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We prove that finitely generated Kleinian groups $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ with small critical exponent are always convex cocompact. We also prove some geometric properties for any complete pinched negatively curved manifold with critical exponent less than 1.

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1 Introduction

A *Kleinian* group is a discrete isometry subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. The study of 3–dimensional finitely generated Kleinian groups dates back to Schottky, Poincaré and Klein. It is only recently that the geometric picture of the associated hyperbolic manifold has been much better understood, after the celebrated work of Ahlfors' finiteness theorem [2], the proof of the tameness conjecture (see Agol [1], Bonahon [10] and Calegari and Gabai [18]), and the unraveling of the ending lamination conjecture; see Bowditch [13], Brock, Canary and Minsky [14], Minsky [36] and Soma [42]. However, such geometric descriptions fail in higher dimensions; see Kapovich [29; 30], Kapovich and Potyagailo [33; 34] and Potyagailo [41; 40].

One way to study higher-dimensional Kleinian groups is to consider the interplay between the group-theoretic properties, the geometry of the quotient manifolds, and the *measure-theoretic size* of the limit set. It was shown by Gusevskii [23] that if the Hausdorff dimension of the entire limit set $\dim_{\mathcal{H}}(\Lambda(\Gamma))$ is less than 1, then Γ is geometrically finite. In this case, the Hausdorff dimension of the entire limit set equals the Hausdorff dimension of the conical limit set (see Bowditch [12]), which is smaller than 1. However, when Γ is geometrically infinite, the size of the entire limit set could a priori be much larger, so $\dim_{\mathcal{H}} \Lambda(\Gamma) > \dim_{\mathcal{H}} \Lambda_c(\Gamma)$. Thus, it is interesting to ask what the relative size of $\Lambda_c(\Gamma)$ is compared to the entire $\Lambda(\Gamma)$, or rather, to what extent is the size of $\Lambda_c(\Gamma)$ able to determine the geometric finiteness of the group. By the work of Bishop and Jones [9], the Hausdorff dimension of the conical limit set $\Lambda_c(\Gamma)$ equals the critical exponent $\delta(\Gamma)$. Hence, Kapovich [31, Problem 1.6] asked:

Question 1.1 *Is every finitely generated Kleinian group* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *with* $\delta(\Gamma) < 1$ *geometrically finite*?

We partly answer this in the affirmative in a slightly more general context.

Theorem 1.2 *For each* n *and* κ *there exists a positive constant* $D(n,\kappa) < \frac{1}{2}$ *with the property that*, *for every* n*–dimensional Hadamard manifold with pinched sectional curvature* $-\kappa^2 \le K \le -1$ *and any finitely generated torsion-free discrete isometry subgroup* $\Gamma < \mathrm{Isom}(X)$, Γ *is convex cocompact if* $\delta(\Gamma) < D(n,\kappa)$.

Remark 1.3 The constant $D(n,\kappa)$ can be obtained from the quantitative version of the Tits alternative for pinched negatively curved manifolds; see Dey, Kapovich and Liu [20].

Remark 1.4 For 3–dimensional finitely generated Kleinian groups Γ of *second kind*, ie $\Lambda(\Gamma) \ne S^2$, Bishop and Jones [9] showed that Γ is geometrically finite if $\delta(\Gamma) < 2$. Hou [25; 26; 27] proved that a 3–dimensional Kleinian group Γ is a classical Schottky group if $\dim_{\mathcal{H}}(\Lambda(\Gamma)) < 1$.

In [31], Kapovich established a relation between the homological dimension and the critical exponent of a Kleinian group. A similar homological vanishing feature has been extended to other rank-one symmetric spaces by Connell, Farb and McReynolds [19]. It is conjectured [31, Conjecture 1.4] that the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ is bounded above by $\delta(\Gamma) + 1$ (assuming Γ has no higher-rank cusps). Under the condition $\delta(\Gamma) < 1$, it is equivalent to ask (see Stallings [43] and also a weaker form by Bestvina [8, Question 5.6]):

Question 1.5 *Is every finitely generated Kleinian group* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *with* $\delta(\Gamma) < 1$ *virtually free*?

In the same paper, Kapovich gave a positive answer to this question under the stronger assumption that Γ is finitely presented. On the other hand, when $\delta(\Gamma)$ is sufficiently small, our Theorem 1.2 automatically implies $\dim_{\mathcal{H}}(\Lambda(\Gamma)) = \delta(\Gamma) < D(n,\kappa) < 1$. This implies that the limit set $\Lambda(\Gamma)$ is a Cantor set since it is perfect. Following the classical result of Kulkarni [35, Theorem 6.11]:

Corollary 1.6 *For each* n *there is a positive constant* $D(n) < \frac{1}{2}$ *such that any finitely generated discrete isometry subgroup* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *is virtually free if* $\delta(\Gamma) < D(n)$.

Remark 1.7 Under the assumption that $\dim_{\mathcal{H}}(\Lambda(\Gamma)) < 1$, Pankka and Souto [39] proved that any torsion-free Kleinian group (not necessarily finitely generated) is free.

The method in [31] also works for discrete isometry subgroups of Hadamard manifolds with negatively pinched sectional curvature $-\kappa^2 \leq K \leq -1$, and Question 1.5 can be asked for this family of groups. If in addition we know Γ is free in Theorem 1.2, then the constant $D(n,\kappa)$ can actually be made effective, and independent of n and κ.

Theorem 1.8 *Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated virtually free discrete isometry subgroup of an n–dimensional Hadamard manifold with pinched negative curvature $-\kappa^2 \leq K \leq -1$. If $\delta(\Gamma) < \frac{1}{16}$, then Γ is convex cocompact.*

Thus, in view of Kapovich's result [31, Corollary 1.5], we obtain:

Corollary 1.9 *A finitely presented Kleinian group with $\delta(\Gamma) < \frac{1}{16}$ is convex cocompact.*

One of the main efforts in our proofs is investigating the geometric properties of the quotient manifold $M = X/\Gamma$ under the condition that δ is small. While these results are only restricted to $\delta < 1$, we still find that they might be of independent interest and worth highlighting. The following theorem is closely related to the classical Plateau's problem, where we obtain a certain type of linear isoperimetric inequality for the quotient manifold $M = X/\Gamma$.

Theorem 1.10 *Suppose that $\mathcal{C}$ is a union of smooth loops in $M = X/\Gamma$ which represents a trivial homology class in $H_1(M,\mathbb{Z})$. If $\delta(\Gamma) = \delta < 1$, then $\mathcal{C}$ bounds a smooth surface $i : \Sigma \to M$* (see Definition 2.6) *whose area satisfies*

$$A(i) \leq \frac{4}{1-\delta}\ell(\mathcal{C}),$$

where $\ell(\mathcal{C})$ denotes the total length of the smooth loops in $\mathcal{C}$.

Finitely generated Kleinian groups in dimension 3 have only finitely many cusps (see Sullivan [44]), but the same result does not hold in higher dimensions; see Kapovich [29]. As an application of Theorem 1.10, we show that, under the assumption $\delta < 1$, the ϵ–thin part of M has only finitely many connected components when ϵ is small enough. In particular, M has only finitely many cusps.

Theorem 1.11 *Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated torsion-free discrete isometry subgroup of an n–dimensional Hadamard manifold with pinched negative curvature $-\kappa^2 \leq K \leq -1$. Suppose that $\delta(\Gamma) < 1$. Then:*

(1) *The number of cusps in* $M = X/\Gamma$ *is at most the first Betti number of* M.

(2) M *has bounded geometry. That is, the noncuspidal part of* M *has a uniform lower bound on its injectivity radius.*

(3) Γ *is convex cocompact if and only if the injectivity radius function* $\operatorname{inj}: M \to \mathbb{R}$ *is proper.*

Remark 1.12 Without the assumption on the critical exponent, Benoist and Hulin [5, Proposition 2.6] showed that Γ is convex cocompact if and only if M is Gromov hyperbolic and the injectivity radius function is proper.

Outline of the proof of Theorem 1.2

We first observe that whenever $\delta < 1$ there is an area-decreasing self-map (the Besson–Courtois–Gallot map) on M. This allows us to prove the linear isoperimetric type inequality as in Theorem 1.10, from which we deduce further that closed geodesics on M asymptotically have uniformly bounded normal injectivity radii. This means that if there is an escaping sequence of closed geodesics on M, then there exists a subsequence on which the normal injectivity radii are uniformly bounded. Next we observe that, given a long closed geodesic with small normal injectivity radius, one can always separate along the normal direction to replace it by a shorter closed geodesic nearby. Then, we use the result by Kapovich and Liu [32] which states that Γ is geometrically infinite if and only if there exists an escaping sequence of closed geodesics. The assumption that $D(n,\kappa)$ is smaller than $\frac{1}{2}$ excludes parabolic elements, so assume for the sake of contradiction that there is one such escaping sequence. Using the idea of infinite descent we can reduce the length of the closed geodesics and find another escaping sequence whose lengths and normal injectivity radii are both uniformly bounded, from which we can find two loxodromic isometries that move a common point within a uniformly bounded distance. This means the nonelementary subgroup generated by the two isometries will have large critical exponent, thus leading to a contradiction if we assume δ is small enough.

Organization of the paper

In Section 2 we review some elementary results of negatively pinched Hadamard manifolds and the Besson–Courtois–Gallot map. In Section 3 we give the proofs of Theorems 1.10 and 1.11. In Section 4 we prove Theorem 4.1, which together with Theorem 1.11 implies Theorems 1.2 and 1.8.

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2 Preliminaries

2.1 Discrete isometry groups

Let X be a complete simply connected n–dimensional Riemannian manifold of pinched negative curvature $-\kappa^2 \le K \le -1$ where $\kappa \ge 1$. The Riemannian metric on X induces the distance function d_X, and (X, d_X) is a uniquely geodesic space. With the curvature assumption, the metric space (X, d_X) is Gromov hyperbolic, where the hyperbolicity constant δ_0 can be chosen as $\cosh^{-1}(\sqrt{2})$, ie every geodesic triangle in X is δ_0–slim.

By the Cartan–Hadamard theorem, X is diffeomorphic to the Euclidean space $\mathbb{R}^n$ via the exponential map at any point in X. We can naturally compactify X by adding the ideal boundary $\partial_\infty X$, thus the compactified space $\overline{X} = X \cup \partial_\infty X$ is homeomorphic to the unit n–ball B^n.

Every isometry $\gamma \in \operatorname{Isom}(X)$ extends the action to the ideal boundary, so it induces a diffeomorphism on $\overline{X}$. Based on its fixed-point set $\operatorname{Fix}(\gamma)$, the isometry γ on X can be classified:

(1) γ is *parabolic* if $\operatorname{Fix}(\gamma)$ is a singleton $\{p\} \subset \partial_\infty X$.

(2) γ is *elliptic* if it has a fixed point in X. In this case, the fixed-point set $\operatorname{Fix}(\gamma)$ is a totally geodesic subspace of X invariant under γ. In particular, the identity map is elliptic.

(3) γ is *loxodromic* if $\operatorname{Fix}(\gamma)$ consists of two distinct points $p, q \in \partial_\infty X$. In this case, γ stabilizes and translates along the geodesic pq, and we call the geodesic pq the *axis* of γ.

One can also use the translation length to classify the isometries on X. For each isometry $\gamma \in \operatorname{Isom}(X)$, we define its *translation length* $\tau(\gamma)$ as

$$\tau(\gamma) := \inf_{x \in X} d_X(x, \gamma(x)).$$

The isometry γ is loxodromic if and only if $\tau(\gamma) > 0$. In this case, the infimum is attained exactly when the points are on the axis of γ. The isometry γ is parabolic if and only if $\tau(\gamma) = 0$ and the infimum is not attained. The isometry γ is elliptic if and only if $\tau(\gamma) = 0$ and the infimum is attained.

Let $\Gamma < \mathrm{Isom}(X)$ be a discrete subgroup which acts on X properly discontinuously. If Γ is torsion-free, then any nontrivial element in Γ is either loxodromic or parabolic. We denote the quotient manifold X/Γ by M, and let $\pi : X \to M$ denote the canonical projection. The geodesic loops $c : [a,b] \to M$ at $p = c(a) = c(b) \in M$ are in one-to-one correspondence with geodesic segments from x to $\gamma(x)$, where $x \in X$ with $\pi(x) = p$ and $\gamma \in \Gamma$. Recall that the injectivity radius at a point $p \in M$ is the largest radius for which the exponential map at p is a diffeomorphism. The injectivity radius at a point $p \in M$ is half the length of shortest geodesic loop at p since there are no conjugate points in M. We use $\mathrm{inj}(p)$ to denote the injectivity radius at p and define

$$d_\Gamma(x) := \min_{\gamma \in \Gamma \setminus \{id\}} d_X(x, \gamma(x))$$

for $x \in X$. Then $d_\Gamma(x) = 2\,\mathrm{inj}(\pi(x))$. We say the injectivity radius function $\mathrm{inj} : M \to \mathbb{R}$ is *proper* if the preimage of a compact set is compact. The injectivity radius function is 1–Lipschitz. To see this, given any two points $p, q \in M$, let $\tilde{p}$ and $\tilde{q}$ be lifts of p and q in X whose distance is the same as the distance $d(p,q)$ of $p, q \in M$. There exists an isometry $\gamma \in \Gamma$ such that $d_X(\tilde{p}, \gamma\tilde{p}) = d_\Gamma(\tilde{p})$, and

$$\begin{aligned} 2\,\mathrm{inj}(q) \le d_X(\tilde{q}, \gamma(\tilde{q})) &\le d_X(\tilde{q}, \tilde{p}) + d_X(\tilde{p}, \gamma(\tilde{p})) + d_X(\gamma(\tilde{p}), \gamma(\tilde{q})) \\ &= 2d(p,q) + 2\,\mathrm{inj}(p). \end{aligned}$$

Hence, $\mathrm{inj}(q) - \mathrm{inj}(p) \le d(p,q)$.

Now recall that the *critical exponent* $\delta(\Gamma)$ of a torsion-free discrete isometry group $\Gamma < \mathrm{Isom}(X)$ is defined to be

$$\delta(\Gamma) := \inf\Big\{ s \;\Big|\; \sum_{\gamma \in \Gamma} \exp\big(-s d_X(p, \gamma(p))\big) < \infty \Big\},$$

where p is a given point in X. Note that $\delta(\Gamma)$ is independent of the choice of p. Alternatively, one can also define the critical exponent $\delta(\Gamma)$ [38] as

$$\delta(\Gamma) = \limsup_{R \to \infty} \frac{\log(N(R))}{R}, \tag{2-1}$$

where $N(R) = \#\{\gamma \in \Gamma \mid d_X(x, \gamma(x)) \le R\}$ for any given point $x \in X$.

We will need to use the following proposition later in the proofs:

Proposition 2.1 [32, Corollary 6.12] *Let $w \in M = X/\Gamma$ be a piecewise geodesic loop which consists of r geodesic segments, and let α be the closed geodesic freely homotopic to w such that $\ell(\alpha) \geq \epsilon > 0$. Then α is contained in the D–neighborhood of the loop w, where*

$$D = \cosh^{-1}(\sqrt{2})\lceil \log_2 r \rceil + \sinh^{-1}\left(\frac{2}{\epsilon}\right) + 2\delta_0.$$

Remark 2.2 The original corollary was stated under the extra assumption that α is simple. However, the proof of [32, Corollary 6.12] does not rely on this fact so we have removed the assumption here.

2.2 Thick–thin decomposition

Given an isometry $\gamma \in \mathrm{Isom}(X)$ and a constant $\epsilon > 0$, we define the *Margulis region* $\mathrm{Mar}(\gamma, \epsilon)$ of γ as

$$\mathrm{Mar}(\gamma, \epsilon) := \{x \in X \mid d_X(x, \gamma(x)) \leq \epsilon\}.$$

It is a convex subset by the convexity of the distance function. Given a point $x \in X$ and a constant $\epsilon > 0$, the set

$$\mathcal{F}_\epsilon(x) := \{\gamma \in \mathrm{Isom}(X) \mid d_X(x, \gamma(x)) \leq \epsilon\}$$

consists of all isometries that translate x by at most ϵ. For any discrete subgroup $\Gamma < \mathrm{Isom}(X)$, we denote by $\Gamma_\epsilon(x)$ the group generated by $\mathcal{F}_\epsilon(x) \cap \Gamma$. The Margulis lemma [3, Theorem 9.5] states that $\Gamma_\epsilon(x)$ is a finitely generated virtually nilpotent group for any $0 < \epsilon < \epsilon(n, \kappa)$, where $\epsilon(n, \kappa)$ is the Margulis constant depending on the dimension n of X and the sectional curvature bound κ.

We define the Γ–invariant set

$$\mathcal{T}_\epsilon(\Gamma) := \{p \in X \mid \Gamma_\epsilon(p) \text{ is infinite}\}.$$

The *thin part* (more precisely, the ϵ–thin part) of the quotient orbifold $M = X/\Gamma$, which we denote by $\mathrm{thin}_\epsilon(M)$, is defined to be $\mathcal{T}_\epsilon(\Gamma)/\Gamma$. The closure of the complement $M \setminus \mathrm{thin}_\epsilon(\Gamma)$ is called the *thick part* of M and is denoted by $\mathrm{thick}_\epsilon(M)$. The thin part consists of bounded and unbounded components. The bounded components are called the *Margulis tubes*, and are neighborhoods of short closed geodesics of length no greater than ϵ. More precisely, for every point x in the closed geodesic and every tangent vector v at x perpendicular to the geodesic, we consider a unit-speed ray ρ emanating from x in the direction of v. There exists R, depending on x and v, such that

$$d_\Gamma(\rho(R)) = \epsilon \quad \text{and} \quad d_\Gamma(\rho(t)) < \epsilon$$

for all $t < R$. We call the arc $\rho([0, R])$ a *maximal radial arc*, and a Margulis tube is the union of all radial arcs emanating from a short closed geodesic. For details, see for example [16].

The unbounded components are called the *Margulis cusps*, and can be described more precisely as follows. Denote the fixed-point set of Γ by

$$\mathrm{Fix}(\Gamma) := \bigcap_{\gamma \in \Gamma} \mathrm{Fix}(\gamma).$$

A discrete subgroup $P < \Gamma$ is called a *parabolic subgroup* if $\mathrm{Fix}(P)$ consists of a single point $\xi \in \partial_\infty X$. Given a constant $0 < \epsilon < \epsilon(n, \kappa)$ and a maximal parabolic subgroup $P < \Gamma$, the set $\mathcal{T}_\epsilon(P) \subset X$ is precisely invariant under P, and we have $\mathrm{stab}_\Gamma(\mathcal{T}_\epsilon(P)) = P$; see [12, Corollary 3.5.6]. In this case, $\mathcal{T}_\epsilon(P)/P$ can be regarded as a subset of M, called a Margulis cusp. The *cuspidal* part of M is the union of all Margulis cusps, denoted by $\mathrm{cusp}_\epsilon(M)$. Note that $\mathrm{cusp}_\epsilon(M) \subset \mathrm{thin}_\epsilon(M)$.

In our context, the parabolic subgroups in Γ (hence also the cuspidal part of M) turn out to be very simple due to the following proposition:

Proposition 2.3 *Let $\Gamma < \mathrm{Isom}(X)$ be a torsion-free discrete isometry group, and $P < \Gamma$ be any parabolic subgroup. If δ is the critical exponent of Γ and P has polynomial growth rate r, then we have $r \le 2\delta$. Thus:*

(1) *If $\delta < 1$, then all parabolic subgroups (if they exist) are isomorphic to $\mathbb{Z}$.*

(2) *If $\delta < \frac{1}{2}$, then all nontrivial isometries in Γ are loxodromic.*

Proof Let $\mathcal{H}$ be a horosphere that P acts on and choose any basepoint $O \in \mathcal{H}$. Denote by $d_{\mathcal{H}}$ the horospherical distance and by $d_{\boldsymbol{P}}$ the Cayley metric with respect to some fixed finite generating set of P. Then there exists a constant $C > 0$ such that

$$d_{\mathcal{H}}(O, \gamma(O)) \le C d_{\boldsymbol{P}}(1, \gamma) \tag{2-2}$$

holds for all $\gamma \in P$. By [24, Theorem 4.6] there exists a constant $C' > 0$ such that, for any $p, q \in \mathcal{H}$ with $d_X(p, q) > C'$, we have

$$d_X(p, q) \le 2 \ln(C' d_{\mathcal{H}}(p, q)). \tag{2-3}$$

By possibly replacing C or C' by a larger constant, we may assume $C' = C$. Therefore we obtain, from the above the asymptotic inequalities (for R large),

$$\begin{aligned} |\{\gamma \in P : d_{\boldsymbol{P}}(1, \gamma) \le R\}| &\le |\{\gamma \in P : d_{\mathcal{H}}(O, \gamma(O)) \le CR\}| && \text{(by (2-2))} \\ &\lesssim |\{\gamma \in P : d_X(O, \gamma(O)) \le 2\ln(C^2 R)\}| && \text{(by (2-3))} \end{aligned}$$

$$\simeq e^{2\ln(C^2R)\delta(P)} \qquad \text{(by (2-1))}$$
$$\simeq R^{2\delta(P)},$$

where $\delta(P)$ is the critical exponent of P. Since $\delta(P) \le \delta$, it follows that $r \le 2\delta$.

In particular, if $\delta < 1$, then $r < 2$ and by the Bass–Guivarc'h formula [4; 22], P must be virtually $\mathbb{Z}$. But since P is torsion-free, it must be $\mathbb{Z}$ [43]. If $\delta < \frac{1}{2}$, then $r < 1$ and P cannot exist. Thus all nontrivial elements in Γ are loxodromic. □

2.3 Geometric finiteness

Recall that the *limit set* $\Lambda(\Gamma)$ of a discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is defined to be the set of accumulation points of the Γ–orbit $\Gamma(p)$ in $\partial_\infty X$, where p is an arbitrary given point in X, and that the definition is independent of the choice of p. If $\Lambda(\Gamma)$ is finite, then Γ is called *elementary*. Otherwise, it is called *nonelementary*. A point $\xi \in \Lambda(\Gamma)$ is called *a conical limit point* if every geodesic ray $\rho\colon \mathbb{R}_+ \to X$ asymptotic to ξ projects to a nonproper map $\pi \circ \rho\colon \mathbb{R}_+ \to M = X/\Gamma$. We denote by $\Lambda_c(\Gamma)$ the set of all conical limit points.

We denote by $\mathrm{Hull}(\Lambda) \subset X$ the closed convex hull of $\Lambda \subset \partial_\infty X$, which is the smallest closed convex subset in X whose accumulation set in $\partial_\infty X$ is Λ, and by $C(\Gamma) = \mathrm{Hull}(\Lambda)/\Gamma$ the *convex core* of Γ.

A discrete isometry subgroup $\Gamma < \mathrm{Isom}\, X$ is *geometrically finite* if the noncuspidal part of the convex core $C(\Gamma)$ in $M = X/\Gamma$ is compact. Otherwise, it is called *geometrically infinite*. If $C(\Gamma)$ is compact, then the discrete subgroup Γ is called *convex cocompact*.

There are various equivalent definitions of geometric finiteness, but we will only mention one of them, proved by Kapovich and the first author. For the other equivalent definitions we refer the readers to [12]. The following theorem is a generalization of a previous result of Bonahon [10]:

Theorem 2.4 [32, Theorem 1.5] *A discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is geometrically infinite if and only if there exists a sequence of closed geodesics $\alpha_i \subset M = X/\Gamma$ which escapes every compact subset of M.*

2.4 Admissible surfaces

In this section, we give a sketch of the existence of smooth admissible surfaces. This can be treated as a smooth version of [17, Section 1.1.5]. In our case, we will need a slightly broader category of admissible surfaces than smooth maps in order to include the gluing of two maps along a smooth boundary. In general the notion of a piecewise smooth map is rather technical (using Whitney stratification), but we only consider maps from a smooth surface with boundary to a smooth manifold. Thus we simplify the notion:

Definition 2.5 Given a smooth surface Σ (possibly with boundary) and a smooth manifold M, we say a map $f\colon \Sigma \to M$ is a *piecewise smooth* map if there is a smooth triangulation $\Delta = \{\sigma_1, \dots, \sigma_m\}$ on Σ (ie edges are all smooth paths) such that:

(1) f is continuous.

(2) f is smooth on the interior of each face σ_i.

(3) If $e = \sigma_i \cap \sigma_j$ is a common edge, then the restriction $f|_p$ is smooth.

Roughly speaking, a piecewise smooth map is just a finite concatenation of smooth maps, possibly pleating along the gluing edges. The singular set forms a piecewise smooth 1–skeleton on Σ. Now we return to our context, where $M = X/\Gamma$ is a complete pinched negatively curved manifold. Suppose $\{\eta_1, \dots, \eta_k\}$ is a collection of k smooth loops in M. If there exists a set of integers $c_1, \dots, c_k$ such that $\sum_{i=1}^k c_i[\eta_i] = 0$ in $H_1(M, \mathbb{Z})$, then we claim that $\bigcup_i c_i\eta_i$ will bound a piecewise smooth surface in the sense explained below.

Choose a basepoint $x_0 \in M$ and connect x_0 to each of the loops η_i by a smooth path p_i. Then the loop $q_i := p_i * (c_i\eta_i) * p_i^{-1}$ is free homotopic to $c_i\eta_i$, which also represents an element $\gamma_i \in \Gamma \cong \pi_1(M, x_0)$. Since $\sum_{i=1}^k c_i[\eta_i] = 0$ in $H_1(M, \mathbb{Z}) \cong \Gamma/[\Gamma, \Gamma]$, it follows that the product $\gamma = \gamma_1 \cdots \gamma_k$ is an element in the commutator subgroup $[\pi_1(M, x_0), \pi_1(M, x_0)]$. Thus we can write

$$\gamma = [a_1, b_1] \cdots [a_g, b_g]$$

for some $a_i, b_i \in \Gamma$. We choose smooth loops α_i and β_i from x_0 that represent a_i and b_i, respectively. Fix a preimage $\tilde{x}_0 \in X$ of x_0 under the projection map $\pi\colon X \to M$. The loop $\sigma = \alpha_1 * \beta_1 * \alpha_1^{-1} * \beta_1^{-1} * \cdots * \alpha_g * \beta_g * \alpha_g^{-1} * \beta_g^{-1} * (q_1 * \cdots * q_k)^{-1}$ is nullhomotopic, thus lifts to a piecewise smooth loop on X. Therefore it bounds a smooth disk on X, that is, there exists a disk $D \subset \mathbb{R}^2$ and a piecewise smooth map $f\colon D \to X$ with $f(\partial D) = \sigma$. Moreover, by identifying D with a $(4g+3k)$–polygon with the label of $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] \bar{p}_1 \ell_1 \bar{p}_1^{-1} \cdots \bar{p}_k \ell_k \bar{p}_k^{-1}$, we can make the map f explicit by sending the edge labels $\bar{a}_i$, $\bar{b}_i$, $\bar{a}_i^{-1}$, $\bar{b}_i^{-1}$, $\bar{p}_i$, ℓ_i and $\bar{p}_i^{-1}$ to α_i, β_i, α_i^{-1}, β_i^{-1}, p_i, $c_i\eta_i$ and p_i^{-1}, respectively. Therefore, after gluing along the edge labels, f descends to a piecewise smooth map from $\Sigma_{g,k}$ (a genus g surface with k boundary components) to M, which sends the boundary components (corresponding to ℓ_i) to $c_i\eta_i$.

In general:

Definition 2.6 Let Σ be a compact oriented (not necessarily connected) surface with k boundary components. Given a collection of k loops $\{\alpha_1,\dots,\alpha_k\}$ on M, we say a map $f\colon\Sigma\to M$ is *admissible* with respect to $\{\alpha_1,\dots,\alpha_k\}$ if the following diagram commutes:

$$\begin{tikzcd} \partial\Sigma \arrow[r, "i"] \arrow[d, "\partial f"'] & \Sigma \arrow[d, "f"] \\ \bigcup_{i=1}^k \alpha_i \arrow[r, "i"] & M \end{tikzcd}$$

Note that α_i could carry multiplicities, and the orientation of the surface Σ induces an orientation on $\partial\Sigma$. In the above commutative diagram we also require ∂f to preserve the orientations. If there exist such Σ and f, then we simply say $\bigcup_{i=1}^k \alpha_i$ bounds a surface f.

By the above discussion:

Proposition 2.7 *Suppose $\{\alpha_1,\dots,\alpha_k\}$ is a collection of k smooth loops in M. If there exists a set of integers $c_1,\dots,c_k$ such that $\sum_{i=1}^k c_i[\alpha_i]=0$ in $H_1(M,\mathbb{Z})$, then there exists a piecewise smooth admissible map with respect to $\{c_1\alpha_1,\dots,c_k\alpha_k\}$, that is, $\bigcup_{i=1}^k c_i\alpha_i$ bounds a piecewise smooth surface $f\colon\Sigma\to M$.*

Given two Riemannian manifolds N and M, a smooth map $F\colon N\to M$ and a positive integer $\mathfrak{p}\le\min\{\dim(N),\dim M\}$, the $\mathfrak{p}$–Jacobian of F at a point $x\in N$ is defined to be

$$\mathrm{Jac}_{\mathfrak{p}}(F)(x)=\sup\|dF_x(e_1)\wedge dF_x(e_2)\wedge\cdots\wedge dF_x(e_{\mathfrak{p}})\|,$$

where the supremum is taken over all orthonormal $\mathfrak{p}$–frames $\{e_1,\dots,e_{\mathfrak{p}}\}$ on T_xN, and the norm is induced by the Riemannian inner product at $T_{F(x)}M$. Note that when $\mathfrak{p}=\dim N\le\dim M$, the $\mathfrak{p}$–Jacobian of F coincides with $\sqrt{\det_{g_N} F^*g_M}$.

Definition 2.8 Given a Riemannian manifold M, a smooth map $f\colon\Sigma\to M$ and a smooth region $U\subset\Sigma$, we define the area of the map on U to be

$$A(f|_U):=\int_U|\mathrm{Jac}_2 f|(x)\,dV_\Sigma,$$

where dV_Σ is the volume form on Σ with respect to some chosen Riemannian metric g_Σ, and it is clear the definition of area is independent of the choice of g_Σ. When $U=\Sigma$, we simply denote it by $A(f)$. The definition naturally extends to a piecewise smooth map. Note that, at the region where df is degenerate, $(\mathrm{Jac}_2\, f)$ vanishes, so it does not contribute to the area.

2.5 Besson–Courtois–Gallot map

In this section, we give a brief introduction to the Besson–Courtois–Gallot map and we refer the readers to [6] for a more detailed exposition. First we recall that, given any discrete subgroup $\Gamma < \mathrm{Isom}(X)$, there exists a family of positive finite Borel measures called the Patterson–Sullivan measures, which satisfy:

(1) μ_x is Γ–equivariant for all $x \in X$.
(2) $d\mu_x(\theta) = e^{-\delta B(x,\theta)} d\mu_o(\theta)$ for all $x \in X$ and $\theta \in \partial_\infty X$.

Here δ is the critical exponent of Γ, o is a basepoint on X, and $B(x,\theta)$ is the Busemann function on X with respect to o. Recall that the Busemann function B is defined by

$$B(x,\theta) = \lim_{t\to\infty} \big(d(x,\alpha_\theta(t)) - t\big),$$

where $\alpha_\theta(t)$ is the unique geodesic ray from o to θ.

We note that the Busemann function $B(x,\theta)$ is convex on X. If μ is any finite Borel measure supported on at least two points on $\partial_\infty X$, then the function

$$x \mapsto \mathcal{B}_\mu(x) := \int_{\partial_\infty X} e^{B(x,\theta)}\, d\mu(\theta)$$

is strictly convex, and one can check it tends to $+\infty$ as $x \to \partial_\infty X$. Hence we can define the barycenter $\mathrm{bar}(\mu)$ of μ to be the unique point in X where the function attains its minimum.

Now we construct the map $\widetilde{F}\colon X \to X$ given by

$$x \mapsto \mathrm{bar}(e^{-B(x,\theta)}\mu_x),$$

where $e^{-B(x,\theta)}\mu_x$ denotes the unique (up to measure zero) Borel measure which is absolutely continuous with respect to μ_x, with the corresponding Radon–Nikodym derivative $e^{-B(x,\theta)}$.

Theorem 2.9 (Besson–Courtois–Gallot [6]) *The map $\widetilde{F}\colon X \to X$ constructed above satisfies:*

(1) *$\widetilde{F}$ is Γ–equivariant, and thus descends to a map $F\colon M \to M$.*
(2) *F is smooth and homotopic to the identity.*
(3) $|\mathrm{Jac}_{\mathfrak{p}}(F)(x)| \le ((1+\delta)/\mathfrak{p})^{\mathfrak{p}}$ *for any integer $\mathfrak{p} \in [1, \dim M]$ and any $x \in M$.*

Remark 2.10 The case of $\mathfrak{p} = 1$ in (3) is not directly stated in the paper, however it is clear from the 2–form equation [6, (4.11)] that $\|dF\| \leq (1+\delta)$. According to the theorem, if $\delta \leq \mathfrak{p}-1$, then $|\mathrm{Jac}_{\mathfrak{p}}(F)| \leq 1$ hence F is a $\mathfrak{p}$–dimensional volume-decreasing map. However, in order to obtain the linear isoperimetric inequality in Section 3.1, we will need an area-decreasing map, which is assured only in the case $\delta < 1$. Thus, we will only apply the theorem to the cases $\mathfrak{p} = 1, 2$.

Notation

Henceforth X always denotes a negatively pinched Hadamard manifold with sectional curvature $-\kappa^2 \leq K \leq -1$, and $\Gamma < \mathrm{Isom}(X)$ denotes a torsion-free discrete isometry subgroup. Let $M = X/\Gamma$ be the quotient manifold, $\pi\colon X \to M$ be the quotient map, and d be the distance on M. Let δ denote the critical exponent of Γ and $C(\delta) = 4/(1-\delta)$. We use ℓ and A to denote the length and area functions, respectively. We let $\mathrm{inj}(x)$ denote the injectivity radius at a point $x \in M$, and let $\mathrm{NJ}(S)$ denote the normal injectivity radius of a submanifold $S \subset M$; see Section 3.2.

3 Geometry with small critical exponent

In this section, we investigate the geometry of the quotient manifold M under the assumption $\delta < 1$.

3.1 Linear isoperimetric type inequality

The study of the isoperimetric problem has a long and significant history. In the classical context, given a region $\Omega \subset \mathbb{R}^2$, it is natural to ask what the optimal relation between its area $A(\Omega)$ and the length of its bounding curve $\ell(\partial\Omega)$ is. It is proved that there is a quadratic relation $A(\Omega) \leq \ell(\partial\Omega)^2/4\pi$, and that equality holds if and only if Ω has a circular boundary. However, our main interest has driven us to work in a slightly different context. Let $M = X/\Gamma$ be a complete quotient manifold and $\mathcal{C} \subset M$ be a union of smooth loops which represents a trivial homology class in M. By the discussion in Section 2.4, $\mathcal{C}$ bounds an admissible surface. Among all admissible surfaces, we find one surface Σ such that $A(\Sigma)$ and $\ell(\partial\Sigma)$ satisfy a linear isoperimetric type inequality.

Definition 3.1 A family of loops $\mathcal{F} = \{\alpha_1, \dots, \alpha_k\}$ in M is *irreducible* if either

(1) $k = 1$ and α_1 represents a trivial or torsion homology class, or

(2) $\mathcal{F}$ consists of linearly dependent loops, and any nontrivial subfamily of $\mathcal{F}$ is linearly independent.

Suppose $\mathcal{F} = \{\alpha_1, \dots, \alpha_k\}$ is an irreducible family of loops. In case (1), $\mathcal{F}$ consists of one homology class $[\alpha]$, so there is a minimal positive integer c such that $c[\alpha] = 0$. In case (2), there exists a unique (up to a sign) set of integers $c_1, \dots, c_k$ such that $\gcd(c_1, \dots, c_k) = 1$ and $\sum_{i=1}^k c_i[\alpha_i] = 0$ in $H_1(M)$. Thus, there exist admissible surfaces in M with respect to $c[\alpha]$ (or $\bigcup_{i=1}^k c_i\alpha_i$) and by irreducibility they are necessarily connected. Note that $c_i\alpha_i$ denotes the c_i multiple of α_i, and c_i being negative corresponds to reversing the orientation of α_i. We call the set of integers $c_1, \dots, c_k$ (or, in case 1, c) the *associated integers* of the irreducible family.

Theorem 3.2 *Let $\mathcal{F} = \{\alpha_1, \dots, \alpha_k\}$ be any family of smooth loops in M which are linearly dependent in $H_1(M, \mathbb{Z})$ such that there are integers $c_1, \dots, c_k$ satisfying $\sum_{i=1}^k c_i[\alpha_i] = 0$ in $H_1(M)$. Suppose the critical exponent δ is less than 1. Then $\bigcup_{i=1}^k c_i\alpha_i$ bounds a smooth surface $f_0\colon \Sigma \to M$ whose area satisfies*

$$A(f_0) \le \frac{4}{1-\delta}\ell(f_0(\partial\Sigma)) = \frac{4}{1-\delta}\left(\sum_{i=1}^k |c_i|\ell(\alpha_i)\right).$$

Proof We may assume $\mathcal{F}$ is irreducible. Otherwise, we decompose $\mathcal{F}$ into irreducible subfamilies and use the additivity of area and length functions on disjoint unions. We consider the set $\mathfrak{S}$ which consists of all piecewise smooth surfaces bounded by $\bigcup_{i=1}^k c_i\alpha_i$, or more precisely, we set $\mathfrak{S}$ equal to

$$\{f\colon \Sigma \to M \mid f \text{ is piecewise smooth admissible with respect to } \{c_1\alpha_1, \dots, c_k\alpha_k\}\}.$$

By Proposition 2.7 it is nonempty. Let $A_0 = \inf\{A(f) : f \in \mathfrak{S}\}$. To avoid possible existence and regularity issues (see the following remark) of minimal surfaces in M, we can choose a piecewise smooth admissible map $f_\epsilon \in \mathfrak{S}$ such that $A(f_\epsilon) \le (1+\epsilon)A_0$ for any $\epsilon > 0$. Composing with the Besson–Courtois–Gallot map F as described in Section 2.5, we obtain a piecewise smooth admissible map $F \circ f_\epsilon$ with respect to $\bigcup_{i=1}^k c_i F(\alpha_i)$. By Theorem 2.9 we have the area estimate

$$\begin{aligned} A(F \circ f_\epsilon) &= \int_\Sigma |\mathrm{Jac}_2(F \circ f_\epsilon)|\, dV_\Sigma \le \int_\Sigma |\mathrm{Jac}_2 F| \cdot |\mathrm{Jac}_2 f_\epsilon|\, dV_\Sigma \\ &\le \left(\tfrac{1}{2}(1+\delta)\right)^2 A(f_\epsilon) \le \left(\tfrac{1}{2}(1+\delta)\right)^2 (1+\epsilon)A_0, \end{aligned}$$

and the length estimate $\ell(F(\alpha_i)) \le (1+\delta)\ell(\alpha_i)$. For each α_i, since $F(\alpha_i)$ is free homotopic to α_i, we can build an (immersed) cylindrical homotopy $\Sigma_i \subset M$ between them by taking the image of the union of two geodesic cones $\mathrm{Cone}_p(\widetilde{F}(\tilde{\alpha}))$ and $\mathrm{Cone}_{\gamma(q)}(\tilde{\alpha})$ under the projection $\pi\colon X \to M$; see Figure 1. Here $\gamma \in \Gamma$ is an element

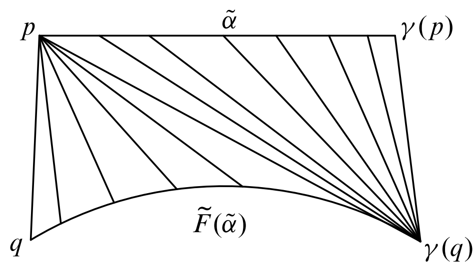

Figure 1

represented by α, $\tilde{\alpha}$ is a lift of α, and p and q as well as $\gamma(p)$ and $\gamma(q)$ are connected by geodesics. To estimate the area of Σ_i, we will need:

Lemma 3.3 *For any $p \in X$ and any smooth curve $\alpha \subset X$, the geodesic cone* $\mathrm{Cone}_p(\alpha)$ *has the area bound*

$$A(\mathrm{Cone}_p(\alpha)) \le \ell(\alpha).$$

Proof We parametrize the smooth curve by $\alpha\colon[0,1]\to X$, and write $D(s)=d(p,\alpha(s))$. The geodesic cone $\mathrm{Cone}_p(\alpha)$ can be parametrized by the smooth map

$$\Phi\colon [0,1]\times[0,D(s)]\to X, \quad (s,t)\mapsto \exp_p(t\beta(s)),$$

where $\beta(s)$ is the unit vector in the direction of the preimage of α under the exponential map, that is, the unique curve in T_pX satisfying $\exp_p(D(s)\beta(s)) = \alpha(s)$. Since $\alpha(s) = \Phi(s, D(s))$, we have

$$\alpha'(s) = \left[\frac{\partial\Phi}{\partial s} + \frac{\partial\Phi}{\partial t} D'(s)\right](s, D(s)).$$

Let $\gamma_s(t) = \Phi(s,t)$. For each s, $\gamma_s(t)$ is a unit-speed geodesic connecting p to $\alpha(s)$, so, at any point $(s,t)\in[0,1]\times[0,D(s)]$,

$$\frac{\partial\Phi}{\partial t} = \gamma_s'(t), \quad \frac{\partial\Phi}{\partial s} = J_s(t),$$

where $J_s(t)$ is the unique Jacobi field along γ_s satisfying $J_s(0)=0$ and

$$J_s(D(s)) = \frac{\partial\Phi}{\partial s}(s, D(s)) = \alpha'(s) - \gamma_s'(D(s))D'(s),$$

which is the projection of $\alpha'(s)$ orthogonal to $\gamma_s'(D(s))$. This implies that $J_s(t)$ is a normal Jacobi field and that $\partial\Phi/\partial t \perp \partial\Phi/\partial s$. Therefore

$$|\mathrm{Jac}(\Phi)| = \left\|\frac{\partial\Phi}{\partial s}\wedge\frac{\partial\Phi}{\partial t}\right\| = \left\|\frac{\partial\Phi}{\partial s}\right\|\cdot\left\|\frac{\partial\Phi}{\partial t}\right\| = \|J_s(t)\|.$$

Using [24, Proposition 2.3] and the curvature assumption $K\le -1$, we can estimate the norm of the Jacobi fields by

$$\|J_s(t)\| \le \frac{\sinh t}{\sinh(D(s))}\|J_s(D(s))\| \le \frac{\sinh t}{\sinh(D(s))}\|\alpha'(s)\|. \tag{3-1}$$

Finally we obtain the area estimate of the geodesic cone:

$$
\begin{aligned}
\text{(3-2)} \qquad A(\mathrm{Cone}_p(\alpha)) &\le \int_0^1 \int_0^{D(s)} |\mathrm{Jac}(\Phi)| \, dt \, ds \\
&\le \int_0^1 \int_0^{D(s)} \frac{\sinh t}{\sinh(D(s))} \|\alpha'(s)\| \, dt \, ds \quad \text{(by (3-1))} \\
&\le \int_0^1 \|\alpha'(s)\| \, ds \le \ell(\alpha). \qquad \square
\end{aligned}
$$

Now we continue with the proof. By the lemma above,

$$\text{(3-3)} \qquad A(\Sigma_i) \le \ell(\alpha_i) + \ell(F(\alpha_i)) \le (2+\delta)\ell(\alpha_i).$$

Here Σ_i is a piecewise immersed surface in M and we can choose any piecewise smooth parametrization $\sigma_i : S^1 \times [0,1] \to M$ to represent Σ_i. If we concatenate each σ_i with $F \circ f_\epsilon$ (glue $\bigcup_{i=1}^k c_i \Sigma_i$ onto $F \circ f_\epsilon(\Sigma)$ on M), we get a new piecewise smooth admissible surface f'_ϵ with respect to $\bigcup_{i=1}^k c_i \alpha_i$, and by assumption $A(f'_\epsilon) \ge A_0$. On the other hand, combining the above inequalities,

$$
\begin{aligned}
A_0 \le A(f'_\epsilon) &= A(F \circ f_\epsilon) + \sum_{i=1}^k |c_i| A(\Sigma_i) \\
&\le \left(\tfrac{1}{2}(1+\delta)\right)^2 (1+\epsilon) A_0 + (2+\delta)\left(\sum_{i=1}^k |c_i| \ell(\alpha_i)\right) \quad \text{(by (3-2) and (3-3)).}
\end{aligned}
$$

Thus, by letting ϵ tend to zero, we obtain

$$A_0 \le \frac{4(2+\delta)}{(1-\delta)(3+\delta)} \left(\sum_{i=1}^k |c_i| \ell(\alpha_i)\right) < \frac{4}{1-\delta} \left(\sum_{i=1}^k |c_i| \ell(\alpha_i)\right).$$

Therefore we can always choose a piecewise smooth map in $\mathfrak{S}$ whose area is arbitrarily close to A_0, and finally we can always smoothen it with an arbitrarily small increase on the area. In particular, there is a smooth admissible map f_0 with area

$$A(f_0) \le \frac{4}{1-\delta} \left(\sum_{i=1}^k |c_i| \ell(\alpha_i)\right). \qquad \square$$

Remark 3.4 The existence and regularity of minimal surfaces for a general complete manifold relate to the generalized Plateau problem, which has been studied in [37]. If there is a uniform lower bound on the injectivity radius on M, then the condition of "homogeneously regular" in [37] is satisfied; hence, the existence and regularity of the area minimizer hold. Although in Theorem 3.7 we manage to show M has bounded geometry, the proof relies on this theorem; hence, using this would fall into circular reasoning.

We do not pursue the optimal bound in the theorem above. Indeed, the linear isoperimetric constant we produce via this method will always tend to infinity as $\delta \to 1$. This stands as an obstacle in improving our main theorems as δ approaches 1.

3.2 Asymptotically uniformly bounded tubular neighborhood

Let S be a closed submanifold of M, $N(S,M) = \{(x,v) \in TM : x \in S \text{ and } v \perp T_x S\}$ be the *normal bundle* of S in M, and $N_r(S,M) = \{(x,v) \in N(S,M) : |v| < r\}$ be the *r–normal bundle* of S in M. The *normal exponential map* $\exp_S$ is defined to be the restriction of the exponential map $\exp\colon TM \to M$ to the normal bundle $N(S,M)$ of S in M. The *normal injectivity radius* $\mathrm{NJ}(S)$ is defined to be the supremum of r such that $\exp_S$ is an embedding on $N_r(S,M)$. In the case where $r \le \mathrm{NJ}(S)$, we say $\exp_S(N_r(S,M)) = \{x \in M \mid d(x,S) < r\}$ is the *r–tubular neighborhood* of S in M, and we denote it by $T_r(S)$. By convention, if the submanifold has a self-intersection, we declare that it has normal injectivity radius zero.

Lemma 3.5 *Let α be a closed geodesic in M with $\mathrm{NJ}(\alpha) = R > 0$, and let $T_R(\alpha)$ be its R–tubular neighborhood in M. If $i\colon \Sigma \to M$ is any smooth admissible map with respect to $\{k\alpha, \alpha'\}$ such that either α' is empty or α' consists of a union of smooth loops outside of $T_R(\alpha)$ (ie $d_M(\alpha',\alpha) > R$), then*

$$A(i|_{i^{-1}(T_R(\alpha))}) \ge kR\ell(\alpha).$$

Proof We choose a Riemannian metric g_0 on Σ, and let ϵ_1 and ϵ_2 be two positive real numbers recognized to be small and to be determined later. First, we perturb the pullback metric i^*g_M to be Riemannian on Σ by setting $g = i^*g_M + \epsilon_1 g_0$ and use this to estimate the area of i. It follows that, for any $\epsilon > 0$ and any region $U \subset \Sigma$,

$$
\begin{aligned}
(3\text{-}4)\qquad |\mathrm{vol}_g(U) - A(i|_U)| &= \left| \int_U 1\, dV_g - \int_U |\mathrm{Jac}_2\, i|\, dV_{g_0} \right| \\
&= \int_U \left(\sqrt{\det_{g_0}(g)} - \sqrt{\det_{g_0}(i^*g_M)} \right) dV_{g_0} \\
&\le \int_\Sigma \left(\sqrt{\det_{g_0}(g)} - \sqrt{\det_{g_0}(i^*g_M)} \right) dV_{g_0} < \epsilon,
\end{aligned}
$$

after choosing ϵ_1 small enough. Note that this follows from the continuity of the determinant function, and that the estimate is uniform on U.

Next, we choose a suitable function on Σ and use the coarea formula to estimate $\mathrm{vol}_g(U)$. Denote by $\sigma \subset \partial\Sigma$ the boundary component which sends to $k\alpha$ under i, and by $\rho_\alpha\colon M \to \mathbb{R}$ the distance function to α on M. Now we construct a function $f\colon \Sigma \to \mathbb{R}$ by setting

$$f = \rho_\alpha \circ i + \epsilon_2 \varphi,$$

where φ is a smooth function on Σ chosen so that:

(1) $\varphi(x) = 0$ on σ and $\varphi(x) > 0$ on $\Sigma \setminus \sigma$.

(2) There exists a collar neighborhood V of σ such that $d\varphi(x) \neq 0$ when $x \in V \setminus \sigma$.

For example, one can choose φ to be the distance function to σ on its local neighborhood and then extend smoothly to any positive function outside. For this choice, it is clear that $f(x) \geq 0$ and $f^{-1}(0) = \sigma$. Since M is negatively curved, there is no conjugate point for M. Thus, for any $y \in T_R(\alpha)$, there is a unique geodesic projection onto α, so ρ_α is smooth on $T_R(\alpha) \setminus \alpha$. It follows that f is smooth on $i^{-1}(T_R(\alpha)) \setminus \sigma \subset \Sigma$. We can estimate the norm of its differential with respect to the metric g by

$$\begin{aligned} \|df\| &= \|d\rho_\alpha \circ di + \epsilon_2 d\varphi\| \\ &\leq \|d\rho_\alpha\| \cdot \|di\| + \epsilon_2 \|d\varphi\| \quad \text{(note that } i \text{ is 1–Lipschitz)} \\ &< (1+\epsilon), \end{aligned} \tag{3-5}$$

after choosing ϵ_2 small enough. This uses the compactness of Σ.

Finally we estimate the area of i on $i^{-1}(T_R(\alpha))$. By the construction of f, we have $f^{-1}([0,R)) \subset i^{-1}(T_R(\alpha))$. Thus, if we set $U = f^{-1}([0,R))$, then

$$\mathrm{vol}_g(U) \leq \mathrm{vol}_g\bigl(i^{-1}(T_R(\alpha))\bigr).$$

On the other hand, by the coarea formula [15, Section 13.4], we obtain from (3-5) that

$$\mathrm{vol}_g(U) > \frac{1}{1+\epsilon} \int_U \|df\|\, dV_g = \frac{1}{1+\epsilon} \int_0^R \ell_g(f^{-1}(t))\, dt. \tag{3-6}$$

Note that in the above formula, $f^{-1}(t)$ might not be a smooth curve if t is a singular value. But by Sard's theorem, almost all values $r \in (0,R)$ are regular, in which case the level sets are unions of smooth circles on Σ, and ℓ_g denotes the total length of the circles. In particular, the above integral makes sense. Other boundary components (if any) of Σ do not intersect with $i^{-1}(T_R(\alpha))$ by assumption, so, given any regular value $t \in [0,R)$, $f^{-1}(t)$ (up to orientation) is homologous to $f^{-1}(0) = \sigma$ on Σ. Hence, taking their images in M, we obtain that $i(f^{-1}(t))$, which is also a union of smooth loops, is homologous to $k\alpha$ on M. Since they are entirely contained in $T_R(\alpha)$, $i(f^{-1}(t))$ is in fact free homotopic to $k\alpha$. More precisely, for almost all $t \in (0, R)$, if we write $i(f^{-1}(t))$ as a disjoint union of circles $\bigcup_{i=1}^m \alpha_i$, then each α_i is a smooth loop free homotopic to $k_i\alpha$ for $k_i \in \mathbb{Z}$, since the fundamental group of the R–neighborhood of α is a cyclic group generated by the loop α. (Some k_i could be zero, in which case α_i is homotopically trivial in M.) Moreover, $\sum_{i=1}^m k_i = k$. Since α is a closed geodesic, we have that $\ell\big(i(f^{-1}(t))\big) = \sum_{i=1}^m \ell(\alpha_i) \geq \sum_{i=1}^m |k_i|\ell(\alpha) \geq k\ell(\alpha)$. Note that i is 1–Lipschitz, so $\ell_g(f^{-1}(t)) \geq \ell\big(i(f^{-1}(t))\big)$. Combining the above inequality with (3-4) and (3-6),

$$A(i|_{i^{-1}(T_R(\alpha))}) > \frac{1}{1+\epsilon} kR\ell(\alpha) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the lemma follows. $\square$

Lemma 3.6 *Assume we have N cusps in M and a constant $\epsilon > 0$ small enough that $\{M_{12\epsilon}^{(i)} : 1 \leq i \leq N\}$ are disjoint components of the cuspidal part $\mathrm{cusp}_{12\epsilon}(M)$. Suppose $\iota\colon \Sigma \to M$ bounds an irreducible collection of smooth loops $\bigcup_{i=1}^N c_i\alpha_i$, where each α_i is contained in the 2ϵ–thinner part $M_{2\epsilon}^{(i)} \subset M_{12\epsilon}^{(i)}$ in each cusp component and is homologically nontrivial. Then*

$$A(\iota) \geq 4\epsilon^2.$$

Proof Since the collection is irreducible and α_1 is homologically nontrivial in its cusp component (which might be homologically trivial in M), $\iota(\Sigma)$ has to leave $M_{12\epsilon}^{(1)}$. We will only focus on the region $U_0 := \iota^{-1}(M_{12\epsilon}^{(1)})$ as shown in Figure 2. If we let $M_{4\epsilon}^{(1)} \subset M_{12\epsilon}^{(1)}$ be the 4ϵ–thinner part and set $T_1 = M_{12\epsilon}^{(1)} \setminus M_{4\epsilon}^{(1)}$, then certainly

$$A(\iota) \geq A(\iota|_{i^{-1}(T_1)}).$$

So it suffices to give a lower bound on the area restricted to the T_1 region.

Similar to the proof of Lemma 3.5, we first choose the same perturbed Riemannian metric on Σ as $g = \iota^* g_M + \epsilon_1 g_0$, and for any $\epsilon' > 0$ the estimate of (3-4) still works after choosing ϵ_1 small enough. Thus, for any $U \subset \Sigma$, we have

$$|\mathrm{vol}_g(U) - A(\iota|_U)| < \epsilon'. \tag{3-7}$$

Denote by $\sigma \subset \partial\Sigma$ the boundary component which maps to $c_1\alpha_1$ under ι, and let φ be, as before, the smooth function on Σ such that:

(1) $\varphi(x) = 0$ on σ and $\varphi(x) > 0$ on $\Sigma \setminus \sigma$.

(2) There exists a collar neighborhood V of σ such that $d\varphi(x) \neq 0$ when $x \in V \setminus \sigma$.

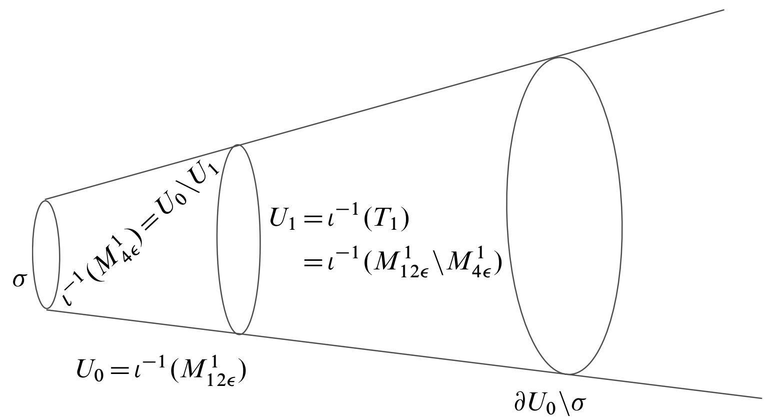

Figure 2

We choose a smooth approximation [21, Proposition 2.1] of the injectivity radius function on a neighborhood of $\iota(\Sigma)$, denoted by j, such that

(1) $j > 0$ on $\iota(\Sigma)$,

(2) j is $(1+\epsilon')$–Lipschitz, and

(3) $|j(y) - \mathrm{inj}(y)| < \epsilon$ on $\iota(\Sigma)$.

Choose a smooth bump function $0 \le \psi \le 1$ on Σ such that $\psi = 1$ on $\iota^{-1}(T_1)$ and $\psi = 0$ on σ. Since Σ is compact, there exists $\mathcal{K} > 0$ such that $\|\varphi\| < \mathcal{K}$ and $\|d\varphi\| < \mathcal{K}$. Choose a positive constant $\epsilon_2 < \min\{\epsilon, \epsilon'\}/\mathcal{K}$. Now define the smooth function $f : \Sigma \to \mathbb{R}$ by

$$f = \epsilon_2 \varphi + \psi (j \circ \iota).$$

By the construction of f, we see that $f(x) \ge 0$ on U_0 and $f^{-1}(0) = \sigma$. When restricting to $U_1 := \iota^{-1}(T_1) = \iota^{-1}(M^{(1)}_{12\epsilon} \setminus M^{(1)}_{4\epsilon})$, the norm of its differential under the metric g can be estimated by

$$\|df\|_{U_1} = \|\epsilon_2 d\varphi + dj \circ d\iota\| \le \epsilon_2 \|d\varphi\| + \|dj\| \cdot \|d\iota\| < 1 + 2\epsilon'. \tag{3-8}$$

The first inequality follows from the fact that $\psi = 1$ on $\iota^{-1}(T_1)$, and the last inequality uses that ι is 1–Lipschitz and also the choice of j and ϵ_2. Now we investigate the value of f on U_0, and apply the coarea formula to give a lower bound for the area of $\iota|_{f^{-1}([4\epsilon, 5\epsilon]) \cap U_0}$.

Claim *The subset $f^{-1}([4\epsilon, 5\epsilon]) \cap U_0$ is contained in U_1, and $f^{-1}([0, 5\epsilon]) \cap U_0$ is disjoint from $\partial U_0 \setminus \sigma$.*

Proof For any $x \in U_0 \setminus U_1 = \iota^{-1}(M^{(1)}_{4\epsilon})$,

$$f(x) = \epsilon_2 \varphi(x) + \psi(x) j(\iota(x)) < \epsilon + j(\iota(x)) < \epsilon + \mathrm{inj}(\iota(x)) + \epsilon < 4\epsilon.$$

This implies that $f^{-1}([4\epsilon, 5\epsilon]) \cap U_0$ is contained in U_1. Next, we notice that ∂U_0 consists of σ and other boundary components on which $\mathrm{inj} = 6\epsilon$. For any $x \in \partial U_0 \setminus \sigma$,

$$f(x) = \epsilon_2 \varphi(x) + \psi(x) j(\iota(x)) > j(\iota(x)) > \mathrm{inj}(\iota(x)) - \epsilon > 5\epsilon.$$

So, for any $t \in [0, 5\epsilon]$, $f^{-1}(t)$, restricted on U_0, does not intersect with ∂U_0. $\square$

As a consequence, for any regular values $t \in [0, 5\epsilon]$, $f^{-1}(t)$ is a union of smooth loops that cobounds with $f^{-1}(0) = \sigma$, and in particular is homologous to σ. Under the image of ι, it shows that $\iota(f^{-1}(t) \cap U_0)$ is homologous to $\iota(\sigma) = c_1[\alpha_1] \neq 0$. Moreover, for regular values $t \in (4\epsilon, 5\epsilon)$ and any point $y \in \iota(f^{-1}(t) \cap U_0)$, we let $x \in f^{-1}(t) \cap U_0 \subset U_1$ be any preimage of y. Then

$$\begin{aligned} \mathrm{inj}(y) = \mathrm{inj}(\iota(x)) \geq j(\iota(x)) - \epsilon = f(x) - \epsilon_2 \varphi(x) - \epsilon \quad (\psi(x) = 1 \text{ since } x \in U_1) \\ \geq t - 2\epsilon > 2\epsilon. \end{aligned}$$

In particular, $\ell\big(\iota(f^{-1}(t) \cap U_0)\big) \geq 2\,\mathrm{inj}(y) \geq 4\epsilon$. Since ι is 1–Lipschitz, we obtain $\ell_g(f^{-1}(t) \cap U_0) \geq 4\epsilon$ for any regular values $t \in (4\epsilon, 5\epsilon)$. Finally, we apply the coarea formula together with (3-7) and (3-8), and obtain

$$\begin{aligned} A(\iota) &\geq A(\iota|_{f^{-1}([4\epsilon,5\epsilon]) \cap U_0}) > \mathrm{vol}_g(f^{-1}([4\epsilon, 5\epsilon]) \cap U_0) - \epsilon' \\ &> \frac{1}{1+2\epsilon'} \int_{f^{-1}([4\epsilon,5\epsilon]) \cap U_0} \|df\| \, dV_g - \epsilon' \\ &= \frac{1}{1+2\epsilon'} \int_{4\epsilon}^{5\epsilon} \ell_g(f^{-1}(t) \cap U_0) \, dt - \epsilon' \geq \frac{1}{1+2\epsilon'} 4\epsilon^2 - \epsilon'. \end{aligned}$$

Since $\epsilon' > 0$ is arbitrary, the lemma follows. $\square$

Now we are ready to prove (1) and (2) of Theorem 1.11.

Theorem 3.7 *Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated torsion-free discrete isometry subgroup of a negatively pinched* (*normalized to $K \leq -1$*) *Hadamard manifold X. Let $N(\Gamma)$ be the number of cusps in M, and $\beta_1(\Gamma)$ be the first Betti number of M. If $\delta < 1$, then*:

(1) $N(\Gamma) \leq \beta_1(\Gamma)$.

(2) *For an integer $k > \beta_1(\Gamma) - N(\Gamma)$ and any family of closed geodesics $\{\alpha_1, \dots, \alpha_k\}$ that are mutually $2C(\delta)+1$ apart, there exists at least one closed geodesic whose normal injectivity radius is $\le C(\delta)$, where $C(\delta) = 4/(1-\delta)$.*

(3) *M has bounded geometry.*

Proof For (1), suppose to the contrary $N(\Gamma) > \beta_1(\Gamma)$, where $N(\Gamma)$ could be infinite. Choose ϵ small enough such that the cuspidal part $\mathrm{cusp}_{12\epsilon}(M)$ consists of $N(\Gamma)$ disjoint components $\bigcup_{i=1}^{N} M_{12\epsilon}^{(i)}$. For each component $M_{12\epsilon}^{(i)}$, the corresponding parabolic subgroup P_i is infinite cyclic by Proposition 2.3, so we can choose $\gamma_i \in P_i < \Gamma$ which represents a nontrivial torsion-free homology class in X/P_i (not necessarily in M). Since $N(\Gamma) > \beta_1(\Gamma)$, we have that $\{[\gamma_1], \dots, [\gamma_{N(\Gamma)}]\}$ is linearly dependent in $H_1(M)$. We can choose an irreducible subfamily containing $[\gamma_1]$ and without loss of generality we assume this to be $\{\gamma_1, \dots, \gamma_k\}$, where $k \le \beta_1(\Gamma) + 1 < \infty$. Let $c_1, \dots, c_k$ be the associated integers such that $\sum_{i=1}^{k} c_i[\gamma_i] = 0$ (with $c_1 \ne 0$). On each component $M_{12\epsilon}^{(i)}$ choose a thinner part $M_{4\epsilon}^{(i)} \subset M_{12\epsilon}^{(i)}$ and let $T_i = M_{12\epsilon}^{(i)} \setminus M_{4\epsilon}^{(i)}$. In particular, the T_i are disjoint and, for any $x \in T_i$, we have $2\epsilon \le \mathrm{inj}(x) \le 6\epsilon$. We choose a loop $\alpha_i \subset M_{2\epsilon}^{(i)}$ representing $[\gamma_i]$ such that $\ell(\alpha_i)$ is small enough that $\sum_{i=1}^{k} |c_i| \ell(\alpha_i) < \epsilon^2 / C(\delta)$; see [12, Proposition 1.1.11]. By Theorem 3.2, $\bigcup_{i=1}^{k} c_i \alpha_i$ bounds a smooth surface $\iota\colon \Sigma \to M$ whose area satisfies

$$A(\iota) \le C(\delta)\left(\sum_{i=1}^{k} |c_i| \ell(\alpha_i)\right) < \epsilon^2. \tag{3-9}$$

However, by Lemma 3.6, $A(\iota) \ge 4\epsilon^2$, which contradicts to (3-9). Hence, $N(\Gamma) \le \beta_1(\Gamma)$.

For (2), suppose there are $k = \beta_1(\Gamma) - N(\Gamma) + 1$ mutually $2C(\delta)+1$ apart simple closed geodesics $\alpha_1, \dots, \alpha_k$ whose normal injectivity radii are greater than $C(\delta)$. To illustrate the idea, we first assume M has no cusps. Then $[\alpha_1], \dots, [\alpha_k]$ are linearly dependent on $H_1(M)$. By Theorem 3.2, there exist integers $c_1, \dots, c_k$ such that $\bigcup_{i=1}^{k} c_i \alpha_i$ bounds a smooth surface $f\colon \Sigma \to M$ whose area satisfies

$$A(f) \le C(\delta)\left(\sum_{i=1}^{k} |c_i| \ell(\alpha_i)\right). \tag{3-10}$$

Let $R_i = \mathrm{NJ}(\alpha_i)$ and, by the assumption $R_i > C(\delta)$, we can pick $\epsilon > 0$ small enough that $\epsilon < \frac{1}{2}$ and $C(\delta) + \epsilon < R_i$ for all i. Denote by T_i the $(C(\delta)+\epsilon)$–tubular neighborhood of α_i, and, since $\{\alpha_i\}$ are mutually $2C(\delta)+1$ apart, $\{T_i\}$ are disjoint, and so are

$\{f^{-1}(T_i)\}$. Therefore, by Lemma 3.5,

$$A(f) \geq \sum_{i=1}^{k} A(f|_{f^{-1}(T_i)}) \geq (C(\delta)+\epsilon)\Bigl(\sum_{i=1}^{k} |c_i|\ell(\alpha_i)\Bigr). \tag{3-11}$$

This contradicts (3-10).

For the general case, pick nontrivial torsion-free homology classes $\{[\gamma_1],\dots,[\gamma_{N(\Gamma)}]\}$ on each cusp component as in (1). This together with $[\alpha_1],\dots,[\alpha_k]$ forms a linearly dependent system on $H_1(M)$. Choose an irreducible system containing $[\alpha_1]$, and without loss of generality assume it to be $\{[\gamma_1],\dots,[\gamma_{N(\Gamma)}],[\alpha_1],\dots,[\alpha_k]\}$. Thus there are integers $b_1,\dots,b_{N(\Gamma)}$ and $c_1,\dots,c_k$ such that $\sum_{i=1}^{N(\Gamma)} b_i[\gamma_i]+\sum_{j=1}^{k} c_j[\alpha_j]=0$. Now choose a loop η_i on each cusp component representing γ_i such that $\ell(\eta_i)$ is small enough that $\sum_{i=1}^{N(\Gamma)} |b_i|\ell(\eta_i)<\epsilon\bigl(\sum_{j=1}^{k}|c_j|\ell(\alpha_j)\bigr)/C(\delta)$, where ϵ is the same constant as above in the noncusp case. By Theorem 3.2, $\bigl(\bigcup_{i=1}^{N(\Gamma)} b_i\eta_i\bigr)\cup\bigl(\bigcup_{j=1}^{k} c_j\alpha_j\bigr)$ bounds a smooth surface $f:\Sigma\to M$ whose area satisfies

$$A(f)\leq C(\delta)\Bigl(\sum_{i=1}^{N(\Gamma)}|b_i|\ell(\eta_i)+\sum_{j=1}^{k}|c_j|\ell(\alpha_j)\Bigr).$$

Thus we have

$$A(f)<C(\delta)\Bigl(1+\frac{\epsilon}{C(\delta)}\Bigr)\Bigl(\sum_{j=1}^{k}|c_j|\ell(\alpha_j)\Bigr)=(C(\delta)+\epsilon)\Bigl(\sum_{j=1}^{k}|c_j|\ell(\alpha_j)\Bigr).$$

However, the area lower bound estimate in (3-11) still holds, which is a contradiction.

For (3), suppose M has unbounded geometry, that is, there exists a sequence of closed geodesics $\{\alpha_i\}$ with $\ell(\alpha_i)\to 0$. When $\ell(\alpha_i)$ is smaller than the Margulis constant, α_i determines a Margulis tube such that the length of every maximal radial arc tends to ∞ as $\ell(\alpha_i)\to 0$; see for example [16, Lemma 2.4]. In particular, the normal injectivity radius $\mathrm{NJ}(\alpha_i)$ goes to ∞. By passing to a subsequence, we can assume that the geodesics α_i are arbitrarily far apart and their normal injectivity radii are all greater than $C(\delta)$, which contradicts (2). □

Remark 3.8 The assumption $\delta<1$ is crucial in Theorem 3.7 (which also traces back to Theorem 3.2). Indeed, the main strategy of the proof is to apply an area-decreasing map on the (approximated) area-minimizing surfaces, which are bounded either by tiny loops in different cusps or by far apart closed geodesics. The existence of such a map follows from a construction of Besson, Courtois and Gallot (Theorem 2.9), where $\delta<1$ has been used to obtain that the area is decreasing.

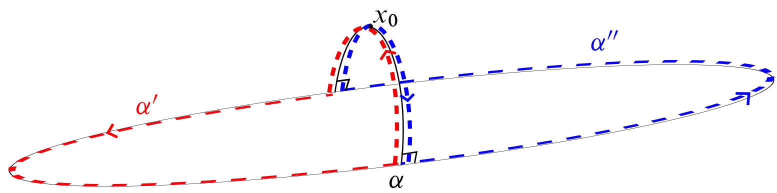

Figure 3

In general, there are examples [29] of finitely generated Kleinian groups $\Gamma < \mathrm{Isom}(\mathbb{H}^4)$ with infinitely many (rank-one) cusps, and by construction it is clear that $\delta \in [2, 3]$. Thus, for every $n \geq 4$, one can construct, via the totally geodesic embedding $\mathbb{H}^4 \to \mathbb{H}^n$, a Kleinian group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ of the same critical exponent which contains infinitely many cusps. Italiano, Martelli and Migliorini [28] constructed new examples of finitely generated Kleinian groups $\Gamma \lhd G < \mathrm{Isom}(\mathbb{H}^n)$ for $5 \leq n \leq 8$ with infinitely many cusps, where G is a lattice and $G/\Gamma \cong \mathbb{Z}$. Hence it follows that $\delta(\Gamma) = \delta(G) = n - 1$. We believe that finitely generated Kleinian groups must have finitely many cusps if $\delta < 2$.

We end this section with a corollary which turns out to be essential to our proofs of the main theorems. It is a direct consequence of Theorem 3.7(2). Roughly speaking, if $\delta < 1$ then closed geodesics asymptotically have uniformly bounded tubular neighborhoods.

Corollary 3.9 *Suppose $\delta < 1$ and M has a sequence of escaping closed geodesics. Then there exists a subsequence of escaping closed geodesics whose normal injectivity radii are $\leq C(\delta)$.*

3.3 Decomposing a closed geodesic

Suppose α is a closed geodesic in M with $\mathrm{NJ}(\alpha) \leq C(\delta)$. By definition, there exists $x_0 \in M$ achieving the normal injectivity radius such that it projects to α in two different geodesic minimizing paths. The two geodesic paths have an angle of π. Thus we can decompose α into two piecewise geodesic loops α' and α'' as shown in Figure 3. It is clear that their lengths satisfy $\ell(\alpha') + \ell(\alpha'') \leq \ell(\alpha) + 4C(\delta)$.

Equivalently, in the universal cover (as shown in Figure 4), there exists an isometry $g \in \Gamma$ and $\tilde{x}_0 \in X$ such that

$$d(\tilde{x}_0, A_\gamma) \leq C(\delta), \quad d(\tilde{x}_0, g^{-1}(A_\gamma)) \leq C(\delta),$$

where A_γ is a lift of α in X. Let $\tilde{x}$ and $\tilde{y}$ be the projections of $\tilde{x}_0$ onto $g^{-1}(A_\gamma)$ and A_γ, respectively, which will realize the shortest distance between $g^{-1}(A_\gamma)$ and A_γ

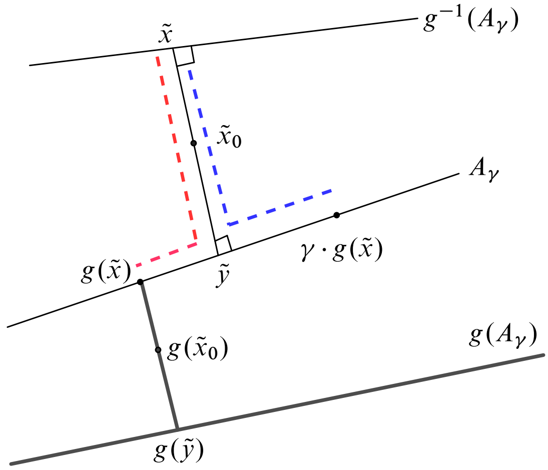

Figure 4

(so $\ell(\tilde{x}\,\tilde{y}) \le 2C(\delta)$). Under the projection map $\pi\colon X \to M$, the consecutive geodesic segments connecting $g(\tilde{x})$, $\tilde{y}$ and $\tilde{x}$ maps to α' and the one connecting $\tilde{x}$, $\tilde{y}$ and $\gamma \cdot g(\tilde{x})$ maps to α'', where γ translates along A_γ and corresponds to α. From Figure 3, we see that α' represents the isometry g and α'' represents the isometry $\gamma \cdot g$; these are nontrivial elements in Γ. We claim that the group $\langle g, \gamma \cdot g\rangle$ is nonelementary. Otherwise, $\langle g, \gamma \cdot g\rangle$ is parabolic or loxodromic. If $\langle g, \gamma \cdot g\rangle$ is parabolic, then both g and $\gamma \cdot g$ are parabolic and they have the same fixed point, which implies that γ has the same fixed point as the one of the parabolic isometry g, which contradicts the assumption that Γ is discrete by [11, Lemma 3.1.2]. (The proof of Lemma 3.1.2 can be applied to the case of negatively pinched Hadamard manifolds directly.) If $\langle g, \gamma \cdot g\rangle$ is loxodromic, then g and $\gamma \cdot g$ are both loxodromic and they preserve an axis setwise, which means that γ will preserve the same axis as g. However, note that γ preserves the axis A_γ, which is not preserved by g.

It is possible that x_0 projects to the same point on α, in which case α' is the entire transverse geodesic loop, and α'' is the concatenation $\alpha'^{-1} * \alpha$. It is also possible that α may have a transverse self-intersection, in which case the above decomposition coincides with the obvious separation at the self-intersection. Note that nontransverse self-intersection of a closed geodesic α can only occur when α is a multiple of some primitive closed geodesic $\bar{\alpha}$, in which case the above decomposition on α can essentially be treated on $\bar{\alpha}$. We remark that in all the abovementioned "exceptional" cases, the decomposition as described always exists.

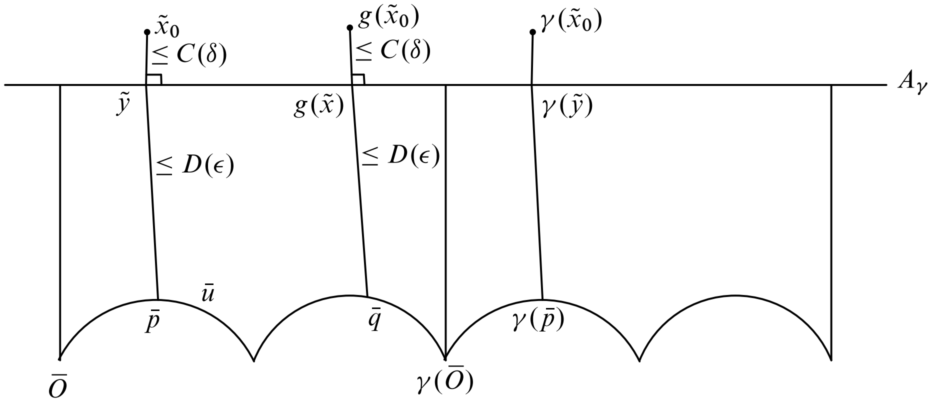

Figure 5

We can extend the above decomposition to a piecewise geodesic loop:

Lemma 3.10 *Let $u \subset M$ be a piecewise geodesic loop consisting of at most two geodesics, and let $\alpha \subset M$ be the closed geodesic free homotopic to u with $\mathrm{NJ}(\alpha) \leq C(\delta)$ and $\ell(\alpha) \geq \epsilon$. Then there exist points $p, q \in u$ (which could be the same) and a geodesic segment ω connecting p and q whose length is bounded above by $C_0 = 2C(\delta) + 2D(\epsilon)$. Here $D(\epsilon)$ is the constant in Proposition 2.1. Moreover, the two piecewise geodesic loops under the decomposition shown in Figure 3 are homotopically nontrivial.*

Proof Write u as the union of two geodesic segments in M which start and end at O. Let $\bar{u}$ be a lift of u in X consisting of two geodesic segments from the lift $\overline{O}$ to $\gamma(\overline{O})$ as in Figure 5, where $\gamma \in \Gamma$ is represented by u. We denote the axis of γ by A_γ, which is a lift of α. Since $\mathrm{NJ}(\alpha) \leq C(\delta)$, by the discussion above there exists a point $\tilde{x}_0 \in X$ and a nontrivial element $g \in \Gamma$ with $g \neq \gamma$ such that $\tilde{x}_0$ and $g(\tilde{x}_0)$ project onto A_γ at two points $\tilde{y}$ and $g(\tilde{x})$ (which could be the same point) satisfying $d(\tilde{x}_0, \tilde{y}) \leq C(\delta)$ and $d(g(\tilde{x}_0), g(\tilde{x})) \leq C(\delta)$; see Figure 4.

By Proposition 2.1 there exist $\bar{p}, \bar{q} \in \bar{u}$ such that $d(\tilde{y}, \bar{p}) \leq D(\epsilon)$ and $d(g(\tilde{x}), \bar{q}) \leq D(\epsilon)$. Thus, the piecewise geodesic consecutively connecting $\bar{p}$, $\tilde{y}$ and $\tilde{x}_0$ together with the one connecting $g(\tilde{x}_0)$, $g(\tilde{x})$ and $\bar{q}$ projects to a piecewise geodesic path connecting $\pi(\bar{p}) = p$ and $\pi(\bar{q}) = q \in M$ with total length $\leq 2C(\delta) + 2D(\epsilon)$. Finally, there is a unique geodesic segment ω connecting p and q which is homotopic to this piecewise geodesic path and it is clear that $\ell(\omega) \leq 2C(\delta) + 2D(\epsilon)$.

The geodesic segment ω divides the piecewise geodesic loop u into two parts, u_1 and u_2. The concatenation of u_i with the geodesic segment ω gives two piecewise

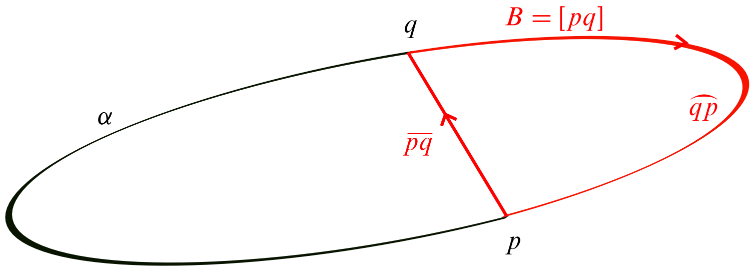

Figure 6

geodesic loops under this decomposition, where $i = 1, 2$. If the two piecewise geodesic loops are homotopically trivial, then $\tilde{x}_0 = g(\tilde{x}_0) = \gamma(\tilde{x}_0)$. By our construction, $g \neq \gamma$ and $g \neq 1$. Hence, they are homotopically nontrivial. $\square$

3.4 Injectivity radius and convex cocompactness

In this section, we prove (3) of Theorem 1.11. We start by introducing the definition of a *bow* which will be used later in the proof.

Definition 3.11 Given a closed geodesic α, we say $B = \overline{pq} * \widehat{qp}$ is a *bow* on α if:

(1) B consists of two edges $\overline{pq}$ and $\widehat{qp}$, where p and q are two distinct points on α.

(2) $\overline{pq}$ is a minimizing geodesic connecting p to q on M, which might not lie on α.

(3) $\widehat{qp}$ is a geodesic segment on α connecting q to p, which might not be length minimizing; see Figure 6.

We say a bow $B = \overline{pq} * \widehat{qp}$ is C*–thin* if $d(p,q) \leq C$, and we say B is *nontrivial* if the loop $\overline{pq} * \widehat{qp}$ of B is homotopically nontrivial in M. The *length* of a bow $B = \overline{pq} * \widehat{qp}$ is the length of the loop $\overline{pq} * \widehat{qp}$.

Lemma 3.12 *Suppose that $\delta < 1$ and the injectivity radius on M is bounded by some constant $\frac{1}{2}\epsilon_0 > 0$ from below. Then there are no closed geodesics α in M satisfying*:

(1) *α has normal injectivity radius at most $C(\delta)$.*

(2) *All points of α have injectivity radii greater than $4C_0 + 1$, where C_0 is the constant in Lemma 3.10.*

Proof Suppose that there exists such a closed geodesic α in M. We consider the set $\mathcal{B} = \mathcal{B}(\alpha, 2C_0)$ that consists of all nontrivial $2C_0$–thin bows on α. The set is never

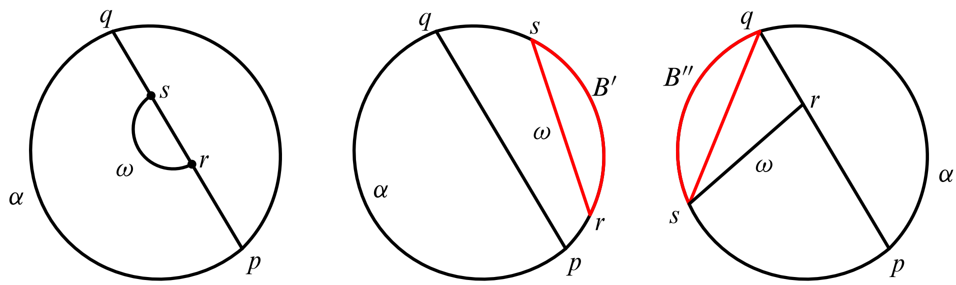

Figure 7

empty. Indeed, choose $p, q \in \alpha$ sufficiently close and choose $\widehat{qp}$ the longer segment on α connecting q to p such that $\ell(\overline{pq}) < \ell(\widehat{qp})$ and $\ell(\overline{pq}) \le 2C_0$. This gives a nontrivial $2C_0$–thin bow on α. Let $t = \inf\{\ell(B) : B \in \mathcal{B}\}$. We choose $B = \overline{pq} * \widehat{qp} \in \mathcal{B}$ to be a bow with length $\le t+1$. Since B is a 2–piecewise geodesic path, by Lemma 3.10 there exist $r, s \in B$ and a geodesic segment $\omega \subset M$ connecting r and s such that

$$\ell(\overline{rs}) = \ell(\omega) \le C_0 \tag{3-12}$$

and that ω splits B nontrivially. Although Lemma 3.10 by itself does not assure that ω is length minimizing, and r and s might even be the same point, we claim this is not the case. Indeed, since $\ell(\overline{pq}) \le 2C_0$, r must be contained in the C_0–neighborhood of α. By the assumption on the injectivity radius, all the points on α have injectivity radius $> 4C_0 + 1$. Since the injectivity radius function is 1–Lipschitz, $\operatorname{inj}(r) > 3C_0 + 1$. This implies that any geodesic segment emanating from r whose length is at most $3C_0 + 1$ must be uniquely length minimizing. In particular, ω is uniquely length minimizing and $r \ne s$.

Based on the positions of r and s, we discuss three cases:

(1) r and s are both on $\overline{pq}$.

(2) r and s are both on $\widehat{qp}$.

(3) $r \in \overline{pq}$ and $s \in \widehat{qp}$.

Observe that (1) is impossible since both ω and $\overline{pq}$ are uniquely length minimizing, so ω has to be entirely contained in $\overline{pq}$, which contradicts the fact that ω splits B nontrivially. Case (2) is also impossible. To see this, we assume without loss of generality that q, s, r and p are in cyclic order in $\widehat{qp}$, as in Figure 7, and r and s cut $\widehat{qp}$ into three geodesic segments, denoted by $\widehat{qs}$, $\widehat{sr}$ and $\widehat{rp}$. By assumption, the bow

$B' = \overline{rs} * \widehat{sr}$ is a nontrivial C_0–thin (of course also $2C_0$–thin) bow on α. So by the choice of B we have $\ell(B') + 1 \geq t + 1 \geq \ell(B)$, hence

$$\ell(\overline{rs}) + 1 \geq \ell(\widehat{rp}) + \ell(\overline{pq}) + \ell(\widehat{qs}). \tag{3-13}$$

Since ω splits B nontrivially, we have obtained a homotopically nontrivial piecewise geodesic loop $\eta = \overline{rs} * \widehat{sq} * \overline{qp} * \widehat{pr}$ whose total length can be estimated as

$$\begin{aligned} \ell(\eta) &= \ell(\overline{rs}) + \ell(\widehat{sq}) + \ell(\overline{qp}) + \ell(\widehat{pr}) \leq 2\ell(\overline{rs}) + 1 \quad &&\text{(by (3-13))} \\ &\leq 2C_0 + 1 &&\text{(by (3-12)).} \end{aligned}$$

This contradicts the assumption on injectivity radius.

For case (3), note that $\ell(\overline{pq}) \leq 2C_0$, so r is C_0 close to either p or q, and without loss of generality we assume it is closer to q. Therefore by the triangle inequality, $d(q,s) \leq \ell(\overline{rq}) + \ell(\omega) \leq 2C_0$. Now we consider the bow $B'' = \overline{sq} * \widehat{qs}$, where $\widehat{qs}$ is the geodesic segment on α. The bow is nontrivial. Otherwise, $\overline{sq}$ coincides with $\widehat{qs}$, which indicates that $\ell(\widehat{qs}) \leq 2C_0$. Then we have a piecewise geodesic loop $\overline{sr} * \overline{rq} * \widehat{qs}$ with length $\leq 4C_0$. By the injectivity radius assumption it must represent a trivial element, which contradicts the fact that ω cuts B_i nontrivially. Hence, $B'' \in \mathcal{B}$. By the choice of B, we have $\ell(B'') + 1 \geq t + 1 \geq \ell(B)$, hence $\ell(\overline{sq}) + 1 \geq \ell(\widehat{sp}) + \ell(\overline{pq})$. So we have obtained a piecewise geodesic loop $\eta' = \overline{qs} * \widehat{sp} * \overline{pq}$ whose total length satisfies

$$\ell(\eta') = \ell(\overline{qs}) + \ell(\widehat{sp}) + \ell(\overline{pq}) \leq 2\ell(\overline{qs}) + 1 \leq 4C_0 + 1.$$

So η' must be homotopically trivial according to the injectivity radius assumption. Since ω splits B_i nontrivially, the piecewise geodesic loop $\overline{rs} * \widehat{sp} * \overline{pr}$ is homotopically nontrivial, and therefore, differing by an η', the geodesic triangle $\eta'' = \overline{rs} * \overline{sq} * \overline{qr}$ is also homotopically nontrivial. On the other hand

$$\ell(\eta'') = \ell(\overline{rs}) + \ell(\overline{sq}) + \ell(\overline{qr}) \leq 4C_0,$$

which contradicts the injectivity radius assumption. $\square$

The following is a restatement of Theorem 1.11(3), which gives an alternative geometric characterization of convex compactness under the assumption that $\delta < 1$.

Theorem 3.13 *If $\delta < 1$, then Γ is convex cocompact if and only if the injectivity radius function* $\mathrm{inj}\colon M \to \mathbb{R}$ *is proper.*

Proof We start with the "only if" part, which does not need the condition $\delta < 1$. Since Γ is convex cocompact, it consists of only loxodromic isometries. Note that all the closed geodesics are in the compact convex core since their lifts in X are in $\mathrm{Hull}(\Lambda(\Gamma))$. Therefore, the length of all closed geodesics in M is uniformly bounded from below. Otherwise, there is an escaping sequence of closed geodesics (whose length tends to 0) inside the convex core, contradicting compactness. Suppose the injectivity radius function is not proper. Then there exists an escaping sequence of points $x_i \in M$ whose injectivity radii are uniformly bounded by some constant R. At each point x_i, we choose a geodesic loop w_i whose length satisfies $\ell(w_i) = 2\,\mathrm{inj}(x_i) \le 2R$. By Proposition 2.1, the closed geodesic free homotopic to w_i is within a D–neighborhood of w_i for some constant D. Hence we get an escaping sequence of closed geodesics in the convex core of M, which contradicts compactness.

To show the "if" part, we first note that properness of the injectivity radius function automatically implies that M has no cusps, and there is a uniform lower bound ϵ_0 on the length of closed geodesics in M. Suppose that Γ is not convex cocompact, ie geometrically infinite. By Theorem 2.4 there is an escaping sequence of closed geodesics $\{\alpha_i\} \subset M$. By Corollary 3.9, there is a subsequence of closed geodesics whose normal injectivity radii are all at most $C(\delta)$. For convenience, we still denote it by $\{\alpha_i\}$. Now we fix a constant $C_0 = 2C(\delta) + 2D(\epsilon_0)$ as in Lemma 3.10. Since the injectivity radius function is proper and the sequence $\{\alpha_i\}$ is escaping, all points on α_i have injectivity radii greater than $4C_0 + 1$ when i is sufficiently large. Hence, there exists a closed geodesic in M whose normal injectivity radius is at most $C(\delta)$, and where all points on the geodesic have injectivity radii greater than $4C_0 + 1$, contradicting Lemma 3.12. Therefore, Γ is convex cocompact. □

4 Proofs of the main theorems

Theorem 4.1 *For each n and κ there exists a positive constant $D(n,\kappa) < \frac{1}{2}$ such that, for any finitely generated torsion-free discrete isometry subgroup $\Gamma < \mathrm{Isom}\, X$, if either*

(1) $\delta < D(n,\kappa)$, *or*

(2) Γ *is free and* $\delta < \frac{1}{16}$,

then the injectivity radius function on M is proper.

Proof Since $D(n,\kappa) < \frac{1}{2}$, there are no parabolic isometries in Γ by Proposition 2.3. Suppose that the injectivity radius function is not proper. By the same argument as in the first paragraph of the proof of Theorem 3.13, there exists an escaping sequence of closed geodesics $\{\alpha_i\}$ of uniformly bounded length in M. Let $\mathcal{G}^\infty$ be the set of all escaping sequences of closed geodesics in M, and let $t = \inf\{\liminf_{i\to\infty} \ell(\alpha_i) : \{\alpha_i\} \in \mathcal{G}^\infty\}$. From the previous discussion, we see that $t < \infty$. On the other hand, M has bounded geometry according to Theorem 3.7, so $t > 0$.

We claim that $t \le 4C(\delta)$. Suppose $t > 4C(\delta)$. Then there exists an escaping sequence of closed geodesics α_i with $\liminf_{i\to\infty} \ell(\alpha_i) = s \in (t, t+\epsilon_0)$, where ϵ_0 is a fixed positive number smaller than $\frac{1}{2}(t-4C(\delta))$. By Corollary 3.9 there exists a subsequence, which by abuse of notation we still denote by $\{\alpha_i\}$, such that $\lim_{i\to\infty} \ell(\alpha_i) = s$ and $\mathrm{NJ}(\alpha_i) \le C(\delta)$ for all i. Without loss of generality, we assume $\ell(\alpha_i) \in (t, t+\epsilon_0)$ for all i. By Section 3.3, each α_i can be decomposed into two nontrivial loops α_i' and α_i'' such that $\ell(\alpha_i') + \ell(\alpha_i'') \le \ell(\alpha_i) + 4C(\delta)$. So the shorter one, which we assume to be α_i', has length $\le \frac{1}{2}\ell(\alpha_i) + 2C(\delta)$, and it represents a nontrivial isometry in Γ. There is a closed geodesic ν_i free homotopic to α_i' with length $\le \frac{1}{2}\ell(\alpha_i) + 2C(\delta)$. Since M has bounded geometry, ν_i is inside a uniformly bounded neighborhood of α_i' by Proposition 2.1. Thus we have found another escaping sequence of closed geodesics ν_i which satisfies

$$\begin{aligned}\ell(\nu_i) \le \ell(\alpha_i') \le \tfrac{1}{2}\ell(\alpha_i)+2C(\delta) \le \tfrac{1}{2}(t+\epsilon_0)+2C(\delta) &< \tfrac{1}{2}\big(t+\tfrac{1}{2}(t-4C(\delta))\big)+2C(\delta)\\ &= \tfrac{3}{4}t+C(\delta).\end{aligned}$$

The last two inequalities follow from the choices of $\{\alpha_i\}$ and ϵ_0. Hence

$$\liminf_{i\to\infty} \ell(\nu_i) \le \tfrac{3}{4}t + C(\delta) < t.$$

This contradicts the choice of t, therefore $t \le 4C(\delta)$.

This means that, for any $\epsilon > 0$, there exists a primitive closed geodesic, denoted by α_0, such that $\ell(\alpha_0) \le t + \epsilon \le 4C(\delta) + \epsilon$ and $\mathrm{NJ}(\alpha_0) \le C(\delta)$. By Section 3.3, α_0 can be decomposed to two nontrivial loops α_0' and α_0'', and again we assume α_0' is the shorter one. So $\ell(\alpha_0') < 4C(\delta) + \epsilon$. Let x_0 be a common point of α_0 and α_0'. Note that α_0 and α_0' represent two loxodromic elements $\gamma_0, \gamma_0' \in \pi_1(M, x_0) \cong \Gamma$, which generate a nonelementary subgroup $\langle \gamma_0, \gamma_0' \rangle = \Gamma_0 < \Gamma$.

Recall that, for any group G with finite generating set S, its entropy is defined as

$$h(G,S) = \lim_{N\to\infty} \frac{\ln |\{g \in G : d_S(1,g) \le N\}|}{N},$$

where d_S is the Cayley graph metric determined by S.

Since Γ is free in (2), Γ_0 must be a free subgroup isomorphic to F_2. So $h(\Gamma_0, S) = \ln 3$ for $S = \{\gamma_0, \gamma_0'\}$. Note that the lengths of geodesic loops from x_0 representing γ_0 and

γ_0' are both bounded by $4C(\delta)+\epsilon$. We conclude that the orbit map $\gamma \mapsto \gamma \cdot x_0$ gives a $(4C(\delta)+\epsilon)$–Lipschitz injection from (Γ_0, d_S) to (X, d). This implies

$$\delta = \delta(\Gamma) \geq \delta(\Gamma_0) \geq \frac{1}{4C(\delta)+\epsilon} h(\Gamma_0, S) = \frac{\ln 3}{4C(\delta)+\epsilon},$$

where the last inequality follows from (2-1). By choosing ϵ small enough and assuming $\delta < \frac{1}{16}$, one can check that the above inequality cannot hold. The contradiction implies that the injectivity radius is proper.

If we are in case (1), then according to [20, Theorem 1.1] there is a free subgroup $\Gamma_0' < \Gamma_0$ generated by two elements g_0 and g_0' whose word lengths measured in (Γ_0, S) are bounded above by some universal constant $C(n,\kappa)$ depending only on the dimension and lower sectional curvature of X. Write $S_0 = \{g_0, g_0'\}$. Therefore, the orbit map $(\Gamma_0', d_{S_0}) \to (X, d)$ through the inclusion $\Gamma_0' \to \Gamma_0$ is a $(4C(\delta)+\epsilon)C(n,\kappa)$–Lipschitz injection. This implies

$$\delta \geq \delta(\Gamma_0) \geq \frac{1}{(4C(\delta)+\epsilon)C(n,\kappa)} h(\Gamma_0', S_0) = \frac{\ln 3}{(4C(\delta)+\epsilon)C(n,\kappa)}.$$

Thus, there exists a constant $D(n,\kappa)$ which is smaller than $\frac{1}{2}$ such that, by choosing ϵ small enough and assuming $\delta < D(n,\kappa)$, the above inequality fails. The contradiction again implies that the injectivity radius is proper. □

Remark 4.2 For case (1), instead of passing to a rank-2 free subgroup, one can also apply the result of [7] to give a uniform lower bound on the entropy of Γ_0.

Now we can finish the proofs of our main results from the introduction.

Proof of Theorems 1.2 and 1.8 Theorem 1.2 follows from Theorems 3.13 and 4.1. For the proof of Theorem 1.8, there exists a finite-index free subgroup $\Gamma' < \Gamma$ such that $\delta(\Gamma') = \delta(\Gamma) < \frac{1}{16}$. Then Γ' is convex cocompact by Theorems 3.13 and 4.1, which implies that Γ is also convex cocompact. □

Proof of Corollary 1.6 Let $D(n)$ be the constant $D(n,\kappa)$ in Theorem 1.2 with $\kappa = 1$. Suppose that $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is a finitely generated discrete isometry subgroup with $\delta(\Gamma) < D(n) < \frac{1}{2}$. By the Selberg lemma, there exists a finite-index torsion-free subgroup $\Gamma' < \Gamma$ with $\delta(\Gamma') = \delta(\Gamma) < D(n) < \frac{1}{2}$. By Theorem 1.2, Γ' is convex cocompact. Hence, the Hausdorff dimension of the limit set equals $\delta(\Gamma')$ [9], which is smaller than 1. Note that since the limit set is a second-countable compact metric space (hence also locally compact and Hausdorff) its topological dimension equals the small inductive dimension, which is bounded above by its Hausdorff dimension, which hence must be zero. This implies that the limit set is totally disconnected (and is in fact a Cantor set). Then we apply a result of Kulkarni [35, Theorem 6.11], which states that if the limit set of a finitely generated Kleinian group is totally disconnected, then the group splits as a free amalgamation of a free group with virtually abelian groups corresponding to the parabolic subgroups. Since the condition $\delta(\Gamma') < 1$ excludes all free abelian factors of higher rank, we conclude Γ' must be free. Therefore, Γ is virtually free. □

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