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Isotopy of the Dehn twist on $K3 \# K3$ after a single stabilization

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Kronheimer and Mrowka recently proved that the Dehn twist along a 3–sphere in the neck of $K3 \# K3$ is not smoothly isotopic to the identity. This provides a new example of self-diffeomorphisms on 4–manifolds that are isotopic to the identity in the topological category but not smoothly so. (The first such examples were given by Ruberman.) We use the Pin(2)–equivariant Bauer–Furuta invariant to show that this Dehn twist is not smoothly isotopic to the identity even after a single stabilization (connected summing with the identity map on $S^2 \times S^2$). This gives the first example of exotic phenomena on simply connected smooth 4–manifolds that do not disappear after a single stabilization.

57R50, 57R52, 57R57; 55P91

1 Introduction

Understanding smooth structures on 4–manifolds remains one of the most difficult topics in low-dimensional topology. In this dimension, many results that hold in the topological category do not hold in the smooth category. Such phenomena are called "exotic phenomena." To motivate our discussion, we list three major instances of exotic phenomena:

- By the groundbreaking work of Donaldson [16; 18] and Freedman [20] (and many subsequent works), there exist many pairs of simply connected closed smooth 4–manifolds that are homeomorphic but not diffeomorphic.

- Ruberman [33] gave the first example of self-diffeomorphisms on 4–manifolds that are isotopic to the identity in the topological category, but not smoothly so. Further examples are given by Auckly, Kim, Melvin and Ruberman[5], Akbulut [3], Baraglia and Konno [8] and Kronheimer and Mrowka [26].

- By the combined work of Wall [36], Perron [31], Quinn [32] and Donaldson [16], there exist pairs of embedded 2–spheres in 4–manifolds with simply connected complement that are topologically isotopic to each other, but not smoothly so; see [3; 5] for explicit families of such examples.

Exotic phenomena appear in each of these three problems, which we call the "diffeomorphism existence problem", the "diffeomorphism isotopy problem" and the "surface isotopy problem". A fundamental principle, discovered by Wall [36; 37] in the 1960s, states that these exotic phenomena will eventually disappear after sufficient many stabilizations on the 4–manifolds. (Here stabilization means taking the connected sum with $S^2 \times S^2$.) More precisely:

- Wall [37] proved that any pair of homotopy equivalent simply connected smooth 4–manifolds are stably diffeomorphic. Namely, they become diffeomorphic after sufficiently many stabilizations.

- Gompf [22] and Kreck [25] further proved that any pair of homeomorphic orientable smooth 4–manifolds (not necessarily simply connected) are stable diffeomorphic. They also proved that nonorientable pairs can be made stably diffeomorphic by first doing a twisted stabilization (ie connected summing a twisted bundle $S^2 \widetilde{\times} S^2$). In fact, for any G with $H^1(G; \mathbb{Z}/2) \neq 0$, Kreck [24] constructed examples of homeomorphic nonorientable smooth 4–manifold pairs with fundamental group G which are not stably diffeomorphic. (Different constructions of such examples were given by Cappell and Shaneson [13] for $G = \mathbb{Z}/2$ and Akbulut [2] for $G = \mathbb{Z}$.) This implies that a twisted stabilization is indeed necessary in the nonorientable case.

- By combining the results of Kreck [23] and Quinn [32], we know that homotopic diffeomorphisms of any simply connected smooth 4–manifold are smoothly isotopic after sufficient many stabilizations. Here stabilization means first isotoping the diffeomorphisms so that they all pointwise fix a small ball B, and then taking the connected sum with the identity map on $S^2 \times S^2$ along B.

- The work of Wall [36], Perron [31] and Quinn [32] shows that any two homologous closed surfaces of the same genus embedded in a 4–manifold with simply connected complement become smoothly isotopic after sufficiently many *external stabilizations*. Here external means that the connected sums with $S^2 \times S^2$ are taken away from the surfaces.

These results motivate the following natural question:

Question 1.1 *How many stabilizations are necessary in each of these three problems?*

There has been speculation that one stabilization is actually enough in all three problems. This is based on several known results:

- It is shown by Baykur and Sunukjian [12] that exotic pairs of nonspin 4–manifolds produced by "standard methods" (logarithmic transforms, knot surgeries, and rational blow-downs) all become diffeomorphic after a single stabilization.

- In the large families of examples (of embedded surfaces and self-diffeomorphisms) established in Akbulut [3] and Auckly, Kim, Melvin and Ruberman [5], exactly one stabilization is needed.

- Auckly, Kim, Ruberman, Melvin and Schwartz [6] proved that any two homologous surfaces F_1 and F_2 of the same genus embedded in a smooth 4–manifold X with simply connected complements are smoothly isotopic after a single stabilization if they are not characteristic (ie $[F_i]$ is not dual to the Stiefel–Whitney class $w_2(X)$). This shows that in the noncharacteristic case, one stabilization is indeed enough in the surface isotopy problem. (When the surfaces are characteristic, they proved a similar result involving a single twisted stabilization.)

We prove the following theorem.

Theorem 1.2 (main theorem) *Let δ be the Dehn twist along a separating 3–sphere in the neck of the connected sum $K3 \# K3$. Then δ is **not** smoothly isotopic to the identity map even after a single stabilization.*

To the author's knowledge, Theorem 1.2 provides the first example that exotic phenomena on simply connected smooth 4–manifolds do not disappear after a single stabilization with respect to $S^2 \times S^2$. In particular, it implies that one stabilization is in general *not* enough in the diffeomorphism isotopy problem.

Note that Kronheimer and Mrowka [26] proved that δ itself is not smoothly isotopic to the identity, using the nonequivariant Bauer–Furuta invariant for spin families. Our result is based on the Kronheimer–Mrowka theorem and makes use of the Pin(2)–equivariant version of the Bauer–Furuta invariant. This invariant was defined in Bauer and Furuta [11] (for a single manifold) and in Szymik [35] and Xu [38] (for families). It has been extensively studied in many papers, including Baraglia [7] and Baraglia and Konno [9], and it is the central tool in Furuta's proof of the $\frac{10}{8}$–theorem [21]. The idea of using gauge-theoretic invariants for families to study the isotopy problem first appears in Ruberman [33]. The idea of using the Pin(2)–equivariant Bauer–Furuta invariant to further study Dehn twists on 4–manifolds was suggested by Kronheimer and Mrowka in [26].

We outline the proof of Theorem 1.2: By taking the mapping torus of δ, we form a smooth bundle N with fiber $K3\#K3$ and base S^1. Then it suffices to show that the bundle $\widetilde{N}$, formed by fiberwise connected sum between N and $(S^2\times S^2)\times S^1$, is not a product bundle. This is proved by showing that the Pin(2)–equivariant Bauer–Furuta invariant $\mathrm{BF}^{\mathrm{Pin}(2)}(\widetilde{N})$ is nonvanishing for both spin structures. Note that $\mathrm{BF}^{\mathrm{Pin}(2)}(\widetilde{N})$ equals the product of $\mathrm{BF}^{\mathrm{Pin}(2)}(N)$ with the Euler class $e_{\widetilde{\mathbb{R}}}$ (a stable homotopy class represented by the inclusion from $S^0=\{0,\infty\}$ to the 1–dimensional representation sphere $S^{\widetilde{\mathbb{R}}}$). We prove this by contradiction, assuming

$$\mathrm{BF}^{\mathrm{Pin}(2)}(N)\cdot e_{\widetilde{\mathbb{R}}}=0. \tag{1}$$

This gives information on $\mathrm{BF}^{\mathrm{Pin}(2)}(N)$ and its S^1–reduction

$$\mathrm{BF}^{S^1}(N)\in\{S^{\mathbb{R}+2\mathbb{H}},S^{6\widetilde{\mathbb{R}}}\}^{S^1}.$$

We can explicitly compute the homotopy group $\{S^{\mathbb{R}+2\mathbb{H}},S^{6\widetilde{\mathbb{R}}}\}^{S^1}$ as $\mathbb{Z}\oplus\mathbb{Z}/2$. Based on this computation, information from (1) and the fact that $\mathrm{BF}^{S^1}(N)$ gives a vanishing family Seiberg–Witten invariant, we can prove that $\mathrm{BF}^{S^1}(N)=0$. This further implies that the nonequivariant Bauer–Furuta invariant $\mathrm{BF}^{\{e\}}(N)$ vanishes, which contradicts Kronheimer and Mrowka's result that $\mathrm{BF}^{\{e\}}(N)$ equals the nonzero element $\eta^3\in\pi_3$. Note that $e_{\widetilde{\mathbb{R}}}$ becomes trivial when reducing to the subgroup $S^1\subset\mathrm{Pin}(2)$. As a consequence, the S^1–equivariant Bauer–Furuta invariant vanishes after a single stabilization (just like the classical Seiberg–Witten invariants and Donaldson's polynomial invariants). This explains why the Pin(2)–equivariance is essential in our proof.

We end this introductory section by remarking that it is still open whether one stabilization is enough to make any pairs of simply connected homeomorphic 4–manifolds diffeomorphic. (See Akbulut, Mrowka and Ruan [4], Donaldson [17] and Fintushel and Stern [19] for a possible approach using the 2–torsion instanton invariants.) It's also unknown whether two homotopic characteristic surfaces with simply connected complements become smoothly isotopic after a single stabilization. The proof of Theorem 1.2 suggests that the Bauer–Furuta invariant could be useful in attacking these problems. As a first step, one needs to establish new examples of spin 4–manifolds with sufficiently interesting higher-dimensional Pin(2)–equivariant Bauer–Furuta invariants. Note that in a recent paper by the author and Mukherjee [29], we use Theorem 1.2 to establish the first pair of orientable exotic surfaces (in a punctured $K3$ surface) which are not smoothly isotopic even after one stabilization.

The paper is organized as follows: In Section 2, we give a brief review of some basic Pin(2)–equivariant stable homotopy theory and recall the definition of the equivariant Bauer–Furuta invariant. We also use this section to set up notation and to adapt some standard results to our setting. The actual proof of Theorem 1.2 is given in Section 3. Experts may directly skip to Section 3 and occasionally refer back to Section 2 for notation and results.

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2 Background

2.1 Pin(2)-equivariant homotopy theory

In this section, we collect some standard results (mostly from [1; 28; 30; 34]) on G–equivariant stable homotopy theory in the case

$$G = \mathrm{Pin}(2) = \{e^{i\theta}\} \cup \{j \cdot e^{i\theta}\} \subset \mathbb{H}.$$

Instead of stating the most general form of these results, we will only focus on the special cases that are actually needed in our argument. We refer to [1; 34] for an introduction to equivariant stable homotopy theory (in the case of finite groups) and to [28; 30] for a more general treatment.

Since all objects we study here are finite G–CW complexes, for simplicity, we will work with the G–equivariant Spanier–Whitehead category [1] (instead of the homotopy category of G–spectra). Of course, there are a lot of drawbacks (eg one cannot always take limits/colimits), but it is enough for our purpose.

2.1.1 Basic facts and definitions Let U be a countably infinite-dimensional G–representation space equipped with a G–invariant inner product, which we call a "universe". We assume that U contains the concrete representation

$$\Big(\bigoplus_{\infty} \mathbb{R}\Big) \oplus \Big(\bigoplus_{\infty} \tilde{\mathbb{R}}\Big) \oplus \Big(\bigoplus_{\infty} \mathbb{H}\Big).$$

Here $\mathbb{R}$ is the trivial representation, $\widetilde{\mathbb{R}}$ is the 1–dimensional representation on which S^1 acts trivially and j acts as -1, and $\mathbb{H}$ is acted upon by G via left multiplication in the quaternions.

To apply the results in [30] directly without checking additional conditions, we further assume that U is "complete". This means that U contains infinitely many copies of all isomorphism classes of irreducible G–representations.[1]

We will use H to denote either the group G or its subgroups S^1 or $\{e\}$. By restricting the G–action on U, we can also use U as a complete H–universe. We use $\boldsymbol{R}_H$ to denote the set of all *finite-dimensional* H–representations contained in U. We will treat $\boldsymbol{R}_G$ as a subset of $\boldsymbol{R}_{S^1}$ and $\boldsymbol{R}_{\{e\}}$ by restricting the G–action.

For any $V \in \boldsymbol{R}_H$, we use S^V to denote the 1–point compactification of V (called the representation sphere) and use $S(V)$ to denote the unit sphere. We set ∞ as the basepoint of S^V and we use $S(V)_+$ to denote the union of $S(V)$ with a disjoint basepoint.

Let X, Y and Z be based finite H–CW complexes; see for example [15, Chapter I] for a definition. We use the notation $[X, Y]^H$ to denote the set of homotopy classes of based H–maps from X to Y (ie maps that preserve the basepoint and are equivariant under H).

Given any $V, W \in \boldsymbol{R}_H$ with $V \subset W$, let $V^\perp$ be the orthogonal complement of V in W. Then smashing with the identity map on $S^{V^\perp}$ provides a map

$$[S^V \wedge X, S^V \wedge Y]^H \to [S^W \wedge X, S^W \wedge Y]^H.$$

One can check that these maps make the collection

$$\{[S^V \wedge X, S^V \wedge Y]^H\}_{V \in \boldsymbol{R}_H}$$

into a direct system. We define $\{X, Y\}^H$ as the direct limit of this system. As in the nonequivariant case, the set $\{X, Y\}^H$ is actually an abelian group. A based H–map

$$S^V \wedge X \to S^V \wedge Y \quad \text{for } V \in \boldsymbol{R}_H$$

will be called a stable H–map from X to Y. An element in the group $\{X, Y\}^H$ will be called a stable homotopy class of H–maps.

[1] Since all G–CW complexes we consider can have only G, S^1 or $\{e\}$ as their isotropy group, all arguments we make actually will still hold for the incomplete universe $(\bigoplus_\infty \mathbb{R}) \oplus (\bigoplus_\infty \widetilde{\mathbb{R}}) \oplus (\bigoplus_\infty \mathbb{H})$, which is more relevant to the geometric setting.

Fact 2.1 *Given any based H–map $f\colon X \to Y$, we form the mapping cone Cf and let $i\colon Y \to Cf$ be the natural inclusion. Then for any Z, the functor $\{*, Z\}^H$ is a generalized cohomology theory* [30, page 157]. *As a result, there is a long exact sequence*

$$\cdots \to \{S^{\mathbb{R}} \wedge X, Z\}^H \xrightarrow{\partial} \{Cf, Z\}^H \xrightarrow{i^*} \{Y, Z\}^H \xrightarrow{f^*} \{X, Z\}^H \xrightarrow{\partial} \{Cf, S^{\mathbb{R}} \wedge Z\}^H \to \cdots \tag{2}$$

associated to the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} Cf$.

Fact 2.2 *Suppose the H–action on X is free away from the basepoint. Then there is a natural map*

$$q_H\colon \{X, Y\}^H \to \{X/H, Y/H\}^{\{e\}} \tag{3}$$

from the equivariant homotopy group to the nonequivariant homotopy group of the quotient space. This map is constructed as follows: Since the H–action on X is free away from the basepoint, any $[f] \in \{X, Y\}^H$ can be represented by an H–map $f\colon S^V \wedge X \to S^V \wedge Y$ such that the H–action on V is trivial; see [1, Proposition 5.5; 28, Theorem 2.8, page 65]. *The map f induces a nonequivariant map between the quotient space,*

$$f/H\colon S^V \wedge (X/H) = (S^V \wedge X)/H \to (S^V \wedge Y)/H = S^V \wedge (Y/H).$$

Then we define $q_H([f])$ as $[f/H]$. One can check that this does not depend on the choice of f and V.

Fact 2.3 [1, Theorem 5.3; 28, Theorem 4.5, page 78] *Suppose the H–action on X is free away from the basepoint and the H–action on Y is trivial. Then the map q_H is an isomorphism.*

For the rest of the section, we assume X and Y are based finite G–CW complexes. The next few facts concern various relations between the G–equivariant homotopy groups and the S^1–equivariant homotopy groups.

Fact 2.4 [1, Theorem 5.1; 28, Theorem 4.7, page 79] *There is a natural isomorphism*

$$\iota\colon \{X, Y\}^{S^1} \xrightarrow{\cong} \{X \wedge (S(\tilde{\mathbb{R}})_+), Y\}^G \tag{4}$$

constructed as follows: Take any $[f] \in \{X, Y\}^{S^1}$ represented by an S^1–map

$$f\colon S^V \wedge X \to S^V \wedge Y.$$

By enlarging V if necessary, we may assume $V \in \boldsymbol{R}_G$. Then we consider the G–map

$$f'\colon S^V \wedge X \wedge (S(\widetilde{\mathbb{R}})_+) = ((S^V \wedge X) \times \{1\}) \vee ((S^V \wedge X) \times \{-1\}) \to Y$$

defined by setting

$$f'(x \times \{1\}) = f(x) \quad \text{and} \quad f'(x \times \{-1\}) = jf(j^{-1}x)$$

for any $x \in S^V \wedge X$. We let $\iota([f]) = [f']$. This map ι turns out to be an isomorphism.

Next, we recall the two operations about changing groups, namely the restriction map

$$\operatorname{Res}^G_{S^1}\colon \{X, Y\}^G \to \{X, Y\}^{S^1} \tag{5}$$

and the transfer map

$$\operatorname{Tr}^G_{S^1}\colon \{X, Y\}^{S^1} \to \{X, Y\}^G. \tag{6}$$

The restriction map is defined by simply ignoring the j–action. To define the transfer map, we consider the Pontryagin–Thom map

$$p\colon S^{\widetilde{\mathbb{R}}} \to S^{\widetilde{\mathbb{R}}} \wedge S(\widetilde{\mathbb{R}})_+$$

that crushes all points outside a normal neighborhood of $S(\widetilde{\mathbb{R}})$ in $S^{\widetilde{\mathbb{R}}}$. (Here we identify the Thom space of the normal bundle of $S(\widetilde{\mathbb{R}})$ as $S^{\widetilde{\mathbb{R}}} \wedge (S(\widetilde{\mathbb{R}})_+)$.) Then the transfer map is defined as the composition

$$\begin{aligned}\{X, Y\}^{S^1} \xrightarrow{\iota} \{(S(\widetilde{\mathbb{R}})_+) \wedge X, Y\}^G = \{S^{\widetilde{\mathbb{R}}} \wedge (S(\widetilde{\mathbb{R}})_+) \wedge X, S^{\widetilde{\mathbb{R}}} \wedge Y\}^G \\ \xrightarrow{p^*} \{S^{\widetilde{\mathbb{R}}} \wedge X, S^{\widetilde{\mathbb{R}}} \wedge Y\}^G = \{X, Y\}^G.\end{aligned} \tag{7}$$

To describe the composition of transfer and restriction, we define the conjugation map

$$c_j\colon \{X, Y\}^{S^1} \to \{X, Y\}^{S^1} \tag{8}$$

as follows: Take any element $[f] \in \{X, Y\}^{S^1}$ represented by an S^1–map $f\colon S^V \wedge X \to S^V \wedge Y$. By enlarging V if necessary, we may assume $V \in \boldsymbol{R}_G$. Then $c_j([f])$ is represented by the composition

$$S^V \wedge X \xrightarrow{j^{-1}} S^V \wedge X \xrightarrow{f} S^V \wedge Y \xrightarrow{j} S^V \wedge Y.$$

Note that when the S^1–action on X is free away from the basepoint, the maps c_j and the map q_{S^1} defined in (3) are compatible. That means

$$q_{S^1}(c_j(\alpha)) = j \circ q_{S^1}(\alpha) \circ j^{-1} \quad \text{for all } \alpha \in \{X, Y\}^{S^1}. \tag{9}$$

Here j and j^{-1} are treated as elements in $\{Y/S^1, Y/S^1\}^{\{e\}}$ and $\{X/S^1, X/S^1\}^{\{e\}}$, respectively.

Next is a special case of the double coset formula [30, Chapter XVIII, Theorem 4.3]. It can be verified directly by unwinding the definitions.

Fact 2.5 *For any $\alpha \in \{X, Y\}^{S^1}$,*

$$\mathrm{Res}^G_{S^1} \mathrm{Tr}^G_{S^1}(\alpha) = \alpha + c_j(\alpha). \tag{10}$$

We end this subsection with an alternative description of the image of $\mathrm{Tr}^G_{S^1}$:

Lemma 2.6 *Let $e_{\tilde{\mathbb{R}}} \in \{S^0, S^{\tilde{\mathbb{R}}}\}^G$ be the element represented by the inclusion map*

$$S^0 = \{0, \infty\} \hookrightarrow S^{\tilde{\mathbb{R}}}. \tag{11}$$

(This element is called the Euler class of $\tilde{\mathbb{R}}$.) *Then the kernel of the map*

$$\{X, Y\}^G \xrightarrow{e_{\tilde{\mathbb{R}}}\cdot} \{X, S^{\tilde{\mathbb{R}}} \wedge Y\}^G \tag{12}$$

equals the image of the transfer map (6).

Proof There is a cofiber sequence $S^0 \hookrightarrow S^{\tilde{\mathbb{R}}} \xrightarrow{p} S^{\tilde{\mathbb{R}}} \wedge S(\tilde{\mathbb{R}})_+$. Smashing this sequence with X and applying the functor $\{*, S^{\tilde{\mathbb{R}}} \wedge Y\}^G$, we get the exact sequence

$$\{(S^{\tilde{\mathbb{R}}} \wedge S(\tilde{\mathbb{R}})_+) \wedge X, S^{\tilde{\mathbb{R}}} \wedge Y\} \xrightarrow{p_*} \{S^{\tilde{\mathbb{R}}} \wedge X, S^{\tilde{\mathbb{R}}} \wedge Y\}^G \xrightarrow{e_{\tilde{\mathbb{R}}}\cdot} \{X, S^{\tilde{\mathbb{R}}} \wedge Y\}^G.$$

So we see that the image of p_* equals the kernel of the map (12). The lemma follows from the definition of $\mathrm{Tr}^G_{S^1}$; see (7). □

2.1.2 The characteristic homomorphism We now define the characteristic homomorphism

$$t\colon \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\tilde{\mathbb{R}}}\}^{S^1} \to \mathbb{Z},$$

following [11], where a, b and c are nonnegative integers with $d \geq a+2$. This homomorphism is of interest to us because the (family) Seiberg–Witten invariant can be obtained by applying t on the Bauer–Furuta invariant. Note that although $\tilde{\mathbb{R}}$ is trivial as an S^1–representation, we still distinguish it with $\mathbb{R}$ in order to keep track of the j–action.

To define t, we take the smash product of the cofiber sequence

$$S^0 \to S^{b\mathbb{H}} \to S^{\mathbb{R}} \wedge S(b\mathbb{H})_+$$

with the sphere $S^{a\mathbb{R}}$ and get the cofiber sequence

$$S^{a\mathbb{R}} \to S^{a\mathbb{R}+b\mathbb{H}} \to S^{(a+1)\mathbb{R}} \wedge S(b\mathbb{H})_+. \tag{13}$$

This induces the long exact sequence

$$(14)\quad \cdots \to \{S^{(a+1)\mathbb{R}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \to \{S^{(a+1)\mathbb{R}} \wedge S(b\mathbb{H})_+, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \to \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \to \{S^{a\mathbb{R}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \to \cdots.$$

Since $d \geq a+2$, the equivariant Hopf theorem [14, Section 8.4] states that the stable homotopy class of an S^1–equivariant stable map from $S^{a\mathbb{R}}$ or $S^{(a+1)\mathbb{R}}$ to $S^{d\widetilde{\mathbb{R}}}$ is determined by its mapping degree on the S^1–fixed point sets. Since this mapping degree is always 0 for dimension reasons,

$$\{S^{a\mathbb{R}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} = \{S^{(a+1)\mathbb{R}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} = 0.$$

Therefore, we get an isomorphism

$$(15)\qquad \xi\colon \{S^{(a+1)\mathbb{R}} \wedge S(b\mathbb{H})_+, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \xrightarrow{\cong} \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1}.$$

Note that the S^1–action on $S^{(a+1)\mathbb{R}} \wedge S(b\mathbb{H})_+$ is free away from the basepoint, with quotient space $S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+$. By composing ξ^{-1} with the isomorphism q_{S^1} given in (3), we get the isomorphism

$$(16)\qquad \psi = q_{S^1} \circ \xi^{-1}\colon \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \xrightarrow{\cong} \{S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+, S^{d\widetilde{\mathbb{R}}}\}^{\{e\}}.$$

Definition 2.7 Suppose $d-a$ is an odd number less than or equal to $4b-1$. Then we define the characteristic homomorphism

$$t\colon \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1} \to \mathbb{Z}$$

by setting $t(\alpha)$ as the image of 1 under the induced map on the reduced cohomology

$$(\psi(\alpha))^*\colon \mathbb{Z} = \widetilde{H}^d(S^{d\widetilde{\mathbb{R}}}) \to \widetilde{H}^d(S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+) \cong \mathbb{Z}.$$

Here we use the standard orientations on $S^{d\widetilde{\mathbb{R}}}$, $S^{(a+1)\mathbb{R}}$ and $\mathbb{C}P^{\frac{1}{2}(d-a-1)}$ to identify the homology groups as $\mathbb{Z}$. If either $d-a$ is even or $d-a > 4b-1$, we simply define t as the zero map.

To discuss the behavior of t under the conjugation map c_j defined in (8), we prove:

Lemma 2.8 *For any $\alpha \in \{S^{a\mathbb{R}+b\mathbb{H}}, S^{d\widetilde{\mathbb{R}}}\}^{S^1}$,*

$$\psi(c_j(\alpha)) = (-1)^d m \circ \psi(\alpha),$$

where $m \in \{\mathbb{C}P^{2b-1}_+, \mathbb{C}P^{2b-1}_+\}^{\{e\}}$ is the "mirror reflection map" defined as

$$m([z_1, z_2, z_3, z_4, \ldots, z_{2b-1}, z_{2b}]) = ([-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3, \ldots, -\bar{z}_{2b}, \bar{z}_{2b-1}]) \quad \textit{for } z_i \in \mathbb{C}.$$

Proof By formula (9), $\psi(c_j(\alpha))$ equals the composition of $\psi(\alpha)$ with the elements

$$j \in \{S^{d\widetilde{\mathbb{R}}}, S^{d\widetilde{\mathbb{R}}}\}^{\{e\}}$$

and

$$j^{-1} \in \{S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+, S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+\}^{\{e\}},$$

which are just $(-1)^d$ and a suspension of m, respectively. □

Corollary 2.9 *When $d-a$ is odd, $t(c_j(\alpha)) = (-1)^{\frac{1}{2}(3d-a-1)} t(\alpha)$ for any α.*

Proof When restricted to $\mathbb{C}P^1$, the map m is just the antipodal map, and so has degree -1. Using the ring structure on $H^*(\mathbb{C}P^{2b-1})$, we see that m has degree $(-1)^{\frac{1}{2}(d-a-1)}$ on $\widetilde{H}^d(S^{(a+1)\mathbb{R}} \wedge \mathbb{C}P^{2b-1}_+)$. The result follows from Lemma 2.8. □

We end this section with the following result, which is essentially the algebraic version of the vanishing result for the Seiberg–Witten invariant of connected sums.

Lemma 2.10 *Given any $\alpha_1 \in \{S^{a_1\mathbb{R}+b_1\mathbb{H}}, S^{d_1\widetilde{\mathbb{R}}}\}^{S^1}$ and $\alpha_2 \in \{S^{a_2\mathbb{R}+b_2\mathbb{H}}, S^{d_2\widetilde{\mathbb{R}}}\}^{S^1}$, we have $t(\alpha_1\alpha_2) = 0$ if $d_1 > a_1$ and $d_2 > a_2$.*

Proof The product $\alpha_1\alpha_2$ belongs to the group

$$\{S^{(a_1+a_2)\mathbb{R}+(b_1+b_2)\mathbb{H}}, S^{(d_1+d_2)\widetilde{\mathbb{R}}}\}^{S^1}.$$

Therefore, $t(\alpha_1\alpha_2)$ can be nonzero only if $d_1 + d_2 - a_1 - a_2$ is odd. Without loss of generality, we may assume $d_1 - a_1$ is odd and $d_2 - a_2$ is even. Since $d_i > a_i$ for $i = 1, 2$, the group $\{S^{a_i\mathbb{R}}, S^{d_i\widetilde{\mathbb{R}}}\}^{S^1}$ vanishes. By the long exact sequence (14), we see that α_i equals the image of some element

$$\beta_i \in \{S^{(a_i+1)\mathbb{R}} \wedge S(b_i\mathbb{H})_+, S^{d_i\widetilde{\mathbb{R}}}\}^{S^1} = \{S^{a_i\mathbb{R}} \wedge (S^{b_i\mathbb{H}}/S^0), S^{d_i\widetilde{\mathbb{R}}}\}^{S^1}.$$

Here we identify $S^{b_i\mathbb{H}}/S^0$ with $S^{\mathbb{R}} \wedge S(b_i\mathbb{H})_+$ by treating $S^{\mathbb{R}}$ as the one-point compactification of $(0, +\infty)$ and sending $v \in \mathbb{H}^{b_i} \setminus \{0\}$ to $(|v|, v/|v|) \in (0, \infty) \times S(b_i\mathbb{H})$.

Next, we consider the commutative diagram

$$\begin{tikzcd} S^{(b_1+b_2)\mathbb{H}} \arrow[r, "q"] \arrow[d, "\cong"] & S^{(b_1+b_2)\mathbb{H}}/S^0 \arrow[d, "\gamma"] \\ S^{b_1\mathbb{H}} \wedge S^{b_2\mathbb{H}} \arrow[r, "q_1 \wedge q_2"] & (S^{b_1\mathbb{H}}/S^0) \wedge (S^{b_2\mathbb{H}}/S^0) \end{tikzcd} \tag{17}$$

where q, q_1, q_2 and

$$\gamma\colon S^{(b_1+b_2)\mathbb{H}}/S^0 \to S^{(b_1+b_2)\mathbb{H}}/((S^0\wedge S^{b_2\mathbb{H}})\cup(S^{b_1\mathbb{H}}\wedge S^0)) = (S^{b_1\mathbb{H}}/S^0)\wedge(S^{b_2\mathbb{H}}/S^0)$$

are all quotient maps. From (17), we see that

$$\alpha_1\wedge\alpha_2 = (\beta_1\wedge\beta_2)\circ(q_1\wedge q_2) = (\beta_1\wedge\beta_2)\circ\gamma\circ q.$$

Therefore, $\xi(\alpha_1\alpha_2) = (\beta_1\beta_2)\circ\gamma$.

Moreover, checking the explicit construction of the map q_{S^1} given in Fact 2.2, we see that q_{S^1} is also natural under the smash product and composition. Therefore,

$$\psi(\alpha_1\alpha_2) = q_{S^1}(\xi(\alpha_1\alpha_2)) = q_{S^1}(\beta_1\beta_2)\circ q_{S^1}(\gamma),$$

and $q_{S^1}(\beta_1\beta_2)$ equals the composition

$$S^{(a_1+a_2+2)\mathbb{R}}\wedge\big((S(b_1\mathbb{H})_+\wedge S(b_2\mathbb{H})_+)/S^1\big) \to (S^{(a_1+1)\mathbb{R}}\wedge(S(b_1\mathbb{H})_+)/S^1))\wedge(S^{(a_2+1)\mathbb{R}}\wedge(S(b_2\mathbb{H})_+)/S^1)) \xrightarrow{q_{S^1}(\beta_1)\wedge q_{S^1}(\beta_2)} S^{d_1\tilde{\mathbb{R}}}\wedge S^{d_2\tilde{\mathbb{R}}}.$$

Because d_2-a_2 is even, the cohomology $\widetilde{H}^{d_2}(S^{(a_2+1)\mathbb{R}}\wedge(S(b_2\mathbb{H})_+)/S^1))$ equals 0. So $q_{S^1}(\beta_2)$ induces the trivial map on the reduced cohomology. This implies that $\psi(\alpha_2\alpha_2)$ induces the trivial map on $\widetilde{H}^{d_1+d_2}(*)$. Hence, $t(\alpha_1\alpha_2)=0$. □

2.2 The Pin(2)-equivariant Bauer–Furuta invariant for spin families

In this section, we briefly summarize the definition and some important properties of the Bauer–Furuta invariant for spin families. This invariant was originally defined in [11] for a single 4–manifold. The family version was first defined in [35; 38] and later extensively studied in [7; 9]. Because we want to construct the Bauer–Furuta invariant as a concrete element in the G–equivariant stable homotopy group of spheres, some care must be taken in the construction.

2.2.1 Spin structures on the circle family of 4–manifolds Let N be a smooth fiber bundle whose fiber is a closed spin 4–manifold M and whose base is another closed manifold B. For simplicity, we will make the following assumption throughout the paper:

Assumption 2.11 The bundle N satisfies:

(i) M is simply connected.

(ii) The signature $\sigma(M)$ is at most 0.

(iii) Let M_x be the fiber over the point $x \in B$. Then the action of $\pi_1(B, x)$ on $H^2(M_x; \mathbb{Z})$ (given by the holonomy of the bundle) is trivial.

We equip N with a Riemannian metric and let $\mathrm{Fr}^v(N)$ be the frame bundle of the *vertical* tangent bundle of N. This is an SO(4)–bundle over N.

Definition 2.12 A spin structure $\mathfrak{s}$ on N is a double covering map $\pi\colon P \to \mathrm{Fr}^v(N)$ that restricts to a nontrivial covering map $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ on each fiber. Two spin structures (π, P) and (π', P') are called isomorphic if there exists a homeomorphism $P \to P'$ that covers the *identity map on* $\mathrm{Fr}^v(N)$.

Definition 2.13 The pair $(N, \mathfrak{s})$ is called a spin family. Two spin families $(N_1, \mathfrak{s}_1)$ and $(N_2, \mathfrak{s}_2)$ over the same base B are called "isomorphic" if there exists a bundle isomorphism $f\colon N_1 \to N_2$ such that $f^*(\mathfrak{s}_2)$ is isomorphic to $\mathfrak{s}_1$.

We are mainly interested in the case that B is a circle or a point. By Assumption 2.11, N has a unique spin structure when B is a point and has two spin structures when B is a circle. We give an explicit description of these two spin structures as follows: Let $\pi_M\colon P_M \to \mathrm{Fr}(M)$ be the covering map given by the unique spin structure on M. Then the bundle N is obtained by gluing the two boundary components of $M \times [0,1]$ via a diffeomorphism $f\colon M \to M$. The diffeomorphism induces a map $f_*\colon \mathrm{Fr}(M) \to \mathrm{Fr}(M)$, which has two lifts $f_*^{\pm}\colon P_M \to P_M$. These lifts differ from each other by the deck transformation $\tau\colon P_M \to P_M$. We use $f_*^{\pm}$ to glue the two boundary components of $P_M \times I$ and form two spin structures on N.

Definition 2.14 When $N = M \times S^1$, the maps $f_*^{\pm}$ are just the identity map and the deck transformation τ. We call the associated spin structures over N the *product spin structure* and the *twisted spin structure*, respectively. Let $\mathfrak{s}$ be the unique spin structure on M. Then we use $\tilde{\mathfrak{s}}$ to denote the former and use $\tilde{\mathfrak{s}}^{\tau}$ to denote the latter.

For general M, the product family and the twisted family are not isomorphic. For example, Kronheimer and Mrowka [26] established:

Example 2.15 The product family $(K3 \times S^1, \tilde{\mathfrak{s}})$ and the twisted family $(K3 \times S^1, \tilde{\mathfrak{s}}^{\tau})$ are not isomorphic, as can be proved by the nonequivariant Bauer–Furuta invariant.

However, for the special case of $S^2 \times S^2$, these two families are indeed isomorphic:

Lemma 2.16 *$((S^2 \times S^2) \times S^1, \tilde{\mathfrak{s}})$ and $((S^2 \times S^2) \times S^1, \tilde{\mathfrak{s}}^\tau)$ are isomorphic.*

Proof There is an S^1–action on S^2 with fixed points $\{0, \infty\}$. We use $\xi\colon S^1 \times S^2 \to S^2$ to denote this action. As x varies from 0 to 2π, the induced map

$$(\mathrm{id}_{S^2} \times \xi(x, \cdot))_*\colon T_{(0,0)}(S^2 \times S^2) \to T_{(0,0)}(S^2 \times S^2)$$

gives an essential loop in SO(4). Using this fact, one can verify that the bundle automorphism

$$f\colon (S^2 \times S^2) \times S^1 \to (S^2 \times S^2) \times S^1$$

defined by $f(y_1, y_2, x) = (y_1, \xi(x, y_2), x)$ satisfies $f^*(\tilde{\mathfrak{s}}) = \tilde{\mathfrak{s}}^\tau$. □

2.2.2 Definition of the Bauer–Furuta invariant As in the case of a single 4–manifold, a spin structure $\mathfrak{s}$ gives rise to two quaternion bundles $S^\pm$ over N. Denote by $S_x^\pm$ the restriction of $S^\pm$ to the fiber M_x. Then the spin Dirac operator

$$D(M_x)\colon \Gamma(S_x^+) \to \Gamma(S_x^-)$$

is a quaternionic linear operator. We form the operator D over N by putting $D(M_x)$ together.

Now we consider four Hilbert bundles $\mathcal{V}^+$, $\mathcal{V}^-$, $\mathcal{U}^+$ and $\mathcal{U}^-$ over B. The fibers of $\mathcal{V}^\pm$ are suitable Sobolev completions of $\Gamma(S_x^\pm)$, and the fibers of $\mathcal{U}^+$ and $\mathcal{U}^-$ are completions of $\Omega^1(M_x)$ and $\Omega^2_+(M_x) \oplus \Omega^0(M_x)/\mathbb{R}$, respectively. We let $G = \mathrm{Pin}(2)$ act on $\mathcal{V}^\pm$ by left multiplication in the quaternions, and we let G act on $\mathcal{U}^\pm$ by setting the S^1–action to be trivial and setting the j–action as multiplication by -1.

The family Seiberg–Witten equations give a fiber-preserving G–equivariant map

$$\mathcal{SW}\colon \mathcal{U}^+ \oplus \mathcal{V}^+ \to \mathcal{U}^- \oplus \mathcal{V}^-.$$

This Seiberg–Witten map can be written as $l + c$, where l is the fiberwise Fredholm operator

$$l := D \oplus (d^+, d^*)$$

and c is a certain 0^{th} order operator. Furthermore, by the boundedness property of the Seiberg–Witten equations [11, Proposition 3.1], $\mathcal{SW}$ extends to a map

$$\mathcal{SW}^+\colon (\mathcal{U}^+ \oplus \mathcal{V}^+)_\infty \to (\mathcal{U}^- \oplus \mathcal{V}^-)_\infty$$

between the one-point completions

$$(\mathcal{U}^\pm \oplus \mathcal{V}^\pm)_\infty := (\mathcal{U}^\pm \oplus \mathcal{V}^\pm) \cup \{\infty\}.$$

To apply the finite-dimensional approximation technique on the map $\mathcal{SW}$, we carefully choose finite-dimensional subspaces of $\mathcal{V}^{\pm}$ and $\mathcal{U}^{\pm}$ as follows: First, we apply Kuiper's theorem [27] to get canonical trivialization of the bundles

$$\mathcal{V}^- \cong B \times L^2(\mathbb{H}^\infty) \quad \text{and} \quad \mathcal{U}^+ \cong B \times L^2(\widetilde{\mathbb{R}}^\infty). \tag{18}$$

Here $L^2(*)$ denotes the completion with respect to the L^2–norm. Choose $m, n \gg 0$ and let $U^+ \subset \mathcal{U}^+$ and $V^- \subset \mathcal{V}^-$ be the subbundles corresponding to the bundles $B \times \mathbb{H}^n$ and $B \times \widetilde{\mathbb{R}}^m$ under the isomorphism (18). Let H_2^+ be the subbundle of $\mathcal{U}^-$ consisting of all self-dual harmonic 2–forms on M_x. We set

$$U^- := H_2^+ \oplus ((d^+, d^*)U^+) \subset \mathcal{U}^-.$$

(Note that (d^+, d^*) is injective by our assumption that $b_1(M) = 0$.) We choose m large enough so that V^- is fiberwise transverse to D and we set $V^+ := D^{-1}(V^-) \subset \mathcal{V}^+$.

Set $W^+ := U^+ \oplus V^+$ and $W^- := U^- \oplus V^-$. As explained in [11], when m and n are large enough,

$$\mathcal{SW}^+(W^+_\infty) \cap S(W^{-,\perp}) = \varnothing,$$

where $S(W^{-,\perp})$ denotes the unit sphere in the orthogonal complement of W^- in $\mathcal{U}^- \oplus \mathcal{V}^-$. Therefore, by composing $\mathcal{SW}^+$ with a specific G–equivariant deformation retraction

$$\rho\colon (\mathcal{U}^- \oplus \mathcal{V}^-)_\infty \setminus S(W^{-,\perp}) \to W^-_\infty,$$

one obtains a G–equivariant map

$$\mathrm{sw}\colon W^+_\infty \to W^-_\infty.$$

Restriction of (18) gives canonical trivializations of the bundles V^- and U^+. By Assumption 2.11, $\pi_1(B)$ acts trivially on $H^2(M_x)$. Therefore, as explained in [26], a homology orientation of M determines a canonical trivialization of H_2^+. At this point, we have obtained canonical trivializations of $U^\pm$ and V^-. Using these trivializations, we get the composition map

$$(S^{m\widetilde{\mathbb{R}}} \wedge V^+_\infty) \cong W^+_\infty \xrightarrow{\mathrm{sw}} W^-_\infty \cong \bigl(S^{(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}} \wedge B_+\bigr) \xrightarrow{pj} S^{(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}}, \tag{19}$$

where pj denotes projection to the first factor.

From now on, we specialize to the case that B is a circle or point. Note that V^+ is a quaternionic bundle of dimension $n - \frac{1}{16}\sigma(M)$ and the group $\mathrm{Sp}\bigl(n - \frac{1}{16}\sigma(M)\bigr)$ has trivial π_i for $i \le 2$. So the bundle V^+ has a trivialization (canonical up to homotopy). This trivialization allows us to fix an identification

$$V^+_\infty \cong (S^{(n-\sigma(M)/16)\mathbb{H}} \wedge B_+)$$

and rewrite the map (19) as a G–map

$$\widetilde{\mathrm{sw}}\colon S^{m\widetilde{\mathbb{R}}+(n-\sigma(M)/16)\mathbb{H}} \wedge B_+ \to S^{(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}}, \tag{20}$$

which represents an element in $[\widetilde{\mathrm{sw}}] \in \{S^{-(\sigma(M)/16)\mathbb{H}} \wedge B_+, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G$. By checking the concrete construction of $\widetilde{\mathrm{sw}}$ in [11], one establishes:

Fact 2.17 *Consider the map $S^{m\widetilde{\mathbb{R}}} \wedge B_+ \to S^{(m+b^+(M))\widetilde{\mathbb{R}}}$ given by restricting $\widetilde{\mathrm{sw}}$ to the S^1–fixed point sets. This map can be explicitly described as the composition*

$$S^{m\widetilde{\mathbb{R}}} \wedge B_+ \xrightarrow{\textit{projection}} S^{m\widetilde{\mathbb{R}}} \xrightarrow{\textit{inclusion}} S^{(m+b^+(M))\widetilde{\mathbb{R}}}.$$

Definition 2.18 Suppose B is a point. Then $M = N$ and $S^{-(\sigma(M)/16)\mathbb{H}} \wedge B_+ = S^{-(\sigma(M)/16)\mathbb{H}}$. In this case, we define the G–equivariant Bauer–Furuta invariant as

$$\mathrm{BF}^G(M, \mathfrak{s}) := [\widetilde{\mathrm{sw}}] \in \{S^{-(\sigma(M)/16)\mathbb{H}}, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G.$$

We will neglect the spin structure $\mathfrak{s}$ in our notation when it is obvious from the context.

Example 2.19 $\mathrm{BF}^G(S^4)$ is an element in $\{S^0, S^0\}^G$ represented by a G–map from the $S^{m\widetilde{\mathbb{R}}+n\mathbb{H}}$ to itself. By the equivariant Hopf theorem [15, Chapter II.4], such a stable homotopy class is determined by its restriction to the S^1–fixed points. Hence, by Fact 2.17, we see that $\mathrm{BF}^G(S^4) = 1$.

Example 2.20 $\mathrm{BF}^G(S^2 \times S^2) \in \{S^0, S^{\widetilde{\mathbb{R}}}\}^G$ is represented by a G–map from $S^{m\widetilde{\mathbb{R}}+n\mathbb{H}}$ to $S^{(m+1)\widetilde{\mathbb{R}}+n\mathbb{H}}$. Such a map is also determined by its restriction on the S^1–fixed points. By Fact 2.17 again, we see that $\mathrm{BF}^G(S^2 \times S^2) = e_{\widetilde{\mathbb{R}}}$. Here $e_{\widetilde{\mathbb{R}}}$ is the Euler class defined in (11).

When B is a circle, we identify it with the unit sphere $S(2\mathbb{R})$ in $S^{2\mathbb{R}}$. Consider the cofiber sequence

$$S(2\mathbb{R}) \cup \{\infty\} \to S^0 \to S^{2\mathbb{R}} \xrightarrow{p} S^{\mathbb{R}} \wedge (S(2\mathbb{R}) \cup \{\infty\}). \tag{21}$$

The map p, which is just the Pontryagin–Thom map for the inclusion $S(2\mathbb{R}) \hookrightarrow S^{2\mathbb{R}}$, can be treated as a stable map from $S^{\mathbb{R}}$ to B_+. This stable map induces the map

$$p^*\colon \{S^{-(\sigma(M)/16)\mathbb{H}} \wedge B_+, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G \to \{S^{\mathbb{R}-(\sigma(M)/16)\mathbb{H}}, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G$$

that sends α to $\alpha \circ (\mathrm{id}_{S^{-(\sigma(M)/16)\mathbb{H}}} \wedge p)$.

Definition 2.21 When $B = S(2\mathbb{R})$ we define the G–equivariant Bauer–Furuta invariant

$$\mathrm{BF}^G(N,\mathfrak{s}) := p^*[\widetilde{\mathrm{sw}}] \in \{S^{\mathbb{R}-(\sigma(M)/16)\mathbb{H}}, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G.$$

In either case, we define both the S^1–equivariant and nonequivariant Bauer–Furuta invariants as the restriction of the G–equivariant Bauer–Furuta invariant:

$$\mathrm{BF}^{S^1}(N,\mathfrak{s}) := \mathrm{Res}^G_{S^1}(\mathrm{BF}^G(N,\mathfrak{s})),$$
$$\mathrm{BF}^{\{e\}}(N,\mathfrak{s}) := \mathrm{Res}^G_{\{e\}}(\mathrm{BF}^G(N,\mathfrak{s})).$$

In [26], Kronheimer and Mrowka gave an alternative definition of $\mathrm{BF}^{\{e\}}(N,\mathfrak{s})$: Take a generic section r of the bundle W^- that is transverse to the map sw. Then the preimage $\mathrm{sw}^{-1}(r)$ is a manifold. When B is a point, the canonical trivializations of the bundles $W^\pm$ determine a stable framing on $\mathrm{sw}^{-1}(r)$. When B is $S(2\mathbb{R})$, we fix a stable framing on B that bounds a framed disk. Then together with the trivializations of $W^\pm$, this determines a stable framing on $\mathrm{sw}^{-1}(r)$. In [26], the family Bauer–Furuta invariant is defined as the framed cobordism class of $\mathrm{sw}^{-1}(r)$.

Recall that the framed cobordism classes of smooth n–manifolds are classified by elements in the n^{th} stable homotopy group of spheres. The following lemma states that our definition of $\mathrm{BF}^{\{e\}}$ is essentially identical to Kronheimer and Mrowka's definition.

Lemma 2.22 *The framed cobordism class of* $\mathrm{sw}^{-1}(r)$ *is classified by the nonequivariant Bauer–Furuta invariant* $\mathrm{BF}^{\{e\}}(N,\mathfrak{s})$.

Proof By Sard's theorem, we can take r to be a constant section that sends the whole B to a generic point $r_0 \in S^{(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}}$. Then $\mathrm{sw}^{-1}(r) = \widetilde{\mathrm{sw}}^{-1}(r_0)$ and it is also the preimage of the point

$$\{0\} \times r_0 \in S^{\mathbb{R}+(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}}$$

under the composition

$$(\mathrm{id}_{S^{\widetilde{\mathbb{R}}}} \wedge \widetilde{\mathrm{sw}}) \circ (\mathrm{id}_{S^{(n-\sigma(M)/16)\mathbb{H}+m\widetilde{\mathbb{R}}}} \wedge p)\colon S^{2\mathbb{R}+m\widetilde{\mathbb{R}}+(n-\sigma(M)/16)\mathbb{H}} \to S^{\mathbb{R}+(m+b^+(M))\widetilde{\mathbb{R}}+n\mathbb{H}}. \tag{22}$$

Because r_0 is a regular value of $\widetilde{\mathrm{sw}}$ and any point in $\{0\} \times B_+$ is a regular value of p, we see that $\{0\} \times r_0$ is indeed a regular value of the map (22). Recall that an element in the stable group of spheres defines a stably framed manifold by taking the preimage of a regular value and taking the induced framing. The proof is finished by observing that the stable framing on B that bounds a framed disk (the one we used to fix the framing on $\mathrm{sw}^{-1}(r)$) is exactly the framing induced by the inclusion $B \hookrightarrow S^{2\mathbb{R}}$. □

2.2.3 Some properties of the Bauer–Furuta invariant In this subsection, we summarize some important properties of the Bauer–Furuta invariant. We start with a vanishing result. Recall from Definition 2.14 that on the trivial bundle $N = M \times S^1$ there are two spin structures: the product spin structure $\tilde{\mathfrak{s}}$ and the twisted spin structure $\tilde{\mathfrak{s}}^\tau$.

Lemma 2.23 *The Bauer–Furuta invariants* BF^G, BF^{S^1} *and* $\mathrm{BF}^{\{e\}}$ *of the product spin structure* $\tilde{\mathfrak{s}}$ *are all vanishing.*

Proof The cofiber sequence (21) induces a long exact sequence

$$\cdots \to \{S^{-(\sigma(M)/16)\mathbb{H}}, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G \xrightarrow{q^*} \{S^{-(\sigma(M)/16)\mathbb{H}} \wedge B_+, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G \xrightarrow{p^*} \{S^{\mathbb{R}-(\sigma(M)/16)\mathbb{H}}, S^{b^+(M)\widetilde{\mathbb{R}}}\}^G \to \cdots,$$

where q^* is induced by the map $q\colon B_+ \to S^0$ that preserves the basepoint and sends B to the other point. By its definition, the map $\widetilde{\mathrm{sw}}$ for $(M \times S^1, \tilde{\mathfrak{s}})$ is just a pullback of the corresponding map for $(M, \mathfrak{s})$ via the map q. So $[\widetilde{\mathrm{sw}}] \in \mathrm{Image}(q^*)$, which implies

$$\mathrm{BF}^G((M \times S^1, \tilde{\mathfrak{s}})) = p^*([\widetilde{\mathrm{sw}}]) = 0.$$

The invariants BF^{S^1} and $\mathrm{BF}^{\{e\}}$ vanish because BF^G vanishes. □

Regarding the Bauer–Furuta invariant of the twisted spin structure, Kronheimer and Mrowka [26] proved the following result by studying the stable framing on the moduli space:

Proposition 2.24 *We have*

$$\mathrm{BF}^{\{e\}}(M \times S^1, \tilde{\mathfrak{s}}^\tau) = \begin{cases} \eta \cdot \mathrm{BF}^{\{e\}}(M, \mathfrak{s}) & \text{when } \sigma(M) \equiv 16 \bmod 32, \\ 0 & \text{when } 32 \mid \sigma(M). \end{cases} \tag{23}$$

Here $\eta \in \{S^{\mathbb{R}}, S^0\}^{\{e\}}$ *denotes the Hopf map.*

Remark It would be interesting to prove a generalization of Proposition 2.24 for $\mathrm{BF}^G(M \times S^1, \tilde{\mathfrak{s}}^\tau)$ and $\mathrm{BF}^{S^1}(M \times S^1, \tilde{\mathfrak{s}}^\tau)$.

Next, we give a connected sum formula for the family Bauer–Furuta invariants. This formula was originally proved by Bauer [10] for a single 4–manifold.

To set up the theorem we let $(N_i, \mathfrak{s}_i)$ for $i = 1, 2$ be two spin families over $B = S(2\mathbb{R})$ with fiber M_i, both satisfying Assumption 2.11. To form the connected sum, we pick sections $\gamma_i\colon B \to N_i$. By Assumption 2.11(i), the section γ_i is unique up to homotopy. We remove small standard 4–balls around these sections to form the family $N_i - D^4 \times S^1$ of 4–manifolds with boundary. Then we can form the fiberwise connected sum by identifying the collars of their boundaries. To fix such an identification, we need to choose a smooth family of orientation reversing isomorphisms

$$\tilde{\phi} := \{\phi_x : T_{\gamma_1(x)}(M_1)_x \xrightarrow{\cong} T_{\gamma_2(x)}(M_2)_x\}_{x \in B}.$$

We use $N_1 \#_{\tilde{\phi}} N_2$ to denote the resulting bundle over B, with fiber $M_1 \# M_2$. In general, the result $N_1 \#_{\tilde{\phi}} N_2$ will depend on the choice of $\tilde{\phi}$ up to homotopy. Because $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}/2$, there are essentially two choices.

Lemma 2.25 *There exists exactly one choice of $\tilde{\phi}$ such that the spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ can be glued together to form a spin structure on $N_1 \#_{\tilde{\phi}} N_2$. We denote this choice by $\tilde{\phi}(\mathfrak{s}_1, \mathfrak{s}_2)$ and denote the resulting spin structure by $\mathfrak{s}_1 \# \mathfrak{s}_2$.*

Proof Denote by $\tilde{\phi}^{\pm}$ the two choices of $\tilde{\phi}$. Then they provide gluing maps

$$f^{\pm} : \partial(N_1 - D^4 \times S^1) \to \partial(N_2 - D^4 \times S^1),$$

which differ from each other by a Dehn twist on $\partial(N_2 - D^4 \times S^1)$. Under any boundary parametrization $\partial(N_2 - D^4 \times S^1) \cong S^3 \times S^1$, this Dehn twist can be written as

$$\iota(v, x) = (\alpha(x)v, x) \quad \text{for } (v, x) \in S^3 \times S^1,$$

where $\alpha : S^1 \to \mathrm{SO}(4)$ is an essential loop. Note that $S^3 \times S^1$, regarded as the product S^3–bundle over S^1, has two family spin structures (the product spin structure and the twisted spin structure), which are related to each other by ι. We see that exactly one of the two maps $f^{\pm}$ sends $\mathfrak{s}_1|_{\partial(N_1 - D^4 \times S^1)}$ to $\mathfrak{s}_2|_{\partial(N_2 - D^4 \times S^1)}$. This finishes the proof. We also note that when $\tilde{\phi} = \tilde{\phi}(\mathfrak{s}_1, \mathfrak{s}_2)$, the gluing map on the boundary has two lifts to the gluing map on the spin bundle, but they give isomorphic spin structures on the connected sum. □

From the discussion above, there is a unique way to take the connected sum of two spin families $(N_i, \mathfrak{s}_i)$. The resulting spin family $(N_1 \#_{\tilde{\phi}(\mathfrak{s}_1, \mathfrak{s}_2)} N_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$ will also be written as $(N_1, \mathfrak{s}_1) \# (N_2, \mathfrak{s}_2)$.

To talk about the Bauer–Furuta invariant of a connected sum, we also need to specify a rule for homology orientation. Given homology orientations on M_1 and M_2, we let the homology orientation on $M_1 \# M_2$ be defined by putting the oriented basis for $H^2_+(M_1)$ in front of the oriented basis for $H^2_+(M_2)$. The following theorem is a family version of Bauer's connected sum formula [10]:

Proposition 2.26 *Let $(M \times S^1, \tilde{\mathfrak{s}})$ be the product family for some spin 4–manifold $(M, \mathfrak{s})$. Then*

$$\mathrm{BF}^{\mathrm{H}}((N_1, \mathfrak{s}_1) \# (M \times S^1, \tilde{\mathfrak{s}})) = \mathrm{BF}^H(N_1, \mathfrak{s}_1) \wedge \mathrm{BF}^H(M, \mathfrak{s})$$

for $H = G$, S^1 or $\{e\}$.

Proof The proof is essentially identical to the single 4–manifold case in [10]; see [26] for a sketch of the proof for the family version (in the nonequivariant setting). A central step is an excision argument that builds a homotopy between the approximated Seiberg–Witten maps $\widetilde{\mathrm{sw}}$ (20) for the bundle

$$N_1 \cup (M \times S^1) \cup (S^4 \times S^1)$$

viewed as a family over S^1 with fiber $M_1 \cup M \cup S^4$, and the bundle

$$(N \# (M \times S^1) \cup (S^4 \times S^1)) \cup (S^4 \times S^1),$$

viewed as a family over S^1 with fiber $(M_1 \# M) \cup S^4 \cup S^4$. This homotopy is constructed by multiplying various sections by scalar-valued real cutoff functions and applying various terms in the Seiberg–Witten map, which are all G–equivariant. Therefore, this homotopy is G–equivariant. □

As a corollary, we get the following result, which computes the Bauer–Furuta invariant under family stabilization:

Corollary 2.27 *Consider the product spin structure $\tilde{\mathfrak{s}}_0$ and the twisted spin structure $\tilde{\mathfrak{s}}_0^\tau$ over the product bundle $((S^2 \times S^2) \times S^1)$. Then, for any spin family $(N, \mathfrak{s})$ that satisfies Assumption 2.11,*

$$\mathrm{BF}^G\big((N, \mathfrak{s}) \# \big(((S^2 \times S^2) \times S^1), \tilde{\mathfrak{s}}_0\big)\big) = \mathrm{BF}^G(N, \mathfrak{s}) \cdot e_{\widetilde{\mathbb{R}}} \tag{24}$$

and

$$\mathrm{BF}^G\big((N, \mathfrak{s}) \# \big(((S^2 \times S^2) \times S^1), \tilde{\mathfrak{s}}_0^\tau\big)\big) = \mathrm{BF}^G(N, \mathfrak{s}) \cdot e_{\widetilde{\mathbb{R}}}. \tag{25}$$

Here $e_{\widetilde{\mathbb{R}}} \in \{S^0, S^{\widetilde{\mathbb{R}}}\}^G$ is the Euler class defined in (11).

Proof The formula (24) follows from Proposition 2.26 and Example 2.20. The formula (25) follows from (24) and Lemma 2.16. □

3 Proof of the main theorem

3.1 The key proposition

In this subsection, we prove the homotopy theoretic Proposition 3.2, which will be the key ingredient in the proof of our main theorem.

Recall that the group $\{S^{\mathbb{R}+2\mathbb{H}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1}$ admits a conjugation action c_j; see (8). The following lemma computes this group and this action:

Lemma 3.1 *The characteristic homomorphism $t\colon \{S^{\mathbb{R}+2\mathbb{H}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} \to \mathbb{Z}$ is surjective and has $\ker t = \mathbb{Z}/2$. The conjugation action c_j acts trivially on $\ker t$.*

Proof Smashing the cofiber sequence $S^0 \to S^{2\mathbb{H}} \to S^{\mathbb{R}} \wedge (S(2\mathbb{H})_+)$ with $S^{\mathbb{R}}$, we get a cofiber sequence $S^{\mathbb{R}} \to S^{\mathbb{R}+2\mathbb{H}} \to S^{2\mathbb{R}} \wedge (S(2\mathbb{H})_+)$, which induces the long exact sequence

$$\cdots \to \{S^{2\mathbb{R}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} \to \{S^{2\mathbb{R}} \wedge (S(2\mathbb{H})_+), S^{6\widetilde{\mathbb{R}}}\}^{S^1} \\ \to \{S^{\mathbb{R}+2\mathbb{H}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} \to \{S^{\mathbb{R}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} \to \cdots$$

By the equivariant Hopf theorem [15, Chapter II.4], $\{S^{\mathbb{R}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} = \{S^{2\mathbb{R}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} = 0$. Hence, we get the isomorphism

$$\{S^{\mathbb{R}+2\mathbb{H}}, S^{6\widetilde{\mathbb{R}}}\}^{S^1} \cong \{S^{2\mathbb{R}} \wedge (S(2\mathbb{H})_+), S^{6\widetilde{\mathbb{R}}}\}^{S^1}.$$

Note that the S^1–action on $S^{2\mathbb{R}} \wedge (S(2\mathbb{H})_+)$ is free away from the basepoint. By Fact 2.3,

$$\{S^{2\mathbb{R}} \wedge (S(2\mathbb{H})_+), S^{6\widetilde{\mathbb{R}}}\}^{S^1} = \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^3_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}}.$$

The cofiber sequence $\mathbb{C}P^1_+ \to \mathbb{C}P^3_+ \to \mathbb{C}P^3/\mathbb{C}P^1$ induces the exact sequence

$$\{S^{3\mathbb{R}} \wedge (\mathbb{C}P^1_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} \to \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^3_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} \\ \to \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^3/\mathbb{C}P^1), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} \to \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^1_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}}.$$

By the cellular approximation theorem,

$$\{S^{3\mathbb{R}} \wedge (\mathbb{C}P^1_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} = \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^1_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} = 0.$$

So we obtain the isomorphism

$$\{S^{2\mathbb{R}} \wedge (\mathbb{C}P^3_+), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}} \cong \{S^{2\mathbb{R}} \wedge (\mathbb{C}P^3/\mathbb{C}P^1), S^{6\widetilde{\mathbb{R}}}\}^{\{e\}}.$$

To understand the stable homotopy type of $\mathbb{C}P^3/\mathbb{C}P^1$ as a nonequivariant space, we let x be the generator of $H^2(\mathbb{C}P^3;\mathbb{Z}/2)$. Then the total Steenrod square is given by

$$\mathrm{Sq}(x)=\mathrm{Sq}^0(x)+\mathrm{Sq}^2(x)=x+x^2.$$

By the Cartan formula,

$$\mathrm{Sq}(x^2)=(x+x^2)^2=x^2\in H^*(\mathbb{C}P^3;\mathbb{Z}/2).$$

In particular, $\mathrm{Sq}^2(x^2)=0$, which implies that the attaching map between the 6–cell and the 4–cell in $\mathbb{C}P^3$, regarded as an element in the stable homotopy group $\pi_1=\mathbb{Z}/2$, is trivial. Therefore, we conclude that $\mathbb{C}P^3/\mathbb{C}P^1$ is stably homotopy equivalent to $S^{6\mathbb{R}}\vee S^{4\mathbb{R}}$. This implies

$$\{S^{2\mathbb{R}}\wedge(\mathbb{C}P^3/\mathbb{C}P^1),S^{6\widetilde{\mathbb{R}}}\}^{\{e\}}=\pi_2\oplus\pi_0=\mathbb{Z}/2\oplus\mathbb{Z}.$$

The projection to the π_0–summand can be alternatively defined as the mapping degree on $H^6(*;\mathbb{Z})$, so it is exactly the characteristic homomorphism t. We have shown that t is surjective with kernel $\mathbb{Z}/2$. By Corollary 2.9, we have $t(c_j(\alpha))=t(\alpha)$ for any $\alpha\in\{S^{\mathbb{R}+2\mathbb{H}},S^{6\widetilde{\mathbb{R}}}\}^{S^1}$. So c_j must send $\ker t$ to $\ker t$. Since $\ker t\cong\mathbb{Z}/2$, c_j must act trivially on it. $\square$

Proposition 3.2 *Let α be an element in $\{S^{\mathbb{R}+2\mathbb{H}},S^{6\widetilde{\mathbb{R}}}\}^G$ that satisfies the conditions*

$$t(\mathrm{Res}^G_{S^1}(\alpha))=0\quad\text{and}\quad\alpha\cdot e_{\widetilde{\mathbb{R}}}=0.$$

Then $\mathrm{Res}^G_{S^1}(\alpha)=0$.

Proof By Lemma 2.6, we see that $\alpha=\mathrm{Tr}^G_{S^1}(\beta)$ for some $\beta\in\{S^{\mathbb{R}+2\mathbb{H}},S^{6\widetilde{\mathbb{R}}}\}^{S^1}$. Therefore, by the double coset formula (10), $\mathrm{Res}^G_{S^1}(\alpha)=\beta+c_j(\beta)$. By Corollary 2.9,

$$0=t(\beta+c_j(\beta))=2t(\beta).$$

So β is in the kernel of t, which is $\mathbb{Z}/2$ by Lemma 3.1. By Lemma 3.1 again, $c_j(\beta)=\beta$. So $\mathrm{Res}^G_{S^1}(\alpha)=2\beta=0$. $\square$

3.2 Proof of Theorem 1.2

Let X_1 be the $K3$ surface and $X_0=S^2\times S^2$. Let $\mathfrak{s}_i$ be the unique spin structure on X_i for $i=0,1$. We consider the Dehn twist

$$\delta\colon X_1\#X_1\to X_1\#X_1$$

along the separating S^3 in the neck. We want to show that δ is not smoothly isotopic to the identity map even after a single stabilization. Without loss of generality, we may assume that the stabilization is done in the first copy of X_1. Then we need to show that the map

$$\delta^s := \mathrm{id}_{X_0} \# \delta \colon X_0 \# X_1 \# X_1 \to X_0 \# X_1 \# X_1$$

is not smoothly isotopic to the identity map. As in [26], we will prove this by forming the mapping torus

$$N_{\delta^s} := ((X_0 \# X_1 \# X_1) \times [0,1])/(x,0) \sim (\delta^s(x),1)$$

and showing that it is a nontrivial smooth bundle over S^1.

By Lemma 2.23, the product spin structure over the trivial bundle has vanishing BF^G. So, it suffices to show that both spin families associated to N_{δ^s} have nontrivial BF^G.

To prove this, we consider the product family $(X_i \times S^1, \tilde{\mathfrak{s}}_i)$ and the twisted family $(X_i \times S^1, \tilde{\mathfrak{s}}_i^\tau)$. By the discussion in [26, begining of Section 5], the mapping torus N_δ can be formed as the fiberwise connected sum

$$(X_1 \times S^1) \#_{\varphi(\tilde{\mathfrak{s}}_1, \tilde{\mathfrak{s}}_1^\tau)} (X_1 \times S^1).$$

Therefore, the bundle N_{δ^s} can formed as the fiberwise connected sum

$$(X_0 \times S^1) \#_{\varphi(\tilde{\mathfrak{s}}_0, \tilde{\mathfrak{s}}_1)} (X_1 \times S^1) \#_{\varphi(\tilde{\mathfrak{s}}_1, \tilde{\mathfrak{s}}_1^\tau)} (X_1 \times S^1)$$

as well as the fiberwise connected sum

$$(X_0 \times S^1) \#_{\varphi(\tilde{\mathfrak{s}}_0^\tau, \tilde{\mathfrak{s}}_1^\tau)} (X_1 \times S^1) \#_{\varphi(\tilde{\mathfrak{s}}_1^\tau, \tilde{\mathfrak{s}}_1)} (X_1 \times S^1).$$

The two spin families associated to N_{δ^s} are

$$(X_0 \times S^1, \tilde{\mathfrak{s}}_0) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)$$

and

$$(X_0 \times S^1, \tilde{\mathfrak{s}}_0^\tau) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1).$$

We will show that

$$\mathrm{BF}^G((X_0 \times S^1, \tilde{\mathfrak{s}}_0) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)) \neq 0,$$

and the other family is similar. We use α to denote the element

$$\mathrm{BF}^G((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)) \in \{S^{\mathbb{R}+2\mathbb{H}}, S^{6\widetilde{\mathbb{R}}}\}^G.$$

By Proposition 2.26, $\mathrm{Res}^G_{S^1}(\alpha)$ can be decomposed as the product of the elements

$$\mathrm{BF}^{S^1}(X_1, \mathfrak{s}_1) \in \{S^{\mathbb{H}}, S^{3\widetilde{\mathbb{R}}}\}^{S^1} \quad \text{and} \quad \mathrm{BF}^{S^1}((X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)) \in \{S^{\mathbb{R}+\mathbb{H}}, S^{3\widetilde{\mathbb{R}}}\}^{S^1}.$$

By Lemma 2.10, the Seiberg–Witten invariant $t(\mathrm{Res}^G_{S^1}(\alpha))$ equals 0. (This can also be proved by checking the explicit description of the Seiberg–Witten moduli space given in [26].)

By Corollary 2.27,

$$\mathrm{BF}^G((X_0 \times S^1, \tilde{\mathfrak{s}}_0) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)) = \alpha \cdot e_{\widetilde{\mathbb{R}}}.$$

For the sake of contradiction, suppose $\alpha \cdot e_{\widetilde{\mathbb{R}}} = 0$. Then, by Proposition 3.2, $\mathrm{Res}^G_{S^1}(\alpha) = 0$, which implies

$$\mathrm{BF}^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_1 \times S^1, \tilde{\mathfrak{s}}_1^\tau)) = \mathrm{Res}^G_{\{e\}}(\alpha) = \mathrm{Res}^{S^1}_{\{e\}} \circ \mathrm{Res}^G_{S^1}(\alpha) = 0.$$

However, Kronheimer and Mrowka [26, Proposition 5.1] computed this nonequivariant Bauer–Furuta invariant as $\eta^3 \neq 0 \in \pi_3$. (The Kronheimer–Mrowka definition of $\mathrm{BF}^{\{e\}}$ coincides with ours because of Lemma 2.22.) This is a contradiction and our proof is finished.

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