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The deformation space of geodesic triangulations and generalized Tutte's embedding theorem

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We prove the contractibility of the deformation space of the geodesic triangulations on a closed surface of negative curvature. This solves an open problem, proposed by Connelly, Henderson, Ho and Starbird (1983), in the case of hyperbolic surfaces. The main part of the proof is a generalization of Tutte's embedding theorem for closed surfaces of negative curvature.

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1 Introduction

We study the deformation space of geodesic triangulations of a surface within a fixed homotopy class. Such a space can be viewed as a discrete analogue of the space of surface diffeomorphisms homotopic to the identity. Our main theorem is:

Theorem 1.1 *For a closed orientable surface of negative curvature, the space of geodesic triangulations in a homotopy class is contractible. In particular, it is connected.*

The group of diffeomorphisms of a smooth surface is a fundamental object in the study of low-dimensional topology. Determining the homotopy types of diffeomorphism groups has profound implications for a wide range of problems in Teichmüller spaces, mapping class groups, and geometry and topology of 3–manifolds. Smale [23] proved that the group of diffeomorphisms of a closed 2–disk which pointwise fix the boundary is contractible. This enabled him to show that the group of orientation-preserving diffeomorphisms of the 2–sphere is homotopy equivalent to SO(3) [23]. Earle and Eells [10] identified the homotopy type of the group of diffeomorphisms homotopic to the identity for any closed surface. In particular, this topological group is contractible for a closed orientable surface with genus greater than one. It is consistent with our Theorem 1.1 for the discrete analogue.

Cairns [6] initiated the investigation of the topology of the space of geodesic triangulations and proved that, if the surface is a geometric triangle in the Euclidean plane, the space of geodesic triangulations with fixed boundary edges is connected. A series of further developments culminated in a discrete version of Smale's theorem, proved by Bloch, Connelly and Henderson [2]:

Theorem 1.2 *The space of geodesic triangulations of a convex polygon with fixed boundary edges is homeomorphic to a Euclidean space. In particular*, *it is contractible.*

A simple proof of the contractibility of the space above is provided in Luo [21] using Tutte's embedding theorem [24]. It also provides examples showing that the homotopy type of this space can be complicated if the boundary of the polygon is not convex. For closed surfaces it is conjectured in Connelly, Henderson, Ho and Starbird [9] that:

Conjecture 1.3 *The space of geodesic triangulations of a closed orientable surface with constant curvature deformation retracts to the group of isometries of the surface homotopic to the identity.*

The connectivity of these spaces has been explored by Cairns [6], Chambers, Erickson, Lin and Parsa [7] and Hass and Scott [18]. Awartani and Henderson [1] identified a contractible subspace in the space of geodesic triangulations of the 2–sphere. Hass and Scott [18] showed that the space of geodesic triangulation of a surface with a hyperbolic metric is contractible if the triangulation contains only one vertex. Recently, the authors [22] and Erickson and Lin [11] proved this conjecture independently in the case of flat tori. Our main result affirms Conjecture 1.3 in the case of hyperbolic surfaces and generalizes its conclusion to surfaces of negative curvatures.

One practical application of our work concerns the graph morphing on higher-genus surfaces. Computing morphs between graphs has a wide range of applications in geometric comparison, animation, and modeling. The 1–skeleton of geodesic triangular mesh is one of the most common graphs on a surface. As a fundamental result, our main theorem implies that, on a closed surface of negative curvature, any two geodesic triangular meshes can be morphed to each other if they have the same combinatorial structure. Furthermore, in the proof of our main theorem, we generalize Tutte's embedding theorem to higher-genus surfaces. Following the idea initiated by Floater and Gotsman [14], we can explicitly construct such morphs by linearly interpolating the nonsymmetric edge weights. A similar idea has been applied for graph morphing on flat tori in work by Chambers, Erickson, Lin and Parsa [7] and Erickson and Lin [11].

1.1 Setup and the main theorem

Assume M is a connected closed orientable smooth surface with a smooth Riemannian metric g of nonpositive Gaussian curvature. A topological triangulation of M can be identified as a homeomorphism ψ from $|T|$ to M, where $|T|$ is the carrier of a 2–dimensional simplicial complex $T=(V,E,F)$ with the vertex set V, the edge set E, and the face set F. For convenience, we label the vertices as $1,2,\dots,n$, where $n=|V|$ is the number of vertices. The edge in E determined by vertices i and j is written ij. Each edge is identified with the closed unit interval $[0,1]$.

Let $T^{(1)}$ be the 1–skeleton of T, and denote by $X=X(M,T,\psi)$ the space of geodesic triangulations homotopic to $\psi|_{T^{(1)}}$. More specifically, X contains all the embeddings $\varphi\colon T^{(1)}\to M$ such that

(i) the restriction φ_{ij} of φ to the edge ij is a geodesic parametrized with constant speed, and

(ii) φ is homotopic to $\psi|_{T^{(1)}}$.

Given an embedding φ in X, φ_{ij} is often identified as a map from $[0,1]$ to M such that $\varphi(0)=i$, $\varphi(1)=j$ and $\varphi_{ij}(t)$ represents the point on the edge ij that is t along the geodesic from i to j parametrized on $[0,1]$.

It has been proved by Colin de Verdière [8] that such $X(M,T,\psi)$ is always nonempty. Further, X is naturally a metric space, with the distance function

$$d_X(\varphi,\phi)=\max_x d_g(\varphi(x),\phi(x)).$$

Then our main theorem is formally stated as follows:

Theorem 1.4 *If (M,g) has strictly negative Gaussian curvature, then $X(M,T,\psi)$ is contractible. In particular, it is connected.*

Here we consider only surfaces of negative curvature since this ensures the uniqueness of the geodesic in a homotopy class, and our estimates using the $\mathrm{CAT}(k)$ comparison theorems of triangles rely on a strictly negative upper bound of the curvature of the surface.

1.2 Generalized Tutte's embedding

Let $\widetilde{X} = \widetilde{X}(M, T, \psi)$ be the superspace of X containing all the continuous maps $\varphi\colon T^{(1)} \to M$ satisfying that

(i) the restriction φ_{ij} of φ to the edge ij is a geodesic parametrized with constant speed, and

(ii) φ is homotopic to $\psi|_{T^{(1)}}$.

The key difference between X and $\widetilde{X}$ is that elements in $\widetilde{X}$ may not be embeddings of $T^{(1)}$ to M. The space $\widetilde{X}$ is also naturally a metric space, with the same distance function

$$d_{\widetilde{X}}(\varphi, \phi) = \max_x d_g(\varphi(x), \phi(x)).$$

We call an element in $\widetilde{X}$ a *geodesic mapping*. A geodesic mapping is determined by the positions $q_i = \varphi(i)$ of the vertices and the homotopy classes of φ_{ij} relative to the endpoints q_i and q_j. In particular, this holds for geodesic triangulations. Since we can perturb the vertices of a geodesic triangulation to generate another, X is a $2n$–dimensional manifold.

Let (i, j) be the directed edge starting from the vertex i and ending at the vertex j. Denote by $\vec{E} = \{(i, j) : ij \in E\}$ the set of directed edges of T. A positive vector $w \in \mathbb{R}^{\vec{E}}_{>0}$ is called a *weight* of T. For any weight w and geodesic mapping $\varphi \in \widetilde{X}$, we say φ is *w–balanced* if, for any $i \in V$,

$$\sum_{j:ij \in E} w_{ij} v_{ij} = 0.$$

Here $v_{ij} \in T_{q_i} M$ is defined with the exponential map $\exp\colon TM \to M$ such that $\exp_{q_i}(t v_{ij}) = \varphi_{ij}(t)$ for $t \in [0, 1]$.

The main part of the proof of Theorem 1.4 is to generalize Tutte's embedding theorem (see Theorem 9.2 in [24] or Theorem 6.1 in Floater [13]) to closed surfaces of negative curvature. Specifically, we prove the following two theorems:

Theorem 1.5 *Assume (M, g) has strictly negative Gaussian curvature. For any weight w there exists a unique geodesic mapping $\varphi \in \widetilde{X}(M, T, \psi)$ that is w–balanced. The induced map $\Phi(w) = \varphi$ is continuous from $\mathbb{R}^{\vec{E}}_{>0}$ to $\widetilde{X}$.*

Theorem 1.6 *If $\varphi \in \widetilde{X}$ is w–balanced for some weight w, then $\varphi \in X$.*

Theorem 1.6 can be regarded as a generalization of the embedding theorems of Colin de Verdière (see Theorem 2 in [8]) and Hass and Scott (see Lemma 10.12 in [18]), which imply that the minimizer of the discrete Dirichlet energy

$$E(\varphi) = \frac{1}{2} \sum_{ij \in E} w_{ij} l_{ij}^2$$

among the maps φ in the homotopy class of $\psi|_{T^{(1)}}$ is a geodesic triangulation. Here l_{ij} is the geodesic length of φ_{ij} in M. The minimizer is a w–balanced geodesic mapping with $w_{ij} = w_{ji}$ for $ij \in E$. Hence, Theorem 1.6 extends the previous results from the cases of symmetric weights to nonsymmetric weights. We believe that the proofs of Colin de Verdière [8] and Hass and Scott [18] could be easily modified to work with our nonsymmetric case. Nevertheless, we give a new proof in Section 3 to make the paper self-contained.

1.3 Mean value coordinates and the proof of Theorem 1.4

Theorems 1.5 and 1.6 give a continuous map Φ from $\mathbb{R}^{\vec{E}}_{>0}$ to X. To map a geodesic embedding to a weight, we use the *mean value coordinates* introduced by Floater [12]. Given $\varphi \in X$ the mean value coordinates are defined to be

$$w_{ij} = \frac{\tan\left(\frac{1}{2}\alpha_{ij}\right) + \tan\left(\frac{1}{2}\beta_{ij}\right)}{|v_{ij}|},$$

where $|v_{ij}|$ equals the geodesic length of $\varphi_{ij}([0,1])$, and α_{ij} and β_{ij} are the inner angles in $\varphi(T^{(1)})$ at $\varphi(i)$ sharing the edge $\varphi_{ij}([0,1])$. The construction of mean value coordinates gives a continuous map Ψ from X to $\mathbb{R}^{\vec{E}}_{>0}$. Further, by Floater's mean value theorem (see Proposition 1 in [12]), any $\varphi \in X$ is $\Psi(\varphi)$–balanced. Namely, $\Phi \circ \Psi = \mathrm{id}_X$. Then Theorem 1.4 is a direct consequence of Theorems 1.5 and 1.6.

Proof of Theorem 1.4 Since $\mathbb{R}^{\vec{E}}_{>0}$ is contractible, $\Psi \circ \Phi$ is homotopic to the identity map. Since $\Phi \circ \Psi = \mathrm{id}_X$, X is homotopy equivalent to the contractible space $\mathbb{R}^{\vec{E}}_{>0}$. □

We will prove Theorem 1.5 in Section 2 and Theorem 1.6 in Section 3.

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2 Proof of Theorem 1.5

Theorem 1.5 consists of three parts: the existence of the w–balanced geodesic mapping, the uniqueness of the w–balanced geodesic mapping and the continuity of the map Φ.

In this section we will first parametrize $\widetilde{X}$ by $\mathcal{M}$, where $\mathcal{M}$ is the product manifold of the n copies of the universal cover $\widetilde{M}$ of M. Then we prove the three parts in Sections 2.1, 2.2 and 2.3, respectively.

For the proof we mainly work on the universal covering space $\widetilde{M}$ instead of the original surface M. This is because a geodesic arc is uniquely determined by its endpoints in $\widetilde{M}$ but not in M, and thus the geodesic triangulation of M in the same homotopy class is naturally parametrized by the lifted vertices in $\widetilde{M}$. The condition of strictly negative curvature is mostly needed in the proof of the existence of the w–balanced mappings, where we frequently compare geodesic triangles in $\widetilde{M}$ with geodesic triangles of constant negative curvature.

Assume that p is the covering map from $\widetilde{M}$ to M, and Γ is the corresponding group of deck transformations of the covering so that $\widetilde{M}/\Gamma = M$. For any $i \in V$, fix a lifting $\tilde{q}_i \in \widetilde{M}$ of $q_i \in M$. For any edge ij, denote by $\tilde{\varphi}_{ij}(t)$ by the lifting of $\varphi_{ij}(t)$ such that $\tilde{\varphi}_{ij}(0) = \tilde{q}_i$. Here $\tilde{\varphi}_{ij}(1)$ may not be equal to $\tilde{q}_j$, but $p(\tilde{\varphi}_{ij}(1)) = \varphi_{ij}(1) = q_j = p(\tilde{q}_j)$, and so there exists a unique deck transformation $A_{ij} \in \Gamma$ such that $\tilde{\varphi}_{ij}(1) = A_{ij}\tilde{q}_j$. Notice that the deck transformation A_{ij} depends on the choice of the lifts $\tilde{q}_i$ and $\tilde{q}_j$ of q_i and q_j, respectively. We can deduce that $A_{ij} = A_{ji}^{-1}$ for any edge ij.

Equip $\widetilde{M}$ with the natural pullback Riemannian metric $\tilde{g}$ of g with negative Gaussian curvature. This metric is equivariant with respect to Γ. For any $x, y \in \widetilde{M}$, there exists a unique geodesic with constant speed parametrization $\gamma_{x,y}\colon [0,1] \to \widetilde{M}$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. We can naturally parametrize $\widetilde{X}$ as follows:

Theorem 2.1 *For any $(x_1, \dots, x_n) \in \mathcal{M}$, define $\varphi = \varphi[x_1, \dots, x_n]$ as*

$$\varphi_{ij}(t) = p \circ \gamma_{x_i, A_{ij}x_j}(t)$$

for any $ij \in E$ and $t \in [0,1]$. Then φ is a well-defined geodesic mapping in $\widetilde{X}$, and the map $(x_1, \dots, x_n) \mapsto \varphi[x_1, \dots, x_n]$ is a homeomorphism from $\mathcal{M}$ to $\widetilde{X}$.

We omit the proof of Theorem 2.1, which is routine but lengthy. In the remainder of this section, for any $x, y, z \in \widetilde{M}$ and $u, v \in T_x\widetilde{M}$:

(i) $d(x, y)$ is the intrinsic distance between x and y in $(\widetilde{M}, \tilde{g})$.

(ii) $v(x, y) = \exp_x^{-1} y \in T_x\widetilde{M}$.

(iii) $\triangle xyz$ is the geodesic triangle in $\widetilde{M}$ with vertices x, y and z, which could possibly be degenerate.

(iv) $\angle yxz$ is the inner angle of $\triangle xyz$ at x if $d(x, y) > 0$ and $d(x, z) > 0$.

(v) $|v|$ is the norm of v under the metric $\tilde{g}_x$.

(vi) $u \cdot v$ is the inner product of u and v under the metric $\tilde{g}_x$.

By scaling the metric if necessary, we may assume that the Gaussian curvatures of (M, g) and $(\widetilde{M}, \tilde{g})$ are bounded above by -1.

2.1 Uniqueness

We first prove Lemma 2.2 using CAT(0) geometry. See Theorem 4.3.5 in [4] and Theorem 1A.6 in [3] for the well-known comparison theorems.

Lemma 2.2 *Assume $x, y, z \in \widetilde{M}$. Then*

(i) $|v(z, x) - v(z, y)| \le d(x, y)$, *and*

(ii) $v(x, y) \cdot v(x, z) + v(y, x) \cdot v(y, z) \ge d(x, y)^2$,

and equality holds if and only if $\triangle xyz$ is degenerate.

Proof If $\triangle xyz$ is degenerate then there exists a geodesic γ in $\widetilde{M}$ such that $x, y, z \in \gamma$, and then the proof is straightforward, so we assume that $\triangle xyz$ is nondegenerate.

(i) Three points $v(z, x)$, $v(z, y)$, and 0 in $T_z \widetilde{M}$ determine a Euclidean triangle, where $|v(z, x)| = d(x, z)$, $|v(z, y)| = d(z, y)$ and the angle between $v(z, x)$ and $v(z, y)$ is equal to $\angle xzy$. Then, by the CAT(0) comparison theorem,

$$|v(z, x) - v(z, y)| < d(x, y).$$

(ii) Let $x', y', z' \in \mathbb{R}^2$ be such that

$$|x' - z'|_2 = |v(x, z)|, \quad |y' - z'|_2 = |v(y, z)| \quad \text{and} \quad |x' - y'|_2 = |v(x, y)|.$$

Then, by the CAT(0) comparison theorem, $\angle yxz < \angle y'x'z'$ and $\angle xyz < \angle x'y'z'$. Hence,

$$\begin{aligned} v(x, y) \cdot v(x, z) + v(y, x) \cdot v(y, z) &> (y' - x') \cdot (z' - x') + (x' - y') \cdot (z' - y') \\ &= |x' - y'|_2^2 = d(x, y)^2. \end{aligned}$$

□

Proof of uniqueness in Theorem 1.5 If φ is not unique, assume $\varphi[x_1, \dots, x_n]$ and $\varphi[x'_1, \dots, x'_n]$ are two different geodesic mappings that are both w–balanced for some weight w. We are going to prove a discrete maximum principle for the function $j \mapsto d(x_j, x'_j)$. Assume $i \in V$ is such that $d(x_i, x'_i) = \max_{j \in V} d(x_j, x'_j) > 0$. By lifting the w–balanced assumption to $\widetilde{M}$, we have that

$$\sum_{\{j : ij \in E\}} w_{ij} v(x_i, A_{ij} x_j) = 0, \tag{1}$$

and

$$\sum_{\{j:ij\in E\}} w_{ij} v(x_i', A_{ij} x_j') = 0. \tag{2}$$

Then, by Lemma 2.2(i) and (1),

$$\begin{aligned}\left|\sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij} x_j')\right| &= \left|\sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij} x_j') - \sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij} x_j)\right| \\ &\le \sum_{\{j:ij\in E\}} w_{ij} d(A_{ij} x_j, A_{ij} x_j') = \sum_{\{j:ij\in E\}} w_{ij} d(x_j, x_j') \\ &\le d(x_i, x_i') \sum_{\{j:ij\in E\}} w_{ij}.\end{aligned}$$

By part (ii) of Lemma 2.2, (2), and the Cauchy–Schwartz inequality,

$$\begin{aligned} d(x_i, x_i')\cdot &\left|\sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij} x_j')\right| \\ &\ge v(x_i, x_i')\cdot \sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij} x_j') + v(x_i', x_i)\cdot \sum_{\{j:ij\in E\}} w_{ij} v(x_i', A_{ij} x_j') \\ &\ge \sum_{\{j:ij\in E\}} w_{ij}\cdot d(x_i, x_i')^2.\end{aligned}$$

Therefore, equality holds in both inequalities above. Then, for any neighbor j of i, $d(x_j, x_j') = d(x_i, x_i') = \max_{k\in V} d(x_k, x_k')$, and $A_{ij} x_j$ is on the geodesic determined by x_i and x_i'. Hence, the one-ring neighborhood of $p(x_i)$ in $\varphi[x_1, \dots, x_n](T^{(1)})$ degenerates to a geodesic arc. By the connectedness of the surface we can repeat the above argument and deduce that $d(x_j, x_j') = d(x_i, x_i')$ for any $j \in V$. Further, $\varphi[x_1, \dots, x_n](\partial\sigma)$ degenerates to a geodesic arc for any triangle $\sigma \in F$.

It is not difficult to extend $\varphi[x_1, \dots, x_n]$ to a continuous map $\tilde\varphi$ from $|T|$ to M such that $\tilde\varphi(\partial\sigma) = \varphi[x_1, \dots, x_n](\partial\sigma)$ is the union of three geodesic arcs for any triangle $\sigma \in F$

It is also not difficult to prove that $\tilde\varphi$ is homotopic to ψ. Therefore, $\tilde\varphi$ is degree one and surjective. This contradicts that $\tilde\varphi(|T|)$ is a finite union of geodesic arcs. □

2.2 Existence

Here we prove a stronger existence result:

Theorem 2.3 *Given a compact subset K of $\mathbb{R}^{\vec{E}}_{>0}$ there exists a compact subset $K' = K'(M, T, \psi, K)$ of $\widetilde{X}$ such that, for any $w \in K$, there exists a w–balanced geodesic mapping $\varphi \in K'$.*

We first introduce the topological Lemma 2.4 and the key Lemma 2.5.

Lemma 2.4 *Suppose $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$ is the unit ball in $\mathbb{R}^n$, and $f : B^n \to \mathbb{R}^n$ is a continuous map such that $x \ne f(x)/|f(x)|$ for any $x \in \partial B^n = S^{n-1}$ with $f(x) \ne 0$. Then f has a zero in B^n.*

Proof If not, $g(x) = f(x)/|f(x)|$ is a continuous map from B^n to ∂B^n. Since B^n is contractible, $g(x)$ is nullhomotopic, and thus $g|_{S^{n-1}}$ is also nullhomotopic. Since $g(x) \ne x$, it is easy to verify that

$$H(x,t) = \frac{tg(x) + (1-t)(-x)}{|tg(x) + (1-t)(-x)|}$$

is a homotopy between $g|_{S^{n-1}}$ and $-\mathrm{id}\,|_{S^{n-1}}$. This contradicts that $-\mathrm{id}\,|_{S_{n-1}}$ is not nullhomotopic. □

Lemma 2.5 *Fix an arbitrary point $q \in \widetilde{M}$. If $w \in \mathbb{R}^{\vec{E}}_{>0}$ and $(x_1, \dots, x_n) \in \mathcal{M}$ satisfies*

$$v(x_i, q) \cdot \sum_{\{j : ij \in E\}} w_{ij} v(x_i, A_{ij} x_j) \le 0 \tag{3}$$

for any $i \in V$, then

$$\sum_{i \in V} d(x_i, q)^2 < R^2$$

for some constant $R > 0$ which depends only on M, T, ψ, q and

$$\lambda_w := \frac{\max_{ij \in E} w_{ij}}{\min_{ij \in E} w_{ij}}.$$

The vector in Figure 1,

$$r_i = \sum_{\{j : ij \in E\}} w_{ij} v(x_i, A_{ij} x_j),$$

is defined as the *residue vector* r_i at x_i of $\varphi[x_1, \dots, x_n]$ with respect to the weight w. Notice that a geodesic mapping φ is w–balanced if and only if all its residue vectors vanish with respect to w. Lemma 2.5 means that, if all the residue vectors are dragging the x_i away from q, then all the x_i must stay a bounded distance from q.

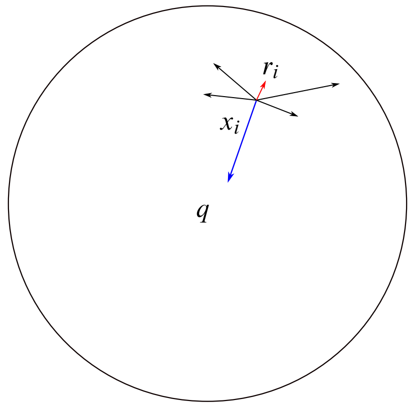

Figure 1: The residue vector and Lemma 2.5.

Our notion of w–balancedness is closely related to the *Riemannian center of mass* developed by Grove and Karcher [17]. In a w–balanced geodesic mapping, each point can be viewed as the weighted center of mass of its neighboring points. The defining formula of our residue vectors also appears in [5; 19]. A survey of Riemannian center of mass by Karcher can be found in [20]. The definition of a residue vector is also similar to the concept of a *discrete tension field* in [15].

Proof of Theorem 2.3 Fix an arbitrary basepoint $q \in \widetilde{M}$. Then by Lemma 2.5 we can pick a sufficiently large constant $R = R(M, T, \psi, K) > 0$ such that, if

$$\sum_{i=1}^{n} d(x_i, q)^2 = R^2,$$

there exists $i \in V$ such that

$$v(x_i, q) \cdot \sum_{\{j:ij \in E\}} w_{ij} v(x_i, A_{ij} x_j) > 0.$$

We will prove that the compact set

$$K' = \left\{ \varphi[x_1, \dots, x_n] \;\middle|\; \sum_{i=1}^{n} d(x_i, q)^2 \le R^2 \right\}$$

is satisfactory.

For $x \in \widetilde{M}$ let $P_x : T_x \widetilde{M} \to T_q \widetilde{M}$ be the parallel transport along the geodesic $\gamma_{x,q}$. Set

$$B = \left\{ (v_1, \dots, v_n) \in (T_q \widetilde{M})^n \;\middle|\; \sum_{i=1}^{n} |v_i|^2 \le 1 \right\},$$

a Euclidean $2n$–dimensional unit ball, and construct a map $F\colon B \to (T_q\widetilde{M})^n$ in three steps: Firstly, we define n points $x_1, \dots, x_n \in \widetilde{M}$ by $x_i(v_1, \dots, v_n) = \exp_q(Rv_i)$. Secondly, we compute the residue vector at each x_i as

$$r_i = \sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij}x_j) \in T_{x_i}\widetilde{M}.$$

Lastly, we pull back the residues to $T_q\widetilde{M}$ via $F(v_1, \dots, v_n) = (P_{x_1}(r_1), \dots, P_{x_n}(r_n))$.

Notice that the map $(v_1, \dots, v_n) \mapsto \varphi[x_1, \dots, x_n]$ is a homeomorphism from B to K', and $F(v_1, \dots, v_n) = 0$ if and only if the corresponding $\varphi[x_1, \dots, x_n]$ in K' is a w–balanced map. Hence, it suffices to prove that F has a zero in B. By Lemma 2.4, it suffices to prove that, for any $(v_1, \dots, v_n) \in \partial B$,

$$(v_1, \dots, v_n) \neq \frac{F(v_1, .., v_n)}{|F(v_1, \dots, v_n)|}.$$

Suppose $(v_1, \dots, v_n)$ is an arbitrary point on ∂B. Then it suffices to prove that there exists $i \in V$ such that $v_i \cdot F_i(v_1, \dots, v_n) = v_i \cdot P_{x_i}(r_i) < 0$.

Notice that $x_1(v_1, \dots, v_n), \dots, x_n(v_1, \dots, v_n)$ satisfy that $\sum_{i=1}^n d(q, x_i)^2 = R^2$, so, by our assumption on R, there exists $i \in V$ such that

$$v(x_i, q) \cdot \sum_{\{j:ij\in E\}} w_{ij} v(x_i, A_{ij}x_j) = v(x_i, q) \cdot r_i > 0,$$

and thus

$$v_i \cdot P_{x_i}(r_i) = -\frac{1}{d(q, x_i)} P_{x_i}(v(x_i, q)) \cdot P_{x_i}(r_i) = -\frac{1}{d(q, x_i)} v(x_i, q) \cdot r_i < 0. \qquad \square$$

In the rest of this subsection we will prove Lemma 2.5 by contradiction. Let us first sketch the idea of the proof. Assume $\sum_{i\in V} d(x_i, q)^2$ is very large. Then by a standard compactness argument there exists a long edge ij in the geodesic mapping $\varphi[x_1, \dots, x_n]$. Assume $d(q, x_i) \geq d(q, x_j)$. Then the corresponding long edge $\gamma_{x_i, A_{ij}x_j}$ in $\widetilde{M}$ is pulling x_i towards q. This implies that there exists another long edge $\gamma_{x_i, A_{ik}x_k}$ dragging x_i away from q, otherwise the residue vector r_i would not drag x_i away from q. It can be shown that $d(q, x_k) > d(q, x_i)$. Repeating the above steps, we can find an arbitrarily long sequence of vertices such that the distance from each of these vertices to q is increasing. This is impossible as we only have finitely many vertices.

Here is a list of useful properties, where (a), (e), (f), (g) and (h) serve directly as building blocks of the proof of Lemma 2.5, (b) and (d) are used to prove (e), and (c) is used to prove (h). The three triangles in Figure 2, from left to right, illustrate the geodesic triangles appearing in (b), (c) and (d) of Lemma 2.6, respectively.

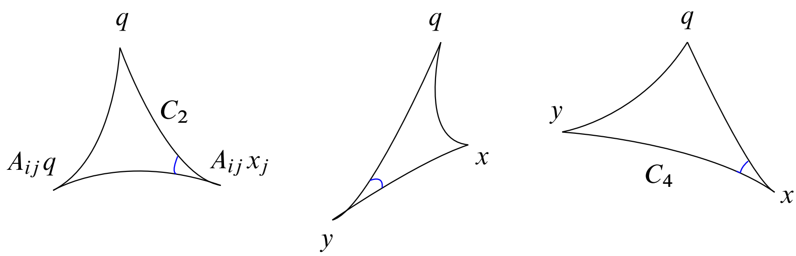

Figure 2: Triangles in (b), (c) and (d).

Lemma 2.6 (a) *For any constant* $C > 0$ *there is a constant* $C_1 = C_1(M, T, \psi, C) > 0$ *such that, if*

$$\sum_{i \in V} d(x_i, q)^2 \geq C_1,$$

then

$$\max_{ij \in E} d(x_i, A_{ij} x_j) \geq C.$$

(b) *There exists a constant* $C_2 = C_2(M, T, \psi) > 0$ *such that, if*

$$d(A_{ij} x_j, q) \geq C_2,$$

then

$$\angle (A_{ji} q) x_j q = \angle q (A_{ij} x_j)(A_{ij} q) \leq \tfrac{1}{8}\pi.$$

(c) *There exists a constant* $C_3 > 0$ *such that, if* $x, y \in \widetilde{M}$ *satisfy*

$$d(y, q) \geq d(x, q) + C_3,$$

then

$$\angle x y q \leq \tfrac{1}{4}\pi.$$

(d) *There exists a constant* $C_4 > 0$ *such that, if* $x, y \in \widetilde{M}$ *satisfy*

$$d(x, y) \geq C_4 \quad \text{and} \quad d(x, q) \geq d(y, q),$$

then

$$\angle y x q \leq \tfrac{1}{8}\pi.$$

(e) *For any constant* $C > 0$ *there is a constant* $C_5 = C_5(M, T, \psi, C) > 0$ *such that, if*

$$\max_{ij \in E} d(x_i, A_{ij} x_j) \geq C_5,$$

then there exists $ij \in E$ *such that*

$$\frac{v(x_i, q)}{|v(x_i, q)|} \cdot v(x_i, A_{ij} x_j) \geq C.$$

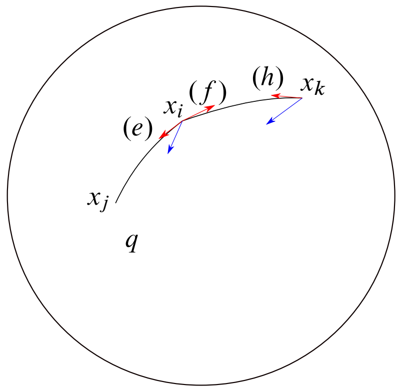

Figure 3: Vertices leaving the point q.

(f) *For any constant* $C > 0$ *there is a constant* $C_6 = C_6(M, T, \psi, \lambda_w, C) > 0$ *such that, if*
$$\frac{v(x_i, q)}{|v(x_i, q)|} \cdot v(x_i, A_{ij} x_j) \geq C_6$$
for some edge $ij \in E$, *then there exists* $ik \in E$ *such that*
$$\frac{v(x_i, q)}{|v(x_i, q)|} \cdot v(x_i, A_{ik} x_k) \leq -C.$$

(g) *For any constant* $C > 0$ *there is a constant* $C_7 = C_7(M, T, \psi, C) > 0$ *such that, if*
$$\frac{v(x_i, q)}{|v(x_i, q)|} \cdot v(x_i, A_{ik} x_k) \leq -C_7,$$
then
$$d(x_k, q) \geq d(x_i, q) + C.$$

(h) *For any constant* $C > 0$ *there is a constant* $C_8 = C_8(M, T, \psi, C) > 0$ *such that, if*
$$d(x_j, q) \geq d(x_i, q) + C_8,$$
then
$$\frac{v(x_j, q)}{|v(x_j, q)|} \cdot v(x_j, A_{ji} x_i) \geq C.$$

Proof of Lemma 2.5 assuming Lemma 2.6 For any $C > 0$ there exists a sufficiently large constant $\widetilde{C} = \widetilde{C}(M, T, \psi, \lambda_w, C)$ determined by (a), (e), (f) and (g) in Lemma 2.6 such that, if
$$\sum_{i \in V} d(x_i, q)^2 \geq \widetilde{C},$$
then there exist three vertices x_i, x_j and x_k, shown in Figure 3, with
$$d(x_k, q) \geq d(x_j, q) + C.$$

Moreover, by (h), (f) and (g) of Lemma 2.6, we can find another vertex x_l such that

$$d(x_l,q) \geq d(x_k,q) + C \geq d(x_j,q) + 2C$$

if the constant $\widetilde{C}(M,T,\psi,\lambda_w,C)$ is sufficiently large.

Inductively, we can find a sequence $i_1,\dots,i_{n+1} \in V$ such that

$$d(x_{i_1},q) > d(x_{i_2},q) > \cdots > d(x_{i_{n+1}},q).$$

This contradicts the fact that V only has n different elements. □

Proof of Lemma 2.6 (a) By a standard compactness argument, the set

$$\{\varphi \in \widetilde{X} : \max_{ij\in E} \operatorname{length}(\varphi_{ij}([0,1])) \leq C\}$$

is a compact subset of $\widetilde{X}$. Notice that $(x_1,\dots,x_n) \mapsto \varphi[x_1,\dots,x_n]$ is a homeomorphism from $\mathcal{M}$ to $\widetilde{X}$, and

$$\operatorname{length}(\varphi_{ij}([0,1])) = d(x_i, A_{ij}x_j).$$

Therefore

$$\Big\{(x_1,\dots,x_n) \in \mathcal{M} : \max_{ij\in E}\; d(x_i, A_{ij}x_j) \leq C\Big\}$$

is compact, and the conclusion follows.

(b) We claim that the constant C_2 determined by

$$\sinh C_2 = \frac{\max_{ij\in E} \sinh d(A_{ij}q,q)}{\sin\frac{1}{8}\pi}$$

is satisfactory. Let $\triangle ABC$ be the hyperbolic triangle with corresponding edge lengths

$$a = d(A_{ij}x_j,q),\quad b = d(A_{ij}x_j, A_{ij}q) \quad\text{and}\quad c = d(A_{ij}q,q).$$

Since $\widetilde{M}$ is a CAT(-1) space, it suffices to show that $\angle C \leq \frac{1}{8}\pi$. By the hyperbolic law of sine,

$$\sin\angle C = \frac{\sinh c\cdot\sin\angle A}{\sinh a} \leq \frac{\max_{ij\in E}\sinh d(A_{ij}q,q)\cdot 1}{\sinh C_2} = \sin\tfrac{1}{8}\pi.$$

(c) We claim that the constant C_3 determined by

$$\sinh C_3 = \frac{1}{\sin\frac{1}{8}\pi}$$

is satisfactory. Let $\triangle ABC$ be the hyperbolic triangle with corresponding edge lengths

$$a = d(x,y),\quad b = d(y,q) \quad\text{and}\quad c = d(x,q).$$

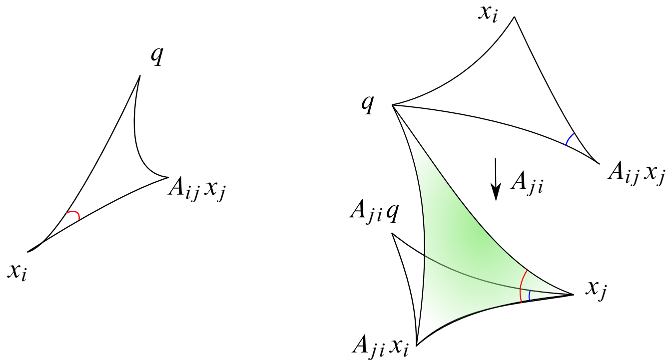

Figure 4: Triangles in (e).

Since $\widetilde{M}$ is a CAT(-1) space it suffices to show that $\angle C \leq \frac{1}{8}\pi$. By the hyperbolic law of sine,

$$\sin \angle C = \frac{\sinh c \cdot \sin \angle B}{\sinh b} \leq \frac{\sinh c}{\sinh b} \leq \frac{\sinh c}{\sinh(c + C_3)} \leq \frac{\sinh c}{\sinh c \cdot \sinh C_3} = \sin \tfrac{1}{8}\pi.$$

(d) We claim that the constant C_4 determined by

$$\sin^2 \tfrac{1}{8}\pi \cdot \cosh C_4 = 2$$

is satisfactory. Let $\triangle ABC$ be the hyperbolic triangle with corresponding edge lengths

$$a = d(x, y), \quad b = d(y, q) \quad \text{and} \quad c = d(x, q).$$

Since $\widetilde{M}$ is a CAT(-1) space it suffices to show that $\angle B \leq \frac{1}{8}\pi$. By the hyperbolic law of cosine,

$$\cos A = -\cos B \cos C + \sin B \sin C \cosh a.$$

Then

$$2 \geq \sin B \sin C \cosh a \geq \sin B \sin C \cosh C_4 = 2 \cdot \frac{\sin B \sin C}{\sin^2 \frac{1}{8}\pi} \geq 2 \cdot \frac{\sin^2 B}{\sin^2 \frac{1}{8}\pi}.$$

Thus, $\angle B \leq \frac{1}{8}\pi$.

(e) We claim that the constant C_5 determined by

$$C_5 = \max\{C_4, 2C_2, \sqrt{2}C\}$$

is satisfactory. Assume $ij \in E$ and $d(x_i, A_{ij}x_j) \geq C_5$. Then we have the two cases shown in Figure 4.

If $d(x_i, q) \geq d(A_{ij}x_j, q)$, then, by (d),

$$\angle(A_{ij}x_j)x_i q \leq \tfrac{1}{8}\pi \leq \tfrac{1}{4}\pi,$$

and

$$\frac{v(x_i,q)}{|v(x_i,q)|}\cdot v(x_i,A_{ij}x_j)=\cos(\angle(A_{ij}x_j)x_iq)\cdot d(x_i,A_{ij}x_j)\geq\frac{1}{\sqrt{2}}C_5\geq C.$$

If $d(x_i,q)\leq d(A_{ij}x_j,q)$, then $d(A_{ij}x_j,q)\geq C_2$. By (b) and (d),

$$\angle(A_{ji}q)x_jq\leq\tfrac{1}{8}\pi\quad\text{and}\quad\angle(A_{ji}q)x_j(A_{ji}x_i)=\angle q(A_{ij}x_j)x_i\leq\tfrac{1}{8}\pi,$$

and $\angle qx_j(A_{ji}x_i)\leq\frac{1}{4}\pi$. Therefore,

$$\frac{v(x_j,q)}{|v(x_j,q)|}\cdot v(x_j,A_{ji}x_i)=\cos(\angle(A_{ji}x_j)x_jq)\cdot d(x_j,A_{ji}x_i)\geq\frac{1}{\sqrt{2}}C_5\geq C.$$

(f) We claim that the constant C_6 determined by

$$C_6=n\lambda_w\cdot C$$

is satisfactory. If not, for any $ik\in E$,

$$\frac{v(x_i,q)}{|v(x_i,q)|}\cdot v(x_i,A_{ik}x_k)>-C.$$

Then

$$\begin{aligned}0\geq\frac{v(x_i,q)}{|v(x_i,q)|}\cdot\sum_{ik\in E}w_{ik}v(x_i,A_{ik}x_k)&>w_{ij}C_6+\sum_{ik\in E}w_{ik}(-C)\\&\geq w_{ij}C_6+\sum_{ik\in E}\lambda_w w_{ij}(-C)\geq w_{ij}(C_6-n\lambda_wC)\geq 0,\end{aligned}$$

which is a contradiction.

(g) We claim that $C_7=C+\max_{ij\in E}d(A_{ij}q,q)$ is satisfactory. Notice that

$$d(A_{ij}x_j,q)=d(x_j,A_{ji}q)\leq d(x_j,q)+d(q,A_{ji}q)\leq d(x_j,q)+\max_{ij\in E}d(A_{ij}q,q).$$

By Lemma 2.2(i),

$$\begin{aligned}d(A_{ij}x_j,q)\geq|v(x_i,A_{ij}x_j)-v(x_i,q)|&\geq-(v(x_i,A_{ij}x_j)-v(x_i,q))\cdot\frac{v(x_i,q)}{|v(x_i,q)|}\\&=C_7+|v(x_i,q)|=C_7+d(x_i,q).\end{aligned}$$

Then

$$d(x_j,q)-d(x_i,q)\geq C_7-\max_{ij\in E}d(A_{ij}q,q)=C.$$

(h) We claim that the constant C_8 determined by

$$C_8=\max\{C_3,\sqrt{2}C\}+\max_{ij\in E}d(A_{ij}q,q)$$

is satisfactory. Notice that

$$\begin{aligned} d(x_j, q) &\geq d(x_i, q) + C_8 \geq d(x_i, A_{ij}q) - d(A_{ij}q, q) + C_8 \\ &\geq d(A_{ji}x_i, q) + \max\{C_3, \sqrt{2}C\}. \end{aligned}$$

Then, by (c), $\angle(A_{ji}x_i)x_jq \leq \frac{1}{4}\pi$, and by the triangle inequality,

$$d(x_j, A_{ji}x_i) \geq d(x_j, q) - d(A_{ji}x_i, q) \geq \sqrt{2}C.$$

Therefore,

$$\frac{v(x_j, q)}{|v(x_j, q)|} \cdot v(x_j, A_{ji}x_i) = \cos(\angle(A_{ji}x_i)x_jq) \cdot d(x_j, A_{ji}x_i) \geq \frac{1}{\sqrt{2}} \cdot \sqrt{2}C = C. \quad \square$$

2.3 Continuity

Proof of continuity in Theorem 1.5 If Φ is not continuous, there exists $\epsilon > 0$, a weight w and a sequence of weights $w^{(k)}$ such that

(i) the $w^{(k)}$ converge to w, and

(ii) $d_{\widetilde{X}}(\Phi(w^{(k)}), \Phi(w)) \geq \epsilon$ for any $k \geq 1$.

By the stronger existence result Theorem 2.3, the sequence $\Phi(w^{(k)})$ is in some fixed compact subset K' of $\widetilde{X}$. By picking a subsequence, we may assume that $\Phi(w^{(k)})$ converges to some $\varphi \in \widetilde{X}$. Since $\Phi(w^{(k)})$ is $w^{(k)}$–balanced, then, by the continuity of the residue vectors r_i, φ is w–balanced and thus $\Phi(w) = \varphi$, which contradicts that $\Phi(w^{(k)})$ does not converge to $\Phi(w)$. $\square$

3 Proof of Theorem 1.6

3.1 Setup and preparation

Assume $\varphi \in \widetilde{X}$ is w–balanced for some weight w. We will prove that φ is an embedding. Recall that $q_i = \varphi(i)$ for each $i \in V$, and denote by l_{ij} the length of $\varphi_{ij}([0,1])$ for any $ij \in E$. It is not difficult to show that φ has a continuous extension $\tilde{\varphi}$ defined on $|T|$ such that for any triangle $\sigma \in F$ a continuous lifting map Φ_σ of $\tilde{\varphi}|_\sigma$ from σ to $\widetilde{M}$ will map σ to

(i) a geodesic triangle in $\widetilde{M}$ homeomorphically if $\varphi(\partial\sigma)$ does not degenerate to a geodesic, and

(ii) $\Phi_\sigma(\partial\sigma)$ if $\varphi(\partial\sigma)$ degenerates to a geodesic.

The main tool we use to prove Theorem 1.6 is the Gauss–Bonnet formula. We will need to define the inner angles for each triangle in $\varphi(T^{(1)})$, even for the degenerate triangles.

A convenient way is to assign a "direction" to each edge, even for the degenerate edges with zero length.

Definition 3.1 A *direction field* is a map $v\colon \vec{E} \to TM$ satisfying

(i) $v_{ij} \in T_{q_i}M$ for any $(i,j) \in \vec{E}$, and
(ii) $|v_{ij}| = 1$ for any $(i,j) \in \vec{E}$.

Given a direction field v, define the inner angle of the triangle $\sigma = \triangle ijk$ at the vertex i as

$$\theta^i_\sigma = \theta^i_\sigma(v) = \angle v_{ij} 0 v_{ik} = \arccos(v_{ij} \cdot v_{ik}),$$

where 0 is the origin and $\angle v_{ij} 0 v_{ik}$ is the angle between v_{ij} and v_{ik} in $T_{q_i}M$.

A direction field v assigns a unit tangent vector in $T_{q_i}M$ to each directed edge starting from i, even if their lengths are zero. It determines the inner angles in T.

Definition 3.2 A direction field v is *admissible* if

(i) $v_{ij} = \varphi'_{ij}(0)/l_{ij}$ if $l_{ij} > 0$,
(ii) $v_{ij} = -v_{ji}$ in $T_{q_i}M = T_{q_j}M$ if $l_{ij} = 0$,
(iii) for a fixed vertex $i \in V$, if $l_{ij} = 0$ for every neighbor j of i, then there exist neighbors k and k' of i such that $v_{ik} = -v_{ik'}$, and
(iv) if $\sigma = \triangle ijk \in F$ and $l_{ij} = l_{jk} = l_{ik} = 0$, then $\theta^i_\sigma(v) + \theta^j_\sigma(v) + \theta^k_\sigma(v) = \pi$.

An admissible direction field encodes the directions of the nondegenerate edges in $\varphi(T^{(1)})$, and the induced angle sum of a degenerate triangle is always π. Then, for any admissible v and triangle $\sigma \in F$, by the Gauss–Bonnet formula,

$$\pi = \sum_{i\in\sigma} \theta^i_\sigma(v) - \int_{\Phi_\sigma(\sigma)} K\, dA \geq \sum_{i\in\sigma} \theta^i_\sigma(v) - \int_{\tilde{\varphi}(\sigma)} K\, d\tilde{A}. \tag{4}$$

Here dA (resp. $d\tilde{A}$) is the area form on (M, g) (resp. $(\widetilde{M}, \tilde{g})$).

The concept of the direction field is similar to the *discrete one-form* defined in [16].

3.2 Proof of Theorem 1.6

The proof of Theorem 1.6 uses the four lemmas below. We will postpone their proofs to the subsequent subsections.

Lemma 3.3 *If v is admissible and $\theta = \theta(v)$, then, for any $i \in V$,*

$$\sum_{\{\sigma : i \in \sigma\}} \theta_\sigma^i = 2\pi,$$

and $\tilde{\varphi}(\sigma) \cap \tilde{\varphi}(\sigma')$ has area 0 for any $\sigma, \sigma' \in F$.

Based on Lemma 3.3, if admissible direction fields exist, the image of the star of each vertex determined by $\tilde{\varphi}$ does not contain any flipped triangles overlapping with each other. If $\tilde{\varphi}(\sigma)$ does not degenerate to a geodesic arc for any triangle $\sigma \in F$, then $\tilde{\varphi}$ is locally homeomorphic and thus globally homeomorphic as a degree-one map. Therefore, we only need to exclude the existence of degenerate triangles.

Define an equivalence relation on V as follows. Two vertices i and j are equivalent if there exists a sequence of vertices $i = i_0, i_1, \dots, i_k = j$ such that $l_{i_0 i_1} = \cdots = l_{i_{k-1} i_k} = 0$. This equivalence relation introduces a partition $V = V_1 \cup \cdots \cup V_m$. Denote by $y_k \in M$ the only point in $\varphi(V_k)$. For any $x \in M$ and $u, v \in T_x M$, write $u \parallel v$ if u and v are parallel, ie there exists $(\alpha, \beta) \neq (0,0)$ such that $\alpha u + \beta v = 0$.

There are plenty of choices of admissible direction fields:

Lemma 3.4 *For any $v_1 \in T_{y_1} M, \dots, v_m \in T_{y_m} M$, there exists an admissible v such that $v_{ij} \parallel v_k$ if $i \in V_k$ and $l_{ij} = 0$.*

For any V_k with at least two vertices, the image of its "neighborhood" lies in a geodesic:

Lemma 3.5 *If $|V_k| \geq 2$, then there exists $v_k \in T_{y_k} M$ such that $v_k \parallel \varphi_{ij}'(0)$ if $i \in V_k$ and $l_{ij} > 0$.*

Now let v_k be as in Lemma 3.5 if $|V_k| \geq 2$, and arbitrary if $|V_k| = 1$. Then construct an admissible direction field v as in Lemma 3.4, with induced inner angles $\theta_\sigma^i = \theta_\sigma^i(v)$. If the image of a triangle σ under φ degenerates to a geodesic, then its inner angles θ_σ^i are π or 0. Let $F' \neq \varnothing$ be the set of degenerate triangles under φ.

Lemma 3.6 *If $\sigma \in F'$, $i \in \sigma$ and $\theta_\sigma^i = \pi$, then $\sigma' \in F'$ for any σ' in the star neighborhood of the vertex i.*

Let Ω be a connected component of the interior of $\bigcup\{\sigma : \sigma \in F'\} \subset |T|$, and $\widetilde{\Omega}$ be the completion of Ω under the natural path metric on Ω. Notice that $\widetilde{\Omega}$ could be different from the closure of Ω in M.

Since $\tilde{\varphi}$ is surjective $F' \neq F$, $\Omega \neq |T|$ and $\widetilde{\Omega}$ has nonempty boundary. Then $\widetilde{\Omega}$ is a connected surface with a natural triangulation $T' = (V', E', F')$, and

$$\chi(\widetilde{\Omega}) = 2 - 2 \times (\text{genus of } \widetilde{\Omega}) - \#\{\text{boundary components of } \widetilde{\Omega}\} \leq 1.$$

Assume V'_I is the set of interior vertices, V'_B is the set of boundary vertices, E'_I is the set of interior edges and E'_B is the set of boundary edges of $\widetilde{\Omega}$. Then $|V'_B| = |E'_B|$ and, by Lemma 3.6, if $i \in V'_B$ and $i \in \sigma$ then $\theta^i_\sigma = 0$. Therefore,

$$\pi|F'| = \sum_{\substack{\sigma \in F' \\ i \in \sigma}} \theta^i_\sigma = \sum_{i \in V'_I} \sum_{\{\sigma \in F' : i \in \sigma\}} \theta^i_\sigma = 2\pi|V'_I|.$$

Thus,

$$\begin{aligned} 1 \geq \chi(\widetilde{\Omega}) &= |V'| - |E'| + |F'| = |V'_I| + |V'_B| - |E'_I| - |E'_B| + |F'| \\ &= |V'_B| - |E'_I| - |E'_B| + \tfrac{3}{2}|F'| = -|E'_I| + \tfrac{3}{2}|F'| \\ &= -|E'_I| + \tfrac{1}{2}(|E'_B| + 2|E'_I|) = \tfrac{1}{2}|E'_B|. \end{aligned}$$

Therefore, $|V'_B| = |E'_B| \leq 2$. Since $\widetilde{\Omega}$ has nonempty boundary, $|E'_B| = 1$ or 2. In either case, it contradicts the fact that T is a simplicial complex. The proof of Theorem 1.6 is now completed.

3.3 Proof of Lemma 3.3

We claim that, for any $i \in V$,

$$\sum_{\{\sigma : i \in \sigma\}} \theta^i_\sigma \geq 2\pi.$$

If $l_{ij} = 0$ for any neighbor j of i, this is a consequence of Definition 3.2(iii). If $l_{ij} \neq 0$, by the w–balanced condition, $\{\varphi'_{ij}(0)/l_{ij} : ij \in E\}$ should not be contained in any open half unit circle, and the angle sum around i should be at least 2π.

By the fact that $\tilde{\varphi}$ is surjective and (4),

$$\sum_{i \in V}\Bigl(2\pi - \sum_{\{\sigma : i \in \sigma\}} \theta^i_\sigma\Bigr) + \sum_{\sigma \in F} \int_{\tilde{\varphi}(\sigma)} K\, dA \leq \sum_{\sigma \in F} \int_{\tilde{\varphi}(\sigma)} K\, dA \leq \int_M K\, dA = 2\pi\chi(M)$$

and

$$\begin{aligned} \sum_{i \in V}\Bigl(2\pi - \sum_{\{\sigma : i \in \sigma\}} \theta^i_\sigma\Bigr) + \sum_{\sigma \in F} \int_{\tilde{\varphi}(\sigma)} K\, dA &\geq 2\pi|V| - \sum_{\sigma \in F}\Bigl(\sum_{i \in \sigma} \theta^i_\sigma - \int_{\tilde{\varphi}(\sigma)} K\, dA\Bigr) \\ &= 2\pi\chi(M). \end{aligned}$$

Hence, the inequalities above are equalities. This fact implies that

$$\sum_{i\in V}\left(2\pi - \sum_{\{\sigma: i\in\sigma\}} \theta^i_\sigma\right) = 0.$$

Since each term in this summation is nonpositive, $\sum_{\{\sigma:i\in\sigma\}}\theta^i_\sigma = 2\pi$. The statement on the area follows similarly.

3.4 Proof of Lemma 3.4

We claim that, for any k, there exists a map $h\colon V_k \to \mathbb{R}$ such that

(i) $h(i) \neq h(j)$ if $i \neq j$, and

(ii) for a fixed $i \in V_k$, if $l_{ij} = 0$ for any neighbor j of i, then there exist neighbors j and j' of i in V_k such that $h(j) < h(i) < h(j')$.

Given such h, set v as

$$v_{ij} = \begin{cases} \operatorname{sgn}[h(j) - h(i)] \cdot v_k & \text{if } i \in V_k \text{ and } l_{ij} = 0, \\ \varphi'_{ij}(0) & \text{if } l_{ij} > 0, \end{cases}$$

where sgn is the sign function. It is easy to verify that v is satisfactory.

To construct h, we prove a more general statement:

Lemma 3.7 *Assume $G = (V', E')$ is a subgraph of the 1–skeleton $T^{(1)}$, and $E' \neq E$. Define*

$$\operatorname{int}(G) = \{i \in V' : ij \in E \Rightarrow ij \in E'\},$$

and $\partial G = V' - \operatorname{int}(G)$. Then there exists $h\colon V' \to \mathbb{R}$ such that

(i) *$h(i) \neq h(j)$ if $i \neq j$, and*

(ii) *any $i \in \operatorname{int}(G)$ has neighbors j and j' in V' such that $h(j) < h(i) < h(j')$.*

Proof We proceed by induction on the size of V'. The case $|V'| = 1$ is trivial. For the case $|V'| \geq 2$, first notice that $|\partial G| \geq 2$ for any proper subgraph G of $T^{(1)}$. Assign distinct values $\bar{h}(i)$ to each $i \in \partial G$, then solve the discrete harmonic equation

$$\sum_{\{j: ij\in E\}} (\bar{h}(j) - \bar{h}(i)) = 0 \quad \text{for all } i \in \operatorname{int}(G)$$

with the given Dirichlet boundary condition on ∂G.

Let $s_1 < \cdots < s_k$ be the distinct values that appear in $\{\bar{h}(i) : i \in V'\}$. Then consider the subgraphs $G_i = (V_i', E_i')$ defined by

$$V_i' = \{j \in V' : h(j) = s_i\} \quad \text{and} \quad E_i' = \{jj' \in E' : j, j' \in V_i'\}.$$

Notice that $|\partial G| \geq 2$, so $k \geq 2$ and $|V_i'| < |V'|$ for any $i = 1, \ldots, k$. By the induction hypothesis, there exists a function $h_i : V_i' \to \mathbb{R}$ such that

(i) $h_i(j) \neq h_i(j')$ if $j \neq j'$, and

(ii) any $j \in \operatorname{int}(G_i)$ has neighbors j' and j'' in V_i' such that $h_i(j') < h_i(j) < h_i(j'')$.

Define $h_i(j) = 0$ if $j \notin V_i'$. Then, for sufficiently small positive $\epsilon_1, \ldots, \epsilon_k$,

$$h = \bar{h} + \sum_{i=1}^{k} \epsilon_i h_i$$

is the desired function. □

3.5 Proof of Lemma 3.5

We must prove that if $i, i' \in V_k$, $ij, i'j' \in E$, $l_{ij} > 0$ and $l_{i'j'} > 0$, then $\varphi_{ij}'(0) \parallel \varphi_{i'j'}'(0)$. Let

$$D = \Big(\bigcup_{i,i',i'' \in V_k} \triangle i i' i'' \Big) \cup \Big(\bigcup_{i,i' \in V_k} i i' \Big),$$

which is a closed path-connected set in $|T|$. For any $i \in V_k$, we have $i \in \partial D$ if and only if there exists $ij \in E$ with $l_{ij} > 0$. Therefore, it suffices to prove that

(i) $\varphi_{ij}'(0) \parallel \varphi_{ij'}'(0)$ for any $i \in V_k$ and edges ij and ij' with $l_{ij} > 0$ and $l_{ij'} > 0$,

(ii) for any $ij \in E$ satisfying $ij \subset \partial D$, there exists $m \in V - V_k$ such that $\triangle ijm \in F$, and thus $\varphi_{im}'(0) = \varphi_{jm}'(0)$, and

(iii) ∂D is connected.

For part (i), if it is not true then there exists $i \in V_k$, $ij \in E$ and $ij' \in E$ such that $l_{ij} > 0$, $l_{ij'} > 0$ and $\varphi_{ij}'(0)$ is not parallel to $\varphi_{ij'}'(0)$. Assuming this claim, by the w–balanced condition, there exists $ij'' \in E$ with $l_{ij''} > 0$, and the three vectors $\varphi_{ij}'(0)$, $\varphi_{ij'}'(0)$ and $\varphi_{ij''}'(0)$ are not contained in any closed half-space in $T_{q_i}M$. Assume $im \in E$, $l_{im} = 0$ and, without loss of generality, ij, im, ij' and ij'' are ordered counterclockwise in the one-ring neighborhood of i in T. By Lemma 3.4, there exists an admissible v such that $v_{im} \parallel \varphi_{ij''}'(0)$. By possibly changing a sign, we may assume that $v_{im} = \varphi_{ij''}'(0)/l_{ij''}$.

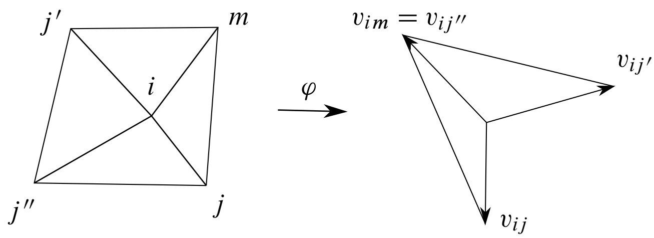

Figure 5: Overlapping triangles lead to angle surplus.

Then, as Figure 5 shows, a contradiction follows:

$$\begin{aligned}2\pi = \sum_{\sigma i \in \sigma} \theta^i_\sigma &\geq \angle v_{ij} 0 v_{im} + \angle v_{im} 0 v_{ij'} + \angle v_{ij'} 0 v_{ij''} + \angle v_{ij''} 0 v_{ij} \\ &= 2\angle v_{ij} 0 v_{ij''} + 2\angle v_{ij''} 0 v_{ij'} > 2\pi.\end{aligned}$$

Part (ii) is straightforward, so we will prove part (iii). By our assumption on the extension $\tilde{\varphi}$, $\tilde{\varphi}(D)$ contains only one point. Then the embedding map $i_D = \psi^{-1} \circ (\psi|_D)$ from D to $|T|$ is homotopic to the constant map $\psi^{-1} \circ (\tilde{\varphi}|_D)$, meaning that D is contractible in $|T|$. If ∂D contains at least two boundary components, then it is not difficult to show that $|T| - D$ has a connected component D' homeomorphic to an open disk. Let $\Phi_D : D \to \widetilde{M}$ be a lifting of $\tilde{\varphi}|_D$. Then $\Phi_D(\partial D') \subset \Phi_D(D)$ contains only a single point $x \in \widetilde{M}$. So, by the w–balanced condition, it is not difficult to derive a maximum principle and show that $\tilde{\varphi}|_{D'}$ equals the constant x. Then, by the definition of D, it is easy to see that D' should be a subset of D, which is a contradiction.

3.6 Proof of Lemma 3.6

Assume ij and ij' are two edges in σ. If the conclusion is not true, then there exists $ik \in E$ such that $l_{ik} > 0$ and v_{ik} is not parallel to v_{ij}. Notice that $v_{ij} = -v_{ij'}$, and

$$2\pi = \sum_{\{\sigma \in F : i \in \sigma\}} \theta^i_\sigma \geq \angle v_{ij} 0 v_{ij'} + \angle v_{ij} 0 v_{ik} + \angle v_{ij'} 0 v_{ik} = 2\pi.$$

Thus, equality holds in the above inequality, and for any $ik' \in E$, $v_{ik'}$ should be on the half circle that contains v_{ij}, v_{ik} and $v_{ij'}$. If v_m is the midpoint of this half circle, then

$$v_m \cdot \sum_{\{j : ij \in E\}} w_{ij} l_{ij} v_{ij} \geq w_{ik} l_{ik} v_m \cdot v_{ik} > 0.$$

This contradicts the fact that φ is w–balanced.

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