, L) = q(\Theta, L) = 1$. As a consequence, $\tau^*\Theta$ and Θ are topologically equivalent on the smooth fibers and hence, as above, numerically equivalent on every fiber. Thus, $\tau^*\Theta$ is π–ample. It follows that τ is a regular morphism.

(3) Let H' be any ample line bundle on J' and let $L' = f^*L = \pi'^*\mathcal{O}(1)$. Then $0 < q(L', H') = q(L, f^*H')$, so by (1) f^*H' is ample and f is an isomorphism. □

In addition to birational automorphisms induced by the automorphisms of X, some examples of birational automorphisms which preserve L are:

(1) The map $\iota\colon J \to J$ induced by the action of (-1) on the smooth fibers of $J \to \mathbb{P}^5$.

(2) The map $t_\alpha\colon J \to J$ induced by the translation of a rational section of $\alpha\colon \mathbb{P}^5 \dashrightarrow J$; cf Section 5.

(3) More generally, any birational automorphism induced by an element of the automorphism group of J_K, the generic fiber of $J \to \mathbb{P}^5$.

Remark 3.12 As already mentioned just below Theorem 1.1, a necessary condition for the irreducibility of the fibers of $J \to \mathbb{P}^5$ is given in [17]. This condition is satisfied if and only if the hyperplane sections Y of X satisfy

$$d(Y) := b_2(Y) - b_4(Y) = 0,$$

where $b_i(Y)$ denotes the i^{th} Betti number of Y and where $d(Y)$ is called the defect of Y. It is easy to see that if Y contains a plane then $d(Y) > 0$.

4 Birational geometry of $J(X)$ for general X

To describe the birational geometry of the intermediate Jacobian fibration we degenerate the underlying cubic to the chordal cubic, following an idea already contained in [41]. There, it was observed that the central fiber of the corresponding family of intermediate Jacobian fibrations can be chosen to be birational to a moduli space of sheaves of OG10–type on a K3 surface of genus two. As in Section 2, by moduli space of OG10–type we mean a moduli space of sheaves on a K3 surface with Mukai vector $2w$, with $w^2 = 2$. We first refine the construction of this degeneration in order to have a central fiber that is actually *isomorphic* to a certain singular moduli space of sheaves on the associated K3 surface. In this way, we can keep track of the limits of the relative theta divisor and of the line bundle inducing the Lagrangian fibration. This is done in Section 4.2. The results of Meachan and Zhang [58], which were recalled in Lemma 2.6, imply that the central fiber of the relative theta divisor can be contracted after a Mukai flop of the zero section. For X general, we then deduce the same result for $J(X)$ and, for very general X, we compute the nef and movable cone of $J(X)$. This is the content of Theorem 4.1.

Theorem 4.1 *Let X be a smooth cubic fourfold and let $J = J(X) \to \mathbb{P}^5$ be a hyper-Kähler compactification of the intermediate Jacobian fibration as in Section 1.*

For very general X:

(1) *There is a unique other hyper-Kähler birational model of J, denoted by N, which is the Mukai flop $p\colon J \dashrightarrow N$ of J along the image of the zero section.*

(2) *There is a divisorial contraction $h\colon N \to \bar{N}$ which contracts the proper transform of Θ onto an 8–dimensional variety which is birational to the LLSvS 8–fold $Z(X)$.*

In other words, we have $\operatorname{Mov}(J) = \langle L, H\rangle = \operatorname{Nef}(J) \cup p^* \operatorname{Nef}(N)$, $\operatorname{Nef}(J) = \langle L, H_0\rangle$ and $p^* \operatorname{Nef}(N) = \langle H_0, H\rangle$, where H_0 is a big and nef line bundle on J which contracts the zero section of $J \to \mathbb{P}^5$ and H is as in Corollary 3.10.

For general X, the relative theta divisor Θ can be contracted after the Mukai flop of the zero section of $J \to \mathbb{P}^5$.

Before the proof of the theorem, which will be given in Section 4.2, we mention, as a consequence of the theorem above, the relation between the intermediate Jacobian fibration and moduli spaces of objects in the Kuznetsov component of X.

4.1 Comparison with moduli spaces of objects in the Kuznetsov component of X

The recent paper [6] establishes the existence and the fundamental properties of moduli spaces of objects in the Kuznetsov component $\mathcal{K}u(X)$ of a smooth cubic fourfold X. We refer the reader to Section 29 of loc. cit. for the relevant definitions and the precise statements of the results.

Given a smooth cubic fourfold X, the extended Mukai lattice $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ is a lattice whose underlying group is the topological K–theory of $\mathcal{K}u(X)$ and whose Mukai pairing and weight-two Hodge structure are induced from those on X. The only classes in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ that are of type $(1,1)$ for very general X are contained in a rank-two lattice A_2, which is spanned by two classes λ_1 and λ_2 that satisfy $\lambda_1^2 = \lambda_2^2 = 2$ and $\lambda_1 \cdot \lambda_2 = -1$; see [6, equation (29.1)]. A description of a full connected component of the space of Bridgeland stability conditions on $\mathcal{K}u(X)$ is also produced; see Theorem 29.1 of loc. cit. It is shown that, for a primitive Mukai vector with $v^2 \geq -2$ and for a v–generic stability condition σ in this component, the moduli space $M_\sigma(\mathcal{K}u(X), v)$ of Bridgeland stable objects in $\mathcal{K}u(X)$ with Mukai vector v is a nonempty smooth projective hyper-Kähler manifold of dimension v^2+2, deformation equivalent to a Hilbert scheme of points on a K3 surface; moreover, the formation of these moduli spaces works in families; see Theorem 29.4 of loc. cit. for the precise statement.

For a Mukai vector of OG10–type in the A_2 lattice, ie of the form $v = 2\lambda$ with $\lambda^2 = 2$, in [51] it is shown that, for a λ–generic stability condition σ, the moduli space $M_\sigma(\mathcal{K}u(X), v)$ is an irreducible normal projective symplectic variety of dimension 10 admitting a symplectic resolution which is deformation equivalent to a manifold of OG10–type. The genericity condition here means that the polystable objects with Mukai vector v are the direct sum of two stable objects with Mukai vector λ. More precisely, the singular locus of $M_\sigma(\mathcal{K}u(X), v)$ is isomorphic $\operatorname{Sym}^2 M_\sigma(\mathcal{K}u(X), \lambda)$.

Moreover, in [51] it is shown that for general X the twisted intermediate Jacobian fibration $J^T(X)$ is birational to $M_\sigma(\mathcal{K}u(X), 2\lambda)$, for $\lambda^2 = 2$. For the nontwisted case we have the following corollary of Theorem 4.1 that goes in the opposite direction.

Corollary 4.2 *For very general X, $J(X)$ is not birational to a moduli space of the form $M_\sigma(\mathcal{K}u(X), v)$.*

Proof First of all, by [6, Remark 29.3], if nonempty, the dimension of a moduli space $M_\sigma(\mathcal{K}u(X), v)$ is v^2+2. This dimension is equal to 10 if and only if either

v is primitive (hence $M_\sigma(\mathcal{K}u(X), v)$ is of K3$^{[5]}$–type and thus cannot be birational to $J(X)$) or else $v = 2\lambda$ with $\lambda^2 = 2$. By the results of [51] cited in the remark above, for $v = 2\lambda$ with $\lambda \in A_2$ and $\lambda^2 = 2$ and σ a λ–generic stability condition, the singular locus of $M_\sigma(\mathcal{K}u(X), v)$ is isomorphic to the second symmetric product of a hyper-Kähler manifold of K3$^{[2]}$–type. By dimension reasons, the symplectic resolution $\widetilde{M}_\sigma(\mathcal{K}u(X), v) \to M_\sigma(\mathcal{K}u(X), v)$ is not a small contraction. Suppose by contradiction that $J(X)$ is birational to $\widetilde{M}_\sigma(\mathcal{K}u(X), v)$. Then by Theorem 4.1, the symplectic resolution has to coincide with $N \to \bar{N}$, and $M_\sigma(\mathcal{K}u(X), v) \cong \bar{N}$. This implies that the singular locus of $M_\sigma(\mathcal{K}u(X), v)$ has to be birational to the Lehn–Lehn–Sorger–van Straten 8–fold $Z(X)$, which gives a contradiction. Indeed, $Z(X)$ cannot be birational to $\mathrm{Sym}^2 M_\sigma(\mathcal{K}u(X), \lambda)$ since, by Proposition 1.7, this would imply that the latter has a symplectic resolution. This, however, is not true because $\mathrm{Sym}^2 M_\sigma(\mathcal{K}u(X), \lambda)$ is a $\mathbb{Q}$–factorial symplectic variety with singular locus of codimension strictly greater than two and hence does not admit a symplectic resolution (since it does not admit a semismall resolution). □

Remark 4.3 We expect the more general statement to hold: for very general X, $J(X)$ is not birational to a Bridgeland moduli space of objects on a 2–CY category that is deformation equivalent to the derived category of a K3 surface. We present a rough sketch of the argument. Assume there is a family of Bridgeland stability conditions on the family of derived categories realizing the deformation. Then, as in [6, Theorem 21.24], a relative moduli space exists as an algebraic space; by a generalization of a theorem of Mukai [68, Theorem 1.4], the stable locus of each fiber is smooth and has a holomorphic symplectic form; the singular locus parametrizing strictly semistable objects of codimension ≥ 2. One then expects such moduli spaces to be normal and irreducible. As in the proof of the projectivity in [6, Theorem 29.4] it follows that these moduli spaces are projective. Finally, a similar argument to the one above shows that the contraction $N \to \bar{N}$ cannot be the symplectic resolution of one of these moduli spaces.

In the next subsection we construct the degeneration of the intermediate Jacobian fibration that will allow us to prove Theorem 4.1. The proof of the theorem will be given at the end of the section.

4.2 Degeneration to the chordal cubic

The secant variety to the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ is a cubic hypersurface isomorphic to $\mathrm{Sym}^2 \mathbb{P}^2$, called the chordal cubic. Such a singular cubic fourfold is unique up to the action of the projective linear group. Given a one-parameter family of cubic fourfolds degenerating to the chordal cubic, it was proved in [41] that, up to a base change, one can fill the corresponding degeneration of intermediate Jacobian fibrations with a smooth central fiber that is birational to $\widetilde{M}_{2v_0} = \widetilde{M}_{2v_0}(S)$, where (S, C) is the degree-two $K3$ surface associated to the degeneration of cubic fourfolds as in [22; 34; 46], and where $v_0 = (0, C, -2)$ is as in (2-2). We will use this degeneration to study the birational properties of the intermediate Jacobian fibration, at least for general X. For this purpose, we need to control what happens to the line bundles L and Θ under the corresponding degeneration of intermediate Jacobian fibrations. We achieve this by constructing a particular degeneration whose central fiber is precisely the singular moduli space M_{2v_0} and is such that the Lagrangian fibrations of the members of this degeneration fit in a relative Lagrangian fibration. This is done in Proposition 4.5. With this degeneration, we are not only able to identify precisely the limits of L and Θ (see Lemma 4.7), but we are also able to deform the results about the birational geometry of M_{2v_0} away from the central fiber (see Proposition 4.9), eventually proving Theorem 4.1.

Let $\mathcal{X} \to \Delta$ be a one-parameter family of cubic fourfolds degenerating to the chordal cubic. By this we will mean that Δ is a small disk or an open affine subset in the base of a pencil of cubic fourfolds with the property that the general fiber is smooth and the central fiber is isomorphic to the chordal cubic. The following facts were proved in [34], see also [46] and [41]:

(a) The monodromy of this family has order two.

(b) To such a degeneration one can associate a degree-two polarized K3 surface (S, C).

(c) For a general pencil, the polarized K3 surface (S, C) is general in moduli.

Suppose that for $t \neq 0$ the cubic fourfold $\mathcal{X}_t$ is general in the sense of LSV — ie in the sense that the construction of the hyper-Kähler compactification of [47] works for $J_U(\mathcal{X}_t)$ — and let $\mathcal{J}^* \to \Delta^*$ be the family of intermediate Jacobians associated to the smooth locus $\mathcal{X}^* \to \Delta^*$ of the pencil, with corresponding family of Lagrangian fibrations $\pi_{\Delta^*} : \mathcal{J}^* \to \mathbb{P}^5_{\Delta^*}$.

Lemma 4.4 [22; 41] *Up to a degree-two base change, we can extend $\pi_{\Delta^*} : \mathcal{J}^* \to \mathbb{P}^5_{\Delta^*}$ to a projective morphism $\pi_{\mathcal{V}} : \mathcal{J}_{\mathcal{V}} \to \mathcal{V}$, where $\mathcal{V} \subset \mathbb{P}^5 \times \Delta$ is an open subset such that $\mathcal{V}_t = \mathbb{P}^5$ for $t \neq 0$ and $\mathcal{V}_0 \subset \mathbb{P}^5$ is nonempty for $t = 0$, and where $\mathcal{J}_0 \to \mathcal{V}_0 \subset \mathbb{P}^5$ is* *identified with the restriction of* $M_{2v_0}(S) \to |2C| = \mathbb{P}^5$ (cf (2-3)) *to an open subset* $V \subset |2C|$. *Moreover,* $\mathcal{J}_{\mathcal{V}} \to \mathcal{V}$ *has a zero section and is polarized by a relative principal polarization.*

Proof Let $H \subset \mathbb{P}^5$ be a general hyperplane. For the degeneration $\mathcal{Y} := \mathcal{X} \cap (H \times \mathbb{P}^5)$ of a single smooth cubic threefold the statement is due to Collino [22]. In Proposition 1.16 of loc. cit., it is also shown that the class of the limit polarization is the theta divisor of the Jacobian of the genus-five hyperelliptic curve, which is the limiting abelian variety. For the statement about the limit of the intermediate Jacobian fibration, this is [41, Section 6.3]. □

We now compactify the projective family $\mathcal{J}_{\mathcal{V}}$ of the lemma above to construct a family of Lagrangian fibered holomorphic symplectic varieties in such a way that the central fiber is exactly $M_{2v_0} = M_{2v_0}(S)$ (or $\widetilde{M}_{2v_0} = \widetilde{M}_{2v_0}(S)$); cf (2-3).

Proposition 4.5 *Let* $\mathcal{X} \to \Delta$ *be as above a general family of smooth cubic fourfolds degenerating to the chordal cubic. Suppose that for very general* $t \in \Delta$, $\mathcal{X}_t$ *is very general. Let* (S, C) *be the corresponding K3 surface of degree two as above. Then, possibly up to a base change, there are two degenerations of the corresponding intermediate Jacobian fibration, fitting in the commutative diagram*

(4-1)
$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "m"] \arrow[dr, "\tilde{f}"'] & \mathcal{M} \arrow[d, "f"] \\ & \Delta \end{tikzcd}$$

where:

(1) $\tilde{f} \colon \widetilde{\mathcal{M}} \to \Delta$ *is a family of smooth hyper-Kähler manifolds, with* $\widetilde{\mathcal{M}}_t = J(\mathcal{X}_t)$ *for* $t \neq 0$ *and* $\widetilde{\mathcal{M}}_0 = \widetilde{M}_{v_0}(S)$. *The family is equipped with a relative Lagrangian fibration* $\widetilde{\mathcal{M}} \to \mathbb{P}^5_\Delta$, *where for each* t *the corresponding Lagrangian fibration is the obvious one.*

(2) $f \colon \mathcal{M} \to \Delta$ *is a degeneration of hyper-Kähler manifolds, with* $\mathcal{M}_t = J(\mathcal{X}_t)$ *for* $t \neq 0$ *and* $\mathcal{M}_0 = M_{v_0}(S)$. *The morphism* $m \colon \widetilde{\mathcal{M}} \to \mathcal{M}$ *is proper and birational, for* $t \neq 0$ *it is an isomorphism and for* $t = 0$ *it is the natural symplectic resolution* $m_0 \colon \widetilde{M}_{2v_0}(S) \to M_{2v_0}(S)$ *of Theorem 2.1. Moreover, there is a relative Lagrangian fibration* $\mathcal{M} \to \mathbb{P}^5_\Delta$ *where for each* t *the corresponding Lagrangian fibration is the obvious one.*

Proof Start from the projective morphism $\pi_{\mathcal{V}}\colon \mathcal{J}_{\mathcal{V}} \to \mathcal{V}$ of Lemma 4.4. There is an isomorphism $\mathcal{J}_{\mathcal{V}_0} \cong (\widetilde{M}_{2v_0})_V$, where $(\widetilde{M}_{2v_0})_V$ is the restriction of the Lagrangian fibration $\widetilde{\pi}\colon \widetilde{M}_{2v_0} \to \mathbb{P}^5$ (cf (2-3)) to an open subset $V \subset \mathbb{P}^5$. Let $\overline{\mathcal{J}_{\mathcal{V}}} \to \mathbb{P}^5_\Delta$ be any projective morphism extending $\pi_{\mathcal{V}}$. Applying Theorem 1.19(2) to $\overline{\mathcal{J}_{\mathcal{V}}} \to \mathbb{P}^5_\Delta$ yields, possibly up to the base change, a family $\widetilde{g}\colon \widetilde{\mathcal{J}} \to \Delta$ of smooth hyper-Kähler manifolds (projective over $(\Delta)^*$), with a relative Lagrangian fibration $\widetilde{\mathcal{J}} \to \mathbb{P}^5_\Delta$. Let $\mathcal{L}$ be the line bundle on $\widetilde{\mathcal{J}}$ inducing it on every fiber. Let

$$\phi_0\colon \widetilde{\mathcal{J}}_0 \dashrightarrow \widetilde{M}_{2v_0} \tag{4-2}$$

be the birational morphism induced by the isomorphism of open subsets $\mathcal{J}_{\mathcal{V}_0} \cong (\widetilde{M}_{2v_0})_V$. Then $(\phi_0)_*\mathcal{L}_0 = \widetilde{\ell} := \widetilde{\pi}^*\mathcal{O}_{\mathbb{P}^5}(1)$.

We now use an argument very similar to that in the proof of [41, Theorems 1.3 and 1.7], to construct a family which is isomorphic to $\mathcal{J}$ over Δ and whose central fiber is actually *isomorphic* to $\widetilde{M}_{2v_0}(S)$. Let Λ be the OG10 lattice. Fixing a marking of the central fiber and trivializing the local system $R^2\widetilde{g}_*\mathbb{Z}$ induces a marking $\eta_t\colon H^2(\widetilde{\mathcal{J}}_t, \mathbb{Z}) \to \Lambda$ of every fiber. Let $\mathcal{D} \subset \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$ be the period domain and let $\mathcal{P}\colon \Delta \to \mathcal{D}$ be the period mapping induced by these markings. Let $\rho_0 = \eta_0(\phi_0)^*\colon H^2(\widetilde{M}_{2v_0}, \mathbb{Z}) \to \Lambda$ be the induced marking on $\widetilde{M}_{2v_0}$. Let $\rho_t\colon H^2(\widetilde{\mathcal{M}}_t, \mathbb{Z}) \to \Lambda$ be markings induced by $\rho_0 = \eta_0(\phi_0)^*$ on fibers of the universal family over $\mathrm{Def}(\widetilde{M})$ and let $\mathcal{P}_{\widetilde{M}}\colon \mathrm{Def}(\widetilde{M}) \to \mathcal{D}$ be the induced period mapping. Since $\mathcal{P}_{\widetilde{M}}$ is a local isomorphism, we can lift $\mathcal{P}$ to a map $\xi\colon \Delta \to \mathrm{Def}(\widetilde{M}_{2v_0})$. Pulling back the universal family gives a family $\widetilde{f}\colon \widetilde{\mathcal{M}} \to \Delta$ with central fiber $\widetilde{\mathcal{M}}_0 = \widetilde{M}_{2v_0}$. As in [41] the two families $\widetilde{g}\colon \widetilde{\mathcal{J}} \to \Delta$ and $\widetilde{f}\colon \widetilde{\mathcal{M}} \to \Delta$ are relatively birational over Δ, since for every $t \in \Delta$, the marked pairs $(\widetilde{\mathcal{J}}_t, \eta_t)$ and $(\widetilde{\mathcal{M}}_t, \rho_t)$ are nonseparated points. To show that the two families $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{M}}$ are isomorphic away from the central fiber, first recall that by [35, Theorem 4.3] (cf also [52, Theorem 3.2]), for every t there exists an effective cycle

$$\Gamma_t = Z_t + \sum W_{i,t}$$

of pure dimension 10 in $\widetilde{\mathcal{M}}_t \times \widetilde{\mathcal{J}}_t$ such that Z_t is the graph of a birational map, the codimension of the images of the $W_{i,t}$ in $\widetilde{\mathcal{M}}_t$ and in $\widetilde{\mathcal{J}}_t$ are equal and positive, and $[\Gamma_t]_*$ is a Hodge isometry and is equal to $\rho_t^{-1} \circ \eta_t\colon H^2(\widetilde{\mathcal{J}}_t, \mathbb{Z}) \to H^2(\widetilde{\mathcal{M}}_t, \mathbb{Z})$. Let $\widetilde{\mathcal{L}}$ be the line bundle on $\mathcal{M}$ such that $\widetilde{\mathcal{L}}_t = \rho_t^{-1}\eta_t(\mathcal{L}) = [\Gamma_t]_*(\mathcal{L}_t)$. Since $\widetilde{\mathcal{L}}_0 = \widetilde{\pi}^*\mathcal{O}_{\mathbb{P}^5}(1)$ induces a Lagrangian fibration on $\widetilde{\mathcal{M}}_0 = \widetilde{M}_{2v_0}$, by [57] $\widetilde{\mathcal{L}}$ induces a Lagrangian fibration on $\mathcal{M}_t$ for every t (maybe up to restricting Δ). For very general t, $\widetilde{\mathcal{L}}_t = [Z_t]_*(\mathcal{L}_t)$, since the isotropic class $[Z_t]_*(\mathcal{L}_t)$ lies in the movable cone of $\widetilde{\mathcal{M}}_t$ and hence by

Corollary 3.10 it has to be equal to $\widetilde{\mathcal{L}}_t$. Corollary 3.4 implies that for very general t, Z_t is the graph of an *isomorphism* between $\widetilde{\mathcal{J}}_t$ and $\widetilde{\mathcal{M}}_t$. The same countability argument as in the proof of [41, Theorems 1.3 and 1.7] shows that there exists a component of the Hilbert scheme parametrizing graphs of such cycles $Z_t \subset \mathcal{J}_t \times \mathcal{M}_t$ that dominates Δ. It follows that there is a cycle $\mathcal{Z}$ in the fiber product $\widetilde{\mathcal{J}} \times_\Delta \widetilde{\mathcal{M}}$ which, maybe up to restricting Δ, induces an isomorphism for $t \neq 0$. The conclusion is that the family $\widetilde{\mathcal{M}} \to \Delta$ is such that central fiber is $\widetilde{\mathcal{M}}_0 \cong \widetilde{M}_{2v_0}$ while for $t \neq 0$, we have $\widetilde{\mathcal{M}}_t \cong J(\mathcal{X}_t)$.

Now we construct the second family. By [62, Theorem 2.2] there is a finite morphism

$$\Xi : \mathrm{Def}(\widetilde{M}_{2v_0}) \to \mathrm{Def}(M_{2v_0})$$

induced by the symplectic resolution $m_0\colon \widetilde{M}_{2v_0} \to M_{2v_0}$ and compatible with the universal families on the two deformation spaces; for more details see Section 2 of loc. cit. Set $\nu = \Xi \circ \xi\colon \Delta \to \mathrm{Def}(M_{2v_0})$ and let

$$\mathcal{M} \to \Delta$$

be the pullback via ν of the universal family on $\mathrm{Def}(M_{2v_0})$. Then the birational map $m\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ over Δ induced by [62, Theorem 2.2] has the desired properties.

Finally, the statement about the Lagrangian fibrations follows from the fact that, since the Lagrangian fibration $\widetilde{M}_{2v_0} \to \mathbb{P}^5$ in the central fiber factors via $\widetilde{M}_{2v_0} \to M_{2v_0}$, the morphism $\widetilde{\mathcal{M}} \to \mathbb{P}^5_\Delta$ factors via a morphism $\mathcal{M} \to \mathbb{P}^5_\Delta$. $\square$

As a consequence of the last part of the proof, notice that there is a line bundle $\mathcal{L}_\mathcal{M}$ on $\mathcal{M}$ with

$$m^*\mathcal{L}_\mathcal{M} = \widetilde{\mathcal{L}}$$

and whose restriction to the central fiber satisfies $\mathcal{L}_{\mathcal{M}_0} = \ell$, where ℓ is as in (2-5).

For any $t \neq 0$, let Θ_t be the relative theta divisor in $\mathcal{M}_t = J(\mathcal{X}_t)$.

Lemma 4.6 *For $\star = \widetilde{\mathcal{M}}$ or $\mathcal{M}$, let $\Theta_\star$ be the divisor defined as the closure of $\bigcup_{t\neq 0} \Theta_t$ in $\star$. Then, $\Theta_\mathcal{M}$ is a Cartier divisor and hence the following compatibility conditions hold* (notation as in diagram (4-1)):

$$\Theta_{\widetilde{\mathcal{M}}_0} \cong (m^*\Theta_\mathcal{M})_{|\widetilde{\mathcal{M}}_0} = m_0^*\Theta_{\mathcal{M}_0}, \tag{4-3}$$

where $\Theta_{\star_0} := (\Theta_\star)_{|0}$ is the fiber of $\Theta_\star$ over $t = 0$.

Proof Let $\mathcal{I}_{\Theta_{\mathcal{M}}} \subset \mathcal{O}_{\mathcal{M}}$ be the ideal sheaf of $\Theta_{\mathcal{M}}$ in $\mathcal{M}$. Since the morphism $\Theta_{\mathcal{M}} \to \Delta$ is flat, it follows that the restriction $(\mathcal{I}_{\Theta_{\mathcal{M}}})_{|\mathcal{M}_0}$ is the ideal sheaf of $\Theta_{\mathcal{M}_0}$ in $\mathcal{M}_0$. By [66] (cf Section 2.2), $\mathcal{M}_0 = M_{2v_0}$ is factorial so $(\mathcal{I}_{\Theta_{\mathcal{M}}})_{|\mathcal{M}_0}$ is locally free. Hence, so is $\mathcal{I}_{\Theta_{\mathcal{M}}}$. It follows that the divisors $\Theta_{\widetilde{\mathcal{M}}}$ and $m^*\Theta_{\mathcal{M}}$ agree and so do their central fibers. □

The next lemma identifies the limit of Θ_t in $M_{2v_0} = \mathcal{M}_0$ and shows that all line bundles on M_{2v_0} deform over $\mathcal{M} \to \Delta$. Recall first that by (2-1), $\mathrm{NS}(M_{2v_0}) = U = \langle \ell, \theta \rangle$, and for every t,

$$\mathrm{NS}(\mathcal{M}_t) \supseteq U_t = \langle \mathcal{L}_t, \Theta_t \rangle,$$

with equality holding for very general t. Here $\Theta_0 = \Theta_{\widetilde{\mathcal{M}}_0}$. In particular, inside $\mathrm{NS}(\widetilde{M}_{2v_0})$ we have the following rank-two sublattices both of which are isomorphic to the hyperbolic lattice U: the limit lattice U_0 spanned by the limits $\mathcal{L}_0 = \widetilde{\ell}$ and Θ_0, and the pullback lattice $m_0^* \mathrm{NS}(M_{2v_0}) = \langle m_0^*\ell, m_0^*\theta \rangle$.

Lemma 4.7 *Let the notation be as above. Then*:

(1) *The two sublattices $U_0 = \langle \widetilde{\ell}, \Theta_{\widetilde{\mathcal{M}}_0} \rangle$ and $\langle m_0^*\ell, m_0^*\theta \rangle$ of $\mathrm{NS}(\widetilde{M}_{2v_0})$ are the same.*

(2) *The limit of the relative theta divisor in $\mathcal{M}_0$ is precisely θ, the relative theta divisor on $M_{2v_0}(S)$ of (2-6).*

Proof By Lemmas 4.4 and 4.6, the limit theta divisor $(\Theta_{\mathcal{M}})_0$ is an effective line bundle on $\mathcal{M}_0 = M_{2v_0}$, which restricts to a theta divisor on the smooth fibers of $M_{2v_0} \to \mathbb{P}^5$. Thus $\Theta_{\mathcal{M}_0}$ is linearly equivalent to an effective line bundle of the form $\theta + a\ell$ for some integer a. We show that $a = 0$. By (4-3), $\Theta_{\widetilde{\mathcal{M}}_0} = m_0^*\Theta_{\mathcal{M}_0} = m_0^*(\theta + a\ell)$ and $\widetilde{\mathcal{L}}_0 = m_0^*\ell$. This is enough to conclude that the two sublattices

$$U = \langle \Theta_{\widetilde{\mathcal{M}}_0}, \widetilde{\mathcal{L}}_0 \rangle \quad \text{and} \quad U = m_0^*\langle \theta, \ell \rangle = m_0^* \mathrm{NS}(M_{2v_0})$$

of $\mathrm{NS}(\widetilde{M}_{2v_0})$ are the same. This proves the first part of the lemma. By Remark 2.5 above, θ does not contain the singular locus of M, thus $m_0^*\theta$ coincides with its proper transform and is irreducible. Since it has negative Beauville–Bogomolov square (cf (2-7)), it is a prime exceptional divisor. By [53, Section 5.1], a prime exceptional divisor deforms where its first Chern class remains algebraic. Thus $m_0^*\theta$ deforms to a relative effective prime exceptional divisor $\theta_{\widetilde{\mathcal{M}}}$ on $\widetilde{\mathcal{M}}$. By Corollary 3.4 and Proposition 3.6, for very general $t \neq 0$, the fiber over t of the two irreducible effective divisors $\theta_{\widetilde{\mathcal{M}}}$ and $\Theta_{\widetilde{\mathcal{M}}}$ have to agree since there is only one prime exceptional divisor on $\mathcal{M}_t$. Thus $\theta_{\widetilde{\mathcal{M}}}$ and $\Theta_{\widetilde{\mathcal{M}}}$ have to be equal for every t. In particular, so are their restrictions to the central fiber. □

Corollary 4.8 *Let X be a general cubic fourfold and let $\pi\colon J(X) \to \mathbb{P}^5$ be the intermediate Jacobian fibration of [47]. The natural rational zero section of π is regular.*

Proof Consider a degeneration of cubic fourfolds to the chordal cubic as in Proposition 4.5 and let $\mathcal{M} \to \Delta$ be the corresponding family. By Lemma 4.6 the divisor $\Theta_{\mathcal{M}}$ is Cartier and by Remark 2.8 it is relatively ample (up to restricting Δ). Since $\Theta_{\mathcal{M}}$ is -1–invariant, it follows that the birational involution -1 is biregular. One component of the fixed locus of this involution has the property that its restriction to every fiber is precisely the closure of the corresponding rational zero section. Since by Remark 2.4, in the central fiber the section is regular, it follows that for general $t \in \Delta$ the corresponding rational section is also regular. □

Consider the family $\mathcal{M} \to \Delta$ of Proposition 4.5, with its relative theta divisor $\Theta_{\mathcal{M}}$. By Druel [26] we know that for every t, the prime exceptional divisor Θ_t can be contracted on a hyper-Kähler projective birational model of $\mathcal{M}_t$. In the central fiber $\mathcal{M}_t = M_{2v_0}$ we have, by Lemma 4.7, that $\Theta_{\mathcal{M}_0} = \theta$. By Lemma 2.6 this divisor can be contracted after a Mukai flop. We now show that the same is true for any $t \neq 0$, namely, that after a Mukai flop the relative theta divisor can be contracted, possibly up to restricting Δ.

Proposition 4.9 *For general X, the relative theta divisor Θ on $J = J(X)$ can be contracted after the Mukai flop of the zero section.*

Proof Let $\mathcal{M} \to \mathbb{P}^5_\Delta$ be as in Proposition 4.5. By Corollary 4.8, there is a relative zero section $s\colon \mathbb{P}^5_\Delta \to \mathcal{M}$. Let T be its image. Then T is contained in the smooth locus of the fibers of $\widetilde{\mathcal{M}} \to \Delta$. Let

$$P\colon \mathcal{M} \dashrightarrow \mathcal{N}$$

be the relative Mukai flop of T in $\mathcal{M}$. By Lemma 2.6, the Mukai flop of the zero section in the central fiber M_{2v_0} can be performed in the projective category. Thus, the central fiber of $\mathcal{N}$ is projective and so are all the fibers of $g\colon \mathcal{N} \to \Delta$ (since by Lemma 4.7 there is an ample class on the central fiber that deforms over Δ). For $t \neq 0$, $\mathcal{N}_t$ is smooth while the central fiber $\mathcal{N}_0$ has the same singularities as $\mathcal{M}_0 = M_{2v_0}$, since they are isomorphic away from the flopped locus, which does not meet the singular locus. Via the birational morphism P, which is a relative isomorphism in codimension one, we can identify the second integral cohomology group of the fibers of the two families. In particular, for every $t \in \Delta$ we have $P_* U_t \subset \mathrm{NS}(\mathcal{N}_t)$, with equality holding for very general t and for $t = 0$. In what follows we freely restrict Δ, if necessary, without any mention. As in Lemma 2.6, let H be the big and nef line bundle on $\mathcal{N}_0$ that contracts θ, ie H is a generator of the ray $\theta^{\perp}$. Since $H \in P_* U_0$, by Lemma 4.7, H deforms to a line bundle $\mathcal{H}$ on $\mathcal{N}$. For very general t, its restriction $\mathcal{H}_t$ is a generator of the one-dimensional space $(P_t)_* \Theta_t^{\perp} \subset \mathrm{NS}(\mathcal{N}_t)$. By [27], $(P_t)_* \Theta_t$ can be contracted on a birational model of $\mathcal{N}_t$. We now show that it can be contracted on $\mathcal{N}_t$ itself. For very general t, the line bundle inducing the divisorial contraction has to be $\mathcal{H}_t$, or rather its proper transform on an appropriate birational model of $\mathcal{N}_t$. It follows that for very general t (and thus for all t) $\mathcal{H}_t$ is big. Moreover, since $\mathcal{H}_0$ is big and nef and $\mathcal{N}_0$ has rational singularities, $H^i(\mathcal{N}_0, \mathcal{H}_0^k) = 0$ for $i > 0$ and any $k \geq 0$. It follows that the locally free sheaf $g_* \mathcal{H}^k$ satisfies base change. Since $\mathcal{H}_0$ is semiample, so is $\mathcal{H}_t$ for all t in Δ. For $k \gg 0$, the regular morphism $\Psi \colon \mathcal{N} \to \mathbb{P}(g^* g_* \mathcal{H}^k)$, relative over Δ, is birational onto its image and contracts Θ_t for very general t and for $t = 0$. Up to further restricting t, we can assume that the locus contracted on $\mathcal{N}_t$ is irreducible, and hence that Ψ_t contracts precisely $(P_t)_* \Theta_t$ for every t. □

The proof of Theorem 4.1 is now complete:

Proof of Theorem 4.1 Let X be general. By Proposition 4.9 the Mukai flop $p \colon J \dashrightarrow N$ of J along the zero section is projective and on J there exists a big and nef line bundle that contracts the zero section. For very general X, H_0 is unique, up to a positive rational multiple, and $\mathrm{Nef}(J) = \langle L, H_0 \rangle$. Moreover, we have shown that for general X there is a divisorial contraction $N \to \bar{N}$, contracting $p_* \Theta$. Since the divisorial contraction $N \to \bar{N}$ contracts the ruling of Θ (cf Proposition 3.6), by (3-2) it follows that the image of Θ in $\bar{N}$ is birational to the LLSvS 8–fold $Z(X)$. For very general X, $\mathrm{Nef}(N) = \langle p_* H_0, p_* H \rangle$, where $p^* H$ is the unique (up to a positive multiple) big and nef line bundle inducing the contraction. By [36, Proposition 4.2], H is the second ray of the movable cone of J, ie $\mathrm{Mov}(J) = \langle L, H \rangle$. □

5 The Mordell–Weil group of $J(X)$

Let $a \colon \mathcal{A} \to B$ be a projective family of abelian varieties over an irreducible basis B and suppose that a admits a zero section. The Mordell–Weil group $\mathrm{MW}(a)$ of $a \colon \mathcal{A} \to B$ is the group of rational sections of $a \colon \mathcal{A} \to B$. Equivalently, if K denotes the function field of B, $\mathrm{MW}(a)$ is the group of K–rational points of the generic fiber $\mathcal{A}_K$. For Lagrangian hyper-Kähler manifolds, the study of the Mordell–Weil group of abelian fibered hyper-Kähler manifolds was started by Oguiso in [65; 64]. The aim of this section is to prove the following theorem.

Theorem 5.1 *Let X be a smooth cubic fourfold and let $\pi\colon J = J(X) \to \mathbb{P}^5$ be as in Theorem 1.6, a smooth projective hyper-Kähler compactification of J_U. Let $\mathrm{MW}(\pi)$ be the Mordell–Weil group of π, ie the group of rational sections of π, and let $H^{2,2}(X,\mathbb{Z})_0$ be the primitive degree-four integral cohomology of X. The natural group homomorphism*

$$\phi_X\colon H^{2,2}(X,\mathbb{Z})_0 \to \mathrm{MW}(\pi)$$

induced by the Abel–Jacobi map (5-1) is an isomorphism.

Corollary 5.2 *The group $\mathrm{MW}(\pi)$ is torsion-free.*

Remark 5.3 In [64] Oguiso proved the existence of Lagrangian fibered hyper-Kähler manifolds whose Mordell–Weil group has rank 20. This is the maximal possible rank among all the known examples of hyper-Kähler manifolds, as follows from the Shioda–Tate formula of [65]; see also Proposition 5.4 below. Oguiso considers deformations of the abelian fibration $\widetilde{M}_{2v_0} \to \mathbb{P}^5$ (cf (2-3)) preserving both the Lagrangian fibration structure and the zero section; among these deformations, Oguiso shows the existence of Lagrangian fibration with rank 20 Mordell–Weil group [65, Theorem 1.4(2)]. The general deformation of $\widetilde{M}_{2v_0} \to \mathbb{P}^5$ for which both the Lagrangian fibration structure and the zero section are preserved (this is a codimension-two condition) is, up to birational isomorphism, $J(X)$; see Remark 3.3. By the theorem above, Lagrangian fibrations of the form $J(X)$, for X with $\operatorname{rk} H^{2,2}(X,\mathbb{Z}) = 22$, satisfy $\operatorname{rk}\mathrm{MW}(\pi) = 20$. Thus, they provide an explicit description of Oguiso's examples.

The following proposition is essentially a reformulation of results from [65; 64].

Proposition 5.4 *Let $\pi: M \to \mathbb{P}^n$ be a projective hyper-Kähler manifold with a fixed (rational) section. Let $K = \mathbb{C}(\mathbb{P}^n)$ be the function field of the base and let M_K be the base change of M to the generic point of $\mathbb{P}^n$. There is a commutative diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}L \oplus \bigoplus_i \mathbb{Z}D_i \arrow[r] \arrow[d, equal] & L^{\perp} \arrow[r] \arrow[d] & \mathrm{Pic}^0(M_K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}L \oplus \bigoplus_i \mathbb{Z}D_i \arrow[r] & \mathrm{NS}(M) \arrow[r, "r_K"] \arrow[d, two heads, "r_b"] & \mathrm{Pic}(M_K) \arrow[r] \arrow[d, two heads] & 0 \\
 & & \mathbb{Z} \arrow[r, equal] & \mathrm{NS}(M_K) &
\end{tikzcd}$$

where $L=\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and where the $D_1,\dots,D_k$ are the irreducible components of the complement of the regular locus of π that do not meet the section. In particular, $\mathrm{rk}(\mathrm{MW}(\pi))=\mathrm{rk}(\mathrm{NS}(M))-\mathrm{rk}\,\mathbb{Z}L\oplus\mathbb{Z}D_i-1=\mathrm{rk}(\mathrm{NS}(M))-k$.

Proof The column on the left is exact by definition. By [76], for b in the locus $U\subset\mathbb{P}^n$ parametrizing smooth fibers of π, $\mathrm{Im}[r_b:\mathrm{NS}(M)\to H^2(M_b)]=\mathbb{Z}$. The same argument as in Lemma 3.5 shows that a line bundle D on M lies in $L^\perp$ if and only if $D^n\cdot L^n=(D_{|M_b})^n=0$. Since $\mathrm{rk}\,r_b=1$, this holds if and only if $D\cdot L^n=D_{|M_b}=0$, which is equivalent to $D\in\ker r_b$. This shows that the central column is exact. The same argument of [64, Theorem 1.1], which was used to show that $\mathrm{rk}\,\mathrm{NS}(M_K)=1$, shows that any element in $\ker(r_b)=L^\perp$ goes to zero in $\mathrm{NS}(M_K)$. Thus there are induced horizontal morphisms $L^\perp\to\mathrm{Pic}^0(M_K)$ and $\mathbb{Z}\to\mathrm{NS}(M_K)$. Since $\mathrm{NS}(M)\to\mathrm{Pic}(M_K)$ is surjective, the bottom horizontal morphism is an isomorphism. The natural morphism $\mathbb{Z}L\oplus\mathbb{Z}D_i\to\mathrm{NS}(M)$ is injective, since by [65, Lemma 2.4] it has maximal rank over $\mathbb{Q}$ and $\mathrm{NS}(M)$ is torsion-free. Clearly, $\mathbb{Z}L\oplus_i\mathbb{Z}D_i\subset\ker(r_K)$. To show the reverse inclusion, let D be any line bundle on M that goes to zero in $\mathrm{Pic}(M_K)$. Then, by what we have already proved, for any smooth fiber we have $r_b(D)=[D_{|M_b}]=0$. It follows that D is a linear combination of $L=\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and boundary divisors, ie $D\in\mathbb{Z}L\oplus_i\mathbb{Z}D_i$. As $\mathrm{rk}(\mathrm{MW}(\pi))=\mathrm{rk}\,\mathrm{Pic}^0(M_K)$, the last statement also follows. □

Remark 5.5 The study of the Mordell–Weil group for the Beauville–Mukai system is being carried out in joint work in progress with Chiara Camere.

Corollary 5.6 *Let $J=J(X)\to\mathbb{P}^5$ be a hyper-Kähler compactification of the intermediate Jacobian fibration. Then*

$$\mathrm{rk}\,\mathrm{MW}(\pi)=\mathrm{rk}\,\mathrm{NS}(J)-2=\mathrm{rk}\,H^{2,2}(X,\mathbb{Z})_0.$$

Proof The discriminant locus of π is irreducible and the fibers of π over the general point of the discriminant are also irreducible; cf Lemma 1.2. Thus, in the notation of the proposition above, $\ker r_K=\mathbb{Z}L$ and the equality $\mathrm{rk}\,\mathrm{MW}(\pi)=\mathrm{rk}\,\mathrm{NS}(J)-2$ follows. The remaining equality follows from Lemma 3.2. □

Remark 5.7 The corollary just proven, which relies on Oguiso's Shioda–Tate formula above, is the only part of this section where we use that J_U admits a hyper-Kähler compactification with a regular Lagrangian fibration extending $J_U\to U$. Indeed, to define the Abel–Jacobi map ϕ_X and to prove that it is injective (Section 5.3), we don't need to assume the existence of a hyper-Kähler compactification. However, we will use this corollary in the proof of the surjectivity (Section 5.4).

Remark 5.8 An interesting problem is to study the action on J of the birational automorphisms induced by translation by a nontrivial element of $\mathrm{MW}(\pi)$, as well as to study the automorphism group of the generic fiber J_K. A consequence of the observations of Section 3.1 is that if $J \to \mathbb{P}^5$ has irreducible fibers, then the birational automorphisms induced by translation are regular morphisms.

5.1 The Abel–Jacobi mapping

This sections uses some ingredients from the theory of normal functions (certain holomorphic sections of intermediate Jacobian fibrations), as developed and used by Griffiths [30; 31], Zucker [83; 84] and Voisin [78]. We refer to these papers, as well as to [77, Sections 7.2.1 and 8.2.2], for the relevant theory.

The first task is to define the morphism $\phi_X\colon H^{2,2}(X,\mathbb{Z})_0 \to \mathrm{MW}(\pi)$. One way to do this is to use relative Deligne cohomology, which allows us to define an algebraic section of the fibration $J_U \to U$. See, for example, [78; 28].

A more geometric way to define the morphism ϕ_X is in terms of algebraic cycles and Abel–Jacobi maps, which is what we use here. This is possible because the integral Hodge conjecture holds for degree-four Hodge classes on X [78; 84]. It allows us to avoid, in the current presentation, defining the normal function associated with a cohomology class. The reader should keep in mind, however, that constructing an algebraic section of the intermediate Jacobian fibrations with a Hodge class on X is a key ingredient in the proof of the Hodge conjecture of [78; 84], so the shortcut is only at the level of our presentation.

As already mentioned, the integral Hodge conjecture holds for degree-four Hodge classes on X. In particular, for every class $\alpha \in H^{2,2}(X,\mathbb{Z})$, there is an algebraic cycle Z such that $[Z]=\alpha$. Let $V \subset \mathbb{P}^5$ be the open subset parametrizing smooth hyperplane sections of X that do not contain any of the components of Z. If α is a primitive cohomology class, then for $b=[Y_b]\in V$, the one-cycle Z_b satisfies

$$[Z_{Y_b}]=0 \quad \text{in } H^4(Y_b,\mathbb{Z})=\mathbb{Z},$$

and hence determines a point $\phi_{Y_b}(Z_b)\in \mathrm{Jac}(Y_b)$ in the intermediate Jacobian of Y_b. By Griffiths [31] (see also [77, Section 7.2.1]) the assignment

$$\sigma_Z\colon V \to J_V, \quad b \mapsto \phi_{Y_b}(Z_b),$$

defines a holomorphic section of the restriction of J to $V \subset \mathbb{P}^5$. By [83], this section is, in fact, algebraic: indeed, consider a Lefschetz pencil $\mathcal{Y}' \to \mathbb{P}^1$ of hyperplanes of X

with $\mathbb{P}^1 \subset V$ and with the property that none of the singular points of the members of the pencil are contained in Z. By [83, Proposition (4.58)] the restriction of σ_Z to the nonempty open subset $V \cap \mathbb{P}^1$ of the pencil extends to a holomorphic function on all of $\mathbb{P}^1$ and is thus algebraic; see also [28]. Since a holomorphic function that is algebraic in each variable is algebraic (see for example [14, Chapter IX, Theorem 5]), it follows that σ_Z actually defines a rational function on $\mathbb{P}^5$, ie

$$\sigma_Z \in \mathrm{MW}(\pi).$$

The holomorphic section σ_Z does not depend on the algebraic cycle representing α. Indeed, since $\mathrm{CH}_0(X) = \mathbb{Z}$, by [79, Theorem 6.24] it follows that the cycle map $\mathrm{CH}^2(X) \to H^{2,2}(X, \mathbb{Z})$ is injective. It follows that if Z and Z' are homologous, then they are rationally equivalent in X and hence so are their restrictions to a general smooth hyperplane section. The conclusion of this discussion is that the Abel–Jacobi map induces a well-defined group homomorphism

$$\phi_X \colon H^{2,2}(X, \mathbb{Z})_0 \to \mathrm{MW}(\pi), \quad \alpha = [Z] \mapsto \sigma_\alpha := \sigma_Z. \tag{5-1}$$

We prove injectivity of ϕ_X in Section 5.3 and surjectivity in Section 5.4. Since we will restrict to general pencils in $\mathbb{P}^5$, we start by recalling a few standard facts about Lefschetz pencils of cubic threefolds.

5.2 Preliminaries on Lefschetz pencils

We start by setting up the notation. Let $\mathbb{P}^1 \subset (\mathbb{P}^5)^\vee$ be a Lefschetz pencil with base locus a smooth cubic surface $\Sigma \subset X$. We have the diagram

$$\begin{tikzcd}
 & \Sigma \times \mathbb{P}^1 \arrow[r, hook, "i"] \arrow[dl, "p_1"'] & \mathcal{Y}' \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\Sigma \arrow[r, hook] & X & & \mathbb{P}^1
\end{tikzcd}$$

where $\mathcal{Y}' = \mathrm{Bl}_\Sigma X$, $q \colon \mathcal{Y}' \to \mathbb{P}^1$ is the fibration of threefolds, and $i \colon \Sigma \times \mathbb{P}^1 \to \mathcal{Y}'$ is the inclusion of the exceptional divisor in $\mathcal{Y}'$. Let $j \colon U' \subset \mathbb{P}^1$ be the open subset parametrizing smooth fibers.

The following lemma is standard. We include a proof for lack of reference.

Lemma 5.9 *The homology and cohomology groups of a cubic threefold which is smooth or has one A_1 singularity have no torsion. Moreover, using notation as above,*

$$R^1 q_* \mathbb{Z} = 0, \quad R^2 q_* \mathbb{Z} = \mathbb{Z}, \quad R^3 q_* \mathbb{Z} = j_* j^* R^3 q_* \mathbb{Z}, \quad R^4 q_* \mathbb{Z} = \mathbb{Z}.$$

Proof The statement about the homology groups of a cubic threefold with at most an A_1 singularity follow from [24, Example 5.3 and Theorem 2.1]; using the universal coefficient theorem, the statements on the cohomology groups then follow. From loc. cit. it also follows that $H^4(Y,\mathbb{Z}) = H_4(Y,\mathbb{Z})^\vee = \mathbb{Z}$, and hence $R^4 q_*\mathbb{Z} = \mathbb{Z}$ follows by proper base change. The first two statements on the higher direct images follow from the Lefschetz hyperplane section theorem. The third equality, which is also known as the "local invariant cycle" property, is well known to hold with $\mathbb{Q}$–coefficients, and we now show it with $\mathbb{Z}$–coefficients, as follows. By adjunction, there is a natural morphism

$$\varepsilon\colon R^3 q_*\mathbb{Z} \to j_* j^* R^3 q_*\mathbb{Z},$$

which is an isomorphism over U. To show ε is an isomorphism over any point of $B := \mathbb{P}^1 \setminus U'$ we restrict, for every $b_0 \in B$, to a small disk Δ centered at b_0. Then ε is an isomorphism around b_0 if and only if the specialization morphism

$$H^3(Y_{b_0},\mathbb{Z}) \cong H^3(\mathcal{Y}',\mathbb{Z}) \to H^3(Y_b,\mathbb{Z})^{\mathrm{inv}} = (j_* j^* R^3 q_*\mathbb{Z})_{b_0}$$

is an isomorphism (cf [69, pages 439–440]), where $b \in \Delta \cap U'$ and $H^3(Y_b,\mathbb{Z})^{\mathrm{inv}} \subset H^3(Y_b,\mathbb{Z})$ are the local monodromy invariants. Let $\delta \in H_3(Y_b,\mathbb{Z})$ be the vanishing cycle of $\mathcal{Y}'_\Delta$. By the Picard–Lefschetz formula, $H^3(Y_b,\mathbb{Z})^{\mathrm{inv}} = \mathbb{Z}\delta^\perp$, where $\perp$ is taken with respect to the intersection product, which is nondegenerate since $H^3(Y_b,\mathbb{Z})$ is torsion-free. By [77, Corollary 2.17], there is a short exact sequence

$$0 \to \mathbb{Z}\delta \to H_3(Y_b,\mathbb{Z}) \to H_3(\mathcal{Y}'_\Delta,\mathbb{Z}) \cong H_3(Y_{b_0},\mathbb{Z}) \to 0,$$

where $0 \neq \delta \in H_3(Y_b,\mathbb{Z})$ is the class of the vanishing cycle. Dualizing, we get a short exact sequence

$$0 \to H^3(Y_{b_0},\mathbb{Z}) \to H^3(Y_b,\mathbb{Z}) \to (\mathbb{Z}\delta)^\vee \to 0.$$

(Recall the absence of torsion in the homology groups of Y_b and Y_{b_0}.) Using the isomorphism $H_3(Y_b,\mathbb{Z}) \cong H^3(Y_b,\mathbb{Z})$ induced by Poincaré duality, we make the identification $\mathbb{Z}\delta^\perp = \ker[H^3(Y_b,\mathbb{Z}) \to \mathbb{Z}\delta^\vee] = \mathrm{Im}[H^3(Y_{b_0},\mathbb{Z}) \to H^3(Y_b,\mathbb{Z})]$. □

It is well known that for a Lefschetz pencil the Leray spectral sequence with coefficients in $\mathbb{Q}$ degenerates at E_2. For a Lefschetz pencil of cubic threefolds, this is true also for $\mathbb{Z}$ coefficients. Again, we include a proof for lack of reference. For the whole family of smooth hyperplane sections of X the Leray spectral sequence with integers coefficients does *not* degenerate at E_2; this is the starting point of the construction of the nontrivial J_U–torsor of [80], cf Remark 1.14.

Lemma 5.10 *Let $q\colon \mathcal{Y}' \to \mathbb{P}^1$ be as above. The Leray spectral sequence with $\mathbb{Z}$ coefficients degenerates at E_2. In particular, the Leray filtration on $H^4(\mathcal{Y}', \mathbb{Z})$ is given by*

$$\begin{gathered}\mathbb{Z} = H^2(\mathbb{P}^1, R^2 f_*\mathbb{Z}) \subset L_1 \subset H^4(\mathcal{Y}', \mathbb{Z}) \twoheadrightarrow H^0(\mathbb{P}^1, R^4 f_*\mathbb{Z}) = \mathbb{Z}, \\ 0 \to H^2(\mathbb{P}^1, R^2 f_*\mathbb{Z}) \to L_1 \xrightarrow{\gamma} H^1(\mathbb{P}^1, R^3 f_*\mathbb{Z}) \to 0.\end{gathered} \tag{5-2}$$

Proof Because of the many vanishings in the E_2–page of the spectral sequence, the only map we need to show is trivial is $H^0(\mathbb{P}^1, R^4 q_*\mathbb{Z}) \to H^2(\mathbb{P}^1, R^3 q_*\mathbb{Z})$. For this, it is enough to show that $H^4(Y_b, \mathbb{Z}) \to H^0(\mathbb{P}^1, R^4 q_*\mathbb{Z})$ is surjective, which is clearly true since both groups are generated by the class of a line. □

Consider the decomposition

$$H^4(\mathcal{Y}', \mathbb{Z}) = H^4(X, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z}) \oplus H^0(\Sigma, \mathbb{Z}) \tag{5-3}$$

given by the blowup formula. The inclusion of the first summand is given by the pullback p^*; we freely omit the symbol p^* when viewing $H^4(X, \mathbb{Z})$ as a subspace of $H^4(\mathcal{Y}', \mathbb{Z})$. The inclusion of the second factor is via the map $H^2(\Sigma, \mathbb{Z}) \to H^4(\mathcal{Y}', \mathbb{Z})$ given by $C \mapsto i_*(C \times \mathbb{P}^1)$. Finally, the inclusion of the last summand is through the map $H^0(\Sigma, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \otimes H^2(\mathbb{P}^1, \mathbb{Z}) \to H^4(\mathcal{Y}', \mathbb{Z})$ that sends $[\Sigma] = [\Sigma \times p] \mapsto i_*([\Sigma \times p])$, where $p \in \mathbb{P}^1$ is a point. We highlight the following results for later use.

Lemma 5.11 *There is a natural isomorphism $H^0(\Sigma, \mathbb{Z}) \cong H^2(\mathbb{P}^1, R^2 q_*\mathbb{Z})$ which allows the identification of the inclusion*

$$H^0(\Sigma, \mathbb{Z}) \cong H^0(\Sigma, \mathbb{Z}) \otimes H^2(\mathbb{P}^1, \mathbb{Z}) \xrightarrow{i_*} H^4(\mathcal{Y}', \mathbb{Z})$$

of (5-3) with the inclusion $H^2(\mathbb{P}^1, R^2 q_*\mathbb{Z}) \to H^4(\mathcal{Y}', \mathbb{Z})$ induced by the Leray filtration of Lemma 5.10.

Proof The closed embedding $i\colon \Sigma \times \mathbb{P}^1 \hookrightarrow \mathcal{Y}'$ determines an isomorphism $p_{2*}\mathbb{Z} \cong R^2 q_*\mathbb{Z}$ of constant local systems. Here, $p_2\colon \Sigma \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the section factor. Since $H^2(\mathbb{P}^1, p_{2*}\mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \otimes H^2(\mathbb{P}^1, \mathbb{Z})$ the lemma follows. □

Via p^*, we identify $H^4(X, \mathbb{Z})_0 \cong L_1 \cap H^4(X, \mathbb{Z})$ and set $L_1^{2,2} = L_1 \cap H^{2,2}(\mathcal{Y}', \mathbb{Z})$. Here $L_1 \subset H^4(\mathcal{Y}', \mathbb{Z})$ denotes the second piece of the Leray filtration; see (5-2).

Corollary 5.12 *The surjective morphism $\gamma\colon L_1 \to H^1(\mathbb{P}^1, R^3 q_*\mathbb{Z})$ of (5-2) restricts to an injection*

$$\bar{\gamma}\colon L_1 \cap (H^{2,2}(X, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z})) \cong L_1^{2,2}/\ker(\gamma) \to H^1(\mathbb{P}^1, R^3 q_*\mathbb{Z}).$$

Proof From Lemmas 5.10 and 5.11 above, it follows that $\ker(\gamma) = H^2(\mathbb{P}^1, R^2q_*\mathbb{Z}) = H^0(\Sigma, \mathbb{Z})$. Thus, by (5-3), it follows that $H^4(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z})) \cap \ker(\gamma) = \{0\}$. Since $H^0(\Sigma,\mathbb{Z}) \subset L_1$ and

$$L_1 \cap \bigl(H^{2,2}(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z}) \oplus H^0(\Sigma,\mathbb{Z})\bigr) = L_1 \cap \bigl(H^{2,2}(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z})\bigr) \oplus H^0(\Sigma,\mathbb{Z}), \tag{5-4}$$

the corollary follows. □

Lemma 5.13 *The restriction morphism $H^1(\mathbb{P}^1, R^3q_*\mathbb{Z}) \to H^1(U', R^3q_{U'*}\mathbb{Z})$ is injective.*

Proof The Leray spectral sequence for the open immersion $j : U' \to \mathbb{P}^1$, applied to the sheaf $j^*R^3q_*\mathbb{Z} = R^3q_{U'*}\mathbb{Z}$, gives a five-term exact sequence starting with

$$0 \to H^1(\mathbb{P}^1, j_*j^*R^3q_*\mathbb{Z}) \to H^1(U', R^3q_{U'*}\mathbb{Z}) \to \cdots .$$

This concludes the proof, since by Lemma 5.9, $R^3q_*\mathbb{Z} = j_*j^*R^3q_*\mathbb{Z}$. □

5.3 Injectivity of ϕ_X

The proof of injectivity uses the Hodge class of a normal function; see [83] and [77, Section 8.2.2].

For a pencil $\mathcal{Y}' \to \mathbb{P}^1$ as above, set

$$H^{2,2}(\mathcal{Y}',\mathbb{Z})_0 := L_1^{2,2} = \ker[H^{2,2}(\mathcal{Y}',\mathbb{Z}) \to H^0(\mathbb{P}^1, R^4q_*\mathbb{Z})] = L_1 \cap H^{2,2}(\mathcal{Y}',\mathbb{Z}),$$

and let

$$\pi' = J' \to \mathbb{P}^1 \quad \text{and} \quad J_{U'} \to U'$$

be the restriction of the intermediate Jacobian fibration to $\mathbb{P}^1$ and to U'. Choosing a set of generators for $H^{2,2}(X,\mathbb{Z})_0$, let $\mathcal{Y}' \to \mathbb{P}^1$ be a general enough pencil that the restriction morphism

$$\phi'_X : H^{2,2}(X,\mathbb{Z})_0 \to \mathrm{MW}(\pi') \tag{5-5}$$

is well-defined. Here, $\mathrm{MW}(\pi')$ is the group of rational sections of π'. Similarly, we get a group homomorphism $\phi'_{\mathcal{Y}'} : H^{2,2}(\mathcal{Y}',\mathbb{Z})_0 \to \mathrm{MW}(\pi')$. Moreover, if $\alpha \in H^{2,2}(X,\mathbb{Z})_0$, then

$$\phi'_{\mathcal{Y}'}(p^*\alpha) = \phi'_X(\alpha) \in \mathrm{MW}(\pi').$$

Recall the Hodge class of a normal function; see for instance [77, Section 8.2.2] and [83, Proposition (3.9)]. Let $\mathcal{H}^3 = R^3q_{U'*}\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{U'}$ be the Hodge bundle associated to the weight-three variation of Hodge structure of the pencil and let $F^*\mathcal{H}^3$ be the Hodge filtration. The sheaf $\mathcal{J}_{U'}$ of holomorphic sections of the intermediate Jacobian fibration fits into the exact sequence

$$0 \to R^3 f'_{U'*}\mathbb{Z} \to \mathcal{H}^3/F^2\mathcal{H}^3 \to \mathcal{J}_{U'} \to 0,$$

and the coboundary morphism

$$H^0(U', \mathcal{J}_{U'}) \xrightarrow{\text{cl}} H^1(U', R^3 f_*\mathbb{Z}), \quad \nu \mapsto \text{cl}(\nu),$$

associates to every holomorphic section ν of $J_{U'} \to U'$ a class $\text{cl}(\nu)$ in $H^1(U', R^3 q_*\mathbb{Z})$, called the Hodge class of ν — in the present context, this class is of Hodge type with respect to the Hodge structure on $H^1(U', R^3 q_*\mathbb{Z})$ induced from that on $H^4(\mathcal{Y}', \mathbb{Z})$ via the degeneracy of the Leray spectral sequence; see [83, Section 3].

Lemma 5.14 *Let $\mathcal{Y}' \to \mathbb{P}^1$ be a general pencil. The homomorphism*

$$\phi'_X : H^{2,2}(X, \mathbb{Z})_0 \xrightarrow{\beta} \text{MW}(\pi')$$

of (5-5) is injective.

Proof By [83, Proposition (3.9)] — see also [77, Lemma 8.20] — the diagram

$$\begin{tikzcd}
H^{2,2}(X,\mathbb{Z})_0 \arrow[r, "p^*"] \arrow[dr, "\phi'_X"'] & H^{2,2}(\mathcal{Y}',\mathbb{Z})_0 \arrow[r, "\gamma"] \arrow[d, "\phi'_{\mathcal{Y}'}"'] & H^1(\mathbb{P}^1, R^3 p_*\mathbb{Z}) \arrow[d, "\varepsilon"] \\
& H^0(U', \mathcal{J}_{U'}) \arrow[r, "\text{cl}"'] & H^1(U', R^3 f'_*\mathbb{Z})
\end{tikzcd} \tag{5-6}$$

is commutative. The map ε is injective by Lemma 5.13, and $p^* \circ \gamma$ is injective by Corollary 5.12. Hence, $\text{cl} \circ \phi'_X$ is injective and thus so is ϕ'_X. □

5.4 Surjectivity of ϕ_X

There are three ingredients in the proof of surjectivity:

- the fact that $\text{rk}\,\text{MW}(\pi) = \text{rk}\, H^{2,2}(X, \mathbb{Z})_0$, as proved in Corollary 5.6;
- the restriction, once again, to Lefschetz pencils;
- the techniques used in [78; 84] for the proof of the integral Hodge conjecture for cubic fourfolds.

We remark that we use their argument in a slightly different way. To prove the Hodge conjecture one starts with a cohomology class, uses it to define a normal function, and then uses the normal function to construct an algebraic cycle representing the cohomology class (possibly up to a multiple of a complete intersection surface). See [78] for more details. Here we start with a rational section of the intermediate Jacobian fibration, we restrict to a general pencil, and use the same method of Voisin to construct an algebraic cycle inducing the section via the Abel–Jacobi map. Then we have to check that the cohomology class representing this cycle is primitive, that it is independent of the pencil, and that it induces, via ϕ_X, the section we started from.

Since by Corollary 5.6 the cokernel of the injection $\phi_X\colon H^{2,2}(X,\mathbb{Z})_0 \to \mathrm{MW}(\pi)$ is finite, for any $\sigma \in \mathrm{MW}(\pi)$ there is an integer N and a cohomology class $\alpha \in H^{2,2}(X,\mathbb{Z})_0$ such that

$$\sigma_\alpha := \phi_X(\alpha) = N\sigma. \tag{5-7}$$

We will show, again using Lefschetz pencils, that given σ and α as above, there exists a $\bar{\beta}' \in H^{2,2}(X,\mathbb{Z})_0$ such that $\alpha = N\bar{\beta}'$. This will give the desired surjectivity. Before we do so, let us introduce some results that we will need.

For a general pencil $\mathcal{Y}' \to \mathbb{P}^1$, let

$$(J^T)' \to \mathbb{P}^1$$

be the restriction of the intermediate Jacobian fibration $J^T \to \mathbb{P}^5$ of [80] (compare with Remark 1.14) to the pencil. For a conic $C \subset \Sigma$, consider the relative one-cycle of degree two in $\mathcal{Y}' \to \mathbb{P}^1$ — any other degree-two relative one-cycle that comes from Σ will do. This defines a section of $(J^T)' \to \mathbb{P}^1$, which trivializes the torsor $(J^T)_{U'}$ inducing an isomorphism $J'_{U'} \cong (J^T)_{U'}$. It is easily seen that this extends to an isomorphism $t_C\colon J' \cong (J^T)'$ over $\mathbb{P}^1$. For any $\sigma' \in H^0(U',\mathcal{J}_{U'})$, we may consider the induced section

$$(\sigma^T)' := t_C \circ \sigma' \in H^0(U',\mathcal{J}^T_{U'}).$$

The following result is proved in Voisin [78]; see also [84, Theorem (3.2)], where the result is proved over $\mathbb{Q}$.

Proposition 5.15 [78, Section 2.3] *For any section $\sigma' \in \mathrm{MW}(\pi')$, there is a relative one-cycle Z on $\mathcal{Y}'$ of degree two such that the cohomology class*

$$\beta' = [Z] - [C \times \mathbb{P}^1] \in H^{2,2}(\mathcal{Y}',\mathbb{Z})_0$$

satisfies $\phi'_{\mathcal{Y}'}(\beta') = \sigma'$ in $\mathrm{MW}(\pi')$.

Proof For the reader's convenience, we give a brief sketch of the argument. By a result of Markushevich and Tikhomirov [54] and Druel [26] there is a relative birational morphism $c_2\colon \mathcal{M}'_{U'} \to J^T_{U'}$, where $\mathcal{M}'_{U'} \to U'$ is the relative moduli space of sheaves on $\mathcal{Y}'_{U'} \to U'$ with $c_1 = 0$ and $c_2 = 2\ell$. The morphism associates to every sheaf corresponding to a point in $\mathcal{M}'_{U'}$ the Abel–Jacobi invariant of its second Chern class. Given a section $(\sigma^T)' \in H^0(U', \mathcal{J}^T_{U'})$ as above, Voisin uses $\mathcal{M}'_{U'} \to U'$ to construct a family $\mathcal{C}_{U'}$ of degree-two curves in the fibers of $\mathcal{Y}'_{U'} \to U'$ with the property that for every $b \in U'$, the curve $\mathcal{C}_b$ represents the c_2 of a sheaf over $(\sigma^T)'(b)$. By construction, letting Z be the closure of $\mathcal{C}_{U'}$ in $\mathcal{Y}'$ and setting $\beta' := [Z] - [C \times \mathbb{P}^1] \in H^{2,2}(\mathcal{Y}', \mathbb{Z})_0$, we have $\phi'_{\mathcal{Y}'}(\beta') = \sigma'$ in $H^0(U', \mathcal{J}_{U'})$. □

Let $\sigma \in \mathrm{MW}(\pi)$. For a general pencil $\mathbb{P}^1 \subset \mathbb{P}^5$, let $\sigma' = \sigma_{|\mathbb{P}^1}$ be the restriction of σ to $\mathbb{P}^1$, and let β' be as in the proposition above so that $\phi'_{\mathcal{Y}'}(\beta') = \sigma'$. It is tempting to say that, via ϕ_X, the class β' induces σ globally and not just on that pencil. This is indeed the case, though we first need to check that β' lies in the primitive cohomology of X and that β' is independent of the pencil as well as of the chosen isomorphism $t_C\colon J' \cong (J^T)'$. More precisely, we need to check that β' induces σ over an open subset of $\mathbb{P}^5$ and not just on the chosen pencil. Before checking this, we have the following proposition.

Recall that we have set $H^{2,2}(\mathcal{Y}', \mathbb{Z})_0 = L_1 \cap H^{2,2}(\mathcal{Y}', \mathbb{Z})$.

Proposition 5.16 [83, Theorem (4.17)] *The Abel–Jacobi morphism*

$$\phi_{\mathcal{Y}'}\colon H^{2,2}(\mathcal{Y}', \mathbb{Z})_0 \to \mathrm{MW}(\pi') \subset H^0(U', \mathcal{J}_{U'})$$

is surjective and defines an isomorphism

$$\bar{\phi}_{\mathcal{Y}'}\colon L_1 \cap (H^{2,2}(X, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z})) \to \mathrm{MW}(\pi').$$

Proof By diagram (5-6) and the fact that ε is injective, $\ker(\phi_{\mathcal{Y}'}) = \ker\gamma$, which by Lemma 5.10 is equal to $H^0(\Sigma, \mathbb{Z})$. Since $\phi_{\mathcal{Y}'}$ is surjective by the proposition above, the induced morphism $\bar{\phi}_{\mathcal{Y}'}\colon H^{2,2}(\mathcal{Y}', \mathbb{Z})_0 / H^0(\Sigma, \mathbb{Z}) \to \mathrm{MW}(\pi')$ is an isomorphism. Finally, by (5-4), $H^{2,2}(\mathcal{Y}', \mathbb{Z})_0 / H^0(\Sigma, \mathbb{Z}) \cong L_1 \cap (H^{2,2}(X, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z}))$. □

We can now end the proof of surjectivity. For $\sigma \in \mathrm{MW}(\pi)$, let $\alpha \in H^{2,2}(X, \mathbb{Z})_0$ be as in (5-7). Restricting to a pencil $\mathcal{Y}' \to \mathbb{P}^1$, set $\sigma' = \sigma_{|\mathbb{P}^1}$ and let β' be as in Proposition 5.15 such that $\phi_{\mathcal{Y}'}(\beta') = \sigma'$. Finally, let $\bar{\beta}'$ be the projection of β' onto

$L_1 \cap (H^{2,2}(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z}))$. In an abuse of notation, we are omitting p^* from the inclusion of $H^4(X,\mathbb{Z})$ in $H^4(\mathcal{Y}',\mathbb{Z})$ and we will write α instead of $p^*\alpha$. We have

$$\phi_{\mathcal{Y}'}(\alpha) = (\phi_X(\alpha))_{|\mathbb{P}^1} = N\sigma' = N\phi_{\mathcal{Y}'}(\beta') = \phi_{\mathcal{Y}'}(N\beta').$$

By Proposition 5.16, $\alpha = N\bar{\beta}' \in L_1 \cap (H^{2,2}(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z}))$. Since

$$\alpha \in H^{2,2}(X,\mathbb{Z})_0 \subset L_1 \cap \big(H^{2,2}(X,\mathbb{Z}) \oplus H^2(\Sigma,\mathbb{Z})\big),$$

it follows that $\bar{\beta}'$, too, has to lie in $H^{2,2}(X,\mathbb{Z})_0 \subset H^{2,2}(\mathcal{Y}',\mathbb{Z})$. Moreover, the class $\bar{\beta}'$, which a priori depends on the chosen Lefschetz pencil, is independent of the pencil. Set $\sigma_{\bar{\beta}'} = \phi_X(\bar{\beta}')$. Then, for *any* sufficiently general Lefschetz pencil $\mathbb{P}^1 \subset \mathbb{P}^5$, we have an equality of sections

$$(\sigma_{\bar{\beta}'})_{|\mathbb{P}^1} = \sigma_{|\mathbb{P}^1},$$

and hence the two rational sections $\sigma_{\bar{\beta}'}$ and σ coincide. This proves surjectivity.

Appendix On the Beauville conjecture for LSV varieties by Claire Voisin

In this appendix we explain a consequence of Corollary 3.10 on the following conjecture made by Beauville in [10].

Conjecture A.1 Let M be a projective hyper-Kähler manifold. Any polynomial cohomological relation $P(d_1,\dots,d_r) = 0$ in $H^*(M,\mathbb{Q})$, where d_i are divisor classes on M, already holds in $\mathrm{CH}(M)$.

Here $\mathrm{CH}(M)$ denotes the Chow groups of M with rational coefficients. Let now $M \to B$ be a projective hyper-Kähler manifold of dimension $2n$ equipped with a Lagrangian fibration, and let $L \in \operatorname{Pic} M = \mathrm{NS}(M)$ be the Lagrangian class pulled back from B; see [55]. We have $q(L) = 0$ by the Beauville–Fujiki relations, since $L^{2n} = 0$. Let also $h \in \operatorname{Pic} M = \mathrm{NS}(M)$ be the class of an ample divisor on M, so that the intersection pairing q restricted to $\langle L, h\rangle$ is nondegenerate by the Hodge index theorem. The same argument as in [15] shows that the polynomial cohomological relations between L and h are generated by the relations

$$\alpha^{n+1} = 0 \quad \text{in } H^{2n+2}(M,\mathbb{Q}) \text{ when } q(\alpha) = 0 \text{ for } \alpha \in \langle L, h\rangle. \tag{A-1}$$

Here we can restrict to rational cohomology classes because we know that there is an isotropic class in $\langle L, h\rangle$. We consider now, more specifically, an LSV variety J

(which is of dimension 10, so $n = 5$) constructed in [47] as a Lagrangian fibration over $\mathbb{P}^5$. The Picard group of a very general such variety is a rank-two lattice which contains as above the Lagrangian class L and an ample class, but we take as a basis the classes L and Θ, where Θ was introduced in [47] and is studied in the present paper. Riess proved in [72] that a hyper-Kähler manifold M which has a Lagrangian fibration and satisfies the "RLF conjecture" characterizing classes associated to Lagrangian fibrations, satisfies Beauville's conjecture. However we do not know that the LSV varieties satisfy the RLF conjecture. We prove here the following result.

Theorem A.2 *The relations (A-1) hold in* $\mathrm{CH}(J)$ *for the lattice* $\langle L, \Theta\rangle$ *of an LSV variety* J. *Conjecture A.1 is thus satisfied by an LSV variety with Picard number two.*

Proof There are, up to multiples, exactly two classes L and L' in $\langle L, \Theta\rangle$ satisfying $q(L) = 0, q(L') = 0$. Obviously $L^6 = 0$ in $\mathrm{CH}(J)$ since L comes from the base which is of dimension 5, so we only have to prove that $L'^6 = 0$ in $\mathrm{CH}(J)$. We use Riess' argument in [72], however, in a different way. As a consequence of fundamental results of Huybrechts in [36], Riess proved the following:

Theorem A.3 [72, Theorem 3.3] *Let* K *be an isotropic class on a projective hyper-Kähler manifold* M *of dimension* $2n$. *Then there exists a cycle* $\Gamma \in \mathrm{CH}^{2n}(M \times M)$ *such that* Γ^* *acts as an automorphism of* $\mathrm{CH}(M)$ *preserving the intersection product*, *the action of* Γ^* *on* $H^2(M)$ *preserves the Beauville–Bogomolov form* q_M, *and* $\Gamma^* K$ *belongs to the boundary of the birational Kähler cone of* M.

Here the birational Kähler cone of M is defined as the union of the Kähler cones of hyper-Kähler manifolds M' bimeromorphic to M (the bimeromorphic map $M' \dashrightarrow M$ inducing an isomorphism on H^2). We apply this theorem to our class L' on J and thus get a correspondence Γ as above. The class $\Gamma^* L'$ is an isotropic class, hence it must be proportional to either L' or L. Furthermore, it belongs to the boundary of the birational Kähler cone. We now have:

Lemma A.4 *The class* L' *does not belong to the boundary of the birational Kähler cone.*

Proof This is proved in Corollary 3.10 of the present paper. □

By Lemma A.4, we conclude that $\Gamma^* L'$ is proportional to L. As $L^6 = 0$ in $\mathrm{CH}^6(J)$ and Γ^* is an automorphism of $\mathrm{CH}(J)$ preserving the intersection product, we conclude that $L'^6 = 0$ in $\mathrm{CH}^6(J)$. □

If we consider the case of Picard rank three, where the Picard lattice N of J is generated by three classes L, Θ and D with $q(L,D)=0$ and $q(\Theta,D)=0$, there are now, according to [15], 13 degree-six cohomological relations between L, Θ and D, generated by the classes $\alpha^6 \in S^6N \subset S^6H^2(J,\mathbb{Q})$, where α belongs to the conic $q(\alpha)=0$. Among these relations, two of them, namely those involving only L and Θ, are established in $\mathrm{CH}(J)$ by Theorem A.2. We also have the relations

$$L^5D=0 \quad \text{and} \quad L'^5D=0 \quad \text{in } H^{12}(J,\mathbb{Q}), \tag{A-2}$$

which are obtained by differentiating the relation (A-1) at $\alpha=L$ or $\alpha=L'$ in the direction given by D, which is tangent to the conic at these points since $q(D,L)=0$ and $q(D,L')=0$. We prove the following:

Theorem A.5 *The relations (A-2) are satisfied in* $\mathrm{CH}^6(J)$.

Proof The first relation is proved by applying the following result from [81], which works in a more general context and needs a mild assumption on the infinitesimal variation of Hodge structure of a family of abelian varieties at the generic point of the base. More generally, let $M \to B$ be a fibration into abelian varieties and let $A \in \operatorname{Pic} M$ be a line bundle whose restriction to the general fiber M_b is topologically trivial.

Proposition A.6 *Assume that at the generic point $t \in B$, there exists a class $\alpha \in H^{1,0}(M_b)$ such that $\overline{\nabla}(\alpha)\colon T_{B,b} \to H^{0,1}(M_b)$ is surjective. Then there exists a point $b \in B$ such that M_b is smooth and $A_{|M_b}$ is a torsion line bundle.*

If all fibers M_b have the same class F in $\mathrm{CH}(M)$, *it thus follows that $F.A=0$ in* $\mathrm{CH}(M)$.

Coming back to our situation, we have to check that the assumption on the infinitesimal variation of Hodge structures is satisfied in our situation. Let J be the LSV variety of a cubic fourfold X. The infinitesimal variation of Hodge structure for the fibers of the Lagrangian fibration $J \to (\mathbb{P}^5)^\vee$ is thus canonically isomorphic to the variation of Hodge structure on the H^3 of the hyperplane section $X_H \subset X$. If Y is a smooth cubic threefold in $\mathbb{P}^4$ defined by an equation $f=0$, Griffiths' theory of IVHS of hypersurfaces says that there are isomorphisms

$$H^{2,1}(Y) \cong R^1_f \quad \text{and} \quad H^{1,2}(Y) \cong R^4_f$$

such that the infinitesimal variation of Hodge structure on $H^3(Y,\mathbb{C})$ is given (using the identification $R^3_f \cong H^1(Y,T_Y)$) by the multiplication map $R^3_f \to \operatorname{Hom}(R^1_f, R^4_f)$. Now consider the case where Y is a hyperplane section X_H, defined by a linear equation H, of the cubic fourfold X. It is immediate to see that the inclusions $X_H \subset X \subset \mathbb{P}^5$ determine a quadratic polynomial $Q_{X,H} \in R_f^2$ such that the natural map $\rho\colon H^0(X_H, \mathcal{O}_{X_H}(1)) \to R_f^3$, defined as the first-order classifying map for the deformations of X_H in X, is given by multiplication by $Q_{X,H}$. Combining these facts, we conclude that the desired infinitesimal criterion for the fibration $J \to (\mathbb{P}^5)^\vee$ holds if there exist a smooth hyperplane section $X_H \subset X$ and a linear form $x \in H^0(X_H, \mathcal{O}_{X_H}(1)) = R_f^1$ such that, with the above notation, the product map

$$xQ_{X,H}\colon R_f^1 \to R_f^4$$

by $xQ_{X,H}$ is an isomorphism. It is quite easy to show that the existence of such a hyperplane section is satisfied by X in codimension one in the moduli space of cubic fourfolds, hence at the generic point of any Hodge locus in this moduli space, or equivalently any Noether–Lefschetz locus for the corresponding LSV variety J. The relation $L^5 D = 0$ in $\mathrm{CH}^6(J)$ is thus satisfied at the generic point of the deformation locus of J preserving the Hodge class D, hence everywhere by specialization.

To conclude the proof of Theorem A.5, we have to prove the relation $L'^5 D = 0$ in $\mathrm{CH}^6(J)$. This follows however from the relation $L^5 D = 0$ in $\mathrm{CH}^6(J)$ by the same argument as in the proof of Theorem A.2, using the specialization of the cycle Γ and observing that Γ^* acts by ± 1 on $H^2(J, \mathbb{Q})^{\perp \langle L, \Theta \rangle}$, hence on D. □

References

[1] **N Addington**, *On two rationality conjectures for cubic fourfolds*, Math. Res. Lett. 23 (2016) 1–13 MR Zbl

[2] **N Addington**, **R Thomas**, *Hodge theory and derived categories of cubic fourfolds*, Duke Math. J. 163 (2014) 1885–1927 MR Zbl

[3] **V Alexeev**, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. 40 (2004) 1241–1265 MR Zbl

[4] **D Arcara**, **A Bertram**, *Bridgeland-stable moduli spaces for K–trivial surfaces*, J. Eur. Math. Soc. 15 (2013) 1–38 MR Zbl

[5] **A Bayer**, *Wall-crossing implies Brill–Noether: applications of stability conditions on surfaces*, from "Algebraic geometry: Salt Lake City 2015" (T de Fernex, B Hassett, M Mustaţă, M Olsson, M Popa, R Thomas, editors), Proc. Sympos. Pure Math. 97, Amer. Math. Soc., Providence, RI (2018) 3–27 MR Zbl

[6] **A Bayer**, **M Lahoz**, **E Macrì**, **H Nuer**, **A Perry**, **P Stellari**, *Stability conditions in families*, Publ. Math. Inst. Hautes Études Sci. 133 (2021) 157–325 MR Zbl

[7] **A Bayer**, **E Macrì**, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, Invent. Math. 198 (2014) 505–590 MR Zbl

[8] **A Bayer**, **E Macrì**, *Projectivity and birational geometry of Bridgeland moduli spaces*, J. Amer. Math. Soc. 27 (2014) 707–752 MR Zbl

[9] **A Beauville**, *Vector bundles on the cubic threefold*, from "Symposium in honor of C H Clemens" (A Bertram, J A Carlson, H Kley, editors), Contemp. Math. 312, Amer. Math. Soc., Providence, RI (2002) 71–86 MR Zbl

[10] **A Beauville**, *On the splitting of the Bloch–Beilinson filtration*, from "Algebraic cycles and motives, II" (J Nagel, C Peters, editors), London Math. Soc. Lecture Note Ser. 344, Cambridge Univ. Press (2007) 38–53 MR Zbl

[11] **A Beauville**, *Vector bundles on Fano threefolds and K3 surfaces*, Boll. Unione Mat. Ital. 15 (2022) 43–55 MR Zbl

[12] **A Beauville**, **R Donagi**, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985) 703–706 MR Zbl

[13] **C Birkar**, **P Cascini**, **C D Hacon**, **J McKernan**, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010) 405–468 MR Zbl

[14] **S Bochner**, **W T Martin**, *Several complex variables*, Princeton Mathematical Series 10, Princeton Univ. Press (1948) MR Zbl

[15] **F A Bogomolov**, *On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky)*, Geom. Funct. Anal. 6 (1996) 612–618 MR Zbl

[16] **T Bridgeland**, *Stability conditions on K3 surfaces*, Duke Math. J. 141 (2008) 241–291 MR Zbl

[17] **P Brosnan**, *Perverse obstructions to flat regular compactifications*, Math. Z. 290 (2018) 103–110 MR Zbl

[18] **M A A de Cataldo**, **A Rapagnetta**, **G Saccà**, *The Hodge numbers of O'Grady 10 via Ngô strings*, J. Math. Pures Appl. 156 (2021) 125–178 MR Zbl

[19] **F Charles**, **G Mongardi**, **G Pacienza**, *Families of rational curves on holomorphic symplectic varieties and applications to zero-cycles*, preprint (2019) arXiv 1907.10970

[20] **C H Clemens**, **P A Griffiths**, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95 (1972) 281–356 MR Zbl

[21] **H Clemens**, *The infinitesimal Abel–Jacobi mapping for hypersurfaces*, from "The Lefschetz centennial conference, I" (D Sundararaman, editor), Contemp. Math. 58, Amer. Math. Soc., Providence, RI (1986) 81–89 MR Zbl

[22] **A Collino**, *The fundamental group of the Fano surface, I, II*, from "Algebraic threefolds" (A Conte, editor), Lecture Notes in Math. 947, Springer (1982) 209–220 MR Zbl

[23] **A Collino**, **J P Murre**, *The intermediate Jacobian of a cubic threefold with one ordinary double point; an algebraic-geometric approach, I*, Nederl. Akad. Wetensch. Proc. Ser. A 81 (= Indag. Math. 40) (1978) 43–55 MR Zbl

[24] **A Dimca**, *On the homology and cohomology of complete intersections with isolated singularities*, Compositio Math. 58 (1986) 321–339 MR Zbl

[25] **R Donagi**, **E Markman**, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, from "Integrable systems and quantum groups" (M Francaviglia, S Greco, editors), Lecture Notes in Math. 1620, Springer (1996) 1–119 MR Zbl

[26] **S Druel**, *Espace des modules des faisceaux de rang 2 semi-stables de classes de Chern $c_1 = 0,\ c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}^4$*, Int. Math. Res. Not. (2000) 985–1004 MR Zbl

[27] **S Druel**, *Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle*, Math. Z. 267 (2011) 413–423 MR Zbl

[28] **F El Zein**, **S Zucker**, *Extendability of normal functions associated to algebraic cycles*, from "Topics in transcendental algebraic geometry" (P Griffiths, editor), Ann. of Math. Stud. 106, Princeton Univ. Press (1984) 269–288 MR Zbl

[29] **D Greb**, **C Lehn**, **S Rollenske**, *Lagrangian fibrations on hyperkähler manifolds—on a question of Beauville*, Ann. Sci. Éc. Norm. Supér. 46 (2013) 375–403 MR Zbl

[30] **P A Griffiths**, *On the periods of certain rational integrals, I, II*, Ann. of Math. 90 (1969) 460–541 MR Zbl

[31] **P A Griffiths**, *Periods of integrals on algebraic manifolds, III: Some global differential-geometric properties of the period mapping*, Inst. Hautes Études Sci. Publ. Math. 38 (1970) 125–180 MR Zbl

[32] **C D Hacon**, **S J Kovács**, *Classification of higher dimensional algebraic varieties*, Oberwolfach Seminars 41, Birkhäuser, Basel (2010) MR Zbl

[33] **J Harris**, **M Roth**, **J Starr**, *Curves of small degree on cubic threefolds*, Rocky Mountain J. Math. 35 (2005) 761–817 MR Zbl

[34] **B Hassett**, *Special cubic fourfolds*, Compositio Math. 120 (2000) 1–23 MR Zbl

[35] **D Huybrechts**, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. 135 (1999) 63–113 MR Zbl

[36] **D Huybrechts**, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. 326 (2003) 499–513 MR Zbl

[37] **D Huybrechts**, *The K3 category of a cubic fourfold*, Compos. Math. 153 (2017) 586–620 MR Zbl

[38] **D Kaledin**, **M Lehn**, **C Sorger**, *Singular symplectic moduli spaces*, Invent. Math. 164 (2006) 591–614 MR Zbl

[39] **J Kollár**, *Rational curves on algebraic varieties*, Ergebnesse der Math. (3) 32, Springer (1996) MR Zbl

[40] **J Kollár**, *Deformations of elliptic Calabi–Yau manifolds*, from "Recent advances in algebraic geometry" (C D Hacon, M Mustaţă, M Popa, editors), London Math. Soc. Lecture Note Ser. 417, Cambridge Univ. Press (2015) 254–290 MR Zbl

[41] **J Kollár**, **R Laza**, **G Saccà**, **C Voisin**, *Remarks on degenerations of hyper-Kähler manifolds*, Ann. Inst. Fourier (Grenoble) 68 (2018) 2837–2882 MR Zbl

[42] **J Kollár**, **S Mori**, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics 134, Cambridge Univ. Press (1998) MR Zbl

[43] **A Kuznetsov**, *Derived categories of cubic fourfolds*, from "Cohomological and geometric approaches to rationality problems" (F Bogomolov, Y Tschinkel, editors), Progr. Math. 282, Birkhäuser, Boston, MA (2010) 219–243 MR Zbl

[44] **M Lahoz**, **M Lehn**, **E Macrì**, **P Stellari**, *Generalized twisted cubics on a cubic fourfold as a moduli space of stable objects*, J. Math. Pures Appl. 114 (2018) 85–117 MR Zbl

[45] **C-J Lai**, *Varieties fibered by good minimal models*, Math. Ann. 350 (2011) 533–547 MR Zbl

[46] **R Laza**, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. 172 (2010) 673–711 MR Zbl

[47] **R Laza**, **G Saccà**, **C Voisin**, *A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4–fold*, Acta Math. 218 (2017) 55–135 MR Zbl

[48] **J Le Potier**, *Faisceaux semi-stables de dimension* 1 *sur le plan projectif*, Rev. Roumaine Math. Pures Appl. 38 (1993) 635–678 MR Zbl

[49] **C Lehn**, **M Lehn**, **C Sorger**, **D van Straten**, *Twisted cubics on cubic fourfolds*, J. Reine Angew. Math. 731 (2017) 87–128 MR Zbl

[50] **M Lehn**, **C Sorger**, *La singularité de O'Grady*, J. Algebraic Geom. 15 (2006) 753–770 MR Zbl

[51] **C Li**, **L Pertusi**, **X Zhao**, *Elliptic quintics on cubic fourfolds, O'Grady 10, and Lagrangian fibrations*, Adv. Math. 408 (2022) art. id. 108584 MR Zbl

[52] **E Markman**, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, from "Complex and differential geometry" (W Ebeling, K Hulek, K Smoczyk, editors), Springer Proc. Math. 8, Springer (2011) 257–322 MR Zbl

[53] **E Markman**, *Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections*, Kyoto J. Math. 53 (2013) 345–403 MR Zbl

[54] **D Markushevich**, **A S Tikhomirov**, *The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold*, J. Algebraic Geom. 10 (2001) 37–62 MR Zbl

[55] **D Matsushita**, *On fibre space structures of a projective irreducible symplectic manifold*, Topology 38 (1999) 79–83 MR Zbl

[56] **D Matsushita**, *On almost holomorphic Lagrangian fibrations*, Math. Ann. 358 (2014) 565–572 MR Zbl

[57] **D Matsushita**, *On deformations of Lagrangian fibrations*, from "K3 surfaces and their moduli" (C Faber, G Farkas, G van der Geer, editors), Progr. Math. 315, Birkhäuser, Cham (2016) 237–243 MR Zbl

[58] **C Meachan**, **Z Zhang**, *Birational geometry of singular moduli spaces of O'Grady type*, Adv. Math. 296 (2016) 210–267 MR Zbl

[59] **Y Miyaoka**, **S Mori**, *A numerical criterion for uniruledness*, Ann. of Math. 124 (1986) 65–69 MR Zbl

[60] **S Mukai**, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. 77 (1984) 101–116 MR Zbl

[61] **N Nakayama**, *Zariski-decomposition and abundance*, MSJ Memoirs 14, Math. Soc. Japan, Tokyo (2004) MR Zbl

[62] **Y Namikawa**, *Deformation theory of singular symplectic n–folds*, Math. Ann. 319 (2001) 597–623 MR Zbl

[63] **K G O'Grady**, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. 512 (1999) 49–117 MR Zbl

[64] **K Oguiso**, *Picard number of the generic fiber of an abelian fibered hyperkähler manifold*, Math. Ann. 344 (2009) 929–937 MR Zbl

[65] **K Oguiso**, *Shioda–Tate formula for an abelian fibered variety and applications*, J. Korean Math. Soc. 46 (2009) 237–248 MR Zbl

[66] **A Perego**, **A Rapagnetta**, *Deformation of the O'Grady moduli spaces*, J. Reine Angew. Math. 678 (2013) 1–34 MR Zbl

[67] **A Perego**, **A Rapagnetta**, *Factoriality properties of moduli spaces of sheaves on abelian and K3 surfaces*, Int. Math. Res. Not. 2014 (2014) 643–680 MR Zbl

[68] **A Perry**, *The integral Hodge conjecture for two-dimensional Calabi–Yau categories*, Compos. Math. 158 (2022) 287–333 MR Zbl

[69] **C A M Peters**, **J H M Steenbrink**, *Mixed Hodge structures*, Ergebnesse der Math. (3) 52, Springer (2008) MR Zbl

[70] **Z Ran**, *Hodge theory and the Hilbert scheme*, J. Differential Geom. 37 (1993) 191–198 MR Zbl

[71] **A Rapagnetta**, *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. 340 (2008) 77–95 MR Zbl

[72] **U Rieß**, *On Beauville's conjectural weak splitting property*, Int. Math. Res. Not. 2016 (2016) 6133–6150 MR Zbl

[73] **J Sawon**, *On the discriminant locus of a Lagrangian fibration*, Math. Ann. 341 (2008) 201–221 MR Zbl

[74] **J Sawon**, *Deformations of holomorphic Lagrangian fibrations*, Proc. Amer. Math. Soc. 137 (2009) 279–285 MR Zbl

[75] **S Takayama**, *On uniruled degenerations of algebraic varieties with trivial canonical divisor*, Math. Z. 259 (2008) 487–501 MR Zbl

[76] **C Voisin**, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, from "Complex projective geometry" (G Ellingsrud, C Peskine, G Sacchiero, S A Strømme, editors), London Math. Soc. Lecture Note Ser. 179, Cambridge Univ. Press (1992) 294–303 MR Zbl

[77] **C Voisin**, *Hodge theory and complex algebraic geometry, II*, Cambridge Studies in Advanced Mathematics 77, Cambridge Univ. Press (2003) MR Zbl

[78] **C Voisin**, *Some aspects of the Hodge conjecture*, Jpn. J. Math. 2 (2007) 261–296 MR Zbl

[79] **C Voisin**, *Chow rings, decomposition of the diagonal, and the topology of families*, Annals of Mathematics Studies 187, Princeton Univ. Press (2014) MR Zbl

[80] **C Voisin**, *Hyper-Kähler compactification of the intermediate Jacobian fibration of a cubic fourfold: the twisted case*, from "Local and global methods in algebraic geometry" (N Budur, T de Fernex, R Docampo, K Tucker, editors), Contemp. Math. 712, Amer. Math. Soc., Providence, RI (2018) 341–355 MR Zbl

[81] **C Voisin**, *Torsion points of sections of Lagrangian torus fibrations and the Chow ring of hyper-Kähler manifolds*, from "Geometry of moduli" (J A Christophersen, K Ranestad, editors), Abel Symp. 14, Springer (2018) 295–326 MR Zbl

[82] **K Yoshioka**, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. 321 (2001) 817–884 MR Zbl

[83] **S Zucker**, *Generalized intermediate Jacobians and the theorem on normal functions*, Invent. Math. 33 (1976) 185–222 MR Zbl

[84] **S Zucker**, *The Hodge conjecture for cubic fourfolds*, Compositio Math. 34 (1977) 199–209 MR Zbl

Department of Mathematics, Columbia University
New York, NY, United States

Institut de mathématiques de Jussieu
Paris, France

gs3032@columbia.edu, claire.voisin@imj-prg.fr

Proposed: Richard P Thomas Received: 5 January 2021
Seconded: Mark Gross, Dan Abramovich Revised: 8 November 2021

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2023 is US \$740/year for the electronic version, and \$1030/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 27 Issue 4 (pages 1273–1655) 2023