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We construct geometric maps from the cyclic homology groups of the (compact or wrapped) Fukaya category to the corresponding S^1–equivariant (Floer/quantum or symplectic) cohomology groups, which are natural with respect to all Gysin and periodicity exact sequences and are isomorphisms whenever the (nonequivariant) open–closed map is. These *cyclic open–closed maps* give constructions of geometric smooth and/or proper Calabi–Yau structures on Fukaya categories, which in the proper case implies the Fukaya category has a cyclic A_∞ model in characteristic 0, and also give a purely symplectic proof of the noncommutative Hodge–de Rham degeneration conjecture for smooth and proper subcategories of Fukaya categories of compact symplectic manifolds. Further applications of cyclic open–closed maps, to counting curves in mirror symmetry and to comparing topological field theories, are the subject of joint projects with Perutz and Sheridan, and with Cohen.

53D12, 53D37; 19D55

1 Introduction

This paper concerns the compatibility between chain level S^1–actions arising in two different types of Floer theory on a symplectic manifold M. The first of these $C_{-*}(S^1)$–actions[1] is induced geometrically on the *Hamiltonian Floer homology chain complex*

[1] We will use a cohomological grading convention, so singular chain complexes are *negatively graded*.

$CF^*(M)$, formally a type of Morse complex for an action functional on the free loop space, through rotating free loops. The homological action of $[S^1]$ is known as the *BV operator* $[\Delta]$, and the $C_{-*}(S^1)$–action can be used to define *S^1–equivariant Floer*[2] *homology theories*; see eg Bourgeois and Oancea [6] and Seidel [51]. The second $C_{-*}(S^1)$–action lies on the *Fukaya category* of M, and has discrete or combinatorial origins, coming from the hierarchy of compatible cyclic $\mathbb{Z}/k\mathbb{Z}$–actions on cyclically composable chains of morphisms between Lagrangians. A (categorical analogue of a) fundamental observation of Connes [13], Tsygan [63] and Loday and Quillen [42] is that such a structure, which exists on any category $\mathcal{C}$, can be packaged into a $C_{-*}(S^1)$–action on the *Hochschild homology chain complex* $\mathrm{CH}_*(\mathcal{C})$ of the category; see also Keller [32] and McCarthy [43]. The associated operation of multiplication by (a cycle representing) $[S^1]$ is frequently called the *Connes B operator*, and the corresponding S^1–equivariant homology theories are called *cyclic homology groups*.

A relationship between the Hochschild homology of the Fukaya category $\mathcal{F}$ and Floer homology on M is provided by the so-called *open–closed string map*

$$\mathcal{OC}\colon \mathrm{CH}_*(\mathcal{F}) \to CF^{*+n}(M); \tag{1-1}$$

see Abouzaid [1]. Our main result is about the compatibility of $\mathcal{OC}$ with $C_{-*}(S^1)$–actions. Namely, we prove — under technical hypotheses detailed below the main result — that $\mathcal{OC}$ can be made (coherently homotopically) $C_{-*}(S^1)$–equivariant:

Theorem 1.1 *Suppose that M, its Fukaya category and $CF^*(M)$ satisfy the technical assumptions ($\star$). Then the map $\mathcal{OC}$ admits a geometrically defined* ***S^1–equivariant enhancement****, to an A_∞ homomorphism of $C_{-*}(S^1)$–modules,*

$$\widetilde{\mathcal{OC}} \in \mathrm{RHom}^n_{C_{-*}(S^1)}(\mathrm{CH}_*(\mathcal{F}), CF^*(M)).$$

Remark 1.2 Theorem 1.1 implies (but is not implied by) the statement (Theorem 5.14) that $[\mathcal{OC}]$ intertwines homological actions of $[S^1]$.

Remark 1.3 In the geometric settings considered here $\mathcal{OC}$ does not a priori strictly intertwine the $C_{-*}(S^1)$–actions (due to a priori nonequivariant perturbations made to moduli spaces to define operations, and further due to the potential nontriviality of Δ, which — as Δ is defined using moduli spaces but B is defined using algebra — imply that

[2] Sometimes S^1–equivariant Floer theory is instead defined as Morse theory of an action functional on the S^1–Borel construction of the loop space. For a comparison between these two definitions, see [6].

$\mathcal{OC}\circ B$ and $\Delta\circ\mathcal{OC}$ involve moduli spaces of maps from differing domains). In particular, the homomorphism $\widetilde{\mathcal{OC}}$ involves extra data recording coherently higher homotopies between the two $C_{-*}(S^1)$–actions. This explains our use of the term "enhancement".

Remark 1.4 It can be shown using usual invariance arguments that the enhancement $\widetilde{\mathcal{OC}}$ we define in this paper is uniquely determined up to homotopy: while the geometric chain-level construction requires a number of auxiliary choices (of perturbation data on moduli spaces), any two sets of such choices produce homotopic enhancements.

To explain the consequences of Theorem 1.1 to cyclic homology and equivariant Floer homology, recall that there are a variety of S^1*–equivariant homology* chain complexes (and homology groups) that one can associate functorially to an A_∞ $C_{-*}(S^1)$–module P. For instance, denote by

$$P_{\mathrm{h}S^1},\quad P^{\mathrm{h}S^1},\quad P^{\mathrm{Tate}} \tag{1-2}$$

the *homotopy orbit complex*, *homotopy fixed-point complex* and *Tate complex* constructions of P, described in Section 2.2. When applied to the Hochschild complex $\mathrm{CH}_*(\mathcal{C})$, the constructions (1-2) by definition recover complexes computing (*positive*) *cyclic homology*, *negative cyclic homology* and *periodic cyclic homology* groups of $\mathcal{C}$, respectively; see Section 3.2. Similarly the group $H^*(CF^*(M)_{\mathrm{h}S^1})$ is the S^1*–equivariant Floer cohomology* studied (for the symplectic homology Floer chain complex); see eg Bourgeois and Oancea [6], Seidel [51] and Viterbo [64]. The groups $H^*(CF^*(M)^{\mathrm{h}S^1})$ and $H^*(CF^*(M)^{\mathrm{Tate}})$ have also been studied in recent work in Floer theory; see Albers, Cieliebak and Frauenfelder [4], Seidel [56] and Zhao [66]. Functoriality of the constructions (1-2) and homotopy-invariance properties of $C_{-*}(S^1)$–modules (see Corollary 2.18 and Proposition 2.19) immediately imply:

Corollary 1.5 *Let $HF^{*,+/-/\infty}_{S^1}(M)$ denote the (cohomology of the) homotopy orbit complex, fixed-point complex, and Tate complex construction applied to $CF^*(M)$, and let $\mathrm{HC}^{+/-/\infty}(\mathcal{C})$ denote the corresponding positive/negative/periodic cyclic homology groups. Under the hypotheses ($\star$) of Theorem 1.1, $\widetilde{\mathcal{OC}}$ induces* ***cyclic open–closed maps***

$$[\widetilde{\mathcal{OC}}^{+/-/\infty}]\colon \mathrm{HC}^{+/-/\infty}_*(\mathcal{F}) \to HF^{*+n,+/-/\infty}_{S^1}(M), \tag{1-3}$$

which are naturally compatible with respect to the various periodicity/Gysin exact sequences, and which are isomorphisms whenever $\mathcal{OC}$ is. □

The map (1-1) is frequently an isomorphism, allowing one to recover in these cases closed string Floer/quantum homology groups from open string, categorical ones; see Abouzaid, Fukaya, Oh, Ohta and Ono [2], Bourgeois, Ekholm and Eliashberg [5], Ganatra [24] and Ganatra, Perutz and Sheridan [27]. In such cases, Theorem 1.1 and Corollary 1.5 allow one to further categorically recover the $C_{-*}(S^1)$ as well as the associated equivariant homology groups (in terms of the cyclic homology groups of the Fukaya category).

Remark 1.6 There are other S^1–equivariant homology functors to which our results apply tautologically as well. For instance, consider the contravariant functor $P \mapsto (P_{\mathrm{h}S^1})^{\vee}$; when applied to $\mathrm{CH}_*(\mathcal{C})$ this produces the *cyclic cohomology* chain complex of $\mathcal{C}$.

We have been deliberately vague about which Fukaya category and which Hamiltonian Floer homology groups Theorem 1.1 applies to, as it applies in several different geometric (compact and noncompact) settings. To keep this paper a manageable length, we implement the map $\widetilde{\mathcal{OC}}$ and prove Theorem 1.1 in the technically simplest of such settings — our technical hypotheses are detailed in $(\star)$ below — for which the moduli spaces appearing in the constructions can be shown to be well behaved by classical methods. That being said, we should remark that our methods and arguments are orthogonal to the usual analytic difficulties faced in constructing Fukaya categories and open–closed maps in more general contexts, and we expect they should extend relatively directly to other settings. For instance, in the setting of relative Fukaya categories of compact projective Calabi–Yau manifolds (not considered here), an adapted version of our construction will appear in joint work with Perutz and Sheridan [26].

$(\star)$ **Assumptions on M, $\mathcal{F}$ and $CF^*(M)$**

In our main results we make technical assumptions, explained in detail in Section 3.3 for M and its Fukaya category and in Sections 4.1.1–4.1.2 for the corresponding Hamiltonian Floer homology chain complexes, which broadly encapsulate the following situations:

(1) If M is compact and satisfies suitable technical hypotheses such as being monotone or symplectically aspherical (see Section 3.3.1), one could take $\mathcal{F}$ to be the usual Fukaya category (or a summand thereof) of those compact Lagrangians also satisfying suitable technical hypotheses such as being monotone or not bounding disks with symplectic area. In this case $CF^*(M)$, the Hamiltonian

Floer complex of any (sufficiently generic) Hamiltonian, is quasi-isomorphic to the *quantum cohomology* ring with its trivial $C_{-*}(S^1)$–action.

(2) If M is noncompact and Liouville, one could take $\mathcal{F} = \mathcal{W}$ to be the *wrapped Fukaya category* and $CF^*(M) = SC^*(M)$ to be the *symplectic cohomology cochain complex* with its (typically highly nontrivial) $C_{-*}(S^1)$–action.

(3) If M is noncompact and Liouville, one could take $\mathcal{F} \subset \mathcal{W}$ to be the *Fukaya category of compact exact Lagrangians*. When restricted to $\mathrm{CH}_*(\mathcal{F})$, the map $\mathcal{OC}$ to $SC^*(M)$ of (2) factors through $H^*(M, \partial^\infty M)$, the *relative* (or *compactly supported*) *cohomology group* with its trivial $C_{-*}(S^1)$–action. In fact, as reviewed in Section 5.6.2, $\mathcal{OC}$ further factors through the symplectic *homology* chain complex $SC_*(M) \cong (SC^*(M))^\vee[-2n]$. One could take any of these groups ($SC^*(M)$, $H^*(M, \partial^\infty M)$ or $SC_*(M)$) to be $CF^*(M)$ here. For the main portion of the paper we use $CF^*(M) := H^*(M, \partial^\infty M)$.

For example, in case (2) above, when the relevant $[\mathcal{OC}]$ map is an isomorphism, Corollary 1.5 computes various S^1–equivariant symplectic cohomology groups[3] in terms of cyclic homology groups of the wrapped Fukaya category.

Remark 1.7 For the Fukaya subcategory of a single Lagrangian in a compact symplectic manifold M over a characteristic-zero (Novikov) field containing $\mathbb{R}$, a variant of the (positive) cyclic open–closed map has also been constructed by Fukaya, Oh, Ohta and Ono [23] (and will be generalized to multiple Lagrangians in Abouzaid, Fukaya, Oh, Ohta and Ono [2]). Their construction, which requires the target group ($H^*(M)$) to have trivial $C_{-*}(S^1)$–action, uses Connes' small ("coinvariants of cyclic group action bar") complex for (in characteristic zero only) positive cyclic homology, along with cyclically symmetric (necessarily virtual) perturbations of all moduli spaces (building on work of Fukaya [20] described in Remark 1.14), to directly construct a geometric map bypassing the higher A_∞ $C_{-*}(S^1)$–action homotopies constructed here. It does not seem possible to generalize the methods of [23] to the (possibly noncompact M with arbitrary coefficients, eg integral/rational/finite characteristic) settings considered here; see for instance the discussion in Remark 1.11. Also, the perspective of $C_{-*}(S^1)$–modules taken here makes it simpler to talk about (and describe) all cyclic homology theories at once, as well to study the compatibility of additional structures, eg exact sequences, semi-infinite/noncommutative Hodge structures.

[3] In particular, it computes the usual equivariant symplectic cohomology $SH^*_{S^1}(M) = H^*(SC^*(M)_{hS^1})$; see Bourgeois and Oancea [6] but note differing conventions, eg regarding homology vs cohomology.

Remark 1.8 There are other settings in which Fukaya categories are now well studied, for instance Fukaya categories of Lefschetz fibrations (and more general LG models), or more generally partially wrapped Fukaya categories (such as wrapped Fukaya categories of *Liouville sectors*). We do not discuss these situations in our paper, but expect suitable versions of Theorem 1.1 to hold in such settings too. We do note however that the target of the open–closed map from Hochschild homology in such settings is usually more subtle than in the cases discussed here, eg it does not typically have the structure of a unital ring.

Remark 1.9 One can consider variations on Theorem 1.1. As a notable example, let M denote a (noncompact) Liouville manifold, and $\mathcal{F}$ the Fukaya category of compact exact Lagrangians in M. Then there is a nontrivial refinement of the map $\mathrm{HH}_*(\mathcal{F}) \to H^*(M, \partial^\infty M)$, which can be viewed as a pairing $\mathrm{HH}_*(\mathcal{F}) \times H^*(M) \to \boldsymbol{k}$, to a pairing

$$\mathrm{CH}_*(\mathcal{F}) \otimes SC^*(M) \to \boldsymbol{k}.$$

(Symplectic cohomology does not satisfy Poincaré duality, so this is *not* equivalent to a map to symplectic cohomology.) Our methods also imply that this pairing admits an S^1–equivariant enhancement, with respect to the diagonal $C_{-*}(S^1)$–action on the left and the trivial action on the right. Passing to adjoints, we obtain cyclic open–closed maps from S^1–equivariant symplectic cohomology to cyclic *cohomology* groups of $\mathcal{F}$, and from cyclic homology of $\mathcal{F}$ to equivariant symplectic *homology*. See Section 5.6.2 for more details.

Beyond computing equivariant Floer cohomology groups in terms of cyclic homology theories, we describe in the following subsection two applications of Theorem 1.1 to the structure of Fukaya categories.

Remark 1.10 We anticipate additional concrete applications of Theorem 1.1 and its homological shadow, Theorem 5.14. For instance, one can study the compatibility of open–closed maps with *dilations* in the sense of Seidel and Solomon [58], which are elements B in $SH^*(M)$ satisfying $[\Delta]B = 1$ — the existence of dilations strongly constrains intersection properties of embedded Lagrangians; see Seidel [55]. Theorem 5.14, or rather the variant discussed in Remark 1.9, implies that *if there exists a dilation, eg an element $x \in SH^1(M)$ with $[\Delta]x = 1$, then on the Fukaya category of compact Lagrangians $\mathcal{F}$, there exists $x' \in (\mathrm{HH}_{n+1}(\mathcal{F}))^\vee$ with $x' \circ [B] = [\mathrm{tr}]$*, where tr is the geometric weak proper Calabi–Yau structure on the Fukaya category; see Section 1.1.

1.1 Calabi–Yau structures on the Fukaya category

Calabi–Yau structures are a type of cyclically symmetric duality structure on a dg or A_∞ category $\mathcal{C}$ generalizing the notion of a nowhere-vanishing holomorphic volume form on a complex algebraic variety X in the case $\mathcal{C} = \mathrm{perf}(X)$. As is well understood, there are two (in some sense dual) types of Calabi–Yau structures on A_∞ categories:

(1) **Proper Calabi–Yau structures** (Kontsevich and Soibelman [37]) These can be associated to *proper* categories $\mathcal{C}$ (those which have cohomologically finite-dimensional morphism spaces), abstract and refine the notion of integration against a nowhere-vanishing holomorphic volume form. For $\mathcal{C} = \mathrm{perf}(X)$ with X a proper n–dimensional variety, the resulting structure in particular induces the Serre duality pairing with trivial canonical sheaf $\mathrm{Ext}^*(\mathcal{E},\mathcal{F}) \times \mathrm{Ext}^*(\mathcal{F},\mathcal{E}) \to \boldsymbol{k}[-n]$. Roughly, a proper Calabi–Yau structure on $\mathcal{C}$ (of dimension n) is a map $[\widetilde{\mathrm{tr}}]\colon \mathrm{HC}^+_*(\mathcal{C}) \to \boldsymbol{k}[-n]$ satisfying a nondegeneracy condition.

(2) **Smooth Calabi–Yau structures** (Kontsevich, Takeda and Vlassopoulos [39]) These can be associated to *smooth* categories $\mathcal{C}$ (those with perfect diagonal bimodule), and abstract the notion of the nowhere-vanishing holomorphic volume form itself, along with the induced identification (by contraction against the volume form) of polyvectorfields with differential forms. Roughly, a smooth Calabi–Yau structure on $\mathcal{C}$ (of dimension n) is a map $[\widetilde{\mathrm{cotr}}]\colon \boldsymbol{k}[n] \to \mathrm{HC}^-_*(\mathcal{C})$, or equivalently an element $[\tilde{\sigma}]$ or "$[\mathrm{vol}_{\mathcal{C}}]$" in $\mathrm{HC}^-_{-n}(\mathcal{C})$, satisfying a nondegeneracy condition.

In both cases, the nondegeneracy condition can be phrased purely in terms of the underlying nonequivariant shadow of the map, eg in the first case on the induced map $[\mathrm{tr}]\colon \mathrm{HH}_*(\mathcal{C}) \to \mathrm{HC}^+(\mathcal{C}) \xrightarrow{[\widetilde{\mathrm{tr}}]} \boldsymbol{k}[-n]$. Precise definitions are reviewed in Section 6. When $\mathcal{C}$ is simultaneously smooth and proper, it is a folk result that the notions are equivalent; see [27, Proposition 6.10].

In general, Calabi–Yau structures may not exist and when they do, there may be a nontrivial space of choices; see Menichi [45] for an example. A Calabi–Yau structure in either form induces nontrivial identifications between Hochschild invariants of the underlying category $\mathcal{C}$.[4] Moreover, categories with Calabi–Yau structures (should) carry induced 2–dimensional chain level TQFT operations on their Hochschild homology

[4] In the proper case, there is an induced isomorphism between Hochschild cohomology and the linear dual of Hochschild homology. In the smooth case, there is an isomorphism between Hochschild cohomology and homology without taking duals.

chain complexes, associated to moduli spaces of Riemann surfaces with marked points; see Costello [14] and Kontsevich and Soibelman [37] in the proper case, and Kontsevich, Takeda, and Vlassopoulos in the smooth case [39; 38]. If the category is proper and nonsmooth (resp. smooth nonproper) the resulting TQFT is incomplete in that every operation must have at least one input (resp. output). In the smooth and proper case in particular, Calabi–Yau structures play a central role in the mirror symmetry motivated question of recovering Gromov–Witten invariants from the Fukaya category and to the related question of categorically recovering Hamiltonian Floer homology with all of its (possibly higher homotopical) operations. See Costello [14; 15] and Kontsevich [35] for work around these questions in the setting of abstract topological field theories, and Ganatra, Perutz and Sheridan [27] for applications of Calabi–Yau structures to recovering genus-0 Gromov–Witten invariants from the Fukaya category.

Remark 1.11 A closely related to (1), and well studied, notion is that of a *cyclic A_∞ category*: this is an A_∞ category $\mathcal{C}$ equipped with a chain level perfect pairing

$$\langle -,-\rangle \colon \hom(X,Y)\times\hom(Y,X)\to \boldsymbol{k}[-n]$$

such that the induced correlation functions

$$\langle \mu^d(-,-,\dots,-),-\rangle$$

are strictly (graded) cyclically symmetric for each d; see for instance Cho and Lee [9], Costello [14] and Fukaya [20]. Although the property of being a cyclic A_∞ structure is not a homotopy-invariant notion (ie not preserved under A_∞ quasi-equivalences), cyclic A_∞ categories and proper Calabi–Yau structures turn out to be weakly equivalent *in characteristic* 0, in the following sense. Any cyclic A_∞ category carries a canonical proper Calabi–Yau structure, and Kontsevich and Soibelman [37, Theorem 10.7] proved that a proper Calabi–Yau structure on any A_∞ category $\mathcal{C}$ determines a (canonical up to quasi-equivalence) quasi-isomorphism between $\mathcal{C}$ and a cyclic A_∞ category $\widetilde{\mathcal{C}}$. When $\mathrm{char}(\boldsymbol{k})\neq 0$, the two notions of proper Calabi–Yau and cyclic A_∞ differ in general, due to group cohomology obstructions to imposing cyclic symmetry. In such instances, it seems that the notion of a proper Calabi–Yau structure is the "correct" one (as it is a homotopy-invariant notion and, by Theorem 1.12, the compact Fukaya category always has one).

As a first application of Theorem 1.1, we verify the longstanding expectation that various compact Fukaya categories possess geometrically defined canonical Calabi–Yau structures.

Theorem 1.12 *The Fukaya category of compact Lagrangians has, under technical hypotheses* $(\star)$, *a canonical geometrically defined proper Calabi–Yau structure over any ground field* $\boldsymbol{k}$ (*over which the Fukaya category and* $\widetilde{\mathcal{OC}}$ *are defined*).

In fact, this proper Calabi–Yau structure is easy to describe in terms of the cyclic open–closed map (cf Corollary 1.5): it is the composition of the map[5]

$$\widetilde{\mathcal{OC}}^{+} \colon \mathrm{HC}_*^{+}(\mathcal{F}) \to H^{*+n}(M, \partial M)(\!(u)\!)/uH^{*+n}(M, \partial M)[\![u]\!]$$

with the linear map to $\boldsymbol{k}$ which sends the top class $\mathrm{PD}(\mathrm{pt}) \cdot u^0 \in H^{2n}(M, \partial M)$ to 1, and all other generators $\alpha \cdot u^{-i}$ to 0. See Section 6 for more details.

As a consequence of the discussion in Remark 1.11, specifically [37, Theorem 10.7], we deduce that

Corollary 1.13 *If* $\mathrm{char}(\boldsymbol{k}) = 0$, *then any Fukaya category of compact Lagrangians satisfying* $(\star)$ *admits a* (*canonical up to equivalence*) *cyclic* A_∞ (*minimal*) *model.*

Remark 1.14 In the case of compact symplectic manifolds and over $\boldsymbol{k} =$ a Novikov field containing $\mathbb{R}$, Fukaya [20] constructed a cyclic A_∞ model of the Floer cohomology algebra of a single compact Lagrangian, which will be extended to multiple objects by Abouzaid, Fukaya, Oh, Ohta and Ono [2].

Remark 1.15 In order to construct (chain level) 2d–TFTs on the Hochschild chain complexes of categories, Kontsevich and Soibelman [37] partly show (on the closed sector) that a proper Calabi–Yau structure can be used instead of the (weakly equivalent in characteristic 0) cyclic A_∞ structures considered in Costello [14]. One might similarly hope that, for applications of cyclic A_∞ structures to disc-counting/open Gromov–Witten invariants developed in Fukaya [21], a proper Calabi–Yau structure is in fact sufficient. See Cho and Lee [9] for related work.

Turning to smooth Calabi–Yau structures, in Section 6.2, we will establish the following existence result for smooth Calabi–Yau structures, which applies to wrapped Fukaya categories of noncompact (Liouville) manifolds as well as to Fukaya categories of compact manifolds.

[5] Recall that $C^*(M, \partial M)$ has the trivial $C_{-*}(S^1)$–module structure; the homology of the associated homotopy orbit complex is $H^{*+n}(M, \partial M)(\!(u)\!)/uH^{*+n}(M, \partial M)[\![u]\!]$, where $|u| = 2$, as described in Section 2.

Theorem 1.16 *Under the technical hypotheses* ($\star$), *suppose further that our symplectic manifold* M *is* ***nondegenerate*** *in the sense of* [24], *meaning that the map* $[\mathcal{OC}]\colon \mathrm{HH}_{*-n}(\mathcal{F}) \to HF^*(M)$ *hits the unit* $1 \in HF^*(M)$. *Then, its* (*compact or wrapped*) *Fukaya category* $\mathcal{F}$ *possesses a canonical, geometrically defined* ***strong smooth Calabi–Yau structure***.

Once more, the cyclic open–closed map gives an efficient description of this structure: it is the unique element $\mathrm{HC}^-_{-n}(\mathcal{F})$ mapping via $\widetilde{\mathcal{OC}}^-$ to the geometrically canonical lift $\tilde{1} \in H^*(CF^*(M)^{\mathrm{h}S^1})$ of the unit $1 \in CF^*(M)$ described in Section 4.4.[6]

Remark 1.17 In contrast to compact Fukaya categories or wrapped Fukaya categories of Liouville manifolds, the Fukaya categories of noncompact Lagrangians discussed in Remark 1.8 are typically not Calabi–Yau in either sense,[7] even if they are smooth or proper categories; indeed they typically arise as homological mirrors to perfect/coherent complexes on non-Calabi–Yau varieties. Instead, one might expect such categories to admit *pre-Calabi–Yau structures* in the sense of Kontsevich, Takeda and Vlassopoulos [38] (see also Yeung [65] and Seidel [57] for a construction of related structures), or *relative Calabi–Yau structures* in the sense of Brav and Dyckerhoff [7].

The notion of a smooth Calabi–Yau structure, or sCY structure, will be studied further in forthcoming joint work with R Cohen [12], and used to compare the wrapped Fukaya category of a cotangent bundle and string topology category of its zero section as *categories with sCY structures* (in order to deduce a comparison of topological field theories on both sides).

1.2 Noncommutative Hodge–de Rham degeneration for smooth and proper Fukaya categories

For a $C_{-*}(S^1)$–module P, there is a canonical Tor spectral sequence converging to $H^*(P_{\mathrm{h}S^1})$ with first page $H^*(P) \otimes_{\boldsymbol{k}} H^*(\boldsymbol{k}_{\mathrm{h}S^1}) \cong H^*(P) \otimes_{\boldsymbol{k}} H_*(\mathbb{CP}^\infty)$. When applied to the Hochschild complex $P = \mathrm{CH}_*(\mathcal{C})$ of a (dg/A_∞) category $\mathcal{C}$, the resulting spectral sequence, from (many copies of) $\mathrm{HH}_*(\mathcal{C})$ to $\mathrm{HC}^+(\mathcal{C})$ is called the *Hochschild-to-cyclic* or *noncommutative Hodge–de Rham* (*ncHDR*) *spectral sequence.* The latter name comes from the fact that when $\mathcal{C} = \mathrm{perf}(X)$ is perfect complexes on a complex

[6]As shown in [24; 27], if $[\mathcal{OC}]$ hits 1, then $[\mathcal{OC}]$ is an isomorphism, and hence by Corollary 1.5, $[\widetilde{\mathcal{OC}}^-]$ is too. Hence one can speak about the unique element.

[7]One manifestation of this is the failure of the target of the open–closed map to have a distinguished unit element, as also discussed in Remark 1.8.

variety X, this spectral sequence is equivalent (via Hochschild–Kostant–Rosenberg (HKR) isomorphisms) to the usual Hodge-to-de Rham spectral sequence from Hodge cohomology to de Rham cohomology

$$H^*(X, \Omega_X^*) \Rightarrow H^*_{\mathrm{dR}}(X),$$

which degenerates (as we are in characteristic 0) whenever X is smooth and proper. Motivated by this, Kontsevich [37; 35] formulated the *noncommutative Hodge–de Rham* (*ncHDR*) *degeneration conjecture*: for any smooth and proper category $\mathcal{C}$ in characteristic 0, its ncHDR spectral sequence degenerates. A general proof of this fact for $\mathbb{Z}$–graded categories was recently given by Kaledin [30; 29], following earlier work establishing it in the coconnective case.

Using the cyclic open–closed map, we can give a purely symplectic proof of the ncHDR degeneration property for those smooth and proper $\mathcal{C}$ arising as Fukaya categories, including in non-$\mathbb{Z}$–graded cases:

Theorem 1.18 *Let $\mathcal{A} \subset \mathcal{F}(M)$ be a smooth and proper subcategory of any Fukaya category of any compact symplectic manifold satisfying the technical assumptions* ($\star$), *over any field $\boldsymbol{k}$* (*over which the Fukaya category and the cyclic open–closed map are defined*). *Then*, *the noncommutative Hodge–de Rham spectral sequence for $\mathcal{A}$ degenerates.*

Proof The noncommutative Hodge–de Rham spectral sequence for $\mathcal{A}$ degenerates at page 1 if and only if $P = CH_*(\mathcal{A})$ is isomorphic (in the category of $C_{-*}(S^1)$–modules) to a trivial $C_{-*}(S^1)$–module, for instance, if the $C_{-*}(S^1)$–action is trivializable; see Dotsenko, Shadrin and Vallette [16, Theorem 2.1]. For compact symplectic manifolds M, recall that $CF^*(M) \cong H^*(M)$ has a canonically trivial(izable) $C_{-*}(S^1)$–action. (See Corollary 4.16; this comes from, for instance, the fact that we can choose a C^2–small Hamiltonian to compute the complex, all of the orbits of which are constant loops on which geometric rotation acts trivially. Or more directly, we can modify the definition of $\widetilde{\mathcal{OC}}$ to give a map directly to $H^*(M)$ with its trivial $C_{-*}(S^1)$–action, as described in Section 5.6.1.)

By earlier work [27; 25], whenever $\mathcal{A}$ is smoooth, $\mathcal{OC}|_{\mathcal{A}}$ is an isomorphism from $\mathrm{HH}_{*-n}(\mathcal{A})$ onto a nontrivial summand S of $HF^*(M) \cong \mathrm{QH}^*(M)$; the $C_{-*}(S^1)$–action on this summand is trivial too. Theorem 1.1 shows that $\widetilde{\mathcal{OC}}|_{\mathcal{A}}$ induces an isomorphism in the category of $C_{-*}(S^1)$–modules between $\mathrm{CH}_*(\mathcal{A})$ and $S[n]$ with its trivial action, so we are done. □

Remark 1.19 Theorem 1.18 holds for a field $\boldsymbol{k}$ of any characteristic over which the Fukaya category and relevant structures (satisfy ($\star$) and) are defined, for any grading structure that can be defined on the given Fukaya category; eg it holds for the $\mathbb{Z}/2$–graded Fukaya category of a monotone symplectic manifold over a field of any characteristic. In contrast, for an arbitrary smooth and proper $\mathbb{Z}/2$–graded dg category in characteristic zero, the noncommutative Hodge–de Rham degeneration is not yet established (though it is expected). And it is not always true in finite characteristic.

An incomplete explanation for the degeneration property holding for finite characteristic smooth and proper Fukaya categories may be that the Fukaya category over a characteristic p field $\boldsymbol{k}$ (whenever Lagrangians are monotone or tautologically unobstructed at least) may always admit a lift to second Witt vectors $W_2(\boldsymbol{k})$.[8]

As is described in joint work (partly ongoing) with Perutz and Sheridan [27; 26], the cyclic open–closed map $\widetilde{\mathcal{OC}}^-$ can further be shown to be a *morphism of semi-infinite Hodge structures*, a key step (along with the above degeneration property and construction of Calabi–Yau structure) in recovering Gromov–Witten invariants from the Fukaya category and enumerative mirror predictions from homological mirror theorems.

1.3 Outline of paper

In Section 2, we recall a convenient model for the category of A_∞–modules over $C_{-*}(S^1)$ and various equivariant homology functors from this category. In Section 3, we review the (compact and wrapped) Fukaya category along with $C_{-*}(S^1)$–action on its (and more generally, any cohomologically unital A_∞ category's) *nonunital Hochschild chain complex* (a variant on usual cyclic bar complex that has usually appeared in the symplectic literature, eg in Abouzaid [1]). In Section 4, we recall the construction of the A_∞ $C_{-*}(S^1)$–module structure on the (Hamiltonian) Floer chain complex, following Bourgeois and Oancea [6] and Seidel [51]; note that our technical setup is slightly different, though equivalent. Then we prove our main results in Section 5. Some technical and conceptual variations on the construction of $\widetilde{\mathcal{OC}}$ (including Remark 1.9) are discussed at the end of this section; see Section 5.6. Finally, in Section 6 we apply our results to construct proper and smooth Calabi–Yau structures, proving Theorems 1.12 and 1.16.

[8]The author wishes to thank Mohammed Abouzaid for discussions regarding this point.

1.4 Conventions

We work over a ground field $\boldsymbol{k}$ of arbitrary characteristic, though we note that all of our geometric constructions are valid over an arbitrary ring, eg $\mathbb{Z}$. All chain complexes will be graded *cohomologically*, including singular chains of any space, which hence have negative the homological grading and are denoted by $C_{-*}(X)$. All gradings are either in $\mathbb{Z}$ or $\mathbb{Z}/2$ (in the latter case, degrees of maps are implicitly mod 2).

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2 Complexes with circle action

In this section, we review a convenient model for the category of A_∞ $C_{-*}(S^1)$–modules, for which the A_∞ $C_{-*}(S^1)$–action can be described by a single hierarchy of maps satisfying equations. We also describe various equivariant homology complexes in this language in terms of simple formulae. This model appears elsewhere in the literature as ∞*–mixed complexes* or S^1*–complexes* or *multicomplexes* (we will sometimes adopt the second term); see eg [6; 66; 16], but note that the first and third references use homological grading conventions.

2.1 Definitions

Let $C_{-*}(S^1)$ denote the dg algebra of chains on the circle with coefficients in $\boldsymbol{k}$, graded cohomologically, with multiplication induced by the Pontryagin product $S^1 \times S^1 \to S^1$. This algebra is *formal*, or quasi-isomorphic to its homology, an exterior algebra on one generator Λ of degree -1 with no differential. Henceforth, by abuse of notation we take this exterior algebra as our working model for $C_{-*}(S^1)$,

$$C_{-*}(S^1) := \boldsymbol{k}[\Lambda]/\Lambda^2, \quad \text{where } |\Lambda| = -1, \tag{2-1}$$

and use the terminology $C_{-*}^{\mathrm{sing}}(S^1)$ to refer to usual singular chains on S^1.

Definition 2.1 A *strict S^1–complex*, or a *chain complex with strict/dg S^1–action*, is a unital differential graded module over $\boldsymbol{k}[\Lambda]/\Lambda^2$.

Let (M,d) be a strict S^1–complex; by definition (M,d) is a cochain complex (recall our conventions for complexes from Section 1.4) and the unital dg $\boldsymbol{k}[\Lambda]/\Lambda^2$–module structure is equivalent to the data of the single additional operation of multiplying by Λ,

$$\Delta = \Lambda \cdot - \colon M_* \to M_{*-1}, \tag{2-2}$$

which must square to zero and anticommute with d. In other words, (M,d,Δ) is what is known as a *mixed complex*; see eg [8; 31; 41].

We will need to work with the weaker notion of an A_∞–action, or rather an A_∞–module structure over $C_{-*}(S^1) = \boldsymbol{k}[\Lambda]/\Lambda^2$. Recall that a *(left) A_∞–module M* [33; 52; 50; 24] over the associative graded algebra $A = \boldsymbol{k}[\Lambda]/\Lambda^2$ is a graded $\boldsymbol{k}$–module M equipped with maps

$$\mu^{k|1} \colon A^{\otimes k} \otimes M \to M, \quad \text{for } k \geq 0, \tag{2-3}$$

of degree $1-k$, satisfying the A_∞–module equations described in [50] or [24, (2.35)]. Since $A = \boldsymbol{k}[\Lambda]/\Lambda^2$ is unital, we can work with modules that are also *strictly unital* (see [50, (2.6)]); this implies that all multiplications by a sequence with at least one unit element is completely specified,[9] and hence the only nontrivial structure maps to define are the operators

$$\delta_k := \mu_M^{k|1}(\underbrace{\Lambda, \dots, \Lambda}_{k \text{ copies}}, -) \colon M \to M[1-2k] \quad \text{for } k \geq 0. \tag{2-4}$$

The A_∞–module equations are equivalent to the relations

$$\sum_{i=0}^{s} \delta_i \delta_{s-i} = 0 \tag{2-5}$$

for (2-4), for each $s \geq 0$. We summarize the discussion so far with the following definition.

[9] More precisely, $\mu^{1|1}(1, \boldsymbol{m}) = \boldsymbol{m}$ and $\mu^{k|1}(\dots, 1, \dots, \boldsymbol{m}) = 0$ for $k > 1$.

Definition 2.2 An S^1*–complex*, or a *chain complex with an A_∞ S^1–action*, is a strictly unital (left) A_∞–module M over $\boldsymbol{k}[\Lambda]/\Lambda^2$. Equivalently, it is a graded $\boldsymbol{k}$–module M equipped with operations $\{\delta_k \colon M \to M[1-2k]\}_{k\geq 0}$ satisfying, for each $s \geq 0$, the hierarchy of equations (2-5).

Remark 2.3 If X is a topological space with S^1–action, then $C_{-*}(X)$ carries a dg $C_{-*}^{\mathrm{sing}}(S^1)$–module structure, with module action induced by the action $S^1 \times X \to X$. Under the A_∞ equivalence $C_{-*}^{\mathrm{sing}}(S^1) \cong \boldsymbol{k}[\Lambda]/\Lambda^2$, it follows that $C_{-*}(X)$ carries an A_∞ (not necessarily dg) $\boldsymbol{k}[\Lambda]/\Lambda^2$–module structure, which can further be made strictly unital, by [40, Theorem 3.3.1.2] or by passing to normalized chains. If one wishes, one can then appeal to abstract strictification results to produce a dg $\boldsymbol{k}[\Lambda]/\Lambda^2$–module which is quasi-isomorphic as A_∞ $\boldsymbol{k}[\Lambda]/\Lambda^2$–modules to $C_{-*}(X)$. More directly, it turns out [12] that one can find an equivalent dg $\boldsymbol{k}[\Lambda]/\Lambda^2$–module by taking a suitable quotient of the normalized singular chain complex $C_{-*}(X)$ to form *unordered normalized singular chains* of X (identifying simplices differing by permuting vertices and quotienting by those that are degenerate).

Remark 2.4 There are multiple sign conventions for A_∞–modules over an A_∞–algebra; the most common two conventions appear in [50, (2.6)] and [52, (1j)], as well as many other places. These conventions are completely irrelevant for strictly unital $A = \boldsymbol{k}[\Lambda]/\Lambda^2$–modules, as the reduced degree of any element in $\bar{A} = \mathrm{span}_{\boldsymbol{k}}(\Lambda)$ is zero; hence the (Koszul) signs in various formulae are $+1$ in either convention.

For $s=0$, equation (2-5) says simply that the differential $d=\delta_0$ squares to 0; for $s=1$, equation (2-5) implies $\delta := \delta_1$ anticommutes with d, and for $s=2$, $(\delta)^2 = -(d\delta_2 + \delta_2 d)$, or that δ^2 is chain-homotopic to zero, but not strictly zero, as measured by the chain homotopy δ_2.

S^1–complexes, as strictly unital A_∞–modules over the augmented algebra $A = \boldsymbol{k}[\Lambda]/\Lambda^2$, are the objects of a dg category, which we will call

$$S^1\text{–mod} := uA\text{–mod}, \tag{2-6}$$

whose morphisms and compositions we now recall.[10] Denote by $\epsilon \colon A \to \boldsymbol{k}$ the augmentation map, and $\bar{A} = \ker\epsilon = \mathrm{span}_{\boldsymbol{k}}(\Lambda)$ the augmentation ideal. Let M and N be two strictly unital A_∞ A–modules. A *unital premorphism of degree k* from M to N

[10] For the definition of this category compare [50, pages 90, 94], where it is called $\mathrm{mod}(A) = \mathrm{mod}(A, \boldsymbol{k})$.

is a collection of maps $F^{d|1}: \bar{A}^{\otimes d} \otimes M \to N$ for $d \geq 0$, of degree $k-d$, or equivalently, since $\dim_{\boldsymbol{k}}(\bar{A}) = 1$ in degree -1, a collection of operators

$$F = \{F^d\}_{d \geq 0}, \quad F^d := F^{d|1}(\underbrace{\Lambda, \dots, \Lambda}_{d \text{ copies}}, -): M \to N[k-2d]. \tag{2-7}$$

If $T(\bar{A}[1]) = \bigoplus_{d \geq 0} \bar{A}[1]^{\otimes d}$ denotes the tensor algebra of $\bar{A}[1]$, then F can be alternatively packaged into the data of a single map $F := \bigoplus_{d \geq 0} F^d : T\bar{A}[1] \otimes M \to N$ of degree k. The space of premorphisms of each degree form the graded space of morphisms in S^1–mod, which we will denote by $\mathrm{Rhom}_{S^1}(-,-)$:

$$\begin{aligned} \mathrm{Rhom}_{S^1}(M, N) &:= \bigoplus_{k \in \mathbb{Z}} \mathrm{Rhom}^k_{S^1}(M, N) \\ &:= \bigoplus_{k \in \mathbb{Z}} \hom_{\mathrm{grVect}}(T(\bar{A}[1]) \otimes M, N[k]) \\ &= \left(\bigoplus_{k \in \mathbb{Z}} \hom_{\mathrm{grVect}} \left(\bigoplus_{d \geq 0} M[2d], N[k] \right) \right). \end{aligned} \tag{2-8}$$

There is a differential ∂ on (2-8) described in [50, page 90]; in terms of the simplified form of premorphisms (2-7), one has

$$(\partial F)^s = \sum_{i=0}^{s} F^i \circ \delta^M_{s-i} - (-1)^{\deg(F)} \sum_{j=0}^{s} \delta^N_{s-j} \circ F^j. \tag{2-9}$$

An A_∞ $\boldsymbol{k}[\Lambda]/\Lambda^2$*–module homomorphism*, or S^1*–complex homomorphism*, is a premorphism $F = \{F^d\}$ which is closed, ie $\partial F = 0$. In particular, F is an A_∞–module homomorphism if the following equations are satisfied for each s:

$$\sum_{i=0}^{s} F^i \circ \delta^M_{s-i} = (-1)^{\deg(F)} \sum_{j=0}^{s} \delta^N_{s-j} \circ F^j. \tag{2-10}$$

Note that the $s = 0$ equation reads $F^0 \circ \delta^M_0 = (-1)^{\deg(F)} \delta^N_0 \circ F^0$, so (if $\partial F = 0$) F^0 induces a cohomology level map $[F^0]$: $H^*(M) \to H^{*+\deg(F)}(N)$. A module homomorphism (or closed morphism) F is said to be a *quasi-isomorphism* if $[F^0]$ is an isomorphism on cohomology. A *strict* module homomorphism F is one for which $F^k = 0$ for $k > 0$.

Remark 2.5 There is an enlarged notion of a *nonunital* premorphism (used for modules which are not necessarily strictly unital), which is a collection of maps $\{\hat{F}^d : A^{\otimes d} \otimes M \to N\}_d$ instead of $\{F^d : \bar{A}^{\otimes d} \otimes M \to N\}_d$. Any premorphism

$F = \{F^d\}_d$ as we have defined it extends to a nonunital premorphism $\widehat{F} = \{\widehat{F}^d\}$ by declaring $\widehat{F}^d(\dots, 1, \dots, \boldsymbol{m}) = 0$. For strictly unital modules, the resulting inclusion from the complex of premorphisms to the complex of nonunital premorphisms is a quasi-isomorphism.

Remark 2.6 When M and N are dg modules, or strict S^1–complexes, the complex $\mathrm{Rhom}_{S^1}(M, N)$ is a *reduced bar model* of the chain complex of derived $\boldsymbol{k}[\Lambda]/\Lambda^2$–module homomorphisms, which is one of the reasons we have adopted the terminology "Rhom". In the A_∞ setting, we recall that there is no sensible "nonderived" notion of a $\boldsymbol{k}[\Lambda]/\Lambda^2$–module map; compare [50].

The composition in the category S^1–mod,

$$\mathrm{Rhom}_{S^1}(N, P) \otimes \mathrm{Rhom}_{S^1}(M, N) \to \mathrm{Rhom}_{S^1}(M, P), \tag{2-11}$$

is defined by

$$(G \circ F)^s = \sum_{j=0}^{s} G^{s-j} \circ F^j. \tag{2-12}$$

Remark 2.7 If M is any S^1–complex, then its endomorphisms $\mathrm{Rhom}_{S^1}(M, M)$, equipped with composition, form a dg algebra. As an example, consider $M = \boldsymbol{k}$, with trivial module structure determined by the augmentation $\epsilon\colon \boldsymbol{k}[\Lambda]/\Lambda^2 \to \boldsymbol{k}$. It is straightforward to compute that, as a dga,

$$\mathrm{Rhom}_{S^1}(\boldsymbol{k}, \boldsymbol{k}) \cong \boldsymbol{k}[u], \quad \text{with } |u| = 2. \tag{2-13}$$

In terms of the definition of morphism spaces (2-8), u corresponds to the unique morphism $G = \{G^d\}_{d \geq 0}$ of degree $+2$ with $G^1 = \mathrm{id}$ and $G^s = 0$ for $s \neq 1$.

In addition to taking the morphism spaces, one can define the (derived) *tensor product* of S^1–complexes N and M: using the isomorphism $A \cong A^{\mathrm{op}}$ coming from commutativity of $A = \boldsymbol{k}[\Lambda]/\Lambda^2$, first view N as a *right A_∞ A–module* (see [50, pages 90, 94], where the category of right A–modules is called $\mathrm{mod}(\boldsymbol{k}, A)$, see also [52, (1j)] and [24, Section 2]), and then take the usual (necessarily derived) tensor product of N and M over A (see [50, page 91] or [24, Section 2.5]). The resulting chain complex — which we will, by abuse of notation, indicate as the derived tensor product over S^1 — has underlying graded vector space

$$N \otimes^{\mathbb{L}}_{S^1} M := N \otimes^{\mathbb{L}}_{A} M := \bigoplus_{d \geq 0} N \otimes \bar{A}[1]^{\otimes d} \otimes M = \bigoplus_{d \geq 0} (N \otimes_{\boldsymbol{k}} M)[2d], \tag{2-14}$$

where the degree s part is $\bigoplus_{d\geq 0}\bigoplus_t N_t\otimes M_{s+2d-t}$. Let us refer to an element $n\otimes m$ of the d^{th} summand of this complex by suggestive notation

$$n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d\text{ times}}\otimes m$$

as in the first line of (2-14). With this notation, the differential on (2-14) acts as

$$\begin{aligned}\partial(n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d}\otimes m)\\ =\sum_{i=0}^{d}\Big((-1)^{|m|}\delta_i^N n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d-i}\otimes m+n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d-i}\otimes\delta_i^M m\Big).\end{aligned}\tag{2-15}$$

Here our sign convention follows [24, Section 2.5] rather than [50], though the sign difference is minimal.

Remark 2.8 Analogously to Remark 2.6, if M and N are unital dg modules over $A=\boldsymbol{k}[\Lambda]/\Lambda^2$, the chain complex described above computes their derived tensor product, whose homology is $\mathrm{Tor}_A(M,N)$. While we have therefore opted for the notation $N\otimes_A^{\mathbb{L}}M$, or rather the abbreviation $N\otimes_{S^1}^{\mathbb{L}}M$, we note that the (derived) tensor product of A_∞–modules is often written in the A_∞ literature without the superscript $\mathbb{L}$ as simply $N\otimes_A M$; compare [50, equation (2.6)].

The pairing (2-14) is suitably functorial with respect to morphisms of the S^1–complexes involved, meaning that $-\otimes_{S^1}N$ and $M\otimes_{S^1}-$ both induce dg functors from S^1–mod to chain complexes; compare [50, page 92]. For instance, if $F=\{F^j\}\colon M_0\to M_1$ is a premorphism of S^1–complexes, then there are induced maps

$$\begin{aligned}&F_\sharp\colon N\otimes_{S^1}^{\mathbb{L}}M_0\to N\otimes_{S^1}^{\mathbb{L}}M_1,\\ &n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d}\otimes m\mapsto\sum_{j=0}^{d}n\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d-j}\otimes F^j(m),\end{aligned}\tag{2-16}$$

$$\begin{aligned}&F_\sharp\colon M_0\otimes_{S^1}^{\mathbb{L}}N\to M_1\otimes_{S^1}^{\mathbb{L}}N,\\ &m\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d}\otimes n\mapsto\sum_{j=0}^{d}(-1)^{\deg(F)\cdot|n|}F^j(m)\otimes\underbrace{\Lambda\otimes\cdots\otimes\Lambda}_{d-j}\otimes n,\end{aligned}\tag{2-17}$$

which are chain maps if $\partial(F)=0$.

Hom and tensor complexes of S^1–complexes, as in any category of A_∞–modules, satisfy the following strong homotopy-invariance properties.

Proposition 2.9 (homotopy invariance) *If $F\colon M \to M'$ is any quasi-isomorphism of S^1–complexes (meaning $\partial(F)=0$ and $[F^0]\colon H^*(M) \xrightarrow{\cong} H^*(M')$ is an isomorphism), then composition with F induces quasi-isomorphisms of hom and tensor complexes:*

$$\begin{aligned} F\circ\cdot\colon \mathrm{Rhom}_{S^1}(M',P) &\xrightarrow{\sim} \mathrm{Rhom}_{S^1}(M,P),\\ \cdot\circ F\colon \mathrm{Rhom}_{S^1}(P,M) &\xrightarrow{\sim} \mathrm{Rhom}_{S^1}(P,M'),\\ F_\sharp\colon N\otimes^{\mathbb{L}}_{S^1} M &\xrightarrow{\sim} N\otimes^{\mathbb{L}}_{S^1} M',\\ F_\sharp\colon M\otimes^{\mathbb{L}}_{S^1} N &\xrightarrow{\sim} M'\otimes^{\mathbb{L}}_{S^1} N. \end{aligned} \tag{2-18}$$

The proof is a standard argument (though we do not know a specific reference): one exhibits acyclicity of the cone of each of the above maps by studying the spectral sequence with respect to the length filtration (with respect to the number of $\bar{A}^{\otimes d}$ factors in the bar model of the complexes); the first page of the associated spectral sequence is the cone of the map associated to the derived homs/tensor products of the associated homology-level modules by the homology level map $[F^0]$, which is acyclic by hypothesis; hence the second page vanishes and the cone is acyclic; compare analogous arguments in [52, Lemma 2.12] or [24, Proposition 2.2].

Let $(P,\{\delta_i^P\})$ and $(Q,\{\delta_j^Q\}_j)$ be S^1–complexes, and $f\colon P\to Q$ a chain map of some degree $\deg(f)$ (with respect to the δ_0^P and δ_0^Q differentials). An *S^1–equivariant enhancement of f* is a degree $\deg(f)$ homomorphism $\boldsymbol{F}=\{\boldsymbol{F}^i\}_{i\geq 0}$ of S^1–complexes — eg a closed morphism, so $\boldsymbol{F}$ satisfies (2-10) — with $[\boldsymbol{F}^0]=[f]$.

Remark 2.10 There are a series of obstructions to the existence of an S^1–equivariant enhancement of a given chain map f; for instance, a first necessary condition is the vanishing of the cohomology class $[f]\circ[\delta_1^P]-[\delta_1^Q]\circ[f]$.

Finally, we note that, just as the product of S^1–spaces $X\times Y$ possesses a diagonal action, the (linear) tensor product of S^1–complexes is again an S^1–complex.

Lemma 2.11 *If*

$$\Bigl(M,\delta_{\mathrm{eq}}^M=\sum_{i=0}^{\infty}\delta_j^M u^j\Bigr) \quad\textit{and}\quad \Bigl(N,\delta_{\mathrm{eq}}^N=\sum_{i=0}^{\infty}\delta_i^N u^i\Bigr)$$

are S^1–complexes, then the graded vector space $M\otimes N$ is naturally an S^1–complex with $\delta_{\mathrm{eq}}^{M\otimes N}=\sum_{i=0}^{\infty}\delta_k^{M\otimes N}u^k$, where

$$\delta_k^{M\otimes N}(\boldsymbol{m}\otimes\boldsymbol{n}) := (-1)^{|\boldsymbol{n}|}\delta_k^M\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{m}\otimes\delta_k^N\boldsymbol{n}. \tag{2-19}$$

We call the resulting S^1–action on $M\otimes N$ the *diagonal S^1–action*.

Proof We compute

$$\begin{aligned}(2\text{-}20)\quad &\delta_j^{M\otimes N}\delta_k^{M\otimes N}(\boldsymbol{m}\otimes\boldsymbol{n})\\ &=\delta_j^M\delta_k^M\boldsymbol{m}\otimes\boldsymbol{n}+(-1)^{|\boldsymbol{n}|+1}\delta_j^M\boldsymbol{m}\otimes\delta_k^N\boldsymbol{n}+(-1)^{|\boldsymbol{n}|}\delta_k^M\boldsymbol{m}\otimes\delta_j^N\boldsymbol{n}+\boldsymbol{m}\otimes\delta_j^N\delta_k^N\boldsymbol{n}.\end{aligned}$$

Summing over all $j+k=s$, the middle two terms cancel in pairs and the sums of the leftmost terms (resp. rightmost) terms respectively vanish because M (resp. N) is an S^1–complex. □

Definition 2.12 Let $M:=(M,d)$ be a chain complex over $\boldsymbol{k}$. The pullback of M along the (augmentation) map $\boldsymbol{k}[\Lambda]/\Lambda^2\to\boldsymbol{k}$ is called the *trivial S^1–complex*, or *chain complex with trivial S^1–action* associated to M, and denoted by $\underline{M}^{\mathrm{triv}}$. Concretely, $\underline{M}^{\mathrm{triv}}:=(M,\delta_0=d,\delta_k=0\text{ for }k>0)$.

2.2 Equivariant homology groups

Let M be an S^1–complex. Let $\boldsymbol{k}=\underline{\boldsymbol{k}}^{\mathrm{triv}}$ denote the strict trivial rank-1 S^1–complex concentrated in degree 0.

Definition 2.13 The *homotopy orbit complex* of M is the (derived) tensor product of M with $\boldsymbol{k}$ over $C_{-*}(S^1)$:

$$M_{\mathrm{h}S^1}:=\boldsymbol{k}\otimes^{\mathbb{L}}_{S^1}M. \tag{2-21}$$

The (strict) morphism of S^1–complexes $\epsilon\colon \boldsymbol{k}[\Lambda]/\Lambda^2\to\boldsymbol{k}$ (here $\boldsymbol{k}[\Lambda]/\Lambda^2$ comes equipped with structure maps $\delta_k=0$ for $k\neq 1$, and $\delta_1=\Lambda\cdot -$) induces by functoriality a chain map from M to $M_{\mathrm{h}S^1}$ called the *projection to homotopy orbits*,

$$\mathrm{pr}\colon M\cong\boldsymbol{k}[\Lambda]/\Lambda^2\otimes^{\mathbb{L}}_{S^1}M\to\boldsymbol{k}\otimes^{\mathbb{L}}_{S^1}M=M_{\mathrm{h}S^1}. \tag{2-22}$$

Remark 2.14 When $M=C_{-*}(X)$, with S^1–complex induced by a topological S^1–action on X as in Remark 2.3, the complex (2-21) computes the Borel equivariant homology of X, by the following reasoning: first, the A_∞ equivalence between $\boldsymbol{k}[\Lambda]/\Lambda^2$ and $C_{-*}^{\mathrm{sing}}(S^1)$ induces an equivalence

$$M_{\mathrm{h}S^1}\simeq C_{-*}(\mathrm{pt})\otimes^{\mathbb{L}}_{C_{-*}^{\mathrm{sing}}(S^1)}C_{-*}(X).$$

Next, one observes that $C_{-*}(ES^1)\to C_{-*}(\mathrm{pt})$ is a quasi-isomorphism of dg $C_{-*}^{\mathrm{sing}}(S^1)$–modules, where the $C_{-*}^{\mathrm{sing}}(S^1)$–actions are induced by the S^1–actions on ES^1 and pt, respectively. Hence, there is a quasi-isomorphism of derived tensor products

$$C_{-*}(\mathrm{pt})\otimes^{\mathbb{L}}_{C_{-*}^{\mathrm{sing}}(S^1)}C_{-*}(X)\simeq C_{-*}(ES^1)\otimes^{\mathbb{L}}_{C_{-*}^{\mathrm{sing}}(S^1)}C_{-*}(X).$$

Finally, it is a standard fact in algebraic topology (used in the construction of Eilenberg–Moore-type spectral sequences, eg [44, Theorem 7.27] and [17, Proposition 6.13]) that, as ES^1 is a principal S^1–bundle,

$$C_{-*}(ES^1) \otimes^{\mathbb{L}}_{C_{-*}^{\mathrm{sing}}(S^1)} C_{-*}(X) \simeq C_{-*}(ES^1 \times_{S^1} X) = C_{-*}(X_{\mathrm{h}S^1}),$$

which is the usual chain complex computing (Borel) equivariant homology. This gives some justification for the usage of the subscript $\mathrm{h}S^1$ notation in Definition 2.13.

Definition 2.15 The *homotopy fixed-point complex* of M is the chain complex of morphisms from $\boldsymbol{k}$ to M in the category of S^1–complexes,

$$M^{\mathrm{h}S^1} := \mathrm{Rhom}_{S^1}(\boldsymbol{k}, M). \tag{2-23}$$

The morphism of modules $\epsilon\colon \boldsymbol{k}[\Lambda]/\Lambda^2 \to \boldsymbol{k}$ induces a chain map $M^{\mathrm{h}S^1} \to M$, called the *inclusion of homotopy fixed points*,

$$\iota\colon M^{\mathrm{h}S^1} = \mathrm{Rhom}_{S^1}(\boldsymbol{k}, M) \to \mathrm{Rhom}_{S^1}(\boldsymbol{k}[\Lambda]/\Lambda^2, M) \cong M. \tag{2-24}$$

Remark 2.16 To motivate the usage "homotopy fixed points", in the topological category, the usual fixed points of a G–action can be described as $\mathrm{Maps}_G(\mathrm{pt}, X)$. When $M = C_{-*}(X)$ for X an S^1–space, there is a canonical map $C_{-*}(X^{\mathrm{h}S^1}) \to (C_{-*}(X))^{\mathrm{h}S^1}$. However, in contrast to the case of homotopy orbits discussed in Remark 2.14, this map need not be an equivalence.

Composition in the category S^1–mod induces a natural action of

$$\begin{aligned} \mathrm{Rhom}_{S^1}(\boldsymbol{k}, \boldsymbol{k}) &= \boldsymbol{k}[u] \quad \text{with } |u| = 2 \\ &= H^*(BS^1) \end{aligned} \tag{2-25}$$

on the homotopy fixed-point complex. There is a third important equivariant homology complex, called the *periodic cyclic*, or *Tate* complex of M, defined as the localization of $M^{\mathrm{h}S^1}$ away from $u = 0$,

$$M^{\mathrm{Tate}} := M^{\mathrm{h}S^1} \otimes_{\boldsymbol{k}[u]} \boldsymbol{k}[u, u^{-1}]. \tag{2-26}$$

The Tate construction sits in an exact sequence between the homotopy orbits and fixed points.

Remark 2.17 (Gysin sequences) It is straightforward from the viewpoint of A_∞ $C_{-*}(S^1)$–modules to explain the appearance of various Gysin and periodicity sequences. Take for instance the *Gysin exact triangle*

$$M \xrightarrow{\mathrm{pr}} M_{\mathrm{h}S^1} \to M_{\mathrm{h}S^1}[2] \xrightarrow{[1]} M.$$

This is a manifestation of a canonical exact triangle of objects in S^1–mod,

$$\boldsymbol{k}[\Lambda]/\Lambda^2 \xrightarrow{\epsilon} \boldsymbol{k} \xrightarrow{u} \boldsymbol{k}[2] \xrightarrow{[1]} \boldsymbol{k}[\Lambda]/\Lambda^2$$

(recall in Remark 2.7 it was shown that $\mathrm{Rhom}_{S^1}(\boldsymbol{k}, \boldsymbol{k}) \cong \boldsymbol{k}[u]$), pushed forward by the functor $(-) \otimes^{\mathbb{L}}_{S^1} M$. The other exact sequences arise similarly.

As a special case of the general homotopy-invariance properties of A_∞–modules stated in Proposition 2.9, we have:

Corollary 2.18 *If $F\colon M \to N$ is a homomorphism of S^1–complexes (meaning a closed morphism), it induces chain maps between equivariant theories:*

$$F^{\mathrm{h}S^1}\colon M^{\mathrm{h}S^1} \to N^{\mathrm{h}S^1}, \tag{2-27}$$

$$F_{\mathrm{h}S^1}\colon M_{\mathrm{h}S^1} \to N_{\mathrm{h}S^1}, \tag{2-28}$$

$$F^{\mathrm{Tate}}\colon M^{\mathrm{Tate}} \to N^{\mathrm{Tate}}. \tag{2-29}$$

If F is a quasi-isomorphism of S^1–complexes (meaning simply that $[F^0]$ is a homology isomorphism), then (2-27)–(2-29) are quasi-isomorphisms of chain complexes. □

Functoriality further tautologically implies:

Proposition 2.19 *If $F\colon M \to N$ is a homomorphism of S^1–complexes, then the various induced maps (2-27)–(2-29) intertwine all of the long exact sequences for (equivariant homology groups of) M with those for N.* □

2.3 u–linear models for S^1–complexes

It is convenient to package the data described in the previous two sections into "u–linear generating functions", in the following way: Let u be a formal variable of degree $+2$. Let us use the abuse of notation

$$M[\![u]\!] := M \mathbin{\widehat{\otimes}_{\boldsymbol{k}}} \boldsymbol{k}[u]$$

for the u–adically completed tensor product in the category of graded vector spaces; in other words $M[\![u]\!] := \bigoplus_k M[\![u]\!]_k$, where $M[\![u]\!]_k = \bigl\{\sum_{i=0}^{\infty} m_i u^i \mid m_i \in M_{k-2i}\bigr\}$. Then, we frequently write an S^1–complex $(M, \{\delta_k\}_{k\geq 0})$ as a $\boldsymbol{k}$–module M equipped with a map

$$\delta_{\mathrm{eq}}^{(M)} = \sum_{i=0}^{\infty} \delta_i^M u^i : M \to M[\![u]\!] \tag{2-30}$$

of total degree 1, satisfying $\delta_{\mathrm{eq}}^2 = 0$. Here we are implicitly conflating δ_{eq} with its u–linear extension to a map $M[\![u]\!] \to M[\![u]\!]$ in order to u–linearly compose and obtain a map $M \to M[\![u]\!]$.

Premorphisms from M to N of degree k can similarly be recast as maps $F_{\mathrm{eq}} = \sum_{i=0}^{\infty} F_i u^i \colon M \to N[\![u]\!]$ of pure degree k (so each F_i has degree $k - 2i$). The differential on premorphisms can be described u–linearly as

$$\partial(F_{\mathrm{eq}}) = F_{\mathrm{eq}} \circ \delta_{\mathrm{eq}}^M - (-1)^{\deg(F)} \delta_{\mathrm{eq}}^N \circ F_{\mathrm{eq}}, \tag{2-31}$$

and composition is simply the u–linear composition $G_{\mathrm{eq}} \circ F_{\mathrm{eq}}$ (again, one implicitly u–linearly extends G_{eq} and then u–linearly composes); explicitly,

$$\Big(\sum_{i\geq 0} G_i u^i\Big) \circ \Big(\sum_{j\geq 0} F_j u^j\Big) = \sum_{k\geq 0} \Big(\sum_{i+j=k} G^i \circ F^j\Big) u^k.$$

With respect to this packaging, the formulae for various equivariant homology chain complexes can be given the more readable forms

$$M_{\mathrm{h}S^1} = (M(\!(u)\!)/uM[\![u]\!], \delta_{\mathrm{eq}}), \tag{2-32}$$

$$M^{\mathrm{h}S^1} = (M[\![u]\!], \delta_{\mathrm{eq}}), \tag{2-33}$$

$$M^{\mathrm{Tate}} = (M(\!(u)\!), \delta_{\mathrm{eq}}), \tag{2-34}$$

where, again, we use the abuse of notation $M(\!(u)\!) = M[\![u]\!] \otimes_{\boldsymbol{k}[u]} \boldsymbol{k}[u, u^{-1}]$. (On the other hand, note that (2-32) is *not* completed.) As before, any homomorphism (that is, closed morphism) of S^1–complexes $F_{\mathrm{eq}} = \sum_{i=0}^{\infty} F^i u^i$ induces a $\boldsymbol{k}[u]$–linear chain map between homotopy fixed-point complexes by u–linearly extended composition, and hence, by tensoring over $\boldsymbol{k}[u]$ with $\boldsymbol{k}(\!(u)\!)/u\boldsymbol{k}[\![u]\!]$ or $\boldsymbol{k}(\!(u)\!)$, chain maps between homotopy orbit and Tate complex constructions. With respect to these explicit complexes, the projection to homotopy orbits (2-22) and inclusion of fixed points (2-24) chain maps have simple explicit descriptions

$$\mathrm{pr}\colon M \to M_{\mathrm{h}S^1}, \qquad \alpha \mapsto \alpha \cdot u^0, \tag{2-35}$$

$$\iota\colon M^{\mathrm{h}S^1} \to M, \qquad \sum_{i=0}^{\infty} \alpha_i u^i \mapsto \alpha_0. \tag{2-36}$$

Remark 2.20 This u–linear lossless packaging of the data describing an S^1–complex is a manifestation of *Koszul duality*; in the case of $A = \boldsymbol{k}[\Lambda]/\Lambda^2$, it posits that there is a fully faithful embedding, $\mathrm{Rhom}(\boldsymbol{k}, -) = (-)^{\mathrm{h}S^1}$ from A–modules into $B := \mathrm{Rhom}_A(\boldsymbol{k}, \boldsymbol{k}) = \boldsymbol{k}[u]$–modules.

From the u–linear point of view, it is easier to observe that the exact triangle of $\boldsymbol{k}[u]$–modules $\boldsymbol{k}[\![u]\!] \to \boldsymbol{k}(\!(u)\!) \to \boldsymbol{k}(\!(u)\!)/\boldsymbol{k}[\![u]\!] = u^{-1}(\boldsymbol{k}(\!(u)\!)/u\boldsymbol{k}[\![u]\!])$ induces an exact triangle (functorial in M) between equivariant homology chain complexes

$$M^{\mathrm{h}S^1} \to M^{\mathrm{Tate}} \to M_{\mathrm{h}S^1}[2] \xrightarrow{[1]} M^{\mathrm{h}S^1}.$$

3 Circle action on the open sector

3.1 The usual and nonunital Hochschild chain complex

Recall that an A_∞ category over $\boldsymbol{k}$, $\mathcal{C}$ is specified by the following data:

- A collection of objects $\operatorname{ob}\mathcal{C}$.
- For each pair of objects X, X', a graded vector space $\hom_{\mathcal{C}}(X, X')$ over $\boldsymbol{k}$.
- For any set of $d+1$ objects $X_0, \dots, X_d$, higher multilinear (over $\boldsymbol{k}$) composition maps

(3-1) $$\mu^d : \hom_{\mathcal{C}}(X_{d-1}, X_d) \times \cdots \times \hom_{\mathcal{C}}(X_0, X_1) \to \hom_{\mathcal{C}}(X_0, X_d)$$

  (sometimes equivalently viewed as a map from the tensor product) of degree $2-d$, satisfying for each $k > 0$ the (quadratic) A_∞ relations

(3-2) $$\sum_{i,l} (-1)^{\maltese_i} \mu_{\mathcal{C}}^{k-l+1}(x_k, \dots, x_{i+l+1}, \mu_{\mathcal{C}}^l(x_{i+l}, \dots, x_{i+1}), x_i, \dots, x_1) = 0,$$

  with sign

(3-3) $$\maltese_i := \|x_1\| + \cdots + \|x_i\|,$$

  where $|x|$ denotes degree and $\|x\| := |x| - 1$ denotes reduced degree.

The first two equations of (3-2) imply that μ^1 is a differential, and the cohomology level maps $[\mu^2]$ are a genuine composition for the (nonunital) category $H^*(\mathcal{C})$ with the same objects and morphisms,

(3-4) $$\operatorname{Hom}_{H^*(\mathcal{C})}(X, Y) := H^*(\hom_{\mathcal{C}}(X, Y), \mu^1).$$

We say that $\mathcal{C}$ is *cohomologically unital* if there exist cohomology-level identity morphisms $[e_X] \in \operatorname{Hom}_{H^*(\mathcal{C})}(X, X)$ for each object X, making $H^*(\mathcal{C})$ into a genuine category. We say that $\mathcal{C}$ is *strictly unital* if there exist elements $e_X^+ \in \hom_{\mathcal{C}}(X, X)$, for every object X, satisfying

(3-5) $$\begin{cases} \mu^1(e_X^+) = 0, & \\ (-1)^{|y|}\mu^2(e_{X_1}^+, y) = y = \mu^2(y, e_{X_0}^+) & \text{for any } y \in \hom_{\mathcal{C}}(X_0, X_1), \\ \mu^d(\dots, e_X^+, \dots) = 0 & \text{for } d > 2. \end{cases}$$

We call such elements the chain-level, or strict, identity elements.

The *Hochschild chain complex*, or *cyclic bar complex*, of an A_∞ category $\mathcal{C}$ is the direct sum of all cyclically composable sequences of morphism spaces in $\mathcal{C}$,

$$\text{(3-6)}\quad \mathrm{CH}_*(\mathcal{C}) := \bigoplus_{\substack{k\geq 0\\ X_{i_0},\dots,X_{i_k}\in \mathrm{ob}\,\mathcal{C}}} \hom_{\mathcal{C}}(X_{i_k},X_{i_0})\otimes \hom_{\mathcal{C}}(X_{i_{k-1}},X_{i_k})\otimes\cdots\otimes \hom_{\mathcal{C}}(X_{i_0},X_{i_1}).$$

The (cyclic bar) differential b acts on Hochschild chains by summing over ways to cyclically collapse elements by any of the A_∞ structure maps:

$$\text{(3-7)}\quad \begin{aligned} &b(\boldsymbol{x}_d\otimes x_{d-1}\otimes\cdots\otimes x_1)\\ &\quad=\sum(-1)^{\#_k^d}\mu^{d-i}(x_k,\dots,x_1,\boldsymbol{x}_d,x_{d-1},\dots,x_{k+i+1})\otimes x_{k+i}\otimes\cdots\otimes x_{k+1}\\ &\qquad\qquad+\sum(-1)^{\maltese_1^s}\boldsymbol{x}_d\otimes\cdots\otimes\mu^j(x_{s+j+1},\dots,x_{s+1})\otimes x_s\otimes\cdots\otimes x_1, \end{aligned}$$

with signs

$$\text{(3-8)}\qquad \maltese_i^k := \sum_{j=i}^{k} \|x_i\|,$$

$$\text{(3-9)}\qquad \#_k^d := \maltese_1^k\cdot(1+\maltese_{k+1}^d)+\maltese_{k+1}^{d-1}+1.$$

In this complex, Hochschild chains are (cohomologically) graded as

$$\text{(3-10)}\quad \deg(\boldsymbol{x}_d\otimes x_{d-1}\otimes\cdots\otimes x_1):=\deg(\boldsymbol{x}_d)+\sum_{i=1}^{d-1}\deg(x_i)-d+1=|\boldsymbol{x}_d|+\sum_{i=1}^{d-1}\|x_i\|.$$

Remark 3.1 Frequently the notation $\mathrm{CH}_*(\mathcal{C},\mathcal{C})$ is used for (3-6) to emphasize that Hochschild homology is taken here with *diagonal coefficients*, rather than coefficients in another bimodule.

If $\mathcal{C}$ is a strictly unital A_∞ category, then the chain complex (3-6) carries a strict S^1–action $B\colon \mathrm{CH}_*(\mathcal{C})\to\mathrm{CH}_{*-1}(\mathcal{C})$, involving summing over ways to cyclically permute chains and insert identity morphisms; see Remark 3.7 below. However, there is a quasi-isomorphic nonunital Hochschild complex of $\mathcal{C}$ which always carries a strict S^1–action (even if $\mathcal{C}$ is not strictly unital), which we will now describe.

As a graded vector space, the *nonunital Hochschild complex* consists of two copies of the cyclic bar complex, the second copy shifted down in grading by 1:

$$\text{(3-11)}\qquad \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C}) := \mathrm{CH}_*(\mathcal{C})\oplus\mathrm{CH}_*(\mathcal{C})[1].$$

With respect to the decomposition (3-11), we sometimes refer to elements as $\sigma := (\check{\alpha}, \widehat{\beta})$, with the notation $\check{\alpha}$ or $\widehat{\beta}$ indicating that a given element α and β belong to the left or right factor respectively. Similarly, we refer to the left and right factors as the *check factor* and the *hat factor*, respectively.

Let b' denote a version of the differential (3-7) omitting the "wrap-around terms" (often simply called the *bar differential*):

$$\begin{aligned}(3\text{-}12)\quad & b'(x_d \otimes x_{d-1} \otimes \cdots \otimes x_1) \\ &= \sum (-1)^{\maltese_1^s} x_d \otimes \cdots \otimes x_{s+j+1} \otimes \mu^j(x_{s+j}, \dots, x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1 \\ &\qquad + \sum (-1)^{\maltese_1^{d-j}} \mu^j(x_d, x_{d-1}, \dots, x_{d-j+1}) \otimes x_{d-j} \otimes \cdots \otimes x_1.\end{aligned}$$

For an element $\widehat{\beta} = x_d \otimes \cdots \otimes x_1$ in the hat (right) factor of (3-11), define an element $d_{\wedge\vee}(\widehat{\beta})$ in the check (left) factor of (3-11) by

$$(3\text{-}13)\quad d_{\wedge\vee}(\widehat{\beta}) := (-1)^{\maltese_2^d + \|x_1\| \cdot \maltese_2^d + 1} x_1 \otimes x_d \otimes \cdots \otimes x_2 + (-1)^{\maltese_1^{d-1}} x_d \otimes \cdots \otimes x_1.$$

In this language, the differential on the nonunital Hochschild complex can be written

$$(3\text{-}14)\qquad b^{\mathrm{nu}} \colon (\check{\alpha}, \widehat{\beta}) \mapsto (b(\check{\alpha}) + d_{\wedge\vee}(\widehat{\beta}), b'(\widehat{\beta})),$$

or equivalently can be expressed via the matrix

$$(3\text{-}15)\qquad b^{\mathrm{nu}} = \begin{pmatrix} b & d_{\wedge\vee} \\ 0 & b' \end{pmatrix}.$$

The left factor $\mathrm{CH}_*(\mathcal{C}) \hookrightarrow \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C})$ is by definition a subcomplex. Moreover, since the quotient complex is the standard A_∞ bar complex with differential b', which is acyclic for cohomologically unital $\mathcal{C}$ (by a standard length-filtration spectral sequence argument, compare [52, Lemma 2.12] or [24, Proposition 2.2]), it follows that:

Lemma 3.2 *The inclusion map $\iota \colon \mathrm{CH}_*(\mathcal{C}) \hookrightarrow \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C})$ is a quasi-isomorphism* (*when $\mathcal{C}$ is cohomologically unital*). □

Remark 3.3 The nonunital Hochschild complex of $\mathcal{C}$ can be conceptually explained in terms of cyclic bar complexes as follows; cf [41, Section 1.4.1; 61, Section 3.5]. First, *augment* the category $\mathcal{C}$ by adjoining strict units; meaning, consider the A_∞ category $\mathcal{C}^+$ with $\mathrm{ob}\, \mathcal{C}^+ = \mathrm{ob}\, \mathcal{C}$ and

$$(3\text{-}16)\qquad \hom_{\mathcal{C}^+}(X, Y) = \begin{cases} \hom_{\mathcal{C}}(X, Y) & \text{when } X \neq Y, \\ \hom_{\mathcal{C}}(X, X) \oplus \boldsymbol{k}\langle e_X^+ \rangle & \text{when } X = Y, \end{cases}$$

whose A_∞ structure maps are completely determined by the fact that $\mathcal{C}$ is an A_∞ subcategory, and the elements e_X^+ act as strict units in the sense of (3-5). Next, consider the *normalized* (*or reduced*) *Hochschild complex* of the strictly unital category $\mathcal{C}^+$, $\mathrm{CH}_*^{\mathrm{red}}(\mathcal{C}^+)$, by definition the quotient of $\mathrm{CH}_*(\mathcal{C}^+)$ by the acyclic subcomplex consisting of e^+ terms in any position but the first. Now, take the further quotient of $\mathrm{CH}_*^{\mathrm{red}}(\mathcal{C}^+)$ by the subcomplex of length one Hochschild chains of the form e_X^+ for some X. The resulting complex, denoted by $\widetilde{\mathrm{CH}}_*(\mathcal{C}^+)$, can be identified as a chain complex with $\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C})$ via the map

$$\begin{aligned} f\colon \widetilde{\mathrm{CH}}_*(\mathcal{C}^+) &\xrightarrow{\cong} \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C}), \\ y_k \otimes \cdots \otimes y_1 &\longmapsto \begin{cases} (0, y_{k-1} \otimes \cdots \otimes y_1) & \text{if } y_k = e_X^+ \text{ for some } X, \\ (y_k \otimes \cdots \otimes y_1, 0) & \text{otherwise.} \end{cases} \end{aligned} \tag{3-17}$$

In particular, the differential in $\mathrm{CH}^{\mathrm{nu}}(\mathcal{C})$ on a Hochschild chain $\widehat{\beta}$ in the right factor (of the decomposition (3-11)) agrees with the (usual cyclic bar) Hochschild differential applied to $e_X^+ \otimes \beta$ under the correspondence f.

3.2 Circle action on the Hochschild complex

The S^1–action on the nonunital (or usual) Hochschild complex is built out of several intermediate operations. First, let $t\colon \mathrm{CH}_*(\mathcal{C}) \to \mathrm{CH}_*(\mathcal{C})$ denote the (signed) cyclic permutation operator on the cyclic bar complex generating the $\mathbb{Z}/k\mathbb{Z}$ cyclic action on the length-k expressions

$$t\colon x_k \otimes \cdots \otimes x_1 \mapsto (-1)^{\|x_1\|\cdot \maltese_2^k + \|x_1\| + \|x_k\|} x_1 \otimes x_k \otimes \cdots \otimes x_2. \tag{3-18}$$

(This is not a chain map.)

Let N denote the *norm* of this operation; that is, the sum of all powers of t (this depends on k, the length of a given Hochschild chain),

$$N\colon x_k \otimes \cdots \otimes x_1 = \sigma \mapsto (1 + t + t^2 + \cdots + t^{k-1})\sigma. \tag{3-19}$$

Let $s^{\mathrm{nu}}\colon \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{CH}_{*-1}^{\mathrm{nu}}(\mathcal{C})$ be the linear map which sends check chains to the corresponding hat chains, and hat chains to zero:

$$s^{\mathrm{nu}}(x_d \otimes \cdots \otimes x_1, y_t \otimes \cdots \otimes y_1) := (-1)^{\maltese_1^d + \|x_d\| + 1}(0, x_d \otimes \cdots \otimes x_1). \tag{3-20}$$

(Again, this is not a chain map.)

Finally define $B^{\mathrm{nu}}\colon \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}) \to \mathrm{CH}^{\mathrm{nu}}_{*-1}(\mathcal{C})$ by

$$\begin{aligned}(3\text{-}21)\quad & B^{\mathrm{nu}}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1) \\ &:= \sum_{i=1}^{k} (-1)^{\maltese_1^i \maltese_{i+1}^k + \|x_k\| + \maltese_1^k + 1}(0, x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1})) \\ &= s^{\mathrm{nu}}(N(x_k \otimes \cdots \otimes x_1), y_l \otimes \cdots \otimes y_1) \\ &= \sum_{i=0}^{k-1} s^{\mathrm{nu}}(t^i(x_k \otimes \cdots \otimes x_1), y_l \otimes \cdots \otimes y_1).\end{aligned}$$

Lemma 3.4 *We have* $(B^{\mathrm{nu}})^2 = 0$ *and* $b^{\mathrm{nu}}B^{\mathrm{nu}} + B^{\mathrm{nu}}b^{\mathrm{nu}} = 0$. *That is,* $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C})$ *is a strict* S^1*–complex, with the action of* $\Lambda = [S^1]$ *given by* B^{nu}. □

Let $b_{\mathrm{eq}} = b^{\mathrm{nu}} + uB^{\mathrm{nu}}$ be the strict S^1–complex structure on the nonunital Hochschild complex $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C})$, u–linearly packaged as in Section 2.3. Using this, we can define cyclic homology groups, as follows.

Definition 3.5 The (*positive*) *cyclic* chain complex, the *negative cyclic* chain complex, and the *periodic cyclic* chain complexes of $\mathcal{C}$ are the *homotopy orbit complex*, *homotopy fixed-point complex*, and *Tate constructions* of the S^1–complex $(\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}), b_{\mathrm{eq}})$, respectively. That is,

$$\mathrm{CC}^+_*(\mathcal{C}) := (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}))_{\mathrm{h}S^1} = (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}) \otimes_{\boldsymbol{k}} \boldsymbol{k}(\!(u)\!)/u\boldsymbol{k}[\![u]\!], b_{\mathrm{eq}}), \tag{3-22}$$

$$\mathrm{CC}^-_*(\mathcal{C}) := (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}))^{\mathrm{h}S^1} = (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}) \mathbin{\widehat{\otimes}}_{\boldsymbol{k}} \boldsymbol{k}[\![u]\!], b_{\mathrm{eq}}), \tag{3-23}$$

$$\mathrm{CC}^\infty_*(\mathcal{C}) := (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}))^{\mathrm{Tate}} = (\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}) \mathbin{\widehat{\otimes}}_{\boldsymbol{k}} \boldsymbol{k}(\!(u)\!), b_{\mathrm{eq}}), \tag{3-24}$$

with grading induced by setting $|u| = +2$, and where, as in Section 2.3, $\widehat{\otimes}$ refers to the u–adically completed tensor product in the category of graded vector spaces. The cohomologies of these complexes, denoted by $\mathrm{HC}^{+/-/\infty}_*(\mathcal{C})$, are called the (*positive*), *negative* and *periodic cyclic homologies* of $\mathcal{C}$, respectively.

The $C_{-*}(S^1)$–module structure on $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C})$ is suitably functorial, in the following sense. Let $\boldsymbol{F}\colon \mathcal{C} \to \mathcal{C}'$ be an A_∞ functor. There is an induced chain map on nonunital Hochschild complexes

$$\boldsymbol{F}^{\mathrm{nu}}_\sharp\colon \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}) \to \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C}', \mathcal{C}'), \quad (x, y) \mapsto (\boldsymbol{F}_\sharp(x), \boldsymbol{F}'_\sharp(y)), \tag{3-25}$$

where

$$F'_\sharp(x_k \otimes \cdots \otimes x_0) := \sum_{i_1,\dots,i_s} F^{i_1}(x_k \cdots) \otimes \cdots \otimes F^{i_s}(\cdots x_0), \tag{3-26}$$

$$\begin{aligned} F_\sharp(x_k \otimes \cdots \otimes x_0) := \sum_{i_1,\dots,i_s,j} & F^{j+1+i_1}(x_j, \dots, x_0, x_k, \dots, x_{k-i_1+1}) \\ & \otimes F^{i_2}(\cdots) \otimes \cdots \otimes F^{i_s}(x_{j+i_s}, \dots, x_{j+1}), \end{aligned} \tag{3-27}$$

which is an isomorphism on homology if F is a quasi-isomorphism (indeed, even a Morita equivalence). This functoriality preserves S^1 structures:

Proposition 3.6 *$F_\sharp^{\mathrm{nu}}$ gives a strict morphism of strict S^1–complexes, meaning*

$$F_\sharp^{\mathrm{nu}} \circ b^{\mathrm{nu}} = b^{\mathrm{nu}} \circ F_\sharp^{\mathrm{nu}} \quad \textit{and} \quad F_\sharp^{\mathrm{nu}} \circ B^{\mathrm{nu}} = B^{\mathrm{nu}} \circ F_\sharp^{\mathrm{nu}}.$$

In other words, the premorphism of A_∞ $\boldsymbol{k}[\Lambda]/\Lambda^2$–modules defined as

$$F_*^d(\underbrace{\Lambda, \dots, \Lambda}_{d}, \sigma) := \begin{cases} F_\sharp^{\mathrm{nu}}(\sigma) & \text{if } d = 0, \\ 0 & \text{if } d \geq 1, \end{cases} \tag{3-28}$$

is closed, ie an A_∞–module homomorphism.

Sketch of proof It is well known that $F_\sharp^{\mathrm{nu}}$ is a chain map, so it suffices to verify that $F_\sharp^{\mathrm{nu}} \circ B^{\mathrm{nu}} = B^{\mathrm{nu}} \circ F_\sharp^{\mathrm{nu}}$, or in terms of (3-25),

$$F'_\sharp \circ s^{\mathrm{nu}} N = s^{\mathrm{nu}} N \circ F_\sharp. \tag{3-29}$$

We leave this an exercise, noting that applying either side to a Hochschild chain $x_k \otimes \cdots \otimes x_1$, the sums match identically. □

Remark 3.7 If $\mathcal{C}$ is strictly unital, one can also define an operator $B\colon \mathrm{CH}_*(\mathcal{C}) \to \mathrm{CH}_{*-1}(\mathcal{C})$ on the usual cyclic bar complex by

$$B = (1-t)sN,$$

where, up to a sign, s denotes the operation of inserting, at the beginning of a chain, the unique strict unit e_X^+ preserving cyclic composability:

$$s\colon x_k \otimes \cdots \otimes x_1 \mapsto (-1)^{\|x_k\| + \maltese_1^k + 1} e^+_{X_{i_k}} \otimes x_k \otimes \cdots \otimes x_1, \tag{3-30}$$

where $x_k \in \hom_{\mathcal{C}}(X_{i_k}, X_{i_0})$. It can be shown that $B^2 = 0$ and $Bb + bB = 0$, $\mathrm{CH}_*(\mathcal{C})$ is a strict S^1–complex; moreover that the quasi-isomorphism $\mathrm{CH}_*(\mathcal{C}) \cong \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C})$ is one of S^1–complexes. In fact, B descends to the *reduced Hochschild complex* $\mathrm{CH}_*^{\mathrm{red}}(\mathcal{C})$ described in Remark 3.3, where it takes the simpler form

$$B^{\mathrm{red}} = sN,$$

as applying tsN results in a Hochschild chain with a strict unit not in the first position, which becomes zero in $\mathrm{CH}_*^{\mathrm{red}}(\mathcal{C})$. If $\mathcal{C}$ was not necessarily strictly unital, following Remark 3.3 one can still consider the quotient of the reduced Hochschild complex of the augmented category $\mathcal{C}^+$, which we called $\widetilde{\mathrm{CH}}_*(\mathcal{C}^+)$. The discussion here equips this complex with an S^1–action $\widetilde{B}^{\mathrm{red}}$. Under the bijection f of (3-17), $\widetilde{B}^{\mathrm{red}}$ is sent to B^{nu} and s is sent to s^{nu}.

Remark 3.8 Continuing Remark 3.3, suppose we have constructed $\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{C})$ as $\widetilde{\mathrm{CH}}_*(\mathcal{C}) := \mathrm{CH}_*^{\mathrm{red}}(\mathcal{C}^+)/\bigoplus_X \boldsymbol{k}\langle e_X^+\rangle$, the quotient of the reduced Hochschild complex of the augmented category $\mathcal{C}^+$. Given any $\boldsymbol{F}$ as above, extend $\boldsymbol{F}$ to an augmented functor $\boldsymbol{F}^+$ by mandating that

$$(\boldsymbol{F}^+)^1(e_X^+) = e_{\boldsymbol{F}X}^+ \quad \text{and} \quad (\boldsymbol{F}^+)^d(\dots, e_X^+, \dots) = 0. \tag{3-31}$$

It is easy to see that the map $(\boldsymbol{F}^+)_*\colon \mathrm{CH}_*(\mathcal{C}^+) \to \mathrm{CH}_*((\mathcal{C}')^+)$ descends to a map $\widetilde{\boldsymbol{F}}\colon \widetilde{\mathrm{CH}}_*(\mathcal{C}^+) \to \widetilde{\mathrm{CH}}_*((\mathcal{C}')^+)$. Under the bijection (3-17), this precisely corresponds to $\boldsymbol{F}_\sharp^{\mathrm{nu}}$ described above. In particular, the fact that strictly unital functors induce strict S^1–morphisms between (usual) Hochschild complexes immediately implies Proposition 3.6.

Remark 3.9 There are options besides the nonunital Hochschild complex for seeing the $C_{-*}(S^1)$–action on a Hochschild complex of the Fukaya category. For instance, one could:

(1) Perform a strictly unital replacement (via homological algebra as in [52, Section 2] or [40, Theorem 3.2.1.1]), and work with the Hochschild complex of the replacement. However, this doesn't retain a relationship between the A_∞ operations and geometric structure, and hence is difficult to use with open–closed maps.

(2) Geometrically construct a strictly unital structure on the Fukaya category via constructing *homotopy units* [22], which roughly involves building a series of geometric higher homotopies between the operation of A_∞ multiplying by a specified geometrically defined cohomological unit, and the operation of A_∞ multiplying by a strict unit (which is algebraically defined, but may also be geometrically characterized in terms of forgetful maps). From this one defines a strictly unital A_∞ category $\mathcal{F}^{\mathrm{hu}}$ with $\hom_{\mathcal{F}^{\mathrm{hu}}}(X,X) = \hom_{\mathcal{F}}(X,X) \oplus \boldsymbol{k}\langle e_X^+, f_X\rangle$ and $\hom_{\mathcal{F}^{\mathrm{hu}}}(X,Y) = \hom_F(X,Y)$ for $X \neq Y$, extending the A_∞ structure on $\mathcal{F}$, such that each e_X^+ is a strict unit and $\mu^1(f_X) = e_X^+ - e_X$ for e_X a chosen a cohomological unit. The geometric higher

homotopies alluded to above give operations used to define for instance, μ^k of a sequence of elements containing one or more f_X terms.

Remark 3.7 then equips the usual Hochschild complex $\mathrm{CH}_*(\mathcal{F}^{\mathrm{hu}}, \mathcal{F}^{\mathrm{hu}})$ with a strict S^1–action. Using this one can construct a cyclic open–closed map with source $\mathrm{CH}_*(\mathcal{F}^{\mathrm{hu}}, \mathcal{F}^{\mathrm{hu}})$, in a manner completely analogous to the construction of $\mathcal{F}^{\mathrm{hu}}$ and the cyclic open–closed map here. This option is equivalent to the one we have chosen (and has some benefits), but requires additional technicalities/moduli spaces beyond the route taken here — both in constructing and defining the category $\mathcal{F}^{\mathrm{hu}}$, and then in defining further "higher homotopies" between inserting a cohomological unit asymptotic and imposing a strict unit (ie forgettable) constraint into the cyclic open–closed map in various places, which give operations that correspond to applying the cyclic open–closed map to a Hochschild chain with one or more f_X terms.

A construction of homotopy units was introduced in the work of Fukaya, Oh, Ohta and Ono [22, Chapter 7, Section 3.1]. See [24] for an implementation in the (possibly wrapped) exact (or otherwise tautologically unobstructed), multiple Lagrangians setting.

3.3 The Fukaya category

The goal of this subsection is to review (under simplifying technical hypotheses) the definition of the Fukaya category of a symplectic manifold. The outcome, a (homologically unital but not necessarily strictly unital) A_∞ category, will in particular carry a circle action on its nonunital Hochschild complex.

In Section 3.3.1 below, we detail a set of simplifying assumptions imposed on all of the moduli spaces of Floer curves considered in this paper (mostly pertaining to transversality and compactness), and recall examples of the variety of geometric situations in which they are satisfied. Such assumptions are in particular satisfied in the technically simplest cases in which (compact or wrapped) Fukaya categories can be defined, namely exact (Liouville) and monotone or aspherical symplectic manifolds. In Section 3.3.2 we will quickly review the construction of the Fukaya category under such hypotheses. The initial thread of discussion will focus on compact Lagrangians, but immediately extends to wrapped Fukaya categories of Liouville manifolds as described in a series of remarks; here we are using the framework of quadratic Hamiltonians as defined in [1] for wrapped Fukaya categories, whose construction is nearly as simple as that of compact Fukaya categories and requires only a few minor modifications.

3.3.1 Geometric setup and assumptions about moduli spaces of Floer trajectories

To simplify technicalities, the main assumption we make about moduli spaces in this paper is as follows.

Assumption 3.10 (main assumption about moduli spaces) All (semistably compactified) moduli spaces of Floer trajectories considered in this paper of virtual dimension ≤ 1 are — for generic choices of complex structure and Hamiltonian ("perturbation data") — compact transversally cut-out manifolds with boundary of dimension equal to virtual dimension. Moreover, the union of any such moduli space with fixed "input" asymptotic conditions over all possible "output" asymptotic conditions remains compact, and in particular is empty for all but finitely many possible output conditions (vacuous when there are only finitely many possible outputs).

Let $M = (M^{2n}, \omega)$ denote our target symplectic manifold and fix a collection of (always properly embedded) Lagrangian submanifolds $\{L_i\}$ in M, which we wish to be the objects of our Fukaya category. We will call any M, $\{L_i\}$, and choices of Floer perturbation data used to define moduli spaces for which Assumption 3.10 holds *admissible*. We will say M and/or $M, \{L_i\}$ are admissible if they possess an ample supply of Floer data for which Assumption 3.10 holds for the moduli spaces considered below involving these targets. Examples of admissible M include:

- Any Liouville manifold (in particular noncompact), which is to say that ω is exact with a fixed choice of primitive λ, such that flowing out by the Liouville vector field Z (defined by $\iota_Z\omega = \lambda$) induces a diffeomorphism

$$M \backslash \mathring{\overline{M}} \cong \partial\overline{M} \times [0,\infty) \tag{3-32}$$

  for some codimension-zero manifold-with-boundary $\overline{M}$, called a Liouville domain whose completion is M.
- Any compact symplectic manifold which is either monotone, ie $[\omega] = 2\tau c_1(M)$ for some constant $\tau > 0$, or symplectically aspherical, ie $\omega(\pi_2(M)) = 0$.

If M is Liouville, we henceforth fix a cylindrical end (3-32), and use r to refer to the corresponding $[0,\infty)$ coordinate. Examples of (properly embedded) admissible Lagrangian submanifolds $L \subset M$ in admissible M include:

- In Liouville M, one can take any exact L, ie with $\lambda_L = df$, equipped with fixed choice of primitive which vanishes outside a compact set, which implies, as in (3-32), that L is modeled on the cone of a Legendrian near infinity.

- In (compact) monotone M, one can take monotone L, in the sense that $\omega(-) = \rho\mu_L(-)\colon H_2(M,L) \to \mathbb{R}$ for some constant $\rho > 0$, where ω is symplectic area and μ_L is the Maslov class.
- In (compact) symplectically aspherical M one can take L to be tautologically unobstructed, ie L bounds no J–holomorphic discs for some J, which holds for all J if $\omega(\pi_2(M,L)) = 0$.

The conditions above on M and L serve to rule out "bad" (unstable) breakings (such as J–holomorphic sphere bubbles in M or J–holomorphic disc bubbles in M with boundary on L) from arising in the limit of a sequence of curves in the moduli spaces considered, which could obstruct compactness and/or simultaneously complicate transversality arguments.

Remark 3.11 (more general examples of admissible M and L) More generally, one could impose that the possible noncompactness of M and (if M is noncompact) L must be of the geometrically tame variety and that M/L have no/bound no J–holomorphic spheres/discs, or if they do, that such spheres/discs can either be shown (using classical methods) either not to arise in the compactifications of virtual one-dimensional moduli spaces, or to arise but only contribute canceling contributions to the resulting algebraic formulae.

For noncompact M and $\{L_i\}$, on any given moduli space of trajectories considered, further (nongeneric) assumptions on the profile of Floer perturbation data near ∞ are required to ensure Assumption 3.10 holds, to preclude sequences of curves escaping to ∞ so that usual Gromov compactness techniques apply, and also to obtain the second finiteness statement of Assumption 3.10, which is trivial in the compact case due to there being a finite list of outputs. We will say a few words about this in Remarks 3.15–3.18; the verification of Assumption 3.10 for the A_∞ structure maps (by citing established works) appears in Lemma 3.19. The verification of Assumption 3.10 for other moduli spaces considered in the paper is identical and hence omitted. However, we will in various places point out that the restrictions are needed on Floer data in noncompact cases to preclude curves escaping to infinity and obtain finiteness along the lines of Lemma 3.19.

3.3.2 Admissible Fukaya categories For an admissible M, we review the definition of the Fukaya category associated to an admissible collection of Lagrangians in M, which we will term an *admissible Fukaya category*. Examples of admissible Fukaya categories, in light of the examples given above, include:

(1) In a compact aspherical M, the Fukaya category of tautologically unobstructed Lagrangians.
(2) In a monotone M, the Fukaya category of monotone Lagrangians.
(3) In Liouville M, the Fukaya category of compact exact Lagrangians.
(4) In Liouville M, the wrapped Fukaya category of exact (cylindrical at infinity) Lagrangians.

Fix first an underlying ground field $\boldsymbol{k}$ and grading structure ($\mathbb{Z}$ or $\mathbb{Z}/2$ here, but see Remark 3.12) that we wish to use when defining the category. If $2c_1(M) = 0$ and we wish to define a $\mathbb{Z}$–graded category, we begin by equipping M with a *grading structure*, which is a trivialization of the square of the canonical bundle $(\Lambda^n_{\mathbb{C}} T^* M)^{\otimes 2}$. Next, one equips the Lagrangian submanifolds under consideration with some extra structure depending on the ground field $\boldsymbol{k}$ and the grading structure. Concretely, we say an *admissible Lagrangian brane* consists of a properly embedded admissible Lagrangian submanifold $L \subset M$ which is equipped with the following extra two optional pieces of data (which are only required if one wants to work with $\operatorname{char} \boldsymbol{k} \neq 2$ or with $\mathbb{Z}$–gradings, respectively, the latter in particular is always excluded in the monotone case):

(3-33) an orientation and Spin structure, and

(3-34) a grading in the sense of [48] (with respect to the fixed grading structure on M).

These choices of extra data respectively require L to be Spin and satisfy $2c_1(M, L) = 0$, where $c_1(M, L) \in H^2(M, L)$ is the relative first Chern class.

Remark 3.12 There are other possible grading structures on M and L that one can use to equip the Fukaya category with suitable gradings (under geometric hypotheses), for instance $\mathbb{Z}/2k$–gradings, homology class gradings or hybrids thereof; cf [48; 59]. We suppress discussion of these, but, seeing as such matters are largely orthogonal to our arguments, note that our results apply in such contexts as well.

Henceforth, by abuse of notation all Lagrangians are implicitly admissible Lagrangian branes. Denote by $\operatorname{ob} \mathcal{F}$ a finite collection of such (admissible) Lagrangians. Choose a (potentially time-dependent) Hamiltonian $H = H_t \colon M \to \mathbb{R}$ satisfying the following genericity condition:

Assumption 3.13 All time-1 chords of X_{H_t} between any pair of Lagrangians in $\operatorname{ob} \mathcal{F}$ are nondegenerate.

Remark 3.14 It is straightforward to adapt all of our constructions to larger collections of Lagrangians by, for instance, choosing a different Hamiltonian H_{L_0,L_1} for each pair of Lagrangians L_0, L_1 (as well as a different H for closed orbits), and by choosing Floer perturbation data depending on corresponding sequences of objects; see eg [52, Section 9j]. We have opted to use a single H_t simply to keep the notation simpler.

Remark 3.15 (admissible Hamiltonians in the Liouville case) When M is Liouville, we need to impose further restrictions on the profile of H near ∞ in order to satisfy Assumption 3.10. If $\operatorname{ob}\mathcal{F}$ consists solely of compact exact Lagrangians, it suffices to impose that H is compactly supported, or more generally of the form $f(r)$ near infinity for some function with nonnegative first and second derivatives. If $\operatorname{ob}\mathcal{F}$ contains any noncompact Lagrangians, we will impose, following [1], that H satisfies the following *quadratic at* ∞ condition: $H = r^2$ on the cylindrical end (3-32), outside a compact subset.

For any pair of Lagrangians $L_0, L_1 \in \operatorname{ob}\mathcal{F}$ the set $\chi(L_0, L_1)$ of time-1 Hamiltonian flow lines of H from L_0 to L_1 can be thought of as the critical points of an action functional $\mathcal{P}_{L_0,L_1}$ on the *path space* from L_0 to L_1; this functional is, a priori, multivalued, but it is certainly $\mathbb{R}$–valued in the presence of primitives λ for ω and f_i for $\lambda|_{L_i}$. Given a choice of grading structure on M and grading for each L_i above, elements of $\chi(L_0, L_1)$ can be graded by the *Maslov index*

$$\deg\colon \chi(L_0, L_1) \to \mathbb{Z}. \tag{3-35}$$

In the absence of grading structures this is always well defined mod 2, provided our Lagrangians are oriented, which is automatic if they are Spin. As a graded $\boldsymbol{k}$–module, the morphism space in the Fukaya category between L_0 and L_1, also known as the (*wrapped* if M is Liouville) *Floer homology cochain complex of* L_0 *and* L_1 *with respect to* H, has one (free) generator for each element of $\chi(L_0, L_1)$; concretely,

$$\hom^i_{\mathcal{F}}(L_0, L_1) = CF^*(L_0, L_1, H_t, J_t) := \bigoplus_{\substack{x\in\chi(L_0,L_1)\\ \deg(x)=i}} |o_x|_{\boldsymbol{k}}, \tag{3-36}$$

where the *orientation line* o_x is the real vector space associated to x by index theory[11] and for any one-dimensional real vector space V and any ring $\boldsymbol{k}$, the $\boldsymbol{k}$*–normalization*

$$|V|_{\boldsymbol{k}} \tag{3-37}$$

[11] See [52, Section 11h]; a priori, o_x depends on a choice of trivialization of x^*TM compatible with the grading structure. However, there is a unique such choice in the presence of a $\mathbb{Z}$–grading, and in the $\mathbb{Z}/2$–graded case any two choices made induce canonically isomorphic orientation lines.

is the $\boldsymbol{k}$–module generated by the two possible orientations on V, with the relationship that their sum vanishes. If one does not want to worry about signs, note that $|V|_{\mathbb{Z}/2} \cong \mathbb{Z}/2$ canonically.

The A_∞ structure maps arise as counts of parametrized families of (suitably coherently perturbed) solutions to Floer's equation with source a disc with d inputs and one output. We will quickly summarize the definition and relevant choices required, referring the reader to standard references for more details. The basic reference we follow is [52] for Fukaya categories of compact exact Lagrangians in Liouville manifolds; see also [60] for the mostly straightforward generalization to the monotone case. In the main body of exposition, we focus on the (slightly simpler) case of compact (admissible) Lagrangians in compact (admissible) symplectic manifolds; we detail the additional data and variations required for Fukaya categories of exact Lagrangians in Liouville manifolds (which are simpler if one is working only with compact exact Lagrangians) in Remarks 3.15–3.18.

For $d \geq 2$, we use the notation $\overline{\mathcal{R}}^d$ for the (Deligne–Mumford compactified) moduli space of discs with $d+1$ marked points modulo reparametrization, with one point z_0^- marked as negative and the remainder $z_1^+, \dots, z_d^+$ (labeled counterclockwise from z_0^-) marked as positive. Orient the open (interior) locus $\mathcal{R}^d$ as in [52, Section 12g] and [1]. $\overline{\mathcal{R}}^d$ can be given the structure of a manifold-with-corners, and its higher strata are trees of stable discs with a total of d exterior positive marked points and 1 exterior negative marked point. Denote the positive and negative semi-infinite strips by

$$Z_+ := [0,\infty) \times [0,1], \tag{3-38}$$

$$Z_- := (-\infty,0] \times [0,1]. \tag{3-39}$$

One first equips the spaces $\overline{\mathcal{R}}^d$ for each d with a *consistent collection of strip-like ends* $\mathfrak{S}$; that is, for each component S of $\overline{\mathcal{R}}^d$, a collection of maps $\epsilon_k^\pm : Z_\pm \to S$ all with disjoint image in S, chosen so that positive/negative strips map to neighborhoods of positively/negatively labeled boundary marked points respectively, smoothly varying with respect to the manifold-with-corners structures and compatible with choices made on boundary and corner strata, which are products of lower-dimensional copies of spaces $\overline{\mathcal{R}}^k$.

In order to associate transversely cut out moduli spaces of such maps, one studies a parametric family of solutions to Floer's equation depending on a choice of "Floer (or perturbation) data" over the parameter space. Concretely, a *Floer datum* for a family of domains (in this case $\overline{\mathcal{R}}^d$) is a choice, for each domain S in the parametric family, of

- an S–dependent (domain-dependent) almost-complex structure J_S and Hamiltonian H_S,
- a one-form α on S,

which depend (smoothly) on the particular domain in $\overline{\mathcal{R}}^d$ (and the position in that domain), and are *compatible with strip-like ends*, meaning α pulls back to dt and (H_S, J_S) pull back to a fixed choice (H_t, J_t) in coordinates (3-38)–(3-39). One inductively chooses a *Floer datum for the A_∞ structure*, which is a choice of Floer data for the collection of domains $\{\overline{\mathcal{R}}^d\}_{d\geq 2}$ which is *consistent*, meaning that the Floer data chosen on a given family of domains $\overline{\mathcal{R}}^d$ agree smoothly along the boundary and corners (which are products of lower-dimensional spaces $\overline{\mathcal{R}}^k$) with previous choices of Floer data made. Such consistent choices exist essentially because spaces of Floer data are contractible.

Remark 3.16 (Floer data for compact exact Lagrangians in Liouville manifolds) If M is Liouville and we are studying the Fukaya category of compact exact Lagrangians, there is an additional requirement imposed on any Floer datum one uses; namely one requires that J_S be of *contact type* in a neighborhood of infinity in the sense of [52, (7.3)], and H_S be either 0 or of the form $f(r)$ near infinity for some function with nonnegative first and second derivatives. The more restrictive types of Floer data chosen for wrapped Fukaya categories in Remark 3.17 of course suffice as well.

Remark 3.17 (Floer data for wrapped Fukaya categories) Following [1], we recall the additional information and constraints appearing in Floer data for wrapped Floer theory (with quadratic Hamiltonians). If M is a Liouville manifold let $\psi^\rho\colon M \to M$ denote the time $\log(\rho)$ (outward) Liouville flow. One fixes for each S, in addition to (H_S, J_S, α_S), a collection of constants $w_k \in \mathbb{R}_{>0}$ for each end, called *weights* (so w_k is the weight associated to the k^{th} end), and a map $\rho_S\colon \partial S \to \mathbb{R}_{>0}$, called the *time-shifting map*, where:

(1) ρ_S should be constant and equal to the weight w_k on the k^{th} strip-like end.

(2) The one-form α_S should be *subclosed* (meaning $d\alpha_S \leq 0$), equal to $w_k\, dt$ in the local coordinates on each strip-like end, and 0 when restricted to ∂S. By Stokes' theorem, this condition implies the sum of weights over all negative ends is greater than or equal to the sum of weights over all positive ends, and there should therefore be at least one negative end always (in this case there is one).

(3) The Hamiltonian should be quadratic at infinity and pull back to $H \circ \psi^{w_k}/w_k^2$ in coordinates on each end. Note that such a Hamiltonian is quadratic if H is, by an elementary computation [1, Lemma 3.1].

(4) The almost-complex structure should be of contact type at infinity and equal to $(\psi^{w_k})^* J_t$ in coordinates on each end.

There is a *rescaling action* by $(\mathbb{R}_{>0}, \cdot)$ on the space of such surface dependent data, which sends

$$(\rho_S, \{w_k\}, \alpha_S, H_S, J_S) \mapsto \left(\lambda\rho_S, \{\lambda w_k\}, \lambda\alpha_S, \frac{H_S \circ \psi^\lambda}{\lambda^2}, (\psi^\lambda)^* J_S\right) \quad \text{for } \lambda \in \mathbb{R}_{>0}.$$

Using this action, one also relaxes the consistency requirement imposed: The Floer datum on $\overline{\mathcal{R}}^d$ must agree smoothly, on a boundary or corner stratum, with *some rescaling* of the previously made choice; compare [1, Definition 4.1].

Given our choices of Floer data, we can define the moduli spaces appearing in the A_∞ operations. First for any pair of objects L_0, L_1, and any pair of chords $x_0, x_1 \in \chi(L_0, L_1)$, define $\widetilde{\mathcal{R}}^1(x_0; x_1)$ to be the moduli space of maps $u\colon \mathbb{R}_s \times [0,1]_t \to M$ with boundary condition and asymptotics $u(s,0) \in L_0$, $u(s,1) \in L_1$, $\lim_{s\to+\infty} u(s,t) = x_1$ and $\lim_{s\to-\infty} u(s,t) = x_0$ satisfying Floer's equation for (H_t, J_t),

$$(du - X \otimes dt)^{0,1} = 0, \tag{3-40}$$

where X is the Hamiltonian vector field associated to H_t and $(0,1)$ is taken with respect to J_t. The translation action on $\mathbb{R}_s$ descends to a map on this moduli space (as the equation satisfied is s–independent), and we define the moduli space of (unparametrized) Floer strips to be

$$\mathcal{R}^1(x_0; x_1) := \widetilde{\mathcal{R}}^1(x_0; x_1)/\mathbb{R}, \tag{3-41}$$

with the added convention that whenever we are in a component of $\widetilde{\mathcal{R}}^1(x_0; x_1)$ with expected dimension 0, this quotient is replaced by the empty set. Now for $d \geq 2$ let $L_0, \dots, L_d$ be objects of $\mathcal{F}$ and fix any sequence of chords $\vec{x} = \{x_k \in \chi(L_{k-1}, L_k)\}$ as well as another chord $x_0 \in \chi(L_0, L_d)$. We write $\mathcal{R}^d(x_0; \vec{x})$ for the space of maps

$$u\colon S \to M$$

with source an arbitrary element $S \in \mathcal{R}^d$, satisfying boundary conditions and asymptotics

$$\begin{cases} u(z) \in L_k & \text{if } z \in \partial S \text{ lies between } z^k \text{ and } z^{k+1}, \\ \lim\limits_{s\to\pm\infty} u \circ \epsilon^k(s,\cdot) = x_k, \end{cases} \tag{3-42}$$

where the limit above is taken as $s \to +\infty$ if the k^{th} end is positive and $-\infty$ if it is negative, and differential equation

$$(du - X_S \otimes \alpha_S)^{0,1} = 0, \tag{3-43}$$

where X_S is the Hamiltonian vector field associated to H_S and where $0, 1$ is taken with respect to the complex structure J_S (for the choice of consistent Floer datum we have fixed).

The consistency condition imposed on Floer data over the abstract moduli spaces $\overline{\mathcal{R}}^d$, along with the compatibility with strip-like ends, implies that the (Gromov-type) compactification of the space of maps $\overline{\mathcal{R}}^d(x_0; \vec{x})$ can be formed by adding the images of the natural inclusions of products of lower-dimensional such moduli spaces,

$$\overline{\mathcal{R}}^{d_2}(y; \vec{x}_2) \times \overline{\mathcal{R}}^{d_1}(x_0; \vec{x}_1) \to \overline{\mathcal{R}}^d(x_0; \vec{x}), \tag{3-44}$$

where y agrees with one of the elements of $\vec{x}_1$ and $\vec{x}$ is obtained by removing y from $\vec{x}_1$ and replacing it with the sequence $\vec{x}_2$. Here, we let d_1 range from 1 to d, with $d_2 = d - d_1 + 1$, with the stipulation that $d_1 = 1$ or $d_2 = 1$ is the semistable case (3-41).

Remark 3.18 (operations for wrapped Fukaya categories) In the setting of the wrapped Fukaya category (continuing Remark 3.17), one needs to incorporate the map ρ_S into the Lagrangian boundary conditions and asymptotics specified in Floer's equation; namely, instead of (3-42), we require the moving boundary condition $u(z) \in (\psi^{\rho_S(z)})^* L_k$ if $z \in \partial S$ lies between z^k and z^{k+1}, where $(\psi^\rho)^* L_i$ denotes the pullback by ψ^ρ (or application of $(\psi^\rho)^{-1} = \psi^{1/\rho}$). We similarly impose that on the k^{th} end, $\lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = (\psi^{\rho_S(z) := w_k})^* x_k$. The point is that Liouville flow for time $\log(\rho)$ defines a canonical identification between Floer complexes,

$$CF^*(L_0, L_1; H, J_t) \simeq CF^*\Big((\psi^\rho)^* L_0, (\psi^\rho)^* L_1; \frac{H}{\rho} \circ \psi^\rho, (\psi^\rho)^* J_t\Big). \tag{3-45}$$

The right-hand object is equivalently the (wrapped) Floer complex for

$$((\psi^\rho)^* L_0, (\psi^\rho)^* L_1)$$

for a strip with one-form $\rho\, dt$ using Hamiltonian $(H/\rho^2) \circ \psi^\rho$ and $(\psi^\rho)^* J_t$. Up to Liouville flow, the Floer equation and boundary conditions satisfied on the k^{th} strip-like end therefore coincides with the usual Floer equation for (H_t, J_t) between L_{k-1} and L_k. In light of this condition and the weakened consistency requirement for Floer data described in Remark 3.17, one can again deduce (3-44), that lower-dimensional strata of the Gromov bordification of the space of maps can be identified (now possibly using a nontrivial Liouville rescaling) with products of previously defined moduli spaces.

In the graded setting, every connected component of the moduli space $\overline{\mathcal{R}}^d(x_0;\vec{x})$ has expected (or virtual) dimension $\deg(x_0)+d-2-\sum_{1\le k\le d}\deg(x_k)$; more generally, this moduli space consists of components of varying expected dimension (a number which can be computed using index theory in terms of the underlying homotopy class of u) all of whose mod 2 reductions are $\deg(x_0)+d-2-\sum_{1\le k\le d}\deg(x_k)$. The following lemma is the prototypical method of verifying Assumption 3.10 for the various moduli spaces considered throughout the paper.

Lemma 3.19 *Assumption 3.10 holds for the moduli spaces $\overline{\mathcal{R}}^d(x_0;\vec{x})$ for admissible M, $\{L_i\}$ and generic choices of a Floer datum for the A_∞ structure* (*satisfying the constraints detailed in Remarks 3.15–3.17 in the Liouville case*). *Namely*: *components of these moduli spaces of virtual dimension ≤ 1 are* (*for generic choices*) *compact manifolds-with-boundary of the given expected dimension. Moreover*, *given a fixed $\vec{x}$ these moduli spaces are empty for all but finitely many x_0*; *this is automatic if there are only finitely many possible x_0 to begin with*, *for instance if all of the Lagrangians being considered are compact.*

Proof If M is compact (and admissible), these assertions (the last of which is automatic) follow from standard Gromov compactness and transversality methods as in [52, (9k), (11h) and Proposition 11.13]. In the case that M and possibly also its Lagrangians are noncompact, there is an additional concern that solutions could escape to infinity in the target. To address this one can, for instance, appeal to the *integrated maximum principle* (compare [3, Lemma 7.2] or [1, Section B]), which implies that elements of $\mathcal{R}(x_0;\vec{x})$ have image contained in a compact subset of M dependent on x_0 and $\vec{x}$, from where one can again appeal to standard Gromov compactness techniques. (This is strongly dependent on the form of H, J and α chosen for our Floer data as in Remarks 3.15–3.17.) The same result can be used to show that solutions do not exist for x_0 of sufficiently negative *action* compared to $\vec{x}$ (with our conventions, action is bounded above and there are finitely many x_0 with action above any fixed level), verifying the last assertion. □

Choose a generic Floer datum for the A_∞ structure satisfying Lemma 3.19 and let $u\in\overline{\mathcal{R}}^d(x_0;\vec{x})$ be a rigid curve, meaning for us an element of the virtual dimension-0 component (which has dimension 0 in this case). By [52, (11h), (12b),(12d)], given the fixed orientation[12] of $\mathcal{R}^d$, any such element $u\in\overline{\mathcal{R}}^d(x_0;\vec{x})$ determines an isomorphism

[12] In the case $d\ge 2$, that is. For $d=1$, one instead needs to "orient the operation of quotienting by $\mathbb{R}$" as in [52, (12f)].

of orientation lines

$$\mathcal{R}_u^d\colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{x_0}. \tag{3-46}$$

Now for any one-dimensional vector spaces $V_1, \dots, V_k$ and W, an isomorphism

$$f\colon V_k \otimes \cdots \otimes V_1 \to W$$

induces a canonical map between $\boldsymbol{k}$–normalizations,

$$|V_1|_{\boldsymbol{k}} \otimes \cdots \otimes |V_k|_{\boldsymbol{k}} \cong |V_1 \otimes \cdots \otimes V_k|_{\boldsymbol{k}} \to |W|_{\boldsymbol{k}},$$

which by abuse of notation, to simplify notation, we also call f (rather than $|f|_{\boldsymbol{k}}$). Using this, for $d \geq 1$ define the d^{th} A_∞ operation

$$\mu^d\colon \hom^*_{\mathcal{F}}(L_{d-1}, L_d) \otimes \cdots \otimes \hom^*_{\mathcal{F}}(L_0, L_1) \to \hom^*_{\mathcal{F}}(L_0, L_d) \tag{3-47}$$

as a sum

$$\mu^d([x_d], \dots, [x_1]) := \sum_{u \in \overline{\mathcal{R}}^d(x_0; \vec{x}) \text{ rigid}} (-1)^{\bigstar_d} \mathcal{R}_u^d([x_d], \dots, [x_1]), \tag{3-48}$$

where $[x_i] \in |o_{x_i}|_{\boldsymbol{k}}$ is an arbitrary element, $\mathcal{R}_u^d$ is the map (on $\boldsymbol{k}$–normalizations induced by) (3-46), and the sign is given by

$$\bigstar_d = \sum_{i=1}^{d} i \cdot \deg(x_i). \tag{3-49}$$

Note that this sum is finite by Lemma 3.19. An analysis of the codimension-1 boundary of one-dimensional moduli spaces along with their induced orientations establishes that the maps μ^d satisfy the A_∞ relations; see [52, Proposition 12.3].

We record here two abuses of notation which will systematically appear in definitions and usage of operations such as μ^d. First, as above, we will frequently use the same symbol for a multilinear map $F\colon V_1 \times \cdots \times V_k \to W$ and its corresponding linear map $F\colon V_1 \otimes \cdots \otimes V_k \to W$. Second, we will frequently use x_i to refer to the arbitrary element $[x_i] \in |o_{x_i}|_{\boldsymbol{k}}$ to simplify expressions; for instance, above we might write $\mu^d(x_d, \dots, x_1)$ in place of $\mu^d([x_d], \dots, [x_1])$.

4 Circle action on the closed sector

4.1 Floer cohomology and symplectic cohomology

Let M be admissible as in Section 3.3.1. Given a (potentially time-dependent) Hamiltonian $H\colon M \to \mathbb{R}$, *Hamiltonian Floer cohomology* when it is defined is formally the

Morse cohomology of the H–perturbed action functional $\mathcal{A}_H\colon \mathcal{L}M \to \mathbb{R}$ on the free loop space $\mathcal{L}M$ of M. If ω is exact and comes with a fixed primitive λ, this functional can be written as

$$x \mapsto -\int_x \lambda + \int_0^1 H_t(x(t))\,dt.$$

In general, $\mathcal{A}_H$ may be multivalued, but $d\mathcal{A}_H$ is always well defined, leading at least to a Morse–Novikov-type theory. Recall that the set of *critical points* of $\mathcal{A}_{H_t}$ (when H_t is implicit) is precisely the set $\mathcal{O}$ of time-1 orbits of the associated (time-dependent) Hamiltonian vector field X_H, and we assume H_t is chosen sufficiently generically that:

Assumption 4.1 The elements of $\mathcal{O}$ are nondegenerate.

Optionally, given the data of a grading structure on M in the sense of Section 3.3.2 one can define an absolute $\mathbb{Z}$–grading on orbits by $\deg(y) := n - \mathrm{CZ}(y)$, where CZ is the Conley–Zehnder index of y (and such a grading is always well defined mod 2).

Fix a (potentially S^1–dependent) almost-complex structure J_t. In the formal picture, this induces a metric on $\mathcal{L}M$. A *Floer trajectory* is formally a gradient flowline of $\mathcal{A}_{H_t}$ using the metric induced by J_t; concretely it is a map $u\colon(-\infty,\infty)\times S^1 \to M$ satisfying Floer's equation (3-40) (which is formally the gradient flow equation for $\mathcal{A}_{H_t}$), and converging exponentially near $\pm\infty$ to a pair of specified orbits $y^\pm \in \mathcal{O}$. In standard coordinates s, t on the cylinder (ie $s \in \mathbb{R}$, $t \in \mathbb{R}/\mathbb{Z} = S^1$) this reads as

$$\partial_s u = -J_t(\partial_t u - X). \tag{4-1}$$

The space of nonconstant Floer trajectories between a fixed y^+ and y^- modulo the free $\mathbb{R}$–action given by translation in the s direction is denoted by $\mathcal{M}(y^-;y^+)$. As in Morse theory, one should compactify this space by allowing *broken trajectories*,

$$\overline{\mathcal{M}}(y^-;y^+) = \coprod \mathcal{M}(y^k;y^+)\times\mathcal{M}(y^{k-1};y^k)\times\cdots\times\mathcal{M}(y^1;y^2)\times\mathcal{M}(y^-;y^1). \tag{4-2}$$

In the graded situation, every component of $\overline{\mathcal{M}}(y^-;y^+)$ has expected/virtual dimension $\deg(y^-)-\deg(y^+)-1$; in general, $\overline{\mathcal{M}}(y^-;y^+)$ has components of varying virtual dimension, of fixed parity $\deg(y^-)-\deg(y^+)-1$, depending on the underlying homotopy class of the cylinder. By Assumption 3.10 for $\overline{\mathcal{M}}(y^-;y^+)$, for generic choices of (time-dependent) J_t, the virtual dimension ≤ 1 components of the moduli spaces $\overline{\mathcal{M}}(y^-;y^+)$ are compact manifolds (with boundary) of the given expected dimension; fix such a J_t.

Putting this all together, the *Floer cochain complex* for (H_t, J_t) over $\boldsymbol{k}$ has generators corresponding to orbits of H_t,

$$CF^i(M) := CF^i(M; H_t, J_t) := \bigoplus_{\substack{y \in \mathcal{O} \\ \deg(y)=i}} |o_y|_{\boldsymbol{k}}, \tag{4-3}$$

where the *orientation line* o_y is a real vector space associated to every orbit in $\mathcal{O}$ via index theory[13] (see eg [1, Section C.6]), and $|V|_{\boldsymbol{k}}$ is the $\boldsymbol{k}$–normalization of V as in (3-37).

The differential $d\colon CF^*(M; H_t, J_t) \to CF^*(M; H_t, J_t)$ counts rigid elements of the compactified moduli spaces (4-2). To fix sign issues, we recall that for a rigid element $u \in \mathcal{M}(y_0; y_1)$ (meaning u belongs to a component of virtual, hence actual, dimension 0) there is a natural isomorphism between orientation lines induced by index theory (see eg [52, (11h), (12b),(12d)] and [1, Lemma C.4]),

$$\mu_u \colon o_{y_1} \to o_{y_0}. \tag{4-4}$$

Then, one defines the differential as

$$d([y_1]) = \sum_{u \in \overline{\mathcal{M}}(y_0; y_1) \text{ rigid}} (-1)^{\deg(y_1)} \mu_u([y_1]), \tag{4-5}$$

where $[y_1] \in |o_{y_1}|_{\boldsymbol{k}}$ is an arbitrary element and μ_u is the map (on $\boldsymbol{k}$–normalizations induced by) (4-4). One can show that $d^2 = 0$ (under the assumptions made), and we call the resulting cohomology group $HF^*(H_t, J_t)$.

If M is compact (and admissible), Assumption 3.10 holds for all (suitably generic) J_t, and all H_t whose time-1 orbits are nondegenerate as in Assumption 4.1. If M is noncompact and admissible then further hypotheses are needed on the profile of (H_t, J_t) at ∞ to obtain admissibility, in particular to prevent curves from escaping to ∞ in M and ensure compactness of $\bigcup_{y^-} \overline{\mathcal{M}}(y^-; y^+)$; we recall the two most relevant possible hypotheses for our purposes in Sections 4.1.1–4.1.2, which can lead to distinct Floer cohomology groups. For simplicity, the discussion in Section 4.1.2 subsumes the case of compact M as well.

Remark 4.2 Our (cohomological) grading convention for Floer cohomology follows [51; 47; 1; 24].

4.1.1 Symplectic cohomology *Symplectic cohomology* [10; 11; 19; 64], the target of the open–closed map for wrapped Fukaya categories, is Hamiltonian Floer cohomology

[13] As before, this index-theoretic definition a priori depends on a choice of trivialization of y^*TM compatible with the grading structure, but any two choices induce isomorphic lines.

for a particular class of Hamiltonians on noncompact convex symplectic manifolds. There are several methods for defining this group. We define it here by making the following specific choices of target, Hamiltonian, and almost-complex structure:

- M is a Liouville manifold *equipped with a conical end*, meaning that it comes equipped with a choice of (3-32). (This serves primarily as a technical device; the resulting invariants are independent of the specific choice.)
- The Hamiltonian term H_t is a sum $H + F_t$ of an *autonomous Hamiltonian* $H\colon M \to \mathbb{R}$ which is *quadratic at* ∞, namely

(4-6) $$H|_{M\setminus\overline{M}}(r, y) = r^2,$$

  and a time-dependent perturbation F_t such that on the collar (3-32) of M, we have:

(4-7) For any $r_0 \gg 0$, there exists an $R > r_0$ such that $F(t, r, y)$ vanishes in a neighborhood of R.

  For instance, F_t could be supported near nontrivial orbits of H, where it is modeled on a Morse function on the circle. We denote by $\mathcal{H}(M)$ the class of Hamiltonians satisfying (4-6).
- The almost-complex structure should belong to the class $\mathcal{J}(M)$ of complex structures which are (*rescaled*) *contact type* on the cylindrical end (3-32), meaning that for some $c > 0$,

(4-8) $$\lambda \circ J = f(r)\,dr,$$

  where f is any function with $f(r) > 0$ and $f'(r) \geq 0$.

A well-known result [47; 1] asserts that Assumption 3.10 holds for the resulting spaces of broken Floer trajectories (4-2). Hence if M, H_t and J_t are as above, one has a well-defined Floer chain complex $CF^*(M, H_t, J_t)$, which we refer to as the *symplectic cochain complex* $SC^*(M)$; this will be the Floer chain complex we use when working with wrapped Fukaya categories. We call the resulting cohomology group *symplectic cohomology* $SH^*(M)$.

4.1.2 Relative cohomology We review here the Floer cohomology group that is the target of the open–closed map for an admissible symplectic manifold M when working with a Fukaya category of *compact* admissible Lagrangian submanifolds in the sense of Section 3.3.1. Fix a (nondegenerate, generic) pair (H_t, J_t) which is arbitrary for compact M and which satisfies the following additional properties if M is Liouville:

- H is linear of very small negative slope near infinity:

$$H_t|_{M\backslash\overline{M}}(r,y) = -\lambda r, \tag{4-9}$$

  where r is the cylindrical coordinate and $\lambda \ll 1$ is a sufficiently small number (smaller than the length of any Reeb orbit on $\partial\overline{M}$).
- J_t is (rescaled) contact type near infinity as before.

It is well known that Assumption 3.10 holds for the moduli spaces (4-2) for generic (H_t, J_t) as above [47], and also that:

Proposition 4.3 *For generic (H_t, J_t) as above, there is an isomorphism*

$$HF^*(H_t, J_t) \cong H^*(\overline{M}, \partial\overline{M}).$$

$H^*(\overline{M}, \partial\overline{M})$ equals $H^*(M)$ in the case that M is compact, using the convention then that $\overline{M} = M$ and $\partial\overline{M} = \varnothing$). □

The isomorphism can be realized in one of two ways:

- Choose H_t as above to be a C^2–small (time-independent) Morse function, in which case a well-known argument of Floer [18] equates $HF^*(H_t, J_t)$ with the Morse complex of H by showing that all Floer trajectories must in fact be Morse trajectories of $H|_{\overline{M}}$ (which in turn, as H is inward pointing near $\overline{M}$, compute the relative cohomology).
- Construct a geometric *PSS morphism* [46]

$$\mathrm{PSS}\colon H^*(\overline{M}, \partial\overline{M}) \cong H_{2n-*}(M) \to HF^*(H_t, J_t).$$

4.2 The cohomological BV operator

The first-order BV operator is a Floer analogue of a natural operator that exists on the Morse cohomology of any manifold with a smooth S^1–action. Like the case of ordinary Morse theory, this operator exists even when the Hamiltonian and complex structure (cf Morse function and metric) are not S^1–equivariant.

For $p \in S^1$, consider the collection of cylindrical ends on $\mathbb{R} \times S^1$

$$\begin{aligned} \epsilon_p^+ &: (s,t) \mapsto (s+1, t+p) && \text{for } s \geq 0, \\ \epsilon_p^- &: (s,t) \mapsto (s-1, t) && \text{for } s \leq 0. \end{aligned} \tag{4-10}$$

Pick $K\colon S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ dependent on p, satisfying

$$(\epsilon_p^{\pm})^* K(p, s, \cdot, \cdot) = H(t, m), \tag{4-11}$$

meaning that

$$K_p(s,t,m) = \begin{cases} H(t+p,m) & \text{if } s \geq 1, \\ H(t,m) & \text{if } s \leq -1, \end{cases} \tag{4-12}$$

so, in the range $-1 \leq s \leq 1$, $K_p(s,t,m)$ interpolates between $H_{t+p}(m)$ and $H_t(m)$ (and outside of this interval is independent of s).

Similarly, pick a family of almost-complex structures $J\colon S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ satisfying

$$(\epsilon_p^{\pm})^* J(p,s,t,m) = J(t,m). \tag{4-13}$$

Now, for $x^+, x^- \in \mathcal{O}$, define

$$\mathcal{M}_1(x^-;x^+) \tag{4-14}$$

to be the *parametrized moduli space* of Floer cylinders

$$\Bigl\{(p,u) \mid p \in S^1, u\colon S \to M \text{ is such that } \lim_{s\to\pm\infty} (\epsilon_p^{\pm})^* u(s,\cdot) = x^{\pm} \text{ and } (du - X_K \otimes dt)^{0,1} = 0\Bigr\}. \tag{4-15}$$

There is a natural bordification by adding broken Floer cylinders to either end,

$$\begin{aligned} \overline{\mathcal{M}}_1(x^-;x^+) = \coprod \mathcal{M}(a_0;x^+) \times \cdots \times \mathcal{M}(a_k;a_{k-1}) \times \mathcal{M}_1(b_1;a_k) \times \mathcal{M}(b_2;b_1) \times \cdots \\ \times \mathcal{M}(x^-;b_l). \end{aligned} \tag{4-16}$$

Remark 4.4 (choices of K and J when M is noncompact) When M is noncompact and Liouville, further constraints on the profile of K and J are required near ∞ (beyond genericity) in order to satisfy Assumption 3.10. In the case of symplectic cohomology described in Section 4.1.1, it suffices to choose K carefully as follows. Given that $H_t(M) = H + F_t$ is a sum of an autonomous term and a time-dependent term that is zero at infinitely many levels tending towards infinity, we can ensure that

(4-17) at infinitely many levels tending towards infinity, $K_p(s,t,m)$ is equal to r^2,

and in particular is autonomous. In the setting of Section 4.1.2 (when M is noncompact), we can similarly ensure a version of (4-17) with r^2 replaced by $-\lambda r$ (in this case we could also more simply ensure that $K_p(s,t,m) = -\lambda r$ outside a compact set). In either case, one can take J to be (rescaled) contact type on the cylindrical end. As usual, the verification of Assumption 3.10 for the moduli spaces (4-16) on Liouville M follows by combining the results [1, Section B] or [3, Lemma 7.2] — which prevent curves escaping to ∞ and ensure $\mathcal{M}_1(x^+, x^-)$ is empty for all but finitely many x^- given the constraints near ∞ fixed in this remark — with classical transversality and compactness arguments.

As before, $\overline{\mathcal{M}}_1(x^-; x^+)$ contains components of varying expected dimension depending on the underlying homotopy class β of a map. Due to the fact that we are studying one-parameter families of domains and not quotienting by $\mathbb{R}$, the relevant expected dimension is 2 more than the expected dimension of the components of $\overline{\mathcal{M}}(x^-; x^+)$ underlying the same homotopy class β. In particular, in the graded case, this expected dimension is $\deg(x^+) - \deg(x^-) + 1$ for every component. By Assumption 3.10 for admissible choices of the above data, ie generic choices satisfying Remark 4.4 in the noncompact case, every component of $\overline{\mathcal{M}}_1(x^-; x^+)$ of virtual dimension ≤ 1 is a compact manifold-with-boundary of dimension equal to its virtual dimension. (In particular, the boundary of the one-dimensional components consists of the once-broken trajectories in (4-16).) In the usual fashion, counting rigid elements of this compactified moduli space of maps with the right sign (explained more carefully in the next section) gives an operation $\delta_1 \colon CF^*(M) \to CF^{*-1}(M)$ satisfying

$$d\delta_1 + \delta_1 d = 0,$$

which comes from the fact that the codimension-1 boundary of $\overline{\mathcal{M}}_1(x^-; x^+)$ is

$$\coprod_y \overline{\mathcal{M}}(y; x^+) \times \overline{\mathcal{M}}_1(x^-; y) \cup \overline{\mathcal{M}}_1(y; x^+) \times \overline{\mathcal{M}}(x^-; y).$$

It would be desirable for δ_1 to square to zero on the chain level, which would give $(CF^*(M), \delta_0 = d, \delta_1)$ the structure of a *strict S^1–complex*, or *mixed complex*. However, the S^1–dependence of our Hamiltonian and almost-complex structure prevent this, in a manner we now explain.

Typically, one attempts to prove that a geometric/Floer-theoretic operation (such as δ_1^2) is zero by exhibiting that the relevant moduli problem has no zero-dimensional solutions (due to, say, extra symmetries in the equation), or otherwise arises as the boundary of a one-dimensional moduli space. To that end, we first describe a moduli space parametrized by $S^1 \times S^1$ which looks like two of the previous parametrized spaces naively superimposed, leading us to call the associated operation δ_2^{naive}. The extra symmetry involved in this definition will allow us to easily conclude:

Lemma 4.5 *The operation δ_2^{naive} is the zero operation.*

For $(p_1, p_2) \in S^1 \times S^1$, consider the collection of cylindrical ends

$$\begin{aligned} \epsilon^{+}_{(p_1,p_2)}&: (s,t) \mapsto (s+1, t+p_1+p_2) && \text{for } s \geq 0, \\ \epsilon^{-}_{(p_1,p_2)}&: (s,t) \mapsto (s-1, t) && \text{for } s \leq 0. \end{aligned} \tag{4-18}$$

Pick $K\colon (S^1 \times S^1) \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$ dependent on (p_1, p_2), satisfying

$$\epsilon^{\pm}_{(p_1,p_2)} K(p_1, p_2, s, \cdot, \cdot) = H(t,m), \tag{4-19}$$

meaning that

$$K_{(p_1,p_2)}(s,t,m) = \begin{cases} H(t+p_1+p_2, m) & \text{if } s \geq 1, \\ H(t,m) & \text{if } s \leq -1, \end{cases} \tag{4-20}$$

so in the range $-1 \leq s \leq 1$, $K_{p_1+p_2}(s,t,m)$ interpolates between $H_{t+p_1+p_2}(m)$ and $H_t(m)$.

Similarly, pick a family of almost-complex structures $J\colon S^1 \times S^1 \times (\mathbb{R} \times S^1) \times M \to \mathbb{R}$,

$$\epsilon^{\pm}_{(p_1,p_2)} J(p_1, p_2, s, t, m) = J(t,m), \tag{4-21}$$

such that

$$J \text{ only depends on the sum } p_1 + p_2. \tag{4-22}$$

Now, for $x^+, x^- \in \mathcal{O}$, define

$$\mathcal{M}_2^{\text{naive}}(x^-; x^+) \tag{4-23}$$

to be the *parametrized moduli space* of Floer cylinders

$$\Big\{(p_1, p_2, u) \mid (p_1, p_2) \in S^1 \times S^1, u\colon S \to M \text{ is such that } \lim_{s \to \pm\infty} (\epsilon^{\pm}_{(p_1,p_2)})^* u(s, \cdot) = x^{\pm} \text{ and } (du - X_K \otimes dt)^{0,1} = 0\Big\}. \tag{4-24}$$

For generic choices of K and J (again bearing in mind the extra impositions of Remark 4.4 in the noncompact case), this moduli space, suitably compactified by adding broken trajectories, will be (for components of virtual dimension ≤ 1) a manifold with boundary of the correct (expected) dimension; the dimension agrees mod 2 in the $\mathbb{Z}/2$–graded case and exactly in the graded case with $\deg(x^+) - \deg(x^-) + 2$. The details are similar to the previous section, and will be omitted. Counts of rigid elements in this moduli space will thus, in the usual fashion, give a map of degree -2, which we call δ_2^{naive}.

Proof of Lemma 4.5 Let (p_1, p_2, u) be an element of $\mathcal{M}_2^{\mathrm{naive}}(x^-; x^+)$. Then, for any $r \in S^1$, $(p_1 - r, p_2 + r, u)$ is an element too, as the equation satisfied by the map u only depends on the sum $p_1 + p_2$. We conclude that elements of $\mathcal{M}_2^{\mathrm{naive}}(x^-; x^+)$ are never rigid, and thus that the resulting operation $\delta_2^{\mathrm{naive}}$ is zero. □

We would like $\delta_2^{\mathrm{naive}}$ to be genuinely equal to δ_1^2, which would imply that $\delta_1^2 = 0$. However, this is only true on the homology level; the lack of S^1 invariance of our Hamiltonian and almost-complex structure, and the corresponding family of choices of homotopy between $\theta^* H_t$ and H_t, over varying $\theta \in S^1$, breaks symmetry and ensures that $\delta_1^2 \neq \delta_2^{\mathrm{naive}}$ as geometric chain maps. However, there is a geometric chain homotopy, δ_2, between δ_1^2 and $\delta_2^{\mathrm{naive}}$, along with a hierarchy of higher homotopies δ_k forming the S^1–complex structure on $CF^*(M)$, which we define in the next section. See in particular Lemma 4.11 for the proof of the S^1–complex equations, one of which recovers the chain homotopy between δ_1^2 and $\delta_2^{\mathrm{naive}} = 0$.

4.3 The A_∞ circle action

We turn to a "coordinate-free" definition of the relevant parametrized moduli spaces, which will help us incorporate the construction into open–closed maps.

Definition 4.6 An *r–point angle-decorated cylinder* consists of a semi-infinite or infinite cylinder $C \subseteq (-\infty, \infty) \times S^1$, along with a collection of auxiliary points $p_1, \dots, p_r \in C$, satisfying

$$(p_1)_s \leq \cdots \leq (p_r)_s, \tag{4-25}$$

where $(a)_s$ denotes the $s \in (-\infty, \infty)$ coordinate. The *heights* associated to this data are the s coordinates

$$h_i = (p_i)_s \quad \text{for } i = 1, \dots, r, \tag{4-26}$$

and the *angles* associated to C are the S^1 coordinates

$$\theta_i := (p_i)_t \quad \text{for } i \in 1, \dots, r. \tag{4-27}$$

The *cumulative rotation* of an r–point angle-decorated cylinder is the first angle:

$$\eta := \eta(C, p_1, \dots, p_r) = \theta_1. \tag{4-28}$$

The *i^{th} incremental rotation* of an r–point angle-decorated cylinder is the difference between the i^{th} and $i-1^{\mathrm{st}}$ angles,

$$\kappa_i^{\mathrm{inc}} := \theta_i - \theta_{i+1}, \quad \text{where } \theta_{r+1} = 0. \tag{4-29}$$

Inductively, each θ_i can be expressed as a sum of all incremental rotations from i to r,

$$\theta_i = \sum_{j=i}^{r} \kappa_j^{\mathrm{inc}}. \tag{4-30}$$

Definition 4.7 The *moduli space of r–point angle-decorated cylinders*

$$\mathcal{M}_r \tag{4-31}$$

is the space of r–point angle-decorated infinite cylinders, modulo translation.

Remark 4.8 (orientation for $\mathcal{M}_r$) The space C_r of all r–point angle-decorated infinite cylinders (not modulo translation) has a canonical complex orientation. Thus, to orient the quotient space $\mathcal{M}_r := C_r/\mathbb{R}$ it is sufficient to give a choice of trivialization of the action of $\mathbb{R}$ on C_r. We choose ∂_s to be the vector field inducing said trivialization.

For an element of this moduli space, the angles and relative heights of the auxiliary points continue to be well defined, so there is a noncanonical isomorphism

$$\mathcal{M}_r \simeq (S^1)^r \times [0,\infty)^{r-1}. \tag{4-32}$$

The moduli space $\mathcal{M}_r$ thus possesses the structure of an open manifold-with-corners, with boundary and corner strata given by the various loci where heights of the auxiliary points p_i are coincident.[14] Given an arbitrary representative C of $\mathcal{M}_r$ with associated heights $h_1, \dots, h_r$, we can always find a translation $\widetilde{C}$ satisfying $\widetilde{h}_r = -\widetilde{h}_1$; we call this the *standard representative* associated to C.

Given a representative C of this moduli space, and a fixed constant δ, we fix a positive cylindrical end around $+\infty$,

$$\epsilon^+ \colon [0,\infty) \times S^1 \to C, \quad (s,t) \mapsto (s + h_r + \delta, t), \tag{4-33}$$

and a negative cylindrical end around $-\infty$ (note the angular rotation in t!),

$$\epsilon^- \colon (-\infty, 0] \times S^1 \to C, \quad (s,t) \mapsto (s - (h_1 - \delta), t + \theta_1). \tag{4-34}$$

These ends are disjoint from the p_i and vary smoothly with C; via thinking of C as a sphere with two points with asymptotic markers removed, these cylindrical ends correspond to the positive asymptotic marker having angle 0 and the negative asymptotic marker having angle $\theta_1 = \kappa_1^{\mathrm{inc}} + \kappa_2^{\mathrm{inc}} + \cdots + \kappa_r^{\mathrm{inc}}$.

[14] We allow the points p_i themselves to coincide; one alternative is to first Deligne–Mumford compactify, and then collapse all sphere bubbles containing multiple points p_i. That the result still forms a smooth manifold-with-corners is a standard local calculation near any such stratum.

There is a compactification of $\mathcal{M}_r$ consisting of *broken r–point angle-decorated cylinders*,

$$\overline{\mathcal{M}}_r = \coprod_s \coprod_{\substack{j_1,\dots,j_s \\ j_i>0, \sum j_i = r}} \mathcal{M}_{j_1} \times \cdots \times \mathcal{M}_{j_s}. \tag{4-35}$$

The stratum consisting of s–fold broken configurations lies in the codimension-s boundary, with the manifold-with-corners structure explicitly defined by local gluing maps using the ends (4-33) and (4-34). The gluing maps, which rotate the bottom cylinder of the gluing in order to match its top end (4-33) with the bottom end (4-34) of the upper cylinder, induce cylindrical ends on the glued cylinders, which agree with the choices of ends made in (4-33)–(4-34). Concretely, for a 1–fold broken configuration of the form $\mathcal{M}_{r-k} \times \mathcal{M}_k$, the gluing map, for any choice of sufficiently small gluing parameter, has the following effect on angles:

$$\big((\theta_1,\dots,\theta_{r-k}),(\overline{\theta}_1,\dots,\overline{\theta}_k)\big) \mapsto \big(\overline{\theta}_1+\theta_1,\overline{\theta}_2+\theta_1,\dots,\overline{\theta}_k+\theta_1,\theta_1,\dots,\theta_{r-k}\big), \tag{4-36}$$

where we have denoted coordinates in the second, bottom factor by $\overline{\theta}_j$ for $1 \le j \le k$, and in the first, top factor by θ_i for $1 \le i \le r-k$; see Figure 1. More simply, in the glued surface, the list of incremental angles $(\kappa_1^{\mathrm{inc,glued}},\dots,\kappa_r^{\mathrm{inc,glued}})$ is equal to the concatenation of the lists of incremental angles of the original bottom and top surfaces, $(\overline{\kappa}_1^{\mathrm{inc}},\overline{\kappa}_2^{\mathrm{inc}},\dots,\overline{\kappa}_k^{\mathrm{inc}},\kappa_1^{\mathrm{inc}},\kappa_2^{\mathrm{inc}},\dots,\kappa_{r-k}^{\mathrm{inc}})$.

The compactification $\overline{\mathcal{M}}_r$ thus has codimension-1 boundary covered by the images of the natural inclusion maps

$$\overline{\mathcal{M}}_{r-k} \times \overline{\mathcal{M}}_k \to \partial\overline{\mathcal{M}}_r \quad \text{for } 0<k<r, \tag{4-37}$$

$$\overline{\mathcal{M}}_r^{i,i+1} \to \partial\overline{\mathcal{M}}_r \quad \text{for } 1 \le i < r, \tag{4-38}$$

where $\overline{\mathcal{M}}_r^{i,i+1}$ denotes the compactification of the locus where i^{th} and $i+1^{\text{st}}$ heights are coincident,

$$\mathcal{M}_r^{i,i+1} := \{C \in \mathcal{M}_r \mid h_i = h_{i+1}\}. \tag{4-39}$$

With regards to the above stratum, for $r>1$ there is a projection map which will be relevant, a version of the forgetful map which remembers only the first of the angles with coincident heights:

$$\begin{aligned} \pi_i : \mathcal{M}_r^{i,i+1} &\to \mathcal{M}_{r-1}, \\ (h_1,\dots,h_i,h_{i+1}=h_i,h_{i+2},\dots,h_r) &\mapsto (h_1,\dots,h_i,h_{i+2},\dots,h_r), \\ (\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_r) &\mapsto (\theta_1,\dots,\theta_{i-1},\theta_i,\widehat{\theta}_{i+1},\theta_{i+2},\dots,\theta_r). \end{aligned} \tag{4-40}$$

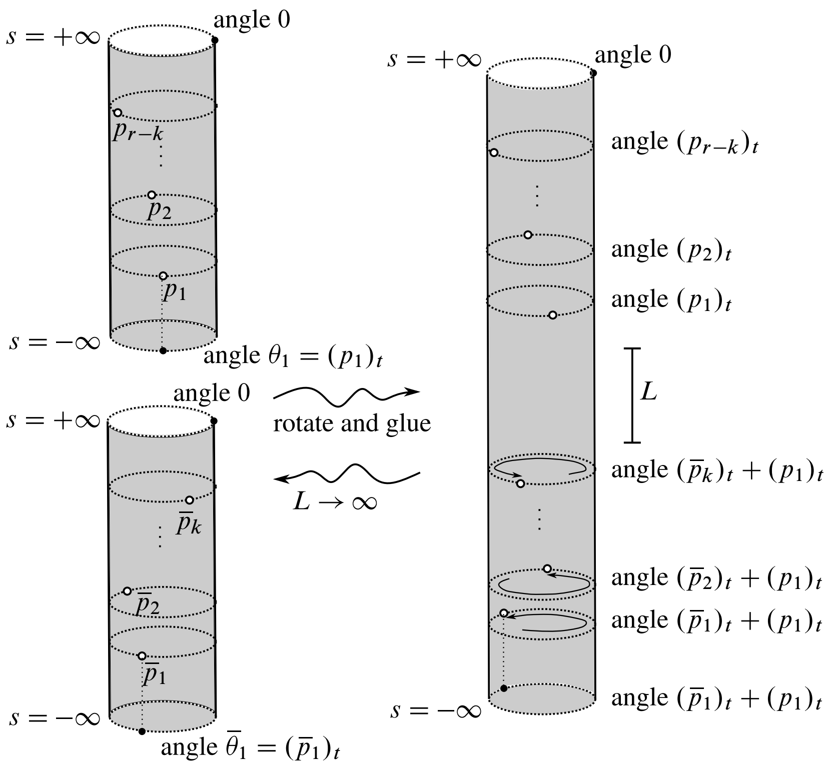

Figure 1: The gluing map for an angle-decorated cylinder rotates all of the angles of the bottom cylinder by the first angle of the top cylinder as in (4-36).

The map π_i is compatible with the choice of positive and negative ends (4-33)–(4-34) and hence π_i extends to compactifications

$$\pi_i \colon \overline{\mathcal{M}}_r^{i,i+1} \to \overline{\mathcal{M}}_{r-1}. \tag{4-41}$$

We will equip each r–point angle-rotated cylinder $\widetilde{C} := (C, p_1, \dots, p_r)$ with perturbation data for Floer's equation or a *Floer datum* in the sense of the last section, which consists of

- the positive and negative cylindrical ends on $\epsilon^{\pm} \colon C^{\pm} \to C$ chosen in (4-33)–(4-34),
- the one-form on C given by $\alpha = dt$,
- a surface-dependent Hamiltonian $H_{\widetilde{C}} \colon C \to \mathcal{H}(M)$ compatible with the positive and negative cylindrical ends, meaning that

$$(\epsilon^{\pm})^* H_{\widetilde{C}} = H_t, \tag{4-42}$$

where H_t was the previously chosen Hamiltonian, and

- a surface-dependent almost-complex structure $J_{\widetilde{C}}\colon C \to \mathcal{J}_1(M)$ also compatible with $\epsilon^{\pm}$, meaning that

$$(\epsilon^{\pm})^* J_{\widetilde{C}} = J_t \tag{4-43}$$

for our previously fixed choice J_t.

A choice of *Floer data for the S^1–action* is an inductive (smoothly varying) choice of Floer data, for each r and each representative $S = (C, p_1, \dots, p_r)$ of $\overline{\mathcal{M}}_r$, satisfying the following consistency conditions at boundary strata:

(4-44) At a boundary stratum (4-37), the datum chosen coincides with the product of Floer data already chosen on lower-dimensional spaces.

(4-45) At a boundary stratum (4-38), the Floer data coincides with the pullback, via the forgetful map π_i defined in (4-41) of the Floer data chosen on $\overline{\mathcal{M}}_{r-1}$.

Inductively, since the space of choices at each level is nonempty and contractible (and since the consistency conditions are compatible along overlapping strata), universal and consistent choices of Floer data exist. From the gluing map, a representative S sufficiently near the boundary strata (4-37) inherits cylindrical regions, also known as *thin parts*, which are the surviving images of the cylindrical ends of lower-dimensional strata. Together with the cylindrical ends of S, this determines a collection of cylindrical regions.

Definition 4.9 Given a fixed positive constant δ, the (δ*–spaced*) *rotated cylindrical regions* for an r–point angle-decorated cylinder $(C, p_1, \dots, p_r)$ consist of the following cylindrical ends and finite cylinders, where $h_i = (p_i)_s$ and $\theta_i = (p_i)_t$:

- The *top cylinder* $\epsilon^+\colon [0, \max(\mathrm{top}(C) - (h_r + \delta), 0)] \times S^1 \to C$, defined by

$$(s, t) \mapsto \big(\min(s + h_r + \delta, \mathrm{top}(C)), t\big). \tag{4-46}$$

- The *bottom cylinder* $\epsilon^-\colon [\min(\mathrm{bottom}(C) - (h_1 - \delta), 0), 0] \times S^1 \to C$, defined by

$$\begin{aligned}(s, t) &\mapsto \big(\max(s - (h_1 - \delta), \mathrm{bottom}(C)), t + \theta_1\big) \\ &= \Bigg(\max(s - (h_1 - \delta), \mathrm{bottom}(C)), t + \sum_{j=1}^{r} \kappa_i\Bigg).\end{aligned}$$

- For any $1 \le i \le r - 1$ satisfying $h_{i+1} - h_i > 2\delta$, the i^{th} *thin part* is

$$\epsilon_i\colon [h_i + \delta, h_{i+1} - \delta] \times S^1 \to C, \quad (s, t) \mapsto (s, t + \eta_i). \tag{4-47}$$

Note that a given r–point angle-decorated cylinder may not have an i^{th} thin part for a given $i \in [1, r-1]$, and indeed may not have any thin parts. The consistency conditions at boundary strata can be ensured in particular by requiring that for any (δ–spaced) rotated cylindrical region $\epsilon\colon C' \to C$ of sufficiently large length (greater than some fixed L, say $L = 3\delta$) associated to $(C, p_1, \dots, p_r)$ and δ, we have that

$$\epsilon^*(K_C, J_C) = (K_t, J_t). \tag{4-48}$$

Given the cylindrical regions of Definition 4.9, this would imply the following condition on (K_C, J_C) (assuming $L \gg 3\delta$): for $z = (s,t) \in C$,

$$(K_z, J_z) = \begin{cases} (K_t, J_t) & \text{for } s > h_r + \delta, \\ (\epsilon^-)^*(K_t, J_t) = (K_{t+\theta_1}, J_{t+\theta_1}) & \text{for } s < h_1 - \delta, \\ \epsilon_i^*(K_z, J_z) = (K_{t+\theta_i}, J_{t+\theta_i}) & \text{if } h_{i+1} - h_i > 3\delta \\ & \quad \text{and } s \in [h_i + \delta, h_{i+1} - \delta]. \end{cases} \tag{4-49}$$

Given a choice of Floer data for the S^1–action and a pair of asymptotics $(x^+, x^-) \in \mathcal{O}$ for each $k \geq 1$, there is an associated parametrized moduli space of Floer cylinders with source an arbitrary element of $S \in \mathcal{M}_r$, where the Floer equation is with respect to the Hamiltonian H_S and J_S, with asymptotics (x^+, x^-):

$$\begin{aligned} &\mathcal{M}_r(x^-; x^+) := \\ &\quad \Bigl\{ S = (C, p_1, \dots, p_r) \in \mathcal{M}_r,\ u\colon C \to M \mid \lim_{s \to \pm\infty} (\epsilon^\pm)^* u(s, \cdot) = x^\pm \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (du - X_{H_S} \otimes dt)^{(0,1)_S} = 0 \Bigr\}. \end{aligned} \tag{4-50}$$

The consistency condition imposes that the boundary of the Gromov bordification $\overline{\mathcal{M}}_r(x^-; x^+)$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_{r-k}(y; x^+) \times \overline{\mathcal{M}}_k(x^-; y) \to \partial \overline{\mathcal{M}}_r(x^-; x^+), \tag{4-51}$$

$$\overline{\mathcal{M}}_r^{i,i+1}(x^-; x^+) \to \partial \overline{\mathcal{M}}_r(x^-; x^+), \tag{4-52}$$

along with the usual semistable strip-breaking boundaries

$$\begin{aligned} \overline{\mathcal{M}}(y; x^+) \times \overline{\mathcal{M}}_r(x^-; y) &\to \partial \overline{\mathcal{M}}_r(x^-; x^+), \\ \overline{\mathcal{M}}_r(y; x^+) \times \overline{\mathcal{M}}(x^-; y) &\to \partial \overline{\mathcal{M}}_r(x^-; x^+). \end{aligned} \tag{4-53}$$

Remark 4.10 (Floer data in the Liouville case) Continuing Remark 4.4, when M is Liouville we impose the following further constraint on Floer data:

(4-54) $H_{\widetilde{C}}$ is equal to r^2 or $-\lambda r$ (depending on whether we are in the setting of Section 4.1.1 or Section 4.1.2) at infinitely many levels of r tending to ∞, and $J_{\widetilde{C}}$ is (rescaled) contact type near ∞.

(In fact, in the setting of Section 4.1.2 we can take $H_{\widetilde{C}}$ to be simply equal to $-\lambda r$ for all r outside of a compact set.) By [3, Lemma 7.2] or [1, Section B], this hypothesis implies that sequences of curves with fixed asymptotics cannot escape to ∞ in M, and that $\mathcal{M}_r(x^-; x^+)$ given a fixed x^+ is nonempty for only finitely many x^-, both necessary inputs to verifying Assumption 3.10.

In the $\mathbb{Z}$–graded case, the virtual dimension of (every component of) $\overline{\mathcal{M}}_r(x^-; x^+)$ is

$$\deg(x^+) - \deg(x^-) + (2r-1), \tag{4-55}$$

while in the $\mathbb{Z}/2$–graded case every component has virtual dimension of the above parity. By Assumption 3.10, for a generic fixed choice of Floer data for the S^1–action (satisfying Remark 4.10 in the Liouville case), the components of virtual dimension ≤ 1 of the moduli spaces $\overline{\mathcal{M}}_r(x^-; x^+)$ are compact manifolds-with-boundary of the correct (expected) dimension. As usual, signed counts of rigid elements of this moduli space for varying x^+ and x^- (using induced maps on orientation lines, twisted as in the differential by $(-1)^{\deg(x_+)}$ — see (4-5)) give the matrix coefficients for the overall map

$$\delta_r \colon CF^*(M) \to CF^{*-2r+1}(M). \tag{4-56}$$

In the degenerate case $r = 0$ we define δ_0 to be the (already defined) differential,

$$\delta_0 := d \colon CF^*(M) \to CF^{*+1}(M). \tag{4-57}$$

Lemma 4.11 *For each r,*

$$\sum_{i=0}^{r} \delta_i \delta_{r-i} = 0. \tag{4-58}$$

Proof The counts of rigid elements associated to the boundary of one-dimensional components of $\partial\overline{\mathcal{M}}_r(x^+; x^-)$, along with a description of this codimension-1 boundary (4-51)–(4-53) immediately implies that

$$\Bigl(\sum_{i=1}^{r} \delta_i \delta_{r-i}\Bigr) + \Bigl(\sum_{i} \delta_r^{i,i+1}\Bigr) + (d\delta_r + \delta_r d) = 0, \tag{4-59}$$

where $\delta_r^{i,i+1}$ for each i is the operation associated to the moduli space of maps (4-52). (Observe that $\delta_2^{1,2}$ is precisely the operation $\delta_2^{\mathrm{naive}}$ from Section 4.2.) Note that the consistency condition (4-45) implies that the Floer datum chosen for any element $S \in \mathcal{M}_r^{i,i+1}$ only depends on $\pi_i(S)$, where the forgetful map $\pi_i \colon \mathcal{M}_r^{i,i+1} \to \mathcal{M}_{r-1}$ has one-dimensional fibers. Hence given an element $(S, u) \in \overline{\mathcal{M}}_r^{i,i+1}(x^-; x^+)$, it follows that $(S', u) \in \overline{\mathcal{M}}_r^{i,i+1}(x^-; x^+)$ for all $S' \in \pi_i^{-1}\pi_i(S)$. In other words, elements of $\overline{\mathcal{M}}_r^{i,i+1}(x^-; x^+)$ are never rigid, so the associated operation $\delta_r^{i,i+1}$ is zero. $\square$

By definition we conclude:

Corollary 4.12 *The pair $(CF^*(M; H_t, J_t), \{\delta_r\}_{r \geq 0})$ as defined above forms an S^1–complex, in the sense of Definition 2.2.* □

By using continuation maps parametrized by various $(S^1)^r \times (0, 1]^r$ (or equivalently, by spaces of angle-decorated cylinders that are not quotiented by overall $\mathbb{R}$–translation), one can prove:

Proposition 4.13 *Any continuation map $f : CF^*(M, H_1) \to CF^*(M, H_2)$ enhances to a homomorphism $\boldsymbol{F}$ of S^1–complexes (which is, in particular, a quasi-isomorphism if f is). Moreover, this homomorphism is canonical up to homotopy, in the sense that any two homomorphisms $\boldsymbol{F}$ and $\boldsymbol{F}'$ enhancing f constructed geometrically from parametrized continuation maps differ by an exact premorphism of S^1–complexes (also constructed geometrically).* □

We omit the proof, which is standard; see eg [66], but note some notational differences. In particular, the S^1–complex defined on the symplectic cochain complex $SC^*(M)$ or the Hamiltonian Floer complex (with small negative slope if M is noncompact) is an invariant of M, up to quasi-isomorphism.

Remark 4.14 (relation to earlier definitions in the literature) In [6], three different definitions of S^1–equivariant symplectic cohomology are considered and shown to be equivalent. One of the definitions involves taking the S^1–equivariant homology associated to a certain S^1–complex defined on $CF^*(M) = SC^*(M)$ [6, Proposition 2.19]. After normalizing for differing conventions (eg homological versus cohomological conventions for Floer theory, and the fact that their u^{-1} is our u), it is direct to see that the S^1–complex constructed therein coincides up to equivalence with the one here — and even agrees on the chain level, seeing as the choices of Floer data chosen in that paper constitute a choice of Floer data for the S^1–action in our sense; compare, for instance, [6, Figure 1] with (4-49).

4.4 The circle action on the interior

From the formal point of view of Floer homology of M as the Morse homology of an action functional on the free loop space $\mathcal{L}M$, one would expect the contributions coming from constant loops to be acted on trivially by the $C_{-*}(S^1)$–action, which comes from rotation of free loops. This is indeed the case, as we now review.

Suppose that the Hamiltonian H_t defining $CF^*(M)$ is chosen to be C^2–small, time-independent, and Morse in the compact region of $\overline{M}$ (which equals M if M is compact). Then, Floer [18] proved that all orbits of H_t inside $\overline{M}$ are (constant orbits at) Morse critical points of H, and all Floer cylinders between such orbits which remain in $\overline{M}$ are in fact Morse trajectories of H.

Let $C_{\mathrm{Morse}}(H)$ denote the Morse complex of H. In the setting where H is as in Section 4.1.2 (M can be Liouville or compact), all contributions to $CF^*(M)$ (both orbits and cylinders) come from $\overline{M}$, so Floer's argument gives an isomorphisms

$$C_{\mathrm{Morse}}(H) \cong CF^*(M). \tag{4-60}$$

In the setting where H is quadratic at infinity (and M is Liouville) as in Section 4.1.1, one can ensure the collection of orbits coming from $\overline{M}$ is an action-filtered subcomplex — and, for instance, the integrated maximum principle will ensure that all cylinders between such orbits lie in $\overline{M}$. Hence, there is an inclusion of subcomplexes

$$C_{\mathrm{Morse}}(H) \to SC^*(M), \tag{4-61}$$

which, under smallness constraints on the Floer data for the S^1–action gives an S^1–*subcomplex* [66, Lemma 5.4], meaning the operators δ_k preserve the subcomplex and in fact the action filtration; hence a morphism of S^1–complexes. We will discuss both of the above cases at once: in either case, by considering a Hamiltonian which is C^2–small on $\overline{M}$, we obtain an inclusion of S^1–subcomplexes

$$C_{\mathrm{Morse}}(H) \to CF^*(M) \tag{4-62}$$

with the understanding that in the former case this inclusion is the whole complex.

Lemma 4.15 *There exists a choice of Floer data for the S^1–action so that $C_{\mathrm{Morse}}(H)$ becomes a trivial S^1–subcomplex; meaning that the various operators δ_r, $r \geq 1$, associated to the $C_{-*}(S^1)$–action strictly vanish on the subcomplex.*

Proof By the integrated maximum principle, any Floer trajectory with asymptotics along two generators in $C_{\mathrm{Morse}}(H)$ remains in the interior of $\overline{M}$. We can choose the Hamiltonian term of our Floer data on $\mathcal{M}_r$ in this region of M to be autonomous (ie t– and s–independent on the cylinder), C^2–small and Morse — in fact equal to H; then Floer's theorem [18] again guarantees that any Floer trajectory in $\mathcal{M}_r(x^-; x^+)$ between Morse critical points $x^\pm$ is in fact a Morse trajectory of H. It follows that for critical points x^+, x^- of H, any element $u = (C, \vec{p})$ in the parametrized moduli space of maps $\overline{\mathcal{M}}_r(x^-; x^+)$ solves an equation that is independent of the choice of parameter

$\vec{p} \in (S^1)^r \times (0,1]^{r-1}$. Namely, u lives in a family of solutions of dimension at least $2r-1$ (given by varying $\vec{p}$), and hence u cannot be rigid. The associated operation δ_r, which counts rigid solutions, is therefore zero. $\square$

By invariance of the S^1–complex structure on $CF^*(M)$ (up to homotopically canonical quasi-isomorphism as in Proposition 4.13), we conclude:

Corollary 4.16 *For M compact and admissible, or Liouville with (H,J) as in Section 4.1.2, $CF^*(M)$ is quasi-isomorphic to a trivial S^1–complex, canonically up to homotopy.*

Corollary 4.17 *For M Liouville with (H,J) as in Section 4.1.1, the inclusion chain map*

$$C^*_{\mathrm{Morse}}(M) \to SC^*(M) \tag{4-63}$$

lifts (cohomologically) canonically to a chain map

$$\begin{aligned}(C_{\mathrm{Morse}}(H)[\![u]\!], d_{\mathrm{Morse}}) = (C_{\mathrm{Morse}}(H))^{\mathrm{h}S^1} \\ \to (SC^*(M))^{\mathrm{h}S^1} = (SC^*(M)[\![u]\!], \delta_{\mathrm{eq}}),\end{aligned} \tag{4-64}$$

inducing a cohomological map

$$H^*(M)[\![u]\!] \to H^*(SC^*(M)^{\mathrm{h}S^1}). \tag{4-65}$$

Remark 4.18 Another possibly more direct way of producing the map $H^*(M)[\![u]\!] \to H^*(SC^*(M)^{\mathrm{h}S^1})$ is via an S^1–equivariant enhancement $\widetilde{\mathrm{PSS}}$ of the PSS morphism $\mathrm{PSS}\colon C^*(M) \to SC^*(M)$. We omit a further description here, and simply note that the resulting map can be shown to coincide cohomologically with the map defined above.

Since the S^1–complex structure on $C^*_{\mathrm{Morse}}(H)$ is trivial, one can (canonically) split the inclusion of homotopy fixed points map (2-36) $H^*(C^*_{\mathrm{Morse}}(H)^{\mathrm{h}S^1}) \to H^*(C^*_{\mathrm{Morse}}(H))$ by the map

$$H^*(M) \xrightarrow{[x \mapsto x \cdot 1]} H^*(M)[\![u]\!]. \tag{4-66}$$

The associated composition

$$H^*(M) \xrightarrow{[x \mapsto x \cdot 1]} H^*(M)[\![u]\!] \to H^*(SC^*(M)^{\mathrm{h}S^1}) \xrightarrow{[\iota]} SH^*(M) \tag{4-67}$$

coincides with the usual map $H^*(M) \to SH^*(M)$. In particular, we note that the homotopy fixed-point complex of $SC^*(X)$ possesses a canonical (geometrically defined) cohomological element,

$$\tilde{1} \in H^*(SC^*(M)^{\mathrm{h}S^1}), \tag{4-68}$$

lifting the usual unit $1 \in SH^*(M)$ (under the map $[\iota]$), defined as the image of 1 under the map (4-65).

5 Cyclic open–closed maps

5.1 Open–closed Floer data

Here we review the sort of Floer perturbation data that needs to be specified on the domains appearing in the open–closed map and their cyclic analogues. The main body of our treatment, following Section 3.3, consists of a (slightly modified) simplification of the setup from [1] tailored to the case of Fukaya categories of compact admissible M; in Remarks 5.1 and 5.2 below we will indicate the modifications we need to make — following [1] and building on Remarks 3.15–3.18 above — in the case of compact Fukaya categories of Liouville manifolds (minor modifications) or wrapped Fukaya categories (slightly more involved modifications). There is one notable deviation from [1], in that we allow our interior marked point to have a varying asymptotic marker and choose Floer data depending on this choice, as is done in constructions of BV-type operations in Hamiltonian Floer theory involving such asymptotic markers; see eg [58; 55].

Let S be a disc with d boundary punctures $z_1, \dots, z_d$ (labeled in counterclockwise order) marked as positive, and an interior marked point p removed, marked as either positive or negative; for the main body of the construction p is negative. We also equip the interior marked point p with an *asymptotic marker*, that is, a half-line $\tau_p \in T_pS$ (or equivalently an element of the unit tangent bundle, defined with respect to some metric). Call any such $S = (S, z_1, \dots, z_d, p, \tau_p)$ an *open–closed framed disc.*

In addition to the notation for semi-infinite strips (3-38)–(3-39), we use the following notation to refer to the positive and negative semi-infinite cylinder:

$$A_+ := [0,\infty) \times S^1, \tag{5-1}$$

$$A_- := (-\infty, 0] \times S^1. \tag{5-2}$$

A *Floer datum* on a stable open–closed framed disc S consists of the following choices on each component:

(1) A collection of *strip-like or cylindrical ends* $\mathfrak{S}$ around each boundary or interior marked point, of sign matching the sign of the marked point; strip-like ends were defined in Section 3.3 and a positive (resp. negative) cylindrical end is a map
$$\delta_j^{\pm} : A_{\pm} \to S.$$
(So for the main body of the construction, we use a negative cylindrical end around p.) All of the strip-like ends around each of the z_i should be positive, and all (strip-like or cylindrical) ends should have disjoint image in S. The cylindrical end around p should further should be *compatible with the asymptotic marker*, meaning the points with angle zero should asymptotically approach the marker,
$$\lim_{s\to\pm\infty} \delta^{\pm}(s,0) = \tau_p. \tag{5-3}$$

(2) A one-form α_S on S, an S–dependent Hamiltonian function H_S on M, and an S–dependent almost-complex structure J_S on M, such that on each strip-like end these data pull back to a given fixed (dt, H_t, J_t), (which we used to define Lagrangian Floer homology chain complexes) and on the cylindrical end this data pulls back to a given fixed $(dt, H_t^{\mathrm{cyl}}, J_t^{\mathrm{cyl}})$ which we used to define our Hamiltonian Floer homology chain complex. (Note that in many cases we could further simplify and choose $(H_t^{\mathrm{cyl}}, J_t^{\mathrm{cyl}}) = (H_t, J_t)$, given a sufficiently generic choice of (H_t, J_t).)

Given a stable open–closed framed disc S equipped with a Floer datum F_S, a collection of Lagrangians $\{L_0, \dots, L_{d-1}\}$ and asymptotics $\{x_1, \dots, x_d; y\}$ with x_i a chord between L_{i-1} and $L_{i \bmod d}$, a map $u : S \to M$ satisfies *Floer's equation for F_S with boundary $\{L_0, \dots, L_{d-1}\}$ and asymptotics $\{x_1, \dots, x_d; y\}$* if
$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \quad \text{using the Floer data given by } F_S \tag{5-4}$$
(meaning X_S is the Hamiltonian vector field associated to H_S, and $0, 1$ parts are taken with respect to J_S), and
$$\begin{cases} u(z)\in L_i \text{ if } z\in\partial S \text{ lies counterclockwise from } z_i, \text{ clockwise from } z_{i+1 \bmod d}, \\ \lim\limits_{s\to+\infty} u\circ\epsilon^k(s,\cdot)=x_k, \\ \lim\limits_{s\to\mp\infty} u\circ\delta(s,\cdot)=y. \end{cases} \tag{5-5}$$

Here ϵ^k denotes the k^{th} strip-like end, δ denotes the cylindrical end, and the sign $\mp$ in the last line is $-$ if δ is a negative end — which is the case for the main body of the construction — and $+$ if δ is a positive end.

Remark 5.1 (Floer data for compact Lagrangians in Liouville manifolds) If M is Liouville and we are studying the Fukaya category of compact exact Lagrangians, then we take H_t^{cyl}, J_t^{cyl} (the data required to define Floer cohomology) as in Section 4.1.2 and we again impose the additional requirements on Floer data described in Remark 3.16, As before the H_t^{cyl}, J_t^{cyl} and more restrictive types of Floer data chosen for wrapped Fukaya categories in Remark 5.2 below would also work. The ability to choose H_t^{cyl} and J_t^{cyl} as in Section 4.1.2 is indicative of a more general freedom in the Floer data here, which also will allow us later to define operations in which the interior marked point (and all boundary marked points) are positive; see Section 5.6.2.

Remark 5.2 (Floer data and Floer's equation for wrapped Fukaya categories) Almost exactly as in Remark 3.17, and following [1], in order to associate operations between the wrapped Fukaya category and symplectic cohomology one needs to make the following modifications to the notion of Floer data. First, one takes H_t^{cyl}, J_t^{cyl} to be the data defining the symplectic cochain complex as in Section 4.1.1. Then one equips S with strip-like and cylindrical ends as above. Let ψ^ρ as before denote the time $\log(\rho)$ Liouville flow on M. The modifications to the Floer data are:

- **Extra choices of weights and time-shifting maps** Exactly as in Remark 3.17, one associates a *weight* $w_k \in \mathbb{R}_{>0}$ to each boundary or interior marked point and a time-shifting map $\rho_S \colon \partial S \to \mathbb{R}_{>0}$ agreeing with w_k near the k^{th} strip-like end.
- **Modified requirements on the one-form** The one-form α_S should be subclosed, meaning $d\alpha_S \leq 0$, should restrict to 0 along ∂S, and restrict to $w_k\,dt$ on each (strip-like or cylindrical) end, as in Remark 3.17(2). It follows by Stokes' theorem that the weight at the (output) cylindrical end should be greater than the sum of weights over all (input) strip-like ends. In particular, it is not possible for α_S to be subclosed and restrict to 0 along ∂S, conditions necessary to appeal to the integrated maximum principle if the interior marked point were also positive. (This is a reflection of the fact that wrapped Fukaya categories do not admit geometric operations with no outputs.)
- **Modified requirements on Hamiltonians**, as in Remark 3.17(3) The Hamiltonian term should pull back to $H \circ \psi^{w_k}/w_k^2$ along any strip-like end, and to $H^{\mathrm{cyl}} \circ \psi^{w_k}/w_k^2$ along the cylindrical end. The Hamiltonian term should also be quadratic at infinitely many levels of (3-32) tending to infinity. (This is a slight weakening of Remark 3.17 coming from the fact that the Hamiltonian used to define $SC^*(X)$ is not quadratic at every level near infinity due to (4-7).)

- **Modified requirements on almost-complex structures**, as in Remark 3.17(4) The almost-complex structure should be contact type at infinity and pull back to $(\psi^{w_k})^* J_t$ along each strip-like end and $(\psi^{w_k})^* J_t^{\mathrm{cyl}}$ along the cylindrical end.

Exactly as in Remark 3.17, there is a rescaling action on the space of such Floer data, and we will relax any consistency requirement imposed on Floer data to allow for an arbitrary rescaling when equating different choices of Floer data. Finally, we note the slight modifications to the boundary and asymptotic conditions of Floer's equation (5-5), following Remark 3.18: on the boundary component of ∂S lying counterclockwise from z_i and clockwise from $z_{i+1 \bmod d}$ we impose the moving boundary condition $u(z) \in (\psi^{\rho_S(z)})^* L_i$, on the k^{th} strip-like end we impose $\lim_{s\to+\infty} u \circ \epsilon^k(s,\cdot) = (\psi^{w_k})^* x_k$, and on the cylindrical end we impose $\lim_{s\to-\infty} u \circ \delta(s,\cdot) = (\psi^{w})^* y$, where w is the weight associated to the interior puncture p.

Exactly as in the proof of Lemma 3.19, the constraints to Floer data in the Liouville case made in the above two remarks help ensure Assumption 3.10 holds for associated moduli spaces.

5.2 Nonunital open–closed maps

We begin by constructing a variant of the open–closed map of [1] with source the nonunital Hochschild complex of (3-11), which we call the *nonunital open–closed map* and indicate by $\mathcal{OC}$ or $\mathcal{OC}^{\mathrm{nu}}$:

$$\mathcal{OC} := \mathcal{OC}^{\mathrm{nu}} \colon \mathrm{CH}^{\mathrm{nu}}_{*-n}(\mathcal{F}) \to CF^*(M). \tag{5-6}$$

This map actually has a straightforward explanation from the perspective of Remark 3.3: we define the map $\mathcal{OC}$ from $\widetilde{\mathrm{CH}}_*(\mathcal{F}^+)$ by counting discs with an arbitrary number of boundary punctures and one interior puncture asymptotic to an orbit, as in [1], with the proviso that we treat the formal elements e_L^+ as "fundamental class $[L]$ point constraints (ie empty constraints)": we fill back in the relevant boundary puncture and impose no constraints on that marked point. With respect to the decomposition (3-11), we define a pair of maps

$$\check{\mathcal{OC}} \oplus \hat{\mathcal{OC}} \colon \mathrm{CH}_*(\mathcal{F}) \oplus \mathrm{CH}_*(\mathcal{F})[1] \to CF^*(M) \tag{5-7}$$

giving the left and right components of the nonunital open–closed map

$$\mathcal{OC} \colon \mathrm{CH}^{\mathrm{nu}}_{*-n}(\mathcal{F}) \to SC(M), \quad (x,y) \mapsto \check{\mathcal{OC}}(x) + \hat{\mathcal{OC}}(y). \tag{5-8}$$

Since the left (check) factor is equal to the usual cyclic bar complex for Hochschild homology, $\check{\mathcal{OC}}$ will be defined exactly as the open–closed map is defined in [1] (briefly recalled below), and the new content is the map $\hat{\mathcal{OC}}$. We will define $\hat{\mathcal{OC}}$ below (and recall the definition of $\check{\mathcal{OC}}$) and prove, extending [1]:

Lemma 5.3 *$\mathcal{OC}$ is a chain map of degree n.*

We note a notational difference from [1], which uses $\mathcal{OC}$ to refer to what we call here $\check{\mathcal{OC}}$; in contrast, in this paper we use $\mathcal{OC}$ exclusively to refer to the (nonunital) open–closed map $\mathcal{OC} = \mathcal{OC}^{\mathrm{nu}} := \check{\mathcal{OC}} \oplus \hat{\mathcal{OC}}$ with domain the nonunital Hochschild complex. Of course, the two maps $\mathcal{OC}$ and $\check{\mathcal{OC}}$ are homologically the same. That is, assuming Lemma 5.3:

Corollary 5.4 *As homology-level maps*, $[\mathcal{OC}] = [\check{\mathcal{OC}}]$.

Proof By construction, the chain level map $\check{\mathcal{OC}}$ constructed in [1] factors as

$$\mathrm{CH}_{*-n}(\mathcal{F}) \subset \mathrm{CH}^{\mathrm{nu}}_{*-n}(\mathcal{F}) \xrightarrow{\mathcal{OC}} CF^*(M). \tag{5-9}$$

The first inclusion is a quasi-isomorphism by Lemma 3.2, since $\mathcal{F}$ is known to be cohomologically unital. □

The moduli space controlling the operation $\check{\mathcal{OC}}$, denoted by

$$\overline{\check{\mathcal{R}}}{}^1_d, \tag{5-10}$$

is the (Deligne–Mumford compactification of the) abstract moduli space of discs with d boundary positive punctures $z_1, \dots, z_d$ labeled in counterclockwise order and one interior negative puncture z_{out}, with an asymptotic marker τ_{out} at z_{out} pointing towards z_d. The space (5-10) has a manifold-with-corners structure, with boundary strata described in [1, Section C.3] — there, the space is called $\overline{\mathcal{R}}{}^1_d$ — in short, codimension-one strata consist of disc bubbles containing any cyclic subsequence of k inputs attached to an element of $\check{\mathcal{R}}^1_{d-k+1}$ at the relative position of this cyclic subsequence. Orient the top stratum $\check{\mathcal{R}}^1_d$ by trivializing it, sending $[S]$ to the unit disc representative S with z_d and z_{out} fixed at 1 and 0, and taking the orientation induced by the (angular) positions of the remaining marked points:

$$-dz_1 \wedge \cdots \wedge dz_{d-1}. \tag{5-11}$$

The moduli space controlling the new map $\hat{\mathcal{OC}}$ is nearly identical to $\check{\mathcal{R}}^1_d$, but there is additional freedom in the direction of the asymptotic marker at the interior puncture z_{out}. The top (open) stratum is easiest to define: let

$$\mathcal{R}^{1,\mathrm{free}}_d \tag{5-12}$$

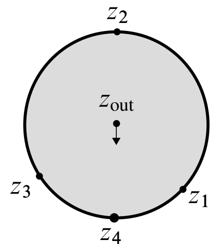

Figure 2: A representative of an element of the moduli space $\check{\mathcal{R}}_4^1$ with special points at 0 (output), $-i$.

be the moduli space of discs with d positive boundary punctures and one interior negative puncture as in $\check{\mathcal{R}}_d^1$, but with the asymptotic marker τ_{out} pointing anywhere between z_1 and z_d.

Remark 5.5 There is a delicate point in naively compactifying $\mathcal{R}_d^{1,\text{free}}$: on any formerly codimension-one stratum in which z_1 and z_d bubble off, the position of τ_{out} becomes fixed too, and so the relevant stratum actually should have codimension two (and hence does not contribute to the codimension-one boundary equation for $\widehat{\mathcal{OC}}$; moreover, there is no nice corner chart near this stratum). For technical convenience, we pass to an alternative, larger (blown-up) model for the compactification in which these strata have codimension one but consist of degenerate contributions.

In light of Remark 5.5, we use (5-12) as motivation and instead define

$$\hat{\mathcal{R}}_d^1 \tag{5-13}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures $z_f, z_1, \ldots, z_d$ and an interior puncture z_{out} with asymptotic marker τ_{out} pointing towards the boundary point z_f, modulo automorphism. We mark z_f as "auxiliary", but otherwise the space is abstractly isomorphic to $\check{\mathcal{R}}_{d+1}^1$. Identifying $\hat{\mathcal{R}}_d^1$ with the space of unit discs with z_{out} and z_f fixed at 1 and 0, the remaining (angular) positions of $z_1, \ldots, z_d$ determine an orientation

$$-dz_1 \wedge \cdots \wedge dz_d. \tag{5-14}$$

The *forgetful map*

$$\pi_f : \hat{\mathcal{R}}_d^1 \to \mathcal{R}_d^{1,\text{free}} \tag{5-15}$$

puts back in the point z_f and forgets it. Since the point z_f is recoverable from the direction of the asymptotic marker at z_{out}, we get:

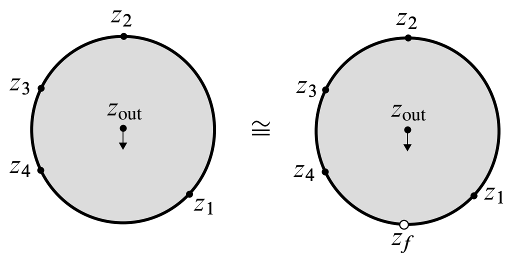

Figure 3: A representative of an element of the moduli space $\mathcal{R}^1_{4,\text{free}}$ and the corresponding element of $\hat{\mathcal{R}}^1_4$.

Lemma 5.6 *The map π_f is a diffeomorphism.* □

The perspective of the former space (5-13) gives us a model for the compactification

$$\overline{\mathcal{R}}_d^{1,\text{free}} \tag{5-16}$$

as the ordinary Deligne–Mumford compactification

$$\overline{\hat{\mathcal{R}}}{}^1_d. \tag{5-17}$$

We call a component T of a representative S of (5-17) the *main component* if it contains the interior marked point, and a *secondary component* if its output is attached to the main component. As a manifold with corners, (5-17) is equal to the compactification $\overline{\check{\mathcal{R}}}{}^1_{d+1}$ except from the point of view of assigning Floer datum, as we will be forgetting the point z_f instead of fixing asymptotics for it. It is convenient therefore (for the purpose of indicating choices of Floer data made) to name components of strata containing z_f differently. At any stratum:

- We treat the main component (containing z_{out} and k boundary marked points) as belonging to $\overline{\hat{\mathcal{R}}}{}^1_{k-1}$ if it contains z_f and $\overline{\check{\mathcal{R}}}{}^1_k$ otherwise.
- If the i^{th} boundary marked point of any nonmain component was z_f, we view it as an element of $\mathcal{R}^{k,f_i}$, the space of discs with one output and k input marked points removed from the boundary, with the i^{th} point marked as "forgotten," constructed in Section A.2.
- We treat any other nonmain component as belonging to $\mathcal{R}^k$ as usual.

Thus, the codimension-one boundary of the Deligne–Mumford compactification is covered by the natural inclusions of the strata

$$\overline{\mathcal{R}}^m \times_i \overline{\hat{\mathcal{R}}}{}^1_{d-m+1}, \quad \text{with } 1 \le i < d-m+1, \tag{5-18}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} \overline{\check{\mathcal{R}}}{}^1_{d-m+1}, \quad \text{with } 1 \le j \le m,\ 1 \le k \le m, \tag{5-19}$$

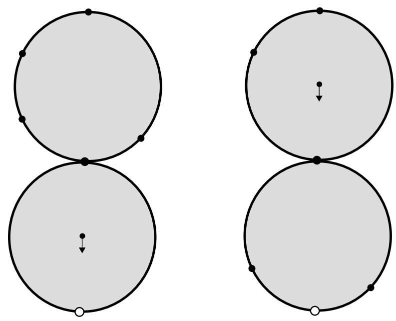

Figure 4: A schematic of the two distinct types of codimension-one boundary strata of (5-17) in codimension one. On the left side, corresponding to (5-18), a disc bubble forms involving any collection of boundary marked points not including z_f. On the right side, corresponding to (5-19), a disc bubble forms involving the point z_f.

where the notation $\times_j$ means that the output of the first component is identified with the j^{th} boundary input of the second. See Figure 4.

The forgetful map π_f extends to a map $\overline{\pi}_f$ from the compactification $\overline{\hat{\mathcal{R}}}{}^1_d$ (to the space of stable framed open–closed discs with d marked points) as follows: $\overline{\pi}_f$ puts the auxiliary point z_f back in, eliminates any component which is not main or secondary and which has only one nonauxiliary marked point p, and labels the positive marked point below this component by p. Given a representative S of $\overline{\hat{\mathcal{R}}}{}^1_d$, we call $\overline{\pi}_f(S)$ the *associated reduced surface*. We will study maps from the associated reduced surfaces $\overline{\pi}_f(S)$, parametrized by S. To this end, define a *Floer datum* on a stable disc S in $\overline{\hat{\mathcal{R}}}{}^1_d$ to consist of a Floer datum for the underlying reduced surface $\overline{\pi}_f(S)$ in the sense of Section 5.1.

First, in Section A.2 we describe an inductive construction of Floer data for (the underlying reduced surfaces of) the compactified moduli space of discs with a forgotten point $\overline{\mathcal{R}}^{d,f_i}$, for every d and i, with the following properties:

$$\begin{cases} \text{For } d > 2, \text{ the choice of Floer datum on } \mathcal{R}^{d,f_i} \text{ should be pulled back from} \\ \text{the forgetful map } \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}. \\ \text{For } d = 2, \text{ the Floer datum on the surface } S \text{ (with } z_i \text{ forgotten) should be} \\ \text{translation invariant.} \end{cases} \tag{5-20}$$

Next, we choose a *Floer datum for the nonunital open–closed map*, which is an inductive set of choices $(\boldsymbol{D}_{\check{\mathcal{OC}}}, \boldsymbol{D}_{\hat{\mathcal{OC}}})$, for each $d \geq 1$ and every representative $S \in \overline{\check{\mathcal{R}}}{}^1_d$, $T \in \overline{\hat{\mathcal{R}}}{}^1_d$, of a Floer datum for S and (the associated reduced surface of) T, respectively. As usual, these choices should be smoothly varying, and restrict smoothly to previously chosen Floer data on boundary strata. Note that for a given d the boundary strata have components that are either $\overline{\check{\mathcal{R}}}^1_{d'}$ or $\overline{\hat{\mathcal{R}}}^1_{d'}$ for $d' < d$, a stratum $\mathcal{R}^{d'}$ (over which we have chosen a Floer datum for the A_∞ structure), or a stratum $\mathcal{R}^{d,f_i}$ where we have chosen a Floer datum in Section A.2 as described above. (As usual for Liouville manifolds we use the notion of Floer data and consistency described in Remark 5.1 or Remark 5.2 in the wrapped case.) Contractibility of the space of choices at every stage (and consistency of the compatibility conditions imposed at corners) ensures as usual that a Floer datum for the nonunital open–closed map exists.

Fixing such a choice, we obtain, for any d–tuple of Lagrangians $L_0, \dots, L_{d-1}$, and asymptotic conditions $\vec{x} = (x_d, \dots, x_1)$ with $x_i \in \chi(L_{i-1}, L_{i \bmod d})$ and $y_{\text{out}} \in \mathcal{O}$, a pair of moduli spaces

$$\check{\mathcal{R}}^1_d(y_{\text{out}}; \vec{x}), \tag{5-21}$$

$$\hat{\mathcal{R}}^1_d(y_{\text{out}}; \vec{x}), \tag{5-22}$$

of parametrized families of solutions to Floer's equation,

$$\big\{(S,u) \mid S \in \check{\mathcal{R}}^1_d, u\colon S \to M \text{ such that } (du - X \otimes \alpha)^{0,1} = 0 \\ \text{using the Floer datum given by } \boldsymbol{D}_{\check{\mathcal{OC}}}(S)\big\}, \tag{5-23}$$

$$\big\{(S,u) \mid S \in \hat{\mathcal{R}}^1_d, u\colon \pi_f(S) \to M \text{ such that } (du - X \otimes \alpha)^{0,1} = 0 \\ \text{using the Floer datum given by } \boldsymbol{D}_{\hat{\mathcal{OC}}}(S)\big\}, \tag{5-24}$$

satisfying asymptotic and boundary conditions (in either case) as in (5-5), with the modification for wrapped Fukaya categories involving Liouville rescalings described in Remark 5.2. The expected dimensions of every component of (5-21) and (5-22), respectively, in the $\mathbb{Z}$–graded case are

$$\deg(y_{\text{out}}) - n + d - 1 - \sum_{k=1}^{d} \deg(x_k), \tag{5-25}$$

$$\deg(y_{\text{out}}) - n + d - \sum_{k=1}^{d} \deg(x_k), \tag{5-26}$$

and mod 2 these in the $\mathbb{Z}/2$–graded case.

As usual there are Gromov-type bordifications

$$\overline{\check{\mathcal{R}}}^1_d(y_{\text{out}}; \vec{x}), \tag{5-27}$$

$$\overline{\hat{\mathcal{R}}}^1_d(y_{\text{out}}; \vec{x}), \tag{5-28}$$

which allow semistable breakings, as well as maps from strata corresponding to the boundary strata of $\overline{\check{\mathcal{R}}}{}^1_d$ and $\overline{\hat{\mathcal{R}}}{}^1_d$.

By Assumption 3.10, for generic choices of Floer datum for the nonunital open–closed map, the components of (5-27) and (5-28) of virtual dimension ≤ 1 are compact manifolds-with-boundary of dimension agreeing with virtual dimension. Fix such a Floer datum. At a rigid element u of each of the above moduli spaces, we obtain, using the fixed orientations of moduli spaces of domains (5-11)–(5-14) and [1, Lemma C.4], isomorphisms of orientation lines

$$(\check{\mathcal{R}}^1_d)_u \colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{\mathrm{out}}}, \tag{5-29}$$

$$(\hat{\mathcal{R}}^1_d)_u \colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{\mathrm{out}}}. \tag{5-30}$$

These isomorphisms in turn define the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of the check and hat components of the nonunital open–closed map with d inputs in the lines $|o_{x_d}|_{\boldsymbol{k}}, \dots, |o_{x_1}|_{\boldsymbol{k}}$, up to a sign twist:

$$\check{\mathcal{OC}}_d([x_d], \dots, [x_1]) := \sum_{u \in \overline{\check{\mathcal{R}}}{}^d_1(y; x_d, \dots, x_1) \text{ rigid}} (-1)^{\check{\star}_d} (\check{\mathcal{R}}^1_d)_u([x_d], \dots, [x_1]), \tag{5-31}$$

where $\check{\star}_d := \deg(x_d) + \sum_{k=1}^d k \deg(x_k)$, and

$$\hat{\mathcal{OC}}_d([x_d], \dots, [x_1]) := \sum_{u \in \overline{\hat{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\hat{\star}_d} (\hat{\mathcal{R}}^1_d)_u([x_d], \dots, [x_1]), \tag{5-32}$$

where $\hat{\star}_d := \sum_{i=1}^d i \cdot \deg(x_i)$.

By analyzing the boundary of one-dimensional components of the moduli spaces $\overline{\check{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}; \vec{x})$, the consistency condition imposed on Floer data, and a sign analysis, in [1] it was proved that:

Lemma 5.7 [1, Lemma 5.4] *The map $\mathcal{OC} := \check{\mathcal{OC}}$ is a chain map of degree n; that is, $(-1)^n d_{CF} \circ \check{\mathcal{OC}} = \check{\mathcal{OC}} \circ b$.* □

Similarly, we prove the following, completing the proof of Lemma 5.3:

Lemma 5.8 *The following equation holds*:

$$(-1)^n d_{CF} \circ \hat{\mathcal{OC}} = \check{\mathcal{OC}} \circ d_{\wedge\vee} + \hat{\mathcal{OC}} \circ b'. \tag{5-33}$$

Proof The consistency condition imposed on Floer data implies that the boundary of the one-dimensional components of $\overline{\check{\mathcal{R}}}^1_d(y;\vec{x})$ are covered by the images of the natural inclusions of the rigid (zero-dimensional) components of the moduli spaces of maps coming from the boundary strata (5-18) and (5-19) along with (the rigid components of) semistable breakings,

$$\overline{\hat{\mathcal{R}}}^1_d(y_1;\vec{x})\times\overline{\mathcal{M}}(y_{\text{out}};y_1)\to\partial\overline{\check{\mathcal{R}}}^1_d(y_{\text{out}};\vec{x}), \tag{5-34}$$

$$\overline{\mathcal{R}}^1(x;x_i)\times\overline{\hat{\mathcal{R}}}^1_d(y_{\text{out}};\tilde{\vec{x}})\to\partial\overline{\check{\mathcal{R}}}^1_d(y_{\text{out}};\vec{x}), \tag{5-35}$$

where $\tilde{\vec{x}}$ denotes the collection of inputs $\vec{x}$ with x_i replaced with x. Let $\mu^{d,i}$ be the operation associated to the space of discs with i^{th} point marked as forgotten $\mathcal{R}^{d,f_i}$, which is described in detail in Section A.2. The operation $\mu^{d,i}$ takes a composable sequence of $d-1$ inputs, separated into an $i-1$ tuple and a $d-i$ tuple; in line with Remark 3.3 we will use the suggestive notation[15]

$$\mu^d(x_d,\dots,x_{i+1},e^+,x_{i-1},\dots,x_1):=\mu^{d,i}(x_d,\dots,x_{i+1};x_{i-1},\dots,x_1). \tag{5-36}$$

(Recall the abuse of notation $x_i:=[x_i]$.) Then, up to sign, by the standard codimension-one boundary principle for Floer-theoretic operations, we have shown that

$$\begin{aligned}0=d_{CF}\hat{\mathcal{OC}}(x_d,\dots,x_1)&\\ -\sum_{i,j}(-1)^{\maltese_1^i}\hat{\mathcal{OC}}(x_d,\dots,x_{i+j+1},\mu^j(x_{i+j},\dots,x_{i+1}),x_i,\dots,x_1)&\\ -\sum_{i,j,k}(-1)^{\sharp_j^k}\check{\mathcal{OC}}\big(\mu^{j+k+1}(x_j,\dots,x_1,e^+,x_d,\dots,x_{d-k+1}),&\\ x_{d-k},\dots,x_{j+1}\big),&\end{aligned} \tag{5-37}$$

with desired signs

$$\maltese_m^n=\sum_{j=m}^n\|x_i\|, \tag{5-38}$$

$$\sharp_j^k=\maltese_1^j\maltese_{j+1}^d+\maltese_{j+1}^d+1. \tag{5-39}$$

However, as shown in Section A.2,

$$\mu^{j+k+1}(x_j,\dots,x_1,e^+,x_d,\dots,x_{d-k+1})=\begin{cases}x_1 & \text{if } j=1,\ k=0,\\ (-1)^{|x_d|}x_d & \text{if } j=0,\ k=1,\\ 0 & \text{otherwise.}\end{cases} \tag{5-40}$$

[15] In fact, when the Fukaya category is equipped with homotopy units, one can ensure that there is a strict unit element e^+ in each self-hom space for which μ^k with an e^+ element admits a geometric description as above. See eg [22] or [24].

(In this manner, e^+, though a formal element, behaves as a strict unit.) So if equation (5-37) held, it would follow that

$$
\begin{aligned}
\text{(5-41)}\quad & d_{CF} \circ \widehat{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1) \\
&= (-1)^{\|x_1\| \maltese_2^d + \maltese_2^d + 1} \check{\mathcal{OC}}(x_1 \otimes x_d \otimes \cdots \otimes x_2) \\
&\qquad + (-1)^{|x_d| + \maltese_1^d + 1} \check{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1) + \widehat{\mathcal{OC}} \circ b'(x_d \otimes \cdots \otimes x_1) \\
&= \check{\mathcal{OC}}((-1)^{\maltese_1^d + \|x_d\|}(1-t)(x_d \otimes \cdots \otimes x_1)) + \widehat{\mathcal{OC}} \circ b'(x_d \otimes \cdots \otimes x_1) \\
&= (\check{\mathcal{OC}} \circ d_{\wedge\vee} + \widehat{\mathcal{OC}} \circ b')(x_d \otimes \cdots \otimes x_1).
\end{aligned}
$$

So we are done if we establish that the signs are exactly (5-38)–(5-39).

Using the notation

$$
\text{(5-42)}\qquad \mathcal{OC}(e^+ \otimes x_d \otimes \cdots \otimes x_1) := \widehat{\mathcal{OC}}(x_d \otimes \cdots \otimes x_1),
$$

where again e^+ is simply a formal symbol referring to the position of the auxiliary (forgotten) input point, we observe that the equation (5-37) is exactly the equation for $\mathcal{OC}$ being a chain map on inputs of the form $(e^+ \otimes x_d \otimes \cdots \otimes x_1)$, where we treat an "$e^+$" input as an auxiliary unconstrained point on our domain. The sign verification therefore follows from that of $\check{\mathcal{OC}}$ being a chain map (in [1, Lemma 5.4]), for we have used identical orientations on the abstract moduli space $\widehat{\mathcal{R}}_d^1$ as on $\check{\mathcal{R}}_{d+1}^1$ (identifying z_f with z_{d+1}), and on $\mathcal{R}^{d,f_i}$ as on $\mathcal{R}^d$, and we can even insert a formal degree zero orientation line o_{e^+} into the procedure for orienting moduli spaces of open–closed maps (see [1, Section C.6]), corresponding to the marked point (obtained by filling in) z_f. Note that o_{e^+}, being of degree zero, commutes with everything, and is just used as a placeholder as if we had an asymptotic condition at z_f. □

Proof of Lemma 5.3 As $\check{\mathcal{OC}}$ is already known to be a chain map by [1, Lemma 5.4], repeated as Lemma 5.7 above, the new part to check is that

$$
(-1)^n d_{CF} \circ \widehat{\mathcal{OC}} = \check{\mathcal{OC}} d_{\wedge\vee} + \widehat{\mathcal{OC}} \circ b'.
$$

This is the content of Lemma 5.8 above. □

5.3 An auxiliary operation

It will be technically convenient to define an auxiliary operation

$$
\text{(5-43)}\qquad \mathcal{OC}^{S^1} : \mathrm{CH}_{*-n}(\mathcal{F}) \to CH^{*+1}(M)
$$

from the left factor of the nonunital Hochschild complex to Floer cochains, in which the asymptotic marker τ_{out} varies freely around the circle. This operation is more easily comparable to the BV operator on Floer cohomology, and moreover, we will show that $\mathcal{OC}^{S^1}$ (and $\hat{\mathcal{OC}}$) can be chosen to satisfy the following crucial identity:

Proposition 5.9 *There is an equality of chain-level operations*,

$$\mathcal{OC}^{S^1} = \hat{\mathcal{OC}} \circ B^{\mathrm{nu}}. \tag{5-44}$$

To define (5-43), let

$$\mathcal{R}_d^{S^1} \tag{5-45}$$

be the abstract moduli space of discs with d boundary positive punctures $z_1, \dots, z_d$ labeled in counterclockwise order and one interior negative puncture z_{out}, with an asymptotic marker τ_{out} at z_{out} (or choice of real half-line in $T_{z_{\mathrm{out}}} D$) which is free to vary. Equivalently,

(5-46) $\mathcal{R}_d^{S^1}$ is the space of discs with $z_1, \dots, z_d$ and z_{out} as before, and an extra auxiliary interior marked point p_1 such that, for a representative with (z_{out}, z_1) fixed at $(0, -i)$, $|p_1| = \frac{1}{2}$, and the asymptotic marker τ_{out} points towards p_1.

By using a representative with fixed (z_{out}, z_1) as above, the argument of p_1 produces an abstract identification

$$\mathcal{R}_d^{S^1} = \check{\mathcal{R}}_d^1 \times S^1. \tag{5-47}$$

Using this identification, fix an orientation of (5-47) given by negative the product orientation of (5-11) with the standard counterclockwise orientation on S^1. The Deligne–Mumford-type compactification can thus be thought of as

$$\overline{\mathcal{R}}_d^{S^1} = \overline{\check{\mathcal{R}}}_d^1 \times S^1. \tag{5-48}$$

Given an element S of $\mathcal{R}_d^{S^1}$ and a choice of marked point z_i on the boundary of S, we say that τ_{out} *points at* z_i, if, when S is reparametrized so that z_1 fixed at $-i$ and z_{out} fixed at 0, the vector τ_{out} is tangent to the straight line from z_{out} to z_i. Equivalently, for this representative, z_{out}, p_1 and z_i are colinear. For each i, the locus where τ_{out} points at z_i forms a codimension-one submanifold, denoted by

$$\mathcal{R}_d^{S_i^1}. \tag{5-49}$$

The notion compactifies well; if z_i is not on the main component of (5-48), we say that τ_{out} *points at* z_i if it points at the root of the bubble tree z_i is on. This compactified locus $\overline{\mathcal{R}}_d^{S_i^1}$ can be identified with $\overline{\mathcal{R}}_d^1$ via the map

$$\tau_i : \overline{\mathcal{R}}_d^{S_i^1} \to \overline{\mathcal{R}}_d^1 \tag{5-50}$$

which cyclically permutes the labels of the boundary marked points so that z_i is now labeled z_d.

In a similar fashion, we have an invariant notion of what it means for τ_{out} to point *between z_i and z_{i+1}*; this is a codimension-zero submanifold with corners of (5-47), denoted by

$$\mathcal{R}_d^{S^1_{i,i+1}}. \tag{5-51}$$

The compactification has some components that are codimension-one submanifolds with corners of (5-48), when z_i and z_{i+1} both lie on a bubble tree.

Finally, there is a *free* $\mathbb{Z}_d$–action generated by the map

$$\kappa\colon \overline{\mathcal{R}}_d^{S^1} \to \overline{\mathcal{R}}_d^{S^1} \tag{5-52}$$

which cyclically permutes the labels of the boundary marked points; for concreteness, κ changes the label z_i to z_{i+1} for $i < d$, and z_d to z_1. Note that if, on a given S, τ_{out} points between z_i and z_{i+1}, then on $\kappa(S)$, τ_{out} points between $z_{i+1 \bmod d}$ and $z_{i+2 \bmod d}$.

Lemma 5.10 *The action generated by* (5-52) *is free and properly discontinuous.*

Sketch The basic observation arises on the level of uncompactified moduli spaces: since any element of $\mathcal{R}_d^{S^1}$ has a unit disk representative with (z_{out}, p_1) fixed at $\bigl(0, \frac{1}{2}\bigr)$, the positions of the remaining points identify $\mathcal{R}_d^{S^1}$ with the space of tuples $(z_1, \dots, z_d)$ of disjoint (cyclically ordered) points on S^1 (without any further quotienting by automorphism). The action of κ, which cyclically permutes the labels $z_1, \dots, z_d$ in this identification, evidently acts freely and properly discontinuously on this locus. Similarly, an element of a boundary stratum consists of an element of $\mathcal{R}_k^{S^1}$ for some $k \le d$ with some collection of stable disc bubble trees attached to some or all of the marked points of $\mathcal{R}_k^{S^1}$, so that there are d leaf (nonnodal) boundary marked points, along with a counterclockwise ordered labeling of these marked points by $z_1, \dots, z_d$ (note that there is a well-defined cyclic counterclockwise ordering of boundary marked points on any such stable configuration). By using a representative of the main component $\mathcal{R}_k^{S^1}$ with (z_{out}, p_1) fixed at $\bigl(0, \frac{1}{2}\bigr)$, an explicit analysis shows that the action of (5-52) remains free and properly discontinuous — for instance, to see free, note that there is a well-defined "first boundary nonnodal marked point at or counterclockwise from the argument of p_1"; the action of (5-52) freely permutes the label of this first boundary marked point hence cannot have a fixed point. □

The quotient of the action of κ consists of the space of discs with z_{out} and p_1 as before,[16] equipped with d *cyclically unordered* or *unlabeled* boundary marked points. Note that on the open-locus $\mathring{\mathcal{R}}_d^{S^1}$, where τ_{out} does not point at a boundary marked point, one can choose a labeling by setting the boundary point immediately clockwise of where τ_{out} points to be z_d. This induces a diffeomorphism

$$\mathring{\mathcal{R}}_d^{S^1}/\kappa \cong \mathcal{R}_d^{1,\mathrm{free}}. \tag{5-53}$$

Similarly, on the complementary locus where τ_{out} points at a boundary marked point, we can similarly choose a labeling by declaring this boundary marked point to be z_d, giving a diffeomorphism (of this locus) with $\check{\mathcal{R}}_d^1$.

We now choose Floer perturbation data for the family of moduli spaces $\mathcal{R}_d^{S^1}$; in fact, it will be helpful to rechoose Floer data for the moduli spaces appearing in the nonunital open–closed map to have extra compatibility. To that end, a *BV compatible Floer datum for the nonunital open–closed map* is an inductive choice $(\boldsymbol{D}_{\check{\mathcal{OC}}}, \boldsymbol{D}_{\hat{\mathcal{OC}}}, \boldsymbol{D}_{S^1})$ of Floer data where $\boldsymbol{D}_{\check{\mathcal{OC}}}$ and $\boldsymbol{D}_{\hat{\mathcal{OC}}}$ is a universal and consistent choice of Floer data for the nonunital open–closed map as before, and $\boldsymbol{D}_{S^1}$ consists of, for each $d \geq 1$ and every representative $S \in \overline{\mathcal{R}}_d^{S^1}$, a Floer datum for S varying smoothly over the moduli space. Again, these satisfy the usual consistency condition with respect to previously made choices along lower-dimensional strata. Moreover, there are two additional inductive constraints on the Floer data chosen:

(5-54) On the codimension-one loci $\overline{\mathcal{R}}_d^{S_i^1}$ where τ_{out} points at z_i, the Floer datum should agree with the pullback by τ_i of the existing Floer datum for the (check) open–closed map.

(5-55) The Floer datum should be κ–equivariant, where κ is the map (5-52).

Also, there is a final a posteriori constraint on the Floer data for the nonunital open–closed map $\boldsymbol{D}_{\hat{\mathcal{OC}}}$: for $S \in \overline{\hat{\mathcal{R}}}_d^1$:

(5-56) The Floer datum on the main component S_0 of $\overline{\pi}_f(S)$ should coincide with the existing datum chosen on $S_0 \in \mathcal{R}_d^{1,\mathrm{free}} \subset \mathcal{R}_d^{S^1}$.

By an inductive argument as before, a BV compatible Floer datum for the nonunital open–closed map exists.

To explain the way choices are made (which ensures both existence at every stage and that the requirements above are satisfied): we choose the data for $\mathcal{R}_d^{S^1}$ prior to

[16] Meaning z_{out} is a negative interior puncture, and p_1 is an auxiliary interior marked point such that for any representative with z_{out} fixed at 0, $|p_1| = \frac{1}{2}$.

choosing that of $\bar{\bar{\mathcal{R}}}_d^1$ and note that the condition (5-56) specifies the Floer datum on $\bar{\bar{\mathcal{R}}}_d^1$ entirely. In particular, the conditions (5-20) required on the latter Floer datum are compatible with consistency and the condition (5-54). With regards to choosing the data for $\mathcal{R}_d^{S^1}$, the equivariance constraint (5-55), which is compatible with both (5-54) (a κ–equivariant condition) and with the consistency condition, is also unproblematic in light of Lemma 5.10: one can pull back a Floer datum from the quotient of $\overline{\mathcal{R}}_d^{S^1}$ by κ.

Fixing a BV compatible Floer datum for the nonunital open–closed map we obtain, for any d–tuple of Lagrangians $L_0, \dots, L_{d-1}$, and asymptotics $\vec{x} = (x_d, \dots, x_1)$ with $x_i \in \chi(L_{i-1}, L_{i \bmod d})$, and $y_{\mathrm{out}} \in \mathcal{O}$, a moduli space

$$\mathcal{R}_d^{S^1}(y_{\mathrm{out}}; \vec{x}) \tag{5-57}$$

of parametrized families of solutions to Floer's equation, with respect to the Floer data chosen,

$$\{(S, u) \mid S \in \mathcal{R}_d^{S^1}, u \colon \pi_f(S) \to M \text{ such that } (du - X \otimes \alpha)^{0,1} = 0\}, \tag{5-58}$$

satisfying asymptotic and boundary conditions as in (5-5) (again with the modifications of Remarks 5.1 or 5.2 for compact or wrapped Fukaya categories of Liouville manifolds). Generically the Gromov–Floer compactifications

$$\overline{\mathcal{R}}_d^{S^1}(y_{\mathrm{out}}; \vec{x}) \tag{5-59}$$

of the components of virtual dimension ≤ 1 are compact manifolds-with-boundary of the expected dimension; this dimension coincides (mod 2 or exactly depending on whether we are in a $\mathbb{Z}/2$– or $\mathbb{Z}$–graded setting) with

$$\deg(y_{\mathrm{out}}) - n + d - \sum_{k=1}^{d} \deg(x_k). \tag{5-60}$$

Each rigid $u \in \overline{\mathcal{R}}_d^{S^1}(y_{\mathrm{out}}; \vec{x})$ gives by the orientation from (5-48) and [1, Lemma C.4] an isomorphism of orientation lines

$$(\mathcal{R}_d^{S^1})_u \colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{\mathrm{out}}}, \tag{5-61}$$

which gives the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of the S^1 open–closed map with d inputs in the lines $|o_{x_d}|_{\boldsymbol{k}}, \dots, |o_{x_1}|_{\boldsymbol{k}}$, up to a sign twist given below: define

$$\mathcal{OC}^{S^1}([x_d], \dots, [x_1]) := \sum_{u \in \overline{\mathcal{R}}_d^{S^1}(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\clubsuit_d} (\mathcal{R}_d^{S^1})_u([x_d], \dots, [x_1]), \tag{5-62}$$

where $\clubsuit_d = \sum_{i=1}^{d} (i+1) \cdot \deg(x_i) + \deg(x_d) + d - 1$.

The proof of Proposition 5.9, which equates $\mathcal{OC}^{S^1}$ with $\hat{\mathcal{OC}} \circ B^{\mathrm{nu}}$, appears below and is composed of two steps. First, we decompose the moduli space $\mathcal{R}_d^{S^1}$ into sectors in which τ_{out} points between a pair of adjacent boundary marked points. It will follow that the sum of the corresponding "sector operations" is exactly $\mathcal{OC}^{S^1}$. The sector operations in turn can be compared to $\hat{\mathcal{OC}}$ via cyclically permuting inputs and an orientation analysis.

We begin by defining the relevant sector operations: For $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, define

$$\hat{\mathcal{R}}^1_{d,\tau_i} \tag{5-63}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures $z_1, \dots, z_i$, z_f, $z_{i+1}, \dots, z_d$ arranged in counterclockwise order and interior puncture z_{out} with asymptotic marker pointing towards the boundary point z_f, which is also marked as "auxiliary". There is a bijection

$$\tau_i \colon \hat{\mathcal{R}}^1_{d,\tau_i} \simeq \hat{\mathcal{R}}^1_d \tag{5-64}$$

given by cyclically permuting labels, inducing a model for the compactification $\bar{\hat{\mathcal{R}}}^1_{d,\tau_i}$. However, we will use a different orientation than the one induced by pullback: on a slice with fixed position of z_d and z_{out}, we take the volume form

$$dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f. \tag{5-65}$$

By construction, the induced "forgetful map"

$$\pi^i_f \colon \hat{\mathcal{R}}^1_{d,\tau_i} \to \mathcal{R}^{S^1_{i,i+1}} \tag{5-66}$$

is an oriented diffeomorphism that extends to a map between compactifications. Note as before that strictly speaking this map does not forget any information, at least on the open locus.

Remark 5.11 In the case $i = 0$, this orientation agrees with the previously chosen orientation (5-14) on $\hat{\mathcal{R}}^1_d$. We previously defined the orientation on $\hat{\mathcal{R}}^1_d$ in terms of a different slice of the group action. To compare the forms $dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f$ (coming from the slice with fixed z_d and z_{out}) and $-dz_1 \wedge \cdots \wedge dz_d$ (coming from the slice with fixed z_f and z_{out}), note that either orientation is induced by the following procedure:

- Fixing an orientation on the space of discs as above with fixed position of z_{out} (but not z_f or z_d): we shall fix the canonical orientation $dz_1 \wedge \cdots \wedge dz_d \wedge dz_f$.

- Fixing a choice of trivializing vector field for the remaining S^1–action on this space of discs with fixed z_{out}: we shall fix $S = (-\partial_{z_f} - \partial_{z_1} - \cdots - \partial_{z_d})$.
- Fixing a convention for contracting orientation forms along slices of the action: to determine the orientation on a slice of an S^1–action, we will contract the orientation on the original space on the right by the trivializing vector field.

Moreover, this data induces an orientation on the quotient by the S^1–action, and also an oriented isomorphism between the induced orientation on any slice and that of the quotient. It follows that on the quotient, the orientation $-dz_1 \wedge \cdots \wedge dz_d$ (from the slice where z_f is fixed) and the orientation $dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_f$ (from the slice where z_d is fixed) agree. We conclude that these two orientations agree. The author thanks Nick Sheridan for relevant discussions about orientations of moduli spaces.

Choose as a Floer datum for each $\overline{\mathcal{R}}^1_{d,\tau_i}$ the Floer datum pulled back from $\overline{\hat{\mathcal{R}}}{}^1_d$ via (5-64); this system of choices is automatically inductively consistent with choices made on lower strata, inheriting this property from the Floer data on the collection of $\overline{\hat{\mathcal{R}}}{}^1_d$. Using this choice, for any d–tuple of Lagrangians $L_0, \dots, L_{d-1}$, and asymptotic conditions $\vec{x} = (x_d, \dots, x_1)$, with $x_i \in \chi(L_{i-1}, L_{i \bmod d})$, and $y_{\mathrm{out}} \in \mathcal{O}$, we obtain a moduli space

$$\mathcal{R}^1_{d,\tau_i}(y_{\mathrm{out}}; \vec{x}) = \hat{\mathcal{R}}^1_d(y_{\mathrm{out}}; (x_{i-1}, \dots, x_1, x_d, \dots, x_i)) \tag{5-67}$$

of parametrized families of solutions to Floer's equation,

$$\bigl\{(S, u) \mid S \in \hat{\mathcal{R}}^1_d,\ u\colon \pi_f(S) \to M \text{ is such that } (du - X \otimes \alpha)^{0,1} = 0 \\ \text{using the Floer datum for } \pi_f(S)\bigr\}, \tag{5-68}$$

satisfying asymptotic and boundary conditions as in (5-5) (with the modifications as in Remarks 5.1 or 5.2 in the Liouville case), as well as its Gromov–Floer compactification

$$\overline{\mathcal{R}}^1_{d,\tau_i}(y_{\mathrm{out}}; \vec{x}) := \overline{\hat{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}; (x_i, \dots, x_1, x_d, \dots, x_{i+1})), \tag{5-69}$$

whose components of virtual dimension ≤ 1 (at least) are compact manifolds-with-boundary of the correct dimension, coinciding (exactly in the graded case and mod 2 in the $\mathbb{Z}/2$–graded case) with $\deg(y_{\mathrm{out}}) - n + d - \sum_{j=0}^{d} \deg(x_j)$.

Each rigid element $u \in \overline{\mathcal{R}}^1_{d,\tau_i}(y_{\mathrm{out}}; \vec{x})$ gives, by (5-65) and [1, Lemma C.4], an isomorphism of orientation lines

$$(\mathcal{R}^1_{d,\tau_i})_u\colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{\mathrm{out}}}, \tag{5-70}$$

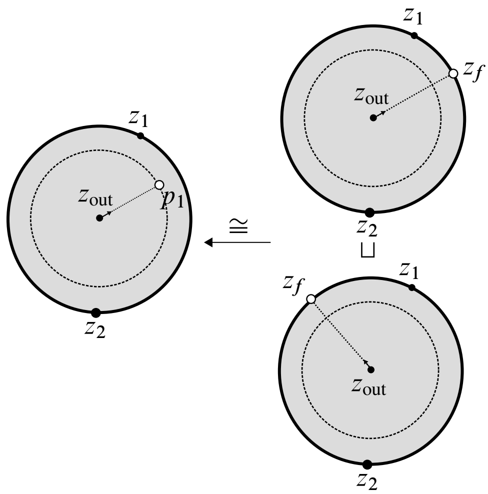

Figure 5: The diffeomorphism between $\widehat{\mathcal{R}}^1_{2,\tau_0} \cup \widehat{\mathcal{R}}^1_{2,\tau_1}$ and the open dense part of $\mathcal{R}_2^{S^1}$ given by $\mathcal{R}_2^{S^1_{0,1}} \cup \mathcal{R}_2^{S^1_{1,2}}$. The former spaces can in turn be compared to $\widehat{\mathcal{R}}^1_2$ via cyclic permutation of labels.

which defines the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of an operation $\widehat{\mathcal{OC}}_{d,\tau_i}$, with d inputs in the lines $|o_{x_d}|_{\boldsymbol{k}}, \dots, |o_{x_1}|_{\boldsymbol{k}}$, up to the following sign twist:

$$\widehat{\mathcal{OC}}_{d,\tau_i}([x_d], \dots, [x_1]) := \sum_{u \in \overline{\widehat{\mathcal{R}}}^1_{d,\tau_i}(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\clubsuit_d} (\widehat{\mathcal{R}}^1_{d,\tau_i})_u([x_d], \dots, [x_1]), \tag{5-71}$$

where $\clubsuit_d = \sum_{i=1}^d (i+1) \cdot \deg(x_i) + \deg(x_d) + d - 1$.

Lemma 5.12 *As chain-level operations,*

$$\mathcal{OC}^{S^1} = \sum_i \widehat{\mathcal{OC}}_{d,\tau_i}. \tag{5-72}$$

Proof For each d, there is an embedding of abstract moduli spaces

$$\coprod_i \widehat{\mathcal{R}}^1_{d,\tau_i} \xrightarrow{\coprod_i \pi_f^i} \coprod_i \mathcal{R}_d^{S^1_{i,i+1}} \hookrightarrow \mathcal{R}_d^{S^1}. \tag{5-73}$$

See Figure 5.

By construction, this map is compatible with Floer data (this uses the fact that the Floer data on $\mathcal{R}^{S^1_{i,i+1}}$ agrees with the data on $\widehat{\mathcal{R}}^1_d$ via the reshuffling map κ^{-i} by (5-55)), and covers all but a codimension-one locus in the target. Since, after perturbation, zero-dimensional solutions to Floer's equation can be chosen to come from the complement of any codimension-one locus in the source abstract moduli space, we conclude that the two operations in the lemma, which arise from either side of (5-73), are identical up to sign. To fix the signs, note that (5-73) is in fact an oriented embedding, and all the sign twists defining the operations $\widehat{\mathcal{OC}}_{d,\tau_i}$ are chosen to be compatible with the sign twist in the operation $\mathcal{OC}^{S^1}$. $\square$

Next, because the Floer data used in the constructions are identical, we have that $\widehat{\mathcal{OC}}_{d,\tau_i}(x_d \otimes \cdots \otimes x_1) := \widehat{\mathcal{OC}}_{d,\tau_i}(x_d, \dots, x_1)$ (recall the abuse of notation $x_i := [x_i]$) agrees with $\widehat{\mathcal{OC}}(x_i \otimes \cdots \otimes x_1 \otimes x_d \otimes \cdots \otimes x_{i+1}) := \widehat{\mathcal{OC}}(x_i, \dots, x_1, x_d, \dots, x_{i+1})$ up to a sign difference coming from orientations of abstract moduli spaces, cyclically reordering inputs, and sign twists. The following proposition computes the sign difference, and hence completes the proof of Proposition 5.9:

Lemma 5.13 *There is an equality*

$$\widehat{\mathcal{OC}}_{d,\tau_i}(x_d \otimes \cdots \otimes x_1) = \widehat{\mathcal{OC}}^d(s^{\mathrm{nu}}(t^i(x_d \otimes \cdots \otimes x_1))), \tag{5-74}$$

where s^{nu} is the operation (3-20) *arising from changing a check term to a hat term with a sign twist.*

Proof It is evident that $\widehat{\mathcal{OC}}_{d,\tau_i}$ agrees with $\widehat{\mathcal{OC}}_d \circ s^{\mathrm{nu}} \circ t^i$ up to sign, as the Floer data used in the two constructions are identical. By an inductive argument it suffices to verify the equalities of signed operations

$$\widehat{\mathcal{OC}}_{d,\tau_0} = \widehat{\mathcal{OC}}_d \circ s^{\mathrm{nu}}, \tag{5-75}$$

$$\widehat{\mathcal{OC}}_{d,\tau_1} = \widehat{\mathcal{OC}}_{d,\tau_0} \circ t, \tag{5-76}$$

the remaining sign changes being entirely incremental. For the equality (5-75), we simply note that the signs appearing in the operations $\widehat{\mathcal{OC}}_{d,\tau_0}([x_d], \dots, [x_1])$ and $\widehat{\mathcal{OC}}_d([x_d], \dots, [x_1])$ differ in the following fashions:

- The abstract orientations on the moduli space of domains agree, as in Remark 5.11.
- The difference in sign twists is given by
$$\clubsuit_d - \hat{\star}_d = \sum_{i=1}^{d} |x_i| + |x_d| + d - 1 = \left(\sum_{i=1}^{d} \|x_i\|\right) + 1 + |x_d| = \maltese_1^d + \|x_d\|.$$

All together, the parity of difference in signs is $\maltese_1^d + \|x_d\|$, which accounts for the sign in the algebraic operation s^{nu} (see (3-20)); this verifies (5-75).

Next, the sign difference between the two operations in the equality (5-76) is a sum of three contributions:

- The two orientations of abstract moduli spaces[17] from $-dz_1 \wedge \cdots \wedge dz_d$ to $dz_2 \wedge \cdots \wedge dz_d \wedge dz_1$ differ by a sign change of parity
$$d-1.$$
- For a given collection of inputs, the change in *sign twisting data* from $\clubsuit_d = \sum_{i=1}^{d}(i+1)\cdot|x_i| + |x_d| + d - 1$ to $\sum_{i=1}^{d-1}(i+1)|x_{i+1}| + (d+1)|x_1| + |x_1| + d-1 = \sum_{i=2}^{d} i|x_i| + d|x_1| + d-1$ ($\clubsuit_d$ for the sequence $(x_2,\dots,x_d,x_1)$) induces a sign change of parity
$$\begin{aligned}\sum_{i=2}^{d}|x_i| + |x_d| + d|x_1| &= \sum_{i=1}^{d}|x_i| + |x_d| + (d-1)|x_1| \\ &= \sum_{i=1}^{d}\|x_i\| + (d-1)\|x_1\| + \|x_d\| \\ &= \maltese_1^d + (d-1)\|x_1\| + \|x_d\|.\end{aligned}$$
- Finally, the reordering of determinant lines of the inputs induces a sign change of parity
$$\begin{aligned}|x_1|\cdot\biggl(\sum_{i=2}^{d}|x_i|\biggr) &= \|x_1\|\cdot\biggl(\sum_{i=2}^{d}\|x_i\|\biggr) + \sum_{i=2}^{d}\|x_i\| + (d-1)\|x_1\| + (d-1) \\ &= \|x_1\|\maltese_2^d + \maltese_1^d + d\|x_1\| + (d-1).\end{aligned}$$

The cumulative sign parity is congruent mod 2 to
$$\|x_1\|\maltese_2^d + \|x_1\| + \|x_d\|,$$
which is precisely the sign appearing in t (see (3-18)). This verifies (5-76). □

Proof of Proposition 5.9 Combine Lemmas 5.12 and 5.13; note the definition of B^{nu} given in (3-21). □

5.4 Compatibility of homology-level BV operators

Before diving into the statement of chain-level equivariance, we prove a homology-level statement. The theorem below is insufficient for studying, say, equivariant homology groups, but may be of independent interest.

[17] On the slice where z_f and z_{out} are fixed; see Remark 5.11.

Theorem 5.14 *The homology-level open–closed map $[\mathcal{OC}]$ intertwines the Hochchild and symplectic cohomology BV operators*; *that is*,

$$[\mathcal{OC}] \circ [B^{\mathrm{nu}}] = [\delta_1] \circ [\mathcal{OC}]. \tag{5-77}$$

Theorem 5.14 is an immediate consequence of the following chain-level statement:

Proposition 5.15 *The following diagram homotopy commutes*:

$$\begin{tikzcd} \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \arrow[r, hook, "\iota", "\sim"'] \arrow[d, "\check{\mathcal{OC}}"'] & \mathrm{CH}^{\mathrm{nu}}_{*-n}(\mathcal{F},\mathcal{F}) \arrow[r, "B^{\mathrm{nu}}"] & \mathrm{CH}^{\mathrm{nu}}_{*-n-1}(\mathcal{F},\mathcal{F}) \arrow[d, "\mathcal{OC}"] \\ CF^*(M) \arrow[rr, "\delta_1"] & & CF^{*-1}(M) \end{tikzcd} \tag{5-78}$$

where ι is the inclusion onto the left factor, *which is a quasi-isomorphism by Lemma 3.2. More precisely*, *there exists an operation $\check{\mathcal{OC}}^1 : \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \to CF^{*-2}(M)$ satisfying*

$$(-1)^{n+1} d\check{\mathcal{OC}}^1 + \check{\mathcal{OC}}^1 b = \hat{\mathcal{OC}} B^{\mathrm{nu}}\iota - (-1)^n \delta_1 \check{\mathcal{OC}}. \tag{5-79}$$

Proof of Theorem 5.14 Proposition 5.15 implies that $[\delta_1] \circ [\check{\mathcal{OC}}] = [\mathcal{OC}] \circ [B^{\mathrm{nu}}] \circ [\iota]$, where $\iota \colon \mathrm{CH}_{*-n}(\mathcal{F},\mathcal{F}) \to \mathrm{CH}^{\mathrm{nu}}_{*-n}(\mathcal{F},\mathcal{F})$ is the inclusion of chain complexes. But by Lemma 3.2, $[\iota]$ is an isomorphism and by Corollary 5.4, $[\check{\mathcal{OC}}] = [\mathcal{OC}]$. □

To define $\check{\mathcal{OC}}^1$, consider

$${}_1\check{\mathcal{R}}^1_d, \tag{5-80}$$

the moduli space of discs with d positive boundary marked points $z_1, \dots, z_d$ labeled in counterclockwise order, one interior negative puncture z_{out} equipped with an asymptotic marker, and one additional interior marked point p_1 (without an asymptotic marker), marked as *auxiliary*. Also, with respect to the unit disc representative of any element of this moduli space fixing z_d at 1 and z_{out} at 0 on the unit disc, p_1 should lie *inside the circle of radius* $\frac{1}{2}$, so

$$0 < |p_1| < \tfrac{1}{2}. \tag{5-81}$$

Using the above representative, one can talk about the *angle*, or *argument* of p_1

$$\theta_1 := \arg(p_1). \tag{5-82}$$

We require that with respect to the above representative:

(5-83) The asymptotic marker on z_{out} points in the direction θ_1.

For every representative $S \in {}_1\check{\mathcal{R}}^1_d$:

(5-84) Fix a negative cylindrical end around z_{out} not containing p_1, compatible with the direction of the asymptotic marker, or *equivalently compatible with the angle* θ_1.

We orient (5-80) as follows: pick, on a slice of the automorphism action which fixes the position of z_d at 1 and z_{out} at 0, the volume form

$$-r_1\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dr_1 \wedge d\theta_1. \tag{5-85}$$

The compactification of (5-80) is a real blow-up of the ordinary Deligne–Mumford compactification, in the sense of [34] (see [58] for a first discussion in the context of Floer theory), reviewed in Section A.1; this is the case $k = 1$ of the more general description therein. The result of this discussion is that the codimension-one boundary of the compactified check moduli space ${}_1\overline{\check{\mathcal{R}}}^1_d$ is covered by the images of the natural inclusions of the following strata:

$$\overline{\mathcal{R}}^s \times {}_1\overline{\check{\mathcal{R}}}^1_{d-s+1}, \tag{5-86}$$

$$\overline{\check{\mathcal{R}}}^1_d \times \overline{\mathcal{M}}_1, \tag{5-87}$$

$$\overline{\check{\mathcal{R}}}^{S^1}_d. \tag{5-88}$$

The stratum (5-88) describes the locus which $|p_1| = \frac{1}{2}$, which is exactly the locus we defined to be the auxiliary moduli space $\mathcal{R}^{S^1}_d$ inducing the operation $\mathcal{OC}^{S^1}$. The strata (5-86)–(5-87) have manifold-with-corners structure given by standard local gluing maps using fixed choices of strip-like ends near the boundary. For (5-86) this is standard, and for (5-87), the local gluing map uses the cylindrical ends (5-84) and (4-33) — in other words, one rotates the 1–pointed angle cylinder by an amount commensurate to the angle of the marked point z_d on the disk before gluing; see Section A.1, particularly (A-12). See Figure 6 for a schematic of (5-80) and two out of the three types of strata (5-87)–(5-88).

We will as usual fix a *Floer datum for the BV homotopy*, meaning an inductive choice, for every $d \geq 1$, of Floer data for every representative $S \in {}_1\overline{\check{\mathcal{R}}}^1_d$ varying smoothly in S, which on boundary strata is smoothly equivalent to a product of Floer data inductively chosen on lower-dimensional moduli spaces. Such a system of choices exist again by a contractibility argument, and for any such choice, one obtains, for any d–tuple of Lagrangians $L_0, \ldots, L_{d-1}$ and asymptotic conditions

$$\vec{x} = (x_d, \ldots, x_1) \ \text{ with } \ x_i \in \chi(L_{i-1}, L_{i \bmod d}) \quad \text{and} \quad y_{\mathrm{out}} \in \mathcal{O}, \tag{5-89}$$

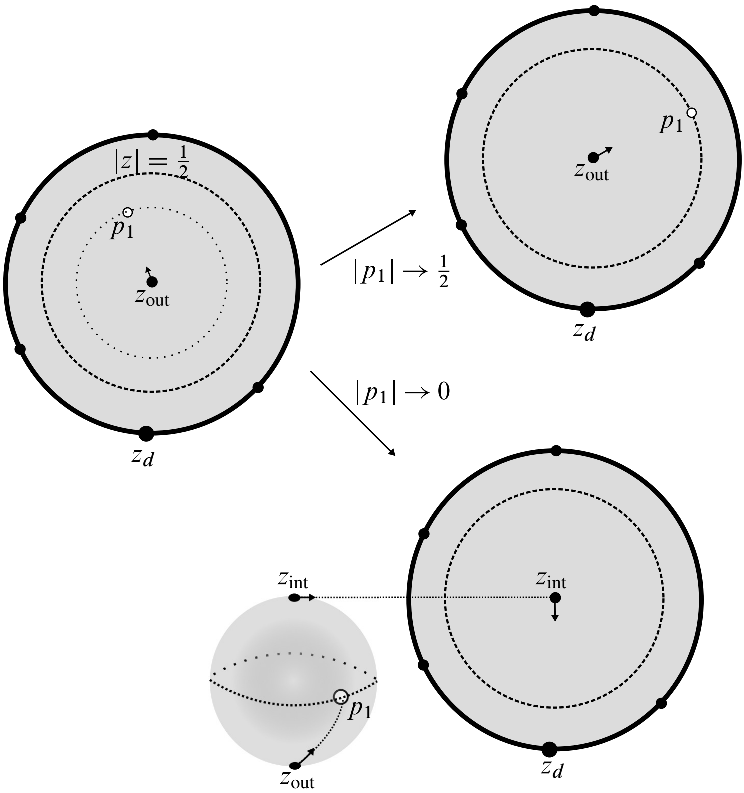

Figure 6: A schematic of an element of (5-80) on the left and a schematic of two of its three types of degenerations on the right, (5-88) (above) and (5-87) (below). The remaining type of degeneration (5-86), omitted from the figure, occurs when some boundary marked points coalesce into a disc bubble.

a compactified moduli space

$$ {}_1\overline{\check{\mathcal{R}}}{}^1_d(y_{\text{out}}, \vec{x}) \tag{5-90} $$

of maps into M with source an arbitrary element S of the moduli space (5-80), satisfying Floer's equation using the Floer datum chosen for the given S as in (5-4) with asymptotics and boundary conditions as in (5-5), with the usual modifications in the Liouville case detailed in Remarks 5.1 and 5.2. The virtual dimension of every component of ${}_1\overline{\check{\mathcal{R}}}{}^1_d(y_{\text{out}}, \vec{x})$ coincides (mod 2 or exactly depending on whether we are in a $\mathbb{Z}/2$– or $\mathbb{Z}$–graded setting) with

$$ \deg(y_{\text{out}}) - n + d + 1 - \sum_{i=1}^{d} \deg(x_i). \tag{5-91} $$

By Assumption 3.10, for generic choices of Floer data, the Gromov–Floer compactification of the components of virtual dimension ≤ 1 of (5-90) are compact manifolds-with-boundary of expected dimension. For rigid elements u of the moduli spaces (5-90), the orientations (5-85) and [1, Lemma C.4] induce isomorphisms of orientation lines

$$({}_1\check{\mathcal{R}}^1_d)_u \colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y. \tag{5-92}$$

As usual "counting rigid elements u", ie summing application of these isomorphisms over all u, defines the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of an operation $\check{\mathcal{OC}}^1$, up to a sign twist which we specify:

$$\check{\mathcal{OC}}^1([x_d], \dots, [x_1]) := \sum_{u \in {}_k\overline{\check{\mathcal{R}}}^1_d(y_{\mathrm{out}};\vec{x}) \text{ rigid}} (-1)^{\check{\star}_d} ({}_k\check{\mathcal{R}}^1_d)_u([x_d], \dots, [x_1]), \tag{5-93}$$

where the sign is given by

$$\check{\star}_d = \deg(x_d) + \sum_i i \cdot \deg(x_i). \tag{5-94}$$

A codimension-one analysis of the moduli spaces (5-90) reveals:

Proposition 5.16 *The following equation is satisfied*:

$$(-1)^n \delta_1 \check{\mathcal{OC}} + (-1)^n d \check{\mathcal{OC}}^1 = \mathcal{OC}^{S^1} + \check{\mathcal{OC}}^1 b. \tag{5-95}$$

Proof The boundary of the one-dimensional components of (5-90) are covered by the rigid components of the following types of strata:

- Spaces of maps with domain lying on the codimension-one boundary of the moduli space, ie in (5-86)–(5-88).
- Semistable breakings, namely those of the form

$${}_1\overline{\check{\mathcal{R}}}^1_d(y_1;\vec{x}) \times \overline{\mathcal{M}}(y_{\mathrm{out}}; y_1), \tag{5-96}$$

$$\overline{\mathcal{R}}^1(x; x_i) \times {}_1\overline{\check{\mathcal{R}}}^1_d(y_{\mathrm{out}}; \tilde{\vec{x}}), \tag{5-97}$$

  where $\tilde{\vec{x}}$ denotes the collection of inputs $\vec{x}$ with x_i replaced with x.

All together, this implies, up to signs, that

$$(-1)^n \delta_1 \check{\mathcal{OC}} + (-1)^n d \check{\mathcal{OC}}^1 = \mathcal{OC}^{S^1} + \check{\mathcal{OC}}^1 b. \tag{5-98}$$

Equation (5-98) is of course a shorthand for saying, for any d and any tuple of d cyclically composable morphisms $x_d, \dots, x_1$, that

$$
\begin{aligned}
(5\text{-}99)\quad & (-1)^n \sum_{i \in \{0,1\}} \delta_i \check{\mathcal{OC}}_d^{k-i}(x_d, \dots, x_1) \\
&= \mathcal{OC}_d^{S^1}(x_d, \dots, x_1) \\
&\quad + \sum_{i,s} (-1)^{\maltese_1^s} \check{\mathcal{OC}}^1_{d-i+1}(x_d, \dots, x_{s+i+1}, \mu^i(x_{s+i}, \dots, x_{s+1}), x_s, \dots, x_1) \\
&\qquad + \sum_{i,j} (-1)^{\sharp_j^i} \check{\mathcal{OC}}^1_{d-i-j}\big(\mu^{i+j+1}(x_i, \dots, x_1, x_d, \dots, x_{d-j}), \\
&\qquad\qquad x_{d-j-1}, \dots, x_{i+1}\big).
\end{aligned}
$$

(Recall the abuse of notation $x_i := [x_i]$.) Thus, it suffices to verify that the signs coming from the codimension-one boundary are exactly those appearing in (5-98) — in particular, that the terms in, for instance, $\check{\mathcal{OC}}^1 b$ appear with the right sign.

Let us recall broadly how the signs are computed. For any operator g defined above, such as $\mathcal{OC}$, $\mathcal{OC}^{S^1}$, μ, d, δ_1 etc, we let g_{ut} denote the *untwisted* version of the same operator, for instance, the operator whose matrix coefficients come from the induced isomorphism on orientation lines, without any sign twists by the degree of the inputs. So, for instance, $\mu^d(x_d, \dots, x_1) = (-1)^{\sum_{i=1}^d i \deg(x_i)} \mu^d_{\mathrm{ut}}(x_d, \dots, x_1)$, and so on. The methods described in [52, Proposition 12.3] and elaborated upon in [1, Section C.3, Lemma 5.3] and [24, Section B], when applied to the boundary of the one-dimensional component of the moduli space of maps, $\overline{\check{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}, \vec{x})$), imply the signed equality

$$
\begin{aligned}
(5\text{-}100)\quad 0 = d_{\mathrm{ut}} \check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \dots, x_1) + (\delta_1)_{\mathrm{ut}} \check{\mathcal{OC}}_{\mathrm{ut}}(x_d, \dots, x_1) - \mathcal{OC}^{S^1}_{\mathrm{ut}}(x_d, \dots, x_1) \\
+ (-1)^{\mathfrak{f}_d} \check{\mathcal{OC}}^1 b(x_d, \dots, x_1),
\end{aligned}
$$

where

$$
(5\text{-}101)\qquad \mathfrak{f}_d := \sum_i (i+1)\deg(x_i) + \deg(x_d) = \check{\star}_d + \maltese_d - d
$$

is an auxiliary sign.

To explain equation (5-100), we note first that the signs appearing in all terms but the last are simply induced by the boundary orientation on the moduli space of domains. The sign appearing in the first term also follows from a standard boundary orientation analysis for Floer cylinders, which we omit (but see eg [52, (12.19-012.20)] for a version close in spirit). The signs for the first two terms are also exactly as in Lemma 4.11. Finally, in the last term, the sign $(-1)^{\mathfrak{f}_d} \check{\mathcal{OC}}^1 b(x_d \otimes \dots \otimes x_1)$ (compare [52, (12.25)] and [24, (B.59)]) appears as a cumulative sum of:

- The sign twists which turn the untwisted operations $\check{\mathcal{OC}}^1_{\mathrm{ut}}$ and μ^s_{ut} into the usual operations $\check{\mathcal{OC}}^1$ and μ^s.
- The Koszul sign appearing in the Hochschild differential b.
- The boundary orientation sign appearing in the relevant (untwisted) term of $\check{\mathcal{OC}}^1 b$, for instance $\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \dots, x_{n+m+1}, \mu^m_{ut}(x_{n+m}, \dots, x_{n+1}), x_n, \dots, x_1)$, which itself is as a sum of two different contributions:
  - (a) The comparison between the boundary (of the chosen) orientation and the product (of the chosen orientation) on the moduli of *domains*.
  - (b) Koszul reordering signs, which measure the signed failure of the method of orienting the moduli of maps (in terms of orientations of the domain and orientation lines of inputs and outputs) to be compatible with passing to boundary strata.

See [52, (12d)] for more details in the case of the A_∞ structure, and [1, Section C] as well as [24, Section C] for the case of these computations for the open–closed map. We note in particular that the forgetful map $F_1 : {}_1\check{\mathcal{R}}^1_d \to \check{\mathcal{R}}^1_d$ which forgets the point p_1 (and changes the direction of the asymptotic marker to point at z_d) has *complex oriented fibers* (in which just the marked point p_1 varies). So the boundary analysis of these "$\check{\mathcal{OC}}^1 \circ b$" strata appearing here is identical to the analysis strata appearing in [1; 24] for the "$\mathcal{OC} \circ b$" strata, which is why we have not repeated it here.

Multiplying all terms of (5-100) by $(-1)^{\check{\star}_d + \maltese_d - d + 1}$ and noting that, for instance, $\maltese_d - d + 1 + n - 2 = \deg(\check{\mathcal{OC}}^1(x_d \otimes \cdots \otimes x_1))$, so that

$$(5\text{-}102)\quad (-1)^{\check{\star}_d + \maltese_d - d + 1}(\delta_1)_{\mathrm{ut}}\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \dots, x_1) = (-1)^{\deg(\check{\mathcal{OC}}^1(x_d, \dots, x_1)) - n}(\delta_1)_{\mathrm{ut}}(-1)^{\check{\star}_d}\check{\mathcal{OC}}^1_{\mathrm{ut}}(x_d, \dots, x_1) = \delta_1\check{\mathcal{OC}}^1(x_d, \dots, x_1),$$

and similarly for the $d \circ \mathcal{OC}^1$ term, it follows that

$$(5\text{-}103)\quad 0 = (-1)^n\delta_1\check{\mathcal{OC}}(x_d, \dots, x_1) + (-1)^n d\check{\mathcal{OC}}^1(x_d, \dots, x_1) - \check{\mathcal{OC}}^1 b(x_d, \dots, x_1) - (-1)^{\check{\star}_d + \maltese_d - d + 1}\mathcal{OC}^{S^1}_{\mathrm{ut}}(x_d, \dots, x_1),$$

but $\check{\star}_d + \maltese_d - d + 1 = \clubsuit_d$, and hence the last term above is $-\mathcal{OC}^{S^1}(x_d, \dots, x_1)$, as desired. □

Proof of Proposition 5.15 The "sector decomposition" performed in Proposition 5.9 which compares $\mathcal{OC}^{S^1}$ to $\hat{\mathcal{OC}} \circ B^{\mathrm{nu}} \circ \iota$, along with Proposition 5.16, immediately implies the result. □

5.5 The main construction

We now turn to the definition of the (closed) morphism of S^1–complexes, and the proof of Theorem 1.1 and Corollary 1.5. The required data takes the form

$$\widetilde{\mathcal{OC}} = \bigoplus_{k\geq 0} \overline{\boldsymbol{k}[\Lambda]/\Lambda^2}^{\otimes k} \otimes \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}) \to CF^*(M)[n], \tag{5-104}$$

which is equivalent, as recalled in Section 2.1, to defining the collection of maps $\widetilde{\mathcal{OC}} = \{\mathcal{OC}^k\}_{k\geq 0}$, or u–linearly (see Section 2.3) $\widetilde{\mathcal{OC}} = \sum_{k=0}^{\infty} \mathcal{OC}^k u^k$, where

$$\mathcal{OC}^k = (\check{\mathcal{OC}}^k + \hat{\mathcal{OC}}^k) := \widetilde{\mathcal{OC}}^{k|1}(\Lambda, \dots, \Lambda, -)\colon \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}) \to CF^{*+n-2k}(M). \tag{5-105}$$

(Recall from Section 2.1 that $\boldsymbol{k}[\Lambda]/\Lambda^2$ is our small model for $C_{-*}(S^1)$, and S^1–complexes are by definition strictly unital A_∞–modules over $\boldsymbol{k}[\Lambda]/\Lambda^2$.) By definition, the case $k=0$ is already covered:

$$\mathcal{OC}^0 = (\check{\mathcal{OC}}^0 \oplus \hat{\mathcal{OC}}^0) = (\check{\mathcal{OC}} \oplus \hat{\mathcal{OC}}) = \mathcal{OC}. \tag{5-106}$$

To handle the general case ($k \geq 0$), for each d we will associate operations, for each d, to compactifications of three moduli spaces of domains, in the order

$${}_k\check{\mathcal{R}}^1_d, \tag{5-107}$$

$${}_k\mathcal{R}^{S^1}_d, \tag{5-108}$$

$${}_k\hat{\mathcal{R}}^1_d. \tag{5-109}$$

The moduli space (5-108) will induce an auxiliary operation useful for the proof, whereas (5-107) and (5-109) will lead to the desired operations. For $k=0$, these moduli spaces are simply $\check{\mathcal{R}}^1_d$, $\mathcal{R}^{S^1}_d$ and $\hat{\mathcal{R}}^1_d$ as defined earlier, and the $k=1$ case of (5-107) was defined in (5-80). Inductively, we will construct and study operations from (5-107) and (5-108) simultaneously, and then finally construct (5-109). Using these moduli spaces, we will construct the maps $\check{\mathcal{OC}}^k$ and $\hat{\mathcal{OC}}^k$, as well as an auxiliary operation $\mathcal{OC}^{S^1,k}$ (which we compare to $\hat{\mathcal{OC}}^{k-1} \circ B^{\mathrm{nu}}$ in Proposition 5.20 below), and then prove:

Proposition 5.17 *The following equations hold, for each $k \geq 0$:*

$$(-1)^n \sum_{i\geq 0}^{k} \delta_i \check{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^{k-1} B^{\mathrm{nu}} + \check{\mathcal{OC}}^k b, \tag{5-110}$$

$$(-1)^n \sum_{i\geq 0}^{k} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k (1-t). \tag{5-111}$$

All at once, writing

$$\mathcal{OC}^k = (\check{\mathcal{OC}}^k + \hat{\mathcal{OC}}^k), \quad \widetilde{\mathcal{OC}} = \sum_{i=0}^{\infty} \mathcal{OC}^i u^i, \quad \delta_{\mathrm{eq}} = \sum_{j=0}^{\infty} \delta_j^{CF} u^j, \quad b_{\mathrm{eq}} = b^{\mathrm{nu}} + u B^{\mathrm{nu}},$$

as in Section 2.3, we have that

$$(-1)^n \delta_{\mathrm{eq}} \circ \widetilde{\mathcal{OC}} = \widetilde{\mathcal{OC}} \circ b_{\mathrm{eq}}. \tag{5-112}$$

This will also directly imply our main theorems, as spelled out at the end of this subsection.

The space (5-107) is the moduli space of discs with d positive boundary marked points $z_1, \dots, z_d$ labeled in counterclockwise order, one interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \dots, p_k$ (without asymptotic markers), marked as *auxiliary*. Also, on the unit disc representative of any element of this moduli space which fixes z_d at 1 and z_{out} at 0, the p_i should be *strictly radially ordered* with norms in $\left(0, \frac{1}{2}\right)$; that is,

$$0 < |p_1| < \dots < |p_k| < \tfrac{1}{2}. \tag{5-113}$$

Using the above representative, one can talk about the *angle*, or *argument*, of each auxiliary interior marked point,

$$\theta_i := \arg(p_i). \tag{5-114}$$

We require that with respect to the above representative:

(5-115) The asymptotic marker on z_{out} points in the direction θ_1 (or towards z_d if $k = 0$).

(Equivalently one could define $\theta_{k+1} = 0$, so that θ_1 is always defined.) See Figure 7 for a depiction. For every representative $S \in {}_k\check{\mathcal{R}}_d^1$:

(5-116) Fix a negative cylindrical end around z_{out} not containing any p_i, compatible with the direction of the asymptotic marker, or *equivalently compatible with the angle θ_1*.

The second moduli space (5-108) is the moduli space of discs with d positive boundary marked points $z_1, \dots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and $k+1$ additional interior marked points $p_1, \dots, p_k, p_{k+1}$ (without asymptotic markers), marked as *auxiliary*. With respect to

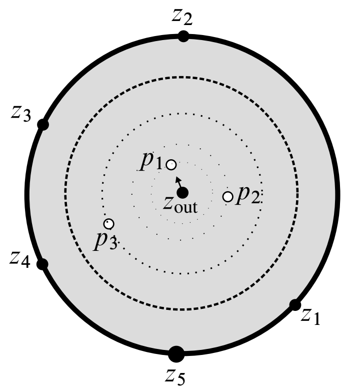

Figure 7: A representative of an element of the moduli space ${}_3\check{\mathcal{R}}^1_5$.

the unit disc representative of any element this moduli space fixing z_d at 1 and z_{out} at 0, the p_i should again be *strictly radially ordered*, this time with norms lying in $\left(0, \frac{1}{2}\right]$ and with p_{k+1} *lying on the circle of radius* $\frac{1}{2}$,

$$0 < |p_1| < \cdots < |p_k| < |p_{k+1}| = \tfrac{1}{2}. \tag{5-117}$$

The asymptotic marker on z_{out} for this representative again satisfies condition (5-115). Abstractly we have that ${}_k\mathcal{R}^{S^1}_d \cong \times_k\check{\mathcal{R}}^1_d \times S^1$, where the S^1 parameter is given by the position of p_{k+1}. See Figure 8 for a depiction of ${}_{k-1}\check{\mathcal{R}}^{S^1}_d$.

The compactification of (5-107) is a real blow-up of the ordinary Deligne–Mumford compactification, in the sense of [34] (see [58] for a first discussion in the context of Floer theory), reviewed in more detail in Section A.1. The result of the discussion there is that the codimension-one boundary of the compactified check moduli space ${}_k\overline{\check{\mathcal{R}}}^1_d$ is

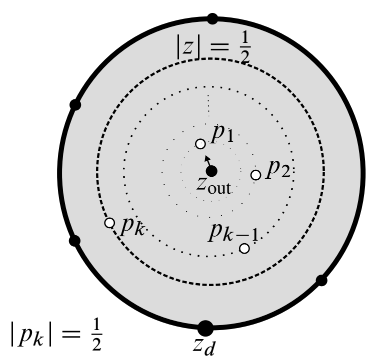

Figure 8: A representative of an element of the moduli space ${}_{k-1}\check{\mathcal{R}}^{S^1}_d$, which also arises as the boundary stratum (5-120) of ${}_k\overline{\check{\mathcal{R}}}^1_d$.

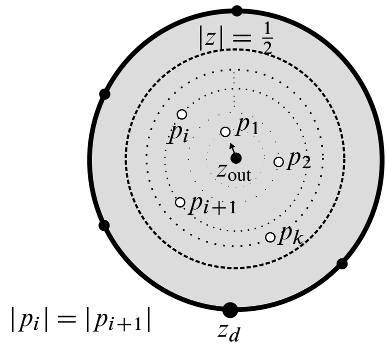

Figure 9: A representative of an element of the stratum (5-121).

covered by the images of the natural inclusions of the strata

$$\overline{\mathcal{R}}^s \times {}_k\check{\overline{\mathcal{R}}}{}^1_{d-s+1}, \tag{5-118}$$

$$ {}_s\check{\overline{\mathcal{R}}}{}^1_d \times \overline{\mathcal{M}}_{k-s}, \tag{5-119}$$

$$ {}_{k-1}\check{\overline{\mathcal{R}}}{}^{S^1}_d, \tag{5-120}$$

$$ {}^{i,i+1}_{k}\check{\overline{\mathcal{R}}}{}^1_d. \tag{5-121}$$

The strata (5-120)–(5-121), in which $|p_k| = \frac{1}{2}$ (Figure 8) and $|p_i| = |p_{i+1}|$ (Figure 9), respectively, describe the boundary loci of the ordering condition (5-113) and hence come equipped with a natural manifold-with-corners structure. The strata (5-118)–(5-119) have manifold-with-corners structure given by standard local gluing maps using fixed choices of strip-like ends near the boundary. For (5-118), depicted in Figure 10, this is standard, and for (5-119), depicted in Figure 11, the local gluing map uses the cylindrical ends (5-116) and (4-33) — in other words, one rotates the $(k-s)$–pointed angle cylinder by an amount commensurate to the angle of the first marked point p_{k-s+1} on the disk before gluing — as also described in Section A.1.

Associated to the stratum (5-121) where p_i and p_{i+1} have coincident magnitudes, there is a forgetful map

$$\check{\pi}_i : {}^{i,i+1}_{k}\check{\overline{\mathcal{R}}}{}^1_d \to {}_{k-1}\check{\overline{\mathcal{R}}}{}^1_d \tag{5-122}$$

which simply forgets the point p_{i+1}. Since the norm of p_{i+1} and p_i agree on this locus, this amounts to forgetting the argument of p_{i+1} (in particular, the fibers of $\check{\pi}_i$ are one-dimensional).

The compactification of the S^1 moduli space (5-108) can be modeled abstractly by ${}_k\check{\overline{\mathcal{R}}}{}^1_d \times S^1$. However, it is again preferable to give an explicit description of the boundary

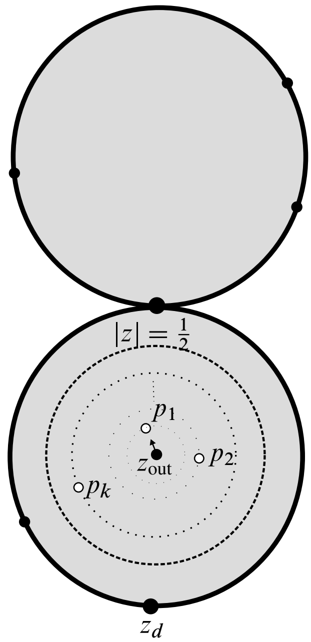

Figure 10: A representative of an element of the boundary stratum (5-118) in which a disc bubble forms (such a disc bubble is allowed to include the "first" point — z_d by our convention — but need not, and does not in the figure).

strata, which are covered in codimension one by the strata

(5-123) $$\overline{\mathcal{R}}^s \times {}_k\overline{\mathcal{R}}^{S^1}_{d-s+1},$$

(5-124) $$_s\overline{\mathcal{R}}^{S^1}_d \times \overline{\mathcal{M}}_{k-s},$$

(5-125) $$^{i,i+1}_{k}\overline{\mathcal{R}}^{S^1}_d.$$

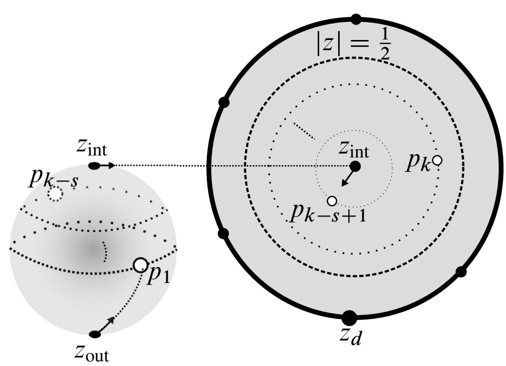

Figure 11: A representative of an element of the boundary stratum (5-119).

Here, (5-123) and (5-124) are just versions of the degenerations (5-118) and (5-119), in which a collection of boundary points bubbles off, or a collection of auxiliary points converges to z_{out} and bubbles off; the fact that the latter occurs in codimension one is part of the "real blow-up phenomenon" already discussed. The stratum (5-125) is the locus where $|p_i| = |p_{i+1}|$, for $i \leq k$; so when $i = k$, $|p_k| = |p_{k+1}| = \frac{1}{2}$.

As in (5-122), on the stratum (5-125), where p_i and p_{i+1} have coincident magnitudes, define the map

$$\pi_i^{S^1} : {}_k^{i,i+1}\overline{\mathcal{R}}_d^{S^1} \to {}_{k-1}\overline{\mathcal{R}}_d^{S^1} \tag{5-126}$$

to be the one forgetting the point p_{i+1}. As before, this map has one-dimensional fibers.

For an element $S \in {}_k\overline{\mathcal{R}}_d^{S^1}$, we say that p_{k+1} *points at a boundary point* z_i if, for any unit disc representative of S with z_{out} at the origin, the ray from z_{out} to p_{k+1} intersects z_i. The locus where p_{k+1} points at z_i is denoted by

$$_k\overline{\mathcal{R}}_d^{S_i^1}. \tag{5-127}$$

Similarly, we say that p_{k+1} *points between z_i and z_{i+1}* (modulo d, so this includes the case of pointing between z_d and z_1) if for such a representative, the ray from z_{out} to p_{k+1} intersects the portion of ∂S between z_i and z_{i+1}. The locus where p_{k+1} points between z_i and z_{i+1} is denoted by

$$_k\overline{\mathcal{R}}_d^{S_{i,i+1}^1}. \tag{5-128}$$

As before in (5-52), there is a free and properly discontinuous $\mathbb{Z}_d$–action

$$\kappa : {}_k\overline{(\mathcal{R}_d^1)^{S^1}} \to {}_k\overline{(\mathcal{R}_d^1)^{S^1}} \tag{5-129}$$

which cyclically permutes the labels of the boundary marked points; as before, κ changes the label z_i to z_{i+1} for $i < d$, and z_d to z_1; compare Lemma 5.10.

Finally, we come to the third moduli space (5-109), the moduli space of discs with $d+1$ positive boundary marked points $z_1, \dots, z_d, z_f$ labeled in counterclockwise order, one interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \dots, p_k$ (without an asymptotic marker), marked as *auxiliary*, satisfying a *strict radial ordering* condition as before: for any representative element with z_f fixed at 1 and z_{out} at 0, we require (5-113) to hold, as well as condition (5-115). The boundary marked point z_f is also marked as auxiliary, but apart from this designation we see, identifying z_f with z_{d+1}, that ${}_k\hat{\mathcal{R}}_d^1 \cong {}_k\check{\mathcal{R}}_{d+1}^1$. See Figure 12.

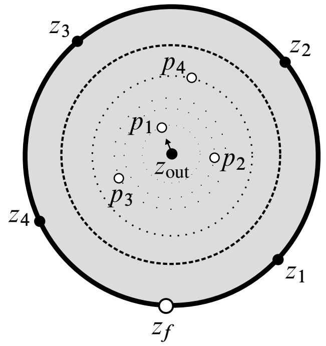

Figure 12: A representative of an element of the moduli space ${}_4\widehat{\mathcal{R}}^1_4$.

In codimension one, the compactification ${}_k\overline{\widehat{\mathcal{R}}}{}^1_d$ has boundary covered by inclusions of the strata

$$\overline{\mathcal{R}}^s \times {}_k\overline{\widehat{\mathcal{R}}}{}^1_{d-s+1}, \tag{5-130}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} {}_k\overline{\check{\mathcal{R}}}{}^1_{d-m+1}, \quad \text{where } 1 \le k \le m, \tag{5-131}$$

$$ {}_s\overline{\widehat{\mathcal{R}}}{}^1_d \times \overline{\mathcal{M}}_{k-s}, \tag{5-132}$$

$$ {}_{k-1}\overline{\widehat{\mathcal{R}}}{}^{S^1}_d, \tag{5-133}$$

$$ {}^{i,i+1}_{k}\overline{\widehat{\mathcal{R}}}{}^1_d. \tag{5-134}$$

Once more, on strata (5-134) where p_i and p_{i+1} have coincident magnitudes, depicted in Figure 13, left, define the map

$$\hat{\pi}_i \colon {}^{i,i+1}_{k}\overline{\widehat{\mathcal{R}}}{}^1_d \to {}_{k-1}\overline{\widehat{\mathcal{R}}}{}^1_d \tag{5-135}$$

to be the one forgetting the point p_{i+1}. Again, this map has one-dimensional fibers. On the stratum (5-133), which is the locus where $|p_k| = \frac{1}{2}$ (Figure 13, right), there is also a map of interest

$$\hat{\pi}_{\mathrm{boundary}} \colon {}_{k-1}\overline{\widehat{\mathcal{R}}}{}^{S^1}_d \to {}_{k-1}\overline{\mathcal{R}}{}^{S^1}_d \tag{5-136}$$

which forgets the position of the auxiliary boundary point z_f. The stratum (5-132), depicted in Figure 14, is the locus where some subcollection of interior auxiliary points $p_1, \dots, p_{k-s}$ tend to zero and split off an angle-decorated cylinder (in the manner again described in Section A.1 for (5-119)). The strata (5-130) and (5-131), depicted in Figure 15 on the left and right, respectively, are the loci where a disc bubble forms involving some boundary marked points (not including or including z_f, respectively).

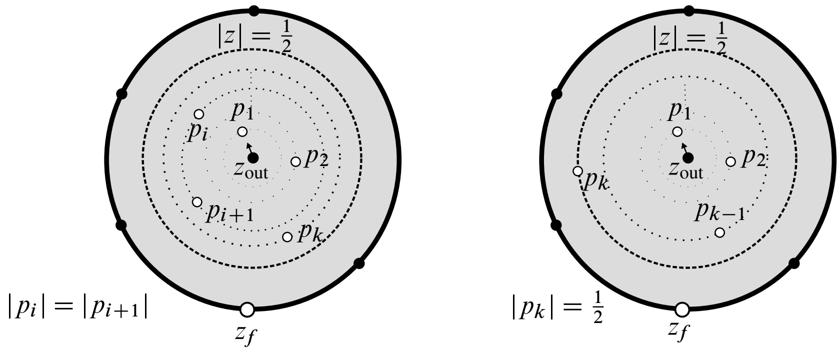

Figure 13: A representative of an element of the stratum (5-134), left, and a representative of an element of the boundary stratum (5-133), right.

Denote by ${}_k\mathcal{R}_d^{1,\text{free}} := {}_k\mathcal{R}_d^{S^1_{d,1}}$ the sector of the moduli space ${}_k\mathcal{R}_d^{S^1}$ where p_{k+1} points between z_d and z_1. The *auxiliary-rescaling map*

$$\pi_f : {}_k\hat{\mathcal{R}}_d^1 \to {}_k\mathcal{R}_d^{1,\text{free}} \tag{5-137}$$

(our replacement of the "forgetful map") can be described as follows: given a representative S in ${}_k\hat{\mathcal{R}}_d^1$ with z_{out} fixed at the origin, there is a unique point p with $|p| = \frac{1}{2}$ between z_{out} and z_f. The element $\pi_f(S)$ is the element of ${}_k\mathcal{R}_d^{S^1}$ obtained from S by setting p_{k+1} equal to this point p and deleting z_f. Of course, z_f is not actually forgotten, because it is determined by the position of p_{k+1}. In particular, (5-137) is a diffeomorphism. We extend this map to a map $\bar{\pi}_f$ from the compactification ${}_k\bar{\hat{\mathcal{R}}}_d^1$ as in Section 5.2, by putting the auxiliary point z_f back in, eliminating any component which is not main or secondary which has only one (nonauxiliary) boundary marked point q, and by labeling the positive marked point below this component by q.

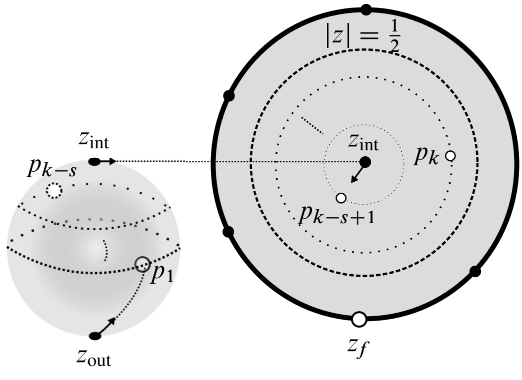

Figure 14: A representative of an element of the boundary stratum (5-132).

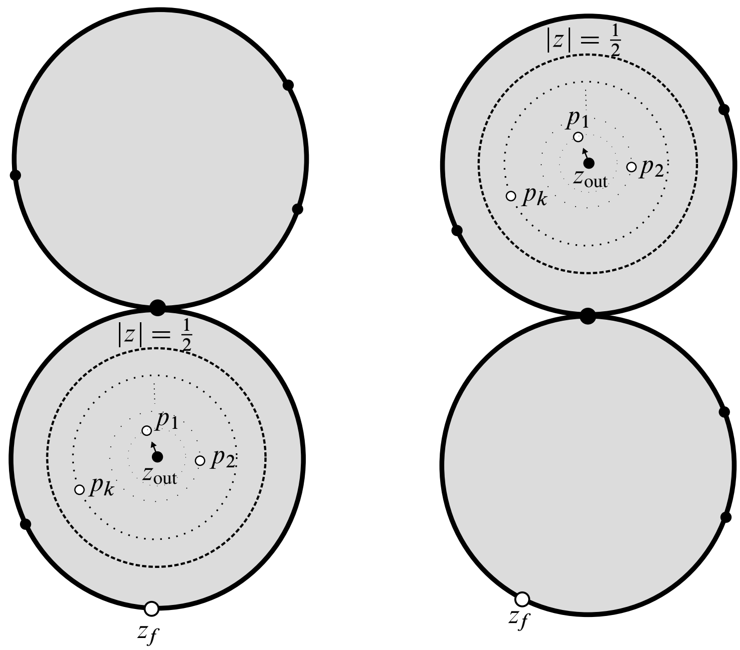

Figure 15: Left: a representative of an element of the boundary stratum (5-130) in which a disc bubble forms not including the auxiliary point z_f. Right: a representative of an element of the boundary stratum (5-131) in which a disc bubble forms including the auxiliary point z_f.

We orient the moduli spaces (5-107)–(5-109) as follows: pick, on a slice of the automorphism action which fixes the position of z_d at 1 and z_{out} at 0, the volume forms

$$-r_1 \cdots r_k \, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k, \tag{5-138}$$

$$r_1 \cdots r_k \, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge d\theta_{k+1} \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k, \tag{5-139}$$

$$r_1 \cdots r_k \, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dz_f \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k. \tag{5-140}$$

Above, (r_i, θ_i) denote the polar coordinate positions of the point p_i. (We could equivalently use Cartesian coordinates (x_i, y_i) and substitute $dx_i \wedge dy_i$ for every instance of $r_i \, dr_i \wedge d\theta_i$, but polar coordinates are straightforwardly compatible with the boundary stratum where $|p_k| = \frac{1}{2}$.)

A *Floer datum* on a stable disc S in ${}_k\overline{\check{\mathcal{R}}}{}^1_d$ or a stable disc S in ${}_k\overline{\mathcal{R}}{}^{S^1}_d$ is simply a Floer datum for S in the sense of Section 5.1. A *Floer datum* on a stable disc $S \in {}_k\overline{\hat{\mathcal{R}}}{}^1_d$ is a Floer datum for $\overline{\pi}_f(S)$.

Again we will make a system of choices of Floer data for the above moduli spaces. A *Floer datum for the cyclic open–closed map* is an inductive sequence of choices, for every $k \geq 0$ and $d \geq 1$, of Floer data for every representative

$$S_0 \in {}_k\overline{\check{\mathcal{R}}}{}^1_d, \quad S_1 \in {}_k\overline{\mathcal{R}}{}^{S^1}_d \quad \text{and} \quad S_2 \in {}_k\overline{\widehat{\mathcal{R}}}{}^1_d,$$

varying smoothly in S_0, S_1 and S_2, which satisfies the usual consistency condition: the choice of Floer datum on any boundary stratum should agree with the previously inductively chosen datum along any boundary stratum for which (it is possibly a product of moduli spaces for) we have already inductively picked data. Moreover, this choice should satisfy a series of additional requirements.

First, for $S_0 \in {}_k\overline{\check{\mathcal{R}}}{}^1_d$:

(5-141) At a boundary stratum of the form (5-121), the Floer datum for S_0 is equivalent to the one pulled back from ${}_{k-1}\overline{\check{\mathcal{R}}}{}^1_d$ via the forgetful map $\check{\pi}_i$.

Next, for $S_1 \in {}_k\overline{\mathcal{R}}{}^{S^1}_d$,

(5-142) On the codimension-one loci ${}_k\overline{\mathcal{R}}{}^{S^1_i}_d$, where p_{k+1} points at z_i, the Floer datum should agree with the pullback by τ_i of the existing Floer datum for the open–closed map.

(5-143) The Floer datum should be κ–equivariant, where κ is the map (5-129).

(5-144) At a boundary stratum of the form (5-125), the Floer datum for S_1 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\mathcal{R}}{}^{S^1}_d$ via the forgetful map $\pi_i^{S^1}$.

Finally, for $S_2 \in {}_k\overline{\widehat{\mathcal{R}}}{}^1_d$:

(5-145) The choice of Floer datum on strata containing $\mathcal{R}^{d,f_i}$ components should be constant along fibers of the forgetful map $\mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}$.

(5-146) The Floer datum on the main component $(S_2)_0$ of $\overline{\pi}_f(S_2)$ should coincide with the Floer datum chosen on $(S_2)_0 \in {}_k\mathcal{R}^{1,\mathrm{free}}_d \subset {}_k\mathcal{R}^{S^1}_d$.

(5-147) At a boundary stratum of the form (5-133), the Floer datum on the main component of S_2 is conformally equivalent to the one pulled back from ${}_k\overline{\mathcal{R}}{}^{S^1}_d$ via the forgetful map $\hat{\pi}_{\mathrm{boundary}}$.

(5-148) At a boundary stratum of the form (5-134), the Floer datum for S_2 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\widehat{\mathcal{R}}}{}^1_d$ via the forgetful map $\hat{\pi}_i$.

The above system of requirements can be split into three broad categories: the first type concerns the compatibility with forgetful maps of Floer data along the lower strata which were not previously constrained, the second type concerns the equivariance (under a free properly discontinuous action) of the Floer data on ${}_k\overline{\mathcal{R}}^{S^1}_d$ as well as the relationship between the Floer datum chosen here and the ones chosen on $\overline{\check{\mathcal{R}}}^1_d$ and ${}_k\overline{\hat{\mathcal{R}}}^1_d$.

Proposition 5.18 *A Floer datum for the cyclic open–closed map exists.*

Proof Since the choices of Floer data at each stage are contractible, this follows from the straightforward verification that, for a suitably chosen inductive order on strata, the conditions satisfied by the Floer data at various strata do not contradict each other. We use the following inductive order: first, say we have chosen a Floer datum for the A_∞ structure as in Section 3.3, along with a BV compatible Floer datum for the nonunital open–closed map following Section 5.3. In particular, we have chosen Floer data for the moduli spaces $\overline{\mathcal{R}}^{d,f_i}$ (per Section A.2), for ${}_0\overline{\check{\mathcal{R}}}_d$, for the auxiliary moduli space ${}_0\overline{\mathcal{R}}^{S^1}_d$, and (using the conditions above) we have induced a particular choice of Floer datum on ${}_0\overline{\hat{\mathcal{R}}}_d$. Next, inductively assuming that we have made all choices at level $k-1$ with $k>0$, we first choose Floer data for ${}_k\overline{\check{\mathcal{R}}}_d$ for each d, then ${}_k\overline{\mathcal{R}}^{S^1}_d$ for each d (by pulling back a choice of Floer datum on the quotient by κ in order to satisfy the equivariance condition), and finally note that a choice is fixed for ${}_k\overline{\hat{\mathcal{R}}}_d$ by the above constraints. □

Fixing a Floer datum for the cyclic open–closed map, we obtain, for any d–tuple of Lagrangians $L_0,\dots,L_{d-1}$, and asymptotic conditions

$$\begin{cases} \vec{x} = (x_d,\dots,x_1) \quad \text{with } x_i \in \chi(L_{i-1}, L_{i \bmod d}), \\ y_{\mathrm{out}} \in \mathcal{O}, \end{cases} \tag{5-149}$$

Gromov–Floer compactified moduli spaces

$$ {}_k\overline{\check{\mathcal{R}}}^1_d(y_{\mathrm{out}}, \vec{x}), \tag{5-150}$$

$$ {}_k\overline{\mathcal{R}}^{S^1}_d(y_{\mathrm{out}}, \vec{x}), \tag{5-151}$$

$$ {}_k\overline{\hat{\mathcal{R}}}^1_d(y_{\mathrm{out}}, \vec{x}), \tag{5-152}$$

of maps into M from an arbitrary element S of the moduli spaces (5-107), (5-108) and (5-109) respectively (or rather from $\pi_f(S)$ in the case of (5-109)) satisfying Floer's equation using the Floer datum chosen for the given S as in (5-4), with asymptotics and Lagrangian boundary conditions as in (5-5), again with the modifications as in Remarks 5.1 or 5.2 for compact or wrapped Fukaya categories of Liouville manifolds. The virtual dimension of each component of these moduli spaces coincides (mod 2 or exactly, depending on whether we are $\mathbb{Z}/2$– or $\mathbb{Z}$–graded) with, respectively,

$$\deg(y_{\mathrm{out}}) - n + d - 1 - \sum_{i=1}^{d} \deg(x_i) + 2k \quad \text{for } {}_k\overline{\check{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}, \vec{x}), \tag{5-153}$$

$$\deg(y_{\mathrm{out}}) - n + d - \sum_{i=1}^{d} \deg(x_i) + 2k \quad \text{for } {}_k\overline{\mathcal{R}}{}^{S^1}_d(y_{\mathrm{out}}, \vec{x}), \tag{5-154}$$

$$\deg(y_{\mathrm{out}}) - n + d - \sum_{i=1}^{d} \deg(x_i) + 2k \quad \text{for } {}_k\overline{\hat{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}, \vec{x}). \tag{5-155}$$

By Assumption 3.10, for generic choices of Floer data, the Gromov–Floer compactifications of the components of virtual dimension ≤ 1 of (5-150)–(5-152) are compact manifolds-with-boundary of the expected dimension. For rigid elements u in the moduli spaces (5-150)–(5-152), which occur for asymptotics $(y, \vec{x})$ satisfying

$$(5\text{-}153) = 0, \quad (5\text{-}154) = 0 \quad \text{or} \quad (5\text{-}155) = 0,$$

respectively, the orientations (5-138)–(5-140) and [1, Lemma C.4] induce isomorphisms of orientation lines

$$({}_k\check{\mathcal{R}}^1_d)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y, \tag{5-156}$$

$$({}_k\mathcal{R}^{S^1}_d)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y, \tag{5-157}$$

$$({}_k\hat{\mathcal{R}}^1_d)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y. \tag{5-158}$$

Summing the application of these isomorphisms over all rigid u (or "counting rigid elements") defines the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of three families of operations $\check{\mathcal{OC}}^k$, $\mathcal{OC}^{S^1,k}$ and $\hat{\mathcal{OC}}^k$, up to a sign twist specified below. Define

$$\check{\mathcal{OC}}^k([x_d], \dots, [x_1]) := \sum_{u \in {}_k\overline{\check{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\check{\star}_d} ({}_k\check{\mathcal{R}}^1_d)_u([x_d], \dots, [x_1]), \tag{5-159}$$

$$\mathcal{OC}^{S^1,k}([x_d], \dots, [x_1]) := \sum_{u \in {}_k\overline{\mathcal{R}}{}^{S^1}_d(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\star^{S^1}_d} ({}_k\mathcal{R}^{S^1}_d)_u([x_d], \dots, [x_1]), \tag{5-160}$$

$$\hat{\mathcal{OC}}^k([x_d], \dots, [x_1]) := \sum_{u \in {}_k\overline{\hat{\mathcal{R}}}{}^1_d(y_{\mathrm{out}}; \vec{x}) \text{ rigid}} (-1)^{\hat{\star}_d} ({}_k\hat{\mathcal{R}}^1_d)_u([x_d], \dots, [x_1]), \tag{5-161}$$

where the signs are given by

$$\check{\star}_d = \deg(x_d) + \sum_i i \cdot \deg(x_i), \tag{5-162}$$

$$\star_d^{S^1} = \clubsuit_d \sum_{i=1}^{d} (i+1) \cdot \deg(x_i) + \deg(x_d) + d - 1 = \check{\star}_d + \maltese_d - 1, \tag{5-163}$$

$$\hat{\star}_d = \sum_i i \cdot \deg(x_i). \tag{5-164}$$

A codimension-one analysis of the moduli spaces (5-150) and (5-152) reveals:

Proposition 5.19 *The following equations hold for each $k \geq 0$:*

$$(-1)^n \sum_{i=0}^{k} \delta_i \check{\mathcal{OC}}^{k-i} = \mathcal{OC}^{S^1,k-1} + \check{\mathcal{OC}}^k b, \tag{5-165}$$

$$(-1)^n \sum_{i=0}^{k} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k (1-t). \tag{5-166}$$

Proof The boundary of the one-dimensional components of (5-150) are covered by the (rigid components of) the following types of strata:

- Spaces of maps with domain lying on the codimension-one boundary of the moduli space, ie in (5-118)–(5-121).
- Semistable breakings, namely those of the form

$$ {}_k\check{\overline{\mathcal{R}}}{}_d^1(y_1; \vec{x}) \times \overline{\mathcal{M}}(y_{\mathrm{out}}; y_1), \tag{5-167}$$

$$\overline{\mathcal{R}}^1(x; x_i) \times {}_k\check{\overline{\mathcal{R}}}{}_d^1(y_{\mathrm{out}}; \tilde{\vec{x}}), \tag{5-168}$$

  where again $\tilde{\vec{x}}$ denotes the collection of inputs $\vec{x}$ with x_i replaced with x.

All together, this implies, up to sign, that

$$(-1)^n \sum_{i=0}^{k} \delta_i \check{\mathcal{OC}}^{k-i} = \mathcal{OC}^{S^1,k-1} + \check{\mathcal{OC}}^k b + \sum_{i=1}^{k-1} \check{\mathcal{OC}}^{k,i,i+1}, \tag{5-169}$$

where $\check{\mathcal{OC}}^{k,i,i+1}$ is an operation corresponding with some sign twist to (5-121). Of course equation (5-169) is a shorthand for saying, for a tuple of d cyclically composable morphisms $x_d, \dots, x_1$ (recalling the abuse of notation $x_i := [x_i]$), that

$$
\begin{aligned}
\text{(5-170)}\quad & (-1)^n \sum_{i=0}^{k} \delta_i \check{\mathcal{OC}}_d^{k-i}(x_d, \dots, x_1) \\
&= \mathcal{OC}_d^{S^1,k-1}(x_d, \dots, x_1) + \sum_{i=1}^{k-1} \check{\mathcal{OC}}_d^{k,i,i+1}(x_d, \dots, x_1) \\
&\quad + \sum_{i,s} (-1)^{\maltese_1^s} \check{\mathcal{OC}}_{d-i+1}^{k}(x_d, \dots, x_{s+i+1}, \mu^i(x_{s+i}, \dots, x_{s+1}), x_s, \dots, x_1) \\
&\quad + \sum_{i,j} (-1)^{\sharp_j^i} \check{\mathcal{OC}}^{k}(\mu^{i+j+1}(x_i, \dots, x_1, x_d, \dots, x_{d-j}), x_{d-j-1}, \dots, x_{i+1}).
\end{aligned}
$$

We first note that in fact the operation $\check{\mathcal{OC}}^{k,i,i+1} = \sum_d \check{\mathcal{OC}}_d^{k,i,i+1}$ is zero, because by condition (5-141), the Floer datum chosen for elements S in (5-121) are constant along the one-dimensional fibers of $\check{\pi}_i$. Hence, elements of the moduli space with source in (5-121) are never rigid; see Lemma 4.11 for an analogous and more detailed explanation.

Thus, it suffices to verify that the signs coming from the codimension-one boundary are exactly those appearing in (5-169). We can safely ignore studying any signs for the vanishing operations such as $\hat{\mathcal{OC}}^{k,i,i+1}$. The remaining sign analysis is exactly as in Proposition 5.16; more precisely, note that the forgetful map $\check{F}_k \colon {}_k\check{\mathcal{R}}_d^1 \to {}_1\check{\mathcal{R}}_d^1$ which forgets $p_1, \dots, p_{k-1}$ has complex oriented fibers, and in particular (since the marked points p_i contribute complex domain orientations and do not introduce any new orientation lines) the sign computations sketched in Proposition 5.16 carry over for any stratum whose domain is pulled back from a boundary stratum of ${}_1\check{\mathcal{R}}_d^1$; in turn, as described in Proposition 5.16, the sign computations for ${}_1\check{\mathcal{R}}_d^1$ largely reduce to those for ${}_0\check{\mathcal{R}}_d^1$. This verifies (5-169).

Similarly, for the hat moduli space, an analysis of the boundary of one-dimensional moduli spaces of maps tells us, up to sign verification,

$$
\text{(5-171)}\quad (-1)^n \sum_{i=0}^{k} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k (1-t) + \hat{\mathcal{OC}}^{k,k,k+1} + \sum_{i=1}^{k-1} \hat{\mathcal{OC}}^{k,i,i+1},
$$

where $\hat{\mathcal{OC}}^{k,k,k+1}$ is an operation corresponding with some sign twist to (5-133), and $\hat{\mathcal{OC}}^{k,i,i+1}$ is an operation corresponding with some sign twist to (5-134). The conditions (5-147)–(5-148) similarly imply that $\hat{\mathcal{OC}}^{k,k,k+1}$ and $\hat{\mathcal{OC}}^{k,i,i+1}$ are zero, so it is not necessary to even establish what the signs for these terms are.

To verify signs for (5-171), we apply the principle discussed in the proof of Lemma 5.8, in which by treating the auxiliary boundary marked point z_f as possessing a "formal unit element asymptotic constraint e_+", therefore viewing $\hat{\mathcal{OC}}^k(x_d \otimes \cdots \otimes x_1) := \hat{\mathcal{OC}}^k(x_d, \dots, x_1)$ formally as $\hat{\mathcal{OC}}^k(e^+ \otimes x_d \otimes \cdots \otimes x_1)$, the signs for (5-171) applied to strings $(x_d \otimes \cdots \otimes x_1)$ of length d follow from the sign computations for $\check{\mathcal{OC}}$ applied to strings $(e^+ \otimes x_d \otimes \cdots \otimes x_1)$ of length $d+1$. This analysis applies to the term $\hat{\mathcal{OC}}^{k,k,k+1}$ as well, which is the hat version of $\mathcal{OC}^{S^1,k}$; however, the former operation happens to be zero because extra symmetries imply the moduli space controlling this operation is never rigid. □

Next, by decomposing the moduli space ${}_k\mathcal{R}_d^{S^1}$ into sectors, we can write the auxiliary operation $\mathcal{OC}^{S^1,j}$ in terms of $\hat{\mathcal{OC}}^j$ and Connes' B operator.

Proposition 5.20 *As chain-level operations*,

$$\mathcal{OC}^{S^1,k} = \hat{\mathcal{OC}}^k \circ B^{\mathrm{nu}}. \tag{5-172}$$

Proof The proof directly emulates Proposition 5.9, and as such we will give fewer details. We begin by defining, for $i \in \mathbb{Z}/d\mathbb{Z}$, operations

$$\hat{\mathcal{OC}}^k_{d,\tau_i} \tag{5-173}$$

associated to various "sectors" of the $k+1^{\text{st}}$ marked point p_{k+1} of ${}_k\mathcal{R}_d^{S^1}$. Once more, to gain better control of the geometry of these sectors in the compactification (when the sector size can shrink to zero), we pass to an alternative model for the compactification. Define

$${}_k\mathcal{R}^1_{d,\tau_i} \tag{5-174}$$

to be the abstract moduli space of discs with $d+1$ boundary punctures, $z_1, \dots, z_i, z_f, z_{i+1}, \dots, z_d$ arranged in counterclockwise order, one interior negative puncture z_{out} with asymptotic marker, and k additional interior auxiliary marked points $p_1, \dots, p_k$ which are *strictly radially ordered* with norms in $\left(0, \frac{1}{2}\right)$ for a representative fixing z_0 at 1 and z_{out} at 0, so

$$0 < |p_1| < \cdots < |p_k| < \tfrac{1}{2}. \tag{5-175}$$

Moreover, as before:

(5-176) The asymptotic marker on z_{out} points in the direction θ_1 (or towards z_f if $k = 0$).

There is a bijection

$$\tau_i \colon {}_k\mathcal{R}^1_{d,\tau_i} \to {}_k\widehat{\mathcal{R}}^1_d \tag{5-177}$$

given by cyclically permuting boundary labels, and in particular we also have an *auxiliary-rescaling map*, as in (5-137),

$${}_k\mathcal{R}^1_{d,\tau_i} \to {}_k\mathcal{R}_d^{S^1_{i,i+1}}, \tag{5-178}$$

which, for a representative with $|z_{\text{out}}| = 0$, adds a point p_{k+1} on the line between z_{out} and z_f with $|p_{k+1}| = \frac{1}{2}$ and deletes z_f. We choose orientations on ${}_k\mathcal{R}^1_{d,\tau_i}$ to be compatible with (5-178); more concretely, for a slice fixing the positions of z_{out} and z_d, consider the top form

$$r_1 \cdots r_k \, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dz_d \wedge dz_f \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k. \tag{5-179}$$

The compactification ${}_k\overline{\mathcal{R}}^1_{d,\tau_i}$ is inherited from the identification (5-177); the salient point is that we treat bubbled-off boundary strata containing the point z_f as coming from $\mathcal{R}^{d,f_i}$, the moduli space of discs with i^{th} marked point forgotten (where the i^{th} marked point is z_f), constructed in Section A.2.

We choose as a Floer datum for ${}_k\overline{\mathcal{R}}^1_{d,\tau_i}$ the pulled-back Floer datum from (5-177); it automatically then exists and is universal and consistent as desired. Moreover we have chosen orientations as in the case $k = 0$ so that the auxiliary rescaling map (5-178) is an oriented diffeomorphism extending to a map between compactifications.

Thus, for a given a Lagrangian labeling $\{L_0, \dots, L_{d-1}\}$ and compatible asymptotics $\{x_1, \dots, x_d; y_{\text{out}}\}$ we obtain a moduli space of maps satisfying Floer's equation with the chosen boundary and asymptotics,

$${}_k\overline{\mathcal{R}}^1_{d,\tau_i}(y_{\text{out}}; \vec{x}) := {}_k\overline{\widehat{\mathcal{R}}}{}^1_d(y_{\text{out}}; x_{i-1}, \dots, x_1, x_d, \dots, x_i), \tag{5-180}$$

which is (for components of virtual dimension ≤ 1) a manifold of dimension equal to the virtual dimension of the right-hand side, namely

$$\deg(y_{\text{out}}) - n + d - \sum_{j=1}^{d} \deg(x_j) + 2k,$$

with $\mathbb{Z}$–gradings or mod 2 if working with $\mathbb{Z}/2$–gradings. The isomorphisms of orientation lines

$$({}_k\mathcal{R}^1_{d,\tau_i})_u \colon o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_{y_{\text{out}}} \tag{5-181}$$

induced by elements u of the zero-dimensional components of (5-180) define the $|o_{y_{\mathrm{out}}}|_{\boldsymbol{k}}$ component of the operation $\hat{\mathcal{OC}}^k_{d,\tau_i}$, up to the sign twist

$$\hat{\mathcal{OC}}^k_{d,\tau_i}([x_d],\dots,[x_1]) := \sum_{u \in {}_k\overline{\hat{\mathcal{R}}}^1_{d,\tau_i}(y_{\mathrm{out}};\vec{x}) \text{ rigid}} (-1)^{\hat{\star}_d} ({}_k\mathcal{R}^1_{d,\tau_i})_u([x_d],\dots,[x_1]), \tag{5-182}$$

where $\star^{S^1}_d = \sum_{i=1}^d (i+1)\cdot\deg(x_i) + \deg(x_d) + d - 1$.

Now, exactly as in Lemma 5.12, there is a chain-level equality of signed operations

$$\mathcal{OC}^{S^1,k}_d = \sum_{i=0}^{d-1} \hat{\mathcal{OC}}^k_{d,\tau_i}. \tag{5-183}$$

We recall the geometric statement underlying this: the point is that by construction there is an oriented embedding

$$\coprod_i {}_k\mathcal{R}^1_{d,\tau_i} \xrightarrow{\coprod_i \pi^i_f} \coprod_i {}_k\mathcal{R}^{S^1_{i,i+1}}_d \hookrightarrow {}_k\mathcal{R}^{S^1}_d, \tag{5-184}$$

compatible with Floer data, covering all but a codimension-one locus in the target, and moreover all the sign twists defining the operations $\mathcal{OC}^k_{d,\tau_i}$ are chosen to be compatible with the sign twist in the operation $\mathcal{OC}^{S^1,k}$ — this uses the fact that the Floer data on ${}_k\mathcal{R}^{S^1_{i,i+1}}$ agrees with the data on ${}_k\hat{\mathcal{R}}^1_d$ via the cyclic permutation map κ^{-i} by (5-55). After perturbation, zero-dimensional solutions to Floer's equation can be chosen to come from the complement of any codimension-one locus in the source abstract moduli space, implying the equality (5-183).

Finally, all that remains is a sign analysis, whose conclusion is that

$$\hat{\mathcal{OC}}^k_{d,\tau_i} = \hat{\mathcal{OC}}^k_d \circ s^{\mathrm{nu}} \circ t^i, \tag{5-185}$$

where s^{nu} is the operation arising from changing a check term to a hat term with a sign twist (3-20). (The equality up to comparing signs is immediate, as the operations are constructed with identical Floer data and hence involve counts of identical moduli spaces.) The details of this sign comparison are exactly the same as in Lemma 5.13, including with signs, since when orienting the moduli of maps, the additional marked points $p_1,\dots p_k$ only contribute complex orientations to the moduli spaces of domains (and no additional orientation line terms). □

Proof of Proposition 5.17 This is an immediate corollary of the previous two propositions. □

We now collect all of this information to finish the proof of our main result.

Proof of Theorem 1.1 The premorphism $\widetilde{\mathcal{OC}} \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}), CF^*(M))$, written u–linearly as $\sum_i \mathcal{OC}^k u^k$, where the $\mathcal{OC}^k = \check{\mathcal{OC}}^k \oplus \hat{\mathcal{OC}}^k$ are as constructed above, satisfies $\partial \widetilde{\mathcal{OC}} = 0$ by Proposition 5.17. Hence $\widetilde{\mathcal{OC}}$ is closed, or an S^1–complex homomorphism, also known as an A_∞ $C_{-*}(S^1)$–module homomorphism; see Section 2.1. Note that $[\mathcal{OC}^0] = [\mathcal{OC}] = [\check{\mathcal{OC}}]$, where the first equality holds by definition and the second holds by Corollary 5.4. Hence $\widetilde{\mathcal{OC}}$ is an enhancement of $\check{\mathcal{OC}}$, as defined in Section 2.1. □

Proof of Corollary 1.5 This is an immediate consequence of Theorem 1.1 and the induced homotopy-invariance properties for equivariant homology groups discussed in Section 2, particularly Corollary 2.18 and Proposition 2.19. □

5.6 Variants of the cyclic open–closed map

5.6.1 Using singular (pseudo)cycles instead of Morse cycles Let M be Liouville or compact and admissible (in which case by our convention $\overline{M} = M$ and $\partial \overline{M} = \varnothing$),[18] and let us consider the version of $\widetilde{\mathcal{OC}}$ with target the relative cohomology $H^*(\overline{M}, \partial \overline{M})$ as in Section 4.1.2. Instead of using a C^2–small Hamiltonian to define the Floer complex computing $H^{*+n}(\overline{M}, \partial \overline{M})$ (which we only did for simultaneous compatibility with the symplectic cohomology case), we can pass to a geometric cycle model for the group, and then build a version of the map $\widetilde{\mathcal{OC}}$ with such a target, which simplifies many of the constructions in the previous section, in the sense that the codimension-one boundary strata of moduli spaces, and hence the equations satisfied by $\widetilde{\mathcal{OC}}$, are strictly a subset of the terms appearing above. As such, it will be sufficient to fix some notation for the relevant moduli spaces, and state the relevant simplified results.

We let

$$ {}_k\check{\mathcal{P}}^1_d, \tag{5-186} $$

$$ {}_k\mathcal{P}^{S^1}_d, \tag{5-187} $$

$$ {}_k\hat{\mathcal{P}}^1_d \tag{5-188} $$

denote copies of the abstract moduli spaces (5-107)– (5-109), where the interior puncture z_{out} is filled in and replaced by a marked point $\bar{z}_{\mathrm{out}}$, *without any asymptotic marker*. The compactifications of these moduli spaces are exactly as before, except that the

[18] Technically we should write $\mathrm{QH}^*(M)$ in the latter case, but additively $\mathrm{QH}^*(M) = H^*(M)$, and correspondingly no sphere bubbling occurs in the moduli spaces we define here, so there is no difference for the purposes of this discussion.

auxiliary points $p_1, \dots, p_k$ are now allowed to coincide with $\overline{z}_{\mathrm{out}}$, without breaking off an angle-decorated cylinder or element of $\mathcal{M}_r$ (in the language of Section 4.3). In other words, the real blow-up of Deligne–Mumford compactifications at z_{out} described in Section A.1, which was responsible for the boundary strata containing $\mathcal{M}_r$ factors, *no longer occurs*, but all other degenerations do occur. Correspondingly the codimension-one boundaries of compactified moduli spaces have all of the strata as before except for strata containing the $\mathcal{M}_r$ factors.

Inductively choose smoothly varying families Floer data as before on these moduli spaces of domains, satisfying all of the requirements and consistency conditions as before, except for any consistency conditions involving $\mathcal{M}_r$ moduli spaces, which no longer occur on the boundary. For a basis $\beta_1, \dots, \beta_s$ of smooth (pseudo)cycles in homology $H_*(M)$ whose Poincaré duals $[\beta_i^\vee]$ generate the cohomology $H^*(\overline{M}, \partial\overline{M})$, one obtains moduli spaces

$$ {}_k\check{\mathcal{P}}_d^1(\beta_i; \vec{x}), \tag{5-189}$$

$$ {}_k\mathcal{P}_d^{S^1}(\beta_i, \vec{x}), \tag{5-190}$$

$$ {}_k\hat{\mathcal{P}}_d^1(\beta_i, \vec{x}) \tag{5-191}$$

of moduli spaces of maps into M with source an arbitrary element of the relevant domain moduli space, satisfying Floer's equation as before, with Lagrangian boundary and asymptotics $\vec{x}$ as before, *with the additional point constraint that $\overline{z}_{\mathrm{out}}$ lie on the cycle β_i*. As before, standard methods ensure that zero- and one-dimensional moduli spaces are (for generic choices of perturbation data and/or β_i) transversely cut-out manifolds of the "right" dimension and boundary, which is all that we need.

Then, define the coefficient of $[\beta_i^\vee] \in H^*(\overline{M}, \partial\overline{M})$ in $\check{\mathcal{OC}}^k(x_d \otimes \cdots \otimes x_1)$ to be given by signed counts (with the same sign twists as before) of the moduli spaces (5-189); similarly for $\hat{\mathcal{OC}}^k$ and $\mathcal{OC}^{S^1,k}$ using the moduli spaces (5-191) and (5-190). A simplification of the arguments already given (in which the δ_k operations no longer occur, but every other part of the argument carries through) implies:

Proposition 5.21 *The premorphism*

$$\widetilde{\mathcal{OC}} = \sum_{i=0}^{\infty} \mathcal{OC}^k u^k \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}, \mathcal{F}), H^*(\overline{M}, \partial\overline{M}))$$

satisfies

$$\widetilde{\mathcal{OC}} \circ b_{\mathrm{eq}} = 0, \tag{5-192}$$

where $b_{\mathrm{eq}} = b^{\mathrm{nu}} + uB^{\mathrm{nu}}$. In other words, $\widetilde{\mathcal{OC}}$ is a homomorphism of S^1–complexes between $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F},\mathcal{F})$ with its strict S^1–action and $H^*(\overline{M},\partial\overline{M})$ with its trivial S^1–action.

As usual this model of $\widetilde{\mathcal{OC}}$ again induces maps $\widetilde{\mathcal{OC}}^{+/-/\infty}$ between homotopy orbit complexes, homotopy fixed-point complexes etc; note that the relevant equivariant homology chain complexes are particularly simple for the latter $H^*(\overline{M},\partial\overline{M})$, seeing as there is no differential and trivial circle action; for instance,

$$H^*(\overline{M},\partial\overline{M})_{\mathrm{h}S^1} = \big(H^*(\overline{M},\partial\overline{M})(\!(u)\!)/uH^*(\overline{M},\partial\overline{M})[\![u]\!], \delta_{\mathrm{eq}} = 0\big).$$

5.6.2 Compact Lagrangians in noncompact manifolds Now let us explicitly restrict to the case of M a Liouville manifold, and denote by $\mathcal{F} \subset \mathcal{W}$ the full subcategory consisting of a finite collection of compact exact Lagrangian branes contained in the compact region $\overline{M}$. By Poincaré duality we may think of the map $\mathcal{OC}$ (and its cyclic analogue, $\widetilde{\mathcal{OC}}$) with target $H^*(\overline{M},\partial\overline{M})$ as a pairing $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F},\mathcal{F}) \otimes C^*(M) \to \boldsymbol{k}[n]$. In this case, there is a nontrivial refinement of this pairing to

$$\mathcal{OC}_{\mathrm{cpct}}\colon \mathrm{CH}_*(\mathcal{F}) \otimes SC^*(M) \to \boldsymbol{k}[-n], \tag{5-193}$$

where $SC^*(M)$ is the *symplectic cohomology* cochain complex.

Remark 5.22 The refinement (5-193) relies on extra flexibility in Floer theory for compact Lagrangians compared to noncompact Lagrangians (compare Remarks 3.16 and 3.17), first alluded to in this form in [54]. This extra flexibility allows us to define operations without outputs — and in particular study a version of the open–closed map where the interior marked point and boundary marked points are all inputs — for instance by Poincaré dually treating some boundary inputs as outputs with "negative weight".

One way to implement such operations, using the type of Floer data discussed in Remark 3.17, is by allowing the *subclosed one-form* α_S used in Floer-theoretic perturbations to have complete freedom along boundary conditions corresponding to compact Lagrangians; in contrast, along possibly noncompact Lagrangian boundary conditions, α_S is required to vanish in order to appeal to the integrated maximum principle. In particular, if we allow α_S to be nonvanishing along boundary components, Stokes' theorem no longer implies that α_S being subclosed implies that the total "output" weights must be greater than the total "input" weights.

Remark 5.23 The existence of a map $SC^*(M) \to \mathrm{CH}_*(\mathcal{F})^\vee[-n]$ is well known. Namely, categories $\mathcal{C}$ with a *weak proper Calabi–Yau structure*[19] of dimension n come equipped with isomorphisms between the dual of Hochschild chains and Hochschild cochains $\mathrm{CH}_*(\mathcal{C})^\vee[-n] \simeq \mathrm{CH}^*(\mathcal{C})$, and the existence of a map $SC^*(M) \to \mathrm{CH}^*(\mathcal{F})$ was observed in [49].

The geometric moduli spaces used to establish our main result apply verbatim in this case, with the interior marked point changed to an input, and the ordering of the auxiliary marked points $p_1, \dots, p_k$ appearing in the cyclic open–closed map reversed. In this case, the operations associated to such moduli spaces imply:

Proposition 5.24 *Consider* $\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}) \otimes SC^*(M)$ *as an* S^1*–complex with its diagonal* S^1*–action* (see Lemma 2.11 in Section 2.1), *and* $\boldsymbol{k} = \underline{\boldsymbol{k}}^{\mathrm{triv}} \in S^1$–mod *with its trivial* S^1*–complex structure. The map from* $\mathrm{CH}_*(\mathcal{F}) \otimes SC^*(M)$ *to* $\boldsymbol{k}$ *can be enhanced to a homomorphism of* S^1*–complexes*

$$\widetilde{\mathcal{OC}}_{\mathrm{cpct}} \in \mathrm{Rhom}^n_{S^1}(\mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}) \otimes SC^*(M), \boldsymbol{k}).$$

For example, $\widetilde{\mathcal{OC}}_{\mathrm{cpct}}$ *satisfies* $\partial \widetilde{\mathcal{OC}}_{\mathrm{cpct}} = 0$. *In other words, in the notation of Section 2.3, there exists a map*

$$\widetilde{\mathcal{OC}}_{\mathrm{cpct,eq}} = \sum_{i=0}^{\infty} \mathcal{OC}_{\mathrm{cpct},i} u^i : \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{F}) \otimes SC^*(M) \to \boldsymbol{k}[[u]]$$

of pure degree n, *with* $[\mathcal{OC}_{\mathrm{cpct},0}] = [\mathcal{OC}_{\mathrm{cpct}}]$, *such that*

$$\widetilde{\mathcal{OC}}_{\mathrm{cpct,eq}} \circ \big((-1)^{\deg(y)} b_{\mathrm{eq}}(\sigma) \otimes y + \sigma \otimes \delta^{SC}_{\mathrm{eq}}(y)\big) = 0.$$

To clarify the relevant moduli spaces used, we define the spaces

$$ {}_k\check{\mathcal{R}}^1_{d,\mathrm{cpct}}, \tag{5-194}$$

$$ {}_k\mathcal{R}^{S^1}_{d,\mathrm{cpct}}, \tag{5-195}$$

$$ {}_k\hat{\mathcal{R}}^1_{d,\mathrm{cpct}}, \tag{5-196}$$

to be copies of the abstract moduli spaces (5-107)–(5-109) where the interior puncture z_{out} is now a *positive* puncture (still equipped with an asymptotic marker), and all of the other inputs and auxiliary points are as before, except we've reversed the order

[19] Such as the Fukaya category of compact Lagrangians; see eg [52, (12j); 53, Proof of Proposition 5.1, Step 1; 60, Section 2.8].

of the labelings $p_1, \dots, p_k$ (for notational convenience), so the ordering constraints all now read as $0 < |p_k| < \cdots < |p_1| < \frac{1}{2}$. The compactified moduli spaces have boundary strata agreeing with the boundary strata of the compactified (5-107)–(5-109), except now the $\mathcal{M}_r$ cylinders break "above" the ${}_{k-r}\mathcal{R}^1_{d,\mathrm{cpct}}$ (equipped with $\check{}$, $\hat{}$ or S^1 decoration) discs instead of "below". The reversal of the ordering of auxiliary marked points is designed to be compatible with the ordering of the auxiliary marked points on the $\mathcal{M}_r$ moduli spaces when it breaks "above" (as in $\mathcal{M}_r$, the label numbers of the auxiliary marked points increase from top to bottom).

Equipping these moduli spaces with perturbation data satisfying the same consistency conditions and other requirements as before, and counting solutions with sign twists as before, defines the terms of the premorphism exactly as in the previous subsections, with identical analysis to show that, for instance, the operation corresponding to ${}_k\mathcal{R}^{S^1}_{d,\mathrm{cpct}}$ is the operation corresponding to ${}_{k-1}\widehat{\mathcal{R}}^1_{d,\mathrm{cpct}}$ composed with Connes' B operator, the boundary strata in which $|p_i|$ and $|p_{i+1}|$ are coincident contributes trivially, and so on.

6 Calabi–Yau structures

6.1 The proper Calabi–Yau structure on the Fukaya category

Here we review the notion of a *proper Calabi–Yau structure*, following Kontsevich and Soibelman [37], and construct proper Calabi–Yau structures on Fukaya categories of compact Lagrangians in a compact admissible or Liouville manifold. A proper Calabi–Yau structure induces chain-level topological-field-theoretic operations on the Hochschild chain complex of the given category, controlled by the open moduli space of curves with marked points equipped with asymptotic markers, at least one of which is an input [14; 37]. Note that Costello's work [14] constructing field-theoretic operations has the (a priori stronger) requirement that the underlying A_∞ category be *cyclic*, but in characteristic zero any proper Calabi–Yau structure determines a unique quasi-isomorphism between the underlying A_∞ category and a cyclic A_∞ category [37, Theorem 10.7]; see Remark 1.11 for more discussion.

We say an A_∞ category $\mathcal{A}$ is *proper* (sometimes called *compact*) if its cohomological morphism spaces $H^*(\hom_{\mathcal{A}}(X, Y))$ have total finite rank over $\boldsymbol{k}$ for each X, Y. Recall that for any object $X \in \mathcal{A}$, there is an inclusion of chain complexes

$$\hom(X, X) \to \mathrm{CH}_*(\mathcal{A}),$$

inducing a map

$$[i]\colon H^*(\hom(X,X)) \to \mathrm{HH}_*(\mathcal{A}).$$

Definition 6.1 Let $\mathcal{A}$ be a proper category. A chain map $\mathrm{tr}\colon \mathrm{CH}_{*+n}(\mathcal{A}) \to \boldsymbol{k}$ is called a *weak proper Calabi–Yau structure*, or *nondegenerate trace* of dimension n if, for any two objects $X, Y \in \mathrm{ob}\,\mathcal{A}$, the composition

$$H^*(\hom_{\mathcal{A}}(X,Y)) \otimes H^{n-*}(\hom_{\mathcal{A}}(Y,X)) \xrightarrow{[\mu^2_{\mathcal{A}}]} H^n(\hom_{\mathcal{A}}(Y,Y)) \xrightarrow{[i]} \mathrm{HH}_n(\mathcal{A}) \xrightarrow{[\mathrm{tr}]} \boldsymbol{k} \tag{6-1}$$

is a perfect pairing; this nondegeneracy property only depends on the homology class [tr]. A chain map from the nonunital Hochschild complex $\mathrm{tr}\colon \mathrm{CH}^{\mathrm{nu}}_{*+n}(\mathcal{A}) \to \boldsymbol{k}$ is called a weak proper Calabi–Yau structure if composition with the inclusion $\mathrm{CH}_{*+n}(\mathcal{A}) \subset \mathrm{CH}^{\mathrm{nu}}_{*+n}(\mathcal{A})$ is a weak proper Calabi–Yau structure in the sense above.

Remark 6.2 In the symplectic literature, weak proper Calabi–Yau structures of dimension n are sometimes defined as bimodule quasi-isomorphisms $\mathcal{A}_\Delta \xrightarrow{\sim} \mathcal{A}^\vee[n]$, where $\mathcal{A}_\Delta$ denotes the *diagonal bimodule* and $\mathcal{A}^\vee$ the *linear dual diagonal bimodule*; see [52, (12j)] and Section 6.2 for brief conventions on A_∞–bimodules, see also [62]. To explain the relationship between this definition and the one above, which has sometimes been called a *weakly cyclic structure* or ∞–*inner product* [62; 60], note that for any compact A_∞ category $\mathcal{A}$, there are quasi-isomorphisms (with explicit chain-level models)

$$(\mathrm{CH}_*(\mathcal{A}))^\vee = \mathrm{CH}^*(\mathcal{A}, \mathcal{A}^\vee) \xleftarrow{\sim} \hom_{\mathcal{A}\text{–}\mathcal{A}}(\mathcal{A}_\Delta, \mathcal{A}^\vee), \tag{6-2}$$

where $\hom_{\mathcal{A}\text{–}\mathcal{A}}$ denotes morphisms in the category of A_∞–bimodules; see eg [50] or [24]. Under this correspondence, nondegenerate morphisms from $\mathrm{HH}_*(\mathcal{A}) \to \boldsymbol{k}$ as defined above correspond precisely (cohomologically) to weak Calabi–Yau structures, for instance, those bimodule morphisms from $\mathcal{A}_\Delta$ to $\mathcal{A}^\vee$ which are cohomology isomorphisms.

Remember that the Hochschild chain complex of an A_∞ category $\mathcal{A}$ comes equipped with a natural chain map to the (positive) cyclic homology chain complex, the *projection to homotopy orbits* (2-22),

$$\mathrm{pr}\colon \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{A}) \to \mathrm{CC}^+_*(\mathcal{A}),$$

modeled on the chain level by the map that sends $\alpha \mapsto \alpha \cdot u^0$ for $\alpha \in \mathrm{CH}^{\mathrm{nu}}(\mathcal{A})$; compare with (2-35).

Definition 6.3 (cf Kontsevich and Soibelman [37]) A (*strong*) *proper Calabi–Yau structure of degree n* is a chain map

$$\widetilde{\mathrm{tr}}\colon \mathrm{CC}_*^+(\mathcal{A}) \to \boldsymbol{k}[-n] \tag{6-3}$$

from the (positive) cyclic homology chain complex of $\mathcal{A}$ to $\boldsymbol{k}$ of degree $-n$, such that the induced map $\mathrm{tr} = \widetilde{\mathrm{tr}} \circ \mathrm{pr}\colon \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \to \boldsymbol{k}[-n]$ — or equivalently the composition $\check{\mathrm{tr}}$ of tr with the inclusion $\mathrm{CH}_*(\mathcal{A}) \subset \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$ — is a weak proper Calabi–Yau structure.

Via the model for cyclic chains given as

$$\mathrm{CC}_*^+(\mathcal{A}) := (\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})(\!(u)\!)/u\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})[\![u]\!], b + uB^{\mathrm{nu}}),$$

such an element $\widetilde{\mathrm{tr}}$ takes the form

$$\widetilde{\mathrm{tr}} := \sum_{i=0}^{\infty} \mathrm{tr}^k\, u^k, \tag{6-4}$$

where

$$\mathrm{tr}^k := (\check{\mathrm{tr}}^k \oplus \hat{\mathrm{tr}}^k)\colon \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \to \boldsymbol{k}[-n-2k]. \tag{6-5}$$

We now complete the proof of Theorem 1.12 described and sketched in Section 1: first, define the putative proper Calabi–Yau structure as the composition

$$\widetilde{\mathrm{tr}}\colon \mathrm{CC}_*^+(\mathcal{F}) \xrightarrow{\widetilde{\mathcal{OC}}^+} C^{*+n}(\overline{M}, \partial\overline{M}) \otimes_{\boldsymbol{k}} \boldsymbol{k}(\!(u)\!)/u\boldsymbol{k}[\![u]\!] \to \boldsymbol{k}, \tag{6-6}$$

where the last map (cohomologically) sends $\mathrm{PD}(\mathrm{pt})\cdot u^0 \in H^{2n}(\overline{M}, \partial\overline{M})$ to 1, and other elements to 0; ie it projects to the u^0 factor then integrates over $[M]$. Instead of using a C^2–small Hamiltonian to define the Floer complex computing $H^{*+n}(\overline{M}, \partial\overline{M})$, which we only did for simultaneous compatibility with the symplectic cohomology case, we can pass to a geometric cycle model for $\widetilde{\mathcal{OC}}^+$ (and therefore $\widetilde{\mathrm{tr}}$), which as described in Section 5.6.1 directly maps (on the chain level) to

$$H^{*+n}(\overline{M}, \partial\overline{M}) \otimes_{\boldsymbol{k}} \boldsymbol{k}(\!(u)\!)/u\boldsymbol{k}[\![u]\!].$$

With respect to this model, the map $\widetilde{\mathrm{tr}}$ involves counts of the moduli spaces described there, where the interior marked point $\overline{z}_{\mathrm{out}}$ is *unconstrained*, eg ${}_k\check{\mathcal{P}}_d^1([M]; \vec{x})$, ${}_k\hat{\mathcal{P}}_d^1([M]; \vec{x})$ and ${}_k\mathcal{P}_d^{S^1}([M]; \vec{x})$; see Figure 16.

The following well-known lemma verifies the nondegeneracy property of the map $\widetilde{\mathrm{tr}}$.

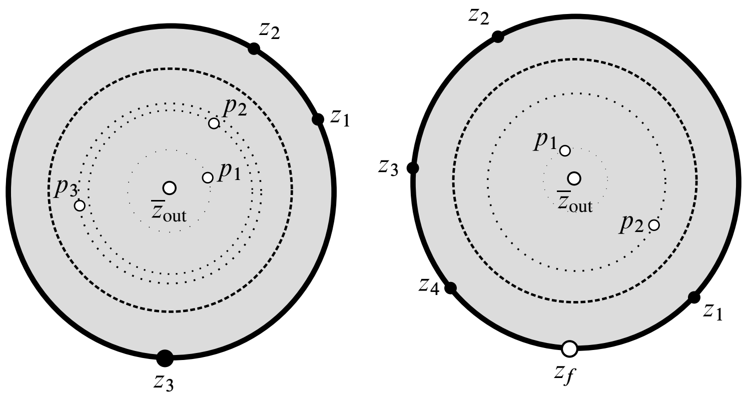

Figure 16: An image of representatives of moduli spaces ${}_3\check{\mathcal{P}}^1_3([M]; \vec{x})$ and ${}_2\widehat{\mathcal{P}}^1_4([M]; \vec{x})$, which appear in the map $\widetilde{\mathrm{tr}}$.

Lemma 6.4 [52, (12j); 60, Lemma 2.4] *The corresponding morphism*

$$[\mathrm{tr}]\colon \mathrm{HH}_{*+n}(\mathcal{F}) \to \boldsymbol{k}$$

is a nondegenerate trace (*or weak proper Calabi–Yau structure*).

Sketch of proof This is an immediate consequence of Poincaré duality in Lagrangian Floer cohomology; see the cited references. As a brief sketch, note that $\check{\mathrm{tr}}^0 \circ \mu^2 = \check{\mathrm{tr}} \circ \mu^2\colon \hom(X,Y) \otimes \hom(Y,X) \to \boldsymbol{k}$ is chain homotopic (and hence equal in cohomology) to a chain map which counts holomorphic discs with an interior marked point satisfying an empty constraint, and two (positive) boundary asymptotics on p and q, with corresponding Lagrangian boundary on x and y. Via a further homotopy of Floer data, one can arrange that the generators of $\hom(X,Y)$ and $\hom(Y,X)$ are in bijection (for instance if one is built out of time-1 flowlines of H and one out of time-1 flowlines of $-H$), and the only such rigid discs are constant discs between p and the corresponding $p^\vee$. □

Proof of Theorem 1.12 The above discussion constructs $\widetilde{\mathrm{tr}}$ and Lemma 6.4 verifies nondegeneracy. □

6.2 The smooth Calabi–Yau structure on the Fukaya category

We give an overview of (a categorical version of) the notion of a (*strong*) *smooth Calabi–Yau structure*, and construct such smooth Calabi–Yau structures on wrapped or compact Fukaya categories under the "nondegeneracy" hypotheses of [24]. Smooth

Calabi-Yau structures were proposed by Kontsevich and Vlassopoulous [36] and later comprehensively studied by Kontsevich, Takeda and Vlassopoulous [39]. Other expositions appear, for instance, in [27] and [7]; in the latter work the terminology "left" is used instead of "smooth", and "right" instead of "proper". A smooth Calabi–Yau structure (analogously to the proper case) induces chain-level topological field theory operations on the Hochschild chain complex of the given category, controlled by the open moduli space of curves with marked points equipped with asymptotic markers, at least one of which is an output [39; 38].[20]

To state the relevant definitions, we make use of some of the theory of *A_∞–bimodules over a category $\mathcal{C}$*. We do so without much explanation, instead referring readers to existing references [50; 62; 24]. An A_∞–bimodule $\mathcal{P}$ over $\mathcal{C}$ is a bilinear A_∞ functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to chain complexes, which is roughly the data of a chain complex $(\mathcal{P}(X,Y), \mu^{0|1|0})$ for every pair of objects $\mathcal{C}$, along with "higher multiplication maps"

$$\mu^{s|1|t} \colon \hom_{\mathcal{C}}(X_{s-1}, X_s) \otimes \cdots \otimes \hom_{\mathcal{C}}(X_0, X_1) \otimes \mathcal{P}(X_0, Y_t) \\ \otimes \hom_{\mathcal{C}}(Y_{t-1}, Y_t) \otimes \cdots \otimes \hom_{\mathcal{C}}(Y_0, Y_1) \to \mathcal{P}(X_s, Y_0)$$

satisfying a generalization of the A_∞ equations. A_∞–bimodules over $\mathcal{C}$ form a dg category $\mathcal{C}$–mod–$\mathcal{C}$, with morphisms denoted by $\hom^*_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{P}, \mathcal{Q})$. (For dg bimodules over a dg category, this chain complex corresponds to a particular chain model for the "derived morphism space" using the bar resolution.) The basic examples of bimodules we require are:

- The *diagonal bimodule* $\mathcal{C}_\Delta$, which associates to a pair of objects (K, L) the chain complex $\mathcal{C}_\Delta(K, L) := \hom_{\mathcal{C}}(L, K)$.
- For any pair of objects A, B, there is a *Yoneda bimodule* $\mathcal{Y}^l_A \otimes_{\boldsymbol{k}} \mathcal{Y}^r_B$, which associates to a pair of objects (K, L) the chain complex $\mathcal{Y}^l_A \otimes_{\boldsymbol{k}} \mathcal{Y}^r_B(K, L) := \hom_{\mathcal{C}}(A, K) \otimes \hom_{\mathcal{C}}(L, B)$.

Yoneda bimodules are the analogues of the free bimodule $A \otimes A^{\mathrm{op}}$ in the category of bimodules over an associative algebra A (which are the same as $A \otimes A^{\mathrm{op}}$–modules). Accordingly, we say a bimodule $\mathcal{P}$ is *perfect* if, in the category $\mathcal{C}$–mod–$\mathcal{C}$, it is split-generated by (ie isomorphic to a retract of a finite complex of) Yoneda bimodules. We say that a category $\mathcal{C}$ is (*homologically*) *smooth* if $\mathcal{C}_\Delta$ is a perfect $\mathcal{C}$–bimodule.

Recall for what follows that for any bimodule $\mathcal{P}$ there is a *cap product action*

$$\cap \colon \mathrm{HH}^*(\mathcal{C}, \mathcal{P}) \otimes \mathrm{HH}_*(\mathcal{C}, \mathcal{C}) \to \mathrm{HH}_*(\mathcal{C}, \mathcal{P}), \tag{6-7}$$

[20] In contrast, note that in the proper case all operations should have at least one *input*.

and hence for any class $[\sigma] \in \mathrm{HH}_*(\mathcal{C},\mathcal{C})$ there is an induced map

$$[\cap\sigma]\colon \mathrm{HH}^*(\mathcal{C},\mathcal{P}) \to \mathrm{HH}_{*+\deg(\sigma)}(\mathcal{C},\mathcal{P}). \tag{6-8}$$

More generally, the cap products acts as $\mathrm{HH}^*(\mathcal{C},\mathcal{P}) \otimes \mathrm{HH}_*(\mathcal{C},\mathcal{Q}) \to \mathrm{HH}_*(\mathcal{C},\mathcal{P}\otimes_{\mathcal{C}}\mathcal{Q})$; here we are considering $\mathcal{Q}=\mathcal{C}_\Delta$, and then composing with the equivalence $\mathcal{P}\otimes_{\mathcal{C}}\mathcal{C}_\Delta \cong \mathcal{P}$. See for instance [24, Section 2.10] for explicit chain-level formulae in the variant case that $\mathcal{P}=\mathcal{C}_\Delta$, which can be straightforwardly adapted to the general case and then specialized to the case here.

Definition 6.5 Let $\mathcal{C}$ be a homologically smooth A_∞ category. A cycle $\sigma \in \mathrm{CH}_{-n}(\mathcal{C},\mathcal{C})$ is said to be a *weak smooth Calabi–Yau structure*, or a *nondegenerate cotrace* if, for any objects K, L, the operation of capping with σ induces a homological isomorphism

$$[\cap\sigma]\colon \mathrm{HH}^*(\mathcal{C},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L) \xrightarrow{\cong} \mathrm{HH}_{*-n}(\mathcal{C},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L) \simeq H^*(\hom_{\mathcal{C}}(K,L)). \tag{6-9}$$

(This nondegeneracy property only depends on the homology class $[\sigma]$.) A cycle in the nonunital Hochschild complex $\sigma \in \mathrm{CH}^{\mathrm{nu}}_{-n}(\mathcal{C})$ is said to be a weak smooth Calabi–Yau structure if again $[\sigma] \in H^*(\mathrm{CH}^{\mathrm{nu}}_{-n}(\mathcal{C})) \cong \mathrm{HH}_{-n}(\mathcal{C})$ is nondegenerate in the sense of (6-9).

Remark 6.6 The second isomorphism $\mathrm{HH}_{*-n}(\mathcal{C},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L) \simeq H^*(\hom_{\mathcal{C}}(K,L))$ always holds for cohomologically unital categories, such as the Fukaya category; the content is in the first.

Remark 6.7 Continuing Remark 6.2, there is an alternative perspective on Definition 6.5 using bimodules. Namely, for any bimodule $\mathcal{P}$, there is a naturally associated *bimodule dual* $\mathcal{P}^!$, defined for a pair of objects (K,L) as the chain complex

$$\mathcal{P}^!(K,L) := \hom^*_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{P},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L).$$

The higher bimodule structure is defined in [24, Definition 2.40]; for an A–bimodule B, it is an A_∞ analogue of defining $B^! := \mathrm{RHom}_{A\otimes A^{\mathrm{op}}}(B, A\otimes A^{\mathrm{op}})$, where RHom is taken with respect to the outer bimodule structure on $A\otimes A^{\mathrm{op}}$, and the bimodule structure on $B^!$ comes from the inner bimodule structure; see eg [28, Section 20.5].

We abbreviate $\mathcal{C}^! := \mathcal{C}^!_\Delta$ and call $\mathcal{C}^!$ the *inverse dualizing bimodule*, following [37]. (Observe that $H^*(\mathcal{C}^!(K,L)) \cong \mathrm{HH}^*(\mathcal{C},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L)$.) For a homologically smooth category $\mathcal{C}$, one notes that there is a quasi-isomorphism $\mathrm{CH}_{*-n}(\mathcal{C}) \simeq \hom^*_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}^!_\Delta[n],\mathcal{C}_\Delta)$ (see [37, Remark 8.11] for the case of A_∞–algebras), where the equivalence associates to any element the bimodule morphism whose cohomology-level map is the cap product operation (6-9). Nondegenerate cotraces in $\mathrm{CH}_{-n}(\mathcal{C})$ then correspond precisely to bimodule quasi-isomorphisms $\mathcal{C}^![n] \xrightarrow{\sim} \mathcal{C}_\Delta$. Further discussion of these structures in the A_∞ categorical setting will appear as part of forthcoming work with Cohen [12].

Let $\iota\colon \mathrm{CC}^-_*(\mathcal{C}) \to \mathrm{CH}^{\mathrm{nu}}_*(\mathcal{C})$ denote the "inclusion of homotopy fixed points" chain map from (2-24); concretely, as described in (2-36), this is the chain map sending $\sum_{i=0}^\infty \alpha_i u^i \mapsto \alpha_0$.

Definition 6.8 Let $\mathcal{C}$ be a homologically smooth A_∞ category. A (*strong*) *smooth Calabi–Yau structure* is a cycle $\tilde{\sigma} \in \mathrm{CC}^-_{-n}(\mathcal{C})$ such that the corresponding element $\iota(\tilde{\sigma}) \in \mathrm{CH}^{\mathrm{nu}}_{-n}(\mathcal{C})$ is a weak smooth Calabi–Yau structure.

Using these definitions and the cyclic open–closed map, we will now restate and prove Theorem 1.16. We adopt the notation of wrapped Fukaya categories in the below result, using $\mathcal{W}$ and $SC^*(M)$ in place of $\mathcal{F}$ and $CF^*(M)$, with the understanding that for a compact symplectic manifold, these are the same.

Theorem 6.9 (Theorem 1.16 above) *Suppose a Liouville* (*or compact admissible symplectic*) *manifold is* ***nondegenerate*** *in the sense of* [24], *meaning that the map* $[\mathcal{OC}]\colon \mathrm{HH}_{*-n}(\mathcal{W}) \to SH^*(M)$ *hits* 1. *Then the Fukaya category* $\mathcal{W}$ *possesses a* (*cohomologically*) *canonical geometrically defined* ***smooth Calabi–Yau structure****.*

Proof In [24] it was proven, assuming nondegeneracy of M, that the map

$$[\mathcal{OC}]\colon \mathrm{HH}_{*-n}(\mathcal{W}) \to SH^*(M)$$

is an isomorphism, $\mathcal{W}$ is homologically smooth, and moreover that the preimage $[\sigma]$ of 1 gives a *weak smooth Calabi–Yau structure* in the sense described above; see [25; 27; 26] for a proof of some of these facts specifically tailored to the case of compact Lagrangians in compact symplectic manifolds. Let us briefly recall how the nondegeneracy condition (6-9) is proven (a fact which is left slightly implicit in [24]). First, a geometric morphism of bimodules $\mathcal{CY}\colon \mathcal{W}_\Delta \to \mathcal{W}^![n]$ is constructed and shown in [24, Theorem 1.3] to be a quasi-isomorphism under the given nondegeneracy hypotheses. Then, it is shown that capping with $[\sigma]$ is a one-sided inverse to the homological map $[\mathcal{CY}]$, and thus an isomorphism also, by the following argument. We establish that the following diagram is commutative (up to an overall sign of $(-1)^{n(n+1)/2}$); it can be thought of as coming from the compatibility of $\mathcal{OC}$ with module structures for Hochschild (co)homology with coefficients in $\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L$, and can be extracted from the holomorphic curve theory appearing in [24, Theorem 13.1]:

$$
\begin{tikzcd}
\begin{matrix}\mathrm{HH}_{*-n}(\mathcal{W},\mathcal{W})\\ \otimes \\ H^*(\hom_{\mathcal{W}}(K,L))\end{matrix} \arrow[r, "{(\mathrm{id},[\mathcal{CY}])}"] \arrow[d, "{([\mathcal{OC}],\mathrm{id})}"] & \begin{matrix}\mathrm{HH}_{*-n}(\mathcal{W},\mathcal{W})\\ \otimes \\ \mathrm{HH}^{*+n}(\mathcal{W},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L)\end{matrix} \arrow[d, "\cap"] \\
\begin{matrix}SH^*(M)\\ \otimes \\ H^*(\hom_{\mathcal{W}}(K,L))\end{matrix} \arrow[r, "{[\mu^2(\mathcal{CO}_0(-),-)]}"] & \begin{matrix}H^*(\hom_{\mathcal{W}}(K,L))\\ = \mathrm{HH}_*(\mathcal{W},\mathcal{Y}^l_K \otimes_{\boldsymbol{k}} \mathcal{Y}^r_L)\end{matrix}
\end{tikzcd} \tag{6-10}
$$

Here $[\mathcal{CO}_0]$ is the length-zero part of the closed open map for the object L, mapping $SH^*(M)$ to $H^*(\hom_{\mathcal{W}}(L,L))$. Plugging $[\sigma]$ into $\mathrm{HH}_*(\mathcal{W},\mathcal{W})$ and noting that $[\mathcal{OC}]([\sigma]) = 1$ and $[\mu^2(\mathcal{CO}_0(1),-)] = [\mu^2]([e_L],-)$ is the identity map establishes, as desired, that $[\sigma \cap (\mathcal{CY}(y))] = [y]$.

To lift the weak smooth Calabi–Yau structure to a (strong) smooth Calabi–Yau structure, first we note that, because $[\mathcal{OC}]$ is an isomorphism, Corollary 1.5 implies that there is a commutative diagram of isomorphisms

$$
\begin{tikzcd}
\mathrm{HC}^-_{*-n}(\mathcal{W}) \arrow[r, "{[\iota]}"] \arrow[d, "{[\widetilde{\mathcal{OC}}^-]}"'] & \mathrm{HH}_{*-n}(\mathcal{W}) \arrow[d, "{[\mathcal{OC}]}"] \\
H^*(SC^*(M)^{\mathrm{h}S^1}) \arrow[r, "{[\iota]}"] & SH^*(M)
\end{tikzcd} \tag{6-11}
$$

where the horizontal maps ι are the "inclusion of homotopy fixed points" maps $\iota\colon P^{\mathrm{h}S^1} \to P$ defined for any S^1–complex P, sending $\sum_{i=0}^{\infty} \alpha_i u^i \mapsto \alpha_0$.

In Section 4.4, and specifically (4-68), it was shown that there is a canonical geometrically defined element $\tilde{1} \in H^*(SC^*(M)^{\mathrm{h}S^1})$ lifting the unit $1 \in SH^*(M)$; essentially this is because the map 1 is in the image of the map $H^*(M) \to SH^*(M)$, which on the chain level (as this map comes from "the inclusion of constant loops into the free loop space" and "constant loops are acted on by S^1 trivially") can be canonically lifted to a map $C^*(M) \to C^*(M)^{\mathrm{h}S^1} = C^*(M)[\![u]\!] \to SC^*(M)^{\mathrm{h}S^1}$.

Since $[\widetilde{\mathcal{OC}}^-]$ is an isomorphism, it follows that there is a unique (cohomological) element $[\tilde{\sigma}] \in \mathrm{HC}^-_{*-n}(\mathcal{W})$ hitting $\tilde{1}$ via $[\widetilde{\mathcal{OC}}]$. By (6-11), $[\iota]([\tilde{\sigma}])] = [\sigma]$, establishing that (any cycle representing) $[\tilde{\sigma}]$ is a smooth Calabi–Yau structure. □

Appendix Moduli spaces and operations

A.1 A real blow-up of Deligne–Mumford space

We review, in a special case, the compactifications of moduli spaces of surfaces where some interior marked points are equipped with asymptotic markers, which are a real blow-up of Deligne–Mumford moduli space as constructed in [34]. In particular, we show how boundary strata of the abstract compactifications in the sense of [34] can be identified with the specific models of the moduli spaces we use in Section 5. The appearance of the compactifications [34] in Floer theory is not new; see eg [58].

To begin, let

$$\mathcal{M}_{2,0} \tag{A-1}$$

denote the space of spheres with two marked points z_1, z_2 removed and asymptotic markers τ_1, τ_2 around the z_1 and z_2, modulo automorphism. Fixing the position of z_1 and z_2 and one of τ_1 or τ_2 gives a diffeomorphism

$$\mathcal{M}_{2,0} \cong S^1.$$

On an arbitrary representative in $\mathcal{M}_{2,0}$, we can think of the map to S^1 as coming from the *difference in angles* between τ_1 and τ_2 — after, say, parallel transporting one tangent space to the other along a geodesic path.

It is convenient to parametrize this difference by a point on the sphere itself, in the following manner (though this will break symmetry between z_1 and z_2). Let

$$\mathcal{M}_{2,1} \tag{A-2}$$

be the space of spheres with two marked points z_1, z_2 removed, an extra marked point p, and asymptotic markers τ_1, τ_2 around the z_1 and z_2, modulo automorphism, such that, for any representative with position of z_1, z_2 and p fixed, τ_2 is pointing towards p. The remaining freedom in τ_1 once more gives a diffeomorphism $\mathcal{M}_{2,1} \cong S^1$.

We can take a different representative for elements of $\mathcal{M}_{2,1}$: up to biholomorphism any element of (A-2) is equal to a cylinder sending z_1 to $+\infty$, z_2 to $-\infty$, with fixed asymptotic direction around $+\infty$ and an extra marked point p at fixed height freely varying around S^1, such that the asymptotic marker at $-\infty$ coincides with the S^1 coordinate of p. Thus, we obtain an identification

$$\mathcal{M}_{2,1} \cong \mathcal{M}_1, \tag{A-3}$$

where $\mathcal{M}_1$ is the space in Definition 4.7, with p_1 corresponding to p here.

Now let

$$ {}_k\mathcal{R}^1_d \tag{A-4}$$

denote the moduli space of discs $(S, z_1, \dots, z_d, z_{\mathrm{out}}, \tau_{z_{\mathrm{out}}}, p_1, \dots, p_k)$ with d boundary marked points $z_1, \dots, z_d$ arranged in counterclockwise order, an interior marked point with asymptotic marker $(z_{\mathrm{out}}, \tau_{z_{\mathrm{out}}})$, and interior marked points with no asymptotic markers $p_1, \dots, p_k$ satisfying two constraints to be described below, modulo automorphism. Up to automorphism, every equivalence class of the unconstrained moduli space of such $(S, z_1, \dots, z_d, z_{\mathrm{out}}, \tau_{z_{\mathrm{out}}}, p_1, \dots, p_k)$ admits a unique unit-disc representative with z_d fixed at 1 and z_{out} at 0; call this the (z_d, z_{out}) *standard representative*, or simply the standard representative. The positions of the asymptotic marker, remaining marked points, and interior marked points identify this unconstrained moduli space with an open subset of $S^1 \times \mathbb{R}^{2k} \times \mathbb{R}^d$. With respect to this identification, the space (A-4) consists of those discs satisfying the (open) "ordering constraint" on the positions of the interior marked points

$$\text{on the standard representative, } 0 < |p_1| < |p_2| < \cdots < |p_k| < \tfrac{1}{2}, \tag{A-5}$$

along with a (codimension-one) condition on the asymptotic marker,

$$\text{on the standard representative, } \tau_{z_{\mathrm{out}}} \text{ points at } p_1. \tag{A-6}$$

The condition (A-5), which cuts out a manifold with corners of the larger space in which the p_i are unconstrained, is technically convenient, as it reduces the types of bubbles that can occur with z_{out}. The compactification of interest, denoted by

$$ {}_k\overline{\mathcal{R}}^1_d, \tag{A-7}$$

differs from the Deligne–Mumford compactification in a couple of respects: firstly, we allow points p_i and p_{i+1} to be coincident without bubbling off (alternatively, we can Deligne–Mumford compactify and collapse the relevant strata). More interestingly, (A-7) is a real blow-up of the usual Deligne–Mumford compactification along any strata in which z_{out} and p_i points bubble off, as in [34]. We will proceed to describe the codimension-one boundary strata of (A-7) along with (after identification with the moduli spaces we introduce in this paper) the boundary chart gluing maps. Let $\Sigma = S_0 \cup_{z^+_{\mathrm{int}} = z^-_{\mathrm{int}}} S_1$ denote a nodal surface, where

- S_0 is a sphere containing interior marked points $(z_{\mathrm{out}}, \tau_{z_{\mathrm{out}}})$, $p_1, \dots, p_j$ and another marked point z^+_{int}, and

- S_1 is a disc with d boundary marked points $z_1, \dots, z_d$ and interior marked points z_{int}^-, $p_{j+1}, \dots, p_k$.

To occur as a possible degenerate limit of (A-4), the relevant points p_i on S_0 and S_1 must satisfy an ordering condition:

(A-8) For any S_0' which is biholomorphic to S_0, with z_{out} and z_{int}^+ at opposite poles, we have $0 < |p_1| < \cdots < |p_j| < |z_{\mathrm{int}}^+|$, where $|p|$ denotes the geodesic distance from z_{out} to p on S_0'.

(A-9) For the $(z_d, z_{\mathrm{int}}^-)$ standard representative of S_1, $0 < |p_{j+1}| < \cdots < |p_k| < \frac{1}{2}$.

Also:

(A-10) For S_0' as in (A-8), the asymptotic marker $\tau_{z_{\mathrm{out}}}$ should point (geodesically) towards p_1.

The relevant codimension-one stratum of (A-7) consists of all (automorphism classes of) such broken configurations $S_0 \cup_{z_{\mathrm{int}}^+ = z_{\mathrm{int}}^-} S_1$ as above, equipped additionally with a *gluing angle* at the node, which is a real positive line $\tau_{z_{\mathrm{int}}^+, z_{\mathrm{int}}^-}$ in $T_{z_{\mathrm{int}}^+} S_0 \otimes T_{z_{\mathrm{int}}^-} S_1$, or equivalently, a pair of asymptotic markers $(\tau_{z_{\mathrm{int}}^+}, \tau_{z_{\mathrm{int}}^-})$ around each of z_{int}^+ and z_{int}^-, modulo the diagonal S^1 rotation action. Note that the set of gluing angles (which is allowed to vary) is S^1, making this stratum codimension-1 (the corresponding stratum in Deligne–Mumford space does not have gluing angles, and hence has real codimension two). The gluing map takes, for a fixed pair of cylindrical ends around z_{int}^+ and z_{int}^- compatible with the pair of asymptotic markers in the sense of (5-3), the usual gluing with respect to the chosen cylindrical ends. Note first that for a given gluing parameter, if the cylindrical ends are chosen to simply rotate as $(\tau_{z_{\mathrm{int}}^+}, \tau_{z_{\mathrm{int}}^-})$ vary, the result of gluing after rotating $\tau_{z_{\mathrm{int}}^+}$ by θ_1 and $\tau_{z_{\mathrm{int}}^-}$ by θ_2 differs from the initial gluing by a rotation of the bottom component by $\theta_2 - \theta_1$. In particular, the glued surface indeed only depends on the gluing angle associated to $(\tau_{z_{\mathrm{int}}^+}, \tau_{z_{\mathrm{int}}^-})$, ie it is unchanged by simultaneously rotating $(\tau_{z_{\mathrm{int}}^+}, \tau_{z_{\mathrm{int}}^-})$.

We can recast this stratum by taking a slice of the quotient by the diagonal S^1–action appearing in the definition of gluing angle: First, note that z_{int}^- on S_1 possesses a canonical asymptotic marker $(\tau_{z_{\mathrm{int}}}^-)_{\mathrm{canon}}$, which (on the standard representative) points towards p_{j+1}; our convention is that $p_{s+1} = z_d$, so $\tau_{z_{\mathrm{int}}}^-$ points at z_d if $j = s$. Choosing the representative $(\tau_{z_{\mathrm{int}}^+}, \tau_{z_{\mathrm{int}}^-})$ of each gluing angle for which $\tau_{z_{\mathrm{int}}^-}$ is the canonical asymptotic marker $(\tau_{z_{\mathrm{int}}}^-)_{\mathrm{canon}}$, we see that the stratum described above can be identified with the space of broken configurations $S_0 \cup_{z_{\mathrm{int}}^+ = z_{\mathrm{int}}^-} S_1$ (up to automorphism) of the form:

- S_1 is as above (ie satisfies (A-9)) but is additionally equipped with $(\tau_{z_{\mathrm{int}}^-})_{\mathrm{canon}}$, ie $S_1 \in {}_{k-j}\mathcal{R}_1^d$.
- S_0 is equipped with interior marked points with asymptotic markers $(z_{\mathrm{out}}, \tau_{z_{\mathrm{out}}})$, $(z_{\mathrm{int}}^+, \tau_{z_{\mathrm{int}}^+})$ and additional marked points $p_1, \dots, p_j$ satisfying (A-8) and (A-10).

Just as in (A-3), the space of such S_0 up to biholomorphism is precisely $\mathcal{M}_j$ as in Definition 4.7, ie given any S_0, there is a one-dimensional space of biholomorphisms to a cylinder sending z_{int} and z_{out} to ∞ and $-\infty$ while fixing the angle of $\tau_{z_{\mathrm{int}}^+}$ to 1; any two such biholomorphisms differ by translation.

Thus, we have identified this stratum with

$$ {}_{k-j}\mathcal{R}_1^d \times \mathcal{M}_j, \tag{A-11} $$

which will be useful in defining the relevant pseudoholomorphic curve counts. From this perspective, the boundary chart gluing maps, defined with respect to the cylindrical ends (4-33) and (4-34) on $\mathcal{M}_j$ and with respect to a smoothly varying choice of cylindrical end over elements of ${}_{k-j}\mathcal{R}_1^d$ compatible with $(\tau_{z_{\mathrm{int}}^-})_{\mathrm{canon}}$, just as in (4-36), rotate the (standard representative of the) angle-decorated cylinder S_0 to match the angle of its top asymptotic marker with the angle of $(\tau_{z_{\mathrm{int}}^-})_{\mathrm{canon}}$, which coincides with the argument of p_{j+1} on the standard representative. In other words, if we denote by θ_i the angle of p_i in S_1 for $j+1 \le i \le k$ — with respect to any standard representative of S_1, with the usual convention that θ_{k+1} is the argument of z_d on the standard representative, so in particular θ_{j+1} is well defined even if $j = k$ — and denote by $\overline{\theta}_s$ the angle of p_s in S_0 for $1 \le s \le j$, the gluing of S_0 and S_1 for small gluing parameter has (on its standard representative) marked points $p_1, \dots, p_k$ with the angles

$$ \begin{aligned} &(\arg(p_1), \dots, \arg(p_k)) \\ &\qquad = (\overline{\theta}_1 + \theta_{j+1}, \overline{\theta}_2 + \theta_{j+1}, \dots, \overline{\theta}_j + \theta_{j+1}, \theta_{j+1}, \theta_{j+2}, \dots, \theta_k). \end{aligned} \tag{A-12} $$

A.2 Operations with a forgotten marked point

We introduce auxiliary degenerate operations that will arise as the codimension-one boundary of the open–closed map and equivariant structure. This subsection is a very special case of the general discussion in [24].

Let $d \geq 2$ and $i \in \{1, \dots, d\}$. The *moduli space of discs with d marked points with i^{th} boundary point forgotten*,

$$\mathcal{R}^{d,f_i}, \tag{A-13}$$

is exactly the moduli space of discs $\mathcal{R}^d$, with i^{th} boundary marked point labeled as auxiliary.

The Deligne–Mumford compactification

$$\overline{\mathcal{R}}^{d,f_i} \tag{A-14}$$

is exactly the usual Deligne–Mumford compactification, along with the data of an *auxiliary label* at the relevant boundary marked point.

For $d > 2$, the *i–forgetful map*

$$\mathcal{F}_{d,i} \colon \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1} \tag{A-15}$$

associates to a surface S the surface obtained by putting the i^{th} point back in and forgetting it. This map admits an extension to the Deligne–Mumford compactification

$$\overline{\mathcal{F}}_{d,i} \colon \overline{\mathcal{R}}^{d,f_i} \to \overline{\mathcal{R}}^{d-1} \tag{A-16}$$

as follows: eliminate any nonmain components with only one nonauxiliary marked point p, and label the positive marked point below this component by p. We say that any component not eliminated is *f–stable* and any component eliminated is *f–semistable*. The above map is only well defined for $d > 2$. In the semistable case $d = 2$, the space $\mathcal{R}^{2,f_i}$ is a point so one can define an ad hoc map

$$\mathcal{F}_i^{\text{ss}} \colon \mathcal{R}^{2,f_i} \to \text{pt}, \tag{A-17}$$

which associates to a surface S the (unstable) strip $\Sigma_1 = (-\infty, \infty) \times [0, 1]$ as follows: take the unique representative of S which, after its three marked points are removed, is biholomorphic to the strip Σ_1 with an additional puncture $(0, 0)$. Then, forget/put back in the point $(0, 0)$.

Definition A.1 A *forgotten Floer datum* for a stable disc with i^{th} point auxiliary $S \in \overline{\mathcal{R}}^{d,f_i}$ consists, for every component T of S, of

- a Floer datum for T, if T does not contain the auxiliary point,
- a Floer datum for $\mathcal{F}_j(T)$, if T is f–stable and contains the auxiliary point as its j^{th} input,
- a Floer datum on $\mathcal{F}_i^{\text{ss}}(T)$ which is *translation invariant* if T is *f–semistable*.

By *translation invariant*, we mean the following: note that Σ_1 has a canonical $\mathbb{R}$–action given by linear translation in the s coordinate. We require H, J and the time-shifting map/weights to be invariant under this $\mathbb{R}$–action, and in particular they should only depend on $t \in [0,1]$ at most.

In particular, this Floer datum should only depend on the point $\overline{\mathcal{F}}_{d,i}(S)$.

Proposition A.2 *Let $i \in \{1,\ldots,d\}$ with $d > 1$. Then the operation associated to $\overline{\mathcal{R}}^{d,f_i}$ is zero if $d > 2$, and the identity operation $I(\cdot)$ (up to a sign) when $d = 2$.*

Sketch Suppose first that $d > 2$, and let u be any solution to Floer's equation over the space $\mathcal{R}^{d,f_i}$ with domain S. Since the Floer data on S only depends on $\mathcal{F}_{d,i}(S)$, we see that maps from S' with $S' \in \mathcal{F}_{d,i}^{-1}(\mathcal{F}_{d,i}(S))$ also give solutions to Floer's equation with the same asymptotics. Moreover, the fibers of the map $\mathcal{F}_{d,i}$ are one-dimensional, implying that u cannot be rigid, and thus the associated operation is zero.

Now suppose that $d = 2$. Then the forgetful map associates to the single point $[S] \in \mathcal{R}^{2,f_i}$ the unstable strip with its translation-invariant Floer datum. Since nonconstant solutions can never be rigid — as, by translating, one can obtain other nonconstant solutions — it follows that the only solutions are constant ones, and the resulting operation is therefore the identity. □

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