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We show that the infimum of the dual volume of the convex core of a convex co-compact hyperbolic 3–manifold with incompressible boundary coincides with the infimum of the Riemannian volume of its convex core, as we vary the geometry by quasi-isometric deformations. We deduce a linear lower bound of the volume of the convex core of a quasi-Fuchsian manifold in terms of the length of its bending measured lamination, with optimal multiplicative constant.

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Introduction

Let M be a complete hyperbolic 3–manifold and let CM be its convex core, namely the smallest nonempty convex subset of M. Then M is said to be *convex cocompact* if CM is a compact subset. The notion of dual volume of the convex core $V_C^*(M)$ arises from the polarity correspondence between the hyperbolic and the de Sitter spaces; see Schlenker [36, Section 1] and Mazzoli [28]. If M is a convex cocompact 3–manifold, then $V_C^*(M)$ coincides with $V_C(M) - \frac{1}{2}\ell_m(\mu)$, where $V_C(M)$ stands for the usual Riemannian volume of the convex core and $\ell_m(\mu)$ denotes the length of the bending measured lamination μ with respect to the hyperbolic metric m of the boundary of the convex core of M. Our aim is to study the infimum of V_C^*, considered as a function over the space $\mathcal{QD}(M)$ of quasi-isometric deformations of a given convex cocompact hyperbolic 3–manifold M with incompressible boundary. In particular, we will prove:

Theorem A *For a convex cocompact hyperbolic 3–manifold M with incompressible boundary,*

$$\inf_{M' \in \mathcal{QD}(M)} V_C^*(M') = \inf_{M' \in \mathcal{QD}(M')} V_C(M').$$

Moreover, $V_C^*(M') = V_C(M')$ if and only if the boundary of the convex core of M' is totally geodesic.

When M is a quasi-Fuchsian manifold, Theorem A can be equivalently stated as

$$(1) \qquad\qquad V_C(M') \geq \tfrac{1}{2}\ell_{m'}(\mu')$$

for every $M' \in \mathcal{QD}(M)$, where $\ell_{m'}(\mu')$ is the length of the bending measure of $\partial CM'$. As a consequence of the variation formulae of V_C (see Bonahon [4]) and of V_C^* (see Mazzoli [29] and see also Krasnov and Schlenker [20]), we will see in Corollary 4.1 that the multiplicative constant $\tfrac{1}{2}$ is optimal, and is realized near the Fuchsian locus.

Theorem A is to the dual volume as the following result of Bridgeman, Brock and Bromberg is to the renormalized volume:

Theorem [9, Theorem 3.11] *For every convex cocompact hyperbolic 3–manifold M with incompressible boundary,*

$$\inf_{M' \in \mathcal{QD}(M)} V_R(M') = \inf_{M' \in \mathcal{QD}(M)} V_C(M').$$

Moreover, $V_R(M') = V_C(M')$ if and only if the boundary of the convex core of M is totally geodesic.

By work of W P Thurston, if the compact 3–manifold with boundary $N := M \cup \partial_\infty M$ is acylindrical, then there exists a unique convex cocompact structure $M_0 \in \mathcal{QD}(M)$ whose convex core has totally geodesic boundary. In [41] (see also [40]), Storm proved that the infimum of the volume of the convex core function $V_C : \mathcal{QD}(M) \to \mathbb{R}$ is equal to half the simplicial volume of the doubled manifold $D(N)$. Moreover, the infimum is realized exactly when N is acylindrical, and it is achieved at M_0. Theorem A and [9, Theorem 3.11] then imply that the same characterization holds true for the infimum of the dual volume and the renormalized volume, respectively. In the case of the renormalized volume V_R, such description of $\inf V_R$ was first established by Pallete [31], without making use of Storm's result. Bridgeman, Brock and Bromberg [10] recently introduced a notion of surgered gradient flow of the renormalized volume in the relatively acylindrical case, which allowed them to obtain new comparisons between the renormalized volume and the Weil–Petersson geometry of the deformation spaces of convex cocompact 3–manifolds, generalizing in particular the works of Brock [12] and Schlenker [38]. In the same work, a new proof of Storm's result in the acylindrical case is obtained as a byproduct of their analysis; see in particular [10, Corollary 6.5].

Dual volume, renormalized volume and Riemannian volume of the convex core are related by the chain of inequalities

$$V_C^*(M) := V_C(M) - \tfrac{1}{2}\ell_m(\mu) \leq V_R(M) \leq V_C(M) - \tfrac{1}{4}\ell_m(\mu) \leq V_C(M).$$

Here the second inequality is due to Schlenker [38], and the lower bound of V_R is proved in [9, Theorem 3.7]. Observe in particular that Theorem A implies [9, Theorem 3.11], which concerns the infimum of the renormalized volume. The requirement on M to have incompressible boundary is necessary; indeed, it has been shown by Pallete [32] that there exist Schottky groups with negative renormalized volume.

Our proof of Theorem A broadly follows the same strategy as the work of Bridgeman, Brock and Bromberg [9], with some necessary differences. The authors of [9] interpret the renormalized volume as a function V_R over the Teichmüller space $\mathcal{T}(\partial_\infty M)$ of the *conformal boundary at infinity* of M (by the works of Bers [1], Kra [18] and Maskit [27]), and they estimate the difference $|V_R - V_C|$ as one follows the (opposite of the) Weil–Petersson gradient flow of V_R on $\mathcal{T}(\partial_\infty M)$. In order to study the dual volume function, the analogy between the variation formula of the renormalized volume (see the work of Krasnov and Schlenker [19, Lemma 5.8], or Section 1.6) and the dual Bonahon–Schläfli formula [29] would tempt us to consider V_C^* as a function of the Teichmüller space $\mathcal{T}(\partial CM)$, seen as the deformation space of *hyperbolic structures* on the boundary of the convex core of M. However, the hyperbolic structure on ∂CM is only conjecturally thought to provide a parametrization of the quasi-isometric deformation space of M. To avoid this difficulty, we rather focus our attention on a family of functions V_k^* approximating V_C^*, for which a similar procedure is possible.

Given k, a real number in the interval $(-1, 0)$, we say that an embedded surface $\Sigma_k \subset M$ is a k–*surface* if its first fundamental form (namely the restriction of the metric of M on the tangent space to Σ_k) is a Riemannian metric with constant Gaussian curvature equal to k. Then, by the work of Labourie [21], the complementary region of the convex core of M is foliated by k–surfaces, which converge to ∂CM as k goes to -1, and tend towards the conformal boundary at infinity $\partial_\infty M$ as k goes to 0. The function $V_k^*(M)$ is then defined to be the dual volume of the region M_k of M enclosed by its k–surfaces, one for each geometrically finite end of M. By the works of Labourie [22] and Schlenker [37], the hyperbolic structures of the k–surfaces do provide a parametrization of $\mathcal{QD}(M)$, a fact that allows us to study V_k^* as a function over the Teichmüller space of ∂M_k. At this point, studying the Weil–Petersson gradient of V_k^* on $\mathcal{T}(\partial M_k)$, we prove that the difference between the dual volume and the standard volume of the regions M_k is well-behaved as one follows the lines of the flow backwards, and finally we deduce the statement of Theorem A by taking a limit for k that goes to -1. While the methods of [9] for the study of the renormalized volume heavily rely on the relations between the geometry of the boundary of the convex

core and the properties of the *Schwarzian at infinity* of $\partial_\infty M$, here we use a more analytical approach to determine the necessary bounds on the geometric quantities of the k–surfaces $\partial_k M$ of M, which will guarantee us the existence and the good behavior of the flow of the Weil–Petersson gradient vector fields of V_k^*.

Outline of the paper

After the first section of background, we suggest the reader move backwards (as with the flow of the gradient of the functions V_k^*) while going through this exposition. In Section 4 the proof of Theorem A is described. Here the analogy with the work of Bridgeman, Brock and Bromberg [9] is manifest; the required technical ingredients (Lemmas 3.4 and 3.7 and Corollary 3.6) are formally very similar to the ones developed for the renormalized volume.

Section 3 focuses on the study of the Weil–Petersson gradient $\mathrm{grad}_{\mathrm{WP}} V_k^*$ of the dual volume functions V_k^* and the proofs of the ingredients mentioned above. In Lemma 3.4 we determine a lower bound of the norm of $\mathrm{grad}_{\mathrm{WP}} V_k^*$ in terms of the integral of the mean curvature of ∂M_k (which replaces the role of the length $\ell_m(\mu)$ in the definition of the dual volume of the regions M_k). In Corollary 3.6 we show that the flow of the vector field $\mathrm{grad}_{\mathrm{WP}} V_k^*$ is defined for all times, and in Lemma 3.7 we prove the existence of a global lower bound of the dual volumes V_k^* over $\mathcal{QD}(M)$. All the proofs in this section rely on differential-geometric methods and are consequences of an explicit description of the Weil–Petersson gradient of V_k^* developed in Proposition 3.2. This presentation of the vector field $\mathrm{grad}_{\mathrm{WP}} V_k^*$ is inspired by an orthogonal decomposition of the space of symmetric tensors due to Fischer and Marsden [15], and it involves the solution u_k of a simple PDE (4) over the k–surface ∂M_k. In particular, the proof of Corollary 3.6 will require us to have uniform control of the $\mathscr{C}^2$–norm of the function u_k. Section 2 (and in particular Lemma 2.3) provides us this last ingredient, and it is essentially based on the classical regularity theory for linear elliptic differential operators (see eg Evans [14]), and on the following property of k–surfaces:

Proposition (see Proposition 2.1) *For any $k \in (0, 1)$ and $n \in \mathbb{N}$ there exists a positive constant $N_{k,n}$ such that, for every convex cocompact hyperbolic 3–manifold M and for every incompressible k–surface Σ_k in M, the $\mathscr{C}^n$–norm of the mean curvature of Σ_k is bounded above by $N_{n,k}$.*

The existence of such a universal upper bound was proved (with weaker assumptions than the ones appearing above) by Bonsante, Danciger, Maloni and Schlenker in

[6, Proposition 3.8] for $n = 0$ (and the same strategy actually shows that the statement holds for any n), and its proof heavily relies on a compactness criterion for isometric immersions of surfaces established by Labourie [21]; see also Bonsante, Danciger, Maloni and Schlenker [6, Proposition 3.6]. As will be manifest in the proof of Proposition 2.1, the constants $N_{n,k}$ that we will produce are unfortunately not explicit.

Acknowledgments

1 Preliminaries

1.1 Hyperbolic 3–manifolds

Let M be an orientable complete hyperbolic 3–manifold, namely a complete Riemannian 3–manifold with constant sectional curvature equal to -1, and let Γ be a discrete and torsion-free group of orientation-preserving isometries of the hyperbolic 3–space $\mathbb{H}^3$, such that M is isometric to $\mathbb{H}^3/\Gamma$. We define the *limit set* of Γ to be

$$\Lambda_\Gamma := \overline{\Gamma \cdot x_0} \cap \partial_\infty \mathbb{H}^3,$$

where $\overline{\Gamma \cdot x_0}$ denotes the closure of the Γ–orbit of x_0 in $\overline{\mathbb{H}}^3 := \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$. It is simple to see that the definition of Λ_Γ does not depend on the choice of basepoint $x_0 \in \mathbb{H}^3$. If Γ is nonelementary (it does not have any finite orbit in $\overline{\mathbb{H}}^3$), then Λ_Γ can be characterized as the smallest closed Γ–invariant subset of $\partial_\infty \mathbb{H}^3$; see eg [33, Chapter 12]. The complementary region Ω_Γ of the limit set in $\partial_\infty \mathbb{H}^3$ is called the *domain of discontinuity* of Γ.

1.2 The convex core

If $\pi\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma \cong M$ denotes the universal cover of M, then a subset C of M is *convex* if and only if $\pi^{-1}(C)$ is convex in $\mathbb{H}^3$. If Γ is nonelementary, then every nonempty Γ–invariant convex subset of $\mathbb{H}^3$ contains the *convex hull* C_Γ of Γ, which consists of the intersection of all half-spaces H of $\mathbb{H}^3$ satisfying $\overline{H} \supseteq \Lambda_\Gamma$ ($\overline{H}$ stands for the closure of H in $\overline{\mathbb{H}}^3$). The image $CM := \pi(C_\Gamma)$ describes a convex subset of M, called the *convex core* of M, which is minimal among the family of nonempty convex subsets of M.

Now let M be a *convex cocompact* hyperbolic 3–manifold, namely a noncompact complete hyperbolic 3–manifold whose convex core is compact. The boundary of the convex core ∂CM of M is the union of a finite collection of connected surfaces, each of which is totally geodesic outside a subset of Hausdorff dimension 1. As described in [13], the hyperbolic metrics on the totally geodesic pieces "merge" together, defining a complete hyperbolic metric m on ∂CM. The locus where the boundary of the convex core is not flat is a geodesic lamination λ, ie a closed subset that is union of disjoint simple geodesics. The surface ∂CM is bent along λ, and the amount of bending can be described by a measured lamination μ called the *bending measure* of ∂CM. The μ–measure along an arc k transverse to λ consists of an integral sum of the exterior dihedral angles along the leaves that k meets. By locally integrating the lengths of the leaves of the lamination in $\mathrm{d}\mu$, we obtain the notion of length of the bending measure with respect to the hyperbolic structure m, which will be denoted by $\ell_m(\mu)$. For a more detailed description we refer to [13, Section II.1.11] or [2].

1.3 Incompressible boundary

When M is convex cocompact and Γ is a discrete and torsion-free subgroup of isometries of $\mathbb{H}^3$ such that $M \cong \mathbb{H}^3/\Gamma$, Γ acts freely and properly discontinuously on the domain of discontinuity Ω_Γ, and the quotient of $\mathbb{H}^3 \cup \Omega_\Gamma$ by Γ determines a natural compactification of M, which will be denoted by $\overline{M} = M \cup \partial_\infty M$. Then M is said to have *incompressible boundary* if the inclusion $S \to \overline{M}$ of each connected component S of $\partial_\infty M$ induces an injection at the level of the fundamental groups. This implies in particular that any lift of the inclusion $S \to \overline{M}$ to the universal covers $\widetilde{S} \to \widetilde{\overline{M}}$ is a homeomorphism onto its image.

1.4 Constant Gaussian curvature surfaces

Definition 1.1 Let S be an immersed surface inside a Riemannian 3–manifold N. The *first fundamental form I* of S is the Riemannian metric of S given by the restriction of

the metric of N to the tangent spaces of S. If S admits a unitary normal vector field $v : S \to T^1 N$, we define its *shape operator* B to be the endomorphism of TS given by $BU := -\mathcal{D}_U v$, for every tangent vector field U of S (here $\mathcal{D}$ denotes the Levi-Civita connection of N). The trace of the shape operator will be called the *mean curvature* of S, and the tensor $\mathit{II} := I(B \cdot, \cdot)$ the *second fundamental form* of S.

Let Σ be a surface immersed in a hyperbolic 3–manifold M, with first and second fundamental forms I and II, and shape operator B. We denote by K_e its *extrinsic curvature*, $K_e = \det B$, and by K_i its *intrinsic curvature*, the Gaussian curvature of the Riemannian metric I. Then the Gauss–Codazzi equations of (Σ, I, II) can be expressed as

$$K_i = K_e - 1, \qquad (\nabla_U B) V = (\nabla_V B) U \quad \text{for all } U, V,$$

where U and V are tangent vector fields to Σ, and ∇ is the Levi-Civita connection of I.

Definition 1.2 Let Σ be an immersed surface inside a hyperbolic 3–manifold, and let $k \in (-1, 0)$. If the intrinsic curvature of Σ is constantly equal to k, it is a *k–surface*.

If Σ is a k–surface, then its extrinsic curvature $K_e = k + 1$ is positive, since $k \in (-1, 0)$. In particular, Σ is a (locally) strictly convex surface.

In every convex cocompact 3–manifold M, the subset $M \setminus CM$ is the disjoint union of a finite number of geometrically finite hyperbolic ends $(E_i)_i$, each of which is homeomorphic to $\Sigma_i \times (0, \infty)$ for some compact orientable surface Σ_i of genus larger than or equal to 2. By the work of Labourie [21], the sets E_i are foliated by embedded k–surfaces $(\Sigma_{i,k})_k$, with k that varies in $(-1, 0)$. The surfaces $\Sigma_{i,k}$ approach the components of the pleated boundary ∂CM of the convex core of M as k goes to -1, and the components of conformal boundary at infinity $\partial_\infty M$ as k goes to 0.

We will denote by M_k the compact region of M whose boundary ∂M_k consists of the union of the surfaces $\bigcup_i \Sigma_{i,k}$, and we will endow ∂M_k with the second fundamental form II_k defined by the normal vector field pointing towards the interior of M_k, so that II_k is positive definite, and H_k is a positive function (observe that the eigenvalues of the shape operator have the same sign since $K_e = \det B > 0$).

1.5 Deformation spaces

Let Σ be a compact orientable surface of genus larger than or equal to 2. The *Teichmüller space* of Σ, denoted by $\mathcal{T}(\Sigma)$, is the space of isotopy classes of hyperbolic metrics

on Σ. Equivalently, in light of the uniformization theorem, $\mathcal{T}(\Sigma)$ can be described as the space of isotopy classes of conformal structures over Σ (compatible with the choice of a fixed orientation on Σ).

Since convex cocompact hyperbolic 3–manifolds are not closed, several different notions of deformation spaces can be introduced. In this exposition we will consider the *quasi-isometric* (or quasiconformal) deformation space.

Definition 1.3 Given M and M' hyperbolic manifolds, a diffeomorphism $M \to M'$ is a *quasi-isometric deformation* of M if it is globally bi-Lipschitz. We denote by $\mathcal{QD}(M)$ the space of quasi-isometric deformations of M, where we identify two deformations $M \to M'$ and $M \to M''$ if their pullback metrics are isotopic to each other.

Remark 1.4 By a theorem of Thurston [42, Proposition 8.3.4], two hyperbolic n–manifolds M and M' are quasi-isometric if and only if their fundamental groups Γ and Γ' (as subgroups of the isometry group of $\mathbb{H}^n$) are quasiconformally conjugated, ie there exists a quasiconformal self-homeomorphism φ of $\partial_\infty \mathbb{H}^n$ such that $\varphi \Gamma \varphi^{-1} = \Gamma'$.

We denote by $m_k(M) \in \mathcal{T}(\partial M_k) = \prod_i \mathcal{T}(\Sigma_i)$ the isotopy class of the hyperbolic metric $(-k)I_k$, where I_k is the first fundamental form of the k–surface $\partial_k M$ of M. Then for every $k \in (-1, 0)$ we have maps

$$T_k \colon \mathcal{QD}(M) \to \mathcal{T}(\partial M_k), \quad M \mapsto m_k(M).$$

The convenience in considering foliations by k–surfaces relies in the following result, based on the works of Labourie [22] and Schlenker [37]:

Theorem 1.5 *If M has incompressible boundary, the map T_k is a $\mathscr{C}^1$–diffeomorphism for every $k \in (-1, 0)$.*

In the compressible case a similar statement can be recovered, replacing the role of the Teichmüller space $\mathcal{T}(\partial M_k)$ with its quotient by the action of a suitable subgroup of the mapping class group of ∂M_k; see eg [26, Theorem 5.1.3] for the corresponding statement concerning the conformal structure of the boundary at infinity.

As mentioned in the introduction it is an open question, asked by Thurston, whether the same statement is true for the hyperbolic structures on the boundary of the convex core, which could be considered as the case $k = -1$ in Theorem 1.5. More precisely, the

map T_{-1} is known to be continuously differentiable by [5] and surjective by the work of Sullivan (described in [13]), but there are no results concerning its global injectivity.

1.6 Dual volume

Let M be a convex cocompact hyperbolic 3–manifold. If N is a compact convex subset of M with smooth boundary, we define the *dual volume* of N to be

$$V^*(N) := V(N) - \frac{1}{2} \int_{\partial N} H \, da,$$

where H stands for the mean curvature of ∂N defined using the inner normal vector field, and $V(N)$ is the Riemannian volume of N. We refer to [28] for a description of the relation between the notion of dual volume and the polarity correspondence between the hyperbolic and de Sitter spaces.

For every $k \in (-1, 0)$ we let $V_k^* : \mathcal{T}(\partial M_k) \to \mathbb{R}$ denote the function that associates, with a hyperbolic structure $m_k \in \mathcal{T}(\partial M_k)$, the dual volume of the region $\partial M_k'$ enclosed by the k–surfaces of the unique convex cocompact hyperbolic 3–manifold $M' = T_k^{-1}(m_k)$ whose k–surfaces have hyperbolic structure m_k.

If $(N_h)_h$ is a sequence of convex compact subsets approaching CM, then the integral of the mean curvature over ∂N_h approaches $\ell_m(\mu)$, the length of the bending measure μ with respect to the hyperbolic structure of ∂CM; see eg [9, Proposition 3.4]. This suggests we should set the *dual volume of the convex core* of M as

$$V_C^*(M) := V(CM) - \frac{1}{2}\ell_m(\mu).$$

In [29], a first-order variation formula for the function V_C^* over $\mathcal{QD}(M)$ is studied, called the *dual Bonahon–Schläfli formula*,

$$dV_C^*(\dot{M}) = -\frac{1}{2} \, dL_\mu(\dot{m}).$$

Here $\dot{m}$ denotes the first-order variation of the hyperbolic metric on ∂CM along $\dot{M}$, and $L_\mu : \mathcal{T}(\partial CM) \to \mathbb{R}$ is the function that associates with every hyperbolic structure m the length of the m–geodesic realization of μ.

A strong similarity between dual and renormalized volumes is displayed by their variation formulae. The renormalized volume satisfies

$$dV_R(\dot{M}) = -\frac{1}{2} \, \text{dext}_{\mathcal{F}_\infty}(\dot{c}_\infty),$$

where $\dot{c}_\infty$ denotes the first-order variation of the conformal structure on $\partial_\infty M$ along $\dot{M}$, and $\text{ext}_{\mathcal{F}_\infty} : \mathcal{T}(\partial_\infty M) \to \mathbb{R}$ is the function that associates with every conformal structure

c the extremal length of the horizontal measured foliation of the Schwarzian at infinity of M with respect to c; see Schlenker [39] for a proof of this relation.

1.7 Norms on $T\mathcal{T}(\Sigma)$

First we introduce the necessary notation for the "Riemannian geometric tools" that will be used in the rest of the paper. Let (N, g) be a Riemannian manifold with Levi-Civita connection ${}^g\nabla$, and consider $(e_i)_i$, a local g–orthonormal frame. Given T, a symmetric 2–tensor on N, we define its g–divergence as the 1–form

$$(\mathrm{div}_g\, T)(X) := \sum_i ({}^g\nabla_{e_i} T)(e_i, X)$$

for every tangent vector field X. Similarly, the g–divergence of a vector field X is the function

$$\mathrm{div}_g\, X = \sum_i g({}^g\nabla_{e_i} X, e_i).$$

The Laplace–Beltrami operator can be expressed as $\Delta_g f = \mathrm{div}_g\, \mathrm{grad}_g\, f$. Given two symmetric 2–tensors T and T', their scalar product is defined as

$$(T, T')_g := g^{ij} g^{hk} T_{ih} T'_{jk} = \mathrm{tr}(g^{-1} T g^{-1} T').$$

In particular, we set $\mathrm{tr}_g\, T := (g, T)_g = \mathrm{tr}(g^{-1} T)$. In the next sections it will also be useful to keep in mind the way that these operators change if we replace g with λg, for some positive constant λ. If $\dim N = n$,

$$(2) \qquad \mathrm{div}_{\lambda g}\, T = \lambda^{-1} \mathrm{div}_g\, T, \quad \Delta_{\lambda g} f = \lambda^{-1} \Delta_g f, \quad \mathrm{d}a_{\lambda g} = \lambda^{n/2} \mathrm{d}a_g,$$

$$(3) \qquad (T, T')_{\lambda g} = \lambda^{-2}(T, T')_g, \quad \mathrm{tr}_{\lambda g}\, T = \lambda^{-1} \mathrm{tr}_g\, T.$$

Now let $\mathcal{M}$ be the set of Riemannian metrics on Σ, and let $\mathcal{H}$ be the subset of hyperbolic ones. The first-order variations $\dot{g}$ of elements of $\mathcal{M}$ identify with smooth symmetric 2–tensors on Σ. The choice of a metric $g \in \mathcal{M}$ determines a scalar product on $T_g\mathcal{M}$, which can be expressed as

$$(\sigma, \tau)_{\mathrm{FT}, g} := \int_\Sigma (\sigma, \tau)_g\, \mathrm{d}a_g,$$

where FT stands for Fischer–Tromba. We define $S_2^{\mathrm{tt}}(\Sigma, g)$ to be the space of those symmetric tensors σ that are traceless with respect to g (meaning $(\sigma, g)_g = 0$) and g–divergence-free (meaning $\mathrm{div}_g\, \sigma = 0$, as defined above). Such tensors are also called *transverse traceless*. A simple way to characterize the space $S_2^{\mathrm{tt}}(\Sigma, g)$ is through

holomorphic quadratic differentials. A holomorphic quadratic differential ϕ on (Σ, g) is a $\mathbb{C}$–valued symmetric tensor that can be locally written as $\phi = f \, dz^2$, where z is a local coordinate conformal to the metric g (and compatible with a given orientation), and $f = f(z)$ is a holomorphic function. Transverse traceless tensors are exactly those 2–tensors that can be written as $\Re\phi$ for some ϕ holomorphic quadratic differential on (Σ, h).

It is shown in [43] that, for every hyperbolic metric h, $S_2^{\mathrm{tt}}(\Sigma, h)$ coincides with

$$T_h \mathcal{H} \cap \big(T_h(\mathrm{Diff}_0(\Sigma) \cdot h) \big)^{\perp},$$

where $T_h(\mathrm{Diff}_0(\Sigma) \cdot h)$ is the tangent space to the orbit of h by the action of the group of diffeomorphisms of Σ isotopic to the identity, and $(\cdot)^{\perp}$ is taken with respect to the scalar product $(\cdot, \cdot)_{\mathrm{FT},h}$ on $T_h \mathcal{M}$. Therefore, if $m = [h]$ denotes the isotopy class of a hyperbolic metric on Σ, we can identify $S_2^{\mathrm{tt}}(\Sigma, h)$ with $T_m \mathcal{T}(\Sigma)$, the tangent space at m to the Teichmüller space $\mathcal{T}(\Sigma) = \mathcal{H}/\mathrm{Diff}_0(\Sigma)$, seen as the space of isotopy classes of hyperbolic metrics on Σ. Moreover, the restriction of the scalar product $(\cdot, \cdot)_{\mathrm{FT},h}$ to $S_2^{\mathrm{tt}}(\Sigma, h)$ coincides with (a multiple of) the *Weil–Petersson metric* $\langle \cdot, \cdot \rangle_{\mathrm{WP}}$ (see Lemma 1.6 for the explicit multiplicative constant).

The Teichmüller space can also be endowed with another Finsler norm that arises from its conformal (or quasiconformal) interpretation, namely the *Teichmüller norm*. The Teichmüller norm $\| \cdot \|_{\mathcal{T}}$ of a tangent vector $\dot{m} \in T_m \mathcal{T}(\Sigma)$ is the infimum of the L^{∞}–norms of the Beltrami differentials representing $\dot{m}$. It is not difficult to see that the Beltrami differential associated to the tangent direction $2\Re\phi$ coincides with ν_{ϕ}, the *harmonic Beltrami differential* associated to ϕ (see eg [16] for a detailed description of these notions, and [30, Lemma 1.2] for a direct computation of this relation). Moreover, the L^{∞}–norm of ν_{ϕ} can be computed as

$$\|\nu_{\phi}\|_{\infty} = \frac{1}{\sqrt{2}} \sup_{\Sigma} \|\Re\phi\|_h.$$

We summarize what we observed:

Lemma 1.6 *For every hyperbolic metric h representing the isotopy class $m \in \mathcal{T}(\Sigma)$, the tangent space $T_m \mathcal{T}(\Sigma)$ identifies with $S_2^{\mathrm{tt}}(\Sigma, h)$. For every $\dot{m} \in T_m \mathcal{T}(\Sigma)$,*

$$\|\dot{m}\|_{\mathrm{WP}} = \frac{1}{\sqrt{2}} \|\Re\phi\|_{\mathrm{FT},h}, \quad \|\dot{m}\|_{\mathcal{T}} \le \frac{1}{\sqrt{2}} \sup_{\Sigma} \|\Re\phi\|_h,$$

where ϕ is a holomorphic quadratic differential such that $2\Re\phi$ represents $\dot{m}$ inside $S_2^{\mathrm{tt}}(\Sigma, h)$.

2 Some useful estimates

In this section we determine estimates for the solution u_k of a certain linear PDEs, defined over a k–surface lying inside an end of a convex cocompact hyperbolic 3–manifold with incompressible boundary. The function u_k will be later used to describe the Weil–Petersson gradient of the dual volume functions V_k^*, and the bounds produced in this section will play an important role in the study of its flow.

Given (N, g) a Riemannian manifold with Levi-Civita connection ${}^g\nabla$ and area form $\mathrm{d}a_g$, we denote by $H^n(N, \mathrm{d}a_g)$ the Sobolev space of real-valued functions f on N with $L^2(N, \mathrm{d}a_g)$–integrable weak derivatives $({}^g\nabla)^i f$ for all $i \le n$. The space $H^n(N, \mathrm{d}a_g)$ is Hilbert if endowed with the scalar product

$$(f, f') := \sum_{i=0}^{n} \int_N (({}^g\nabla)^i f, ({}^g\nabla)^i f')_g \, \mathrm{d}a_g \quad \text{for } f, f' \in H^n(N, \mathrm{d}a_g),$$

where $(\,\cdot\,, \cdot\,)_g$ denotes the scalar product induced by g on the space of i–tensors over N. Given $f \colon N \to \mathbb{R}$ a $\mathscr{C}^n$–function, we define its $\mathscr{C}^n(N, g)$–norm as

$$\|f\|_{\mathscr{C}^n(N, g)} := \sum_{i=0}^{n} \sup_{p \in N} \|({}^g\nabla)^i f|_p\|_g,$$

where $\|T\|_g = \sqrt{(T, T)_g}$.

Now let h_k denote the hyperbolic metric $(-k)I_k$ on the k–surface ∂M_k, with Levi-Civita connection ${}^k\nabla$ and Laplace–Beltrami operator Δ_k (here we consider $\Delta_k u$ to be the trace of the Hessian of u). We define the linear differential operator L_k to be

$$L_k u := (\Delta_k - 2\mathbb{1})u = \Delta_k u - 2u.$$

Let A be the symmetric bilinear form on $H^1(\partial M_k, \mathrm{d}a_k)$ with quadratic form

$$A(u, u) := -(L_k u, u) = \int_\Sigma (\|\mathrm{d}u\|_k^2 + 2u^2) \, \mathrm{d}a_k,$$

where $\|\cdot\|_k$ and $\mathrm{d}a_k$ denote the norm and the area form of h_k, respectively. By the Lax–Milgram theorem (see eg [7, Corollary 5.8]) applied to the Sobolev space $H^1(\partial M_k, \mathrm{d}a_k)$ and to the coercive symmetric bilinear form A, we have that, for every $f \in L^2(\partial M_k, \mathrm{d}a_k)$, there exists a unique weak solution $u \in H^1(\partial M_k, \mathrm{d}a_k)$ of the equation $L_k u = f$. We will in particular denote by u_k the function satisfying

$$(4) \qquad\qquad L_k u_k = -k^{-1} H_k \iff \Delta_{I_k} u_k + 2k u_k = H_k,$$

where H_k denotes the mean curvature of the k–surface ∂M_k. By the classical regularity theory for linear elliptic PDEs (see eg [14, Section 6.3]), the smoothness of the mean curvature H_k and the compactness of ∂M_k imply that the function u_k is smooth and it is a strong solution of (4).

By the work of Rosenberg and Spruck [35, Theorem 4], for every Jordan curve c in $\partial_\infty \mathbb{H}^3$ there exist exactly two k–surfaces $\widetilde{\Sigma}_k^\pm(c)$ asymptotic to c. A fundamental property of k–surfaces, which will be crucial in Lemma 2.3, is:

Proposition 2.1 [6, Proposition 3.8] *Let* $k \in (-1, 0)$ *and* $n \in \mathbb{N}$. *Then there exists a constant* $N_{k,n} > 0$ *such that, for every Jordan curve* c *in* $\partial_\infty \mathbb{H}^3$, *the mean curvature* $H_{c,k}$ *of the* k*–surface* $\widetilde{\Sigma}_k(c) = \widetilde{\Sigma}_k^+(c) \sqcup \widetilde{\Sigma}_k^-(c)$ *asymptotic to* c *satisfies*

$$\|H_{c,k}\|_{\mathscr{C}^n(\widetilde{\Sigma}_k(c))} \le N_{n,k}.$$

Proof We briefly recall here the proof of this statement (which was stated in [6] for $n = 0$). First, recall that k–surfaces satisfy the following compactness criterion:

Proposition 2.2 [6, Proposition 3.6] *Let* $k \in (-1, 0)$, *and consider* $f_n : \mathbb{H}_k^2 \to \mathbb{H}^3$ *a sequence of proper isometric embeddings of the hyperbolic plane* $\mathbb{H}_k^2$ *with constant Gaussian curvature* k. *If there exists a point* $p \in \mathbb{H}^2$ *such that* $(f_n(p))_n$ *is precompact, then there exists a subsequence of* $(f_n)_n$ *that converges* $\mathscr{C}^\infty$*–uniformly on compact sets to an isometric immersion* $f : \mathbb{H}_k^2 \to \mathbb{H}^3$.

Fixing $k \in (-1, 0)$ and $n \in \mathbb{N}$, assume by contradiction that there exists a sequence of Jordan curves $(c_m)_m$ such that the mean curvatures $H_m = H_{c_m,k}$ of the k–surfaces $\widetilde{\Sigma}_k(c_m)$ satisfy $\|H_m\|_{\mathscr{C}^n(\widetilde{\Sigma}_k(c_m))} > m$. Up to extracting a subsequence, there exists an $i \le n$ such that for every $m \in \mathbb{N}$

$$\sup_{\widetilde{\Sigma}_k(c_m)} \|({}^k\nabla)^i H_m\| > \frac{m}{n+1} = C_n m.$$

Now choose $q_m \in \widetilde{\Sigma}_k(c_m)$ for which the norm of $({}^k\nabla)^i H_m$ at q_m is at least $C_n m$. Since each component of $\widetilde{\Sigma}_k(c_m)$ is embedded and isometric to the hyperbolic plane $\mathbb{H}_k^2$ (which is homogeneous), we can find a sequence of proper isometric embeddings $f_m : \mathbb{H}_k^2 \to \mathbb{H}^3$, parametrizing a component of $\widetilde{\Sigma}_k(c_m)$, such that $f_m(\bar{p}) = q_m$ for some fixed basepoint $\bar{p} \in \mathbb{H}_k^2$. Up to postcomposing f_m by an isometry of $\mathbb{H}^3$, we can also assume that $f_m(\bar{p}) = \bar{q}$ is fixed. In this way, we have found a sequence of proper isometric embeddings $f_m : \mathbb{H}_k^2 \to \mathbb{H}^3$ satisfying

- $f_m(\bar{p}) = \bar{q} \in \mathbb{H}^3$ is independent of $m \in \mathbb{N}$,

- the mean curvature of the surfaces $f_m(\mathbb{H}_k^2)$ at $\bar{q}$ has some i^{th}–order derivative that is unbounded as m goes to ∞.

This clearly contradicts the compactness criterion mentioned above. $\square$

From this result we can now obtain a uniform control on u_k:

Lemma 2.3 *Let M be a convex cocompact hyperbolic 3–manifold. Then the function $u_k \colon \partial M_k \to \mathbb{R}$, the solution of (4), satisfies*

$$\frac{\max_{\partial M_k} H_k}{2k} \le u_k \le \frac{\min_{\partial M_k} H_k}{2k} = \frac{\sqrt{k+1}}{k} < 0.$$

Moreover, if M has incompressible boundary, then there exists a constant $C_k > 0$ depending only on the intrinsic curvature $k \in (-1, 0)$, and in particular not on the hyperbolic structure of M, such that

$$\max_{\partial M_k} \|^k \nabla^2 u_k\|_k \le C_k.$$

Proof The first assertion is an immediate consequence of the maximum principle applied to u_k as a solution of the PDE (4). Moreover, since the product of the principal curvatures (the eigenvalues of the shape operator) of a k–surface is everywhere equal to $k + 1$, the trace of the shape operator is bounded from below by $2\sqrt{k+1}$ (see Remark 2.5 for an explanation of the equality $\min_{\partial M_k} H_k = 2\sqrt{k+1}$).

The proof of the second assertion requires more care. Let Σ_k be a connected component of the k–surface ∂M_k, and let $\widetilde{M} \cong \mathbb{H}^3$ denote the universal cover of M. Since M is a convex cocompact hyperbolic 3–manifold with incompressible boundary, every component $\widetilde{\Sigma}_k$ of the preimage of Σ_k in $\widetilde{M}$ is stabilized by a subgroup $\Gamma \cong \pi_1(\Sigma_k)$ of the fundamental group of M, acting by isometries on $\widetilde{M}$. Each of these subgroups Γ is quasi-Fuchsian (see eg [17, Corollary 4.112 and Theorem 8.17] for a proof of this assertion), and the surface $\widetilde{\Sigma}_k$ is a k–surface asymptotic to some Jordan curve in $\partial_\infty \widetilde{M} \cong \partial_\infty \mathbb{H}^3$. In particular, by Proposition 2.1, we can find a universal constant $N_k = N_{2,k} > 0$ that satisfies

$$(5) \qquad \qquad \|\widetilde{H}_k\|_{\mathscr{C}^2(\widetilde{\Sigma}_k)} \le N_k.$$

Here we stress that the constant N_k does not depend on the hyperbolic structure of M or Σ_k, but only on the value of $k \in (-1, 0)$.

Our goal is now to make use of this control to obtain a uniform bound of the norm of the Hessian of u_k. For this purpose, we will need a classical result of regularity for linear elliptic differential equations:

Theorem 2.4 [14, Theorem 2, page 314] *Let $m, n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be a bounded open set. We consider a differential operator L of the form*

$$Lf = -\sum_{i,j=1}^{n} a^{ij}(x)\partial^2_{x_i,x_j} f + \sum_{i=0}^{n} b^i(x)\partial_{x_i} f + c(x)f,$$

where $a^{ij} = a^{ji}, b^i, c \in \mathscr{C}^{m+1}(U, \mathbb{R})$. Assume that L is uniformly elliptic, ie there exists a constant $\varepsilon > 0$ such that $\sum_{i,j} a^{ij}(x)v_i v_j \geq \varepsilon \|v\|^2$ for all $v \in \mathbb{R}^n$ and $x \in U$. If $f \in H^1(U)$ is a weak solution of the equation $Lf = \lambda$ for some $\lambda \in H^m(U)$, then for every bounded open set V with closure contained in U there exists a constant C, depending only on m, U and V and the functions a^{ij}, b^i and c, such that

$$\|f\|_{H^{m+2}(V)} \leq C(\|\lambda\|_{H^m(U)} + \|f\|_{L^2(U)}),$$

where the Sobolev spaces $H^{m+2}(V)$, $H^m(U)$ and $L^2(U)$ are defined with respect to the Euclidean metric of $U \subset \mathbb{R}^n$.

The surface $\widetilde{\Sigma}_k$ endowed with the lift of the hyperbolic metric h_k of Σ_k is isometric to the hyperbolic plane $\mathbb{H}^2$. In the rest of the proof we will identify $\widetilde{\Sigma}_k$ with the Poincaré disk model $\mathbb{H}^2 := (B_1, g)$, where B_1 is the Euclidean ball of radius 1 and center 0 in $\mathbb{C}$, and g is the Riemannian metric

$$g = \left(\frac{2}{1 - |z|^2}\right)^2 |dz|^2.$$

Now we choose U and V to be the g–geodesic balls of center $0 \in B_1$ and hyperbolic radius equal to 2 and 1, respectively. The lift of the operator $-L_k$ over U is clearly uniformly elliptic because of the compactness of $\overline{U}$ and its expression in coordinates

$$-L_k f = -g^{ij}(\partial^2_{ij} f - \Gamma^h_{ij}(g)\partial_h f) + 2f,$$

where the $\Gamma^h_{ij}(g)$ denote the Christoffel symbols of g. Again by the compactness of $\overline{U}$ and $\overline{V}$, the norms of the Sobolev spaces $\|\cdot\|_{H^j(U)}$ and $\|\cdot\|_{H^j(V)}$, computed with respect to the flat connection of $B_1 \subset \mathbb{R}^2$ and the Euclidean volume form, are equivalent to the norms of the corresponding Sobolev spaces defined using the Levi-Civita connection of g and the g–volume form. Moreover, the bi-Lipschitz constants involved in the equivalence only depend on a bound of the $\mathscr{C}^{j+1}$–norm of g over U, so they can be chosen to depend only on $j \in \mathbb{N}$. From now on, we will consider the norms on the spaces $H^j(U)$ and $H^j(V)$ to be defined using the metric g and its connection.

Now we apply Theorem 2.4 to $m = n = 2$, the operator $-L_k$ and the functions $f = \tilde{u}_k$ and $\lambda = -k^{-1}\widetilde{H}_k$, where $\widetilde{F}$ denotes the lift of the function F over $\widetilde{\Sigma}_k$. We can find

a universal constant $C > 0$ (depending only on the open sets U and V, and on the metric $g|_U$) such that

$$\|\tilde{u}_k\|_{H^4(V)} \le C(-k^{-1}\|\tilde{H}_k\|_{H^2(U)} + \|\tilde{u}_k\|_{L^2(U)}).$$

By the first part of Lemma 2.3, $\|\tilde{u}_k\|_{\mathscr{C}^0(U)} \le -(2k)^{-1}\|\tilde{H}_k\|_{\mathscr{C}^0(\mathbb{H}^2)}$. In addition,

$$\|\tilde{u}_k\|_{L^2(U)} \le \operatorname{Area}(U, g)^{1/2}\|\tilde{u}_k\|_{\mathscr{C}^0(U)} \le -(2k)^{-1}\operatorname{Area}(U, g)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^0(\mathbb{H}^2)},$$

and

$$\|\tilde{H}_k\|_{H^2(U)} \le \operatorname{Area}(U, g)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

In conclusion, we deduce that

$$\|\tilde{u}_k\|_{H^4(V)} \le -2k^{-1}C\operatorname{Area}(U, g)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

By the Sobolev embedding theorem (see eg [7, Corollary 9.13, page 283]), given W an open set satisfying $0 \in W \subset \overline{W} \subset V$, the $\mathscr{C}^2(W)$–norm of $\tilde{u}_k$ (again, computed with respect to the Levi-Civita connection of g) is controlled by a multiple of its H^4–norm over V, and the multiplicative factor depends only on W and V. Therefore, if we choose for instance $W = B_{\mathbb{H}^2}(0, \frac{1}{2})$,

$$\|^k\nabla^2\tilde{u}_k\|_{\mathscr{C}^0(W)} \le C'(k)\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

Now the desired statement easily follows. From relation (5) and the last inequality, we obtain a uniform bound of the Hessian of $\tilde{u}_k$ over $W \ni 0$. Now let q be any other point of $\mathbb{H}^2$, and choose a g–isometry $\varphi_q \colon B_1 \to B_1$ such that $\varphi_q(0) = q$. If we replace $\tilde{u}_k$ and $\tilde{H}_k$ with $\tilde{u}_k \circ \varphi_q$ and $\tilde{H}_k \circ \varphi_q$, respectively, the exact same argument above applies, since the operator L_k and the norms $\|\cdot\|_{H^j}$ and $\|\cdot\|_{\mathscr{C}^l}$ are invariant under the action of the isometry group of $\mathbb{H}^2$ (and since $\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)} = \|\tilde{H}_k \circ \varphi_q\|_{\mathscr{C}^2(\mathbb{H}^2)}$). In particular, this gives us a control on the norm of $^k\nabla^2\tilde{u}_k$ over $\varphi_q(W)$ for any point $q \in \mathbb{H}^2$, and the last part of our assertion follows. □

Remark 2.5 The minimum of the mean curvature $2\sqrt{k+1}$ is always realized. As described by Labourie in [23], whenever we have a k–surface Σ_k with first and second fundamental forms I_k and II_k, respectively, the identity map $\operatorname{id}\colon (\Sigma_k, \mathit{II}_k) \to (\Sigma_k, I_k)$ is harmonic, with Hopf differential ψ_k satisfying

$$2\Re\psi_k = I_k - \frac{H_k}{2(k+1)}\mathit{II}_k.$$

Its squared norm with respect to II_k can be expressed as

$$\|2\Re\psi_k\|_{\mathit{II}_k}^2 = \frac{H_k^2 - 4(k+1)}{(k+1)^2}.$$

In particular, at each zero of ψ_k (which necessarily exist because $\chi(\Sigma_k) < 0$), we have $H_k = 2\sqrt{k+1}$.

We stress that, even if the maximum of the mean curvature H_k will clearly depend on the hyperbolic structure of M, Proposition 2.1 guarantees that $\max H_k$ is controlled by a function of k independent of the geometry of M, as long as ∂M is incompressible.

We will make use of the upper bound $u_k \leq \sqrt{k+1}/k$ in Lemma 3.4, where we will determine a lower bound of the Weil–Petersson norm of the differential of V_k^* in terms of the integral of the mean curvature.

3 The gradient of the dual volume

The aim of this section is to describe the gradient of the dual volume function V_k^* with respect to the Weil–Petersson metric on the Teichmüller space of ∂M_k in terms of the function u_k studied in the previous section.

The first-order variation of the dual volume of M_k as we vary the convex cocompact hyperbolic structure of M can be computed by applying the *differential Schläfli formula* due to Rivin and Schlenker [34]. In particular:

Proposition 3.1 *We have*

$$\mathrm{d}(V_k^* \circ T_k)(\dot{M}) = \frac{1}{4} \int_{\partial M_k} (\dot{I}_k, I\!I_k - H_k I_k)_{I_k} \, \mathrm{d}a_{I_k}$$

$$= \frac{1}{4} \int_{\partial M_k} (\dot{h}_k, I\!I_k + k^{-1} H_k h_k)_{h_k} \, \mathrm{d}a_{h_k},$$

where $\dot{I}_k = -k^{-1}\dot{h}_k$ is the first-order variation of the first fundamental form on ∂M_k along the variation $\dot{M}$, and $T_k \colon \mathcal{QD}(M) \to \mathcal{T}(\partial M_k)$ is the diffeomorphism introduced in Section 1.5.

A proof of this relation based on the results of Rivin and Schlenker can be found in [29, Proposition 2.5]. From its variation formula, we can give an explicit description of the Weil–Petersson gradient of the dual volume function V_k^*, which will turn out to be useful for the study of its flow.

Proposition 3.2 *The vector field $\mathrm{grad}_{\mathrm{WP}} V_k^*$ is represented by the symmetric 2–tensor $2\Re\phi_k$, where ϕ_k is the (unique) holomorphic quadratic differential satisfying*

$$\Re\phi_k = I\!I_k - {}^k\nabla^2 u_k + u_k h_k,$$

where u_k denotes the solution of (4).

Proof Let $\dot{m}_k$ denote a tangent vector to the Teichmüller space of ∂M_k at m_k. As described in Section 1.5, given any hyperbolic metric h_k representing the isotopy class $m_k \in \mathcal{T}(\partial M_k)$, we can find a unique transverse traceless tensor $\dot{h}_k \in S_2^{\mathrm{tt}}(\Sigma, h_k)$ representing $\dot{m}_k$. Assume for a moment that we can find a decomposition of the symmetric tensor $I\!I_k + k^{-1} H_k h_k$ of the form

$$I\!I_k + k^{-1} H_k h_k = S_{\mathrm{tt}} + \mathcal{L}_X h_k + \lambda h_k,$$

where S_{tt} is a transverse traceless tensor with respect to h_k, $\mathcal{L}_X h_k$ is the Lie derivative of h_k with respect to a vector field X, and λ is a smooth function on ∂M_k. Then, by Proposition 3.1, we can express the variation of the dual volume V_k^* along a transverse traceless variation $\dot{h}_k$:

$$\mathrm{d}V_k^*(\dot{h}_k) = \frac{1}{4} \int_{\partial M_k} (\dot{h}_k, S_{\mathrm{tt}} + \mathcal{L}_X h_k + \lambda h_k)_{h_k} \, \mathrm{d}a_{h_k}.$$

Since $\dot{h}_k$ is traceless, the scalar product $(\dot{h}_k, h_k)_{h_k} = \mathrm{tr}_{h_k}(\dot{h}_k)$ vanishes identically. The L^2–scalar product between $\dot{h}_k$ and $\mathcal{L}_X h_k$ vanishes too, because $\mathcal{L}_X h_k$ is tangent to the orbit of h_k by the action of $\mathrm{Diff}_0(\Sigma)$; see Section 1.7. In particular,

$$\mathrm{d}V_k^*(\dot{h}_k) = \frac{1}{4} \int_{\partial M_k} (\dot{h}_k, S_{\mathrm{tt}})_{h_k} \, \mathrm{d}a_{h_k} = \frac{1}{8}(\dot{h}_k, 2S_{tt})_{\mathrm{FT},h_k}.$$

In light of Lemma 1.6, by varying the tangent vector $\dot{m}_k \in T_{m_k} \mathcal{T}(\partial M_k)$, we deduce that the tensor $2S_{\mathrm{tt}}$ is the element of $S_2^{\mathrm{tt}}(\Sigma, h_k)$ that represents $\mathrm{grad}_{\mathrm{WP}} V_k^*$.

In conclusion, this argument shows us that, in order to prove our assertion, we need to determine a decomposition of the tensor $I\!I_k + k^{-1} H_k h_k$ of the form we described above, with $S_{\mathrm{tt}} = I\!I_k - {}^k\nabla^2 u_k + u_k h_k$. For this purpose, we consider the expression

$$I\!I_k + k^{-1} H_k h_k = (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + {}^k\nabla^2 u_k + (k^{-1} H_k - u_k) h_k$$

$$= (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + \tfrac{1}{2} \mathcal{L}_{\mathrm{grad}_{h_k} u_k} h_k + (k^{-1} H_k - u_k) h_k,$$

where we used the relation $\mathcal{L}_{\mathrm{grad}_{h_k} u_k} h_k = 2({}^k\nabla^2 u_k)$. In this expression, the second term of the sum is of the type $\mathcal{L}_X h_k$, while the third term has the form λh_k. Therefore, by the argument above, it is enough to show that the first term is h_k–traceless and h_k–divergence-free. The trace of $I\!I_k - {}^k\nabla^2 u_k + u_k h_k$ satisfies

$$\mathrm{tr}_{h_k}(I\!I_k - {}^k\nabla^2 u_k + u_k h_k) = -k^{-1} H_k - \Delta_k u_k + 2u_k.$$

This expression vanishes because u_k is a solution of (4). In order to compute the divergence of our tensor, we will need the relations

$$\mathrm{div}_{h_k} I\!I_k = -k^{-1} \mathrm{d}H_k, \quad \mathrm{div}_g({}^g\nabla^2 f) = \mathrm{d}(\Delta_g f) + \mathrm{Ric}_g(\mathrm{grad}_g f, \cdot).$$

The first equality follows from the Codazzi equation $(^{k}\nabla_X B_k)Y = (^{k}\nabla_Y B_k)X$ satisfied by the shape operator B_k of ∂M_k (the Levi-Civita connections of h_k and the first fundamental form I_k are the same, since they differ by a multiplicative constant). The second relation is true for any Riemannian metric g, and we will apply it in the case $g = h_k$ and $f = u_k$. Since h_k is a hyperbolic metric on a 2–manifold, we have $\mathrm{Ric}_{h_k} = -h_k$. Therefore

$$\mathrm{div}_{h_k}(II_k - \nabla_k^2 u_k + u_k h_k) = -k^{-1}\mathrm{d}H_k - \mathrm{d}(\Delta_k u_k) + \mathrm{d}u_k + \mathrm{d}u_k$$
$$= \mathrm{d}(-k^{-1}H_k - \Delta_k u_k + 2u_k),$$

where we used the relation $\mathrm{div}_g(fg) = \mathrm{d}f$. Again, the expression above vanishes because u_k solves (4). Then we have shown that $II_k - {}^k\nabla^2 u_k + u_k h_k$ is a transverse traceless tensor, as desired. $\square$

Remark 3.3 In fact, the decomposition we presented for the tensor $II_k + k^{-1}H_k h_k$ is related to the orthogonal decomposition of the space of symmetric tensors due to Fischer and Marsden [15]. Given g, a hyperbolic metric, every symmetric tensor S admits an orthogonal decomposition of the form

$$S = S_{\mathrm{tt}} + \mathcal{L}_X g + ((-\Delta_g f + f)g + {}^g\nabla^2 f),$$

where:

- S_{tt} is transverse traceless with respect to g.

- $S_{\mathrm{tt}} + \mathcal{L}_X g$ is tangent to the space of Riemannian metrics with constant Gaussian curvature equal to -1. That is, if $g' \mapsto K(g')$ denotes the operator that associates to the Riemannian metric g' its Gaussian curvature, then $S_{\mathrm{tt}} + \mathcal{L}_X g \in \ker \mathrm{d}K_g$.

- $(-\Delta_g f + f)g + {}^g\nabla^2 f$ is L^2–orthogonal to $\ker \mathrm{d}K_g$.

Then, the expression

$$II_k + k^{-1}H_k h_k = (II_k - {}^k\nabla^2 u_k + u_k h_k) + 0 + ((k^{-1}H_k - u_k)h_k + {}^k\nabla^2 u_k)$$
$$= (II_k - {}^k\nabla^2 u_k + u_k h_k) + 0 + ((-\Delta_k u_k + u_k)h_k + {}^k\nabla^2 u_k)$$

is the Fischer–Marsden decomposition of $II_k + k^{-1}H_k h_k$, where $f = u_k$, $X = 0$ and $S_{\mathrm{tt}} = (II_k - {}^k\nabla^2 u_k + u_k h_k)$.

Using this explicit description of the Weil–Petersson gradient of the dual volume function V_k^*, we can determine a lower bound of its norm in terms of the integral of the mean curvature:

Lemma 3.4 *For every $k \in (-1, 0)$,*

$$\|dV_k^*\|_{\mathrm{WP}}^2 \geq -\frac{\sqrt{k+1}}{2k} \int_{\partial M_k} H_k \, da_{I_k} - \frac{2\pi(k+1)}{k^2} |\chi(\partial M)|.$$

Proof In what follows, we will prove the expression

$$(6) \qquad \|I\!I_k - \nabla_k^2 u_k + u_k h_k\|_{I_k}^2 = k u_k H_k - 2(k+1) + \mathrm{div}_{I_k} W$$

for some tangent vector field W on ∂M_k. Assuming for the moment this relation,

$$
\begin{aligned}
\|dV_k^*\|_{\mathrm{WP}}^2 &= \frac{1}{2} \int_{\partial M_k} \|\Re\phi_k\|_{h_k}^2 \, da_{h_k} && \text{(Proposition 3.2 and Lemma 1.6)} \\
&= \frac{1}{2} \int_{\partial M_k} (-k)^{-2} \|\Re\phi_k\|_{I_k}^2 (-k) \, da_{I_k} \\
&= -\frac{1}{2k} \int_{\partial M_k} (k u_k H_k - 2(k+1)) \, da_{I_k} && \text{(relation (6))},
\end{aligned}
$$

where we used that $h_k = (-k) I_k$, relations (2) and (3), and that the integral of the term $\mathrm{div}_{I_k} W$ vanishes by the divergence theorem. By Lemma 2.3 $u_k \leq \sqrt{k+1}/k$, so

$$\|dV_k^*\|_{\mathrm{WP}}^2 \geq -\frac{\sqrt{k+1}}{2k} \int_{\partial M_k} H_k \, da_{I_k} - \frac{2\pi(k+1)}{k^2} |\chi(\partial M)|,$$

where we applied the Gauss–Bonnet theorem to say that the area of ∂M_k with respect to I_k is equal to $-2\pi k^{-1}|\chi(\partial M)|$.

The only ingredient left to prove is relation (6). For this computation, we will use *Bochner's formula* (see eg [25, page 223]),

$$(7) \quad \tfrac{1}{2}\Delta_g \|df\|_g^2 = \|{}^g\nabla^2 f\|_g^2 + g(\mathrm{grad}_g f, \mathrm{grad}_g \Delta_g f) + \mathrm{Ric}_g(\mathrm{grad}_g f, \mathrm{grad}_g f),$$

and the expressions

$$(8) \qquad\qquad\qquad \mathrm{div}_g(fX) = g(\mathrm{grad}_g f, X) + f \, \mathrm{div}_g X,$$

$$(9) \qquad\qquad\qquad \tfrac{1}{2}(\mathcal{L}_X g, T)_g = -(\mathrm{div}_g T)(X) + \mathrm{div}_g Y,$$

where X is a tangent vector field, f is a smooth function, T is a symmetric 2–tensor, and $Y = T(X, \cdot)^\sharp$ is the vector field defined by requiring that $g(Y, Z) = T(X, Z)$ for all vector fields Z. From now on, we will omit everywhere the dependence of the connections, norms, gradients, and the Laplace–Beltrami operator on the Riemannian

metric g, and everything has to be interpreted as associated to $g = I_k$. Observe also that the Levi-Civita connections of I_k and h_k are equal, since these metrics differ by multiplication by a constant and, in particular, the h_k- and I_k-Hessians coincide. Then

$$(10) \quad \|I\!I_k - \nabla^2 u_k + u_k h_k\|^2 = \|I\!I_k - \nabla^2 u_k - ku_k I_k\|^2$$
$$= \|I\!I_k\|^2 + \|\nabla^2 u_k\|^2 + k^2 u_k^2 \|I_k\|^2 - 2(I\!I_k, \nabla^2 u_k)$$
$$- 2ku_k(I\!I_k, I_k) + 2ku_k(\nabla^2 u_k, I_k).$$

First, we focus our attention on the terms $\|\nabla^2 u_k\|^2$ and $(I\!I_k, \nabla^2 u_k)$. In order to simplify the notation, we say that two functions a and b on ∂M_k are equal "modulo divergence", and we write $a \equiv_{\mathrm{div}} b$, if their difference coincides with the divergence of some smooth vector field. Then

$$\|\nabla^2 u_k\|^2 = \tfrac{1}{2}\Delta\|\mathrm{d}u_k\|^2 - \langle \mathrm{grad}\, u_k, \mathrm{grad}\, \Delta u_k\rangle - k\|\mathrm{d}u_k\|^2 \quad \text{(relation (7))}$$
$$\equiv_{\mathrm{div}} -\langle \mathrm{grad}\, u_k, \mathrm{grad}\, \Delta u_k\rangle - k\|\mathrm{d}u_k\|^2 \quad (\Delta_g f = \mathrm{div}_g \,\mathrm{grad}_g\, f)$$
$$= -\mathrm{div}(\Delta u_k\, \mathrm{grad}\, u_k) + (\Delta u_k)^2 - k\|\mathrm{d}u_k\|^2 \quad \text{(relation (8))}$$
$$\equiv_{\mathrm{div}} (\Delta u_k)^2 - k\,\mathrm{div}(u_k \,\mathrm{grad}\, u_k) + ku_k\Delta u_k \quad \text{(relation (8))}$$
$$\equiv_{\mathrm{div}} \Delta u_k(\Delta u_k + ku_k),$$

and

$$(I\!I_k, \nabla^2 u_k) = \tfrac{1}{2}(I\!I_k, \mathcal{L}_{\mathrm{grad}\, u_k} I_k) \quad (\mathcal{L}_{\mathrm{grad}_g f}\, g = 2^g\nabla^2 f)$$
$$\equiv_{\mathrm{div}} -(\mathrm{div}\, I\!I_k)(\mathrm{grad}\, u_k) \quad \text{(relation (9))}$$
$$= -\langle \mathrm{grad}\, H_k, \mathrm{grad}\, u_k\rangle \quad (\mathrm{div}\, I\!I_k = \mathrm{d}H_k)$$
$$= -\mathrm{div}(H_k \,\mathrm{grad}\, u_k) + H_k\Delta u_k \quad \text{(relation (8))}$$
$$\equiv_{\mathrm{div}} H_k\Delta u_k.$$

The other terms in (10) are simpler to handle. In particular,

$$\|I\!I_k\|^2 = H_k^2 - 2(k+1), \qquad \|I_k\|^2 = 2,$$
$$(I\!I_k, I_k) = H_k, \qquad (\nabla^2 u_k, I_k) = \Delta u_k.$$

Replacing all the relations we found in (10), we obtain

$$\|I\!I_k - \nabla^2 u_k + u_k h_k\|^2$$
$$\equiv_{\mathrm{div}} H_k^2 - 2(k+1) + \Delta u_k(\Delta u_k + ku_k) + 2k^2 u_k^2$$
$$- 2H_k\Delta u_k - 2ku_k H_k + 2ku_k\Delta u_k$$
$$= H_k^2 - 2(k+1) + 2k^2 u_k^2 - 2ku_k H_k + \Delta u_k(\Delta u_k + 3ku_k - 2H_k).$$

Finally, by replacing $\Delta u_k = \Delta_{I_k} u_k$ using (4) in the equality above, we find that

$$\|I\!I_k - \nabla^2 u_k + u_k h_k\|^2 \equiv_{\text{div}} k u_k H_k - 2(k+1),$$

which is equivalent to relation (6). □

Since the Weil–Petersson metric of the Teichmüller space is noncomplete, a control from above of the quantity $\|dV_k^*\|_{\text{WP}}$ would not suffice to guarantee the existence of the flow for every time. For this purpose, we rather study the L^∞–norm of the Beltrami differentials equivalent to $\text{grad}_{\text{WP}} V_k^*$, which gives a control with respect to the Teichmüller metric (that is complete). At this point, the estimates determined in Lemma 2.3 will play an essential role.

Proposition 3.5 *There exists a constant $D_k > 0$ depending only on the intrinsic curvature $k \in (-1, 0)$ such that*

$$\|\text{grad}_{\text{WP}} V_k^*\|_{\mathcal{T}} \le D_k,$$

where $\|\cdot\|_{\mathcal{T}}$ denotes the Teichmüller norm on $T\mathcal{T}(\partial M_k)$.

Proof Let m_k be a point of the Teichmüller space $\mathcal{T}(\partial M_k)$ and let h_k be a hyperbolic metric in the isotopy class m_k. In Proposition 3.2, we showed that the vector field $\text{grad}_{\text{WP}} V_k^*$ at a point $m_k \in \mathcal{T}(\partial M_k)$ is represented by the transverse traceless tensor $2\Re\phi_k \in S_2^{\text{tt}}(\partial M_k, h_k)$. Therefore by Lemma 1.6,

$$\|\text{grad}_{\text{WP}} V_k^*\|_{\mathcal{T}} \le \frac{1}{\sqrt{2}} \sup_{\partial M_k} \|\Re\phi_k\|_{h_k}.$$

So it is enough to show that the norm $\|I\!I_k - {}^k\nabla^2 u_k + u_k h_k\|_{h_k}$ is uniformly bounded by a constant depending only on k. The norm of $I\!I_k$ is equal to $-k^{-1}\sqrt{H_k^2 - 2(k+1)}$, and $\|u_k h_k\|_{h_k} = \sqrt{2}|u_k|$. Therefore

$$\|I\!I_k - {}^k\nabla^2 u_k + u_k h_k\|_{h_k} \le -k^{-1}\sqrt{\|H_k\|_{\mathscr{C}^0}^2 - 2(k+1)} + \|{}^k\nabla^2 u_k\|_{h_k} + \sqrt{2}\|u_k\|_{\mathscr{C}^0}.$$

Our assertion is now an immediate consequence of Proposition 2.1 and Lemma 2.3. □

Corollary 3.6 *The flow Θ_t of the vector field $-\text{grad}_{\text{WP}} V_k^*$ over $\mathcal{T}(\partial M_k)$ is defined for all times $t \in \mathbb{R}$.*

Proof The assertion follows from the fact that the Teichmüller distance is complete, and the bound shown in Proposition 3.5. □

The last ingredient that we will need for the proof of Theorem A is the existence of some lower bound for the dual volume function V_k^*. To find one, we will make use of the properties of the dual volume proved in [29], and of an upper bound for the length of the bending measure of the boundary of the convex core of a convex cocompact manifold with incompressible boundary, whose existence was first proved by Bridgeman [8] and improved in later works; see [11]. We will make use of the best result currently known in this direction for convex cocompact manifolds with incompressible boundary, which is due to Bridgeman, Brock and Bromberg [9].

Lemma 3.7 *For every $k \in (-1, 0)$ and for every convex cocompact hyperbolic 3–manifold M with incompressible boundary,*

$$V_k^*(M) \geq F(k, \chi(\partial M)),$$

where F is an explicit function of the curvature $k \in (-1, 0)$ and the Euler characteristic of ∂M.

Proof Since the k–surfaces foliate the complement of the convex core CM, a simple application of the geometric maximum principle (see for instance [24, lemme 2.5.1]) shows that the k–surface ∂M_k is contained in $N_{\varepsilon_k} CM$, the ε_k–neighborhood of the convex core CM, for $\varepsilon_k = \operatorname{arctanh} \sqrt{k+1}$. The dual volume of a convex set is a decreasing function with respect to inclusion (see [29, Proposition 2.6] for a proof of this assertion), therefore the quantity $V_k^*(M)$ is bounded from below by the dual volume of the ε_k–neighborhood of the convex core. It is not difficult to show that, for every $\varepsilon > 0$,

$$V^*(N_\varepsilon CM) = V(CM) - \tfrac{1}{4}\ell_m(\mu)(\cosh 2\varepsilon + 1) - \tfrac{1}{2}\pi|\chi(\partial CM)|(\sinh 2\varepsilon - 2\varepsilon),$$

where $\ell_m(\mu)$ denotes the length of the bending measured lamination on the boundary of the convex core of M; see eg [29, Proposition 2.4]. By [9, Theorem 2.16], the term $\ell_m(\mu)$ is less than or equal to $6\pi|\chi(\partial M)|$. Combining these observations,

$$
\begin{aligned}
V_k^*(M) &\geq V^*(N_{\varepsilon_k} CM) \\
&= V(CM) - \tfrac{1}{4}\ell_m(\mu)(\cosh 2\varepsilon_k + 1) - \tfrac{1}{2}\pi|\chi(\partial CM)|(\sinh 2\varepsilon_k - 2\varepsilon_k) \\
&\geq -\tfrac{1}{4}\ell_m(\mu)(\cosh 2\varepsilon_k + 1) - \tfrac{1}{2}\pi|\chi(\partial CM)|(\sinh 2\varepsilon_k - 2\varepsilon_k) \\
&\geq -\tfrac{1}{2}\pi|\chi(\partial M)|(3\cosh \varepsilon_k + 3 + \sinh 2\varepsilon_k - 2\varepsilon_k),
\end{aligned}
$$

which proves the desired inequality. $\qquad\square$

4   The proof of Theorem A

This section is dedicated to the proof of our main theorem, and to the proof of the optimality of the multiplicative constant appearing in (1).

Proof of Theorem A   Let M be a convex cocompact hyperbolic 3–manifold with incompressible boundary. We denote by $M_t := \Theta_t(M)$ the hyperbolic 3–manifold obtained by following the flow of the vector field $-\operatorname{grad}_{\mathrm{WP}} V_k^*$, which is defined for every $t \in \mathbb{R}$ in light of Corollary 3.6. In order to simplify the notation, we will continue to denote by V_k^* the k–dual volume as a function over the space of quasi-isometric deformations of M. This abuse is justified by the fact that, for every $k \in (-1, 0)$, a convex cocompact manifold is uniquely determined by the hyperbolic structures on its k–surfaces (see Theorem 1.5). We have

$$V_k^*(M) - V_k^*(M_t) = \int_0^t \|\mathrm{d} V_k^*\|_{M_s}^2 \, \mathrm{d}s.$$

By Lemma 3.7, the left side of the relation is bounded from above with respect to t. In particular, the integral on the right side has to converge as t goes to $+\infty$. Therefore we can find an unbounded increasing sequence $(t_n)_n$ for which the Weil–Petersson norm $\|\mathrm{d} V_k^*\|^2$ evaluated at M_{t_n} goes to 0 as n goes to ∞. Then, by Lemma 3.4,

$$\limsup_{n \to \infty} \int_{\partial M_{t_n,k}} H_k \, \mathrm{d}a_{I_k} \leq -4\pi k^{-1} \sqrt{k+1} |\chi(\partial M)|,$$

where $M_{t_n,k}$ stands for $(M_{t_n})_k$, the region of M_{t_n} enclosed by its k–surfaces. Therefore

$$V_k^*(M) \geq \lim_{n \to \infty} V_k^*(M_{t_n}) = \lim_{n \to \infty} \left(V_k(M_{t_n}) - \frac{1}{2} \int_{\partial M_{t_n,k}} H_k \, \mathrm{d}a_{I_k} \right)$$

$$\geq \inf_{M' \in \mathcal{QD}(M)} V_k(M') - \frac{1}{2} \limsup_{n \to \infty} \int_{\partial M_{t_n,k}} H_k \, \mathrm{d}a_{I_k}$$

$$\geq \inf_{M' \in \mathcal{QD}(M)} V_k(M') + 2\pi k^{-1} \sqrt{k+1} |\chi(\partial M)|,$$

where $V_k(M')$ denotes the Riemannian volume of the region M_k' of M' enclosed by its k–surface. Observe that the term $2\pi k^{-1} \sqrt{k+1} |\chi(\partial M)|$ is equal to $-\frac{1}{2} \int_{\partial M_k'} H_k \, \mathrm{d}a_{I_k}$ when the boundary of the convex core of M' is totally geodesic.

Finally, by taking the limit as k goes to $(-1)^+$, we obtain that $V_C^*(M) \geq \inf_{M'} V_C(M')$ for every convex cocompact structure M. This proves that

$$\inf_{\mathcal{QD}(M)} V_C^* \geq \inf_{\mathcal{QD}(M)} V_C.$$

On the other hand, the dual volume $V_C^*(M) := V_C(M) - \frac{1}{2}\ell_m(\mu)$ is always smaller than or equal to $V_C(M)$, so the other inequality between the infima is clearly satisfied.

If $V_C^*(M) = V_C(M)$, then the length of the bending measured lamination μ of the convex core of M has to vanish. Therefore $\mu = 0$ or, in other words, ∂CM is totally geodesic. $\qquad\square$

Corollary 4.1 *For every quasi-Fuchsian manifold M, $V_C(M) \geq \frac{1}{2}\ell_m(\mu)$, where $m = m(M)$ and $\mu = \mu(M)$ denote the hyperbolic metric and the bending measure of the boundary of the convex core of M, respectively. Moreover, for every positive ε and for every neighborhood U of a Fuchsian manifold M_0 inside $\mathcal{QD}(M_0) = \mathcal{QD}(M)$, there exists a quasi-Fuchsian manifold M_ε in U that satisfies $V_C(M_\varepsilon) < \left(\frac{1}{2} + \varepsilon\right)\ell_{m_\varepsilon}(\mu_\varepsilon)$, where $m_\varepsilon = m(M_\varepsilon)$ and $\mu_\varepsilon = \mu(M_\varepsilon)$.*

Proof If M is quasi-Fuchsian, the infimum of the volume of the convex core over the space of quasi-isometric deformations $\mathcal{QD}(M)$ is equal to 0, and it is realized on the Fuchsian locus.

For the second part of the statement, consider M_0 a Fuchsian manifold whose convex core is a totally geodesic surface homeomorphic to Σ with hyperbolic metric m_0. Let $\alpha : [0, 1] \to \mathcal{QD}(M)$ be a path starting at $\alpha(0) = M_0$ and for which the right derivative of the bending measure $\dot{\mu}_0^+$ exists and it is equal to a nonzero measured lamination on $\Sigma \sqcup \Sigma$. A fairly explicit way to produce such a path is to choose a measured lamination $\lambda \in \mathcal{ML}(\Sigma)$ and consider the deformation of M_0 given by the holonomies of pleated surfaces with bending Hölder cocycle equal to $t\lambda$ and hyperbolic metric m_0, as t varies in $[0, 1]$; compare with [3]. Then, for every $\varepsilon > 0$, we define

$$f_\varepsilon(t) := V_C(\alpha(t)) - \left(\tfrac{1}{2} + \varepsilon\right)\ell_{m_t}(\mu_t) = V_C^*(\alpha(t)) - \varepsilon\ell_{m_t}(\mu_t) \quad \text{for } t \in [0, 1],$$

where $m_t = m(M_t)$ and $\mu_t = \mu(M_t)$ denote the hyperbolic metric and the bending measure of the boundary of the convex core of $M_t = \alpha(t)$. As shown in [20, (4)],

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell_{m_t}(\mu_t)\bigg|_{t=0^+} = \mathrm{d}(L_{\mu_0})(\dot{m}_0) + \ell_{m_0}(\dot{\mu}_0^+) = \ell_{m_0}(\dot{\mu}_0^+),$$

where we use that $\mu_0 = 0$ (here $L_{\mu_0} : \mathcal{T}(\partial CM) \to \mathbb{R}$ is the function that associates with every hyperbolic structure m the length of the m–geodesic realization of μ_0). Then

$$\begin{aligned}
f_\varepsilon(t) &= f_\varepsilon(0) + f_\varepsilon'(0)t + o(t; \varepsilon) \\
&= 0 + (\mathrm{d}(V_C{}^*)_{M_0}(v) - \varepsilon\ell_{m_0}(\dot{\mu}_0^+))t + o(t; \varepsilon) \\
&= -\varepsilon\ell_{m_0}(\dot{\mu}_0^+)t + o(t; \varepsilon) \qquad\qquad (V_C^* \in \mathscr{C}^1 \text{ and } M_0 \text{ minimum}).
\end{aligned}$$

This proves that $f_\varepsilon(t) < 0$ for t sufficiently small (depending on ε), and therefore the existence of a quasi-Fuchsian manifold M_ε satisfying the desired properties.   □

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