AGT relations for sheaves on surfaces

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We consider a natural generalization of the Carlsson–Okounkov Ext operator on the K–theory groups of the moduli spaces of stable sheaves on a smooth projective surface. We compute the commutation relations between the Ext operator and the action of the deformed W–algebra on K–theory, which was developed by the author in previous work. The conclusion is that the Ext operator is closely related to a vertex operator, thus giving a mathematical incarnation of the Alday–Gaiotto–Tachikawa correspondence for a general algebraic surface.

1 Introduction

1.1 Fix a smooth projective surface S over an algebraically closed field of characteristic zero (henceforth denoted by $\mathbb{C}$), and invariants $(r, c_1) \in \mathbb{N} \times H^2(S, \mathbb{Z})$. An important object in algebraic geometry is the moduli space

$$(1\text{-}1) \qquad \mathcal{M} = \bigsqcup_{c_2 = \lceil ((r-1)/2r)c_1^2 \rceil}^{\infty} \mathcal{M}_{c_2}$$

of H–stable sheaves on S with invariants (r, c_1, c_2) for any $c_2 \in \mathbb{Z}$. The reason that c_2 is bounded below is called Bogomolov's inequality, which states that there are no H–stable sheaves if $c_2 < ((r-1)/2r)c_1^2$. We make the same assumptions as in our earlier work [15; 17; 16]:

- **Assumption A** $\gcd(r, c_1 \cdot H) = 1$.

- **Assumption S** Either $\omega_S \cong \mathcal{O}_S$, or $c_1(\omega_S) \cdot H < 0$.

Assumption A implies that $\mathcal{M}$ is proper and there exists a universal sheaf[1]

(1-2)
$$
\begin{array}{c}
\mathcal{U} \\
\vdots \\
\downarrow \\
\mathcal{M} \times S
\end{array}
$$

Assumption S implies that $\mathcal{M}$ is smooth.

1.2 The enumerative geometry of the moduli space of stable sheaves is quite rich, as evidenced by Donaldson invariants arising as certain integrals of cohomology classes on $\mathcal{M}$. In the present paper, we will consider algebraic K–theory instead of cohomology, a process which accounts for the adjective "deformed" in the representation-theoretic structures explained in Section 1.6. Explicitly, we consider the following algebraic K–theory groups with $\mathbb{Q}$ coefficients:

(1-3)
$$
K_{\mathcal{M}} = \bigoplus_{c_2 = \lceil ((r-1)/2r)c_1^2 \rceil}^{\infty} K_0(\mathcal{M}_{c_2}) \otimes_{\mathbb{Z}} \mathbb{Q}.
$$

Let $m \in \operatorname{Pic}(S)$, and consider two copies $\mathcal{M}$ and $\mathcal{M}'$ of the moduli space (1-1). These two copies may be defined with respect to a different c_1 and stability condition H, but we assume that the rank r of the sheaves parametrized by $\mathcal{M}$ and $\mathcal{M}'$ is the same. In this paper, we will mostly be concerned with the virtual vector bundle

(1-4)
$$
\begin{array}{c}
\mathcal{E}_m \\
\vert \\
\downarrow \\
\mathcal{M} \times \mathcal{M}'
\end{array}
$$

$$
\mathcal{M} \underset{\pi_1}{\swarrow} \qquad \underset{\pi_2}{\searrow} \mathcal{M}'
$$

(a straightforward generalization of the construction of Carlsson and Okounkov [7]) given by

(1-5)
$$
\mathcal{E}_m = \mathrm{R}\Gamma(m) - \mathrm{R}\pi_*(\mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{U} \otimes m)).
$$

The $\mathrm{R}\mathscr{H}om$ is computed on $\mathcal{M} \times \mathcal{M}' \times S$: the notation $\mathcal{U}, \mathcal{U}'$ and m stands for the pullback of the universal sheaf from $\mathcal{M} \times S$ and $\mathcal{M}' \times S$, respectively, as well as the pullback of the line bundle m from S. Similarly, $\pi \colon \mathcal{M} \times \mathcal{M}' \times S \to \mathcal{M} \times \mathcal{M}'$ is the standard projection, so $\mathcal{E}_m$ is a complex of coherent sheaves on $\mathcal{M} \times \mathcal{M}'$.

---

[1]We require the universal sheaves on the various connected components of $\mathcal{M}$ to be constructed as in [15, Section 5.9], which will ensure that they lift in a compatible way to the moduli spaces $\mathfrak{Z}_1$ and $\mathfrak{Z}_2^\bullet$ of Section 2.4.

1.3 Any Schur functor applied to $\mathcal{E}_m$ gives rise to a K–theory class on $\mathcal{M} \times \mathcal{M}'$, which in turn induces an operator from $K_{\mathcal{M}'}$ to $K_{\mathcal{M}}$ via the usual formalism of correspondences. With this in mind, let us consider the following immediate generalization of Carlsson and Okounkov [7, Equation (3)] and Carlsson, Nekrasov and Okounkov [6, Equation (19)].

Definition 1.4 Consider the so-called *Ext operator* $K_{\mathcal{M}'} \xrightarrow{A_m} K_{\mathcal{M}}$ given by

$$(1\text{-}6) \qquad A_m = \pi_{1*}(\wedge^\bullet \mathcal{E}_m \cdot \pi_2^*),$$

with π_1 and π_2 as in (1-4). The pushforward and pullback maps are well-defined due to the properness and smoothness of $\mathcal{M}$ and $\mathcal{M}'$, respectively.

In (1-6), the symbol $\wedge^\bullet \mathcal{E}_m$ denotes the total exterior power of $\mathcal{E}_m$; as $\mathcal{E}_m$ is in general a complex of coherent sheaves, some explanation is in order. Specifically, consider

$$(1\text{-}7) \qquad \wedge^\bullet\left(\frac{\mathcal{E}_m}{t}\right) = \sum_{k \geq 0}(-t)^{-k}[\wedge^k \mathcal{E}_m] \in K_{\mathcal{M} \times \mathcal{M}'}[\![t^{-1}]\!],$$

where the right-hand side is the power series expansion of a rational function in t; see Section 3.1 for details. Then the quantity $\wedge^\bullet \mathcal{E}_m$ in (1-6) denotes the $t = 1$ specialization of (1-7). If this specialization is not well-defined, then all the results in Sections 1.6 and 1.9 hold with m replaced by m/t, and with all formulas being equalities of rational functions in t; see Section 3.1 for details.

Example 1.5 Let $\mathcal{M} = \mathcal{M}'$ and $m = \mathcal{O}_S/t$, with t being a formal parameter. Then Assumption S implies that $\mathcal{E}_{\mathcal{O}_S}$ is locally free (up to a constant sheaf) and that

$$\mathcal{E}_{\mathcal{O}_S}|_\Delta \cong \operatorname{Tan}_{\mathcal{M}},$$

where $\Delta \subset \mathcal{M} \times \mathcal{M}'$ denotes the diagonal. By a simple computation involving correspondences, the isomorphism above implies that

$$\operatorname{Tr}(A_{\mathcal{O}_S/t}) = \sum_{k \geq 0}(-t)^{-k}\chi(\mathcal{M}, \wedge^k \operatorname{Tan}_{\mathcal{M}})$$

(up to a constant rational function in t). The right-hand side is the χ_t–genus of the moduli space $\mathcal{M}$, as considered for example in Göttsche and Kool [10].

1.6 In the present paper, we will seek to determine the Ext operator A_m using the representation-theoretic properties of the vector space $K_{\mathcal{M}}$. To this end, we need

to make $K_{\mathcal{M}}$ into a representation of an appropriate algebra which is "big" enough, in order to constrain the operator A_m as much as possible. A candidate for such an algebra is $\mathcal{A}_r$, namely a particular integral form of the deformed W–algebra of type $\mathfrak{gl}_r$ (initially defined in Awata, Kubo, Odake and Shiraishi [1] and Feigin and Frenkel [8]). The main purpose of our work in [15; 17; 16] is to construct an action $\mathcal{A}_r \curvearrowright K_{\mathcal{M}}$; we will recall the construction in Section 2, but let us summarize the main idea here. In [17, Section 6.7], we construct certain geometric operators

$$(1\text{-}8) \qquad K_{\mathcal{M}} \xrightarrow{W_{n,k}} K_{\mathcal{M}\times S} \quad \text{for all } (n,k) \in \mathbb{Z} \times \mathbb{N}.$$

Under Assumptions A and S, we show in [16, Theorem 4.15] that the operators $W_{n,k}$ satisfy the quadratic commutation relations developed in [1] and [8]; see (2-28) for the specific form of these relations in our language. In [17, Theorem 6.9], we further show that $W_{n,k} = 0$ for all $n \in \mathbb{Z}$ and $k > r$, which tautologically implies that the operators (1-8) yield an action $\mathcal{A}_r \curvearrowright K_{\mathcal{M}}$. Write

$$(1\text{-}9) \qquad q = [\omega_S] \in K_S := K_0(S) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Given two copies $\mathcal{M}$ and $\mathcal{M}'$ of the moduli space of stable sheaves, each with its own universal sheaf $\mathcal{U}$ and $\mathcal{U}'$, respectively, we may write

$$(1\text{-}10) \qquad u = \det\mathcal{U} \quad \text{and} \quad u' = \det\mathcal{U}'$$

for the determinant line bundles on $\mathcal{M} \times S$ and $\mathcal{M}' \times S$, respectively. We set

$$(1\text{-}11) \qquad \gamma = \frac{m^r u}{q^r u'},$$

which is the class of a line bundle on $\mathcal{M} \times \mathcal{M}' \times S$ (it is implicit that m and q are pulled back from S). Our main result, which will be proved in Section 3, is:

Theorem 1.7 *We have the following interaction between the Ext operator* (1-6) *and the generators* (1-8) *of the W–algebra action*:

$$(1\text{-}12) \qquad A_m W_k(x)(1-x) = m^k W_k(x\gamma) A_m \left(1 - \frac{x}{q^k}\right),$$

where $W_k(x) = \sum_{n\in\mathbb{Z}} W_{n,k}/x^n$. The series coefficients of the two sides of (1-12) *are maps $K_{\mathcal{M}'} \to K_{\mathcal{M}\times S}$ which arise from certain correspondences in $K_{\mathcal{M}\times\mathcal{M}'\times S}$.*

Remark See Section 2.1 for a review of correspondences as K–theoretic operators. In particular, the composition of operators depends on which of A_m and $W_k(x)$ is on

the left of the other:

$$A_m W_{n,k} : K_{\mathcal{M}'} \xrightarrow{W_{n,k}} K_{\mathcal{M}' \times S} \xrightarrow{A_m \times \mathrm{Id}_S} K_{\mathcal{M} \times S},$$

$$W_{n,k} A_m : K_{\mathcal{M}'} \xrightarrow{A_m} K_{\mathcal{M}} \xrightarrow{W_{n,k}} K_{\mathcal{M} \times S}.$$

The expressions above are actually given by certain correspondences in $K_{\mathcal{M} \times \mathcal{M}' \times S}$. Then the factors q and γ on the right-hand side of (1-12) indicate multiplication of the aforementioned correspondences by various powers of the line bundles (1-9) and (1-11).

1.8 A major motivation for the study of the Ext operator A_m stems from mathematical physics: as explained in Carlsson, Nekrasov and Okounkov [6], the operator A_m encodes the contribution of bifundamental matter to partition functions of 5d supersymmetric gauge theory on the algebraic surface S times a circle. Moreover, the deformed $W-$algebra $\mathcal{A}_r$ encodes symmetries of Toda conformal field theory. In this language, (1-12) becomes a mathematical manifestation of the Alday–Gaiotto–Tachikawa (AGT) correspondence between gauge theory and conformal field theory, by describing the Ext operator A_m in terms of its commutation with $W-$algebra generators. To the author's knowledge, the present paper is the first mathematical treatment of AGT over general algebraic surfaces in rank $r > 1$ (the reference [6] used different techniques from ours to describe the Ext operator in the $r = 1$ case).

However, we note that formulas (1-12) are not enough to completely determine A_m for a general smooth projective surface S, and one should instead work with a deformed vertex operator algebra which properly contains several deformed $W-$algebras $\mathcal{A}_r$. In the nondeformed case, a potential candidate for such a larger algebra was studied in Feigin and Gukov [9], where the authors expect that it contains operators which modify sheaves on S along entire curves, on top of our operators $W_{n,k}$ which modify sheaves at individual points. While we give a complete algebrogeometric description of the latter operators, we do not have such a description for the former operators. Once such a description is available, we hope that one can extend Theorem 1.7 to a bigger vertex operator algebra properly containing $\mathcal{A}_r$.

There is a situation where formulas (1-12) do indeed determine the Ext operator A_m completely: this corresponds to taking $S = \mathbb{A}^2$, replacing $\mathcal{M}$ by the moduli space of framed rank r sheaves on the projective plane, and working with torus equivariant K–theory; see Section 4.1 for details. In this particular case, we showed in [14] that $K_{\mathcal{M}}$ is isomorphic to the universal Verma module of $\mathcal{A}_r$. Theorem 1.7 holds in the situation at hand, and we will show in Theorem 4.5 that our formulas completely

determine the Ext operator A_m. This precisely yields the AGT correspondence for 5d supersymmetric gauge theory on $\mathbb{A}^2 \times S^1$; see for instance Braverman, Finkelberg and Nakajima [4], Bruzzo, Pedrini, Sala and Szabo [5], Maulik and Okounkov [12] and Schiffmann and Vasserot [18] for the history of this correspondence in mathematical language.

1.9 Alongside the operators (1-8), we constructed in [16, Theorem 4.15] K–theory lifts of the operators introduced by Grojnowski and Nakajima [11; 13] for $r = 1$, and generalized by Baranovsky [2] for any r, in cohomology:

$$(1\text{-}13) \qquad\qquad K_{\mathcal{M}} \xrightarrow{P_n} K_{\mathcal{M}\times S} \quad \text{for all } n \in \mathbb{Z}\backslash 0.$$

These operators satisfy the Heisenberg commutation relation (2-29), and interact with the deformed W–algebra generators according to relation (2-30).

Recall the line bundles q and γ of (1-9) and (1-11), respectively, and the footnote in Theorem 1.7 to properly interpret compositions of the operators A_m and $P_{\pm n}$.

Theorem 1.10 *We have the following interaction between the Ext operator (1-6) and the Heisenberg operators $P_{\pm n}$ for all $n > 0$:*

$$(1\text{-}14) \qquad\qquad A_m P_{-n} - P_{-n} A_m \gamma^n = A_m(1 - \gamma^n),$$

$$(1\text{-}15) \qquad\qquad A_m P_n - P_n A_m \gamma^{-n} = A_m(\gamma^{-n} - q^{rn}).$$

In $\mathcal{A}_r$, the series $W_r(x)$ matches the normal-ordered exponential of the generating series of the P_n; see Theorem 2.8. With this in mind, it is straightforward to show that the $k = r$ case of Theorem 1.7 follows from Theorem 1.10.

For any $\alpha \in K_S$, we will write $P_n\{\alpha\}$ for the composition

$$P_n\{\alpha\}\colon K_{\mathcal{M}} \xrightarrow{P_n} K_{\mathcal{M}\times S} \xrightarrow{\text{multiplication by } \mathrm{proj}_2^*(\alpha)} K_{\mathcal{M}\times S} \xrightarrow{\mathrm{proj}_{1*}} K_{\mathcal{M}},$$

where proj_1 and proj_2 are the projections from $\mathcal{M} \times S$ to $\mathcal{M}$ and S, respectively. Let q_1 and q_2 denote the Chern roots of the cotangent bundle Ω_S^1. Any symmetric Laurent polynomial in q_1 and q_2 gives rise to a well-defined element of K_S, via

$$q_1 + q_2 = [\Omega_S^1] \quad \text{and} \quad q = q_1 q_2 = [\omega_S].$$

Define

$$(1\text{-}16) \qquad\qquad \Phi_m = A_m \exp\left[\sum_{n=1}^{\infty} \frac{P_n}{n} \left\{ \frac{(q^n - 1)q^{-nr}}{[n]_{q_1}[n]_{q_2}} \right\} \right],$$

where $[n]_x = 1 + x + \cdots + x^{n-1}$. The expression in curly brackets is an element of K_S because $[n]_{q_1}[n]_{q_2}$ is a unit in the ring K_S.

Remark To see that $[n]_{q_1}[n]_{q_2}$ is a unit in the ring K_S, since the Chern character gives us an isomorphism $K_S \cong A^*(S, \mathbb{Q})$, we have $q_1 + q_2 = [\Omega_S^1] \in 2 + \mathcal{N}$ and $q = [\omega_S] \in 1 + \mathcal{N}$, where $\mathcal{N} \subset K_S$ denotes the nilradical. Therefore $[n]_{q_1}[n]_{q_2} \in n^2 + \mathcal{N}$, and is thus invertible in the ring K_S.

Corollary 1.11 *Formulas* (1-12), (1-14) *and* (1-15) *imply*

$$(1\text{-}17) \qquad [\Phi_m W_k(x) - m^k W_k(x\gamma)\Phi_m]\left(1 - \frac{x}{q^k}\right) = 0,$$

$$(1\text{-}18) \qquad \Phi_m P_{\pm n} - P_{\pm n}\Phi_m \gamma^{\mp n} = \pm \Phi_m(\gamma^{\mp n} - q^{\pm rn})$$

for all $k, n > 0$. *An operator* Φ_m *satisfying* (1-17) *and* (1-18) *is called a* **vertex operator**.

2 The moduli space of sheaves

2.1 Throughout the present paper, we will work with smooth projective varieties over the field $\mathbb{C}$. For such varieties X, we let

$$K_X = K_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the Grothendieck group of the category of coherent sheaves on X, with scalars extended to $\mathbb{Q}$. Derived tensor product yields a ring structure on K_X, and we have pullback and pushforward maps for any proper l.c.i. morphism $X \to Y$.

Definition 2.2 Given smooth projective varieties X and Y, any class $\Gamma \in K_{X \times Y}$ (called a "correspondence" in this setup) defines an operator

$$(2\text{-}1) \qquad K_Y \xrightarrow{\Psi_\Gamma} K_X, \quad \Psi_\Gamma = \mathrm{proj}_{X*}(\Gamma \cdot \mathrm{proj}_Y^*),$$

where $\mathrm{proj}_X, \mathrm{proj}_Y$ denote the projection maps from $X \times Y$ to X and Y, respectively.

The composition of operators (2-1) can also be described as a correspondence

$$(2\text{-}2) \qquad \Psi_\Gamma \circ \Psi_{\Gamma'} = \Psi_{\Gamma''} \colon K_Z \to K_X$$

for any $\Gamma \in K_{X \times Y}$ and $\Gamma' \in K_{Y \times Z}$, where

(2-3) $$\Gamma'' = \text{proj}_{X \times Z *}\big(\text{proj}_{X \times Y}^*(\Gamma) \otimes \text{proj}_{Y \times Z}^*(\Gamma')\big),$$

where $\text{proj}_{X \times Y}$, $\text{proj}_{Y \times Z}$ and $\text{proj}_{X \times Z}$ are the standard projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$ and $X \times Z$. Throughout the present paper, all operators $K_Y \to K_X$ arise from explicit correspondences. While we will use the language of composition of operators for convenience, what is really happening behind the scenes is composition of correspondences under the operation $(\Gamma, \Gamma') \mapsto \Gamma''$ of (2-3).

2.3 In Section 1.6, we referred to various operators $K_\mathcal{M} \to K_{\mathcal{M} \times S}$ as defining an action of a certain algebra on $K_\mathcal{M}$, and we will now explain the meaning of this notion. Given two arbitrary homomorphisms (of abelian groups)

(2-4) $$K_\mathcal{M} \xrightarrow{x,y} K_{\mathcal{M} \times S},$$

their "product" $xy|_\Delta$ is defined as the composition

$$xy|_\Delta : K_\mathcal{M} \xrightarrow{y} K_{\mathcal{M} \times S} \xrightarrow{x \times \text{Id}_S} K_{\mathcal{M} \times S \times S} \xrightarrow{\text{Id}_\mathcal{M} \times \Delta^*} K_{\mathcal{M} \times S}$$

where $S \xrightarrow{\Delta} S \times S$ is the diagonal. It is easy to check that $(xy|_\Delta)z|_\Delta = x(yz|_\Delta)|_\Delta$, hence the aforementioned notion of product is associative, and it makes sense to define $x_1 \cdots x_n|_\Delta$ for arbitrarily many operators $x_1, \ldots, x_n : K_\mathcal{M} \to K_{\mathcal{M} \times S}$.

Similarly, given two operators (2-4), we may define their commutator

$$K_\mathcal{M} \xrightarrow{[x,y]} K_{\mathcal{M} \times S \times S}$$

as the difference of the two compositions

$$K_\mathcal{M} \xrightarrow{y} K_{\mathcal{M} \times S} \xrightarrow{x \times \text{Id}_S} K_{\mathcal{M} \times S \times S},$$

$$K_\mathcal{M} \xrightarrow{x} K_{\mathcal{M} \times S} \xrightarrow{y \times \text{Id}_S} K_{\mathcal{M} \times S \times S} \xrightarrow{\text{Id}_\mathcal{M} \times \text{swap}^*} K_{\mathcal{M} \times S \times S},$$

where swap: $S \times S \to S \times S$ is the permutation of the two factors. In all cases studied in the present paper, we will have[2]

$$[x, y] = \Delta_*(z)$$

for some $K_\mathcal{M} \xrightarrow{z} K_{\mathcal{M} \times S}$ which is uniquely determined (the diagonal embedding Δ_* is injective because it has a left inverse), and which will be denoted by $z = [x, y]_{\text{red}}$. We leave it as an exercise to the interested reader to prove that the commutator satisfies

---

[2]Here we abuse notation by writing Δ_* instead of $(\text{Id}_\mathcal{M} \times \Delta)_*$ for the diagonal map $K_{\mathcal{M} \times S} \to K_{\mathcal{M} \times S \times S}$.

the Leibniz rule in the form $[xy|_\Delta, z]_{red} = x[y, z]_{red}|_\Delta + [x, z]_{red} y|_\Delta$, and the Jacobi identity in the form $\big[[x, y]_{red}, z\big]_{red} + \big[[y, z]_{red}, x\big]_{red} + \big[[z, x]_{red}, y\big]_{red} = 0$.

Finally, we consider the ring homomorphism $\mathbb{K} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{Sym} \to K_S$ given by sending q_1 and q_2 to the Chern roots of the cotangent bundle of S (therefore, $q = q_1 q_2$ goes to the class of the canonical line bundle). We will often abuse notation, and write q_1, q_2, q for the images of the indeterminates in the ring K_S. For any $\lambda \in \mathbb{K}$ and any operator (2-4), we may define their product as the composition

$$\lambda \cdot x \colon K_{\mathcal{M}} \xrightarrow{x} K_{\mathcal{M} \times S} \xrightarrow{\mathrm{Id}_{\mathcal{M}} \times (\text{multiplication by } \lambda)} K_{\mathcal{M} \times S},$$

where we identify $\lambda \in \mathbb{K}$ with its image in K_S. With this in mind, the ring K_S can be thought of as the "ring of constants" for the algebra of operators (2-4).

2.4 Recall the universal sheaf (1-2), and consider the derived scheme

$$(2\text{-}5) \qquad \mathfrak{Z}_1 = \mathbb{P}_{\mathcal{M} \times S}(\mathcal{U}) \to \mathcal{M} \times S.$$

Since $\mathcal{U}$ is isomorphic to a quotient $\mathcal{V}/\mathcal{W}$ of vector bundles on $\mathcal{M} \times S$ (Proposition 2.2 of [15]), the projectivization in (2-5) is defined as the derived zero locus of a section of a vector bundle on the projective bundle $\mathbb{P}_{\mathcal{M} \times S}(\mathcal{V})$. However, it was shown in [15, Proposition 2.10] that under Assumption S, the derived zero locus is actually a smooth scheme

$$\mathfrak{Z}_1 = \bigsqcup_{c = \lceil ((r-1)/2r)c_1^2 \rceil}^{\infty} \mathfrak{Z}_{c+1,c},$$

whose connected components are given by

$(2\text{-}6)\ \ \mathfrak{Z}_{c+1,c} = \{(\mathcal{F}_{c+1}, \mathcal{F}_c) \text{ such that } \mathcal{F}_{c+1} \subset_x \mathcal{F}_c \text{ for some } x \in S\} \subset \mathcal{M}_{c+1} \times \mathcal{M}_c,$

and $\mathcal{F}' \subset_x \mathcal{F}$ means that $\mathcal{F}' \subset \mathcal{F}$ and the quotient $\mathcal{F}/\mathcal{F}'$ is isomorphic to the length one skyscraper sheaf at the point $x \in S$. This scheme comes with projection maps

$$(2\text{-}7)$$

$$
\begin{array}{ccc}
 & \mathfrak{Z}_{c+1,c} & \\
 \swarrow^{p_+} & \downarrow^{p_S} & \searrow^{p_-} \\
 \mathcal{M}_{c+1} & S & \mathcal{M}_c
\end{array}
$$

More generally, we defined a derived scheme $\mathfrak{Z}_2^\bullet$ in [17, Definition 4.17], which was shown (under Assumption S, in [17, Proposition 4.21]) to be a smooth scheme

$$\mathfrak{Z}_2^\bullet = \bigsqcup_{c = \lceil ((r-1)/2r)c_1^2 \rceil}^{\infty} \mathfrak{Z}_{c+2,c}^\bullet,$$

whose connected components are given by

(2-8) $\mathfrak{Z}^{\bullet}_{c+2,c} = \{(\mathcal{F}_{c+2} \subset_x \mathcal{F}_{c+1} \subset_x \mathcal{F}_c) \text{ for some } x \in S\} \subset \mathcal{M}_{c+2} \times \mathcal{M}_{c+1} \times \mathcal{M}_c.$

This scheme is equipped with projection maps as in (2-9) below, but we observe that the rhombus is not derived Cartesian (and this is key to our construction):

(2-9)

$$
\begin{array}{ccc}
 & \mathfrak{Z}^{\bullet}_{c+2,c} & \\
\pi_+ \swarrow & & \searrow \pi_- \\
\mathfrak{Z}_{c+2,c+1} & & \mathfrak{Z}_{c+1,c} \\
{}_{p_- \times p_S} \searrow & & \swarrow {}_{p_+ \times p_S} \\
 & \mathcal{M}_{c+1} \times S &
\end{array}
$$

Note that all of the maps in the diagram above are proper, l.c.i. morphisms. Define

(2-10)
$$
\mathfrak{Z}^{\bullet}_n = \bigsqcup_{c=\lceil((r-1)/2r)c_1^2\rceil}^{\infty} \mathfrak{Z}^{\bullet}_{c+n,c},
$$

whose connected components are given by derived fiber products

(2-11) $\mathfrak{Z}^{\bullet}_{c+n,c} = \mathfrak{Z}^{\bullet}_{c+n,c+n-2} \underset{\mathfrak{Z}_{c+n-1,c+n-2}}{\times} \cdots \underset{\mathfrak{Z}_{c+2,c+1}}{\times} \mathfrak{Z}^{\bullet}_{c+2,c} \to \mathcal{M}_{c+n} \times \cdots \times \mathcal{M}_c.$

While $\mathfrak{Z}^{\bullet}_n$ is a derived scheme, we note that its closed points are all of the form

(2-12) $\mathfrak{Z}^{\bullet}_{c+n,c} = \{(\mathcal{F}_{c+n}, \ldots, \mathcal{F}_c) \text{ sheaves with } \mathcal{F}_{c+n} \subset_x \cdots \subset_x \mathcal{F}_c \text{ for some } x \in S\}.$

Therefore, we have the following projection maps, which only remember the smallest and the largest sheaf in a flag (2-12):

(2-13)

$$
\begin{array}{ccc}
 & \mathfrak{Z}^{\bullet}_{c+n,c} & \\
p_+ \swarrow & \downarrow {}_{p_S} & \searrow p_- \\
\mathcal{M}_{c+n} & S & \mathcal{M}_c
\end{array}
$$

(the notation generalizes (2-7)). In diagram (2-13), the maps $p_{\pm}$ are l.c.i. morphisms, and the maps $p_{\pm} \times p_S$ are proper (they inherit these properties from the maps in (2-9)). Finally, we consider the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\mathfrak{Z}^{\bullet}_n$, whose fibers are given by

(2-14) $\mathcal{L}_i|_{(\mathcal{F}_{c+n}, \ldots, \mathcal{F}_c)} = \mathcal{F}_{c+n-i,x} / \mathcal{F}_{c+n-i+1,x}$

on the connected component $\mathfrak{Z}^{\bullet}_{c+n,c} \subset \mathfrak{Z}^{\bullet}_n.$

2.5 Using the derived scheme (2-11) and the maps (2-13), define for all $n, k \in \mathbb{N}$

$$(2\text{-}15) \qquad K_{\mathcal{M}} \xrightarrow{L_{n,k}} K_{\mathcal{M} \times S}, \qquad L_{n,k} = (-1)^{k-1}(p_+ \times p_S)_*(\mathcal{L}_n^k \cdot p_-^*),$$

$$(2\text{-}16) \qquad K_{\mathcal{M}} \xrightarrow{U_{n,k}} K_{\mathcal{M} \times S}, \qquad U_{n,k} = \frac{(-1)^{rn+k-1}u^n}{q^{(r-1)n}}(p_- \times p_S)_*\left(\frac{\mathcal{L}_n^k}{\mathcal{Q}^r} \cdot p_+^*\right),$$

where $\mathcal{Q} = \mathcal{L}_1 \cdots \mathcal{L}_n$, and u is the determinant of the universal sheaf on $\mathcal{M} \times S$, as in (1-10).[3] Implicit in the definitions (2-15) and (2-16) is that we define the operators therein for all components $\mathcal{M}_c$ of the moduli space $\mathcal{M}$. We also set

$$(2\text{-}17) \qquad L_{n,0} = U_{n,0} = \delta_n^0 \quad \text{and} \quad L_{0,k} = U_{0,k} = \delta_k^0.$$

Finally, consider for all $k \in \mathbb{N} \sqcup 0$ the operators

$$(2\text{-}18) \qquad E_k \colon K_{\mathcal{M}} \xrightarrow{\text{pullback}} K_{\mathcal{M} \times S} \xrightarrow{\text{multiplication by } \wedge^k \mathcal{U}} K_{\mathcal{M} \times S}.$$

Since $\mathcal{U} \cong \mathcal{V}/\mathcal{W}$ is a coherent sheaf of projective dimension one on $\mathcal{M} \times S$ (see [15, Proposition 2.2]), the class $\wedge^k \mathcal{U}$ in (2-18) is defined by setting

$$(2\text{-}19) \qquad \wedge^\bullet\left(\frac{\mathcal{U}}{z}\right) = \frac{\wedge^\bullet\left(\frac{\mathcal{V}}{z}\right)}{\wedge^\bullet\left(\frac{\mathcal{W}}{z}\right)}$$

and picking out the coefficient of z^{-k} when expanding in negative powers of z. The reason for our notation for the operators (2-15), (2-16) and (2-18) is that these three operators are respectively lower triangular, upper triangular, and diagonal with respect to the grading on $K_{\mathcal{M}}$ by the second Chern class; see (1-3).

Definition 2.6 [17, Section 6.7] For any $(n, k) \in \mathbb{Z} \times \mathbb{N}$, consider the operators

$$(2\text{-}20) \qquad W_{n,k} = \sum_{\substack{k_0+k_1+k_2=k}}^{n_2-n_1=n} q^{(k-1)n_2} \cdot L_{n_1,k_1} E_{k_0} U_{n_2,k_2}\Big|_\Delta$$

as k_0, k_1, k_2, n_1, n_2 run over $\mathbb{N} \sqcup 0$ (recall the convention (2-17)).

Note that (2-20) is an infinite sum, but its action on $K_{\mathcal{M}}$ is well-defined because the operators $L_{n,k}$ (resp. $U_{n,k}$) increase (resp. decrease) the c_2 of stable sheaves by n, and Bogomolov's inequality ensures that the moduli space of stable sheaves is empty if c_2 is small enough.

---

[3]Note that u parametrizes the determinant of any one of the sheaves $\mathcal{F}_{c+n}, \ldots, \mathcal{F}_c$ in a flag (2-12), since these sheaves have canonically isomorphic determinants; see Proposition 3.4.

2.7 Similarly with (2-15) and (2-16), for all $n \in \mathbb{N}$ we have the operators

$$(2\text{-}21) \quad K_{\mathcal{M}} \xrightarrow{P_{-n}} K_{\mathcal{M} \times S}, \quad P_{-n} = (p_+ \times p_S)_* \left(\sum_{i=0}^{n-1} \frac{q^i \mathcal{L}_n}{\mathcal{L}_{n-i}} \cdot p_-^* \right),$$

$$(2\text{-}22) \quad K_{\mathcal{M}} \xrightarrow{H_{-n}} K_{\mathcal{M} \times S}, \quad H_{-n} = (p_+ \times p_S)_*(p_-^*),$$

$$(2\text{-}23) \quad K_{\mathcal{M}} \xrightarrow{P_n} K_{\mathcal{M} \times S}, \quad P_n = (-1)^{rn} u^n (p_- \times p_S)_* \left(\sum_{i=0}^{n-1} \frac{q^i \mathcal{L}_n}{\mathcal{Q}^r \mathcal{L}_{n-i}} \cdot p_+^* \right),$$

$$(2\text{-}24) \quad K_{\mathcal{M}} \xrightarrow{H_n} K_{\mathcal{M} \times S}, \quad H_n = (-1)^{rn} u^n (p_- \times p_S)_*(\mathcal{Q}^{-r} \cdot p_+^*).$$

As a consequence of [17, formulas (2.15) and (5.18)–(5.21)], the operators $H_{\pm n}$ are to the operators $P_{\pm n}$ as complete symmetric functions are to power sum functions

$$(2\text{-}25) \qquad \sum_{n=0}^{\infty} \frac{H_{\pm n}}{z^{\pm n}} = \exp \left(\sum_{n=1}^{\infty} \frac{P_{\pm n}}{n z^{\pm n}} \right) \Big|_{\Delta}$$

or, explicitly,

$$H_0 = \text{proj}_1^*,$$

where $\text{proj}_1 \colon \mathcal{M} \times S \to \mathcal{M}$ is the usual projection, and

$$H_{\pm 1} = P_{\pm 1},$$
$$H_{\pm 2} = \tfrac{1}{2}(P_{\pm 1} P_{\pm 1}|_{\Delta} + P_{\pm 2}),$$
$$H_{\pm 3} = \tfrac{1}{6}(P_{\pm 1} P_{\pm 1} P_{\pm 1}|_{\Delta} + 3 P_{\pm 1} P_{\pm 2}|_{\Delta} + 2 P_{\pm 3}),$$

and so on.

Theorem 2.8 [17, Theorem 6.9] *The operators* (2-20) *satisfy*

$$(2\text{-}26) \qquad W_{n,r} = u \sum_{\substack{n_1, n_2 \geq 0}}^{n_2 - n_1 = n} H_{-n_1} H_{n_2} \Big|_{\Delta} \quad \text{for all } n \in \mathbb{Z},$$

$$(2\text{-}27) \qquad W_{n,k} = 0 \quad \text{for all } k > r.$$

2.9 We will now present the interaction of the operators (2-20), (2-21) and (2-23). Recall the commutator construction from Section 2.3.

The following theorem was stated in [17, Theorem 3.13 and Proposition 3.15] and proved in [16, Theorem 4.15] under Assumption S.

Theorem 2.10 *We have the following formulas for all $n, n' \in \mathbb{Z}$ and $k, k' \in \mathbb{N}$:*

$$(2\text{-}28) \quad [W_{n,k}, W_{n',k'}] = \Delta_* \left(\sum_{\substack{k+k'=l+l' \\ m+m'=n+n'}}^{\substack{m/l \le m'/l' \\ \min(l,l') \le \min(k,k')}} c_{n,n',k,k'}^{m,m',l,l'}(q_1, q_2) \cdot W_{m,l} W_{m',l'} \Big|_{\Delta} \right),$$

$$(2\text{-}29) \quad [P_{n'}, P_n] = \Delta_* \begin{cases} 0 & \text{if } \operatorname{sign}(n) = \operatorname{sign}(n'), \\ \delta_{n+n'}^0 n [n]_{q_1} [n]_{q_2} [r]_{q^n} \cdot \operatorname{proj}_{\mathcal{M}}^* & \text{if } n' < 0 < n, \end{cases}$$

$$(2\text{-}30) \quad [W_{n',k'}, P_{\pm n}] = \Delta_*(\pm [n]_{q_1} [n]_{q_2} [k']_{q^n} q^{n(r-k')\delta_{\pm}^{\mp}} \cdot W_{\pm n+n',k'}),$$

where the coefficients $c_{n,n',k,k'}^{m,m',l,l'}(q_1, q_2) \in K_S$ were computed algorithmically in [17]. *They are certain universal symmetric Laurent polynomials in q_1 and q_2.*

Indeed, we show in [17, Theorem 3.13] that (2-28) is equivalent to the defining relation in the deformed W–algebra $\mathcal{A}_r$ (with Δ_* replaced by $(1-q_1)(1-q_2)$). Similarly, relation (2-29) is the defining relation in the deformed Heisenberg algebra. As we explained in [17, Definition 5.2 and formulas (5.20)–(5.21)] and proved in [16, Theorem 4.15], the fact that the operators (2-20), (2-21) and (2-23) satisfy the relations in Theorem 2.10 is precisely what we mean when we say that the deformed W–algebra $\mathcal{A}_r$ and the deformed Heisenberg algebra act on the groups $K_{\mathcal{M}}$.

2.11 Let us consider the operators of Section 2.5 and form the generating series

$$(2\text{-}31) \qquad L_n(y) = \sum_{k=1}^{\infty} \frac{L_{n,k}}{(-y)^k} \quad \text{and} \quad U_n(y) = \sum_{k=1}^{\infty} \frac{U_{n,k}}{(-y)^k}.$$

In other words, these power series are considered as operators

$$K_{\mathcal{M}} \xrightarrow{L_n(y)} K_{\mathcal{M} \times S}\left[\!\left[\frac{1}{y}\right]\!\right], \quad L_n(y) = (p_+ \times p_S)_* \left(\frac{1}{1-(y/\mathcal{L}_n)} \cdot p_-^* \right),$$

$$K_{\mathcal{M}} \xrightarrow{U_n(y)} K_{\mathcal{M} \times S}\left[\!\left[\frac{1}{y}\right]\!\right], \quad U_n(y) = \frac{(-1)^{rn} u^n}{q^{(r-1)n}} (p_- \times p_S)_* \left(\frac{\mathcal{Q}^{-r}}{1-(y/\mathcal{L}_n)} \cdot p_+^* \right).$$

We will also consider the operators

$$E(y) \colon K_{\mathcal{M}} \xrightarrow{\text{pullback}} K_{\mathcal{M} \times S} \xrightarrow{\text{multiplication by } \wedge^{\bullet}(\mathcal{U}/y)} K_{\mathcal{M} \times S}\left[\!\left[\frac{1}{y}\right]\!\right].$$

Furthermore, we will consider the generating series

$$(2\text{-}32) \qquad L(x, y) = 1 + \sum_{n=1}^{\infty} L_n(y) x^n \quad \text{and} \quad U(x, y) = 1 + \sum_{n=1}^{\infty} \frac{U_n(y)}{x^n},$$

and also set

$$(2\text{-}33) \qquad\qquad W_k(x) = \sum_{n=-\infty}^{\infty} \frac{W_{n,k}}{x^n},$$

$$(2\text{-}34) \qquad\qquad W(x,y) = 1 + \sum_{k=1}^{\infty} \frac{W_k(x)}{y^k}.$$

The definition of the W–algebra generators in (2-20) is equivalent to

$$(2\text{-}35) \qquad\qquad W(x, yD_x) = L(x, yD_x)E(yD_x)U(xq, yD_x)|_\Delta,$$

where D_x is the q–difference operator in the variable x, ie $D_x(f(x)) = f(xq)$. In formula (2-35), we place all powers of D_x to the right (resp. left) of all powers of x when writing down the power series $L(x, yD_x)$ (resp. $U(xq, yD_x)$). In terms of generating series, formula (2-30) reads

$$(2\text{-}36) \qquad\qquad [W_k(x), P_{\pm n}] = \Delta_* \big(\pm [n]_{q_1} [n]_{q_2} [k]_{q^n} q^{n(r-k)\delta_\pm^+} \cdot x^{\pm n} W_k(x) \big).$$

2.12 Given a rational function $F(z)$, whose set of simple poles is partitioned into two disjoint sets $\mathcal{P}_1 \sqcup \mathcal{P}_2$ (which may be empty), we will write

$$(2\text{-}37) \qquad\qquad \int_{\mathcal{P}_1 \prec z \prec \mathcal{P}_2} F(z) = \sum_{c \in \mathcal{P}_1} \operatorname*{Res}_{z=c} \frac{F(z)}{z} = -\sum_{c \in \mathcal{P}_2} \operatorname*{Res}_{z=c} \frac{F(z)}{z}.$$

The first equality is a definition, and the second equality is the residue theorem. If $F(z_1, \ldots, z_n)$ is a rational function with simple poles of the form $z_i = c$ and $z_i = \gamma z_j$ for various $c \in \mathcal{P}_1 \sqcup \mathcal{P}_2$ and various scalars γ in some set $\mathcal{Q}$, then we set

$$(2\text{-}38) \qquad\qquad \int_{\mathcal{P}_1 \prec z_1 \prec \cdots \prec z_n \prec \mathcal{P}_2} F(z_1, \ldots, z_n)$$

as the result of the n–step process which starts with $F(z_1, \ldots, z_n)/z_1 \cdots z_n$, and at the ith step replaces a rational function in $z_i, \ldots, z_n$ by the sum of its residues of the form $z_i = c\gamma_1 \cdots \gamma_{i-1}$ for various $c \in \mathcal{P}_1$ and $\gamma_1, \ldots, \gamma_{i-1} \in \mathcal{Q} \cup \{1\}$. Just like in (2-37), the residue theorem implies that the answer is the same as $(-1)^n$ times the result of the n–step process which starts with $F(z_1, \ldots, z_n)/z_1 \cdots z_n$, and at the ith step replaces a rational function in $z_1, \ldots, z_{n+1-i}$ by the sum of its residues of the form $z_{n+1-i} = c\gamma_1 \cdots \gamma_{i-1}$ for various $c \in \mathcal{P}_2$ and $\gamma_1, \ldots, \gamma_{i-1} \in \mathcal{Q} \cup \{1\}$.

Proposition 2.13 [17, following the proof of Proposition 5.12] *We have the following formulas for the maps* (2-13):

$$(2\text{-}39) \quad (p_+ \times p_S)_* r(\mathcal{L}_1, \dots, \mathcal{L}_n)$$

$$= \int_{\{0,\infty\} \sqcup \mathcal{P} \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}} \frac{r(z_1, \dots, z_n) \prod_{i=1}^{n} \wedge^\bullet \left(\dfrac{z_i q}{\mathcal{U}} \right)}{\left(1 - \dfrac{z_2 q}{z_1} \right) \cdots \left(1 - \dfrac{z_n q}{z_{n-1}} \right) \prod_{1 \le i < j \le n} \zeta \left(\dfrac{z_j}{z_i} \right)},$$

$$(2\text{-}40) \quad (p_- \times p_S)_* r(\mathcal{L}_1, \dots, \mathcal{L}_n)$$

$$= \int_{\mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{P}} \frac{r(z_1, \dots, z_n) \prod_{i=1}^{n} \wedge^\bullet \left(-\dfrac{\mathcal{U}}{z_i} \right)}{\left(1 - \dfrac{z_2 q}{z_1} \right) \cdots \left(1 - \dfrac{z_n q}{z_{n-1}} \right) \prod_{1 \le i < j \le n} \zeta \left(\dfrac{z_j}{z_i} \right)},$$

where

$$\zeta(x) = \frac{(1 - x q_1)(1 - x q_2)}{(1 - x)(1 - x q)} \in K_S(x)$$

and $r(z_1, \dots, z_n)$ *is a rational function with coefficients in* $(p_\pm \times p_S)^*(K_{\mathcal{M} \times S})$ *whose poles are all of the form* $z_i = c$, *where* $c \in \{0, \infty\} \sqcup \mathcal{P}$ *for some finite set* $\mathcal{P}$.

Note that the integrands in (2-39)–(2-40) have poles when z_i equals $q^{1 \text{ or } 0}$ times one of the Chern roots of $\mathcal{U}$. Thus, the location of the symbol $\mathcal{U}$ in the subscripts of the integrals (2-39)–(2-40) indicates whether these poles are thought to lie in the set $\mathcal{P}_1$ or $\mathcal{P}_2$ for the sake of the notation (2-37).

3 Computing the Ext operator

3.1 To properly define the Ext operator (1-6), note that the complex $\mathcal{E}_m$ of (1-4) can be written as a difference $\mathcal{V}_1 - \mathcal{V}_2$ of vector bundles. Then we define

$$(3\text{-}1) \qquad \wedge^\bullet \left(\frac{\mathcal{E}_m}{t} \right) = \frac{\wedge^\bullet \left(\dfrac{\mathcal{V}_1}{t} \right)}{\wedge^\bullet \left(\dfrac{\mathcal{V}_2}{t} \right)} = \frac{\sum_{k=0}^{\operatorname{rank} \mathcal{V}_1} (-t)^{-k} [\wedge^k \mathcal{V}_1]}{\sum_{k=0}^{\operatorname{rank} \mathcal{V}_2} (-t)^{-k} [\wedge^k \mathcal{V}_2]}$$

and interpret it as a rational function in t, with coefficients in $K_{\mathcal{M} \times \mathcal{M}'}$. Strictly speaking, the object $\wedge^\bullet \mathcal{E}_m$ in (1-6) refers to the specialization of this rational function at $t = 1$. If

this specialization is not well-defined, ie if

$$\sum_{k=0}^{\operatorname{rank} \mathcal{V}_2} (-1)^k [\wedge^k \mathcal{V}_2]$$

is not a unit in $K_{\mathcal{M} \times \mathcal{M}'}$, then we employ the following artifice: replace m by m/t in formulas (1-11), (1-12), (1-17) and throughout the current section. Once one does this, our main Theorems 1.7, 1.10 and Corollary 1.11 will be equalities of operator-valued rational functions in t. Moreover, we will often use the notation

$$\wedge^{\bullet}\left(\frac{t}{\mathcal{U}}\right) \quad \text{instead of} \quad \wedge^{\bullet}(\mathcal{U}^{\vee} t)$$

for any coherent sheaf $\mathcal{U}$ (all our coherent sheaves have finite projective dimension).

3.2 The main goal of the present section is to compute the commutation relations between the Ext operator $A_m : K_{\mathcal{M}'} \to K_{\mathcal{M}}$ of (1-6) and the operators

$$(3\text{-}2) \qquad\qquad W_{n,k}, P_{\pm n'} : K_{\mathcal{M}} \to K_{\mathcal{M} \times S}$$

of (2-20), (2-21) and (2-23) for all $n \in \mathbb{Z}$ and $n', k \in \mathbb{N}$. One must be careful what one means by "commutation relation". While the operator

$$P_{\pm n} A_m \quad \text{unambiguously refers to} \quad K_{\mathcal{M}'} \xrightarrow{A_m} K_{\mathcal{M}} \xrightarrow{P_{\pm n}} K_{\mathcal{M} \times S},$$

$$A_m P_{\pm n} \quad \text{henceforth refers to} \quad K_{\mathcal{M}'} \xrightarrow{P_{\pm n}} K_{\mathcal{M}' \times S} \xrightarrow{A_m \times \operatorname{Id}_S} K_{\mathcal{M} \times S},$$

and analogously for $W_{n,k}$ instead of $P_{\pm n}$. As opposed to the operators (3-2), the operator A_m acts nontrivially between all components of the moduli space

$$(3\text{-}3) \qquad\qquad A_m|_c^{c'} : K_{\mathcal{M}_{c'}} \to K_{\mathcal{M}_c}.$$

In principle, the moduli spaces of sheaves in the domain and codomain can correspond to different choices of first Chern class and stability condition, but we always require them to have the same rank r. Therefore, there are two universal sheaves

$$\begin{array}{ccc} \mathcal{U} & & \mathcal{U}' \\ \vdots & \text{and} & \vdots \\ \downarrow & & \downarrow \\ \mathcal{M} \times S & & \mathcal{M}' \times S \end{array}$$

of the same rank r, where $\mathcal{M}$ (resp. $\mathcal{M}'$) is the union of the moduli spaces that appear in the codomain (resp. domain) of (3-3). The determinants of these universal sheaves are denoted by u and u', respectively, as in (1-10).

3.3 We must explain how to make sense of the symbols q, m, γ in (1-12), (1-14) and (1-15). In the language of correspondences from Section 2.1, the operators

$$K_{\mathcal{M}'} \xrightarrow{z} K_{\mathcal{M} \times S}$$

studied in the present paper (such as the compositions $W_{n,k} A_m$ or $P_{\pm n} A_m$ that appear in (1-12), (1-14) and (1-15)) arise from K–theory classes Γ on $\mathcal{M} \times \mathcal{M}' \times S$. Then the product qz refers to the operator corresponding to the class $\text{proj}_S^*(q) \cdot \Gamma$, while the product γz refers to the operator corresponding to the class

$$\text{proj}_S^* \left(\frac{m}{q} \right)^r \cdot \frac{\text{proj}_{\mathcal{M} \times S}^*(\det \mathcal{U})}{\text{proj}_{\mathcal{M}' \times S}^*(\det \mathcal{U}')} \cdot \Gamma,$$

where $\mathcal{M} \times \mathcal{M}' \times S \xrightarrow{\text{proj}_{\mathcal{M} \times S}, \ \text{proj}_{\mathcal{M}' \times S}, \ \text{proj}_S} \mathcal{M} \times S, \ \mathcal{M}' \times S, \ S$ are the projections.

Proposition 3.4 *We have the equality of correspondences* $K_{\mathcal{M}_{c \pm n}} \to K_{\mathcal{M}_c \times S}$

$$(3\text{-}4) \qquad\qquad P_{\pm n} \cdot (\det \mathcal{U}_{c \pm n}) = (\det \mathcal{U}_c) \cdot P_{\pm n}$$

for all $c \in \mathbb{Z}$. *Formula* (3-4) *also holds with* $P_{\pm n}$ *replaced by* $W_{n,k}$ *or* $H_{\pm n}$.

Equation (3-4) is best restated in the language of correspondences from Section 2.1. In these terms, $P_{\pm n}$ is given by a K–theory class supported on the locus

$$\mathfrak{C} = \{ (\mathcal{F}_{c+n} \subset_{nx} \mathcal{F}_c) \text{ for some } x \in S \} \subset \mathcal{M}_{c+n} \times \mathcal{M}_c \times S,$$

where $\mathcal{F}' \subset_{nx} \mathcal{F}$ means that $\mathcal{F}' \subset \mathcal{F}$ and that $\mathcal{F}/\mathcal{F}'$ is a length n sheaf supported at x. Then (3-4) merely states that the universal sheaves $\mathcal{U}_{c+n}$ and $\mathcal{U}_c$ have isomorphic determinants when restricted to $\mathfrak{C}$. This is just the version "in families" of the well-known statement that a codimension-2 modification of a torsion-free sheaf does not change its determinant. As a consequence of Proposition 3.4, γ of (1-11) will behave just like a constant in all our computations, ie it will not matter where we insert γ in any product of operators among $P_{\pm n}$, $H_{\pm n}$ and $W_{n,k}$.

3.5 Our main intersection-theoretic computation is the following:

Lemma 3.6 *We have the following relations involving the* **Ext** *operator* A_m

$$(3\text{-}5) \qquad\qquad A_m(H_{-n} - H_{-n+1}) = \gamma^n (H_{-n} - H_{-n+1}) A_m,$$

$$(3\text{-}6) \qquad\qquad A_m(H_n - H_{n-1}\gamma^{-1}) = (H_n \gamma^{-n} - H_{n-1} q^r \gamma^{-n+1}) A_m$$

for all $n \in \mathbb{N}$. *(Recall that* $H_0 = \text{proj}_1^*$, *where* $\mathcal{M} \times S \xrightarrow{\text{proj}_1} \mathcal{M}$ *is the usual projection.)*

Proof Consider the following diagrams of spaces and arrows, for all $c, c' \in \mathbb{Z}$:

(3-7)

(3-8)

Recall that $H_{-n} = (p_+ \times p_S)_* p_-^*$, in the notation of (2-13). Then the rule for composition of correspondences in (2-2) gives us the formulas

$$(3\text{-}9) \qquad A_m H_{-n} = (\pi_1 \times \mathrm{Id}_S)_* (\Upsilon_n \cdot \pi_2^*),$$

$$(3\text{-}10) \qquad H_{-n} A_m = (\pi_1' \times \mathrm{Id}_S)_* (\Upsilon_n' \cdot \pi_2'^*),$$

where, in the notation of (3-7) and (3-8),

$$(3\text{-}11) \qquad \Upsilon_n = (\mathrm{Id} \times p_S \times p_-)_* \big[\wedge^\bullet \big((\mathrm{Id} \times p_+)^* \mathcal{E}_m \big) \big],$$

$$(3\text{-}12) \qquad \Upsilon_n' = (p_+' \times p_S' \times \mathrm{Id})_* \big[\wedge^\bullet \big((p_-' \times \mathrm{Id})^* \mathcal{E}_m \big) \big]$$

are certain classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$, which we will now compute.

Claim 3.7 *In K–theory we have the equalities*

$$(3\text{-}13) \qquad (\mathrm{Id} \times p_+)^* \mathcal{E}_m = (\mathrm{Id} \times p_-)^* \mathcal{E}_m + \Big(\frac{1}{\mathcal{L}_1} + \cdots + \frac{1}{\mathcal{L}_n} \Big) (\mathrm{Id} \times p_S)^* \Big(\frac{\mathcal{U}m}{q} \Big)$$

on $\mathcal{M}_c \times \mathfrak{Z}_{c'+n,c'}^\bullet$, where $\mathcal{U}$ denotes the universal sheaf on $\mathcal{M}_c \times S$, and

$$(3\text{-}14) \qquad (p_-' \times \mathrm{Id})^* \mathcal{E}_m = (p_+' \times \mathrm{Id})^* \mathcal{E}_m - (\mathcal{L}_1 + \cdots + \mathcal{L}_n)(p_S' \times \mathrm{Id})^* (\mathcal{U}'^\vee m)$$

on $\mathfrak{Z}_{c,c-n}^\bullet \times \mathcal{M}_{c'}$, where $\mathcal{U}'$ denotes the universal sheaf on $\mathcal{M}_{c'} \times S$.

Proof To prove (3-13), consider the diagram

$$(3\text{-}15)$$

where the vertical maps are the natural projections (we use the notation ρ for all of them). We have the short exact sequence of sheaves over $\mathfrak{Z}^{\bullet}_{c'+n,c'} \times S$

$$(3\text{-}16) \qquad 0 \to \mathcal{U}'_+ \to \mathcal{U}'_- \to \Gamma_*(\mathcal{L}_1 \text{ "$\oplus$" } \cdots \text{ "$\oplus$" } \mathcal{L}_n) \to 0,$$

where $\mathcal{U}'_{\pm} = (p^*_{\pm} \times \mathrm{Id}_S)(\text{universal sheaf})$, while $\mathcal{L}_1, \ldots, \mathcal{L}_n$ denote the tautological line bundles on $\mathfrak{Z}^{\bullet}_{c'+n,c'}$ that were defined in (2-14), and

$$(3\text{-}17) \qquad \Gamma \colon \mathfrak{Z}^{\bullet}_{c'+n,c'} \to \mathfrak{Z}^{\bullet}_{c'+n,c'} \times S$$

is the graph of the map p_S. The notation "$\oplus$" in (3-16) refers to a coherent sheaf which is filtered by the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$; since we work in K–theory, we henceforth make no distinction between this coherent sheaf and its associated graded object. We may also pull back the short exact sequence (3-16) to $\mathcal{M}_c \times \mathfrak{Z}^{\bullet}_{c'+n,c'} \times S$. Now apply the functor $\mathrm{R}\mathscr{H}om(-, \mathcal{U} \otimes m)$ to the short exact sequence (3-16), where $\mathcal{U}$ is the universal sheaf pulled back from $\mathcal{M}_c \times S$:

$$\mathrm{R}\mathscr{H}om(\mathcal{U}'_+, \mathcal{U} \otimes m) = \mathrm{R}\mathscr{H}om(\mathcal{U}'_-, \mathcal{U} \otimes m) - \sum_{i=1}^{n} \frac{1}{\mathcal{L}_i} \mathrm{R}\mathscr{H}om(\mathcal{O}_\Gamma, \mathcal{U} \otimes m).$$

Now recall that the line bundles $\mathcal{L}_i$ come from $\mathfrak{Z}^{\bullet}_{c'+n,c'}$, and so they are unaffected by the derived pushforward map ρ_*,

$$\rho_* \mathrm{R}\mathscr{H}om(\mathcal{U}'_+, \mathcal{U} \otimes m) = \rho_* \mathrm{R}\mathscr{H}om(\mathcal{U}'_-, \mathcal{U} \otimes m) - \sum_{i=1}^{n} \frac{1}{\mathcal{L}_i} \rho_* \mathrm{R}\mathscr{H}om(\mathcal{O}_\Gamma, \mathcal{U} \otimes m).$$

Recalling (1-5), the formula above reads

$$(3\text{-}18) \qquad (\mathrm{Id} \times p_+)^* \mathcal{E}_m = (\mathrm{Id} \times p_-)^* \mathcal{E}_m + \sum_{i=1}^{n} \frac{1}{\mathcal{L}_i} \rho_* \mathrm{R}\mathscr{H}om(\mathcal{O}_\Gamma, \mathcal{U} \otimes m).$$

Then (3-13) follows from the fact that

$$(3\text{-}19) \quad \rho_* \mathrm{R}\mathscr{H}om(\mathcal{O}_\Gamma, \mathcal{U} \otimes m) = \overbrace{\rho_* \circ \Gamma_*}^{\mathrm{Id}}\big(\mathrm{R}\mathscr{H}om(\mathcal{O}, \Gamma^!(\mathcal{U} \otimes m))\big) = \mathcal{U}m|_\Gamma \otimes \Gamma^! \mathcal{O}.$$

(The first equality is coherent duality, and the second equality holds for any closed embedding Γ.) The right-hand side of (3-19) matches $(\mathrm{Id} \times p_S)^*(\mathcal{U}m/q)$ because the map $\Gamma \colon \mathfrak{Z}_n^\bullet \to \mathfrak{Z}_n^\bullet \times S$ is obtained by base change from the diagonal map $S \to S \times S$, and the ratio of dualizing objects on S and $S \times S$ is precisely $q = [\omega_S]$.

As for (3-14), consider the diagram

(3-20)

$$
\begin{array}{c}
\mathfrak{Z}_{c,c-n}^\bullet \times \mathcal{M}_{c'} \times S \\
\end{array}
$$

with maps $p'_+ \times \mathrm{Id} \times \mathrm{Id}_S$, ρ, $p'_- \times \mathrm{Id} \times \mathrm{Id}_S$ to
$\mathcal{M}_c \times \mathcal{M}_{c'} \times S$,
$\mathfrak{Z}_{c,c-n}^\bullet \times \mathcal{M}_{c'}$,
$\mathcal{M}_{c-n} \times \mathcal{M}_{c'} \times S$; then ρ, $p'_+ \times \mathrm{Id}$, $p'_- \times \mathrm{Id}$, ρ to
$\mathcal{M}_c \times \mathcal{M}_{c'}$,
$\mathcal{M}_{c-n} \times \mathcal{M}_{c'}$

and consider the following analogue of (3-16):

$$0 \to \mathcal{U}_+ \to \mathcal{U}_- \to \Gamma'_*(\mathcal{L}_1 \text{“}\oplus\text{”} \cdots \text{“}\oplus\text{”} \mathcal{L}_n) \to 0,$$

where $\mathcal{U}_\pm = (p'^*_\pm \times \mathrm{Id}_S)(\mathcal{U})$, and Γ' denotes the graph of the map $p_S \colon \mathfrak{Z}_{c,c-n}^\bullet \to S$. Let us apply the functor $\mathrm{R}\mathscr{H}om(\mathcal{U}', - \otimes m)$ to the short exact sequence above:

$$\mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{U}_- \otimes m) = \mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{U}_+ \otimes m) + \sum_{i=1}^n \mathcal{L}_i \otimes \mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{O}_{\Gamma'} \otimes m).$$

Let us apply ρ_* to the equality above, and recall the definition of $\mathcal{E}_m$ in (1-5):

$$(p'_- \times \mathrm{Id})^* \mathcal{E}_m = (p'_+ \times \mathrm{Id})^* \mathcal{E}_m - \sum_{i=1}^n \mathcal{L}_i \otimes \rho_* \mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{O}_{\Gamma'} \otimes m).$$

By adjunction, we have

$$\rho_* \mathrm{R}\mathscr{H}om(\mathcal{U}', \mathcal{O}_{\Gamma'} \otimes m) = \overbrace{\rho_* \circ \Gamma'_*}^{\mathrm{Id}} \mathrm{R}\mathscr{H}om(\mathcal{U}'|_{\Gamma'}, p'^*_S m) = (\mathcal{U}'^\vee m)|_{\Gamma'}. \qquad \square$$

Armed with (3-13) and (3-14), we may rewrite (3-11) and (3-12) as

$$\Upsilon_n = [\wedge^\bullet \mathcal{E}_m] \cdot (\mathrm{Id} \times p_S \times p_-)_* \left[\bigotimes_{i=1}^n \wedge^\bullet \left(\frac{\mathcal{U}m}{\mathcal{L}_i q} \right) \right],$$

$$\Upsilon'_n = [\wedge^\bullet \mathcal{E}_m] \cdot (p'_+ \times p'_S \times \mathrm{Id})_* \left[\bigotimes_{i=1}^n \wedge^\bullet \left(-\frac{\mathcal{L}_i m}{\mathcal{U}'} \right) \right].$$

Henceforth, "$\mathcal{U}, \mathcal{U}'$" in the subscript of the integrals are simply shorthand for "the set of Chern roots of $\mathcal{U}, \mathcal{U}'$", respectively, and Proposition 2.13 implies

$$(3\text{-}21) \qquad \Upsilon_n = [\wedge^\bullet \mathcal{E}_m] \int\limits_{\mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}} \frac{\prod\limits_{i=1}^n \dfrac{\wedge^\bullet(\mathcal{U}m/(z_i q))}{\wedge^\bullet(\mathcal{U}'/z_i)}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)},$$

$$(3\text{-}22) \qquad \Upsilon'_n = [\wedge^\bullet \mathcal{E}_m] \int\limits_{\{0,\infty\} \sqcup \mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}} \frac{\prod\limits_{i=1}^n \dfrac{\wedge^\bullet(z_i q/\mathcal{U})}{\wedge^\bullet(z_i m/\mathcal{U}')}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)}.$$

Consider the rational function with coefficients in $K_{\mathcal{M}_c \times S \times \mathcal{M}_{c'}}$ given by

$$(3\text{-}23) \qquad I_n(z_1, \ldots, z_n) = \frac{\prod\limits_{i=1}^n \dfrac{\wedge^\bullet(\mathcal{U}m/(z_i q))}{\wedge^\bullet(\mathcal{U}'/z_i)}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)}.$$

One may then rewrite (3-21) and (3-22) as

$$\Upsilon_n = [\wedge^\bullet \mathcal{E}_m] \int\limits_{\mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}} I_n(z_1, \ldots, z_n),$$

$$\Upsilon'_n = [\wedge^\bullet \mathcal{E}_m] \int\limits_{\{0,\infty\} \sqcup \mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}} I_n(z_1 m, \ldots, z_n m) \cdot \gamma^{-n}.$$

Changing the variables $z_i \mapsto z_i/m$ in the second formula, we conclude that

$$(3\text{-}24) \quad \Upsilon_n - \Upsilon'_n \cdot \gamma^n = [\wedge^\bullet \mathcal{E}_m] \left[\int\limits_{\mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}} I_n - \int\limits_{\{0,\infty\} \sqcup \mathcal{U}' \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}} I_n \right].$$

The only difference between the two integrals is the location of the poles $\{0, \infty\}$ with respect to the variables $z_1, \ldots, z_n$. Therefore, we conclude that the difference above

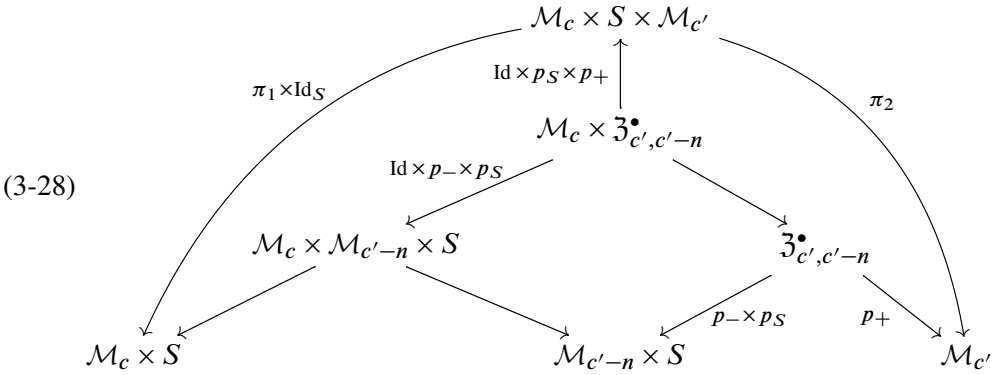picks up the residues at 0 and ∞ in the various variables. However, all such residues vanish, except for

$$(3\text{-}25) \qquad \operatorname*{Res}_{z_1=\infty} \frac{I_n(z_1,\ldots,z_n)}{z_1} = -I_{n-1}(z_2,\ldots,z_n),$$

$$(3\text{-}26) \qquad \operatorname*{Res}_{z_n=0} \frac{I_n(z_1,\ldots,z_n)}{z_n} = \gamma \cdot I_{n-1}(z_1,\ldots,z_{n-1}).$$

Therefore, formula (3-24) implies that

$$(3\text{-}27) \qquad \Upsilon_n - \Upsilon_n' \cdot \gamma^n = \Upsilon_{n-1} - \Upsilon_{n-1}' \cdot \gamma^n$$

which, as an equality of classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$, precisely encodes (3-5). Let us run the analogous computation for (3-6) (we will recycle all of our notation):

$$(3\text{-}28)$$

$$(3\text{-}29)$$
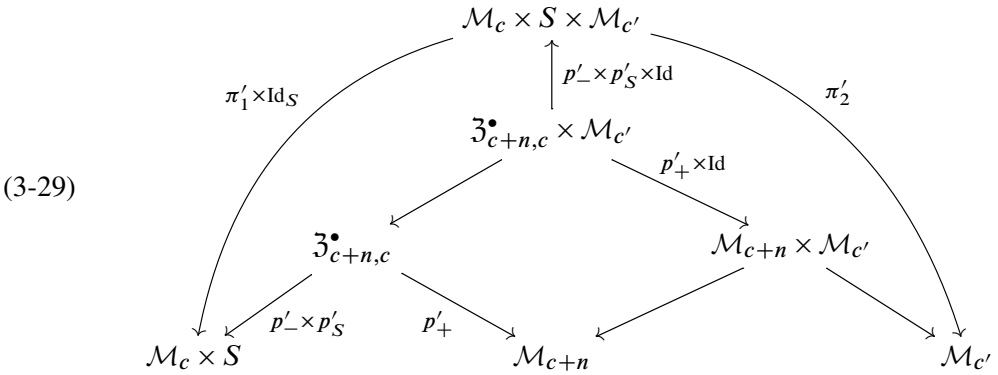
Recall that $H_n = (-1)^{rn} u^n (p_- \times p_S)_*(\mathcal{Q}^{-r} \cdot p_+^*)$, in the notation of (2-13). Then the rule for composition of correspondences in (2-2) gives us

$$(3\text{-}30) \qquad A_m H_n = (\pi_1 \times \mathrm{Id}_S)_*(\Upsilon_n \cdot \pi_2^*),$$

$$(3\text{-}31) \qquad H_n A_m = (\pi_1' \times \mathrm{Id}_S)_*(\Upsilon_n' \cdot \pi_2'^*),$$

where

$$(3\text{-}32) \qquad \Upsilon_n = (-1)^{rn} u'^n (\mathrm{Id} \times p_S \times p_+)_* \big[\mathcal{Q}^{-r} \cdot \wedge^\bullet \big((\mathrm{Id} \times p_-)^* \mathcal{E}_m \big) \big],$$

$$(3\text{-}33) \qquad \Upsilon'_n = (-1)^{rn} u^n (p'_- \times p'_S \times \mathrm{Id})_* \big[\mathcal{Q}^{-r} \cdot \wedge^\bullet \big((p'_+ \times \mathrm{Id})^* \mathcal{E}_m \big) \big]$$

are certain classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$. As a consequence of (3-13) and (3-14), which continue to hold as stated in the new setup, we may rewrite (3-32) and (3-33) as

$$\Upsilon_n = (-1)^{rn} u'^n [\wedge^\bullet \mathcal{E}_m] (\mathrm{Id} \times p_S \times p_+)_* \Big[\mathcal{Q}^{-r} \bigotimes_{i=1}^n \wedge^\bullet \Big(-\frac{\mathcal{U}m}{\mathcal{L}_i q} \Big) \Big],$$

$$\Upsilon'_n = (-1)^{rn} u^n [\wedge^\bullet \mathcal{E}_m] (p'_- \times p'_S \times \mathrm{Id})_* \Big[\mathcal{Q}^{-r} \bigotimes_{i=1}^n \wedge^\bullet \Big(\frac{\mathcal{L}_i m}{\mathcal{U}'} \Big) \Big].$$

Therefore, Proposition 2.13 implies

$$(3\text{-}34) \quad \Upsilon_n = [\wedge^\bullet \mathcal{E}_m] \int_{\{0,\infty\} \sqcup \mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}'} \frac{(-1)^{rn} u'^n z_1^{-r} \ldots z_n^{-r} \prod\limits_{i=1}^n \dfrac{\wedge^\bullet (z_i q / \mathcal{U}')}{\wedge^\bullet (\mathcal{U}m / (z_i q))}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)},$$

$$(3\text{-}35) \quad \Upsilon'_n = [\wedge^\bullet \mathcal{E}_m] \int_{\mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}'} \frac{(-1)^{rn} u^n z_1^{-r} \ldots z_n^{-r} \prod\limits_{i=1}^n \dfrac{\wedge^\bullet (z_i m / \mathcal{U}')}{\wedge^\bullet (\mathcal{U}/z_i)}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)}.$$

Consider the rational function with coefficients in $K_{\mathcal{M}_c \times S \times \mathcal{M}_{c'}}$ given by

$$(3\text{-}36) \qquad I_n(z_1, \ldots, z_n) = \frac{q^{rn} \prod\limits_{i=1}^n \dfrac{\wedge^\bullet (\mathcal{U}'/(z_i q))}{\wedge^\bullet (\mathcal{U}m/(z_i q))}}{\prod\limits_{i=1}^{n-1} (1 - (q z_{i+1}/z_i)) \prod\limits_{i<j} \zeta(z_j/z_i)}$$

One may then rewrite (3-34) and (3-35) as

$$\Upsilon_n = [\wedge^\bullet \mathcal{E}_m] \int_{\{0,\infty\} \sqcup \mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}'} I_n(z_1, \ldots, z_n),$$

$$\Upsilon'_n = [\wedge^\bullet \mathcal{E}_m] \int_{\mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}'} I_n \Big(\frac{z_1 m}{q}, \ldots, \frac{z_n m}{q} \Big) \cdot \gamma^n.$$

Changing the variables $z_i \mapsto z_i q / m$ in the second formula, we conclude that

$$(3\text{-}37) \quad \Upsilon_n - \Upsilon'_n \cdot \gamma^{-n} = [\wedge^\bullet \mathcal{E}_m] \Big[\int_{\{0,\infty\} \sqcup \mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \mathcal{U}'} I_n - \int_{\mathcal{U} \prec z_n \prec \cdots \prec z_1 \prec \{0,\infty\} \sqcup \mathcal{U}'} I_n \Big].$$

The only difference between the two integrals is the location of the poles $\{0, \infty\}$ with respect to the variables $z_1, \ldots, z_n$. Therefore, we conclude that the difference above picks up the residues at 0 and ∞ in the various variables. However, all such residues vanish, except for

$$\operatorname*{Res}_{z_n=0} \frac{I_n(z_1, \ldots, z_n)}{z_n} = \gamma^{-1} \cdot I_{n-1}(z_1, \ldots, z_{n-1}),$$

$$\operatorname*{Res}_{z_1=\infty} \frac{I_n(z_1, \ldots, z_n)}{z_1} = -q^r \cdot I_{n-1}(z_2, \ldots, z_n).$$

Therefore, formula (3-37) implies that

$$(3\text{-}38) \qquad \Upsilon_n - \Upsilon'_n \cdot \gamma^{-n} = \Upsilon_{n-1} \cdot \gamma^{-1} - \Upsilon'_{n-1} \cdot q^r \gamma^{-n+1},$$

which, as an equality of classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$, precisely encodes (3-6). □

3.8 We will now show how Lemma 3.6 allows us to prove Theorem 1.10.

Proof of Theorem 1.10 We will only prove (1-14), since (1-15) is analogous. We will use formulas (2-25), which say that the H operators are to the P operators as complete symmetric functions are to power sum functions. Then let us place (3-5) into a generating series that goes over all $n \in \mathbb{N}$,

$$(3\text{-}39) \qquad \sum_{n=0}^{\infty} A_m H_{-n}(z^n - z^{n+1}) = \sum_{n=0}^{\infty} ((\gamma z)^n - (\gamma z)^{n+1}) H_{-n} A_m.$$

If we write $H_-(z)$ for the power series (2-25) (with sign $\pm = -$), then (3-39) reads

$$(3\text{-}40) \qquad A_m H_-(z)(1 - z) = H_-(z\gamma)(1 - \gamma z) A_m.$$

If P is an operator $K_\mathcal{M} \to K_{\mathcal{M} \times S}$ which commutes with two line bundles ℓ and ℓ' (in the sense of Proposition 3.4, and the discussion after it), then

$$(3\text{-}41) \quad A \exp(P) \exp(\ell')|_\Delta = \exp(P) \exp(\ell)|_\Delta A \iff AP + A\ell' = PA + \ell A.$$

(This claim uses the associativity of the operation $x, y \rightsquigarrow xy|_\Delta$, as discussed in Section 2.3.) With this in mind, formula (3-40) implies

$$A_m P_-(z) - \sum_{n=1}^{\infty} \frac{A_m}{nz^{-n}} = P_-(z\gamma) A_m - \sum_{n=1}^{\infty} \gamma^n \frac{A_m}{nz^{-n}},$$

where $P_-(z) = \sum_{n=1}^{\infty} P_{-n}/(nz^{-n})$. Extracting the coefficient of z^n yields precisely equation (1-14). □

3.9 Having proved [Lemma 3.6](), we will now perform the analogous computations for the commutator of A_m with the operators of [Section 2.5]().

Lemma 3.10 *We have the following relations involving the Ext operator A_m:*

$$(3\text{-}42) \quad A_m L_n(y) - A_m L_{n-1}(y)$$
$$= L_n\left(\frac{y}{m}\right) A_m \cdot \gamma^n - L_{n-1}\left(\frac{yq}{m}\right) E\left(\frac{yq}{m}\right) A_m E(y)^{-1}\bigg|_{\Delta} \cdot \gamma^{n-1},$$

$$(3\text{-}43) \quad U_n\left(\frac{yq}{m}\right) A_m \cdot \gamma^{-n} - U_{n-1}\left(\frac{yq}{m}\right) A_m \cdot q\gamma^{-n+1}$$
$$= A_m U_n(y) - E\left(\frac{yq}{m}\right)^{-1} A_m E(yq) U_{n-1}(yq)\bigg|_{\Delta} \cdot q.$$

The two sides of [(3-42)]() and [(3-43)]() map $K_{\mathcal{M}'}$ to $K_{\mathcal{M} \times S}[\![y^{-1}]\!]$. The symbol $|_{\Delta}$ applied to any term that involves three of the series L, E, U means that we restrict a certain operator $K_{\mathcal{M}'} \to K_{\mathcal{M} \times S \times S \times S}[\![y^{-1}]\!]$ to the small diagonal.

Proof In order to prove [(3-42)](), we will closely follow the proof of [Lemma 3.6](). With the notation therein, one needs to replace [(3-11)]() and [(3-12)]() by

$$\Upsilon_{n,y} = (\mathrm{Id} \times p_S \times p_-)_* \left[\frac{1}{1 - (y/\mathcal{L}_n)} \wedge^{\bullet} \left((\mathrm{Id} \times p_+)^* \mathcal{E}_m \right) \right],$$
$$\Upsilon'_{n,y} = (p'_+ \times p'_S \times \mathrm{Id})_* \left[\frac{1}{1 - (y/\mathcal{L}_n)} \wedge^{\bullet} \left((p'_- \times \mathrm{Id})^* \mathcal{E}_m \right) \right].$$

This has the effect of inserting

$$\left(1 - \frac{y}{z_n}\right)^{-1}$$

into the right-hand sides of formulas [(3-21)]() and [(3-22)](). Therefore, the function $I_n(z_1, \ldots, z_n)$ defined in [(3-23)]() should be replaced by

$$I_{n,y}(z_1, \ldots, z_n) = \frac{I_n(z_1, \ldots, z_n)}{1 - (y/z_n)}.$$

It is easy to see that the nonzero residues of $I_{n,y}$ are

$$\operatorname*{Res}_{z_1 = \infty} \frac{I_{n,y}(z_1, \ldots, z_n)}{z_1} = -I_{n-1,y}(z_2, \ldots, z_n),$$
$$\operatorname*{Res}_{z_n = y} \frac{I_{n,y}(z_1, \ldots, z_n)}{z_n} = \frac{\wedge^{\bullet}(\mathcal{U}m/(yq))}{\wedge^{\bullet}(\mathcal{U}'/y)} \cdot \frac{I_{n-1,yq}(z_1, \ldots, z_{n-1})}{\prod_{i=1}^{n-1} \zeta(y/z_i)}.$$

Therefore, the analogue of identity (3-27) is

$$\Upsilon_{n,y} - \Upsilon'_{n,y/m} \cdot \gamma^n = \Upsilon_{n-1,y} - \Upsilon'_{n-1,yq/m} \cdot \gamma^{n-1} \frac{\wedge^\bullet(\mathcal{U}m/(yq))}{\wedge^\bullet(\mathcal{U}'/y)}.$$

This equality of classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$ precisely underlies equality (3-42).

As for (3-43), we proceed analogously. One needs to replace (3-32) and (3-33) by

$$\Upsilon_n = \frac{(-1)^{rn} u'^n}{q^{(r-1)n}} (\mathrm{Id} \times p_S \times p_+)_* \left[\frac{\mathcal{Q}^{-r}}{1 - y/\mathcal{L}_n} \cdot \wedge^\bullet\big((\mathrm{Id} \times p_-)^* \mathcal{E}_m\big) \right],$$

$$\Upsilon'_n = \frac{(-1)^{rn} u^n}{q^{(r-1)n}} (p'_- \times p'_S \times \mathrm{Id})_* \left[\frac{\mathcal{Q}^{-r}}{1 - y/\mathcal{L}_n} \cdot \wedge^\bullet\big((p'_+ \times \mathrm{Id})^* \mathcal{E}_m\big) \right].$$

This has the effect of inserting

$$q^{n(1-r)}\left(1 - \frac{y}{z_n}\right)^{-1}$$

into the right-hand sides of formulas (3-34) and (3-35). Therefore, the function I_n defined in (3-36) should be replaced by

$$I_{n,y}(z_1, \ldots, z_n) = \frac{I_n(z_1, \ldots, z_n)}{q^{(r-1)n}(1 - y/z_n)}.$$

It is easy to see that the nonzero residues of $I_{n,y}$ are

$$\operatorname*{Res}_{z_n=y} \frac{I_{n,y}(z_1, \ldots, z_n)}{z_n} = q \frac{\wedge^\bullet(\mathcal{U}'/(yq))}{\wedge^\bullet(\mathcal{U}m/(yq))} \cdot \frac{I_{n-1,yq}(z_1, \ldots, z_{n-1})}{\prod\limits_{i=1}^{n-1} \zeta(y/z_i)},$$

$$\operatorname*{Res}_{z_1=\infty} \frac{I_{n,y}(z_1, \ldots, z_n)}{z_1} = -q \cdot I_{n-1,y}(z_2, \ldots, z_n).$$

Therefore, the analogue of identity (3-38) is

$$\Upsilon_{n,y} - \Upsilon_{n,yq/m} \cdot \gamma^{-n} = \Upsilon_{n-1,yq} \cdot q \frac{\wedge^\bullet(\mathcal{U}'/(yq))}{\wedge^\bullet(\mathcal{U}m/(yq))} - \Upsilon_{n-1,yq/m} \cdot q\gamma^{-n+1}.$$

This equality of classes on $\mathcal{M}_c \times S \times \mathcal{M}_{c'}$ precisely underlies equality (3-43). □

3.11 In all formulas below, whenever one encounters a product of several L, E, U operators, one needs to place the symbol $|_\Delta$ next to it, eg $L(\ldots)E(\ldots)U(\ldots)|_\Delta$ as in (2-20). From now on, we will suppress the notation $|_\Delta$ from our formulas for brevity.

Proof of Theorem 1.7 In terms of the generating series (2-32), formulas (3-42) and (3-43) take the form

$$(1-x)A_m L(x,y) = L\left(x\gamma, \frac{y}{m}\right)A_m - xL\left(x\gamma, \frac{yq}{m}\right)E\left(\frac{yq}{m}\right)A_m E(y)^{-1},$$

$$U\left(x\gamma, \frac{yq}{m}\right)A_m\left(1-\frac{q}{x}\right) = A_m U(x,y) - \frac{q}{x}E\left(\frac{yq}{m}\right)^{-1}A_m E(yq)U(x,yq).$$

Change the variables $x \mapsto xq$, $y \mapsto y/q$ in the second equation, and multiply the first equation by $E(y)$ and the second equation by $E(y/m)$. Thus we obtain

$$(1-x)A_m L(x,y)E(y) = L\left(x\gamma, \frac{y}{m}\right)A_m E(y)$$
$$- xL\left(x\gamma, \frac{yq}{m}\right)E\left(\frac{yq}{m}\right)A_m,$$

$$E\left(\frac{y}{m}\right)U\left(xq\gamma, \frac{y}{m}\right)A_m\left(1-\frac{1}{x}\right) = E\left(\frac{y}{m}\right)A_m U\left(xq, \frac{y}{q}\right) - \frac{1}{x}A_m E(y)U(xq,y).$$

Now let us replace the variable y by the symbol yD_x, where D_x denotes the $q-$difference operator $D_x(f(x)) = f(xq)$. However, we make the following prescription: in the first equation above, the D_x's are placed to the right of all x's, while in the second equation, the D_x's are placed to the left of all the x's. We thus obtain

$$(1-x)A_m L(x,yD_x)E(yD_x)$$
$$= L\left(x\gamma, \frac{yD_x}{m}\right)A_m E(yD_x) - xL\left(x\gamma, \frac{yD_x q}{m}\right)E\left(\frac{yD_x q}{m}\right)A_m,$$

$$E\left(\frac{yD_x}{m}\right)U\left(xq\gamma, \frac{yD_x}{m}\right)A_m(1-x)$$
$$= A_m E(yD_x)U(xq, yD_x) - E\left(\frac{yD_x}{m}\right)A_m U\left(xq, \frac{yD_x}{q}\right)x.$$

Now let us multiply the first equation on the right by $U(qx, yD_x)$ (with the D_x's placed to the left of all the x's) and the second equation on the left by $L(x\gamma, yD_x/m)$ (with the D_x's placed to the right of all the x's):

$$(1-x)A_m L(x,yD_x)E(yD_x)U(xq, yD_x)$$
$$= L\left(x\gamma, \frac{yD_x}{m}\right)A_m E(yD_x)U(xq, yD_x)$$
$$- xL\left(x\gamma, \frac{yD_x q}{m}\right)E\left(\frac{yD_x q}{m}\right)A_m U(xq, yD_x)$$

and

$$L\left(x\gamma, \frac{yD_x}{m}\right)E\left(\frac{yD_x}{m}\right)U\left(xq\gamma, \frac{yD_x}{m}\right)A_m(1-x)$$

$$= L\left(x\gamma, \frac{yD_x}{m}\right)A_m E(yD_x)U(xq, yD_x)$$

$$- L\left(x\gamma, \frac{yD_x}{m}\right)E\left(\frac{yD_x}{m}\right)A_m U\left(xq, \frac{yD_x}{q}\right)x.$$

The two terms in the right-hand sides of the above equations are pairwise equal to each other (this is not manifestly obvious for the second term, because y differs from yq, but this is a consequence of commuting D_x past x). We conclude that

$$(1-x)A_m L(x, yD_x)E(yD_x)U(xq, yD_x)$$

$$= L\left(x\gamma, \frac{yD_x}{m}\right)E\left(\frac{yD_x}{m}\right)U\left(xq\gamma, \frac{yD_x}{m}\right)A_m(1-x).$$

Recalling the definition (2-35), this implies

$$(1-x)A_m W(x, yD_x) = W\left(x\gamma, \frac{yD_x}{m}\right)A_m(1-x).$$

Taking the coefficient of $(yD_x)^{-k}$ implies (1-12). In doing so, the right-most factor $1-x$ changes into $1-x/q^k$ due to the fact that the operators $1/D_x^k$ must pass over it. □

3.12 Finally, we recall the operator $\Phi_m \colon K_{\mathcal{M}'} \to K_{\mathcal{M}}$ defined in (1-16),

$$\Phi_m = A_m \exp\left[\sum_{n=1}^{\infty} \frac{P_n}{n}\left\{\frac{(q^n-1)q^{-rn}}{[n]_{q_1}[n]_{q_2}}\right\}\right],$$

and let us translate (1-12), (1-14) and (1-15) into commutation relations involving Φ_m.

Proof of Corollary 1.11 Since P_n commutes with $P_{n'}$ for all $n, n' > 0$, (1-15) implies (1-18) when the sign is $+$. Let us now prove (1-18) when the sign is $-$. We write

$$\Phi_m = A_m \cdot \exp,$$

where exp is shorthand for

$$\exp\left[\sum_{n=1}^{\infty} \frac{P_n}{n}\left\{\frac{(q^n-1)q^{-rn}}{[n]_{q_1}[n]_{q_2}}\right\}\right].$$

Then (1-14) reads

$$\Phi_m \cdot \exp^{-1} \cdot P_{-n} - P_{-n} \cdot \Phi_m \cdot \exp^{-1}\gamma^n = \Phi_m \cdot \exp^{-1}(1-\gamma^n).$$

The relation above will establish (1-18) for $\pm = -$ once we prove that

$$(3\text{-}44) \qquad [\exp^{-1}, P_{-n}] = (1 - q^{-rn}) \exp^{-1}.$$

If we take the logarithm of (3-44), it boils down to

$$(3\text{-}45) \qquad \left[P_{-n}, \frac{P_n}{n} \left\{ \frac{(q^n - 1)q^{-nr}}{[n]_{q_1}[n]_{q_2}} \right\} \right] = 1 - q^{-rn}.$$

Relation (3-45) is an equality of operators $K_{\mathcal{M}} \to K_{\mathcal{M} \times S}$ (the right-hand side denotes pullback multiplied by $\text{proj}_S^*(1 - q^{-rn})$), and it is proved as follows. Take equality (2-29) of operators $K_{\mathcal{M}} \to K_{\mathcal{M} \times S \times S}$, multiply it by

$$(3\text{-}46) \qquad \text{proj}_3^* \left(\frac{1}{n} \cdot \frac{(q^n - 1)q^{-nr}}{[n]_{q_1}[n]_{q_2}} \right) \in K_{\mathcal{M} \times S \times S}$$

and then apply proj_{12*} to the result (above, we write $\mathcal{M} \times S \times S \xrightarrow{\text{proj}_{12}, \ \text{proj}_3} \mathcal{M} \times S, \ S$ for the obvious projection maps). The outcome of this procedure is precisely (3-45).

Now let us prove (1-12) $\Longrightarrow$ (1-17). To do so, we must take formula (2-36) for $\pm = +$ (which is a priori an equality of operators $K_{\mathcal{M}} \to K_{\mathcal{M} \times S \times S}$), multiply it by (3-46) and then apply proj_{12*} to the result. The resulting equality reads

$$\left[W_k(x), \frac{P_n}{n} \left\{ \frac{(q^n - 1)q^{-nr}}{[n]_{q_1}[n]_{q_2}} \right\} \right] = \frac{(1 - q^{-nk})x^n}{n} W_k(x).$$

It is easy to show that $[W, P] = cW$ implies that $\exp(-P)W = \exp(c) \cdot W \exp(-P)$ as long as c commutes with both W and P. Therefore, we infer that

$$\exp^{-1} W_k(x) = \exp\left[\sum_{n=1}^{\infty} \frac{(1 - q^{-nk})x^n}{n} \right] W_k(x) \exp^{-1}$$

$$\Longrightarrow \qquad \exp^{-1} W_k(x) = \frac{1 - (x/q^k)}{1 - x} \cdot W_k(x) \exp^{-1}$$

$$\Longrightarrow \quad \Phi_m \exp^{-1} W_k(x) \cdot (1 - x) = \Phi_m W_k(x) \exp^{-1} \cdot \left(1 - \frac{x}{q^k} \right).$$

With this in mind, (1-12) and the fact that $\Phi_m \exp^{-1} = A_m$ imply that

$$m^k W_k(x\gamma) \Phi_m \exp^{-1} \cdot \left(1 - \frac{x}{q^k} \right) = \Phi_m W_k(x) \exp^{-1} \cdot \left(1 - \frac{x}{q^k} \right)$$

Multiplying on the right with $\exp$ yields (1-17). $\qquad \square$

4 The Verma module

4.1 Let us now specialize to $S = \mathbb{A}^2$, and explain all the necessary modifications to the constructions in the present paper; we refer the reader to [14, Section 3] for details. From here on, let $\mathcal{M}$ be the moduli space parametrizing rank r torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$, together with a trivialization along a fixed line $\infty \subset \mathbb{P}^2$:

$$\mathcal{M} = \{\mathcal{F}, \mathcal{F}|_\infty \stackrel{\phi}{\cong} \mathcal{O}_\infty^r\}.$$

The c_1 of such sheaves is forced to be 0, but c_2 is free to vary over the nonnegative integers, and so the moduli space breaks up into connected components as before:

$$\mathcal{M} = \bigsqcup_{c=0}^{\infty} \mathcal{M}_c.$$

The space $\mathcal{M}$ is acted on by the torus $T = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r$, where the first two factors act by scaling $\mathbb{A}^2$, and the latter r factors act on the framing ϕ. Note that

$$K_0^T(\mathrm{pt}) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}],$$

where $q_1, q_2, u_1, \ldots, u_r$ are the standard elementary characters of the torus T. We note that q_1 and q_2 are the equivariant weights of $\Omega_{\mathbb{A}^2}^1$, and the determinant of the universal sheaf $\mathcal{U}$ is the equivariant constant $u = u_1 \cdots u_r$. Consider the group

$$K_{\mathcal{M}} = \bigoplus_{c=0}^{\infty} K_0^T(\mathcal{M}_c) \otimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}]} \mathbb{Q}(q_1, q_2, u_1, \ldots, u_r)$$

The main goal of loc. cit. was to define operators akin to (2-20), (2-21) and (2-23),

$$(4\text{-}1) \qquad\qquad W_{n,k}, P_{\pm n'} : K_{\mathcal{M}} \to K_{\mathcal{M}}$$

for all $n \in \mathbb{Z}$ and $k, n' \in \mathbb{N}$, and then show that these operators satisfy the relations in the deformed W–algebra of type $\mathfrak{gl}_r$ (since $S = \mathbb{A}^2$, $K_{\mathcal{M}} \cong K_{\mathcal{M} \times S}$ naturally).

Definition 4.2 [14, Definition 2.28] Let $q_1, q_2, u_1, \ldots, u_r$ be formal symbols. The universal Verma module $M_{u_1, \ldots, u_r}$ is the $\mathbb{Q}(q_1, q_2, u_1, \ldots, u_r)$–vector space with basis

$$(4\text{-}2) \qquad\qquad W_{n_1, k_1} \ldots W_{n_s, k_s} |\varnothing\rangle$$

as the pairs (n_i, k_i) range over $-\mathbb{N} \times \{1, \ldots, r\}$ and are ordered in nondecreasing order of the slope n_i/k_i. We make $M_{u_1, \ldots, u_r}$ into a deformed W–algebra module as follows.

The action of an arbitrary generator $W_{n,k}$ on the basis vector (4-2) is prescribed by the commutation relations (2-28), together with the relations

$$W_{n,k}|\varnothing\rangle = 0 \qquad \text{if } n > 0 \text{ or } k > r,$$
$$W_{0,k}|\varnothing\rangle = e_k(u_1,\ldots,u_r)|\varnothing\rangle \quad \text{for all } k,$$

where e_k denotes the kth elementary symmetric polynomial.

Theorem 4.3 [14, Theorem 3.12] *We have an isomorphism of modules for the deformed W–algebra of type $\mathfrak{gl}_r$ (the action on the left-hand side is given by* (4-1)*)*

(4-3) $$K_{\mathcal{M}} \cong M_{u_1,\ldots,u_r},$$

induced by sending the K–theory class of the structure sheaf of $\mathcal{M}_0 \subset \mathcal{M}$ to $|\varnothing\rangle$.

4.4 The Ext (respectively vertex) operator A_m (respectively Φ_m) for $S = \mathbb{A}^2$ was studied in [14, Section 4], where we obtained an analogue of Theorem 1.7 in the case $k = 1$ (some coefficients in the formulas of loc. cit. differ from those of the present paper, because their operator A_m differs from ours by an equivariant constant). However, having only proved the case $k = 1$ in loc. cit. led to weaker formulas than (1-12). Thus, the present paper strengthens the results of loc. cit.; see Remark 4.8 therein. Specifically, Corollary 1.11 completely determines the operator Φ_m (hence also A_m) in the case $S = \mathbb{A}^2$, due to Theorems 4.3 and 4.5.

Theorem 4.5 *Given two Verma modules $M_{u_1,\ldots,u_r}$ and $M_{u'_1,\ldots,u'_r}$, there is a unique (up to constant multiple in $\mathbb{Q}(q_1, q_2, u_1,\ldots,u_r, u'_1,\ldots,u'_r)$) linear map*

$$\Phi_m\colon M_{u'_1,\ldots,u'_r} \to M_{u_1,\ldots,u_r}$$

satisfying (1-17) *for all $k \geq 1$.*

Proof The existence of such a linear map follows from the very fact that the operator (1-16) satisfies (1-17). To show uniqueness, it is enough to prove $\langle\varnothing|\Phi_m|\varnothing\rangle = 0$ implies $\Phi_m = 0$, for any operator that satisfies the following relations for all n, k:

(4-4) $$\Phi_m W_{n,k} - \Phi_m W_{n+1,k} \cdot q^{-k} = W_{n,k} \Phi_m \cdot m^k \gamma^{-nk} - W_{n+1,k} \Phi_m \cdot \frac{m^k}{q^k} \gamma^{-(n+1)k},$$

where m and γ are certain nonzero constants.

Claim 4.6 *For any parameters $u_1, \ldots, u_r$, there exists a nondegenerate pairing*

$$M_{u_1,\ldots,u_r} \otimes M_{u_1,\ldots,u_r} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Q}(q_1, q_2, u_1, \ldots, u_r)$$

such that the adjoint of $W_{n,k}$ is $W_{-n,k}$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Proof Using Theorem 4.3, the required pairing is provided by the equivariant Euler characteristic pairing on $K_{\mathcal{M}}$ (renormalized as in [14, Section 3.14]). The operators $W_{n,k}$ and $W_{-n,k}$ are adjoint with respect to this pairing [14, formula (3.39)]. □

Let us now complete the proof of Theorem 4.5. Because Verma modules are generated by $W_{n,k}$ acting on $\varnothing$, then we must show that $\langle \varnothing | \Phi_m | \varnothing \rangle = 0$ implies

$$(4\text{-}5) \qquad \langle \varnothing | W_{-n_s,k_s} \ldots W_{-n_1,k_1} \Phi_m W_{n'_1,k'_1} \ldots W_{n'_t,k'_t} | \varnothing \rangle = 0$$

for all collections of indices $(n_i, k_i), (n'_i, k'_i) \in \mathbb{Z}_{\leq 0} \times \{1, \ldots, r\}$, ordered by slope

$$\frac{n_1}{k_1} \leq \cdots \leq \frac{n_s}{k_s} \quad \text{and} \quad \frac{n'_1}{k'_1} \leq \cdots \leq \frac{n'_t}{k'_t}.$$

The matrix coefficient (4-5) is nonzero only if the n_i and the n'_j are all nonpositive, so we will prove formula (4-5) by induction on the nonpositive integer $\delta = \sum n_i + \sum n'_i$. We may assume that $n_s, n'_t < 0$ because $W_{0,k} | \varnothing \rangle$ is a multiple of $| \varnothing \rangle$ for any k. The base case $\delta = 0$ of the induction is simply the assumption $\langle \varnothing | \Phi_m | \varnothing \rangle = 0$. As for the induction step, let us iterate relation (4-4) to obtain

$$\Phi_m W_{n'_1,k'_1} \cdots W_{n'_t,k'_t} \in \text{span} \begin{cases} \Phi_m W_{n'_1+\varepsilon_1,k'_1} \cdots W_{n'_t+\varepsilon_t,k'_t}, \\ W_{n'_1+\varepsilon'_1,k'_1} \cdots W_{n'_t+\varepsilon'_t,k'_t} \Phi_m, \end{cases}$$

where $\varepsilon_1, \ldots, \varepsilon_t \in \{0, 1\}$ are not all 0, and $\varepsilon'_1, \ldots, \varepsilon'_t \in \{0, 1\}$. That means that the left-hand side of (4-5) is a linear combination of

$$\langle \varnothing | W_{-n_s,k_s} \cdots W_{-n_1,k_1} \Phi_m W_{n'_1+\varepsilon_1,k'_1} \cdots W_{n'_t+\varepsilon_t,k'_t} | \varnothing \rangle,$$

which is 0 by the induction hypothesis, because the ε_i are not all 0, and

$$(4\text{-}6) \qquad \langle \varnothing | W_{-n_s,k_s} \cdots W_{-n_1,k_1} W_{n'_1+\varepsilon'_1,k'_1} \cdots W_{n'_t+\varepsilon'_t,k'_t} \Phi_m | \varnothing \rangle.$$

The induction step will be complete once we show that (4-6) is 0. As a consequence of (2-28), the product of W's in (4-6) can be written as a linear combination of

$$W_{-n''_r,k''_r} \cdots W_{-n''_1,k''_1} \quad \text{with} \quad \frac{n''_1}{k''_1} \leq \cdots \leq \frac{n''_r}{k''_r},$$

and $\sum n_i'' = \sum n_i - \sum n_i' - \sum \varepsilon_i'$ for degree reasons. If $n_r'' > 0$, then the product of W's above annihilates $\langle \varnothing |$. Thus, we may assume $n_r'' \le 0$, in which case the fact that

$$\sum n_i'' = \sum n_i - \sum n_i' - \sum \varepsilon_i' > \sum n_i + \sum n_i'$$

(recall that $n_i' < 0$ by assumption, while $\varepsilon_i' \in \{0, 1\}$) means that we can apply the induction hypothesis to conclude that (4-6) is 0. $\qquad\square$

We note that the identification of A_m (in the case $S = \mathbb{A}^2$) with a vertex operator was also achieved in [3], which computed relations (3-42) and (3-43) for $n = 1$ in the basis of fixed points. This uniquely determines the operator A_m due to certain features of the Ding–Iohara–Miki algebra, but does not directly establish the connection with the generating currents of the deformed W–algebra of $\mathfrak{gl}_r$. From a geometric point of view, this is because the Nakajima-type simple correspondences only describe the operators $L_{1,k}$ and $U_{1,k}$. As we have seen in Section 2.4, in order to define the operators $L_{n,k}$ and $U_{n,k}$ for all n (with the ultimate goal of defining the W–algebra generators $W_{n,k}$ in (2-20)), one needs to introduce the more complicated correspondences (2-11).

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