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Higher genus FJRW invariants of a Fermat cubic

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We reconstruct all-genus Fan–Jarvis–Ruan–Witten invariants of a Fermat cubic Landau–Ginzburg space $(x_1^3 + x_2^3 + x_3^3 \colon [\mathbb{C}^3/\mu_3] \to \mathbb{C})$ from genus-one primary invariants, using tautological relations and axioms of cohomological field theories. The genus-one primary invariants satisfy a Chazy equation by the Belorousski–Pandharipande relation. They are completely determined by a single genus-one invariant, which can be obtained from cosection localization and intersection theory on moduli of three-spin curves.

We solve an all-genus *Landau–Ginzburg/Calabi–Yau correspondence conjecture* for the Fermat cubic Landau–Ginzburg space using Cayley transformation on quasimodular forms. This transformation relates two nonsemisimple CohFT theories: the Fan–Jarvis–Ruan–Witten theory of the Fermat cubic polynomial and the Gromov–Witten theory of the Fermat cubic curve. As a consequence, Fan–Jarvis–Ruan–Witten invariants at any genus can be computed using Gromov–Witten invariants of the elliptic curve. They also satisfy nice structures, including holomorphic anomaly equations and Virasoro constraints.

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1 Introduction

Let $(d;\delta)$ be a *weight system* such that $\delta=(\delta_1,\dots,\delta_N)\in\mathbb{Z}_+^N$ is a primitive N–tuple with $w_i:=d/\delta_i\in\mathbb{Z}_+$. We say the system is of *Calabi–Yau* (*CY*) *type* if

$$d=\delta_1+\cdots+\delta_N,\quad \text{ie}\ \sum_{i=1}^N\frac{1}{w_i}=1. \tag{1-1}$$

The *dimension* of the CY-type weight system $(d;\delta)$ is defined to be

$$\hat{c}=\sum_{i=1}^N\left(1-\frac{2\delta_i}{d}\right)=N-2.$$

Let μ_d be the multiplicative group consisting of d^{th} roots of unity and

$$J_\delta=(\zeta_d^{\delta_1},\dots,\zeta_d^{\delta_N})\in\mu_d\quad\text{for } \zeta_d:=\exp\left(\frac{2\pi\sqrt{-1}}{d}\right).$$

We call the data $([\mathbb{C}^N/\langle J_\delta\rangle],W)$ a Landau–Ginzburg (LG) space, where W is a nondegenerate quasihomogeneous polynomial on $\mathbb{C}^N$ satisfying

$$W(\lambda^{\delta_1}x_1,\dots,\lambda^{\delta_N}x_N)=\lambda^dW(x_1,\dots,x_N)\quad\text{for all } \lambda\in\mathbb{C}^*.$$

The polynomial W is assumed to have only an isolated critical point at the origin and not involve quadratic terms x_ix_j for $i\neq j$. In general, we can consider Landau–Ginzburg spaces $([\mathbb{C}^N/G],W)$ for a group G which is a subgroup of the group of diagonal symmetries with $J_\delta\in G$; see Chang, J Li and W-P Li [6] and Fan, Jarvis and Ruan [20]. Two enumerative theories can be associated to such an LG space:

- The first is the Gromov–Witten (GW) theory of the $G/\langle J_\delta\rangle$–quotient of the hypersurface defined by the vanishing of W in the corresponding weighted projective space $\mathbb{P}^{N-1}(\delta_1,\dots,\delta_N)$. The quotient space is a CY $(N-2)$–orbifold by the CY condition in (1-1).
- The second is the Fan–Jarvis–Ruan–Witten (FJRW) theory of the pair (W,G) as introduced by Fan, Jarvis and Ruan [19; 20].

Both the GW theory and the FJRW theory associated to a CY-type weight system are cohomological field theories (CohFT, for short) in the sense of Kontsevich and Manin [32].

We shall focus on the theories arising from one-dimensional CY-type weight systems. These systems are classified by

$$(d;\delta) = (3;1,1,1), (4;1,1,2), (6;1,2,3). \tag{1-2}$$

The LG space we consider is $([\mathbb{C}^3/\langle J_\delta\rangle], W)$, with W the Fermat polynomials

$$W = x_1^{d/\delta_1} + x_2^{d/\delta_2} + x_3^{d/\delta_3}. \tag{1-3}$$

On the CY side, the hypersurface $W=0$ in the weighted projective space $\mathbb{P}^2(\delta_1,\delta_2,\delta_3)$ is an elliptic curve, denoted by $\mathcal{E}_d$ or $\mathcal{E}$ (when the degree d is implicit or unimportant in the discussion) for simplicity. We focus on the GW theory of $\mathcal{E}$. The GW state space is then defined to be $\mathcal{H}_{\mathcal{E}} := H^*(\mathcal{E},\mathbb{C})$. Let $\overline{\mathcal{M}}_{g,n}(\mathcal{E},\beta)$ be the moduli stack of degree-β stable maps from a connected genus-g curve with n markings to the target $\mathcal{E}$. Let ev_k for $k=1,2,\dots,n$ be the evaluation morphisms, π be the forgetful morphism, and $[\overline{\mathcal{M}}_{g,n}(\mathcal{E},\beta)]^{\mathrm{vir}}$ be the virtual fundamental cycle of $\overline{\mathcal{M}}_{g,n}(\mathcal{E},\beta)$. The *ancestor GW invariants* are given by

$$\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle^{\mathcal{E}}_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{E},\beta)]^{\mathrm{vir}}} \prod_{k=1}^{n} \mathrm{ev}_k^*(\alpha_k)\pi^*\psi_k^{\ell_k}.$$

The *ancestor GW correlation function* is the formal q–series

$$\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q) = \sum_{d\ge 0} q^{\beta}\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle^{\mathcal{E}}_{g,n,\beta}. \tag{1-4}$$

By the virtual degree counting of $[\overline{\mathcal{M}}_{g,n}(\mathcal{E},\beta)]^{\mathrm{vir}}$, if the series

$$\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q)$$

in (1-4) is nontrivial, then

$$\sum_{k=1}^{n}\left(\tfrac{1}{2}\deg\alpha_k + \ell_k\right) = (3-\dim_{\mathbb{C}}\mathcal{E})(g-1)+n = 2g-2+n. \tag{1-5}$$

On the LG side, we consider the FJRW theory of the pair $(W,\langle J_\delta\rangle)$ as originally constructed in [19; 20]. The main ingredients consist of a CohFT

$$(\mathcal{H}_{(W,\langle J_\delta\rangle)}, \langle\,,\rangle, \mathbf{1}, \Lambda^{(W,\langle J_\delta\rangle)})$$

and FJRW invariants (see Section 2.1 for details)

$$\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle^{(W,\langle J_\delta\rangle)}_{g,n},$$

with α_i elements in the vector space $\mathscr{H}_{(W,\langle J_\delta\rangle)}$. The space $\mathscr{H}_{(W,\langle J_\delta\rangle)}$ contains a canonical degree-2 element, denoted by ϕ below. We assemble the FJRW invariants into an *ancestor FJRW correlation function* (as a formal series in s)

$$\langle\!\langle \alpha_1\psi_1^{\ell_1},\ldots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{(W,\langle J_\delta\rangle)}(s) := \sum_{m=0}^{\infty}\frac{1}{m!}\langle \alpha_1\psi_1^{\ell_1},\ldots,\alpha_n\psi_n^{\ell_n},\underbrace{s\phi,\ldots,s\phi}_{m}\rangle_{g,n+m}^{(W,\langle J_\delta\rangle)}. \tag{1-6}$$

1.1 LG/CY correspondence via modularity

One of the motivations for constructing the FJRW invariants [19; 20] is to understand mathematically the so-called *Landau–Ginzburg/Calabi–Yau correspondence* proposed by physicists; see Greene, Vafa and Warner [24; 54], Martinec [39] and Witten [56]. The *Landau–Ginzburg/Calabi–Yau correspondence conjecture* (see Chiodo and Ruan [12; 48] and Fan, Jarvis and Ruan [20]) predicts that for a CY-type weight system the corresponding GW and FJRW theories are related. In the past decade, a lot of effort has been made to formulate and solve this conjecture:

- An LG/CY correspondence between the vector spaces was solved by Chiodo and Ruan [13].
- Genus-zero LG/CY correspondence for various pairs (W, G) has been studied using Givental's I–functions; see Basalaev and Priddis [1], Chiodo, Iritani and Ruan [10; 11], Clader [14] and Lee, Priddis and Shoemaker [36; 37].
- For the quintic 3–fold, the correspondence has been pushed to genus one; see Guo and Ross [25].
- For higher genera, the only known examples in the literature (see Iritani, Milanov, Ruan and Shen [29; 40; 41], Krawitz and Shen [34] and Shen and Zhou [50]) are all generically semisimple, and therefore the correspondence at higher genus is a consequence of the genus-zero correspondence, based on Givental [23] and Teleman's [52] classification of semisimple CohFTs.

One of our main results is to solve this conjecture at all genera for the Fermat cubic pair ($W = x_1^3 + x_2^3 + x_3^3$, J_δ), using the properties of moduli spaces and quasimodular forms. We remark that the GW CohFT and the FJRW CohFT for such a pair are not generically semisimple, and therefore this case is beyond the scope of Givental and Teleman's results.

1.1.1 Quasimodular forms and the Chazy equation Specializing to the cases of one-dimensional CY-type weight systems, it is known (see Bloch and Okounkov [3] and Okounkov and Pandharipande [43]) that the GW correlation functions for an elliptic curve are quasimodular forms; see Kaneko and Zagier [30]. The key of this work is to relate the generating series in (1-4) and (1-6) using transformations on quasimodular forms.

Consider the Eisenstein series

$$E_{2k}(\tau) := \frac{1}{2\zeta(2k)} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(c\tau+d)^{2k}} \quad \text{for } \tau\in\mathbb{H}, \tag{1-7}$$

where ζ is the Riemann zeta function. These are holomorphic functions on the upper half-plane $\mathbb{H}$, of which E_{2k} for $k\geq 2$ are modular under the group $\Gamma := \mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$, while E_2 is *quasimodular* [30]. To be more precise, E_2 is not modular, but its nonholomorphic modification $\widehat{E}_2(\tau,\bar\tau)$ is modular, where

$$\widehat{E}_2(\tau,\bar\tau) := E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)}.$$

The set of quasimodular forms (we regard modular forms as special cases of quasi-modular forms) for Γ form a ring [30]:

$$\widetilde{M}_*(\Gamma) := \mathbb{C}[E_2(\tau), E_4(\tau), E_6(\tau)]. \tag{1-8}$$

The set of almost-holomorphic modular forms as introduced in [30] also gives rise to a ring that is isomorphic to $\widetilde{M}_*(\Gamma)$:

$$\widehat{M}_*(\Gamma) := \mathbb{C}[\widehat{E}_2(\tau,\bar\tau), E_4(\tau), E_6(\tau)]. \tag{1-9}$$

Let $q=\exp(2\pi\sqrt{-1}\tau)$. The GW invariants of elliptic curves are (see [43]) Fourier coefficients expanded around the infinity cusp $\tau=\sqrt{-1}\infty$ of certain quasimodular forms. For example,[1] let $\omega\in H^2(\mathcal{E})$ be the Poincaré dual of the point class. Then

$$-24\langle\!\langle\omega\rangle\!\rangle_{1,1}^{\mathcal{E}}(q) = E_2(q) = 1-24\sum_{n=1}^{\infty} n\frac{q^n}{1-q^n}. \tag{1-10}$$

For any $f\in\widetilde{M}_*(\Gamma)$, we define

$$f'(\tau) := \frac{1}{2\pi\sqrt{-1}}\frac{df}{d\tau}.$$

[1]We are sometimes sloppy about the argument for a quasimodular form when no confusion should arise. For instance, we shall occasionally write $E_k(q)$ for $E_k(\tau)$.

The Eisenstein series E_2, E_4 and E_6 satisfy the so-called Ramanujan identities

$$E_2' = \tfrac{1}{12}(E_2^2 - E_4), \quad E_4' = \tfrac{1}{3}(E_2E_4 - E_6), \quad E_6' = \tfrac{1}{2}(E_2E_6 - E_4^2). \tag{1-11}$$

Eliminating E_4 and E_6, we see that E_2 is a solution to the so-called Chazy equation,

$$2f''' - 2ff'' + 3(f')^2 = 0. \tag{1-12}$$

Our key observation is that the Chazy equation (1-12) appears in both GW and FJRW theory for one-dimensional CY weight systems, thanks to the Belorousski–Pandharipande relation discovered in [2].

Proposition 1 *Consider the LG space $([\mathbb{C}^3/\langle J_\delta\rangle], W)$ given by (1-2) and (1-3). Then both the genus-one GW correlation function $-24\langle\!\langle\omega\rangle\!\rangle_{1,1}^{\mathcal{E}}(q)$ and the genus-one FJRW correlation function $-24\langle\!\langle\phi\rangle\!\rangle_{1,1}^{(W,\langle J_\delta\rangle)}(s)$ are solutions to the Chazy equation (1-12).*

Here for a function $f(q)$ in q, we use the convention $f'(q) = q\partial_q f$; for a function $f(s)$ in s, $f'(s) = \partial_s f$.

Further, using more tautological relations discovered by Faber and Pandharipande [17] and Ionel [28], we can show that both the GW and FJRW correlation functions in (1-4) and (1-6) are determined by the genus-one correlation functions in Proposition 1.

Proposition 2 *Consider the LG space $([\mathbb{C}^3/\langle J_\delta\rangle], W)$ given by (1-2) and (1-3). Let*

$$f = -24\langle\!\langle\omega\rangle\!\rangle_{1,1}^{\mathcal{E}} \quad \text{or} \quad f = -24\langle\!\langle\phi\rangle\!\rangle_{1,1}^{(W,\langle J_\delta\rangle)}.$$

Then the GW correlation functions in (1-4) (or the FJRW correlation functions in (1-6)) are determined from f by tautological relations and are elements in the ring $\mathbb{C}[f, f', f'']$.

1.1.2 LG/CY correspondence via Cayley transformation By direct calculation, we can show $\langle\!\langle\omega\rangle\!\rangle_{1,1}^{\mathcal{E}}(q)$ and $\langle\!\langle\phi\rangle\!\rangle_{1,1}^{(W,\langle J_\delta\rangle)}(s)$ are expansions of the same quasimodular form $-\frac{1}{24}E_2(\tau)$ at two different points on the upper half-plane. In particular, the GW functions are Fourier expansions around the cusp $\tau = \sqrt{-1}\infty$. This viewpoint allows us to relate the GW functions in (1-4) and the FJRW functions in (1-6) by a variant of the Cayley transformation which we now briefly review, following Shen and Zhou [50].

For any point $\tau_* \in \mathbb{H}$, there exists a Cayley transform that maps a point τ on the upper half-plane $\mathbb{H}$ to a point $s(\tau)$ in the unit disk $\mathbb{D}$, namely

$$s(\tau) = (\tau_* - \bar{\tau}_*)\frac{\tau - \tau_*}{\tau - \bar{\tau}_*}.$$

This transform is biholomorphic, and we denote its inverse by $\tau(s)$. Following Zagier [57] and [50], there exists a Cayley transformation that maps a weight-k almost-holomorphic modular form

$$\hat{f} \in \widehat{M}_*(\Gamma) = \mathbb{C}[\widehat{E}_2(\tau,\bar{\tau}), E_4(\tau), E_6(\tau)]$$

to

$$\left(\frac{\tau(s)-\bar{\tau}_*}{\tau_*-\bar{\tau}_*}\right)^k \hat{f}(\tau(s), \overline{\tau(s)}). \tag{1-13}$$

The Taylor expansion of the image gives a natural way to expand the almost-holomorphic modular form $\hat{f}$ near $\tau = \tau_*$, where the local complex coordinate is $s(\tau)$.

Using the fact that $\widetilde{M}_*(\Gamma)$ and $\widehat{M}_*(\Gamma)$ are isomorphic differential rings, a *holomorphic Cayley transformation* $\mathscr{C}^{\mathrm{hol}}_{\tau_*}$ (see Section 4) can then be defined [50]. This turns out to be the correct transformation to relate the GW correlation functions in (1-4) and the FJRW correlation functions in (1-6), both of which are holomorphic, and it allows us to solve the LG/CY correspondence conjecture for the Fermat cubic pair.

Theorem 3 *Consider the Fermat cubic polynomial $W = x_1^3 + x_2^3 + x_3^3$ and the LG space $([\mathbb{C}^3/\mu_3], W)$. There exists a degree- and grading-preserving vector space isomorphism*

$$\Psi\colon \mathscr{H}_{\mathcal{E}} = H^*(\mathcal{E}) \to \mathscr{H}_{(W,\mu_3)}$$

and a holomorphic Cayley transformation $\mathscr{C}^{\mathrm{hol}}_{\tau_*}$ with

$$\tau_* = -\frac{\sqrt{-1}}{\sqrt{3}} \exp\left(\frac{2\pi\sqrt{-1}}{3}\right) \in \mathbb{H},$$

such that

$$\mathscr{C}^{\mathrm{hol}}_{\tau_*}(\langle\!\langle \alpha_1\psi_1^{\ell_1}, \dots, \alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q)) = \langle\!\langle \Psi(\alpha_1)\psi_1^{\ell_1}, \dots, \Psi(\alpha_n)\psi_n^{\ell_n}\rangle\!\rangle^{(W,\mu_3)}_{g,n}(s).$$

The explicit construction of Ψ and $\mathscr{C}^{\mathrm{hol}}_{\tau_*}$ will be given in Section 4.

It is straightforward to generalize Theorem 3 to the rest of the one-dimensional CY-type weight systems in (1-2); the only difference lies in the technical computations on the initial genus-one FJRW invariants. This approach of using modular forms was previously introduced in [50] for elliptic orbifold curves.

It is worthwhile to mention that for one-dimensional CY-type weight systems, our approach of the LG/CY correspondence is compatible with the I–function approach introduced by Chiodo and Ruan [11] and Milanov and Ruan [40]. In fact, the automorphy factor in the Cayley transformation (1-13) provides equivalent information to the symplectic transformation that appears in [11, Corollary 4.2.4].

1.2 Applications: higher-genus FJRW invariants and their structures

The higher-genus FJRW invariants are very difficult to compute in general. In our example, with the identification of the correlation functions with quasimodular forms, various results from the GW side can be transformed into the LG side via the holomorphic Cayley transformation, which respects the differential ring structure of quasimodular forms. In particular, higher-genus FJRW invariants can be computed easily, and nice structures of the FJRW correlation functions can be obtained for free.

Indeed, higher-genus FJRW invariants are determined from the results on descendent GW invariants of elliptic curves, given by Bloch and Okounkov [3], whose generating series admit very concrete and beautiful formulae. The following gives a sample of the computations.

Corollary 4 *For the ancestor FJRW correlation functions, when $d=3$,*

$$\langle\!\langle \phi\psi_1^{2g-2}\rangle\!\rangle_{g,1}^{(W,\mu_3)} = \sum_{\substack{\ell,m,n\geq 0\\ \ell+2m+3n=g}} \frac{b_{m,n}}{\ell!}\left(-\frac{\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2)}{24}\right)^{\ell}\left(\frac{\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_4)}{24}\right)^{m}\left(-\frac{\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_6)}{108}\right)^{n},$$

where $\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_{2i})$ for $i=1,2,3$ are holomorphic Cayley transformations of the Eisenstein series E_2, E_4, E_6 whose expansions can be computed explicitly, while $\{b_{m,n}\}_{m,n}$ are rational numbers that can be obtained recursively.

The holomorphic anomaly equations (HAEs) discovered by Oberdieck and Pixton [42] and the Virasoro constraints discovered by Okounkov and Pandharipande [44] for the GW theory of elliptic curves also carry over to the corresponding FJRW theory. See Corollaries 23 and 24 for the explicit statements.

Outline In Section 2 we review the basic construction of CohFTs and use tautological relations, in particular the Belorousski–Pandharipande relation, to prove Propositions 1 and 2. In Section 3 we calculate a genus-one FJRW invariant for the $d=3$ case using cosection localization. In Section 4 we prove Theorem 3 using properties of quasimodular forms. In Section 5 we review some results on GW invariants for the elliptic curve and discuss the ancestor/descendent correspondence. In Section 6 we give some applications of the quasimodularity of the GW and FJRW theory for the $d=3$ case, such as the explicit computations of higher-genus FJRW invariants based on the results on the GW invariants of the elliptic curve, the derivation of holomorphic anomaly equations and Virasoro constraints they satisfy.

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2 The Belorousski–Pandharipande relation and the Chazy equation

We study the two cohomological field theories (GW and FJRW) for the one-dimensional CY-type weight systems using tautological relations and axioms of CohFTs. The key is the identification between the Belorousski–Pandharipande relation and the Chazy equation.

2.1 Cohomological field theories

Both the GW theory and the FJRW theory of the LG space $([\mathbb{C}^N/G], W)$ satisfy axioms of cohomological field theories (CohFT) in the sense of [32], which we briefly recall.

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford moduli stack of genus-g stable (ie $2g-2+n>0$) curves with n markings. A *cohomological field theory with a flat identity* is a quadruple

$$(\mathcal{H}, \eta, \mathbf{1}, \Lambda),$$

where the *state space*

$$\mathcal{H} := \mathcal{H}^{\text{even}} \oplus \mathcal{H}^{\text{odd}}$$

is a $\mathbb{Z}_2$–graded finite-dimensional $\mathbb{C}$–vector space (called a superspace in [32]), η is a nondegenerate pairing on $\mathcal{H}$, $\mathbf{1} \in \mathcal{H}$ is the *flat identity*, and

$$\Lambda := \{\Lambda_{g,n} \in \operatorname{Hom}(\mathcal{H}^{\otimes n}, H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}))\}$$

is a set of multilinear maps satisfying the CohFT axioms below:

(i) Let $|\cdot|$ be the grading. The maps $\Lambda_{g,n}$ satisfy

$$\Lambda_{g,n}(\dots,\alpha_1,\alpha_2,\dots) = (-1)^{|\alpha_1||\alpha_2|}\Lambda_{g,n}(\dots,\alpha_2,\alpha_1,\dots). \tag{2-1}$$

(ii) The maps in Λ are compatible with the gluing and the forgetful morphisms

- $\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g-1,n+2}\to\overline{\mathcal{M}}_{g,n}$,
- $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ forgetting one of the markings.

For example, the compatibility with the forgetful morphism is

$$\Lambda_{g,n+1}(\alpha_1,\dots,\alpha_n,\mathbf{1}) = \pi^*\Lambda_{g,n}(\alpha_1,\dots,\alpha_n). \tag{2-2}$$

(iii) The pairing η is compatible with $\Lambda_{0,3}$:

$$\int_{\overline{\mathcal{M}}_{0,3}}\Lambda_{0,3}(\alpha_1,\alpha_2,\mathbf{1}) = \eta(\alpha_1,\alpha_2).$$

Let $\psi_k\in H^2(\overline{\mathcal{M}}_{g,n})$ be the cotangent line class at the k^{th} marking. For each CohFT $(\mathcal{H},\eta,\mathbf{1},\Lambda)$, one defines the quantum invariants from Λ by

$$\langle\alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle_{g,n}^{\Lambda} := \int_{\overline{\mathcal{M}}_{g,n}}\Lambda_{g,n}(\alpha_1,\dots,\alpha_n)\prod_{k=1}^{n}\psi_k^{\ell_k} \quad \text{for } \alpha_k\in\mathcal{H}. \tag{2-3}$$

Such invariants are called the ancestor GW invariants for the GW CohFT, and FJRW invariants for the LG CohFT. Our focus is the relation between these two types of invariants arising from the same CY-type LG space $([\mathbb{C}^N/G], W)$.

Fix a basis $\mathcal{B}$ for $\mathcal{H}$. It is convenient to choose the elements α_k from $\mathcal{B}$ and parametrize α_k by s_k. We introduce the *genus-zero primary potential of the CohFT* as a formal power series

$$\mathcal{F}_0^{\Lambda} := \sum_{n\geq 0}\sum_{\alpha_k\in\mathcal{B}}\frac{1}{n!}\langle\alpha_1,\dots,\alpha_n\rangle_{0,n}^{\Lambda}\prod_{k=1}^{n}s_k. \tag{2-4}$$

Here primary means all $\ell_k=0$ in (2-3).

2.1.1 FJRW invariants The CohFTs arising from GW theories have become a familiar topic since [32]. Here we only recall some basics on the LG CohFT constructed from the FJRW invariants defined in [19; 20]. See also [4; 6; 31; 46] for various CohFT constructions for LG models.

As G acts on $\mathbb{C}^N$, for any $\gamma \in G$, the fixed-point set $\mathrm{Fix}(\gamma)$ is an N_γ–dimensional subspace of $\mathbb{C}^N$. Let W_γ be the restriction of W on $\mathrm{Fix}(\gamma)$. Following [20], one considers the graded vector space (called the FJRW state space)

$$\mathscr{H}_{(W,G)} = \bigoplus_{\gamma \in G} \mathscr{H}_\gamma, \tag{2-5}$$

where each $\mathscr{H}_\gamma$ is the space of G-invariants of the middle-dimensional relative cohomology in $\mathrm{Fix}(\gamma)$. There is a natural pairing $\langle\, ,\rangle$ and an isomorphism (see [20, Section 5.1])

$$(\mathscr{H}_{(W,G)}, \langle\, ,\rangle) \cong \Bigl(\bigoplus_{\gamma \in G} (\mathrm{Jac}(W_\gamma)\Omega_{\mathrm{Fix}(\gamma)})^G, \mathrm{Res}\Bigr). \tag{2-6}$$

Here $\mathrm{Jac}(W_\gamma)$ is the *Jacobi algebra* of W_γ, $\Omega_{\mathrm{Fix}(\gamma)}$ is the standard holomorphic volume form on $\mathrm{Fix}(\gamma)$, and Res is the residue pairing.

In [19; 20], Fan, Jarvis and Ruan constructed the virtual fundamental cycle over the moduli space of W–spin structures, and a corresponding CohFT

$$(\mathscr{H}_{(W,G)}, \langle\, ,\rangle, \mathbf{1}, \Lambda^{(W,G)}),$$

which defines the so-called *FJRW invariants* $\langle \alpha_1\psi_1^{\ell_1}, \dots, \alpha_n\psi_n^{\ell_n}\rangle_{g,n}^{(W,G)}$ through (2-3).

We now specialize to a pair (W,G) given in (1-3) with $G = \langle J_\delta\rangle$. For a set of homogeneous elements $\alpha_k \in \mathscr{H}_{\gamma_k}$ for $k = 1, 2, \dots, n$, the dimension formula in [20, Theorem 4.1.8] shows that, if $\langle \alpha_1\psi_1^{\ell_1}, \dots, \alpha_n\psi_n^{\ell_n}\rangle_{g,n}^{(W,\langle J_\delta\rangle)}$ is nontrivial, then

$$2g - 2 + n = \sum_{k=1}^{n} \tfrac{1}{2} \deg \alpha_k + \sum_{k=1}^{n} \ell_k. \tag{2-7}$$

We remark that both $\mathscr{H}_{J_\delta}$ and $\mathscr{H}_{J_\delta^{-1}}$ are one-dimensional: $\mathscr{H}_{J_\delta}$ is spanned by the flat identity $\mathbf{1} \in \mathscr{H}_{J_\delta}$ and $\mathscr{H}_{J_\delta^{-1}}$ by a canonical degree-2 element $\phi \in \mathscr{H}_{J_\delta^{-1}}$. We let s be the corresponding linear coordinate of the space $\mathscr{H}_{J_\delta^{-1}}$. The constraint (2-7) allows us to define the *ancestor FJRW correlation function* (as a formal series in s)

$$\langle\!\langle \alpha_1\psi_1^{\ell_1}, \dots, \alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{(W,\langle J_\delta\rangle)}(s) := \sum_{m=0}^{\infty} \frac{1}{m!} \langle \alpha_1\psi_1^{\ell_1}, \dots, \alpha_n\psi_n^{\ell_n}, \underbrace{s\phi, \dots, s\phi}_{m}\rangle_{g,n+m}^{(W,\langle J_\delta\rangle)}. \tag{2-8}$$

In the following, we will use the subscript d to label the CY-type weight systems in (1-2). Let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. For each polynomial W_d, when $d = 3$ (resp. 4, 6), we consider the element

$$h(W_d) = \tfrac{1}{27} x_1x_2x_3 \quad \bigl(\text{resp. } \tfrac{1}{32} x_1^2x_2^2,\ \tfrac{1}{36} x_1^4x_2\bigr). \tag{2-9}$$

According to (2-6), the FJRW state space is

$$\mathcal{H}_{(W_d,G_d)} = \mathcal{H}_{J_\delta} \oplus \mathcal{H}_{J_\delta^{-1}} \oplus \mathcal{H}_{1\in G_d} = \mathbb{C}\{\mathbf{1}, \phi, \mathfrak{b}_1, \mathfrak{b}_2\}. \tag{2-10}$$

Here the even part is spanned by $\mathbf{1} \in \mathcal{H}_{J_\delta}$ and $\phi \in \mathcal{H}_{J_\delta^{-1}}$, while the odd part is spanned by

$$\mathfrak{b}_1 = h(W_d)\Omega \quad \text{and} \quad \mathfrak{b}_2 = \Omega \in (\mathrm{Jac}(W_d)\Omega)^G \subseteq \mathcal{H}_{1\in G_d}.$$

The degrees are

$$\deg \mathbf{1} = 0, \quad \deg \mathfrak{b}_1 = \deg \mathfrak{b}_2 = 1, \quad \deg \phi = 2. \tag{2-11}$$

2.1.2 Genus-zero comparison We begin with a comparison between the genus-zero parts of the two theories. On the GW side, recall the state space for the elliptic curve $\mathcal{E}_d$ is $H^*(\mathcal{E}_d, \mathbb{C})$. Let $\mathbf{1} \in H^0$ be the identity of the cup product, and $\omega \in H^2$ be the Poincaré dual of the point class. We choose a symplectic basis $\{e_1, e_2\}$ of H^1 such that

$$e_1 \cup e_2 = -e_2 \cup e_1 = \omega.$$

We define a linear map $\Psi\colon H^*(\mathcal{E}_d) \to \mathcal{H}_{(W_d,\langle J_\delta\rangle)}$ by

$$\Psi(\mathbf{1}) = \mathbf{1}, \qquad \Psi(\omega) = \phi, \qquad \Psi(e_i) = \mathfrak{b}_i \quad \text{for } i = 1, 2. \tag{2-12}$$

Let (t_0, t_1, t_2, t) be the coordinates with respect to the basis $\{\mathbf{1}, e_1, e_2, \omega\}$. Similarly we let (u_0, u_1, u_2, u) be the coordinates with respect to the basis $\{\mathbf{1}, \mathfrak{b}_1, \mathfrak{b}_2, \phi\}$.

The moduli stack $\overline{\mathcal{M}}_{g,n}(\mathcal{E}_d, \beta)$ is empty when $g = 0$ and $\beta > 0$. Then according to (2-4), the *genus-zero primary GW potential* is

$$\mathcal{F}_0^{\mathcal{E}_d} = \tfrac{1}{2} t_0^2 t + t_0 t_1 t_2.$$

A calculation on residue shows that

$$\langle \mathbf{1}, \mathbf{1}, \phi\rangle_{0,3}^{W_d} = \langle \mathbf{1}, \mathfrak{b}_1, \mathfrak{b}_2\rangle_{0,3}^{W_d} = 1 \quad \text{and} \quad \langle \mathbf{1}, \mathfrak{b}_2, \mathfrak{b}_1\rangle_{0,3}^{W_d} = -1. \tag{2-13}$$

Thus the *genus-zero primary FJRW potential* is

$$\mathcal{F}_0^{W_d} = \tfrac{1}{2} u_0^2 u + u_0 u_1 u_2 + \text{quantum corrections}.$$

These quantum corrections vanish as shown below. This was first observed by Francis [21, Section 4.2] using WDVV equations.

Proposition 5 *The map Ψ in (2-12) is a degree- and grading-preserving ring isomorphism, and*

$$\mathcal{F}_0^{W_d} = \tfrac{1}{2} u_0^2 u + u_0 u_1 u_2. \tag{2-14}$$

Proof It is easy to see that Ψ preserves the degree and grading. To show Ψ is a ring isomorphism, it is enough to prove (2-14). The compatibility condition (2-2) implies the string equation in FJRW theory. Combining the degree constraints (2-11) and (2-7), we find that the quantum corrections are encoded in $C_i(s)$, where $C_i(s)$ is the correlation function with i copies of $\mathfrak{b}_1$–insertions and $4-i$ copies of $\mathfrak{b}_2$–insertions. For example,

$$C_0(s) = \langle\!\langle \mathfrak{b}_1, \mathfrak{b}_1, \mathfrak{b}_1, \mathfrak{b}_1 \rangle\!\rangle_{0,4}^{W_d} \quad \text{and} \quad C_3(s) = \langle\!\langle \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_2, \mathfrak{b}_2 \rangle\!\rangle_{0,4}^{W_d}.$$

The $\mathbb{Z}_2$–grading (2-1) shows $C_i(s) = 0$, because for $\alpha = \mathfrak{b}_1$ or $\mathfrak{b}_2$,

$$\langle\!\langle \alpha, \alpha, \dots \rangle\!\rangle_{g,n}^{W_d} = (-1)^{|\alpha||\alpha|} \langle\!\langle \alpha, \alpha, \dots \rangle\!\rangle_{g,n}^{W_d} = -\langle\!\langle \alpha, \alpha, \dots \rangle\!\rangle_{g,n}^{W_d}. \qquad \square$$

2.2 The Belorousski–Pandharipande relation and g–reduction

The tautological rings $RH(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ are defined (see [17] for example) as the smallest system of subrings of $H^*(\overline{\mathcal{M}}_{g,n})$ stable under pushforward and pullback by the gluing and forgetful morphisms. Thus pulling back the tautological relations in $RH(\overline{\mathcal{M}}_{g,n})$ via the CohFT maps $\Lambda_{g,n}$ gives relations among quantum invariants. We use this technique to prove Propositions 1 and 2.

2.2.1 The Belorousski–Pandharipande relation for a genus-one correlation function The degree constraints (2-11) and (2-7) show that the nonvanishing genus-one primary FJRW invariants could only come from the coefficients in $\langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s)$. We determine this series and the GW correlation function $\langle\!\langle \omega \rangle\!\rangle_{1,1}^{\mathcal{E}_d}(q)$, up to some initial values, by using the tautological relation found by Belorousski and Pandharipande [2, Theorem 1]. The relation is a nontrivial rational equivalence among codimension-2 descendent stratum classes in $\overline{\mathcal{M}}_{2,3}$, shown in Figure 1.

Each stratum in the relation is represented by the topological type of the stable curve corresponding to the generic moduli point in the stratum. The markings on the stratum are unassigned. The geometric genera of the components are underlined. The cotangent line class ψ always appears on the genus-two component.

Proof of Proposition 1 On the FJRW side, we integrate

$$\Lambda_{2,3}^{W_d}(\phi, \phi, \phi) \in H^4(\overline{\mathcal{M}}_{2,3})$$

over the Belorousski–Pandharipande relation. We read off one term from each stratum.

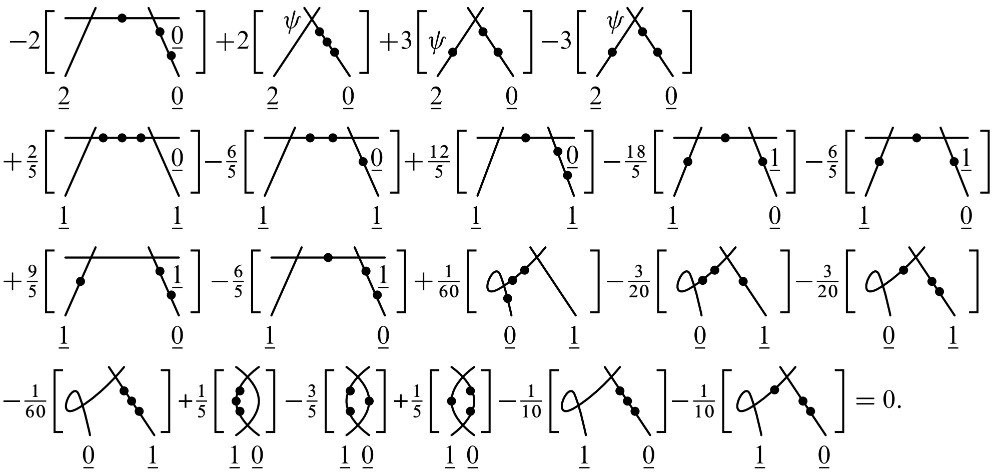

Figure 1: The Belorousski–Pandharipande relation.

Strata in the first row of Figure 1 Let us consider the first stratum in the first row. Integration over this stratum gives the term

$$-2\sum_{\alpha,\alpha',\beta,\beta'\in\mathscr{H}_{W_d}}\langle\!\langle\alpha\rangle\!\rangle_{2,1}^{W_d}(s)\eta^{\alpha,\alpha'}\langle\!\langle\alpha',\phi,\beta\rangle\!\rangle_{0,3}^{W_d}(s)\eta^{\beta,\beta'}\langle\!\langle\beta',\phi,\phi\rangle\!\rangle_{0,3}^{W_d}(s).$$

Here the notation $\eta^{\alpha,\alpha'}$ stands for the (α,α') component of the inverse of the paring η, etc. For any homogeneous element $\alpha\in\mathscr{H}_{W_d}$, the degree constraint (2-7) implies that if $\langle\!\langle\alpha\rangle\!\rangle_{2,1}^{W_d}(s)$ is nonzero, then

$$2(2-1)+1=\tfrac{1}{2}\deg\alpha.$$

This contradicts (2-11), where we have $\deg\alpha=0,1,2$. Thus $\langle\!\langle\alpha\rangle\!\rangle_{2,1}^{W_d}(s)=0$, and hence the contribution from this stratum is zero. Similar arguments imply that the contribution from all the strata in the first row of Figure 1 vanish, since the contribution from each stratum must contain one of the following terms as a factor:

$$\langle\!\langle\alpha\rangle\!\rangle_{2,1}^{W_d}(s)=\langle\!\langle\alpha\psi_1\rangle\!\rangle_{2,1}^{W_d}(s)=\langle\!\langle\phi\psi_1,\alpha\rangle\!\rangle_{2,2}^{W_d}(s)=\langle\!\langle\phi,\alpha\psi_2\rangle\!\rangle_{2,2}^{W_d}(s)=0.$$

Other vanishing strata Now we look at the first, second and fifth strata in the second row, the third, fourth and fifth strata in the third row, and the second, third, fifth and sixth strata in the last row. Each stratum has a genus-zero component with at least four markings (including the nodes). According to Proposition 5, for the primary invariants,

$$\langle\!\langle\cdots\rangle\!\rangle_{0,n}^{W_d}=0\quad\text{for all }\ n\geq 4.$$

Thus the integral of $\Lambda_{2,3}^{W_d}(\phi,\phi,\phi)\in H^4(\overline{\mathcal{M}}_{2,3})$ over each of these strata vanishes.

For the first and second strata in the third row, the genus-zero component only contains three markings, but at least two of the markings are labeled with the class ϕ. Again by Proposition 5, we have

$$\langle\!\langle \phi, \phi, \alpha \rangle\!\rangle_{0,3}^{W_d} = 0 \quad \text{for all } \alpha \in \mathcal{H}_{W_d}.$$

So the contribution from these two strata also vanishes.

Finally, the integral on the first stratum in the fourth row also vanishes. This is a consequence of the $\mathbb{Z}_2$–grading. In fact, we apply the degree constraint (2-7) to the genus-one component and find that the nonvanishing contribution from this stratum, if it exists, should be of the form

$$-\frac{1}{60} \sum_{\alpha,\alpha'} \langle\!\langle \phi, \phi, \phi, \phi \rangle\!\rangle_{1,4}^{W_d}(s) \eta^{\phi,\mathbf{1}} \langle\!\langle \mathbf{1}, \alpha, \alpha' \rangle\!\rangle_{0,3}^{W_d} \eta^{\alpha',\alpha}.$$

The vanishing of this term is a direct consequence of the formula (2-13), where

$$\eta^{\phi,\mathbf{1}} = \eta^{\mathfrak{b}_1,\mathfrak{b}_2} = 1 \quad \text{and} \quad \eta^{\mathfrak{b}_2,\mathfrak{b}_1} = -1.$$

Nonvanishing terms Now we see that all the possibly nonvanishing terms are from the third and fourth strata in the second row, and the fourth stratum in the last row. Let us calculate them term by term. The third stratum of the second row gives a possibly nonvanishing term

$$\tfrac{12}{5} \langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s) \eta^{\phi,\mathbf{1}} \langle\!\langle \mathbf{1}, \phi, \mathbf{1} \rangle\!\rangle_{0,3}^{W_d}(s) \eta^{\mathbf{1},\phi} \langle\!\langle \phi, \phi, \phi \rangle\!\rangle_{1,3}^{W_d}(s) = \tfrac{12}{5} g g''.$$

The fourth stratum of the second row gives a possibly nonvanishing term

$$-\tfrac{18}{5} \langle\!\langle \phi, \phi \rangle\!\rangle_{1,2}^{W_d}(s) \eta^{\phi,\mathbf{1}} \langle\!\langle \mathbf{1}, \phi, \mathbf{1} \rangle\!\rangle_{0,3}^{W_d}(s) \eta^{\mathbf{1},\phi} \langle\!\langle \phi, \phi \rangle\!\rangle_{1,2}^{W_d}(s) = -\tfrac{18}{5} g' g'.$$

The fourth stratum of the last row gives a possibly nonvanishing term

$$\tfrac{1}{5} \cdot \tfrac{1}{2} \langle\!\langle \mathbf{1}, \phi, \mathbf{1} \rangle\!\rangle_{0,3}^{W_d}(s) \eta^{\mathbf{1},\phi} \langle\!\langle \phi, \phi, \phi, \phi \rangle\!\rangle_{1,4}^{W_d}(s) \eta^{\phi,\mathbf{1}} = \tfrac{1}{5} \cdot \tfrac{1}{2} g'''.$$

Here the denominator 2 in the term above comes from the automorphism of the graph.

Putting all these together, the Belorousski–Pandharipande relation in Figure 1 allows us to verify by brute-force computation that the correlation function $g := \langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s)$ is a solution to

$$\tfrac{12}{5} g g'' - \tfrac{18}{5} g' g' + \tfrac{1}{5} \cdot \tfrac{1}{2} g''' = 0. \tag{2-15}$$

Thus $-24 \langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s)$ is a solution of the Chazy equation (1-12).

By integrating the GW cycle $\Lambda_{2,3}^{\mathcal{E}_d}(\omega,\omega,\omega)$ over the Belorousski–Pandharipande relation in Figure 1, we similarly see that $-24\langle\!\langle\phi\rangle\!\rangle_{1,1}^{\mathcal{E}_d}(q)$ is a solution of the Chazy equation (1-12). $\square$

The identity (2-15) is independent of the specific form $\mathcal{E}_d$, as should be the case since the GW invariants are independent of the choice of complex structures put on the elliptic curve.

Remark 6 For the elliptic orbifold curve $\mathcal{X}_N := \mathcal{E}^{(N)}/\mu_N$ for some particular elliptic curve $\mathcal{E}^{(N)}$ that admits μ_N as its automorphism group, the first stratum in the fourth line does not vanish. Let μ be the rank of the Chen–Ruan cohomology $H^*_{\mathrm{CR}}(\mathcal{X}_N)$, which satisfies

$$1-\frac{\mu}{12}=\frac{1}{N}.$$

Similarly, define $g=\langle\!\langle\mathcal{P}\rangle\!\rangle_{1,1}^{\mathcal{X}_N}$, where $\mathcal{P}$ is the point class on $\mathcal{X}_N$. The Belorousski–Pandharipande relation now gives

$$\tfrac{12}{5}gg''-\tfrac{18}{5}(g')^2+\bigl(-\tfrac{1}{60}\mu+\tfrac{1}{5}\bigr)\cdot\tfrac{1}{2}g'''=0,$$

where $'=Q\partial_Q$ is now the derivative with respect to the parameter for the point class $\mathcal{P}$. Then $f=-24g$ satisfies

$$2ff''-3(f')^2-2\bigl(1-\tfrac{1}{12}\mu\bigr)f'''=0.$$

Its solutions coincide with those of (2-15) via the relation $Q=q^N$; see [49] for more details.

2.2.2 g–reduction for higher-genus correlation functions We prove Proposition 2 using the *g–reduction technique* introduced in [18], first recalling:

Lemma 7 [17; 28] *Let $M(\psi,\kappa)$ be a monomial of ψ–classes and κ–classes $\overline{\mathcal{M}}_{g,n}$. Assume* deg $M\geq g$ *when $g\geq 1$, and* deg $M\geq 1$ *when $g=0$,. Then $M(\psi,\kappa)$ is equal to a linear combination of dual graphs on the boundary of $\overline{\mathcal{M}}_{g,n}$.*

Proof of Proposition 2 Consider the GW or FJRW correlation function of the form

$$\langle\!\langle\alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{\clubsuit},\quad \text{where } \clubsuit=\mathcal{E}_d \text{ or } W_d.$$

Using that the cohomology classes have $0\leq\deg\alpha_k\leq 2$, and using (1-5) and (2-7), we deduce that the correlation function is trivial if

$$\sum_{k=1}^{n}\ell_k<2g-2.$$

Assuming it is nontrivial and $\sum_{k=1}^{n} \ell_k \geq 1$, we must have

$$\deg\left(\prod_{k=1}^{n} \psi_k^{\ell_k}\right) = \sum_{k=1}^{n} \ell_k \geq \begin{cases} 2g-2 \geq g & \text{if } g \geq 2, \\ 1 & \text{if } g = 0, 1. \end{cases} \tag{2-16}$$

Then, $\prod_{k=1}^{n} \psi_k^{\ell_k}$ is a monomial satisfying the condition in Lemma 7, thus we can apply this technique and use the splitting axiom in GW/FJRW theory to rewrite the function as a linear combination of products of other correlation functions, with smaller genera.

We then repeat the process for nontrivial correlation functions with smaller genera and eventually rewrite the correlation function as a linear combination of products of primary (all $\ell_k = 0$) correlation functions in genus zero (which are just constants) and in genus one, which must be $f_d^{(n-1)} = \langle\!\langle \omega, \ldots, \omega \rangle\!\rangle_{1,n}^{\mathcal{E}_d}$ or $\langle\!\langle \phi, \ldots, \phi \rangle\!\rangle_{1,n}^{W_d}$. Thus we have

$$\langle\!\langle \alpha_1 \psi_1^{\ell_1}, \ldots, \alpha_k \psi_n^{\ell_n} \rangle\!\rangle_{g,n}^{\clubsuit} \in \mathbb{C}[f_d, f_d', f_d'', \ldots] = \mathbb{C}[f_d, f_d', f_d''].$$

The last equality follows from (2-15). □

3 A genus-one FJRW invariant

Throughout this section, we consider the $d = 3$ case, with $W_3 = x_1^3 + x_2^3 + x_3^3$ and $G = \mu_3$. We focus on the following genus-one FJRW invariant (see (1-6)) with $n = 3$:

$$\Theta_{1,n} := \langle \underbrace{\phi, \ldots, \phi}_{n} \rangle_{1,n}^{(W_3,\mu_3)}.$$

Combining the computations in [38], we will prove:

Proposition 8 [38, Theorem 1.1] *For the* (W_3, μ_3) *case, one has the FJRW invariant*

$$\Theta_{1,3} = \langle \phi, \phi, \phi \rangle_{1,3}^{(W_3,\mu_3)} = \tfrac{1}{108}. \tag{3-1}$$

We first obtain a formula that expresses the Witten top Chern class for $\Theta_{1,3}$ in terms of a Witten top Chern class of three-spin curves in Lemma 9. Then in Proposition 15 and Corollary 17, we analyze the latter virtual class explicitly by cosection localization. Finally, we deduce Proposition 8 from these results and explicit computations in [38].

3.1 Witten top Chern class

We begin with a formula for a Witten top Chern class of the moduli of three-spin curves. The relevant moduli $\overline{\mathcal{M}}_{g=1,2^3}(W_3, \mu_3)$ (defined in [6]) is the moduli of families

$$\xi = [\Sigma \subset \mathscr{C}, (\mathscr{L}_i, \rho_i)_{i=1}^{3}] \tag{3-2}$$

such that $\Sigma \subset \mathscr{C}$ is a family of genus-one 3–pointed twisted nodal curves, each marking is a stacky point of automorphism group μ_3, $\rho_i : \mathscr{L}_i^{\otimes 3} \cong \omega_{\mathscr{C}}^{\log}$ are isomorphisms together with isomorphisms $\mathscr{L}_i \cong \mathscr{L}_1$ for $i = 2$ and 3 understood, and the monodromy of $\mathscr{L}_1$ along $\Sigma_i \subset \Sigma$ is $\frac{3-1}{3}$.[2] Because of the isomorphisms $\mathscr{L}_i \cong \mathscr{L}_1$, we have the canonical isomorphism

$$\mathcal{W}_3 := \overline{\mathcal{M}}^{1/3}_{1,2^3} \cong \overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3),$$

where we recall that $\mathcal{W}_3$ parametrizes families of $\xi = [\Sigma \subset \mathscr{C}, \mathscr{L}, \rho]$ with objects Σ, $\mathscr{C}$, $\mathscr{L}$ and ρ as before.

Let

$$[\overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3)^p]^{\mathrm{vir}} \in A_* \overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3)$$

be the FJRW invariant of the pair (W_3, μ_3), which is defined in [6] as the cosection localized virtual cycles of the moduli stack $\overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3)^p$, parametrizing

$$\xi = \{(\mathscr{C}, \Sigma, \mathscr{L}_1, \dots, \varphi_1, \varphi_2, \varphi_3) \mid (\mathscr{C}, \Sigma, \mathscr{L}_1, \dots) \in \overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3) \text{ and } \varphi_i \in \Gamma(\mathscr{L}_i)\}.$$

We let

$$[\overline{\mathcal{M}}^{1/3,p}_{1,2^3}]^{\mathrm{vir}} \in A_* \overline{\mathcal{M}}^{1/3}_{1,2^3}$$

be the similarly defined cosection localized virtual cycle.

Lemma 9 *We have the identity*

$$[\overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3)^p]^{\mathrm{vir}} = ([\overline{\mathcal{M}}^{1/3,p}_{1,2^3}]^{\mathrm{vir}})^3 \in A^3 \mathcal{W}_3 \equiv A_0 \mathcal{W}_3. \tag{3-3}$$

Proof First, we have the Cartesian product

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3)^p \times \overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3)^p \arrow[d] & \overline{\mathcal{M}}_{1,2^3}(x^3 + y^3, (\mu_3)^2)^p \arrow[l] \arrow[d] \\
\overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3) \times \overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3) & \overline{\mathcal{M}}_{1,2^3}(x^3 + y^3, (\mu_3)^2) \arrow[l, "f"']
\end{tikzcd}$$

where the morphism f sends $(\mathscr{C}, \Sigma, \mathscr{L}_1, \mathscr{L}_2)$ to

$$((\mathscr{C}, \Sigma, \mathscr{L}_1), (\mathscr{C}, \Sigma, \mathscr{L}_2)).$$

Applying [6, Theorem 4.11], we get that

$$[\overline{\mathcal{M}}_{1,2^3}(x^3+y^3, (\mu_3)^2)^p]^{\mathrm{vir}} = f^*\big([\overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3)^p]^{\mathrm{vir}} \times [\overline{\mathcal{M}}_{1,2^3}(x^3, \mu_3)^p]^{\mathrm{vir}}\big). \tag{3-4}$$

[2] Our convention is that for $\mathscr{C} = [\mathbb{A}^1/\mu_r]$ and an invertible sheaf of $\mathscr{O}_{\mathscr{C}}$–modules having monodromy $a/r \in [0,1)$ at $[0]$, locally the sheaf takes the form $\mathscr{O}_{\mathbb{A}^1}(a[0])/\mu_r$.

Now let

$$g\colon \overline{\mathcal{M}}_{1,2^3}(x^3+y^3,\mu_3)=\overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)\to \overline{\mathcal{M}}_{1,2^3}(x^3+y^3,(\mu_3)^2)$$

be the diagonal morphism. Then

$$f\circ g\colon \overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)\to \overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)\times \overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)$$

is the diagonal morphism. As g is étale and proper, we conclude

$$[\overline{\mathcal{M}}_{1,2^3}(x^3+y^3,\mu_3)^p]^{\mathrm{vir}}=g^*[\overline{\mathcal{M}}_{1,2^3}(x^3+y^3,(\mu_3)^2)^p]^{\mathrm{vir}}. \tag{3-5}$$

Combined with (3-4) and (3-5), we obtain

$$[\overline{\mathcal{M}}_{1,2^3}(x^3+y^3,\mu_3)^p]^{\mathrm{vir}}=(f\circ g)^*\big([\overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)^p]^{\mathrm{vir}}\times[\overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)^p]^{\mathrm{vir}}\big),$$

which is $\big([\overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)^p]^{\mathrm{vir}}\big)^2$. Here we have used that $\overline{\mathcal{M}}_{1,2^3}(x^3,\mu_3)$ is smooth. Repeating the same argument to go from x^3+y^3 to W_3 proves the lemma. □

3.1.1 Cosection localized virtual cycles Let $\mathcal{W}$ be a smooth DM stack, with a complex of locally free sheaves of $\mathcal{O}_{\mathcal{W}}$–modules

$$\mathcal{E}^\bullet := [\mathcal{O}_{\mathcal{W}}(E_0)\xrightarrow{s}\mathcal{O}_{\mathcal{W}}(E_1)] \tag{3-6}$$

of rank a_0 and $a_1=a_0+1$, respectively. Let $\pi\colon E_0\to\mathcal{W}$ be the projection; the section s induces a section $\tilde{s}\in\Gamma(\widetilde{E}_1)$ of the pullback bundle $\widetilde{E}_1:=\pi^*E_1$. We define

$$\mathcal{M}:=(\tilde{s}=0)\subset E_0. \tag{3-7}$$

Assumption 10 We assume $\mathcal{D}=(\ker s\neq 0)\subset\mathcal{W}$ is a smooth Cartier divisor; $\mathrm{Im}(s|_{\mathcal{D}})$ is a rank-(a_0-1) subbundle of $E_1|_{\mathcal{D}}$.

Because $\mathcal{D}$ is a smooth Cartier divisor, we can find a vector bundle F on $\mathcal{W}$ fitting into

$$\mathcal{O}_{\mathcal{W}}(E_0)\xrightarrow{\eta_1}\mathcal{O}_{\mathcal{W}}(F)\xrightarrow{\eta_2}\mathcal{O}_{\mathcal{W}}(E_1) \tag{3-8}$$

so that $\eta_1|_{\mathcal{W}-\mathcal{D}}=s|_{\mathcal{W}-\mathcal{D}}$ is an isomorphism, $F\to E_1$ is a subvector bundle, and $s=\eta_2\circ\eta_1$.

We let $\mathcal{A}=H^1(\mathcal{E}^\bullet)$. By Assumption 10, it fits into the exact sequence

$$0\to\mathcal{O}_{\mathcal{W}}(E_0)\xrightarrow{\phi}\mathcal{O}_{\mathcal{W}}(F)\to\mathcal{A}\to 0. \tag{3-9}$$

Further, there is a line bundle A on $\mathcal{D}$ such that $\mathcal{A}=\mathcal{O}_{\mathcal{D}}(A)$. In the following, we will view $c_1(A)$ as an element of $A^1\mathcal{D}$. Then for the inclusion $\iota\colon\mathcal{D}\to\mathcal{W}$, we have $\iota_*c_1(A)\in A^2\mathcal{W}$. Since A is a line bundle on $\mathcal{D}$, we have $c_1(\mathcal{A})=[\mathcal{D}]$, thus:

Lemma 11 $$c_1(E_1 - F) = c_1(E_1 - E_0) - [\mathcal{D}].$$

We let $J \subset E_0|_{\mathcal{D}}$ be the kernel of $s|_{\mathcal{D}}$; by our assumption it is a line bundle on $\mathcal{D}$. We relate A to J:

Lemma 12 *Let the situation be as stated and assume Assumption 10. Then $A \cong J(\mathcal{D})$.*

Proof Let $\mathcal{J} = \mathcal{O}_{\mathcal{D}}(J)$ and let $\eta = \ker\{\mathcal{O}_{\mathcal{D}}(F) \to \mathcal{A}\}$. Then η fits into the exact sequences

$$0 \to \mathcal{O}_{\mathcal{D}}(J) \to \mathcal{O}_{\mathcal{D}}(E_0) \to \eta \to 0 \quad \text{and} \quad 0 \to \eta \to \mathcal{O}_{\mathcal{D}}(F) \to \mathcal{O}_{\mathcal{D}}(A) \to 0.$$

Let $\xi \in \mathcal{O}_{\mathcal{D}}(J)$ be any (local) section. Let $\tilde{\xi} \in \mathcal{O}_{\mathcal{W}}(E_0)$ be a lift of the image of ξ in $\mathcal{O}_{\mathcal{D}}(E_0)$. Then $\phi(\tilde{\xi}) \in \mathcal{O}_{\mathcal{W}}(F)$, where ϕ is as in (3-9). Clearly, $\phi(\tilde{\xi})|_{\mathcal{D}} = 0$. Let $t \in \mathcal{O}_{\mathcal{W}}(\mathcal{D})$ be the defining equation of $\mathcal{D}$. Then $t^{-1}\phi(\tilde{\xi}) \in \mathcal{O}_{\mathcal{W}}(F)(-\mathcal{D})$. We define $\varphi(\xi)$ to be the image of $t^{-1}\phi(\tilde{\xi})$ in $\mathcal{O}_{\mathcal{D}}(A(-\mathcal{D}))$ under the composition

$$\mathcal{O}_{\mathcal{W}}(F)(-\mathcal{D}) \to \mathcal{O}_{\mathcal{D}}(F(-\mathcal{D})) \to \mathcal{O}_{\mathcal{D}}(A(-\mathcal{D})).$$

It is direct to check that $\varphi\colon \mathcal{O}_{\mathcal{D}}(J) \to \mathcal{O}_{\mathcal{D}}(A(-\mathcal{D}))$ is a well-defined homomorphism of sheaves, and is an isomorphism. □

This way, $\mathcal{M}$ (see (3-7)) is a union of $\mathcal{W} \subset E_0$ (the 0–section) and the subbundle $J \subset E_0|_{\mathcal{D}} \subset E_0$. As $\mathcal{M} \subset E_0$ is defined by the vanishing of $\tilde{s}$, it comes with a normal cone

$$\boldsymbol{C} := \lim_{t \to 0} \Gamma_{t^{-1}\tilde{s}} \subset \widetilde{E}_1|_{\mathcal{M}}. \tag{3-10}$$

Lemma 13 *With Assumption 10, the cone $\boldsymbol{C} \subset \widetilde{E}_1|_{\mathcal{M}}$ is a union of two subvector bundles $\eta_2(F) \subset E_1$ and $\pi^*\eta_2(F)|_J \subset \widetilde{E}_1|_J$.*

Proof This is local, thus without loss of generality we can assume $a_0 = 1$. Since $\mathcal{D} = (s = 0)$ is a smooth divisor in $\mathcal{W}$, near a point at $\mathcal{D}$ we can give $\mathcal{W}$ an analytic neighborhood U with chart (u, x), where u is a multivariable, so that $\mathcal{D} = (x = 0)$ and $s|_U\colon E_0|_U \to E_1|_U$ takes the form

$$s|_U = (x, 0)\colon \mathcal{O}_U \to \mathcal{O}_U \oplus \mathcal{O}_U^{\oplus(a_1-1)} \cong \mathcal{O}_U(E_1).$$

We let y be the fiber-direction coordinate of $E_0|_U$. Then $\pi^{-1}(U) \subset E_0$ has the chart (u, x, y), with $\tilde{s}|_{\pi^{-1}(U)} = (xy, 0)$. Therefore, the cone $\boldsymbol{C} \subset E_0$ over $\pi^{-1}(U)$ is the line bundle

$$\mathcal{O}_{\pi^{-1}(U)\cap\mathcal{M}} \subset \mathcal{O}_{\pi^{-1}(U)\cap\mathcal{M}} \oplus \mathcal{O}_{\pi^{-1}(U)\cap\mathcal{M}}^{\oplus(a_1-1)} \cong \mathcal{O}_{\pi^{-1}(U)\cap\mathcal{M}}(\widetilde{E}_1).$$ □

Assumption 14 We assume that there is a homomorphism (cosection)

$$\sigma\colon \widetilde{E}_1|_{\mathcal{M}} \to \mathscr{O}_{\mathcal{M}}$$

such that $\sigma|_{\mathcal{W}} = 0$, and $\pi^*\eta_2(F)|_J$ lies in the kernel of σ.

Let

$$[\mathcal{M}]^{\mathrm{vir}}_{\sigma} := 0^!_{\sigma}[\boldsymbol{C}] \in A^{a_1-a_0}\mathcal{W}$$

be the image of $[\boldsymbol{C}]$ under the cosection localized Gysin map.

Proposition 15 *Let the situation be as mentioned, and suppose the cosection σ is fiberwise homogeneous of degree e. Then*

$$[\mathcal{M}]^{\mathrm{vir}}_{\sigma} = -c_1(E_0 - E_1) - (e+1)[\mathcal{D}] \in A^1\mathcal{W} \quad \textit{when } a_1 - a_0 = 1.$$

Proof Following the discussion leading to [5, Lemma 6.4], we compactify $\mathcal{M}$ by compactifying J by $\mathcal{P} := \mathbb{P}_{\mathcal{D}}(J \oplus 1)$. Let $\mathcal{D}_\infty = \mathbb{P}_{\mathcal{D}}(J \oplus 0) \subset \mathbb{P}_{\mathcal{D}}(J \oplus 1)$. Then $\mathcal{P} = J \cup \mathcal{D}_\infty$, and $\overline{\mathcal{M}} = \mathcal{P} \cup \mathcal{W}$. Let $\bar{\pi}\colon \mathcal{P} \to \mathcal{D}$ be the tautological projection. Then $\pi^*F|_J \subset \widetilde{E}_1|_J$ extends to $\bar{\pi}^*F \subset \bar{\pi}^*E_1$, a subbundle. Because σ is fiberwise homogeneous of degree e, we see that $\sigma|_J\colon \widetilde{E}_1|_J = \bar{\pi}^*E_1|_J \to \mathscr{O}_J$ extends to a homomorphism

$$\bar{\sigma}\colon \bar{\pi}^*E_1(-e\mathcal{D}_\infty) \to \mathscr{O}_{\mathcal{P}},$$

which is surjective along $\mathcal{D}_\infty = \overline{\mathcal{M}} - \mathcal{M}$.

We let $\bar{\pi}^*F(-e\mathcal{D}_\infty) \subset \bar{\pi}^*E_1(-e\mathcal{D}_\infty)$ be the associated twisting of the subbundle $\bar{\pi}^*F \subset \bar{\pi}^*E_1$. Applying [5, Lemma 6.4], we conclude that

$$0^!_{\sigma}[\boldsymbol{C}] = 0^!_{E_1}[F] + \bar{\pi}_*\big(0^!_{\bar{\pi}^*E_1(-e\mathcal{D}_\infty)}[\bar{\pi}^*F(-e\mathcal{D}_\infty)]\big). \tag{3-11}$$

When $a_1 - a_0 = 1$,

$$0^!_{\bar{\pi}^*E_1(-e\mathcal{D}_\infty)}[\bar{\pi}^*F(-e\mathcal{D}_\infty)] = c_1(\bar{\pi}^*(E_1/F)(-e\mathcal{D}_\infty)) = \bar{\pi}^*c_1(E_1/F) - e[\mathcal{D}_\infty].$$

Thus $\bar{\pi}_*\big(0^!_{\bar{\pi}^*E_1(-2\mathcal{D}_\infty)}[\bar{\pi}^*F(-e\mathcal{D}_\infty)]\big) = -e[\mathcal{D}]$. Combined with Lemma 11, the proposition follows. □

3.2 Applying to the FJRW invariant

We let $\mathcal{M} = \overline{\mathcal{M}}^{1/3,p}_{1,2^3}$. We claim that there is a complex of vector bundle as in (3-6) so that $\mathcal{M}$ is defined as in (3-7), and there is a cosection σ satisfying Assumption 14.

Indeed, let $\overline{\mathcal{M}}_{1,2^3}$ be the moduli of 3–pointed genus-one twisted curves where all markings are μ_3 stacky. Then the forgetful morphism $q\colon \overline{\mathcal{M}}^{1/3}_{1,2^3} \to \overline{\mathcal{M}}_{1,2^3}$ is finite and smooth. Furthermore, let $(\Sigma \subset \mathscr{C}, \mathscr{L})$ be the universal family of $\overline{\mathcal{M}}^{1/3}_{1,2^3}$. Then $(\Sigma \subset \mathscr{C})$ is the pullback of the universal family of $\overline{\mathcal{M}}_{1,2^3}$, and a standard method shows that we can find a complex $\mathcal{E}^\bullet = [s\colon \mathscr{O}_{\mathscr{C}}(E_0) \to \mathscr{O}_{\mathscr{C}}(E_1)]$ of locally free sheaves such that $\mathcal{E}^\bullet = R^\bullet\pi_*\mathscr{L}$, in the derived category. Here $\pi\colon \mathscr{C} \to \overline{\mathcal{M}}^{1/3}_{1,2^3}$ is the projection. Then a standard argument shows that this complex $\mathcal{E}^\bullet$ is the desired one, giving a canonical embedding of $\mathcal{M} = \overline{\mathcal{M}}^{1/3,p}_{1,2^3}$ into the total space of E_0, as the vanishing locus of $\tilde{s}$.

The choice of cosection σ is induced by $\mathscr{O}_{\mathcal{W}}(E_1) \to H^1(\mathcal{E}^\bullet)$, following that in [6], and satisfies Assumption 14. Finally, following the construction of $[\overline{\mathcal{M}}^{1/3,p}_{1,2^3}]^{\mathrm{vir}}$, we see that

$$[\mathcal{M}]^{\mathrm{vir}}_\sigma = [\overline{\mathcal{M}}^{1/3,p}_{1,2^3}]^{\mathrm{vir}}.$$

We skip the details here.

We next check that Assumption 10 holds in this case.

Lemma 16 *Let $\mathcal{D} \subset \mathcal{W}$ $(= \overline{\mathcal{M}}^{1/3}_{1,2^3})$ be the locus where $R^0\pi_*\mathscr{L}$ is nontrivial. Then it is a smooth divisor of $\mathcal{W}$.*

Proof Let $(\mathscr{C}, \Sigma, \mathscr{L}) \in \mathcal{W}$ be a closed point such that $H^0(\mathscr{L}) \neq 0$. Then a direct calculation shows that $\mathscr{C}$ has a node $q \in \mathscr{C}$ that separates $\mathscr{C}$ into two irreducible components $\mathscr{E}$ and $\mathscr{R}$, so that $q \subset \mathscr{E}$ is a 1–pointed (twisted) elliptic curve with $h^0(\mathscr{L}|_{\mathscr{E}}) = 1$, and $q \cup \Sigma \subset \mathscr{R}$ is a 4–pointed (twisted) rational curve. The same argument shows that the converse is also true. Therefore, letting $\mathcal{D} \subset \overline{\mathcal{M}}^{1/3}_{1,2^3}$ be the closed locus (see Figure 2) where $R^0\pi_*\mathscr{L}$ is nontrivial, $R^0\pi_*\mathscr{L}$ is a locally free sheaf of $\mathscr{O}_{\mathcal{D}}$–modules. Equivalently, letting

$$\pi_{\mathcal{D}}\colon \mathscr{C}_{\mathcal{D}} = \mathscr{C} \times_{\overline{\mathcal{M}}^{1/3}_{1,2^3}} \mathcal{D} \to \mathcal{D}$$

be the projection, this says that $\pi_{\mathcal{D}*}(\mathscr{L}|_{\mathscr{C}_{\mathcal{D}}})$ is a rank-one locally free sheaf of $\mathscr{O}_{\mathcal{D}}$–modules. Let t be a local section of this sheaf. Then $(t = 0) \subset \mathscr{C}_{\mathcal{D}}$ becomes a family of rational curves, the family that contains all those $q \cup \Sigma \subset \mathscr{R}$ mentioned. This shows that $\mathscr{C}_{\mathcal{D}} \to \mathcal{D}$ is exactly the subfamily in $\overline{\mathcal{M}}^{1/3}_{1,2^3}$ that can be decomposed into 1–pointed twisted elliptic curves $q \subset \mathscr{E}$ with $h^0(\mathscr{L}|_{\mathscr{E}}) = 1$, and 4–pointed twisted rational curves $q \cup \Sigma \subset \mathscr{R}$. This implies that $\mathcal{D}$ is a smooth divisor of $\mathcal{W} = \overline{\mathcal{M}}^{1/3}_{1,2^3}$. □

We illustrate the divisor $\mathcal{D}$ by a decorated graph in Figure 2. A generic point in $\mathcal{D}$ consists a nodal curve with a genus-one component $(g = 1)$ and a genus-zero component

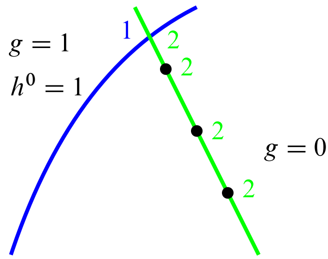

Figure 2: The divisor $\mathcal{D}$ on the moduli stack $\mathcal{W} = \overline{\mathcal{M}}^{1/3}_{1,2^3}$.

($g = 0$). The monodromy along the node is $\frac{1}{3}$ on the genus-one component and $\frac{2}{3}$ on the genus-zero component. Here $h^0 = 1$ is the rank of $R^0\pi_*\mathscr{L}$ restricted to the genus-one component.

Finally, to apply Proposition 15 we need to show that the cosection is fiberwise homogeneous of degree $e = 2$. This follows from the definition of the cosection in [6], and the degree e is $3 - 1$, where 3 is the denominator of $\frac{1}{3}$. Applying Proposition 15, we obtain:[3]

Corollary 17 *The Witten top Chern class of the moduli of three-spin curves $\overline{\mathcal{M}}^{1/3}_{1,2^3}$ is*

$$\left[\overline{\mathcal{M}}^{1/3,p}_{1,2^3}\right]^{\mathrm{vir}} = -c_1(R^\bullet\pi_*\mathscr{L}) - 3[\mathcal{D}]. \tag{3-12}$$

Applying Lemma 9, we get

$$\Theta_{1,3} = \deg[\overline{\mathcal{M}}_{1,2^3}(W_3, \mu_3)^p]^{\mathrm{vir}} = \deg([\overline{\mathcal{M}}^{1/3,p}_{1,2^3}]^{\mathrm{vir}})^3. \tag{3-13}$$

Thus the FJRW invariant $\Theta_{1,3}$ in Proposition 8 can be calculated explicitly from the triple self-intersection of the cycle (3-12). Note that the first term in (3-12) can be calculated by Chiodo's formula [9]. The calculation is subtle and lengthy, and the details are given in [38]. An alternative approach to computing this invariant using the mixed-spin-P fields method developed in [7; 8] is also presented in [38].

4 LG/CY correspondence for the Fermat cubic

This section is devoted to proving Theorem 3. We shall show that the GW/FJRW correlation functions as Fourier/Taylor expansions of the same quasimodular form

[3]This formula is a special case of a sequence of formulae for moduli of r–spin curves, conjectured by Janda (personal communication, 2019).

around different points (the infinity cusp and an interior point on the upper half-plane) which are related by the so-called holomorphic Cayley transformation that we shall introduce.

4.1 Cayley transformation and elliptic expansions of quasimodular forms

It is well known that the Eisenstein series $E_2(\tau)$ is not modular; however, its non-holomorphic modification

$$\widehat{E}_2(\tau,\bar{\tau}) := E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)} \tag{4-1}$$

is modular. The map (called modular completion) sending E_2 to $\widehat{E}_2$, and E_4 and E_6 to themselves, is an isomorphism from $\widetilde{M}_*(\Gamma)$ to the *ring of almost-holomorphic modular forms*

$$\widehat{M}_*(\Gamma) := \mathbb{C}[\widehat{E}_2, E_4, E_6]. \tag{4-2}$$

More precisely, for any quasimodular form $f(\tau) \in \widetilde{M}_*(\Gamma)$ of weight k, we denote by $\hat{f}(\tau,\bar{\tau}) \in \widehat{M}_*(\Gamma)$ its modular completion. The function $\hat{f}$ can be regarded as a polynomial in the formal variable $1/\mathrm{Im}(\tau)$,

$$\hat{f} = f + \sum_{j=1}^{k} f_j \left(\frac{1}{\mathrm{Im}(\tau)}\right)^j, \tag{4-3}$$

with coefficients some holomorphic functions f_j for $j = 1, 2, \ldots, k$ in τ. We call the inverse of the modular completion the *holomorphic limit*. It maps the almost-holomorphic modular form $\hat{f}$ in (4-3) to its degree-zero term f in the formal variable $1/\mathrm{Im}(\tau)$.

For any point $\tau_* \in \mathbb{H}$, we form the Cayley transform from $\mathbb{H}$ to a disk $\mathbb{D}$ (of appropriate radius determined by τ_* and $c \neq 0$),

$$\tau \mapsto s(\tau) := c2\pi\sqrt{-1}(\tau_* - \bar{\tau}_*)\frac{\tau - \tau_*}{\tau - \bar{\tau}_*} \in \mathbb{D}. \tag{4-4}$$

It is biholomorphic and we denote its inverse by $\tau(s)$.

Following [57], in [50] we defined a Cayley transformation $\mathscr{C}_{\tau_*}$ based on the action (4-4) on the space of almost-holomorphic modular forms; it maps the almost-holomorphic modular form $\hat{f} \in \widehat{M}_*(\Gamma)$ to

$$\mathscr{C}_{\tau_*}(\hat{f})(s,\bar{s}) := (2\pi\sqrt{-1}c)^{-k/2}\left(\frac{\tau(s) - \bar{\tau}_*}{\tau_* - \bar{\tau}_*}\right)^k \hat{f}(\tau(s), \bar{\tau}(s)). \tag{4-5}$$

This gives a natural way to expand an almost-holomorphic modular form near $\tau = \tau_*$.

$$\begin{tikzcd}
\widetilde{M}(\Gamma) \arrow[rr, shift left, "\text{modular completion}"] \arrow[d, dotted, "\mathscr{C}^{\text{hol}}_{\tau_*}"] & & \widehat{M}(\Gamma) \arrow[ll, shift left, "\text{constant term map}"] \arrow[d, "\mathscr{C}_{\tau_*}"] \\
\mathscr{C}^{\text{hol}}_{\tau_*}(\widetilde{M}(\Gamma)) \arrow[rr, shift left, "\text{modular completion}"] & & \mathscr{C}_{\tau_*}(\widehat{M}(\Gamma)) \arrow[ll, shift left, "\text{holomorphic limit}"]
\end{tikzcd}$$

Figure 3: Cayley transformation on quasimodular and almost-holomorphic modular forms.

A similar notion of holomorphic limit can be defined near the interior point τ_*. Computationally, this amounts to taking the degree-zero term in the $\bar{s}$–expansion of (4-5) (now regarded as a real-analytic function in s and $\bar{s}$) using the structure (4-3). This procedure induces a transformation $\mathscr{C}^{\text{hol}}_{\tau_*}$ on quasimodular forms. We will call the transformation $\mathscr{C}^{\text{hol}}_{\tau_*}$ the *holomorphic Cayley transformation*. This transformation can be shown to respect the differential ring isomorphism between the differential ring of quasimodular forms and the differential ring of almost-holomorphic modular forms. We illustrate the construction by the commutative diagram in Figure 3. See [50] for details.

We are mainly concerned with the expansions of the quasimodular form E_2 around the infinity cusp $\sqrt{-1}\infty$ and the elliptic points

$$\tau_* = -\frac{1}{2\pi\sqrt{-1}}\Gamma\Big(\frac{1}{d}\Big)\Gamma\Big(1-\frac{1}{d}\Big)e^{-\pi\sqrt{-1}/d} \quad \text{for } d \in \{3,4,6\}. \tag{4-6}$$

For the Fermat cubic polynomial case $d = 3$, in (4-4) we take

$$c = \frac{1}{2\pi\sqrt{-1}}\frac{\Gamma(1/d)}{\Gamma(1-1/d)^2}e^{-\pi\sqrt{-1}/d}. \tag{4-7}$$

The choices in (4-6) and (4-7) then lead to the rational expansion of E_2 around τ_*:

$$\mathscr{C}^{\text{hol}}_{\tau_*}(E_2) = -\tfrac{1}{9}s^2 - \tfrac{1}{1215}s^5 - \tfrac{1}{459\,270}s^8 + \cdots. \tag{4-8}$$

The other cases, $d = 4, 6$, are similar. All of these computations are easy following those in [50].

4.2 LG/CY correspondence

We consider the elliptic points (4-6) and the value (4-7) for c in (4-4). Theorem 3 then follows from:

Theorem 18 *Consider the LG space* $([\mathbb{C}^3/\langle J_\delta\rangle], W)$ *given by* (1-2) *and* (1-3), *with* $d=3$.

(i) *The genus-one GW correlation function is*

$$-24\langle\!\langle \omega\rangle\!\rangle_{1,1}^{\mathcal{E}_d}(q) = E_2(q). \tag{4-9}$$

(ii) *The GW correlation functions* $\langle\!\langle \cdots\rangle\!\rangle_{g,n}^{\mathcal{E}_d}$ *are quasimodular forms in the ring* $\mathbb{C}[E_2, E_2', E_2'']$.

(iii) *The genus-one FJRW correlation function* $\langle\!\langle \phi\rangle\!\rangle_{1,1}^{W_d}(s)$ *is the Taylor expansion of* $-\frac{1}{24}E_2$ *around the elliptic point*

$$\tau_* = -\frac{\sqrt{-1}}{\sqrt{3}}\exp\Big(\frac{2\pi\sqrt{-1}}{3}\Big) \in \mathbb{H};$$

that is,

$$\langle\!\langle \phi\rangle\!\rangle_{1,1}^{W_d}(s) = \mathscr{C}_{\tau_*}^{\mathrm{hol}}(\langle\!\langle \omega\rangle\!\rangle_{1,1}^{\mathcal{E}_d}(q)). \tag{4-10}$$

(iv) *The FJRW correlation functions* $\langle\!\langle \cdots\rangle\!\rangle_{g,n}^{W_d}$ *are holomorphic Cayley transformations of quasimodular forms in the ring*

$$\mathbb{C}[\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2), \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2'), \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2'')]$$

such that

$$\mathscr{C}_{\tau_*}^{\mathrm{hol}}(\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{\mathcal{E}_d}(q)) = \langle\!\langle \Psi(\alpha_1)\psi_1^{\ell_1},\dots,\Psi(\alpha_n)\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{W_d}(s).$$

Proof Part (i) is a well-known result in the literature; see eg [43]. We give a new proof based on the Chazy equation. In order to get (4-9), it suffices to check[4]

$$\langle\omega\rangle_{1,1,0}^{\mathcal{E}_d} = -\tfrac{1}{24} \quad\text{and}\quad \langle\omega\rangle_{1,1,1}^{\mathcal{E}_d} = 1.$$

Both invariants can be obtained by analyzing the virtual fundamental classes explicitly.

Part (ii) is a consequence of (i), the Ramanujan identities (1-11), and Proposition 2.

For (iii), the selection rule [20, Proposition 2.2.8] implies $\Theta_{1,1} = \Theta_{1,2} = 0$, as the corresponding moduli spaces are empty. On the other hand, according to Proposition 8,

$$\Theta_{1,3} = \tfrac{1}{108}.$$

Now we see that as a formal power series in s, the first three terms of $\langle\!\langle \phi\rangle\!\rangle_{1,1}^{W_d}(s)$ match those obtained from $\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2)$ in (4-8). Since both $\langle\!\langle \phi\rangle\!\rangle_{1,1}^{W_d}(s)$ and $\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2)$ satisfy the

[4] Note that only two initial conditions are needed to determine a solution from the space of formal power series in $q = e^{2\pi i\tau}$.

Chazy equation (1-12), we conclude that

$$\langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s) = -\tfrac{1}{24}\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2).$$

For (iv), we recall that, by g–reduction, in either theory all nontrivial correlation functions are differential polynomials in the building block $\langle\!\langle \omega \rangle\!\rangle_{1,1}^{\mathcal{E}_d}(q)$ or $\langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_d}(s)$. Since the holomorphic Cayley transformation respects the differential ring structure and the g–reduction is independent of the CohFT in consideration, (iv) is a consequence of (iii), the Ramanujan identities (1-11), and Proposition 2. □

Remark 19 Propositions 1 and 2 hold for all of the one-dimensional CY weight systems in (1-2) and (1-3). Provided the analogue of Proposition 8 for the $d = 4$ or 6 case is obtained, the same argument as in the proof of Theorem 18 generalizes straightforwardly.

5 Ancestor GW invariants for elliptic curves

The tautological relations used in establishing Proposition 2 are not constructive, and hence not so useful for actual calculation of higher-genus invariants. For this reason, we make use of the beautiful formulae for the descendent GW invariants of elliptic curves given by Bloch and Okounkov [3] and reviewed below. For later use we also discuss the ancestor/descendent correspondence.

5.1 Higher-genus descendent GW invariants of elliptic curves

In [43], Okounkov and Pandharipande proved a correspondence between the stationary GW invariants and Hurwitz covers, called Gromov–Witten/Hurwitz correspondence. To be more precise, let $\langle \prod_{i=1}^{N} \omega \widetilde{\psi}_i^{\ell_i} \rangle_{g,d}^{\bullet\mathcal{E}}$ be the disconnected, *stationary*, descendent GW invariant of genus g and degree d (the number N of markings is self-explanatory in the notation). Here $\widetilde{\psi}_i$ is the descendent cotangent line class attached to the i^{th} marking, and the symbol • stands for disconnected counting. The invariant is called *stationary* as the insertions only involve the descendents of ω.

Following [43], we define the *N–point generating function* by

$$F_N(z_1, \dots, z_N, q) := \sum_{\ell_1,\dots,\ell_N \geq -2} \left\langle\!\!\left\langle \prod_{i=1}^{N} \omega \widetilde{\psi}_i^{\ell_i} \right\rangle\!\!\right\rangle_g^{\bullet\mathcal{E}} \prod_{i=1}^{N} z_i^{\ell_i+1}, \tag{5-1}$$

with the convention

$$\langle\!\langle \omega \widetilde{\psi}^{-2} \rangle\!\rangle_0^{\bullet\mathcal{E}}(q) = 1.$$

The GW/Hurwitz correspondence [43, Theorem 5] allows one to rewrite the N–point generating function $F_N(z_1,\dots,z_N,q)$ by a beautiful character formula from [3]:

$$F_N(z_1,z_2,\dots,z_N,q)=(q)_\infty^{-1}\sum_{\text{all permutations of } z_1,\dots,z_N}\frac{\det M_N(z_1,z_2,\dots,z_N)}{\Theta(z_1+z_2+\dots+z_N)}. \tag{5-2}$$

Here $M_N(z_1,z_2,\dots,z_N)$ is the matrix where the (i,j) entry is zero if $j\neq N$ and $i>j+1$ and otherwise is given by

$$\frac{\Theta^{(j-i+1)}(z_1+\dots+z_{N-j})}{(j-i+1)!\,\Theta(z_1+\dots+z_{N-j})}\quad\text{if } j\neq N\qquad\text{and}\qquad\frac{\Theta^{(N-i+1)}(0)}{(N-i+1)!}\quad\text{if } j=N.$$

Recall that Θ is defined to be the prime form

$$\begin{aligned}\Theta(z)&=\frac{\vartheta_{(1/2,1/2)}(z,q)}{\partial_z\vartheta_{(1/2,1/2)}(z,q)|_{z=0}}=2\pi\sqrt{-1}\,\frac{\vartheta_{(1/2,1/2)}(z,q)}{-2\pi\eta^3}\\&=2\pi\sqrt{-1}\,e^{E_2z^2/24}\sigma(z),\end{aligned}\tag{5-3}$$

with:

(i) The Euler function

$$(q)_\infty:=\prod_{n=1}^{\infty}(1-q^n)$$

is related to the Dedekind eta function by $\eta=q^{1/24}(q)_\infty$.

(ii) The Jacobi theta function

$$\vartheta_{(1/2,1/2)}(z,q):=\sum_{n\in\mathbb{Z}}q^{(1/2)(n+1/2)^2}e^{(n+1/2)z}$$

has characteristic $\left(\frac12,\frac12\right)$.

(iii) The Weierstrass σ–function $\sigma(z)$ satisfies the well-known formula[5] (see [51])

$$\sigma(z)=\frac{z}{2\pi\sqrt{-1}}\exp\Bigl(\sum_{k=2}^{\infty}\frac{B_{2k}}{2k(2k)!}z^{2k}E_{2k}\Bigr), \tag{5-4}$$

where B_{2k} for $k\geq 1$ are Bernoulli numbers determined from

$$\frac{x}{e^x-1}=1-\tfrac12 x+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}x^{2k}.$$

[5]Note that the z–variable here differs from the usual one by a $2\pi\sqrt{-1}$ factor.

Note that we often omit the subscript g in the correlation function

$$\left\langle\!\!\left\langle \prod_{i=1}^{N} \omega \widetilde{\psi}_i^{\ell_i} \right\rangle\!\!\right\rangle_g^{\bullet \mathcal{E}},$$

which can be read off from the degree of the insertion according to the dimension axiom. We shall also omit the argument q in the functions for ease of notation.

The formula (5-2) provides an effective algorithm for computing the stationary descendent GW invariants. For example, as already computed in [3], one has

$$\begin{aligned} F_1(z_1) &= \frac{1}{(q)_\infty \Theta(z_1)}, \\ F_2(z_1, z_2) &= \frac{1}{(q)_\infty \Theta(z_1+z_2)}(\partial_{z_1} \ln \Theta(z_1) + \partial_{z_2} \log \Theta(z_2)), \\ &\vdots \end{aligned} \tag{5-5}$$

Remark 20 Let $\langle\!\langle \omega \rangle\!\rangle^{\circ \mathcal{E}}$ be the generating series of stable maps with connected domains with neither descendent nor ancestor classes. Then one has the well-known formula

$$\langle\!\langle \omega \rangle\!\rangle^{\circ \mathcal{E}} = -\tfrac{1}{24} E_2. \tag{5-6}$$

It is easy to see that

$$\langle\!\langle \omega \rangle\!\rangle^{\bullet \mathcal{E}} = \langle\!\langle \omega \rangle\!\rangle^{\circ \mathcal{E}} \exp(G(q)) \quad \text{and} \quad G = \sum_{d \geq 1} \langle\!\langle \ \rangle\!\rangle^{\circ \mathcal{E}}_{g=1,d} q^d. \tag{5-7}$$

In this case, by enumerating stable maps with connected domains, one can show that

$$q \frac{d}{dq} G = \sum_{d \geq 1} \langle\!\langle \omega \rangle\!\rangle^{\circ \mathcal{E}}_{g=1,d} q^d = -q \frac{d}{dq} \log (q)_\infty. \tag{5-8}$$

Solving this equation and using the initial terms of G, which can be easily computed, one obtains

$$G = -\log (q)_\infty. \tag{5-9}$$

This then gives

$$\langle\!\langle \omega \rangle\!\rangle^{\bullet \mathcal{E}} = (q)_\infty^{-1} \langle\!\langle \omega \rangle\!\rangle^{\circ \mathcal{E}} = (q)_\infty^{-1} \left(-\tfrac{1}{24}\right) E_2. \tag{5-10}$$

More generally, for the one-point GW correlation function, the same reasoning implies that

$$\langle\!\langle \omega \widetilde{\psi}^k \rangle\!\rangle^{\bullet \mathcal{E}} = (q)_\infty^{-1} \langle\!\langle \omega \widetilde{\psi}^k \rangle\!\rangle^{\circ \mathcal{E}}.$$

The result (5-9) indicates that one can add an extra contribution from the degree-zero part to G, whose corresponding moduli is an Artin stack. This contribution can be defined to be $\log q^{-\frac{1}{24}}$. In this way, after applying the divisor equation, it yields the contribution $-\frac{1}{24}$ for the degree-zero part in $\langle\!\langle\omega\rangle\!\rangle^{\circ\mathcal{E}}$. This definition of the extra contribution for the Artin stack changes $(q)_\infty$ to η. What one gains from the inclusion of this is the quasimodularity of the GW generating functions. The discrepancy will be further discussed from the viewpoint of ancestor/descendent correspondence below.

It is shown in [3] by manipulating the series expansions that the descendent GW correlation functions are essentially (modulo the issue discussed in Remark 20) quasimodular forms. By induction, the weight of $(q)_\infty\langle\!\langle\prod_{i=1}^N \omega\widetilde{\psi}_i^{k_i}\rangle\!\rangle^{\bullet\mathcal{E}}$ is $\sum(k_i+2)$. This can also be seen easily by using (5-3) and (5-4).

5.2 Ancestor/descendent correspondence

Since explicit formulae in [3] are available only for descendent GW invariants, while we are mainly concerned with ancestor GW invariants, we shall first exhibit the relation between these two types of GW invariants. The relation between the descendent GW invariants and the ancestor GW invariants are described for general targets in [33, Theorem 1.1]. This is the so-called *ancestor/descendent correspondence*. This correspondence is written down elegantly using a quantization formula of quadratic Hamiltonians in [22, Theorem 5.1].

We summarize some basics of quantization of quadratic Hamiltonians from [22]. Let H be a vector space of finite rank, equipped with a nondegenerating pairing $\langle -,-\rangle$. Let $H((z))$ be the loop space of the vector space H, equipped with a symplectic form Ω defined by

$$\Omega(f(z),g(z)) := \mathrm{Res}_{z=0}\langle f(-z),g(z)\rangle.$$

Let $\boldsymbol{t}_k$ be the collection of variables $\boldsymbol{t}_k=\{t_k^\alpha\}_\alpha$ where α runs over a basis of H, and $\boldsymbol{t}$ be the collection

$$\boldsymbol{t}=\{\boldsymbol{t}_0,\boldsymbol{t}_1,\dots\}.$$

We organize the collection $\boldsymbol{t}_k$ into a formal series $\boldsymbol{t}_k$:

$$\boldsymbol{t}_k(z)=\sum_i t_k^i\alpha_i z^k.$$

Similar notation is used for $\boldsymbol{s}_k$ and $\boldsymbol{s}$ below. Introduce the *dilaton shift*

$$\boldsymbol{q}(z)=\boldsymbol{t}(z)-z\mathbf{1}. \tag{5-11}$$

We consider an upper-triangular symplectic operator on $H((z))$, defined by

$$S(z^{-1}) := 1 + \sum_{i=1}^{\infty} z^{-i} S_i \quad \text{for } S_i \in \mathrm{End}(H).$$

Given an element $\mathcal{G}(\boldsymbol{q})$ in a certain Fock space, the quantization operator $\widehat{S}$ of a symplectic operator S gives another Fock space element

$$(\widehat{S}^{-1}\mathcal{G})(\boldsymbol{q}) = e^{W(\boldsymbol{q},\boldsymbol{q})/2\hbar}\mathcal{G}([S\boldsymbol{q}]_+), \tag{5-12}$$

where $[S\boldsymbol{q}]_+$ is the power series truncation of the function $S(z^{-1})\boldsymbol{q}(z)$, and the quadratic form $W = \sum(W_{k\ell}\boldsymbol{q}_k, \boldsymbol{q}_\ell)$ is defined by

$$\sum_{k,\ell\geq 0} \frac{W_{k,\ell}}{w^k z^\ell} := \frac{S^*(w^{-1})S(z^{-1}) - \mathrm{Id}}{w^{-1} + z^{-1}}.$$

Here Id is the identity operator on $H((z))$ and S^* is the adjoint operator of S.

Following Givental [22, Section 5], for the descendent theory we define a particular symplectic operator S_t by

$$(a, S_t b) := \left\langle a, \frac{b}{z-\psi}\right\rangle =: (a,b) + \sum_{k=0}^{\infty} \langle\!\langle a, b\tilde{\psi}^k \rangle\!\rangle_{0,2}^{\circ\mathcal{E}} z^{-1-k}. \tag{5-13}$$

Now we specialize to the elliptic curve case and write down the quantization formula for the ancestor/descendent correspondence explicitly. Henceforward, we use the following convention:

- Recall $\{\mathbf{1}, \mathfrak{b}_1, \mathfrak{b}_2, \phi\}$ is a basis of the FJRW state space $\mathscr{H}_{(W_d,G_d)}$ given in (2-10). We parametrize the ancestor classes $\mathbf{1}\psi^\ell$, $\mathfrak{b}_1\psi^\ell$, $\mathfrak{b}_2\psi^\ell$ and $\phi\psi^\ell$ by

$$s_\ell^0, \quad s_\ell^1, \quad s_\ell^2 \quad \text{and} \quad s_\ell^3. \tag{5-14}$$

- Recall $\{\mathbf{1}, e_1, e_2, \omega\}$ is a basis of the cohomology space $H^*(\mathcal{E})$. We parametrize the ancestor classes $\mathbf{1}\psi^\ell$, $e_1\psi^\ell$, $e_2\psi^\ell$ and $\omega\psi^\ell$, and descendent classes $\mathbf{1}\tilde{\psi}^\ell$, $e_1\tilde{\psi}^\ell$, $e_2\tilde{\psi}^\ell$ and $\omega\tilde{\psi}^\ell$ by

$$t_\ell^0, t_\ell^1, t_\ell^2, t_\ell^3, \quad \text{and} \quad \tilde{t}_\ell^0, \tilde{t}_\ell^1, \tilde{t}_\ell^2, \tilde{t}_\ell^3, \tag{5-15}$$

  respectively.

The total descendent potential of the GW theory of $\mathcal{E}$ is defined by

$$\mathcal{D}^{\mathcal{E}}(\tilde{\boldsymbol{t}}) := \exp\Big(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g^{\circ\mathcal{E}}(\tilde{\boldsymbol{t}})\Big) := \exp\Big(\sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}\langle \tilde{\boldsymbol{t}}, \dots, \tilde{\boldsymbol{t}}\rangle_{g,n}^{\circ\mathcal{E}}\Big). \tag{5-16}$$

The total ancestor potential of the GW theory of $\mathcal{E}$ is defined by

$$\mathcal{A}^{\mathcal{E}}(\boldsymbol{t}) := \exp\biggl(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_g^{\circ\mathcal{E}}(\boldsymbol{t})\biggr) := \exp\biggl(\sum_{g\geq 0}\hbar^{g-1}\sum_{\substack{n\geq 0\\ 2g-2+n>0}}\langle \boldsymbol{t},\dots,\boldsymbol{t}\rangle_{g,n}^{\circ\mathcal{E}}\biggr).$$

The total ancestor FJRW potential is defined similarly.

The quantity $\mathcal{F}_1^{\circ\mathcal{E}}(t)$ is the genus-one primary potential of the GW theory of $\mathcal{E}$ appearing in $\mathcal{A}^{\mathcal{E}}$, with the parameter $q=e^t$ keeping track of the degree. By [22, Theorem 5.1], the ancestor/descendent correspondence of the elliptic curve is given by

$$\mathcal{D}^{\mathcal{E}} = e^{\mathcal{F}_1^{\circ\mathcal{E}}(t)}\widehat{S}_t^{-1}\mathcal{A}^{\mathcal{E}}, \tag{5-17}$$

under the identification $\tilde{t}_\ell^i = t_\ell^i$.

According to (5-9), the genus-one potential is

$$\mathcal{F}_1^{\circ\mathcal{E}}(t) = G(q) = \sum_{d\geq 1}\langle\,\rangle_{1,0,d}^{\circ\mathcal{E}}q^d = -\log(q)_\infty \quad \text{for } q=e^t.$$

Thus we obtain

$$\widehat{S}_t^{-1}\mathcal{A}^{\mathcal{E}} = e^{-\mathcal{F}_1^{\circ\mathcal{E}}(t)}\mathcal{D}^{\mathcal{E}} = (q)_\infty\mathcal{D}^{\mathcal{E}} = (q)_\infty\sum_{g,n\in\mathbb{Z}}\hbar^{g-1}\langle\!\langle\tilde{\boldsymbol{t}},\dots,\tilde{\boldsymbol{t}}\rangle\!\rangle_{g,n}^{\bullet\mathcal{E}}.$$

A direct calculation of (5-13) shows the restriction of S_t to the odd cohomology is the identity operator, and the restriction to even cohomology is given by

$$S_t\begin{pmatrix}\mathbf{1}\\ \omega\end{pmatrix} = \begin{pmatrix}1 & t/z\\ & 1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \omega\end{pmatrix}.$$

Now, we write down an explicit formula for the quantization operator (5-12). The symplectic operator S_t is given in terms of infinitesimal symplectic operator $h(t)/z$:

$$S_t = \exp\biggl(\frac{h(t)}{z}\biggr).$$

Here $h(t)\in\operatorname{End}(H)$ is such that $h(t)(\mathbf{1}) = t\omega$ if $h(t)(\omega)=0$, and $h(t)(e_i)=0$ otherwise. In terms of the Darboux coordinates $\tilde{q}_k^i$ and $\tilde{p}_k^i$, the corresponding quadratic Hamiltonian has the form (see [35, Section 3], for example)

$$P\biggl(\frac{h(t)}{z}\biggr) = -t\cdot\tfrac{1}{2}(\tilde{q}_0^0)^2 - t\sum_{k\geq 0}\tilde{q}_{k+1}^0\tilde{p}_k^0.$$

Applying the quantization formula, we get

$$\widehat{S}_t = \exp\biggl(\widehat{P\biggl(\frac{h(t)}{z}\biggr)}\biggr) = \exp\biggl(-t\cdot\tfrac{1}{2}(\tilde{q}_0^0)^2 - t\sum_{k\geq 0}\tilde{q}_{k+1}^0\frac{\partial}{\partial\tilde{q}_k^0}\biggr). \tag{5-18}$$

As a consequence, we observe that this operator has no influence on the parameter $\tilde{q}_k^3$ for the descendent $\omega\tilde{\psi}^k$. Thus we obtain:

Proposition 21 *The relation between the stationary descendent invariants and the corresponding ancestor invariants is given by*

$$(q)_\infty \left\langle\!\!\left\langle \prod_{i=1}^{N} \omega\tilde{\psi}_i^{\ell_i} \right\rangle\!\!\right\rangle_g^{\bullet\mathcal{E}} = \left\langle\!\!\left\langle \prod_{i=1}^{N} \omega\psi_i^{\ell_i} \right\rangle\!\!\right\rangle_g^{\bullet\mathcal{E}}. \tag{5-19}$$

Quasimodularity for the correlation functions in the disconnected theory is equivalent to quasimodularity for the connected theory, as one can see by examining the generating series. Hence our Theorem 18(ii) is consistent with the results in [3; 43] about the quasimodularity via the above proposition.

6 Higher-genus FJRW invariants for the Fermat cubic

In this section we give several applications of Theorem 3. With the help of the Bloch–Okounkov formula [3], Cayley transformation allows us to compute the FJRW invariants of the Fermat elliptic polynomials at all genera. It also transforms various structures for the GW theory of elliptic curves, such as the holomorphic anomaly equations [42; 43] and Virasoro constraints [44], to those in the corresponding FJRW theory.

6.1 Higher-genus ancestor FJRW invariants for the cubic

Consider the Laurent expansion of the N–point generating function

$$F_N(z_1, z_2, \dots, z_N, q).$$

The Laurent expansion of $\partial^m \ln\Theta$ is clear from (5-4), while that of $1/\Theta$ or $1/\sigma$ can be obtained by applying the Faá di Bruno formula to the exponential term in $1/\sigma$, which in the current case is determined by the Bell polynomials in $-B_{2k}E_{2k}/2k$ for $k \geq 2$. However, this only gives the Laurent coefficients in terms of the generators E_{2k} for $k \geq 2$, for the ring of modular forms. The expansions obtained are not particularly useful for our later purpose, which prefers a finite set of generators only.

We proceed as follows. First, the Taylor expansion of the Weierstrass σ–function is given by the classical result [55]

$$\sigma = \sum_{m,n\geq 0} \frac{a_{m,n}}{(4m+6n+1)!} \left(\frac{2\pi^4}{3}E_4\right)^m \left(\frac{16\pi^6}{27}E_6\right)^n \left(\frac{z}{2\pi\sqrt{-1}}\right)^{4m+6n+1}, \tag{6-1}$$

where the coefficients $a_{m,n}$ are complex numbers determined from the Weierstrass recursion

$$a_{m,n}=3(m+1)a_{m+1,n-1}+\tfrac{16}{3}(n+1)a_{m-2,n+1}-\tfrac{1}{6}(4m+6n-1)(4m+6n-2)a_{m-1,n},$$

with the initial values $a_{0,0}=1$ and $a_{m,n}=0$ if either of m or n is strictly negative. The Laurent expansion of $1/\sigma$ is then obtained from the above. It takes the form

$$\text{(6-2)}\qquad \frac{1}{\sigma}=\sum_{m,n\geq 0} b_{m,n}\left(\frac{2\pi^4}{3}E_4\right)^m\left(\frac{16\pi^6}{27}E_6\right)^n\left(\frac{z}{2\pi\sqrt{-1}}\right)^{4m+6n-1}$$

for some $b_{m,n}$ that can also be obtained recursively. The formula in (6-1) also gives rise to the Laurent expansion of $\partial\ln\sigma$, and hence of $\partial\ln\Theta$, in terms of the generators E_2, E_4 and E_6. Together with that of $\partial\ln\Theta$ it can be used to compute the Laurent expansion of $F_N(z_1,z_2,\dots,z_N,q)$.

Consider the $N=1$ case first. According to (5-5), the Laurent expansion of F_1 is given by

$$\begin{aligned}F_1(z,q)&=\frac{1}{2\pi\sqrt{-1}(q)_\infty}e^{-E_2/24z^2}\sigma^{-1}\\&=\frac{1}{z(q)_\infty}\sum_{\ell,m,n\geq 0}\frac{b_{m,n}}{\ell!}\left(-\tfrac{1}{24}E_2\right)^\ell\left(\tfrac{1}{24}E_4\right)^m\left(-\tfrac{1}{108}E_6\right)^n z^{2\ell+4m+6n}.\end{aligned}$$

We therefore arrive at the following relation for the descendent GW correlation functions when $k\geq -2$:

$$\text{(6-3)}\qquad (q)_\infty\langle\!\langle\omega\widetilde{\psi}^k\rangle\!\rangle^{\bullet\mathcal{E}}=\sum_{\substack{\ell,m,n\geq 0\\ 2\ell+4m+6n=k+2}}\frac{b_{m,n}}{\ell!}\left(-\tfrac{1}{24}E_2\right)^\ell\left(\tfrac{1}{24}E_4\right)^m\left(-\tfrac{1}{108}E_6\right)^n.$$

As explained in Proposition 21, this is the corresponding ancestor GW correlation function and is indeed a quasimodular form of weight $k+2$. The first few Laurent coefficients are

$$\text{(6-4)}\qquad 1,\quad -\tfrac{1}{24}E_2,\quad \tfrac{1}{2^6 3^2}\left(\tfrac{1}{5}E_4+\tfrac{1}{2}E_2^2\right),\quad \dots.$$

The other cases are similar. For example, for the $N=2$ case from (5-5) we write

$$(q)_\infty F_2(z_1,z_2)=\frac{z_1+z_2}{\Theta(z_1+z_2)}\,\frac{\partial_{z_1}\ln\Theta(z_1)+\partial_{z_2}\log\Theta(z_2)}{z_1+z_2}.$$

The first term on the right-hand side is expanded as in the $N=1$ case, while the second term is expanded using (5-3) and (5-4).

Recall that the derivative on the level of generating series corresponds to the divisor equation in GW theory, and that taking derivatives commutes with Cayley transformations, as shown in [50]. The generators of the differential ring of quasimodular forms are E_2, E_4 and E_6. To deal with the differential structure, it is in fact more convenient to use the generators E_2, E_2' and E_2'' for the ring of quasimodular forms as opposed to E_2, E_4 and E_6. By Theorem 18, the ancestor GW correlation functions satisfy

$$\left\langle\!\!\left\langle \prod_{i=1}^{N} \omega\psi_i^{k_i} \right\rangle\!\!\right\rangle^{\circ\mathcal{E}} \in \mathbb{C}[E_2, E_2', E_2'']. \tag{6-5}$$

Theorem 3 applies to the disconnected invariants (by examining the relation between the generating series), and we have

$$\langle\!\langle \phi\psi_1^{k_1}, \dots, \phi\psi_N^{k_N} \rangle\!\rangle_g^{\bullet W_d} = \mathscr{C}_{\tau_*}^{\mathrm{hol}}(\langle\!\langle \omega\psi_1^{k_1}, \dots, \omega\psi_N^{k_N} \rangle\!\rangle_g^{\bullet\mathcal{E}_d}). \tag{6-6}$$

Now we can apply Cayley the transformation directly to the disconnected, ancestor GW correlation functions and obtain the disconnected, ancestor FJRW correlation functions. As computed in (4-8), for the $d = 3$ case we have

$$\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2) = -\tfrac{1}{9}s^2 - \tfrac{1}{1215}s^5 - \tfrac{1}{459270}s^8 + \cdots. \tag{6-7}$$

Since $\mathscr{C}_{\tau_*}^{\mathrm{hol}}$ respects the product and the differential structure [50], the differential equations (1-11) imply

$$\begin{cases} \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_4) = \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2^2 - 12E_2') = \tfrac{8}{3}s + \tfrac{5}{81}s^4 + \tfrac{2}{5103}s^7 + \cdots, \\ \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_6) = \mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2E_4 - 3E_4') = -8 - \tfrac{28}{27}s^3 - \tfrac{7}{405}s^6 + \cdots. \end{cases} \tag{6-8}$$

From (6-3), Proposition 21, Theorem 3 and the degree formula (1-5), we immediately obtain

$$\langle\!\langle \phi\psi_1^{2g-2} \rangle\!\rangle_{g,1}^{\bullet W_3} = \sum_{\substack{\ell,m,n\geq 0 \\ \ell+2m+3n=g}} \frac{b_{m,n}}{\ell!}\bigl(-\tfrac{1}{24}\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_2)\bigr)^{\ell}\bigl(\tfrac{1}{24}\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_4)\bigr)^{m}\bigl(-\tfrac{1}{108}\mathscr{C}_{\tau_*}^{\mathrm{hol}}(E_6)\bigr)^{n}.$$

Now Corollary 4 follows from the fact that the disconnected and connected one-point ancestor functions are the same.

6.2 Holomorphic anomaly equations

We now describe holomorphic anomaly equations for the FJRW correlation functions. In the rest of the paper we shall only discuss connected invariants, and hence omit the superscript $\circ$ from the notation.

6.2.1 HAEs for ancestor GW correlation functions In [42], Oberdieck and Pixton use the polynomiality of double ramification cycles to prove that the GW cycles $\Lambda^{\mathcal{E}}_{g,n}(\alpha_1,\dots,\alpha_n)$ of the elliptic curves are cycle-valued quasimodular forms. Taking the derivative of those cycles with respect to the second Eisenstein series $E_2(q)$, they obtain a holomorphic anomaly equation [42, Theorem 3]. As a consequence, intersecting the corresponding GW cycles with $\prod_k \psi_k^{\ell_k}$ on $\overline{\mathcal{M}}_{g,n}$ leads to a holomorphic anomaly equation for the ancestor GW functions

$$\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q)\in\mathbb{C}[E_2,E_4,E_6].$$

For each subset $I\subseteq\{1,\dots,n\}$, we use the convention

$$\alpha_I:=\{\alpha_i\psi_i^{\ell_i}\mid i\in I\}.$$

For convenience, we introduce the normalized Eisenstein series

$$C_2(q)=-\tfrac{1}{24}E_2(q).$$

It is a classical fact that the Eisenstein series E_2, E_4 and E_6 are algebraically independent. We have [42], for the ancestor GW correlation functions,

$$\begin{aligned}(6\text{-}9)\quad &\frac{\partial}{\partial C_2}\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q)\\ &=\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n},\mathbf{1},\mathbf{1}\rangle\!\rangle^{\mathcal{E}}_{g-1,n+2}(q)\\ &\qquad+\sum_{\substack{g_1+g_2=g\\ \{1,\dots,n\}=I_1\sqcup I_2}}\langle\!\langle \alpha_{I_1},\mathbf{1}\rangle\!\rangle^{\mathcal{E}}_{g_1}(q)\langle\!\langle \mathbf{1},\alpha_{I_2}\rangle\!\rangle^{\mathcal{E}}_{g_2}(q)\\ &\qquad\qquad-2\sum_{i=1}^{n}\biggl(\int_{\mathcal{E}}\alpha_i\biggr)\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\mathbf{1}\psi_i^{\ell_i+1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{\mathcal{E}}_{g,n}(q).\end{aligned}$$

Remark 22 This equation can also be proved using only the combinatorial results reviewed in Section 5.1; see Pixton [45].

6.2.2 HAEs for ancestor FJRW correlation functions Recall that the holomorphic Cayley transformation $\mathscr{C}^{\mathrm{hol}}_{\tau_*}$ respects the differential ring structure of the set of quasimodular forms. Applying the holomorphic Cayley transformation to (6-9), using Theorem 18 we immediately obtain the following HAE for the ancestor FJRW correlation functions:

Corollary 23 *Let the notation be as in Theorem 3. For the $d=3$ case, the ancestor FJRW correlation function*

$$\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle^{W_d}_{g,n}\in\mathbb{C}[\mathscr{C}^{\mathrm{hol}}_{\tau_*}(C_2),\mathscr{C}^{\mathrm{hol}}_{\tau_*}(E_4),\mathscr{C}^{\mathrm{hol}}_{\tau_*}(E_6)]\quad \textit{for } C_2=-\tfrac{1}{24}E_2$$

satisfies

$$\text{(6-10)}\quad \frac{\partial}{\partial \mathscr{C}^{\mathrm{hol}}_{\tau_*}(C_2)} \langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{W_d}$$
$$= \langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\alpha_n\psi_n^{\ell_n},\mathbf{1},\mathbf{1}\rangle\!\rangle_{g-1,n+2}^{W_d} + \sum_{\substack{g_1+g_2=g\\ \{1,\dots,n\}=I_1\amalg I_2}} \langle\!\langle \alpha_{I_1},\mathbf{1}\rangle\!\rangle_{g_1}^{W_d}\langle\!\langle \mathbf{1},\alpha_{I_2}\rangle\!\rangle_{g_2}^{W_d}$$
$$-2\sum_{i=1}^{n}\langle\!\langle \alpha_1\psi_1^{\ell_1},\dots,\delta_{\alpha_i}^{\phi}\mathbf{1}\psi_i^{\ell_i+1},\dots,\alpha_n\psi_n^{\ell_n}\rangle\!\rangle_{g,n}^{W_d},$$

where $\delta_{\alpha_i}^{\phi}$ is the Kronecker symbol.

6.3 Virasoro constraints

Virasoro operators in Gromov–Witten theory were proposed by Eguchi, Hori and Xiong [16] for Fano manifolds, and later extended to more general targets [15; 22]. The famous *Virasoro conjecture* predicts that the total descendent potentials in GW theory are annihilated by the Virasoro operators. It is one of the most fascinating conjectures in GW theory. Despite significant developments in the literature, it remains open for a large category of targets.

The Virasoro conjecture for nonsingular target curves is solved by Okounkov and Pandharipande [44]. In particular, when the target is an elliptic curve, the formulae are particularly simple. To be more explicit, using the coordinates induced by (5-15) and letting

$$(\ell)_n := \ell(\ell+1)\cdots(\ell+n-1)$$

be the Pochhammer symbol with the convention $(\ell)_0 := 1$, the Virasoro operators $\{L_k^{\mathcal{E}} \mid k\in\mathbb{Z} \text{ and } k\ge -1\}$ are given by

$$L_k^{\mathcal{E}} = -(k+1)!\frac{\partial}{\partial \tilde{t}^0_{k+1}} + \sum_{\ell\ge 0}\left((\ell)_{k+1}\tilde{t}^0_\ell\frac{\partial}{\tilde{t}^0_{k+\ell}} + (\ell+1)_{k+1}\tilde{t}^3_\ell\frac{\partial}{\partial\tilde{t}^3_{k+\ell}}\right)$$
$$+\sum_{\ell\ge 0}\left((\ell+1)_{k+1}\tilde{t}^1_\ell\frac{\partial}{\tilde{t}^1_{k+\ell}} + (\ell)_{k+1}\tilde{t}^2_\ell\frac{\partial}{\partial\tilde{t}^2_{k+\ell}}\right).$$

According to [44, Theorem 1], the total descendent GW potential defined in (5-16) is annihilated by these Virasoro operators:

$$L_k^{\mathcal{E}}\mathcal{D}^{\mathcal{E}}(\tilde{\boldsymbol{t}}) = 0.$$

Recently in [27], using Givental's quantization formula of quadratic Hamiltonians [22], the second author and his collaborator study Virasoro operators in FJRW theory and conjecture that the total ancestor FJRW potential of any admissible LG pair (W, G) is annihilated by the defining Virasoro operators. Besides various generically semisimple cases, they also verified the conjecture for the nonsemisimple Fermat cubic pair (W_3, μ_3), using Theorem 3. More explicitly, using the coordinates induced in (5-14), the Virasoro operators $\{L_k^{W_3,\mu_3} \mid k \in \mathbb{Z} \text{ and } k \geq -1\}$ for the Fermat cubic pair (W_3, μ_3) are

$$L_k^{W_3,\mu_3} := -(k+1)!\frac{\partial}{\partial t_k^0} + \sum_{\ell\geq 0}^{\infty}\left((\ell)_{k+1}s_\ell^0\frac{\partial}{\partial s_{k+\ell}^0} + (\ell+1)_{k+1}s_\ell^3\frac{\partial}{\partial s_{k+\ell}^3}\right) + \sum_{\ell\geq 0}^{\infty}\left((\ell+1)_{k+1}s_\ell^1\frac{\partial}{\partial s_{k+\ell}^1} + (\ell)_{k+1}s_\ell^2\frac{\partial}{\partial s_{k+\ell}^2}\right).$$

It is not hard to see that these operators commute with the quantization operator $\widehat{S}_t^{-1}$ in the ancestor/descendent correspondence formula (5-17) and the holomorphic Cayley transformation $\mathscr{C}_{\tau_*}^{\text{hol}}$ in Theorem 3. Therefore, Virasoro constraints for the FJRW theory are a consequence of Theorem 3.

Corollary 24 [27] *The total ancestor FJRW potential of the pair (W_3, μ_3) is annihilated by the Virasoro operators $\{L_k^{W_3,\mu_3}\}$:*

$$L_k^{W,\mu_3}\mathcal{A}^{W_3,\mu_3}(\boldsymbol{s}) = 0.$$

Appendix

A.1 A genus-one formula for the Fermat cubic polynomial

For the examples we study, the connection between modular forms and periods of families of elliptic curve gives rise to nice formulae for the holomorphic Cayley transformation of quasimodular forms in terms of hypergeometric series and Givental's I–functions. In the following, we shall only consider the $d = 3$ case, as an example, the other cases are similar.

Let us first recall some facts of quasimodular forms following the exposition in [49]. Let $\Gamma(3)$ be the level-3 principal congruence subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$. It is well known that the ring of quasimodular forms (with a certain Dirichlet character) for $\Gamma(3)$ is generated by

$$A = \theta_{A_2}(2\tau)$$

and

$$E = \tfrac{1}{4}(3E_2(3\tau) + E_2(\tau)), \tag{A-1}$$

where θ_{A_2} is the theta function for the A_2–lattice. Further, define the quantities (where η is the Dedekind eta function)

$$C = 3\frac{\eta(3\tau)^3}{\eta(\tau)} \quad \text{and} \quad \alpha = \frac{C^3}{A^3}. \tag{A-2}$$

These quantities satisfy

$$A = {}_2F_1\bigl(\tfrac{1}{3}, \tfrac{2}{3}; 1; \alpha\bigr), \tag{A-3}$$

and furthermore

$$\begin{aligned} A^2 &= \tfrac{1}{2}(3E_2(3\tau) - E_2(\tau)) = \frac{1}{2\pi\sqrt{-1}}\frac{1}{\alpha(1-\alpha)}\frac{\partial}{\partial\tau}\alpha, \\ E &= \frac{6}{2\pi\sqrt{-1}}\frac{\partial}{\partial\tau}\log A - \frac{2C^3 - A^3}{A}. \end{aligned} \tag{A-4}$$

Using (A-1), (A-2) and (A-4), we can rewrite the quasimodular form E_2 as

$$\begin{aligned} E_2(\tau) &= \frac{12}{2\pi\sqrt{-1}}\frac{\partial}{\partial\tau}\log A - (4\alpha - 1)A^2 \\ &= \frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\tau}\bigl(12\log A + \log(\alpha(1-\alpha)^3)\bigr). \end{aligned} \tag{A-5}$$

In [50] the following was obtained from period calculation. Taking $\tau_* = 1/(1-\zeta_3)$ as given in (4-6) and c as in (4-7), one has

$$\begin{aligned} s(\tau) &= 2\pi\sqrt{-1}c(\tau_* - \bar{\tau}_*)\frac{\tau - \tau_*}{\tau - \bar{\tau}_*} \\ &= -2\pi\sqrt{-1}c(\tau_* - \bar{\tau}_*)\frac{\Gamma\bigl(-\frac{1}{3}\bigr)\Gamma\bigl(\frac{2}{3}\bigr)^2}{\Gamma\bigl(\frac{1}{3}\bigr)^3}(-\alpha)^{-1/3}\frac{{}_2F_1\bigl(\frac{2}{3}, \frac{2}{3}; \frac{4}{3}; \alpha^{-1}\bigr)}{{}_2F_1\bigl(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; \alpha^{-1}\bigr)}. \end{aligned}$$

Also,

$$\mathscr{C}_{\tau_*}(A) = (2\pi\sqrt{-1}c)^{-1/2}\frac{\Gamma\bigl(\frac{1}{3}\bigr)}{\Gamma\bigl(\frac{2}{3}\bigr)^2}(-\alpha)^{-1/3}{}_2F_1\bigl(\tfrac{1}{3}, \tfrac{1}{3}; \tfrac{2}{3}; \alpha^{-1}\bigr)$$

and

$$\mathscr{C}_{\tau_*}(C) = (2\pi\sqrt{-1}c)^{-1/2}\frac{\Gamma\bigl(\frac{1}{3}\bigr)}{\Gamma\bigl(\frac{2}{3}\bigr)^2}(-1)^{-1/3}{}_2F_1\bigl(\tfrac{1}{3}, \tfrac{1}{3}; \tfrac{2}{3}; \alpha^{-1}\bigr).$$

Combining the properties of the holomorphic Cayley transformation, Theorem 18 and (A-5), we immediately get

$$\langle\!\langle \phi \rangle\!\rangle_{1,1}^{W_3} = \mathscr{C}_{\tau_*}^{\mathrm{hol}}(\langle\!\langle \omega \rangle\!\rangle_{1,1}^{\mathcal{E}}) = c^{-1}\frac{\partial}{\partial s}\bigl(-\tfrac{1}{2}\log{}_2F_1\bigl(\tfrac{1}{3}, \tfrac{1}{3}; \tfrac{2}{3}; \alpha^{-1}\bigr) - \tfrac{1}{8}\log(1-\alpha^{-1})\bigr).$$

In the above GW generating series, the divisor class ω which corresponds to the first Chern class of a degree-one line bundle on $\mathcal{E}$ is used as the insertion. According to the divisor axiom,

$$\langle\!\langle\ \rangle\!\rangle_{1,0}^{\mathcal{E}} = -\log \eta(\tau),$$

up to an additive constant. Results derived for a plane cubic curve $\mathcal{E}_3$, such as those in Givental's formalism, use the pullback of the hyperplane class on the ambient space $\mathbb{P}^2$ as the insertion. The corresponding class H is related to the one ω above by $H = 3\omega$. Hence we have, up to an additive constant,

$$\langle\!\langle\ \rangle\!\rangle_{1,0}^{\mathcal{E}_3} = -\log \eta(3\tau),$$

and thus

$$\langle\!\langle H\rangle\!\rangle_{1,0}^{\mathcal{E}_3} = -\tfrac{1}{24}\cdot 3E_2(3\tau).$$

Using (A-1), (A-2) and (A-4), one can rewrite it as

$$\langle\!\langle H\rangle\!\rangle_{1,0}^{\mathcal{E}_3} = \frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\tau}\bigl(-\tfrac{1}{2}\log {}_2F_1\bigl(\tfrac{1}{3},\tfrac{2}{3};1;\alpha\bigr) - \tfrac{1}{24}\log(\alpha^3(1-\alpha))\bigr).$$

This matches the results in [47; 58] obtained using virtual localization. Its holomorphic Cayley transformation is

$$\mathscr{C}_{\tau_*}^{\mathrm{hol}}(\langle\!\langle H\rangle\!\rangle_{1,0}^{\mathcal{E}_3}) = c^{-1}\frac{\partial}{\partial s}\bigl(-\tfrac{1}{2}\log {}_2F_1\bigl(\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3};\alpha^{-1}\bigr) - \tfrac{1}{24}\log(1-\alpha^{-1})\bigr).$$

This agrees with the result derived using the wall-crossing method in Guo and Ross [26].

A.2 Cayley transformation and I–functions

Now we discussion the connection between our formulation of LG/CY correspondence and the original formulation in [11, Conjecture 3.2.1] using I–functions.

A.2.1 I–functions and analytic continuation Following [11, Section 4.2], the cohomology-valued Givental I–function for the GW theory of the cubic hypersurface

$$\{W_3 = x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{P}^2$$

is given by[6]

$$I_{\mathrm{GW}}(\mathrm{q}, z) := \sum_{d\geq 0} z\mathrm{q}^{H/z+d}\frac{\prod_{k=1}^{3d}(3H+kz)}{\prod_{k=1}^{d}(H+kz)^3} = I_0^{\mathrm{GW}}(\mathrm{q})z\mathbf{1} + I_1^{\mathrm{GW}}(\mathrm{q})H, \tag{A-6}$$

[6]Here the variable q should not be confused with the variable $q = e^{2\pi i\tau}$ in modular forms.

where H is the hyperplane class of $\mathbb{P}^2$. The I–function for the FJRW theory of the pair (W_3, μ_3) is given by

$$I_{\mathrm{FJRW}}(t,z) := z\sum_{k=1}^{2}\frac{1}{\Gamma(k)}\sum_{\ell\geq 0}\frac{\left(\left(\frac{k}{3}\right)_\ell\right)^3 t^{k+3\ell}}{(k)_{3\ell}z^{k-1}}\phi_{k-1} = I_0^{\mathrm{FJRW}}(t)z\mathbf{1} + I_1^{\mathrm{FJRW}}(t)\phi, \tag{A-7}$$

where $\phi_0 = \mathbf{1}$ and $\phi_1 = \phi$ are nontrivial degree-zero and degree-two elements in the state space, respectively. The genus-zero LG/CY correspondence [11] relates these two I–functions by analytic continuation via $\mathrm{q} = t^{-3}$. To be more explicit, one has the analytic continuation

$$\begin{aligned}\begin{pmatrix}\frac{1}{3}I_1^{\mathrm{FJRW}}(t)\\ \frac{1}{3}I_0^{\mathrm{FJRW}}(t)\end{pmatrix} &= \begin{pmatrix}\frac{(-1)}{\Gamma^3\left(\frac{2}{3}\right)}\frac{2\pi\sqrt{-1}\zeta_3}{1-\zeta_3} & -\frac{(-1)}{\Gamma^3\left(\frac{2}{3}\right)}\frac{(2\pi\sqrt{-1})^2\zeta_3}{(1-\zeta_3)^2}\\ -\frac{(-1)^2}{\Gamma^3\left(\frac{1}{3}\right)}\frac{2\pi\sqrt{-1}\zeta_3^2}{1-\zeta_3^2} & \frac{(-1)^2}{\Gamma^3\left(\frac{1}{3}\right)}\frac{(2\pi\sqrt{-1})^2\zeta_3^2}{(1-\zeta_3^2)^2}\end{pmatrix}\begin{pmatrix}I_1^{\mathrm{GW}}(t(\mathrm{q}))\\ I_0^{\mathrm{GW}}(t(\mathrm{q}))\end{pmatrix},\end{aligned} \tag{A-8}$$

where the normalization factor $\frac{1}{3}$ on the basis $\{I_0^{\mathrm{FJRW}}, I_1^{\mathrm{FJRW}}\}$ is introduced so that the connection matrix lies in $\mathrm{SL}_2(\mathbb{C})$. In particular, define

$$t_{\mathrm{GW}} := \frac{I_1^{\mathrm{GW}}(\mathrm{q})}{I_0^{\mathrm{GW}}(\mathrm{q})} \quad \text{and} \quad t_{\mathrm{FJRW}} := \frac{I_1^{\mathrm{FJRW}}(t)}{I_0^{\mathrm{FJRW}}(t)}. \tag{A-9}$$

Then one has

$$t_{\mathrm{FJRW}} = -e^{\pi i/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)^2}\frac{t_{\mathrm{GW}} - 2\pi i\tau_*}{t_{\mathrm{GW}} - 2\pi i\bar{\tau}_*}. \tag{A-10}$$

A.2.2 Cayley transformation Following the computations in [50] as in Section A.1, we can relate the above I–functions to modular forms. In particular, we see that

$$t_{\mathrm{GW}} := \frac{I_1^{\mathrm{GW}}(\mathrm{q})}{I_0^{\mathrm{GW}}(\mathrm{q})} = 2\pi i\tau, \quad t_{\mathrm{FJRW}} := \frac{I_1^{\mathrm{FJRW}}(t)}{I_0^{\mathrm{FJRW}}(t)} = e^{2\pi i/3}\left(-\frac{\sqrt{3}}{i}\right)\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(-\frac{1}{3}\right)}s. \tag{A-11}$$

Here s is the coordinate given in (4-4), again with $\tau_* = 1/(1-\zeta_3)$ as given in (4-6) and c as in (4-7). Analytical continuations on the I–functions, induced by (A-10), coincide with Cayley transformations on them induced by (4-4), by construction [50].

Through the connection to modular forms, LG/CY correspondence on I–functions can be restated as follows. Let $\mathcal{M} = \Gamma(3)\backslash\mathbb{H}^*$ be the modular curve as the global moduli space, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. Denote its canonical bundle by $K_{\mathcal{M}}$. Then I^{GW} and I^{FJRW} correspond to descriptions of the same holomorphic section of the line bundle that is isomorphic to $K_{\mathcal{M}}^{\otimes 1/2}$, but on different patches of the moduli space. Their coordinate expressions I_0^{GW} and I_0^{FJRW}, with respect to the trivializations $(d\tau)^{1/2}$ and $(ds)^{1/2}$, respectively, are modular forms related by Cayley transformation.

A.2.3 Stationary correlation functions At higher genera, consider the stationary correlation function

$$\langle\!\langle \alpha_1 \psi_1^{\ell_1}, \dots, \alpha_n \psi_n^{\ell_n} \rangle\!\rangle_{g,n}^{\clubsuit},$$

with $\alpha_i = \omega$ when $\clubsuit = \mathcal{E}_3$ and $\alpha_i = \phi$ when $\clubsuit = W_3$. By applying the g–reduction technique in Lemma 7 inductively, we see that under the map (A-11) this correlation function on the GW side is the Fourier expansion of a quasimodular form of weight $2g-2+2n$ near the cusp, and on the FJRW side is the Taylor expansion (in terms of the parameter s) of the same quasimodular form near the point τ_*.

According to standard facts in the theory of modular forms (see eg [53; 57]) on the transition between quasimodular forms and almost-holomorphic modular forms, we see that on the level of GW correlation functions the modular completion is induced by the transformation mapping of the frame of $H^{\mathrm{even}}(\mathcal{E}_3, \mathbb{C})$ from $\{\mathbf{1} + 2\pi i \tau H, 2\pi i H\}$ to $\{\mathbf{1} + 2\pi i \tau H, (1/(\bar\tau - \tau))(\mathbf{1} - 2\pi i \bar\tau H)\}$. This transformation also induces the modular completion on the FJRW correlation functions by composing with the aforementioned transformation that relates I^{GW} with I^{FJRW}.

Then we have a succinct way to reformulate our higher-genus LG/CY correspondence result on $\langle\!\langle \alpha_1 \psi_1^{\ell_1}, \dots, \alpha_n \psi_n^{\ell_n} \rangle\!\rangle_{g,n}^{\clubsuit}$. Denote its modular completion by

$$\langle\!\langle \alpha_1 \psi_1^{\ell_1}, \dots, \alpha_n \psi_n^{\ell_n} \rangle\!\rangle_{g,n}^{\clubsuit,\widehat{\ }}.$$

Let $I_0^{\clubsuit} = I_0^{\mathrm{GW}}$ and $d\tau^{\clubsuit} = d\tau$ for $\clubsuit = \mathcal{E}_3$, and $I_0^{\clubsuit} = I_0^{\mathrm{FJRW}}$ and $d\tau^{\clubsuit} = ds$ for $\clubsuit = W_3$. Then the quantity

$$(I_0^{\clubsuit})^{2-2g} \langle\!\langle \alpha_1 \psi_1^{\ell_1}, \dots, \alpha_n \psi_n^{\ell_n} \rangle\!\rangle_{g,n}^{\clubsuit,\widehat{\ }} (d\tau^{\clubsuit})^{\otimes n}$$

is a global (smooth with holomorphic pole) section of the holomorphic line bundle $K_{\mathcal{M}}^{\otimes n}$ on the modular curve $\mathcal{M} = \Gamma(3)\backslash\mathbb{H}^*$.

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