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Sunhyuk Lim

Facundo Mémoli

Zane Smith

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The Gromov–Hausdorff distance between spheres

SUNHYUK LIM
FACUNDO MÉMOLI
ZANE SMITH

We provide general upper and lower bounds for the Gromov–Hausdorff distance $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ between spheres $\mathbb{S}^m$ and $\mathbb{S}^n$ (endowed with the round metric) for $0 \le m < n \le \infty$. Some of these lower bounds are based on certain topological ideas related to the Borsuk–Ulam theorem. Via explicit constructions of (optimal) correspondences, we prove that our lower bounds are tight in the cases of $d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n)$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^\infty)$, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2)$, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^3)$ and $d_{\mathrm{GH}}(\mathbb{S}^2, \mathbb{S}^3)$. We also formulate a number of open questions.

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1 Introduction

Despite being widely used in Riemannian geometry — see, for example, Burago, Burago and Ivanov [4] and Petersen [30] — very little is known in terms of the *exact* value of the Gromov–Hausdorff distance between two given spaces. In a closely related vein, Gromov [16, page 141] poses the question of computing/estimating the value of the *box distance* $\square_1(\mathbb{S}^m, \mathbb{S}^n)$ (a close relative of d_{GH}) between spheres (viewed as metric measure spaces). In [14], Funano provides asymptotic bounds for this distance via an idea due to Colding (see the discussion preceding Proposition 1.2).

The Gromov–Hausdorff distance is also a natural choice for expressing the stability of invariants in applied algebraic topology — see Carlsson and Mémoli [5; 6; 7] — and has also been invoked in applications related to shape matching — see Bronstein, Bronstein and Kimmel [3] and Mémoli and Sapiro [25; 27] — as a notion of dissimilarity between shapes.

We consider the problem of estimating the Gromov–Hausdorff distance $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ between spheres (endowed with their round/geodesic distance). In particular we show that in some cases, topological ideas related to the Borsuk–Ulam theorem yield lower bounds which turn out to be tight.

1.1 Basic definitions

The Gromov–Hausdorff distance — see Edwards [12] and Gromov [16] — between two bounded metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{\mathrm{GH}}(X, Y) := \inf d_{\mathrm{H}}(f(X), g(Y)),$$

where d_{H} denotes the Hausdorff distance between subsets of the ambient space Z and the infimum is taken over all isometric embeddings f and g of X and Y, respectively, into Z, and over all metric spaces Z. We will henceforth denote by $\mathcal{M}_b$ the collection of all bounded metric spaces.

It is known that d_{GH} defines a metric on compact metric spaces up to isometry [16]. A standard reference is [4]. A useful property is that whenever (X, d_X) is a compact metric space and, for some $\delta > 0$, a subset $A \subseteq X$ is a δ–net for X, then $d_{\mathrm{GH}}\big((X, d_X), (A, d_X|_{A\times A})\big) \le \delta$.

Given two sets X and Y, a correspondence between them is any relation $R \subseteq X \times Y$ such that $\pi_X(R) = X$ and $\pi_Y(R) = Y$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$

are the canonical projections. Given two bounded metric spaces (X, d_X) and (Y, d_Y), and any nonempty relation $R \subseteq X \times Y$, its distortion is defined as

$$\mathrm{dis}(R) := \sup_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|.$$

Remark 1.1 In particular, the graph of any map $\psi\colon X \to Y$ is a relation $\mathrm{graph}(\psi)$ between X and Y and this relation is a correspondence whenever ψ is surjective. The distortion of the relation induced by ψ will be denoted by $\mathrm{dis}(\psi)$.

A theorem of Kalton and Ostrovskii [18] proves that the Gromov–Hausdorff distance between any two bounded metric spaces (X, d_Y) and (Y, d_Y) is equal to

$$d_{\mathrm{GH}}(X,Y) := \tfrac{1}{2} \inf_R \mathrm{dis}(R), \tag{1}$$

where R ranges over all correspondences between X and Y. It was also observed in [18] that

$$d_{\mathrm{GH}}(X,Y) = \tfrac{1}{2} \inf_{\varphi,\psi} \max\{\mathrm{dis}(\varphi), \mathrm{dis}(\psi), \mathrm{codis}(\varphi,\psi)\}, \tag{2}$$

where $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ are any (not necessarily continuous) maps, and

$$\mathrm{codis}(\varphi,\psi) := \sup_{x\in X,\, y\in Y} |d_X(x,\psi(y)) - d_Y(\varphi(x),y)|$$

is the *codistortion* of the pair (φ,ψ).

Known results on $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ We will find it useful to refer to the infinite matrix $\mathfrak{g}$ such that for $m, n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$,

$$\mathfrak{g}_{m,n} := d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n);$$

see Figure 2.

The following lower bound for $\mathfrak{g}_{m,n}$, obtained via simple estimates for covering and packing numbers based on volumes of balls, is in the same spirit as a result by Colding [10, Lemma 5.10].[1] By $v_n(\rho)$ we denote the *normalized volume* of an open ball of radius $\rho \in (0,\pi]$ on $\mathbb{S}^n$ (so that the entire sphere has volume 1). Colding's approach yields:

Proposition 1.2 *For all integers $0 < m < n$,*

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \mu_{m,n} := \tfrac{1}{2} \sup_{\rho\in(0,\pi]} \bigl(v_n^{-1} \circ v_m\bigl(\tfrac{1}{2}\rho\bigr) - \rho\bigr).$$

[1] Funano used a similar idea in [14] to estimate Gromov's box distance between metric measure space representations of spheres.

We relegate the proof of this proposition to Section 3.

Example 1.3 (lower bound for $\mathfrak{g}_{1,2}$ via Colding's idea) In this case, $m=1$ and $n=2$, the lower bound provided by Proposition 1.2 above is $\sup_{\rho\in(0,\pi]}(\arccos(1-\rho/\pi)-\rho)$, which is approximately equal to and bounded below by 0.1605. Thus, $\mathfrak{g}_{1,2}\geq 0.0802$. See Remark 1.9 for a comparison with a new lower bound which also arises from covering/packing arguments via the Lyusternik–Schnirelmann theorem.

In contrast, in this paper, via techniques which include both certain topological ideas leading to lower bounds and the precise construction of correspondences with matching (and hence optimal) distortion, we prove results which imply (see Proposition 1.16 below) that, in particular, $\mathfrak{g}_{1,2}=\frac{\pi}{3}\simeq 1.0472$, which is about 13 times larger than the value obtained by the method above. In [26, Example 5.3] the lower bound $\mathfrak{g}_{1,2}\geq\frac{\pi}{12}$ was obtained via a calculation involving Gromov's *curvature sets* $\boldsymbol{K}_3(\mathbb{S}^1)$ and $\boldsymbol{K}_3(\mathbb{S}^2)$. Finally, via considerations based on Katz's precise calculation [19] of the filling radius of spheres — see Lim, Mémoli and Okutan [21, Corollary 9.3] — yields that $\mathfrak{g}_{1,n}\geq\frac{\pi}{6}$ for all $n\geq 2$ as well as other lower bounds for $\mathfrak{g}_{m,n}$ for general $m<n$ which are not tight. In a related vein, in [17] Ji and Tuzhilin determine the precise value of $d_{\mathrm{GH}}([0,\lambda],\mathbb{S}^1)$ between an interval of length $\lambda>0$ and the circle (with geodesic distance).

1.2 Overview of our results

The diameter of a bounded metric space (X,d_X) is the number

$$\operatorname{diam}(X):=\sup_{x,x'\in X} d_X(x,x').$$

For $m\in\overline{\mathbb{N}}$ we view the m–dimensional sphere,

$$\mathbb{S}^m:=\{(x_1,\dots,x_{m+1})\in\mathbb{R}^{m+1}\mid x_1^2+\cdots+x_{m+1}^2=1\},$$

as a metric space by endowing it with the geodesic distance: for any two points $x,x'\in\mathbb{S}^m$,

$$d_{\mathbb{S}^m}(x,x'):=\arccos(\langle x,x'\rangle)=2\arcsin\bigl(\tfrac{1}{2}d_{\mathrm{E}}(x,x')\bigr),$$

where d_{E} denotes the canonical Euclidean metric inherited from $\mathbb{R}^{m+1}$.

Note that for $m=0$ this definition yields that $\mathbb{S}^0$ consists of two points at distance π, and that $\mathbb{S}^\infty$ is the unit sphere in ℓ^2 with distance given in the expression above.

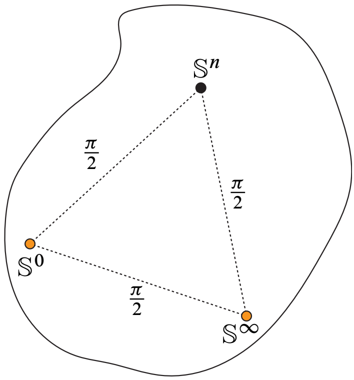

Figure 1: Propositions 1.5 and 1.6 encode the peculiar fact that all triangles in $(\mathcal{M}_b, d_{\mathrm{GH}})$ with vertices $\mathbb{S}^0, \mathbb{S}^\infty$, and $\mathbb{S}^n$ (for $0 < n < \infty$) are equilateral.

Remark 1.4 First recall [4, Chapter 7] that, for any two bounded metric spaces X and Y, one always has $d_{\mathrm{GH}}(X,Y) \le \frac{1}{2}\max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$. This means that

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \le \tfrac{\pi}{2} \quad \text{for all } 0 \le m \le n \le \infty. \tag{3}$$

We first prove the following two propositions, which establish that the above upper bound is tight in certain extremal cases:

Proposition 1.5 (distance to $\mathbb{S}^0$; Chowdhury and Mémoli [9, Proposition 1.2]) *For any integer* $n \ge 1$,

$$d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n) = \tfrac{\pi}{2}.$$

Proposition 1.6 (distance to $\mathbb{S}^\infty$) *For any integer* $m \ge 0$,

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^\infty) = \tfrac{\pi}{2}.$$

Proposition 1.5 can be proved as follows: any correspondence between $\mathbb{S}^0$ and $\mathbb{S}^n$ induces a closed cover of $\mathbb{S}^n$ by two sets; then, by the Lyusternik–Schnirelmann theorem, one of these blocks must contain two antipodal points. Proposition 1.6 can be proved in a similar manner; see Figure 1.

Remark 1.7 When taken together, Remark 1.4 and Propositions 1.5 and 1.6 above might suggest that the Gromov–Hausdorff distance between *any* two spheres of different dimension is $\frac{\pi}{2}$. In fact, this is true for the following *continuous* version of d_{GH}:

$$d_{\mathrm{GH}}^{\mathrm{cont}}(X,Y) := \tfrac{1}{2} \inf_{\varphi', \psi'} \max\{\mathrm{dis}(\varphi'), \mathrm{dis}(\psi'), \mathrm{codis}(\varphi', \psi')\},$$

where $\varphi' \colon X \to Y$ and $\psi' \colon Y \to X'$ are *continuous* maps.

Indeed, suppose that $n > m \geq 1$. Then, by the Borsuk–Ulam theorem — see Munkholm [28, Theorem 1] or Matoušek [24, page 29] — for any continuous $\varphi' \colon \mathbb{S}^n \to \mathbb{S}^m$, there must be two antipodal points with the same image under φ'; that is, there is an $x \in \mathbb{S}^n$ such that $\varphi'(x) = \varphi'(-x)$. This implies that $\mathrm{dis}(\varphi') = \pi$, and consequently $d_{\mathrm{GH}}^{\mathrm{cont}}(\mathbb{S}^n, \mathbb{S}^m) \geq \frac{\pi}{2}$. The reverse inequality can be obtained by choosing constant maps φ' and ψ' in the above definition; thus implying that

$$d_{\mathrm{GH}}^{\mathrm{cont}}(\mathbb{S}^m, \mathbb{S}^n) = \tfrac{\pi}{2}.$$

In contrast, we prove the following result for the standard Gromov–Hausdorff distance:

Theorem A *$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) < \frac{\pi}{2}$ for all $0 < m < n < \infty$.*

The Borsuk–Ulam theorem implies that, for any positive integers $n > m$ and for any given continuous function $\varphi \colon \mathbb{S}^n \to \mathbb{S}^m$, there exist two antipodal points in the higher dimensional sphere which are mapped to the *same* point in the lower dimensional sphere. This forces the distortion of any such continuous map to be π. In contrast, in order to prove Theorem A, we exhibit, for every pair of positive numbers m and n with $m < n$, a *continuous antipode-preserving surjection* from $\mathbb{S}^m$ to $\mathbb{S}^n$ with distortion *strictly* bounded above by π, which implies the claim since the graph of any such surjection is a correspondence between $\mathbb{S}^m$ and $\mathbb{S}^n$; see Remark 1.1. The proof relies on ideas related to space-filling curves and spherical suspensions.

The standard Borsuk–Ulam theorem is however still useful for obtaining additional information about the Gromov–Hausdorff distance between spheres. Indeed, via Lemma 3.2 and the triangle inequality for d_{GH}, one can prove the following general lower bound:

Proposition 1.8 *For any $1 \leq m < n < \infty$,*

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \nu_{m,n} := \tfrac{\pi}{2} - \mathrm{cov}_{\mathbb{S}^m}(n+1).$$

Above, for any integer $k \geq 1$, and any compact metric space X, $\mathrm{cov}_X(k)$ denotes the k^{th} *covering radius* of X,

$$\mathrm{cov}_X(k) := \inf\{d_{\mathrm{H}}(X, P) \mid P \subset X \text{ such that } |P| \leq k\}. \tag{4}$$

Remark 1.9 Both of the lower bounds, $\mu_{m,n}$ and $\nu_{m,n}$, from Propositions 1.2 and 1.8, respectively, implement covering/packing ideas and as such it is interesting to compare them:

(1) Note that, since $\operatorname{cov}_{\mathbb{S}^1}(3) = \frac{\pi}{3}$, we have $\nu_{1,2} = \frac{\pi}{6}$, which is about 6.5 times larger than $\mu_{1,2} \approx 0.0802$ (see Example 1.3).

(2) Computing $\nu_{m,n}$ in general requires knowledge of the covering radius $\operatorname{cov}_{\mathbb{S}^m}(k)$ of spheres which is currently only known for $k \le m+2$; see Cho [8, Theorem 3.2]. In contrast, computing $\mu_{m,n}$ can be done (in principle) for any $m < n$ given that we have the explicit formula $v_m(\rho) = (\mathbf{Vol}(\mathbb{S}^{m-1})/\mathbf{Vol}(\mathbb{S}^m)) \int_0^\rho (\sin\theta)^{m-1}\, d\theta$, which is valid for every positive integer m and $\rho \in [0,\pi]$; see Gray [15].

(3) The lower bound $\mu_{m,n}$ is more widely applicable than $\nu_{m,n}$, which originates from the Lyusternik–Schnirelman theorem (see below) and the underlying ideas are in principle only applicable when one of the two metric spaces is a sphere.[2] Indeed, see Colding [10] and Furano [14] for estimates of the Gromov–Hausdorff distance between Riemannian manifolds satisfying upper and lower bounds on curvature obtained by combining volume comparison theorems with techniques similar to those used in proving Proposition 1.2.

(4) Through [8, Theorem 3.2] it is known that $\operatorname{cov}_{\mathbb{S}^m}(m+2) = \pi - \arccos(-1/(m+1))$ for $m \ge 1$. Therefore, when $n = m+1$, the lower bound $\nu_{m,m+1}$ given by Proposition 1.8 becomes $\arccos(-1/(m+1)) - \frac{\pi}{2}$ for $m \ge 1$, which tends to zero as m goes to infinity. It is not known whether or not $\mu_{m,m+1}$ has the same behavior.

As an immediate corollary, we obtain the following result, which complements both Proposition 1.6 and Theorem A:

Corollary 1.10 *Given any positive integer m and $\varepsilon > 0$, there exists an integer $n = n(m,\varepsilon) > m$ such that*

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \ge \frac{\pi}{2} - \varepsilon.$$

Remark 1.11 For small $\varepsilon > 0$ one can estimate the value of n above as

$$n = n(m,\varepsilon) = O(\varepsilon^{-m}).$$

The results above motivate the following two questions:

Question I *Is it true that, for fixed $m \ge 1$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ is nondecreasing for all $n \ge m$?*

[2] This can be ascertained by inspecting the proof of Proposition 1.8 in Section 3.2.

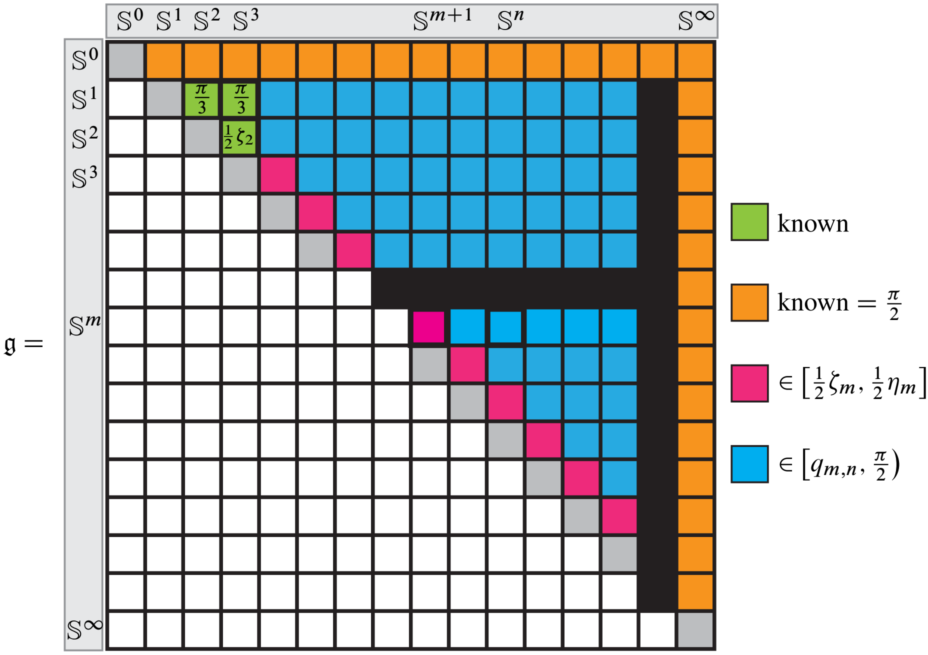

Figure 2: The matrix $\mathfrak{g}$ such that $\mathfrak{g}_{m,n} := d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$. According to Remark 1.4 and Corollary 1.12, all nonzero entries of the matrix $\mathfrak{g}$ are in the range $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. In the figure, $\zeta_m = \arccos(-1/(m+1))$ is the edge length of the regular geodesic simplex inscribed in $\mathbb{S}^m$, η_m is the diameter of a face of the regular geodesic simplex in $\mathbb{S}^m$ — see (5) — and $q_{m,n} = \max\left\{\frac{1}{2}\zeta_m, \frac{\pi}{2} - \mathrm{cov}_{\mathbb{S}^m}(n+1)\right\}$.

Question II *Fix $m \geq 1$ and $\varepsilon > 0$. Find (optimal) estimates for*

$$k_m(\varepsilon) := \inf\left\{k \geq 1 \mid d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+k}) \geq \tfrac{\pi}{2} - \varepsilon\right\}.$$

Via the Lyusternik–Schnirelmann theorem, Proposition 1.8 above depends on the classical Borsuk–Ulam theorem which, in one of its guises [24, Theorem 2.1.1], states that there is no *continuous* antipode-preserving map $g\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$. As a consequence, if $g\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ is any antipode-preserving map, then g cannot be continuous. A natural question is *how discontinuous* is any such g forced to be. This question was actually tackled in 1981 by Dubins and Schwarz [11], who proved that the *modulus of discontinuity* $\delta(g)$ of any such g needs to be suitably bounded below. These results are instrumental for proving Theorem B below; see Section 5 and Appendix A for details and for a concise proof of the main theorem from [11] (following a strategy outlined by Matoušek in [24]).

For each $m \in \mathbb{N}$ let ζ_m denote the edge length (with respect to the geodesic distance) of a regular $m+1$ simplex inscribed in $\mathbb{S}^m$,

$$\zeta_m := \arccos\left(\frac{-1}{m+1}\right),$$

which is monotonically decreasing in m. For example,

$$\zeta_0 = \pi, \quad \zeta_1 = \tfrac{2\pi}{3}, \quad \zeta_2 = \arccos\left(-\tfrac{1}{3}\right) \approx 0.608\pi, \quad \lim_{m\to\infty} \zeta_m = \tfrac{\pi}{2}.$$

Then we have the following lower bound which will turn out to be optimal in some cases:

Theorem B (lower bound via geodesic simplices) *For all integers* $0 < m < n$,

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \tfrac{1}{2}\zeta_m.$$

Moreover, for any map $\varphi\colon \mathbb{S}^n \to \mathbb{S}^m$, *we have that* $\operatorname{dis}(\varphi) \geq \zeta_m$.

From the above, we have the following general lower bound:

Corollary 1.12 *For all integers* $0 < m < n$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{\pi}{4}$.

This corollary of course implies that the sequence of compact metric spaces $(\mathbb{S}^n)_{n\in\overline{\mathbb{N}}}$ is not Cauchy.

Remark 1.13 The lower bound for $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ given by Theorem B coincides with the *filling radius* of $\mathbb{S}^m$; see Katz [19, Theorem 2]. This lower bound is twice the one obtained via the stability of Vietoris–Rips persistent homology [21, Corollary 9.3].

Note that $\operatorname{cov}_{\mathbb{S}^1}(k) \leq \pi/k$, which can be seen by considering the vertices of a regular polygon inscribed in $\mathbb{S}^1$ with k sides. Combining this fact with Proposition 1.8, Theorem B, and the fact that $\zeta_1 = \frac{2\pi}{3}$, one obtains the following special claim for the entries in the first row of the matrix $\mathfrak{g}$:

Corollary 1.14 *For all* $n > 1$, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^n) \geq \pi \cdot \max\left\{\frac{1}{3}, \frac{1}{2}(n-1)/(n+1)\right\}$.

Remark 1.15 This implies that, whereas $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^n) \geq \frac{\pi}{3}$ for $n \in \{2, 3, 4, 5\}$, one has the larger lower bound $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^6) \geq \frac{5\pi}{14} > \frac{\pi}{3}$. Propositions 1.16 and 1.18 below establish that actually $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) = d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^3) = \frac{\pi}{3}$.

Finally, in order to prove that $d_{\mathrm{GH}}(\mathbb{S}^1,\mathbb{S}^2)=\frac{\pi}{3}$, we combine Theorem B with an explicit construction of a correspondence between $\mathbb{S}^1$ and $\mathbb{S}^2$ as follows. Let $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)$ denote the closed upper hemisphere of $\mathbb{S}^2$. Then the following proposition shows that there exists a correspondence between $\mathbb{S}^1$ and $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)$ with distortion at most $\frac{2\pi}{3}$. A correspondence between $\mathbb{S}^1$ and $\mathbb{S}^2$ (see Figure 7) with the same distortion is then obtained via a certain *odd* (ie antipode-preserving) extension of the aforementioned correspondence (see Lemma 5.7):

Proposition 1.16 *There exists*

(1) *a correspondence between $\mathbb{S}^1$ and $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)$, and*

(2) *a correspondence between $\mathbb{S}^1$ and $\mathbb{S}^2$,*

both of which have distortion at most $\frac{2\pi}{3}$. In particular, together with Theorem B, this implies $d_{\mathrm{GH}}(\mathbb{S}^1,\mathbb{S}^2)=\frac{\pi}{3}$.

Even though we do not state it explicitly, in a manner similar to Proposition 1.16, all correspondences constructed in Propositions 1.18, 1.19 and 1.20 below also arise from odd extensions of correspondences between the lower dimensional sphere and the upper hemisphere of the larger dimensional sphere (see their respective proofs).

Remark 1.17 Also, by combining the first claim of Proposition 1.16 and Example 1.24(4) below (which is analogous to the claim of Theorem B but tailored to the case of $\mathbb{S}^m$ versus $\boldsymbol{H}_{\geq 0}(\mathbb{S}^m)$), one concludes that $d_{\mathrm{GH}}(\mathbb{S}^1,\boldsymbol{H}_{\geq 0}(\mathbb{S}^2))=\frac{1}{2}\zeta_1=\frac{\pi}{3}$.

Via a construction somewhat reminiscent of the Hopf fibration, we prove that there exists a correspondence between the 3–dimensional sphere and the 1–dimensional sphere with distortion at most $\frac{2\pi}{3}$. By applying suitable rotations in $\mathbb{R}^4$, the proof of the following proposition extends the (a posteriori) optimal correspondence between $\mathbb{S}^1$ and $\mathbb{S}^2$ constructed in the proof of Proposition 1.16 (see Figure 10):

Proposition 1.18 *There exists a correspondence between $\mathbb{S}^1$ and $\mathbb{S}^3$ with distortion at most $\frac{2\pi}{3}$. In particular, together with Theorem B, this implies $d_{\mathrm{GH}}(\mathbb{S}^1,\mathbb{S}^3)=\frac{\pi}{3}$.*

Finally, we were able to compute the exact value of the distance between $\mathbb{S}^2$ and $\mathbb{S}^3$ by producing a correspondence whose distortion matches the one implied by the lower bound in Theorem B. This correspondence is structurally different from the ones constructed in Propositions 1.16 and 1.18 and arises by partitioning $\mathbb{S}^3$ into 32 regions whose diameter is (necessarily) bounded above by ζ_2 and which satisfy suitable pairwise constraints (see Section 2.2):

Proposition 1.19 *There exists a correspondence between $\mathbb{S}^2$ and $\mathbb{S}^3$ with distortion at most ζ_2. In particular, together with Theorem B, this implies $d_{\mathrm{GH}}(\mathbb{S}^2, \mathbb{S}^3) = \frac{1}{2}\zeta_2$.*

Keeping in mind Remark 1.15 and Propositions 1.16 and 1.18, we pose the following:

Question III *Is it true that $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^n) = \frac{\pi}{3}$ for $n \in \{4, 5\}$?*

Theorem B and Propositions 1.16 and 1.19 lead to formulating the following conjecture:

Conjecture 1 *For all $m \in \mathbb{N}$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+1}) = \frac{1}{2}\zeta_m$.*

Note that when $m = 1$ and $m = 2$, Conjecture 1 reduces to Propositions 1.16 and 1.19, respectively. Moreover, the conjecture would imply that $\lim_{m\to\infty} d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+1}) = \frac{\pi}{4}$.

While trying to prove Conjecture 1, we were able to prove the following weaker result via an explicit construction of a certain correspondence generalizing the one constructed in the proof of Proposition 1.16:

Proposition 1.20 *For any positive integer $m > 0$, there exists a correspondence between $\mathbb{S}^m$ and $\mathbb{S}^{m+1}$ with distortion at most η_m, where*

$$\eta_m := \begin{cases} \arccos(-(m+1)/(m+3)) & \text{if } m \text{ is odd,} \\ \arccos(-\sqrt{m/(m+4)}) & \text{if } m \text{ is even.} \end{cases} \tag{5}$$

Here, η_m is the diameter of a face of the regular geodesic m–simplex in $\mathbb{S}^m$; see Figure 8 and the discussion in Section 6.2.

This correspondence arises from a partition of $\mathbb{S}^{m+1}$ into $2(m+2)$ regions which are induced by two antipodal regular simplices inscribed in $\mathbb{S}^m$, the equator of $\mathbb{S}^{m+1}$ (see Figure 7 for the case $m = 1$, a case in which this correspondence turns out to be optimal).

Corollary 1.21 *For any positive integer $m > 0$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+1}) \le \frac{1}{2}\eta_m$.*

Remark 1.22 Note that $\eta_m \ge \zeta_m$ for any $m > 0$ and $\eta_1 = \zeta_1$, so Proposition 1.20 generalizes Proposition 1.16. However, since $1.9106 \approx \zeta_2 < \eta_2 \approx 2.1863$, by Proposition 1.19 we see that Corollary 1.21 is not tight when $m = 2$. Also, since $\eta_m < \pi$, Corollary 1.21 gives a quantitative version of the claim in Theorem A when $n = m + 1$.

Remark 1.23 Combining Theorem B and Proposition 1.8, we obtain a generalization of the bound given in Corollary 1.14: for all $1 \le m < n$,

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \ge \max\left\{\tfrac{1}{2}\zeta_m, \tfrac{\pi}{2} - \mathrm{cov}_{\mathbb{S}^m}(n+1)\right\} =: q_{m,n}. \tag{6}$$

Question IV *Formula (6) and Remark 1.15 motivate the following question: for $m \geq 1$ large, find the* **rate** *at which the number*[3]

$$n_{\mathrm{diag}}(m) := \max\Big\{n > m \;\Big|\; \mathrm{cov}_{\mathbb{S}^m}(n+1) \geq \tfrac{1}{2}\arccos\Big(\frac{1}{m+1}\Big)\Big\}$$

grows with m. The reason for the notation $n_{\mathrm{diag}}(m)$ is that this number provides an estimate for a band around the principal diagonal of the matrix $\mathfrak{g}$ (see Figure 2) inside of which one would hope to prove that

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) = \tfrac{1}{2}\zeta_m \quad \text{for all } n \in \{m+1, \dots, n_{\mathrm{diag}}(m)\}.$$

1.3 Additional results and questions

Besides what we have described so far, this paper includes a number of other results about Gromov–Hausdorff distances between spaces closely related to spheres.

1.3.1 Spheres with Euclidean distance Some of the ideas described above (for spheres with geodesic distance) can be easily adapted to provide bounds for the distance between half spheres with geodesic distance, and between spheres with Euclidean distance. However, there is evidence that this phenomenon is subtle and to the best of our knowledge, there is no complete translation between the geodesic and Euclidean cases. This is exemplified by the following.

Let $\mathbb{S}^n_{\mathrm{E}}$ denote the unit sphere with the Euclidean metric d_{E} inherited from $\mathbb{R}^{n+1}$. Then, via Remark 1.17 and Corollary 9.8(2) (which provides a bridge between geodesic distortion and Euclidean distortion via the sine function), we have that

$$d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, \boldsymbol{H}_{\geq 0}(\mathbb{S}^2_{\mathrm{E}})) \leq \sin\big(d_{\mathrm{GH}}(\mathbb{S}^1, \boldsymbol{H}_{\geq 0}(\mathbb{S}^2))\big) = \tfrac{\sqrt{3}}{2}.$$

Despite this, in Proposition 9.10 we were able to construct a correspondence between these two spaces with distortion *strictly smaller* than $\sqrt{3}$. This suggests that Euclidean analogues of Theorem B may not be direct consequences; see Section 9 for other related results.

This motivates posing the following question:

Question V *Determine $\mathfrak{g}^{\mathrm{E}}_{m,n} := d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}}, \mathbb{S}^n_{\mathrm{E}})$ for all integers $1 \leq m < n$.*

It should however be noted that by Corollary 9.8 we have $\mathfrak{g}^{\mathrm{E}}_{m,n} \leq \sin(\mathfrak{g}_{m,n})$, which renders Proposition 1.20 immediately applicable, yielding $\mathfrak{g}^{\mathrm{E}}_{m,m+1} \leq \sin\big(\tfrac{1}{2}\eta_m\big)$.

[3] Note that $\zeta_m = \pi - \arccos(1/(m+1))$.

1.3.2 A stronger version of Theorem B By inspecting the proof of Theorem B, we actually have Theorem C which subsumes these results in a much greater degree of generality. Indeed, via this theorem one can obtain the following estimates:

Example 1.24 The following lower bounds hold:

(1) $d_{\mathrm{GH}}([0,\pi],\mathbb{S}^n) \geq \frac{\pi}{3}$ for any $n \geq 2$.

(2) $d_{\mathrm{GH}}(\mathbb{S}^1,\mathbb{S}^2\times\cdots\times\mathbb{S}^2) \geq \frac{\pi}{3}$ for any number of $\mathbb{S}^2$ factors.

(3) $d_{\mathrm{GH}}(\mathbb{S}^m,\boldsymbol{H}_{\geq 0}(\mathbb{S}^n)) \geq \frac{1}{2}\zeta_m$ whenever $0 < m < n < \infty$.

(4) $d_{\mathrm{GH}}(\boldsymbol{H}_{\geq 0}(\mathbb{S}^m),\boldsymbol{H}_{\geq 0}(\mathbb{S}^n)) \geq \frac{1}{2}\zeta_m$ whenever $0 < m < n < \infty$.

(5) $d_{\mathrm{GH}}(P,\mathbb{S}^2) \geq \frac{\pi}{3}$ for any finite $P \subset \mathbb{S}^1$. Compare to the $\frac{\pi}{2}$ lower bound given in Lemma 3.2.

(6) $d_{\mathrm{GH}}(P_3,\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)) = \frac{\pi}{3}$, where P_3 is the three-point metric space with all interpoint distances equal to $\frac{2\pi}{3}$. Also $d_{\mathrm{GH}}(P_6,\mathbb{S}^2) = \frac{\pi}{3}$, where P_6 is the six-point metric space corresponding to a regular hexagon inscribed in $\mathbb{S}^1$. These are consequences of (5) and small modifications of the correspondences constructed in Proposition 1.16.

Theorem C *Let bounded metric spaces X and Y be such that, for some positive integer m,*

(i) *X can be isometrically embedded into $\mathbb{S}^m$, and*

(ii) *$\boldsymbol{H}_{\geq 0}(\mathbb{S}^{m+1})$ can be isometrically embedded into Y.*

Then:

(1) $d_{\mathrm{GH}}(X,Y) \geq \frac{1}{2}\zeta_m$.

(2) *Moreover,* $\mathrm{dis}(\phi) \geq \zeta_m$ *for any map* $\phi\colon Y \to X$.

Remark 1.25 Example 1.24(1) also holds for $n = 1$, albeit this is not implied by Theorem C. The fact that $d_{\mathrm{GH}}([0,\pi],\mathbb{S}^1) \geq \frac{\pi}{3}$ follows from [17, Theorem 4.10] and it also follows from the proof of [20, Lemma 2.3]; see Appendix B.

Organization

In Section 2 we review some preliminaries.

The proof of Proposition 1.2 on a lower bound for $\mathfrak{g}_{m,n}$ involving the normalized volume of open balls is given in Section 3.1, whereas those of Propositions 1.5 (establishing the precise value of $\mathfrak{g}_{0,n}$), 1.6 (establishing the precise value of $\mathfrak{g}_{m,\infty}$), and 1.8 (on a lower bound for $\mathfrak{g}_{m,n}$ involving the covering radius) are given in Section 3.2.

The proof of Theorem A, establishing that $\mathfrak{g}_{m,n} < \frac{\pi}{2}$ (for any $0 < m < n < \infty$), is given in Section 4, whereas those of Theorem B, on a lower bound for $\mathfrak{g}_{m,n}$ deduced from a discontinuous version of the Borsuk–Ulam theorem, and Theorem C (a generalization of Theorem B) are given in Section 5.

The proofs of Propositions 1.16, establishing the precise value of $\mathfrak{g}_{1,2}$, and 1.20, on an upper bound involving the diameter of a face of a geodesic simplex, are given in Section 6.

Proposition 1.18, establishing the precise value of $\mathfrak{g}_{1,3}$, is proved in Section 7, and Proposition 1.19, establishing the precise value of $\mathfrak{g}_{2,3}$, is proved in Section 8.

The case of spheres with Euclidean distance is discussed in Section 9.

Finally, this paper has three appendices. Appendix A provides a succinct and self contained proof of the version of Borsuk–Ulam's theorem due to Dubins and Schwarz [11] which is instrumental for proving Theorem B and related results. Appendix B establishes that the Gromov–Hausdorff distance between the n–dimensional sphere and an interval is always bounded below by $\frac{\pi}{3}$, and Appendix C provides some results about the Gromov–Hausdorff distance between regular polygons.

Additional aspects of this project (such as computational experiments and further constructions of correspondences) are described in [22; 23].

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2 Preliminaries

Given a metric space (X, d_X) and $\delta > 0$, a δ–net for X is any $A \subset X$ such that for all $x \in X$ there exists $a \in A$ with $d_X(x, a) \leq \delta$. The diameter of X is $\mathrm{diam}(X) := \sup_{x,x' \in X} d_X(x, x')$.

Recall [4, Chapter 2] that complete metric space (X, d_X) is a *geodesic space* if and only if it admits midpoints: for all $x, x' \in X$ there exists $z \in X$ such that

$$d_X(x, z) = d_X(x', z) = \tfrac{1}{2} d_X(x, x').$$

We henceforth use the symbol $*$ to denote the one point metric space. It is easy to check that $d_{\mathrm{GH}}(*, X) = \frac{1}{2}\operatorname{diam}(X)$ for any bounded metric space X. From this, and the triangle inequality for the Gromov–Hausdorff distance, it then follows that for all bounded metric spaces X and Y,

$$d_{\mathrm{GH}}(X,Y) \geq \tfrac{1}{2}|\operatorname{diam}(X) - \operatorname{diam}(Y)|. \tag{7}$$

2.1 Notation and conventions about spheres

Finally, let us collect and introduce important notation and conventions which will be used throughout this paper (except for Section 7). For each nonnegative integer $m \in \mathbb{N}$, we define

- $\mathbb{S}^m := \{(x_1,\dots,x_{m+1}) \in \mathbb{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 = 1\}$ (m–sphere);
- $\boldsymbol{H}_{\geq 0}(\mathbb{S}^m) := \{(x_1,\dots,x_{m+1}) \in \mathbb{S}^m \mid x_{m+1} \geq 0\}$ (closed upper hemisphere);
- $\boldsymbol{H}_{>0}(\mathbb{S}^m) := \{(x_1,\dots,x_{m+1}) \in \mathbb{S}^m \mid x_{m+1} > 0\}$ (open upper hemisphere);
- $\boldsymbol{H}_{\leq 0}(\mathbb{S}^m) := \{(x_1,\dots,x_{m+1}) \in \mathbb{S}^m \mid x_{m+1} \leq 0\}$ (closed lower hemisphere);
- $\boldsymbol{H}_{<0}(\mathbb{S}^m) := \{(x_1,\dots,x_{m+1}) \in \mathbb{S}^m \mid x_{m+1} < 0\}$ (open lower hemisphere);
- $\boldsymbol{E}(\mathbb{S}^m) := \{(x_1,\dots,x_{m+1}) \in \mathbb{S}^m \mid x_{m+1} = 0\}$ (equator of sphere);
- $\mathbb{B}^{m+1} := \{(x_1,\dots,x_{m+1}) \in \mathbb{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 \leq 1\}$ (unit closed ball);
- $\widehat{\mathbb{B}}^{m+1} := \{(x_1,\dots,x_{m+1}) \in \mathbb{R}^{m+1} \mid |x_1| + \cdots + |x_{m+1}| \leq 1\}$ (unit cross-polytope).

Also, $\mathbb{S}^m$, $\boldsymbol{H}_{\geq 0}(\mathbb{S}^m)$, $\boldsymbol{H}_{>0}(\mathbb{S}^m)$, $\boldsymbol{H}_{\leq 0}(\mathbb{S}^m)$, $\boldsymbol{H}_{<0}(\mathbb{S}^m)$ and $\boldsymbol{E}(\mathbb{S}^m)$ are all equipped with the geodesic metric $d_{\mathbb{S}^m}$. Observe that $\mathbb{S}^m$ and $\boldsymbol{E}(\mathbb{S}^{m+1})$ are isometric. We will denote by

$$\iota_m \colon \mathbb{S}^m \to \mathbb{S}^{m+1}, \quad (x_1,\dots,x_{m+1}) \mapsto (x_1,\dots,x_{m+1},0), \tag{8}$$

the canonical isometric embedding from $\mathbb{S}^m$ into $\mathbb{S}^{m+1}$.

2.2 A general construction of correspondences

Assume X and Y are compact metric spaces such that $X \overset{\phi}{\hookrightarrow} Y$ isometrically, eg $\mathbb{S}^m \hookrightarrow \mathbb{S}^n$ for $m \leq n$.

As mentioned in Remark 1.1, any surjection $\psi \colon Y \twoheadrightarrow X$ gives rise to a correspondence between X and Y. The following simple construction of such a ψ will be used throughout this paper. Given $k \in \mathbb{N}$, assume $P_k = \{B_1,\dots,B_i,\dots,B_k\}$ is any partition of $Y \setminus \phi(X)$ and $\mathbb{X}_k = \{x_1, \dots, x_i, \dots, x_k\}$ are any k points in X. Then define $\psi\colon Y \twoheadrightarrow X$ by $\psi|_{\phi(X)} := \phi^{-1}$ and $\psi|_{B_i} := x_i$ for each $1 \le i \le k$. It then follows that the distortion of this correspondence is

$$\operatorname{dis}(\psi) = \max\{A, B, C\},$$

where

- $A := \max_i \operatorname{diam}(B_i)$,
- $B := \max_{i \neq j} \max_{y \in B_i, y' \in B_j} |d_X(x_i, x_j) - d_Y(y, y')|$, and
- $C := \max_i \max_{x \in X, y \in B_i} |d_X(x, x_i) - d_Y(\phi(x), y)|$.

This pattern will be used several times in this paper.

3 Some general lower bounds

3.1 The proof of Proposition 1.2

For a metric space X and $\rho > 0$, let $N_X(\rho)$ denote the minimal number of open balls of radius ρ needed to cover X. Also, let $C_X(\rho)$ denote the maximal number of pairwise disjoint open balls of radius $\frac{1}{2}\rho$ that can be placed in X. N_X and C_X are usually referred to as the *covering number* and the *packing number*, respectively.

Note that the covering radius cov_X — see (4) — and the covering number N_X are related by

$$\operatorname{cov}_X(k) = \inf\{\rho > 0 : N_X(\rho) \le k\}.$$

The following *stability* property of $N_X(\cdot)$ and $C_X(\cdot)$ is classical and can be used to obtain estimates for the Gromov–Hausdorff distance between spheres:

Proposition 3.1 [30, page 299] *If X and Y are metric spaces and $d_{\mathrm{GH}}(X, Y) < \eta$ for some $\eta > 0$, then for all $\rho \ge 0$,*

(1) $N_X(\rho) \ge N_Y(\rho + 2\eta)$, *and*

(2) $C_X(\rho) \ge C_Y(\rho + 2\eta)$.

Recall that $v_n(\rho)$ is the normalized volume of an open ball or radius ρ on $\mathbb{S}^n$.

Proof of Proposition 1.2 The proof that

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \ge \mu_{m,n} := \tfrac{1}{2} \sup_{\rho \in (0,\pi]} \bigl(v_n^{-1} \circ v_m\bigl(\tfrac{1}{2}\rho\bigr) - \rho\bigr)$$

for any $0 < m < n < \infty$ is by contradiction. We first state two claims that we prove at the end.

Claim 1 *For any* $\rho > 0$ *and* $n \geq 1$, *the packing number satisfies* $C_{\mathbb{S}^n}(\rho) \leq \big(v_n\big(\frac{1}{2}\rho\big)\big)^{-1}$.

Claim 2 *For any* $\rho > 0$ *and* $n \geq 1$, *the covering number* $N_{\mathbb{S}^n}(\rho)$ *satisfies*

$$1 \leq N_{\mathbb{S}^n}(\rho) \cdot v_n(\rho).$$

Assuming the claims above, suppose that $n > m \geq 1$ and $\eta := d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) < \mu_{m,n}$. Pick $\varepsilon > 0$ small enough such that $\eta + \frac{1}{2}\varepsilon < \mu_{m,n}$.

Since $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) < \eta + \frac{1}{2}\varepsilon$, from Proposition 3.1, the fact that for $N_X(\rho) \leq C_X(\rho)$ for any compact metric space X and any $\rho > 0$, and Claim 1, we have that

$$N_{\mathbb{S}^n}(\rho + 2\eta + \varepsilon) \leq N_{\mathbb{S}^m}(\rho) \leq C_{\mathbb{S}^m}(\rho) \leq \big(v_m\big(\tfrac{1}{2}\rho\big)\big)^{-1}.$$

Now, from Claim 2 we obtain that, for all $\rho \in [0, \pi]$,

$$1 \leq N_{\mathbb{S}^n}(\rho + 2\eta + \varepsilon) \cdot v_n(\rho + 2\eta + \varepsilon) \leq \frac{v_n(\rho + 2\eta + \varepsilon)}{v_m\big(\frac{1}{2}\rho\big)}.$$

Then, for all $\rho \in [0, \pi]$, we must have

$$\eta + \tfrac{1}{2}\varepsilon \geq \tfrac{1}{2}\big(v_n^{-1} \circ v_m\big(\tfrac{1}{2}\rho\big) - \rho\big).$$

Then, in particular, $\eta + \frac{1}{2}\varepsilon \geq \mu_{m,n}$, a contradiction. □

Proof of Claim 1 Let $k = C_{\mathbb{S}^n}(\rho)$ and let $x_1, \dots, x_k \in \mathbb{S}^n$ be such that

$$B\big(x_i, \tfrac{1}{2}\rho\big) \cap B\big(x_j, \tfrac{1}{2}\rho\big) = \varnothing$$

for all $i \neq j$. Thus, $\bigcup_{i=1}^k B(x_i, \frac{1}{2}\rho) \subset \mathbb{S}^n$, and

$$\mathbf{Vol}(\mathbb{S}^n) \geq \mathbf{vol}_{\mathbb{S}^n}\bigg(\bigcup_{i=1}^k B\big(x_i, \tfrac{1}{2}\rho\big)\bigg) = k \cdot v_n\big(\tfrac{1}{2}\rho\big) \cdot \mathbf{Vol}(\mathbb{S}^n).$$ □

Proof of Claim 2 Fix $N = N_{\mathbb{S}^n}(\rho)$ and $x_1, \dots, x_N \in \mathbb{S}^n$ such that $\bigcup_{i=1}^N B(x_i, \rho) = \mathbb{S}^n$. Then

$$\mathbf{Vol}(\mathbb{S}^n) \leq \mathbf{vol}_{\mathbb{S}^n}\bigg(\bigcup_{i=1}^N B(x_i, \rho)\bigg) \leq N \cdot v_n(\rho) \cdot \mathbf{Vol}(\mathbb{S}^n).$$ □

3.2 Other lower bounds and the proofs of Propositions 1.5 and 1.6

Recall the following corollary to the Borsuk–Ulam theorem [24]:

Theorem D (Lyusternik–Schnirelmann) *Let* $n \in \mathbb{N}$ *and* $\{U_1, \dots, U_{n+1}\}$ *be a closed cover of* $\mathbb{S}^n$. *Then there is an* $i_0 \in \{1, \dots, n+1\}$ *such that* U_{i_0} *contains two antipodal points.*

The lemma below will be useful in what follows:

Lemma 3.2 *For any integer $m \geq 1$ and any finite metric space P with cardinality at most $m+1$, we have $d_{\mathrm{GH}}(\mathbb{S}^m, P) \geq \frac{\pi}{2}$.*

Remark 3.3 Lemma 3.2 and Remark 1.4 imply that for each integer $n \geq 1$, we have $d_{\mathrm{GH}}(\mathbb{S}^n, P) = \frac{\pi}{2}$ for any finite metric space P with $|P| \leq n+1$ and $\mathrm{diam}(P) \leq \pi$.

Proof Suppose $m \geq 1$ is given. We prove that $d_{\mathrm{GH}}(\mathbb{S}^m, P) \geq \frac{\pi}{2}$ for any finite set P of size at most $m+1$. Assume that R is an arbitrary correspondence between $\mathbb{S}^m$ and P. We claim that $\mathrm{dis}(R) \geq \pi$, from which the proof will follow. For each $p \in P$, let $R(p) := \{z \in \mathbb{S}^m \mid (z, p) \in R\}$. Then $\{\overline{R(p)} \subseteq \mathbb{S}^m \mid p \in P\}$ is a closed cover of $\mathbb{S}^m$. Since $|P| \leq m+1$, Theorem D yields that $\mathrm{diam}(R(p_0)) = \pi$ for some $p_0 \in P$. Finally, the claim follows since $\mathrm{dis}(R) \geq \max_{p \in P} \mathrm{diam}(R(p))$. □

By a refinement of the proof of Lemma 3.2 above one obtains:

Corollary 3.4 *Let R be any correspondence between a finite metric space P and $\mathbb{S}^\infty$. Then $\mathrm{dis}(R) \geq \pi$. In particular, $d_{\mathrm{GH}}(P, \mathbb{S}^\infty) \geq \frac{\pi}{2}$.*

Proof As in the proof of Lemma 3.2, the correspondence R induces a closed cover of $\mathbb{S}^\infty$. Thus, it induces a closed cover of any finite dimensional sphere $\mathbb{S}^{|P|-1} \subset \mathbb{S}^\infty$. The claim follows from Theorem D. □

Notice that if P has diameter at most π, then $d_{\mathrm{GH}}(P, \mathbb{S}^\infty) = \frac{\pi}{2}$ (see Remarks 1.4 and 3.3). In Appendix C we consider a scenario which is thematically connected with Remark 3.3 and Corollary 3.4, namely the determination of the Gromov–Hausdorff distance between a finite metric space and a sphere. Appendix C fully resolves this question for the case of $\mathbb{S}^1$ and (the vertex set of) inscribed regular polygons.

By a small modification of the proof of Corollary 3.4, we obtain the following stronger claim:

Proposition 3.5 *Let X be any totally bounded metric space. Then $d_{\mathrm{GH}}(X, \mathbb{S}^\infty) \geq \frac{\pi}{2}$.*

Proof Fix any $\varepsilon > 0$ and let $P_\varepsilon \subset X$ be a finite ε–net for X. Then, by the triangle inequality for d_{GH} and Corollary 3.4,

$$d_{\mathrm{GH}}(X, \mathbb{S}^\infty) \geq d_{\mathrm{GH}}(\mathbb{S}^\infty, P_\varepsilon) - d_{\mathrm{GH}}(X, P_\varepsilon) \geq \frac{\pi}{2} - \varepsilon,$$

which implies the claim since $\varepsilon > 0$ was arbitrary. □

Proof of Proposition 1.5 That $d_{\mathrm{GH}}(\mathbb{S}^0,\mathbb{S}^n)=\frac{\pi}{2}$ for any integer $n\geq 1$ follows from Lemma 3.2 and Remark 1.4. □

Proof of Proposition 1.6 That $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^\infty)=\frac{\pi}{2}$ for any nonnegative integer $m<\infty$ follows from Proposition 3.5 and Remark 1.4. □

Proof of Proposition 1.8 We prove that $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)\geq \nu_{m,n}:=\frac{\pi}{2}-\mathrm{cov}_{\mathbb{S}^m}(n+1)$ for any $1\leq m<n<\infty$.

Let P be any subset $\mathbb{S}^m$ with cardinality not exceeding $n+1$. Since the Hausdorff distance satisfies $d_{\mathrm{H}}(P,\mathbb{S}^m)\geq d_{\mathrm{GH}}(P,\mathbb{S}^m)$, and by the triangle inequality for the Gromov–Hausdorff distance, we have

$$d_{\mathrm{H}}(P,\mathbb{S}^m)+d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)\geq d_{\mathrm{GH}}(P,\mathbb{S}^m)+d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)\geq d_{\mathrm{GH}}(P,\mathbb{S}^n).$$

Since $\mathrm{diam}(P)\leq\pi$, by Remark 3.3 we have that $d_{\mathrm{GH}}(P,\mathbb{S}^n)=\frac{\pi}{2}$. Hence, from the above,

$$d_{\mathrm{H}}(P,\mathbb{S}^m)+d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)\geq\frac{\pi}{2}$$

for *any* $P\subset\mathbb{S}^m$ with $|P|\leq n+1$. By the definition of the covering radius (see (4)), we obtain the claim by taking the infimum over all possible such choices of P. □

4 The proof of Theorem A

The Borsuk–Ulam theorem implies that, for any positive integers $n>m$ and for any given continuous map $\varphi\colon\mathbb{S}^n\to\mathbb{S}^m$, there exists two antipodal points in the higher dimensional sphere which are mapped to the same point in the lower dimensional sphere.

We now prove that, in contrast, there always exists a *surjective*, antipode-preserving, and continuous map $\psi_{m,n}$ from the lower dimensional sphere to the higher dimensional sphere.

Theorem E *For all integers $0<m<n<\infty$, there exists an* ***antipode-preserving*** *continuous surjection*

$$\psi_{m,n}\colon\mathbb{S}^m\twoheadrightarrow\mathbb{S}^n,$$

ie $\psi_{m,n}(-x)=-\psi_{m,n}(x)$ for every $x\in\mathbb{S}^m$.

With this theorem, the proof of Theorem A, stating that $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)<\frac{\pi}{2}$ for all $0<m<n<\infty$, now follows:

Proof of Theorem A Let $\psi_{m,n}\colon \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n$ be the map given in Theorem E. Recall that the graph of a surjective map can be seen as a correspondence and let $R_{m,n} := \operatorname{graph}(\psi_{m,n})$. In order to prove the claim, it is enough to verify that

$$\operatorname{dis}(R_{m,n}) = \operatorname{dis}(\psi_{m,n}) < \pi.$$

Since $\psi_{m,n}$ is continuous and $\mathbb{S}^m$ is compact, the supremum in the definition of distortion is a maximum,

$$\operatorname{dis}(\psi_{m,n}) = \max_{x,x'\in\mathbb{S}^m} |d_{\mathbb{S}^m}(x,x') - d_{\mathbb{S}^n}(\psi_{m,n}(x),\psi_{m,n}(x'))|.$$

Let $x_0, x_0' \in \mathbb{S}^m$ attain the maximum above. Note that we may assume that $x_0 \neq x_0'$, for otherwise we would have $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n) \leq \frac{1}{2}\operatorname{dis}(R_{m,n}) = \frac{1}{2}\operatorname{dis}(\psi_{m,n}) = 0$, which would imply that $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n) = 0$, ie that $\mathbb{S}^m$ and $\mathbb{S}^n$ are isometric, which is a contradiction since $m \neq n$.

Assume first that $x_0' \neq -x_0$. In this case,

$$0 < d_{\mathbb{S}^m}(x_0,x_0') < \pi \quad \text{and} \quad 0 \leq d_{\mathbb{S}^n}(\psi_{m,n}(x_0),\psi_{m,n}(x_0')) \leq \pi.$$

Thus,

$$|d_{\mathbb{S}^m}(x_0,x_0') - d_{\mathbb{S}^n}(\psi_{m,n}(x_0),\psi_{m,n}(x_0'))| < \pi.$$

Assume now that $x_0' = -x_0$. In this case, $d_{\mathbb{S}^m}(x_0,x_0') = d_{\mathbb{S}^n}(\psi_{m,n}(x_0),\psi_{m,n}(x_0')) = \pi$ since $\psi_{m,n}$ is antipode-preserving. Thus, in this case we also have

$$0 = |d_{\mathbb{S}^m}(x_0,x_0') - d_{\mathbb{S}^n}(\psi_{m,n}(x_0),\psi_{m,n}(x_0'))| < \pi. \qquad \square$$

Remark 4.1 The antipode-preserving property of $\psi_{m,n}$ given in Theorem E is stronger than what we need for the purpose of proving Theorem A. Indeed, all one needs is that $\psi_{m,n}(x) \neq \psi_{m,n}(-x)$ for every $x \in \mathbb{S}^m$.

The goal for the rest of this section is to prove Theorem E.

Spherical suspensions and space-filling curves are key technical tools, which we now review.

Space-filling curves

The existence of the space-filling curves is well known [29]:

Theorem F (space-filling curve) *There exists a continuous and surjective map*

$$H\colon [0,1] \twoheadrightarrow [0,1]^2.$$

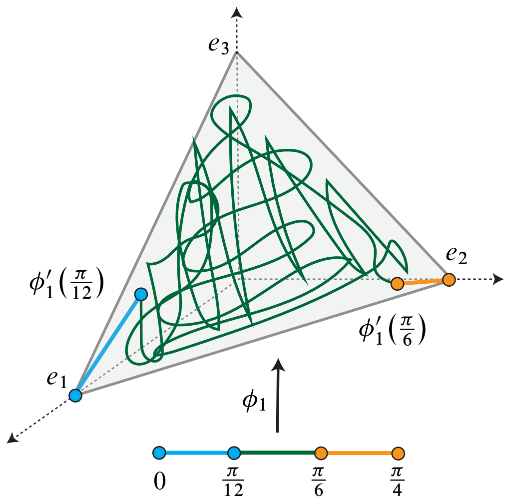


Figure 3: The continuous surjection $\phi_1\colon [0, \frac{\pi}{4}] \twoheadrightarrow \mathrm{Conv}(e_1, e_2, e_3)$.

In the sequel, we will use the notation $\mathrm{Conv}(v_1, v_2, \dots, v_d)$ to denote the convex hull of vectors $v_1, v_2, \dots, v_d$.

By resorting to space-filling curves, one can prove the following proposition, which will be crucial in the sequel:

Proposition 4.2 *There exists an antipode-preserving continuous surjection*

$$\psi_{1,2}\colon \mathbb{S}^1 \twoheadrightarrow \mathbb{S}^2.$$

Proof Recall the definition of the 3–dimensional cross-polytope,

$$\widehat{\mathbb{B}}^3 := \mathrm{Conv}(e_1, -e_1, e_2, -e_2, e_3, -e_3) \subset \mathbb{R}^3,$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Then its boundary $\partial \widehat{\mathbb{B}}^3$, which consists of eight triangles

$$\mathrm{Conv}(e_1, e_2, e_3), \quad \mathrm{Conv}(e_1, e_2, -e_3), \quad \dots, \mathrm{Conv}(-e_1, -e_2, -e_3),$$

is homeomorphic to $\mathbb{S}^2$.

Now, divide $\mathbb{S}^1$ into eight closed circular arcs with equal length $\frac{\pi}{4}$. In other words, let

$$\left[0, \tfrac{\pi}{4}\right], \ \left[\tfrac{\pi}{4}, \tfrac{\pi}{2}\right], \ \left[\tfrac{\pi}{2}, \tfrac{3\pi}{4}\right], \ \left[\tfrac{3\pi}{4}, \pi\right], \ \left[\pi, \tfrac{5\pi}{4}\right], \ \left[\tfrac{5\pi}{4}, \tfrac{3\pi}{2}\right], \ \left[\tfrac{3\pi}{2}, \ \tfrac{7\pi}{4}\right], \ \left[\tfrac{7\pi}{4}, 2\pi\right]$$

be those eight regions. Of course, we are identifying 0 and 2π here.

Note that we are able to build a continuous and surjective map

$$\phi_1\colon \bigl[0,\tfrac{\pi}{4}\bigr] \twoheadrightarrow \mathrm{Conv}(e_1,e_2,e_3) \quad \text{such that } \phi_1(0)=e_1 \text{ and } \phi_1\bigl(\tfrac{\pi}{4}\bigr)=e_2$$

as follows: since $\mathrm{Conv}(e_1,e_2,e_3)$ is homeomorphic to $[0,1]^2$, by Theorem F there exists a continuous and surjective map ϕ_1' from $\bigl[\frac{\pi}{12},\frac{\pi}{6}\bigr]$ to $\mathrm{Conv}(e_1,e_2,e_3)$; then, we extend its domain by using linear interpolation between e_1 and $\phi_1'\bigl(\frac{\pi}{12}\bigr)$, and e_2 and $\phi_1'\bigl(\frac{\pi}{6}\bigr)$ to give rise to ϕ_1 — see Figure 3.

By using an analogous procedure, we construct continuous and surjective maps

$$\begin{aligned}
&\phi_2\colon \bigl[\tfrac{\pi}{4},\tfrac{\pi}{2}\bigr] \twoheadrightarrow \mathrm{Conv}(-e_1,e_2,e_3) && \text{such that } \phi_2\bigl(\tfrac{\pi}{4}\bigr)=e_2 \text{ and } \phi_2\bigl(\tfrac{\pi}{2}\bigr)=e_3,\\
&\phi_3\colon \bigl[\tfrac{\pi}{2},\tfrac{3\pi}{4}\bigr] \twoheadrightarrow \mathrm{Conv}(e_1,-e_2,e_3) && \text{such that } \phi_3\bigl(\tfrac{\pi}{2}\bigr)=e_3 \text{ and } \phi_3\bigl(\tfrac{3\pi}{4}\bigr)=-e_2,\\
&\phi_4\colon \bigl[\tfrac{3\pi}{4},\pi\bigr] \twoheadrightarrow \mathrm{Conv}(-e_1,-e_2,e_3) && \text{such that } \phi_4\bigl(\tfrac{3\pi}{4}\bigr)=-e_2 \text{ and } \phi_4(\pi)=-e_1.
\end{aligned}$$

Next, we construct the remaining continuous and surjective maps by suitably reflecting the ones already constructed,

$$\begin{aligned}
&\phi_5\colon \bigl[\pi,\tfrac{5\pi}{4}\bigr] \twoheadrightarrow \mathrm{Conv}(-e_1,-e_2,-e_3) && \text{such that } \phi_5(x):=-\phi_1(-x),\\
&\phi_6\colon \bigl[\tfrac{5\pi}{4},\tfrac{3\pi}{2}\bigr] \twoheadrightarrow \mathrm{Conv}(e_1,-e_2,-e_3) && \text{such that } \phi_6(x):=-\phi_2(-x),\\
&\phi_7\colon \bigl[\tfrac{3\pi}{2},\tfrac{7\pi}{4}\bigr] \twoheadrightarrow \mathrm{Conv}(e_1,e_2,-e_3) && \text{such that } \phi_7(x):=-\phi_3(-x),\\
&\phi_8\colon \bigl[\tfrac{7\pi}{4},2\pi\bigr] \twoheadrightarrow \mathrm{Conv}(-e_1,e_2,-e_3) && \text{such that } \phi_8(x):=-\phi_4(-x).
\end{aligned}$$

Finally, by gluing all the eight maps ϕ_i, we build an antipode-preserving continuous and surjective map $\overline{\psi}_{1,2}\colon \mathbb{S}^1 \twoheadrightarrow \partial\widehat{\mathbb{B}}^3$. Using the canonical (closest point projection) homeomorphism between $\partial\widehat{\mathbb{B}}^3$ and $\mathbb{S}^2$, we finally have the announced $\psi_{1,2}\colon \mathbb{S}^1 \twoheadrightarrow \mathbb{S}^2$. It is clear from its construction that the map $\psi_{1,2}$ is continuous, surjective, and antipode-preserving. Figure 4 depicts the overall structure of the map $\psi_{1,2}$. □

Spherical suspensions

Suppose $m,n\in\mathbb{N}$ and a map $f\colon \mathbb{S}^m \to \mathbb{S}^n$ are given. Then one can lift this map f to a map from $\mathbb{S}^{m+1}$ to $\mathbb{S}^{n+1}$ in the following way: observe that an arbitrary point in $\mathbb{S}^{m+1}$ can be expressed as $(p\sin\theta,\cos\theta)$ for some $p\in\mathbb{S}^m$ and $\theta\in[0,\pi]$; then the *spherical suspension of f* is the map

$$Sf\colon \mathbb{S}^{m+1}\to\mathbb{S}^{n+1},\quad (p\sin\theta,\cos\theta)\mapsto(f(p)\sin\theta,\cos\theta).$$

Lemma 4.3 *If the map $f\colon \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n$ is continuous, surjective and antipode-preserving, then $Sf\colon \mathbb{S}^{m+1} \twoheadrightarrow \mathbb{S}^{n+1}$ is also continuous, surjective and antipode-preserving.*

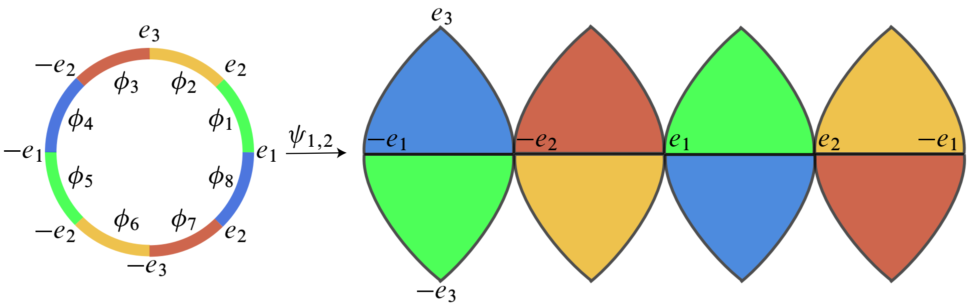


Figure 4: Structure of the map $\psi_{1,2}$ constructed in Proposition 4.2. Inside each arc, the map is defined via a space-filling curve. For simplicity, $\mathbb{S}^2$ is "cartographically" depicted.

Proof Continuity and surjectivity are clear from the construction. Since f is antipode-preserving, we know that $f(-p) = -f(p)$ for every $p \in \mathbb{S}^m$. Hence,

$$\begin{aligned} Sf(-p\sin\theta, -\cos\theta) &= Sf(-p\sin(\pi-\theta), \cos(\pi-\theta)) \\ &= (f(-p)\sin(\pi-\theta), \cos(\pi-\theta)) \\ &= (-f(p)\sin\theta, -\cos\theta) \\ &= -(f(p)\sin\theta, \cos\theta) \\ &= -Sf(p\sin\theta, \cos\theta) \end{aligned}$$

for any $p \in \mathbb{S}^m$ and $\theta \in [0,\pi]$. Thus, Sf is also antipode-preserving. □

We now use induction to obtain:

Corollary 4.4 *For any integer $m > 0$, there exists a continuous, surjective, and antipode-preserving map*

$$\psi_{m,(m+1)} \colon \mathbb{S}^m \twoheadrightarrow \mathbb{S}^{m+1}.$$

Proof Proposition 4.2 guarantees the existence of $\psi_{1,2}$. For general m, it suffices to apply Lemma 4.3 inductively. □

The following lemma is obvious:

Lemma 4.5 *Suppose that $l, m, n \in \mathbb{N}$, and maps $f \colon \mathbb{S}^l \twoheadrightarrow \mathbb{S}^m$ and $g \colon \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n$ are given such that both f and g are continuous, surjective, and antipode-preserving. Then their composition $g \circ f \colon \mathbb{S}^l \twoheadrightarrow \mathbb{S}^n$ is also continuous, surjective, and antipode-preserving.*

The proof of Theorem E

We are now ready to prove Theorem E, which states that there exists an antipode-preserving continuous surjection $\psi_{m,n}\colon \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n$ for any $0 < m < n < \infty$.

Proof of Theorem E By Corollary 4.4, there are continuous, surjective, and antipode-preserving maps $\psi_{m,(m+1)}, \psi_{(m+1),(m+2)}, \dots, \psi_{(n-1),n}$. Then, by Lemma 4.5, the map

$$\psi_{m,n} := \psi_{(n-1),n} \circ \cdots \circ \psi_{(m+1),(m+2)} \circ \psi_{m,(m+1)}$$

is also continuous, surjective, and antipode-preserving. □

5 A Borsuk–Ulam theorem for discontinuous functions and the proof of Theorem B

Definition 5.1 (modulus of discontinuity) Let X be a topological space, Y be a metric space, and $f\colon X \to Y$ be any function. Then we define $\delta(f)$, the *modulus of discontinuity of* f, by

$$\delta(f) := \inf\{\delta \geq 0 \mid \text{each } x \in X \text{ has an open neighborhood } U_x \text{ with } \operatorname{diam}(f(U_x)) \leq \delta\}.$$

Remark 5.2 Of course, $\delta(f) = 0$ if and only if f is continuous.

It turns out that the modulus of discontinuity is a lower bound for distortion:

Proposition 5.3 *Let $\phi\colon (X, d_X) \to (Y, d_Y)$ be a map between two metric spaces. Then*

$$\delta(\phi) \leq \operatorname{dis}(\phi).$$

Proof If $\operatorname{dis}(\phi) = \infty$, then the proof is trivial, so suppose $\operatorname{dis}(\phi) < \infty$. Now, fix arbitrary $x \in X$ and $\varepsilon > 0$. Consider the open ball $U_x := B\big(x, \frac{1}{2}\varepsilon\big)$. Then, for any $x', x'' \in U_x$,

$$d_Y(\phi(x'), \phi(x'')) \leq d_X(x', x'') + |d_X(x', x'') - d_Y(\phi(x'), \phi(x''))| < \operatorname{dis}(\phi) + \varepsilon,$$

so $\operatorname{diam}(\phi(U_x)) \leq \operatorname{dis}(\phi) + \varepsilon$. Since x is arbitrary, this implies $\delta(\phi) \leq \operatorname{dis}(\phi) + \varepsilon$. Since ε is arbitrary, we have the required inequality. □

The following variant of the Borsuk–Ulam theorem, due to Dubins and Schwarz, is the main tool used in the proof of Theorem B.

Theorem G [11, Theorem 1] *For each integer $n > 0$, the modulus of discontinuity of any function $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ that maps every pair of antipodal points on the boundary of $\mathbb{B}^n$ onto antipodal points on $\mathbb{S}^{n-1}$ is not less than ζ_{n-1}.*

In Appendix A we provide a concise self-contained proof of this result based on ideas by Arnold Waßmer; see Matoušek [24, page 41].

We immediately have:

Corollary 5.4 [11, Corollary 3] *For each integer $n > 0$, the modulus of discontinuity of any function $g : \mathbb{S}^n \to \mathbb{S}^{n-1}$ which maps every pair of antipodal points on $\mathbb{S}^n$ onto antipodal points on $\mathbb{S}^{n-1}$ is not less than ζ_{n-1}.*

We provide a detailed proof of this result for completeness.

Proof Consider the map

$$\Phi : \mathbb{B}^n \to \mathbb{S}^n, \quad (x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \sqrt{1 - (x_1^2 + \cdots + x_n^2)}\right).$$

Obviously, Φ is continuous and its image is $\boldsymbol{H}_{\geq 0}(\mathbb{S}^n)$. Now, fix an arbitrary $\delta \geq 0$ such that for every $x \in \mathbb{S}^n$, there exists an open neighborhood U_x of x with $\operatorname{diam}(g(U_x)) \leq \delta$.

Now, fix an arbitrary $x' \in \mathbb{B}^n$. Then $\Phi^{-1}(U_{\Phi(x')})$ is an open neighborhood of x', and

$$\operatorname{diam}\left(g \circ \Phi(\Phi^{-1}(U_{\Phi(x')}))\right) \leq \operatorname{diam}(g(U_{\Phi(x')})) \leq \delta.$$

Since x' is arbitrary, this means that $\delta \geq \delta(g \circ \Phi)$. Moreover, since $g \circ \Phi$ is antipode-preserving, $\delta(g \circ \Phi) \geq \zeta_{n-1}$ by Theorem G. Hence, we conclude that $\delta \geq \zeta_{n-1}$. Finally, since δ was arbitrary, by taking the infimum we conclude that

$$\delta(g) \geq \zeta_{n-1}.$$

□

Corollary 5.5 *For each integer $n > 0$, any function $g : \mathbb{S}^n \to \mathbb{S}^{n-1}$ which maps every pair of antipodal points on $\mathbb{S}^n$ onto antipodal points on $\mathbb{S}^{n-1}$ satisfies $\operatorname{dis}(g) \geq \zeta_{n-1}$.*

Proof Apply Corollary 5.4 and Proposition 5.3. □

5.1 The proof of Theorem B

We are almost ready to prove Theorem B, which establishes $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$ for any $0 < m < n < \infty$.

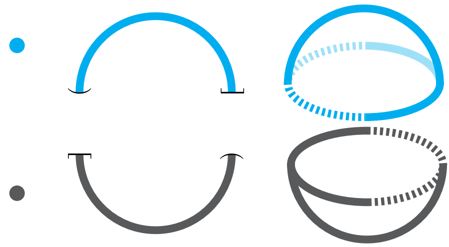


Figure 5: From left to right, the blue sets represent $\boldsymbol{A}(\mathbb{S}^0)$, $\boldsymbol{A}(\mathbb{S}^1)$ and $\boldsymbol{A}(\mathbb{S}^2)$. The figure also shows their antipodes in dark gray. See Definition 5.6.

For each integer $n \geq 1$, recall the natural isometric embedding of $\mathbb{S}^{n-1}$ to the equator $\boldsymbol{E}(\mathbb{S}^n)$ of $\mathbb{S}^n$,

$$\iota_{n-1}\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n, \quad (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0).$$

Also, let us define the sets $\boldsymbol{A}(\mathbb{S}^n) \subset \mathbb{S}^n$ (which we will sometimes refer to as "helmets") for $n \in \mathbb{N}$:

Definition 5.6 (definition of $\boldsymbol{A}(\mathbb{S}^n)$) Let

$$\boldsymbol{A}(\mathbb{S}^0) := \{1\} \quad \text{and} \quad \boldsymbol{A}(\mathbb{S}^1) := \{(\cos\theta, \sin\theta) \in \mathbb{S}^1 \mid \theta \in [0,\pi)\}.$$

Moreover, for general $n \geq 1$, define, inductively,

$$\boldsymbol{A}(\mathbb{S}^n) := \boldsymbol{H}_{>0}(\mathbb{S}^n) \cup \iota_{n-1}(\boldsymbol{A}(\mathbb{S}^{n-1})).$$

See Figure 5 for an illustration. Observe that, for any $n \geq 0$,

$$\boldsymbol{A}(\mathbb{S}^n) \cap (-\boldsymbol{A}(\mathbb{S}^n)) = \varnothing \quad \text{and} \quad \boldsymbol{A}(\mathbb{S}^n) \cup (-\boldsymbol{A}(\mathbb{S}^n)) = \mathbb{S}^n.$$

The following lemma is simple but critical. Given any map $\phi\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ it will permit constructing an antipode-preserving map ϕ^* with at most the same distortion.

Lemma 5.7 *For any $m, n \geq 0$, let $\varnothing \neq C \subseteq \mathbb{S}^n$ satisfy $C \cap (-C) = \varnothing$ and let $\phi\colon C \to \mathbb{S}^m$ be any map. Then the extension ϕ^* of ϕ to the set $C \cup (-C)$ defined by*

$$\phi^*\colon C \cup (-C) \to \mathbb{S}^m, \qquad x \mapsto \phi(x), \quad -x \mapsto -\phi(x) \quad \text{for } x \in C,$$

is antipode-preserving and satisfies $\operatorname{dis}(\phi^*) = \operatorname{dis}(\phi)$.

Proof By definition, ϕ^* is antipode-preserving. Now, fix arbitrary $x, x' \in C$. Then

$$\begin{aligned} |d_{\mathbb{S}^n}(x,-x') - d_{\mathbb{S}^m}(\phi^*(x),\phi^*(-x'))| &= |(\pi - d_{\mathbb{S}^n}(x,x')) - (\pi - d_{\mathbb{S}^m}(\phi(x),\phi(x')))| \\ &= |d_{\mathbb{S}^n}(x,x') - d_{\mathbb{S}^m}(\phi(x),\phi(x'))| \\ &\le \operatorname{dis}(\phi) \end{aligned}$$

and

$$|d_{\mathbb{S}^n}(-x,-x') - d_{\mathbb{S}^m}(\phi^*(-x),\phi^*(-x'))| = |d_{\mathbb{S}^n}(x,x') - d_{\mathbb{S}^m}(\phi(x),\phi(x'))| \le \operatorname{dis}(\phi).$$

This implies $\operatorname{dis}(\phi^*) = \operatorname{dis}(\phi)$. □

Corollary 5.8 *For each $n \in \mathbb{Z}_{>0}$ and any map $\phi\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$, there exists an antipode-preserving map $\phi^*\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ such that $\operatorname{dis}(\phi^*) \le \operatorname{dis}(\phi)$.*

Proof Consider the restriction of ϕ to $\boldsymbol{A}(\mathbb{S}^n)$ and apply Lemma 5.7. □

Finally, we are ready to prove Theorem B.

Proof of Theorem B Let $0 < m < n < \infty$. We first prove the second claim of Theorem B that $\operatorname{dis}(\phi) \ge \zeta_m$ for any map $\phi\colon \mathbb{S}^n \to \mathbb{S}^m$. Suppose to the contrary, so that there is a map $\tilde{g}\colon \mathbb{S}^n \to \mathbb{S}^m$ with $\operatorname{dis}(\tilde{g}) < \zeta_m$. By restriction, this map induces a map $g\colon \mathbb{S}^{m+1} \to \mathbb{S}^m$ such that $\operatorname{dis}(g) < \zeta_m$. By applying Corollary 5.8, one can modify g into an antipode-preserving map $\hat{g}\colon \mathbb{S}^{m+1} \to \mathbb{S}^m$ with $\operatorname{dis}(\hat{g}) < \zeta_m$, which contradicts Corollary 5.5. This yields the proof of the second claim of Theorem B.

Now, in order to prove the first claim of Theorem B that $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \ge \frac{1}{2}\zeta_m$, suppose that Γ is a correspondence between $\mathbb{S}^m$ and $\mathbb{S}^n$ with $\operatorname{dis}(\Gamma) < \zeta_m$. Pick any function $g\colon \mathbb{S}^n \to \mathbb{S}^m$ such that $(g(x), x) \in \Gamma$ for every $x \in \mathbb{S}^n$. This implies that

$$\operatorname{dis}(g) \le \operatorname{dis}(\Gamma) < \zeta_m,$$

which contradicts the second claim. This proves the first claim. □

5.2 The proof of Theorem C

By carefully inspecting the proof of Theorem B, one can extract the much stronger Theorem C.

Proof of Theorem C We will actually prove slightly stronger result. Suppose

(i) X can be isometrically embedded into $\mathbb{S}^m$, and

(ii) $\boldsymbol{A}(\mathbb{S}^{m+1})$ (note that $\boldsymbol{A}(\mathbb{S}^{m+1}) \subset \boldsymbol{H}_{\ge 0}(\mathbb{S}^{m+1})$) can be isometrically embedded into Y.

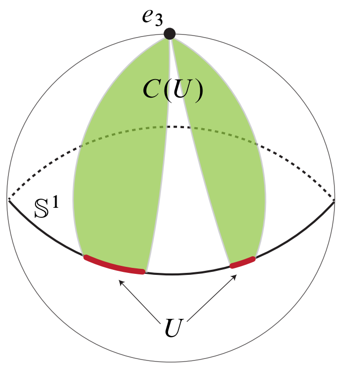


Figure 6: The cone $\mathcal{C}(U)$ for a subset U of $\mathbb{S}^1$.

Now, we prove that $d_{\mathrm{GH}}(X,Y) \geq \frac{1}{2}\zeta_m$. Moreover, $\mathrm{dis}(\phi) \geq \zeta_m$ for any map $\phi\colon Y \to X$.

We first prove the second claim. Suppose to the contrary, so that there is a map $\tilde{g}\colon Y \to X$ with $\mathrm{dis}(\tilde{g}) < \zeta_m$. Then, since $\boldsymbol{A}(\mathbb{S}^{m+1})$ is isometrically embedded in Y and X is isometrically embedded in $\mathbb{S}^m$ by the assumption, one can construct a map $g\colon \boldsymbol{A}(\mathbb{S}^{m+1}) \to \mathbb{S}^m$ with $\mathrm{dis}(g) < \zeta_m$. Hence, with the aid of Lemma 5.7, one can modify this g into an antipode-preserving map $\hat{g}\colon \mathbb{S}^{m+1} \to \mathbb{S}^m$ with $\mathrm{dis}(\hat{g}) < \zeta_m$, which contradicts Corollary 5.5. This yields the proof of the second claim.

Now, in order to prove the first claim, use the same argument used in the proof of Theorem B. □

6 The proofs of Propositions 1.16 and 1.20

To prove Propositions 1.16 and 1.20, we need to define a few notions.

Definition 6.1 For any nonempty $U \subseteq \mathbb{S}^{n-1}$, we define *the cone of U*, as the following subset of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$:

$$\mathcal{C}(U) := \left\{\cos\theta \cdot e_{n+1} + \sin\theta \cdot \iota_{n-1}(u) \in \boldsymbol{H}_{\geq 0}(\mathbb{S}^n) \mid u \in U \text{ and } \theta \in \left[0, \tfrac{\pi}{2}\right]\right\},$$

where $e_{n+1} = (0, 0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ is the north pole of $\mathbb{S}^n$. See Figure 6.

Lemma 6.2 *For any nonempty $U \subseteq \mathbb{S}^{n-1}$,*

$$\mathrm{diam}(\mathcal{C}(U)) = \begin{cases} \frac{\pi}{2} & \textit{if } \mathrm{diam}(U) \leq \frac{\pi}{2}, \\ \mathrm{diam}(U) & \textit{if } \mathrm{diam}(U) \geq \frac{\pi}{2}. \end{cases}$$

Proof Recall that

$$\mathcal{C}(U) := \left\{\cos\theta\cdot e_{n+1} + \sin\theta\cdot\iota_{n-1}(u) \in \boldsymbol{H}_{\geq 0}(\mathbb{S}^n) \mid u \in U \text{ and } \theta \in \left[0, \tfrac{\pi}{2}\right]\right\}.$$

Now, for $u, v \in U$ and $\theta, \theta' \in \left[0, \frac{\pi}{2}\right]$, consider the inner product

$$\langle \cos\theta\cdot e_{n+1} + \sin\theta\cdot\iota_{n-1}(u), \cos\theta'\cdot e_{n+1} + \sin\theta'\cdot\iota_{n-1}(v)\rangle$$
$$= \cos\theta\cos\theta' + \langle u, v\rangle\cdot\sin\theta\sin\theta'.$$

Hence, if $\langle u, v\rangle \geq 0$,

$$\langle \cos\theta\cdot e_{n+1} + \sin\theta\cdot\iota_{n-1}(u), \cos\theta'\cdot e_{n+1} + \sin\theta'\cdot\iota_{n-1}(v)\rangle \geq 0,$$

so $d_{\mathbb{S}^n}(\cos\theta\cdot e_{n+1} + \sin\theta\cdot u, \cos\theta'\cdot e_{n+1} + \sin\theta'\cdot v) \leq \frac{\pi}{2}$.

If $\langle u, v\rangle \leq 0$, $\cos\theta\cos\theta' + \langle u, v\rangle\cdot\sin\theta\sin\theta$ becomes decreasing in θ and θ'. Hence, it is minimized for $\theta = \theta' = \frac{\pi}{2}$. Therefore,

$$\langle \cos\theta\cdot e_{n+1} + \sin\theta\cdot\iota_{n-1}(u), \cos\theta'\cdot e_{n+1} + \sin\theta'\cdot\iota_{n-1}(v)\rangle \geq \langle u, v\rangle,$$

so $d_{\mathbb{S}^n}(\cos\theta\cdot e_{n+1} + \sin\theta\cdot\iota_{n-1}(u), \cos\theta'\cdot e_{n+1} + \sin\theta'\cdot\iota_{n-1}(v)) \leq d_{\mathbb{S}^{n-1}}(u, v)$, which completes the proof. □

Definition 6.3 (geodesic convex hull) Given a nonempty subset $A \subset \mathbb{S}^n$, its *geodesic convex hull* $\mathrm{conv}_{\mathbb{S}^n}(A)$ is defined to be the smallest subset of $\mathbb{S}^n$ containing A such that for any two points in the set, all minimizing geodesics between them are also contained in the set. It is clear that when A is contained in an open hemisphere,

$$\mathrm{conv}_{\mathbb{S}^n}(A) = \{\Pi_{\mathbb{S}^n}(c) \mid c \in \mathrm{conv}(A)\},$$

where $\Pi_{\mathbb{S}^n}(p) := p/\|p\|$ for $p \neq 0$ and $\Pi_{\mathbb{S}^n}(p) := 0$ otherwise.

In what follows we will prove Proposition 1.20 after proving Proposition 1.16. The proof of the former proposition generalizes the construction used in the proof of the latter one, and as a consequence Proposition 1.16 (which exhibits a correspondence between $\mathbb{S}^2$ and $\mathbb{S}^1$) is a special case of Proposition 1.20 (which constructs a correspondence between $\mathbb{S}^{m+1}$ and $\mathbb{S}^m$).

With the goal of making the construction more understandable, we have however decided to first present a detailed proof of Proposition 1.16 since the optimal $R_{2,1}$ correspondence constructed therein is used in the proof of Proposition 1.18 in order to construct an optimal correspondence $R_{3,1}$. After this we provide a streamlined proof of Proposition 1.20.

6.1 The proof of Proposition 1.16

We will find an upper bound for $d_{\mathrm{GH}}(\mathbb{S}^1, \boldsymbol{H}_{\geq 0}(\mathbb{S}^2))$ (resp. $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2)$) by constructing a specific correspondence between $\mathbb{S}^1$ and $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)$ (resp. $\mathbb{S}^1$ and $\mathbb{S}^2$). This correspondence is inspired by the case $m = 1$ of certain surjective maps from $\mathbb{S}^{m+1}$ to $\mathbb{S}^m$ [11, Scholium 1] developed in the course of the authors' study of the modulus of discontinuity of antipode-preserving maps between spheres. In spite of the fact that these maps will in general fail to yield tight upper bounds for distortion, they still permit giving nontrivial upper bounds for $\mathfrak{g}_{m,m+1}$. This will be explained in Section 6.2.

Proof of Proposition 1.16 We will prove both claims: that there exists

(1) a correspondence between $\mathbb{S}^1$ and $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2)$, and

(2) a correspondence between $\mathbb{S}^1$ and $\mathbb{S}^2$,

both of which have distortion at most $\frac{2\pi}{3}$ in an intertwined way.

In order to prove the first claim, it is enough to find a surjective map $\tilde{\phi}_{2,1} : \boldsymbol{H}_{\geq 0}(\mathbb{S}^2) \twoheadrightarrow \mathbb{S}^1$ (resp. $\phi_{2,1} : \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$) such that $\mathrm{dis}(\tilde{\phi}_{2,1}) \leq \zeta_1 = \frac{2\pi}{3}$ (resp. $\mathrm{dis}(\phi_{2,1}) \leq \zeta_1 = \frac{2\pi}{3}$) since this map gives rise to a correspondence $\widetilde{R}_{2,1} := \mathrm{graph}(\tilde{\phi}_{2,1})$ (resp. $R_{2,1} := \mathrm{graph}(\phi_{2,1})$) with $\mathrm{dis}(\widetilde{R}_{2,1}) = \mathrm{dis}(\tilde{\phi}_{2,1}) \leq \zeta_1$ (resp. $\mathrm{dis}(R_{2,1}) = \mathrm{dis}(\phi_{2,1}) \leq \zeta_1$).

Let

$$u_1 := (1, 0, 0), \quad u_2 := \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right), \quad u_3 := \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, 0\right).$$

Note that $\{u_1, u_2, u_3\}$ are the vertices of a regular triangle inscribed in $\boldsymbol{E}(\mathbb{S}^2)$. We divide the open upper hemisphere $\boldsymbol{H}_{>0}(\mathbb{S}^2)$ into three regions by using the Voronoi

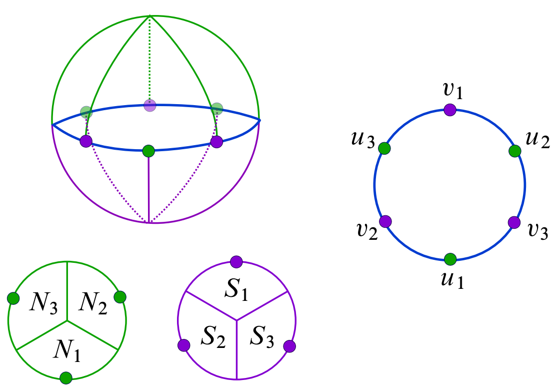


Figure 7: The surjection $\phi_{2,1} : \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$ constructed in Proposition 1.16. In the figure, $S_i := -N_i$ and $v_i := -u_1$ for $i = 1, 2, 3$. The equator of $\mathbb{S}^2$ is mapped to itself under the map (via the identity map).

partitions induced by these three points. Precisely, for each $i=1,2,3$ we define the set

$$N_i := \{x \in \boldsymbol{H}_{>0}(\mathbb{S}^2) \mid d_{\mathbb{S}^2}(x,u_i) \le d_{\mathbb{S}^2}(x,u_j) \text{ if } j \ne i \text{ and } d_{\mathbb{S}^2}(x,u_i) < d_{\mathbb{S}^2}(x,u_j) \text{ if } j < i\}.$$

See Figure 7 for an illustration of the construction.

Observe that $\overline{N}_i = \mathcal{C}\big(\mathrm{Conv}_{\mathbb{S}^1}(\{\iota_1^{-1}(-u_j) \in \mathbb{S}^1 \mid j \ne i\})\big)$ for each $i=1,2,3$. Since $\mathrm{Conv}_{\mathbb{S}^1}(\{\iota_1^{-1}(-u_j) \in \mathbb{S}^1 \mid j \ne i\})$ is just the shortest geodesic between the two points $\{\iota_1(-u_j) \in \mathbb{S}^1 \mid j \ne i\}$ with length $\zeta_1 = \frac{2\pi}{3}$, $\mathrm{diam}(\overline{N}_i) \le \zeta_1$ by Lemma 6.2 for any $i=1,2,3$.

We now construct a map $\tilde{\phi}_{2,1} \colon \boldsymbol{H}_{\ge 0}(\mathbb{S}^2) \to \mathbb{S}^1$,

$$\tilde{\phi}_{2,1}(p) := \begin{cases} \iota_1^{-1}(u_i) & \text{if } p \in N_i, \\ \iota_1^{-1}(p) & \text{if } p \in \boldsymbol{E}(\mathbb{S}^2). \end{cases}$$

Let us prove that the distortion of $\tilde{\phi}_{2,1}$ is less than or equal to ζ_1. We break the study of the value of

$$|d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q))|$$

for $p,q \in \boldsymbol{H}_{\ge 0}(\mathbb{S}^2)$ into several cases:

(1) Suppose $p \in N_i$ and $q \in N_j$. If $i=j$, then $0 \le d_{\mathbb{S}^2}(p,q) \le \zeta_1$ and $\tilde{\phi}_{2,1}(p) = \tilde{\phi}_{2,1}(q) = \iota_m^{-1}(u_i)$, so $d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q)) = 0$. Hence,

$$|d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q))| \le \zeta_1.$$

If $i \ne j$, then $0 \le d_{\mathbb{S}^2}(p,q) \le \pi$ and $d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q)) = \zeta_1$, so

$$|d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q))| \le \zeta_1.$$

(2) Suppose $p \in N_i$ and $q \in \boldsymbol{E}(\mathbb{S}^2)$. Then

$$\begin{aligned} |d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q))| &= |d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\iota_1^{-1}(u_i), \iota_1^{-1}(q))| \\ &= |d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^2}(u_i,q)| \\ &\le d_{\mathbb{S}^2}(p,u_i) \le \zeta_1. \end{aligned}$$

(3) Suppose $p,q \in \boldsymbol{E}(\mathbb{S}^2)$. Then $\tilde{\phi}_{2,1}(p) = \iota_1^{-1}(p)$ and $\tilde{\phi}_{2,1}(q) = \iota_1^{-1}(q)$. Hence,

$$|d_{\mathbb{S}^2}(p,q) - d_{\mathbb{S}^1}(\tilde{\phi}_{2,1}(p), \tilde{\phi}_{2,1}(q))| = 0 \le \zeta_1.$$

This implies that $\mathrm{dis}(\tilde{\phi}_{2,1}) \le \zeta_1$. Observe that $\tilde{\phi}_{2,1}$ is the identity on $\boldsymbol{E}(\mathbb{S}^2)$, so $\tilde{\phi}_{2,1}$ is surjective.

For the second claim, by applying Lemma 5.7 to $\tilde{\phi}_{2,1}|_{A(\mathbb{S}^2)}$, we construct a map $\phi_{2,1}\colon \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$ such that $\operatorname{dis}(\phi_{2,1}) = \operatorname{dis}(\tilde{\phi}_{2,1}) \le \zeta_1$. Moreover, by construction, $\phi_{2,1}$ is obviously surjective and antipode-preserving. □

Remark 6.4 The antipode-preserving property of $\phi_{2,1}$ will be useful for the proof of Proposition 1.18.

6.2 The proof of Proposition 1.20

One can prove Proposition 1.20 using a generalization of the approach used in the proof of Proposition 1.16.

Remark 6.5 (diameter of faces of geodesic simplices) Let $\{u_1,\dots,u_{m+2}\}$ be the vertices of a regular $(m+1)$–simplex inscribed in $\mathbb{S}^m$. Let

$$F_m := \operatorname{Conv}_{\mathbb{S}^m}(\{u_1,\dots,u_{m+1}\}).$$

In other words, F_m is just a *face* of the geodesic regular simplex inscribed in $\mathbb{S}^m$, where the length of each edge is $\zeta_m = \arccos(-1/(m+1))$.

The diameter of F_m can be determined by applying a result by Santaló [31, Lemma 1]:

$$\operatorname{diam}(F_m) = \eta_m := \begin{cases} \arccos(-(m+1)/(m+3)) & \text{if } m \text{ is odd,} \\ \arccos(-\sqrt{m/(m+4)}) & \text{if } m \text{ is even.} \end{cases}$$

As proved by Santaló, this diameter is realized either by the distance between the circumcenter of the geodesic convex hull of $A_m^{\text{odd}} := \{u_1,\dots,u_{(m+1)/2}\}$ and the circumcenter of the geodesic convex hull of $B_m^{\text{odd}} := \{u_{(m+3)/2},\dots,u_{m+1}\}$ if m is odd, or by the distance between the circumcenter of the geodesic convex hull of $A_m^{\text{even}} := \{u_1,\dots,u_{m/2}\}$ and the circumcenter of the geodesic convex hull of $B_m^{\text{even}} := \{u_{(m+2)/2},\dots,u_{m+1}\}$ if m is even. See Figure 8.

Observe that, in general,

$$\zeta_m \le \eta_m \le 2(\pi - \zeta_m).$$

Note that as m goes to infinity, ζ_m goes to $\frac{\pi}{2}$, η_m goes to π, and $2(\pi-\zeta_m)$ also goes to π.

Remark 6.6 Let $\{u_1,\dots,u_{m+2}\} \subset \mathbb{S}^m$ be the vertices of a regular $(m+1)$–simplex inscribed in $\mathbb{S}^m$. Let $V_1,\dots,V_{m+2}$ be the Voronoi partition of $\mathbb{S}^m$ induced by these vertices. Then $\overline{V_i} = \operatorname{Conv}_{\mathbb{S}^m}(\{-u_j : j \ne i\})$ (so $\overline{V_i}$ is congruent to F_m in Remark 6.5) for each $i = 1,\dots,m+2$. Here is a proof:

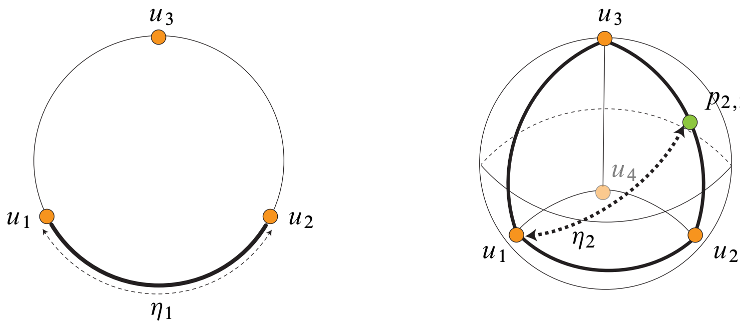


Figure 8: The diameter of a face F_m of a geodesic simplex; the cases $m = 1$ and $m = 2$. When $m = 1$, $A_1^{\text{odd}} = \{u_1\}$ and $B_1^{\text{odd}} = \{u_2\}$. When $m = 2$ (on the right), $A_2^{\text{even}} = \{u_1\}$, $B_2^{\text{even}} = \{u_2, u_3\}$ and the circumcenter of the geodesic convex hull of B_2^{even} is the point $p_{2,3}$, ie $\operatorname{diam}(F_2) = \eta_2 = d_{\mathbb{S}^2}(u_1, p_{2,3})$.

Without loss of generality, one can assume $i = 1$. Observe that

$$\overline{V}_1 = \{x \in \mathbb{S}^m \mid d_{\mathbb{S}^m}(x, u_1) \le d_{\mathbb{S}^m}(x, u_j) \text{ for all } j \neq 1\}.$$

Now fix arbitrary $x \in \operatorname{Conv}_{\mathbb{S}^m}(\{-u_j \mid j \neq 1\})$. Then $x = v/\|v\|$ where

$$v = \sum_{j=2}^{m+2} \lambda_j(-u_j)$$

and the λ_j are nonnegative coefficients such that $\sum_{j=2}^{m+2} \lambda_j = 1$. Then

$$\langle x, u_1\rangle = \frac{1}{\|v\|} \cdot \frac{1}{m+1} \cdot \sum_{j=2}^{m+2} \lambda_j = \frac{1}{\|v\|} \cdot \frac{1}{m+1}$$

and, for any $k \neq 1$,

$$\langle x, u_k\rangle = \frac{1}{\|v\|} \cdot \left(-1 + \frac{1}{m+1} \cdot \sum_{\substack{2 \le j \le m+2 \\ j \neq k}} \lambda_j\right).$$

Hence, this implies $\langle x, u_1\rangle \ge \langle x, u_k\rangle$, so $d_{\mathbb{S}^m}(x, u_1) \le d_{\mathbb{S}^m}(x, u_k)$ for any $k \neq 1$. Therefore, $x \in \overline{V}_1$ and $\operatorname{Conv}_{\mathbb{S}^m}(\{-u_j : j \neq 1\}) \subseteq \overline{V}_1$.

For the other direction, fix an arbitrary $x \in \overline{V}_1$. Since $\{-u_2, \dots, -u_{m+2}\}$ is a basis of $\mathbb{R}^{m+1}$, there is a unique set of coefficients $\{c_i\}_{i=2}^{m+2}$ such that $x = \sum_{i=2}^{m+2} c_i(-u_i)$. Then one can check $c_i = ((m+1)/(m+2))(\langle x, u_1\rangle - \langle x, u_i\rangle)$ for $i = 2, \dots, m+2$ by using the fact $\sum_{i=1}^{m+2} \langle x, u_i\rangle = \left\langle x, \sum_{i=1}^{m+2} u_i\right\rangle = \langle x, 0\rangle = 0$, and [13, Theorem 5.27] (the fact that $\sum_{i=1}^{m+2} u_i = 0$ can be easily checked by the induction on m). Note that $c_i \geq 0$ since $\langle x, u_1\rangle \geq \langle x, u_i\rangle$. Hence, if we define

$$\lambda_i := \frac{c_i}{\sum_{j=2}^{m+2} c_j} = \frac{1}{m+2}\left(1 - \frac{\langle x, u_i\rangle}{\langle x, u_1\rangle}\right)$$

for each $i = 2, \dots, m+2$ and $v := \sum_{i=2}^{m+2} \lambda_i(-u_i)$, then $x = v/\|v\|$. Therefore, $x \in \mathrm{Conv}_{\mathbb{S}^m}(\{-u_j \mid j \neq 1\})$ and $\overline{V}_1 \subseteq \mathrm{Conv}_{\mathbb{S}^m}(\{-u_j \mid j \neq 1\})$. Hence, $\overline{V}_1 = \mathrm{Conv}_{\mathbb{S}^m}(\{-u_j : j \neq 1\})$, as we claimed.

Proof of Proposition 1.20 We construct a surjective and antipode-preserving map

$$\phi_{(m+1),m} \colon \mathbb{S}^{m+1} \twoheadrightarrow \mathbb{S}^m$$

with

$$\mathrm{dis}(\phi_{(m+1),m}) \leq \eta_m.$$

Let $\{u_1, \dots, u_{m+2}\}$ be the vertices of a regular $(m+1)$–simplex inscribed in $\boldsymbol{E}(\mathbb{S}^{m+1})$. We divide open upper hemisphere $\boldsymbol{H}_{>0}(\mathbb{S}^{m+1})$ into $m+2$ regions by using the Voronoi partitions induced by these $m+2$ vertices. Precisely, for each $i = 1, \dots, m+2$ we define the set

$$N_i := \big\{p \in \boldsymbol{H}_{>0}(\mathbb{S}^{m+1}) \mid d_{\mathbb{S}^{m+1}}(p, u_i) \leq d_{\mathbb{S}^{m+1}}(p, u_j) \text{ for all } j \neq i \\ \text{and } d_{\mathbb{S}^{m+1}}(p, u_i) < d_{\mathbb{S}^{m+1}}(p, u_j) \text{ for all } j < i\big\}.$$

Observe that $\overline{N}_i = \mathcal{C}(\overline{V}_i)$, where $\{V_1, \dots, V_{m+2}\}$ is the Voronoi partition of $\mathbb{S}^m$ induced by

$$\{\iota_m^{-1}(u_1), \dots, \iota_m^{-1}(u_{m+2})\}.$$

Hence, by Lemma 6.2 and Remarks 6.5 and 6.6, one concludes that $\mathrm{diam}(\overline{N}_i) \leq \eta_m$ for any $i = 1, \dots, m+2$.

We now construct a map $\tilde{\phi}_{(m+1),m} \colon \boldsymbol{A}(\mathbb{S}^{m+1}) \to \mathbb{S}^m$ by

$$\tilde{\phi}_{(m+1),m}(p) := \begin{cases} \iota_m^{-1}(u_i) & \text{if } p \in N_i, \\ \iota_m^{-1}(p) & \text{if } p \in \iota_m(\boldsymbol{A}(\mathbb{S}^m)). \end{cases}$$

In order to prove that the distortion of $\tilde{\phi}_{(m+1),m}$ is less than or equal to η_m we break the study of the value of

$$|d_{\mathbb{S}^{m+1}}(p, q) - d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q))|$$

for $p, q \in \boldsymbol{A}(\mathbb{S}^{m+1})$ into several cases:

(1) Suppose $p \in N_i$ and $q \in N_j$. If $i = j$, then $d_{\mathbb{S}^{m+1}}(p,q) \le \eta_m$ and $\tilde{\phi}_{(m+1),m}(p) = \tilde{\phi}_{(m+1),m}(q) = \iota_m^{-1}(u_i)$, so $d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q)) = 0$. Hence,

$$|d_{\mathbb{S}^{m+1}}(p,q) - d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q))| \le \eta_m.$$

If $i \neq j$, then $d_{\mathbb{S}^{m+1}}(p,q) \le \pi$ and $d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q)) = \zeta_m$, so that

$$|d_{\mathbb{S}^{m+1}}(p,q) - d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q))| \le \zeta_m \le \eta_m.$$

(2) Suppose $p \in N_i$ and $q \in \iota_m(\boldsymbol{A}(\mathbb{S}^m))$. Then

$$\begin{aligned}
|d_{\mathbb{S}^{m+1}}(p,q) &- d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q))| \\
&= |d_{\mathbb{S}^{m+1}}(p,q) - d_{\mathbb{S}^m}(\iota_m^{-1}(u_i), \iota_m^{-1}(q))| \\
&= |d_{\mathbb{S}^{m+1}}(p,q) - d_{\mathbb{S}^{m+1}}(u_i, q)| \\
&\le d_{\mathbb{S}^{m+1}}(p, u_i) \le \eta_m.
\end{aligned}$$

(3) Suppose $p, q \in \iota_m(\boldsymbol{A}(\mathbb{S}^m))$. Then $\tilde{\phi}_{(m+1),m}(p) = p$ and $\tilde{\phi}_{(m+1),m}(p) = q$. Hence,

$$|d_{\mathbb{S}^{m+1}}(p,q) - d_{\mathbb{S}^m}(\tilde{\phi}_{(m+1),m}(p), \tilde{\phi}_{(m+1),m}(q))| = 0 \le \eta_m.$$

This implies that $\operatorname{dis}(\tilde{\phi}_{(m+1),m}) \le \eta_m$. Finally, by applying Lemma 5.7 to $\tilde{\phi}_{(m+1),m}$, we construct the map $\phi_{(m+1),m} : \mathbb{S}^{m+1} \twoheadrightarrow \mathbb{S}^m$ such that

$$\operatorname{dis}(\phi_{(m+1),m}) = \operatorname{dis}(\tilde{\phi}_{(m+1),m}) \le \eta_m.$$

Moreover, by construction, $\phi_{(m+1),m}$ is obviously surjective and antipode-preserving. Therefore,

$$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+1}) \le \tfrac{1}{2}\eta_m. \qquad \square$$

Remark 6.7 Even though during the proof of Proposition 1.20 we only established the fact that $\operatorname{dis}(\phi_{(m+1),m}) \le \eta_m$, one can check that $\operatorname{dis}(\phi_{(m+1),m})$ is *exactly equal* to η_m, since one can choose two points $p, q \in N_i$ such that $d_{\mathbb{S}^{m+1}}(p,q)$ is arbitrarily close to η_m.

7 The proof of Proposition 1.18

In this section, we will prove Proposition 1.18 by constructing a specific correspondence between $\mathbb{S}^1$ and $\mathbb{S}^3$ with distortion less than or equal to $\zeta_1 = \frac{2\pi}{3}$. The construction of this correspondence is based on the optimal correspondence $R_{2,1} = \operatorname{graph}(\phi_{2,1})$ between $\mathbb{S}^1$ and $\mathbb{S}^2$ identified in the proof of Proposition 1.16 given in Section 6.1 and some ideas reminiscent of the Hopf fibration. We will define a surjective map $\phi_{3,1} : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^1$ by suitably "rotating" the (optimal) surjection $\phi_{2,1} : \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$; see Figure 9.

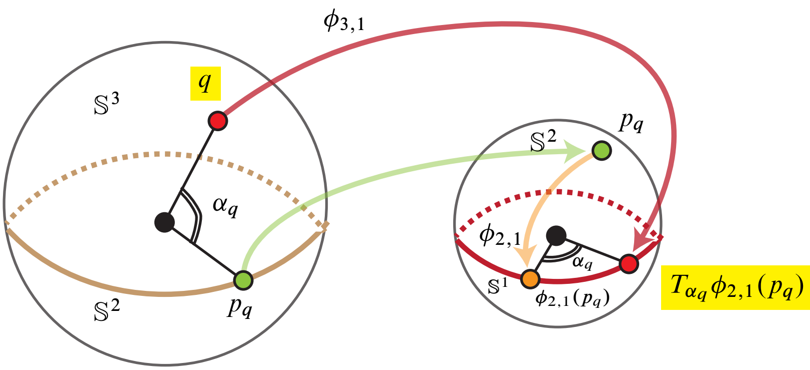


Figure 9: The definition of $\phi_{3,1}$: given $q \in \mathbb{S}^3 \setminus \mathbb{S}^2$ there exists a unique angle $\alpha_q \in (0,\pi)$ and unique point $p_q \in \mathbb{S}^2 \setminus \mathbb{S}^1$ such that $q = T_{\alpha_q} p_q$; then we consider the point $\phi_{2,1}(p_q) \in \mathbb{S}^1$ and define $\phi_{3,1}(q) := T_{\alpha_q}\phi_{2,1}(p_q)$. That $\phi_{3,1}(q) \in \mathbb{S}^1$ follows from Lemma 7.2(2).

Overview of the construction of $\boldsymbol{\phi_{3,1}}$ The diagram below describes the construction of the map $\phi_{3,1}$ at a high level:

$$\begin{array}{ccc} \mathbb{S}^3 & \overset{\phi_{3,1}}{\dashrightarrow\!\!\!\!\rightarrow} & \mathbb{S}^1 \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle T_\bullet} \\ \mathbb{S}^2 \times [0,\pi) & \xrightarrow{\phi_{2,1}\times \mathrm{id}} & \mathbb{S}^1 \times [0,\pi) \end{array}$$

To an arbitrary $q \in \mathbb{S}^3$, we will be able to assign both a corresponding point $p_q \in \mathbb{S}^2$ and an angle $\alpha_q \in [0,\pi)$ giving rise to a map $h\colon \mathbb{S}^3 \to \mathbb{S}^2 \times [0,\pi)$ such that $h(q) := (p_q, \alpha_q)$. Also, $T_\bullet\colon \mathbb{S}^1 \times [0,\pi) \twoheadrightarrow \mathbb{S}^1$ will be a map such that for each $\alpha \in [0,\pi)$, T_α is a rotation of $\mathbb{S}^1$ by an angle α. Then, as described in the diagram, for $q \in \mathbb{S}^3$, $\phi_{3,1}(q)$ will be defined as $T_{\alpha_q}(\phi_{2,1}(p_q))$. Figures 9 and 10 illustrate the construction.

Note that there is a certain degree of similarity between the map $\pi_1 \circ h\colon \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$ (where π_1 is the canonical projection from $\mathbb{S}^2 \times [0,\pi)$ to $\mathbb{S}^2$) and the "Hopf fibration", in the sense that the set $(\pi_1 \circ h)^{-1}(\{p,-p\})$ is isometric to $\mathbb{S}^1$ for $p \in \mathbb{S}^2 \setminus \mathbb{S}^1$ (whereas $(\pi_1 \circ h)^{-1}(\{p\}) = \{p\}$ for $p \in \mathbb{S}^1$).

Details The following coordinate representations will be used throughout this section:[4]

- $\mathbb{S}^1 := \{(x,y,0,0) \in \mathbb{R}^4 : x^2 + y^2 = 1\}$,

[4]In comparison to the coordinate representation specified in Section 2, here we are embedding $\mathbb{S}^1$, $\mathbb{S}^2$, and $\mathbb{S}^3$ into $\mathbb{R}^4$ in such a way that the embeddings $\mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \hookrightarrow \mathbb{S}^3$ are also specific.

- $\mathbb{S}^2 := \{(x, y, z, 0) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = 1\}$,
- $\mathbb{S}^3 := \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$.

Also, we will use the map $\phi_{2,1} : \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$ and the regions $N_1, N_2, N_3 \subset \mathbb{S}^2$ constructed in the proof of Proposition 1.16; see Section 6.1.

Remark 7.1 The following simple observations will be useful later. See Figure 7.

(1) $\operatorname{diam}(\overline{N}_i) \le \zeta_1 = \frac{2\pi}{3}$ for any $i = 1, 2, 3$. (This fact has been already mentioned during the proof of Proposition 1.20.)

(2) If $p = (x, y, z, 0) \in N_i$ and $q = (a, b, c, 0) \in N_j$ for $(i, j) = (1, 2)$, $(2, 3)$ or $(3, 1)$ (resp. $(i, j) = (2, 1)$, $(3, 2)$ or $(1, 3)$), then $bx - ay \ge 0$ (resp. ≤ 0) and $\phi_{2,1}(p)$ and $\phi_{2,1}(q)$ are in clockwise (resp. counterclockwise) order.

Now, for any $\alpha \in \mathbb{R}$, consider the rotation matrix

$$T_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

For any $p \in \mathbb{S}^3$, $T_\alpha p$ denotes the result of matrix multiplication by viewing p as a 4 by 1 column vector according to the coordinate system described at the beginning of this section.

The following basic properties of these rotation matrices will be useful soon.

Lemma 7.2 *Let $\alpha, \beta \in \mathbb{R}$. Then*:

(1) *For any $q \in \mathbb{S}^3 \setminus \mathbb{S}^1$, there is a unique $p_q \in \mathbb{S}^2 \setminus \mathbb{S}^1$ and a unique $\alpha_q \in [0, \pi)$ such that $q = T_{\alpha_q} p_q$. In particular, $\alpha_q = 0$ if and only if $q \in \mathbb{S}^2 \setminus \mathbb{S}^1$.*

(2) *$\mathbb{S}^1$ and $\mathbb{S}^3$ are invariant with respect to the action of the rotation matrices T_α.*

(3) *$T_\alpha T_\beta = T_{\alpha+\beta}$.*

(4) *$d_{\mathbb{S}^3}(T_\alpha\, p, T_\alpha\, q) = d_{\mathbb{S}^3}(p, q)$ for any $p, q \in \mathbb{S}^3$.*

(5) *$d_{\mathbb{S}^3}(T_\alpha\, p, p) = \alpha$ for any $p \in \mathbb{S}^3$ and $\alpha \in [0, \pi]$.*

(6) *$d_{\mathbb{S}^3}(T_\alpha(-p), p) = \pi - \alpha$ for any $p \in \mathbb{S}^3$ and $\alpha \in [0, \pi]$.*

Proof (1) Let $q=(x',y',z',w')\in\mathbb{S}^3\setminus\mathbb{S}^1$. Since q is not in $\mathbb{S}^1$, we know that $(z')^2+(w')^2>0$. Then there exists a unique $\alpha_q\in[0,\pi)$ and $z\in\mathbb{R}\setminus\{0\}$ such that

$$\begin{pmatrix} z' \\ w' \end{pmatrix}=\begin{pmatrix} \cos\alpha_q & -\sin\alpha_q \\ \sin\alpha_q & \cos\alpha_q \end{pmatrix}\begin{pmatrix} z \\ 0 \end{pmatrix};$$

ie $z^2=(z')^2+(w')^2$. Then this α_q is the required angle and we choose the unique point $p_q=(x,y,z,0)\in\mathbb{S}^2\setminus\mathbb{S}^1$ such that

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}=\begin{pmatrix} \cos\alpha_q & -\sin\alpha_q & 0 & 0 \\ \sin\alpha_q & \cos\alpha_q & 0 & 0 \\ 0 & 0 & \cos\alpha_q & -\sin\alpha_q \\ 0 & 0 & \sin\alpha_q & \cos\alpha_q \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}.$$

Since T_{α_q} is the identity matrix when $\alpha_q=0$, it is clear that $\alpha_q=0$ if and only if $q\in\mathbb{S}^2\setminus\mathbb{S}^1$.

(2) Obvious.

(3) Obvious.

(4) This item is equivalent to the condition $\langle T_\alpha p, T_\alpha q\rangle=\langle p,q\rangle$, and it can be easily checked by direct computation.

(5) This item is equivalent to the condition $\langle T_\alpha p, p\rangle=\cos\alpha$, and it can be easily checked by direct computation.

(6) This item is equivalent to the condition $\langle T_\alpha(-p), p\rangle=-\cos\alpha$, and it can be easily checked by direct computation. □

Additional details and the proof of Proposition 1.18 We need a few more definitions and technical lemmas for the proof of Proposition 1.18. We in particular make the following definitions for notational convenience:

- For any $p,q\in\mathbb{S}^2$,

$$E_{p,q}\colon[0,\pi]\to[-1,1],\quad \alpha\mapsto\langle T_\alpha p,q\rangle.$$

- For any $p,q\in\mathbb{S}^2$,

$$F_{p,q}\colon[0,\pi]\to\mathbb{R},\quad \alpha\mapsto d_{\mathbb{S}^3}(T_\alpha p,q)-\alpha.$$

- For any $p,q\in\mathbb{S}^2$,

$$G_{p,q}\colon[0,\pi]\to\mathbb{R},\quad \alpha\mapsto d_{\mathbb{S}^3}(T_\alpha\, p,q)+\alpha.$$

Lemma 7.3 *For any* $p = (x, y, z, 0) \in \mathbb{S}^2 \setminus \mathbb{S}^1$ *and* $q = (a, b, c, 0) \in \mathbb{S}^2$:

(1) $E_{p,q}(\alpha) \in (-1, 1)$ *for any* $\alpha \in (0, \pi)$.

(2) $(E'_{p,q}(\alpha))^2 + (E_{p,q}(\alpha))^2 \le 1$ *for any* $\alpha \in [0, \pi]$.[5]

(3) $F_{p,q}$ *is a nonincreasing function. Thus,* $-d_{\mathbb{S}^2}(p, q) \le F_{p,q}(\alpha) \le d_{\mathbb{S}^2}(p, q)$ *for any* $\alpha \in [0, \pi]$.

(4) $G_{p,q}$ *is a nondecreasing function. Thus,* $d_{\mathbb{S}^2}(p, q) \le G_{p,q}(\alpha) \le 2\pi - d_{\mathbb{S}^2}(p, q)$ *for any* $\alpha \in [0, \pi]$.

Proof (1) Suppose not, so that $E_{p,q}(\alpha) = \pm 1$. This implies that $T_\alpha p = q$ or $-q \in \mathbb{S}^2$. This cannot be true because $\alpha \in (0, \pi)$ and, by Lemma 7.2(1), $T_\alpha p \in \mathbb{S}^3 \setminus \mathbb{S}^2$. So this is a contradiction; hence, $E_{p,q}(\alpha) \in (-1, 1)$.

(2) As a result of direct computation, we know that

$$E_{p,q}(\alpha) = \langle p, q\rangle \cos\alpha + (bx - ay)\sin\alpha.$$

Here, observe that $bx - ay$ is the 3rd coordinate of the cross product $(x, y, z) \times (a, b, c)$. In particular, this implies $|bx - ay| \le \|(x, y, z) \times (a, b, c)\| = \sin\beta$ where $\langle p, q\rangle = \cos\beta$. Therefore,

$$(E'_{p,q}(\alpha))^2 + (E_{p,q}(\alpha))^2 = \langle p, q\rangle^2 + (bx - ay)^2 \le \cos^2\beta + \sin^2\beta = 1.$$

(3) Note that $F_{p,q}(\alpha) = \arccos(E_{p,q}(\alpha)) - \alpha$. Hence, for any $\alpha \in (0, \pi)$,

$$F'_{p,q}(\alpha) = -\frac{E'_{p,q}(\alpha)}{\sqrt{1 - (E_{p,q}(\alpha))^2}} - 1.$$

Observe that this expression is well defined by (1). Also, by (2),

$$\begin{aligned}(E'_{p,q}(\alpha))^2 + (E_{p,q}(\alpha))^2 \le 1 &\iff -E'_{p,q}(\alpha) \le \sqrt{1 - (E_{p,q}(\alpha))^2}\\ &\iff F'_{p,q}(\alpha) = -\frac{E'_{p,q}(\alpha)}{\sqrt{1 - (E_{p,q}(\alpha))^2}} - 1 \le 0.\end{aligned}$$

Hence, $F_{p,q}$ is a nonincreasing function. Also, since $F_{p,q}(0) = d_{\mathbb{S}^2}(p, q)$ and

$$F_{p,q}(\pi) = d_{\mathbb{S}^3}(T_\pi p, q) - \pi = d_{\mathbb{S}^2}(-p, q) - \pi = (\pi - d_{\mathbb{S}^2}(p, q)) - \pi = -d_{\mathbb{S}^2}(p, q),$$

we have that

$$-d_{\mathbb{S}^2}(p, q) \le F_{p,q}(\alpha) \le d_{\mathbb{S}^2}(p, q).$$

[5] Here E'_{pq} denotes the derivative of $E_{p,q}$.

(4) Note that $G_{p,q}(\alpha) = \arccos(E_{p,q}(\alpha)) + \alpha$. Hence, for any $\alpha \in (0,\pi)$,

$$G'_{p,q}(\alpha) = -\frac{E'_{p,q}(\alpha)}{\sqrt{1-(E_{p,q}(\alpha))^2}} + 1.$$

Observe that this expression is well defined by (1). Also, by (2),

$$\begin{aligned}(E'_{p,q}(\alpha))^2 + (E_{p,q}(\alpha))^2 \le 1 &\iff E'_{p,q}(\alpha) \le \sqrt{1-(E_{p,q}(\alpha))^2} \\ &\iff G'_{p,q}(\alpha) = -\frac{E'_{p,q}(\alpha)}{\sqrt{1-(E_{p,q}(\alpha))^2}} + 1 \ge 0.\end{aligned}$$

Hence, $G_{p,q}$ is nondecreasing function. Also, since $G_{p,q}(0) = d_{\mathbb{S}^2}(p,q)$ and

$$G_{p,q}(\pi) = d_{\mathbb{S}^3}(T_\pi p, q) + \pi = d_{\mathbb{S}^2}(-p,q) + \pi = (\pi - d_{\mathbb{S}^2}(p,q)) + \pi = 2\pi - d_{\mathbb{S}^2}(p,q),$$

we have that

$$d_{\mathbb{S}^2}(p,q) \le G_{p,q}(\alpha) \le 2\pi - d_{\mathbb{S}^2}(p,q).$$

□

Lemma 7.4 *For any $p = (x,y,z,0), q = (a,b,c,0) \in \mathbb{S}^2 \setminus \mathbb{S}^1$:*

(1) *If $p \in N_i$ and $q \in N_j$ for $(i,j) = (1,2)$, $(2,3)$ or $(3,1)$, then*

$$d_{\mathbb{S}^3}(T_{2\pi/3} p, q) \le \tfrac{2\pi}{3}.$$

(2) *If $p \in N_i$ and $q \in N_j$ for $(i,j) = (2,1)$, $(3,2)$ or $(1,3)$, then*

$$d_{\mathbb{S}^3}(T_{\pi/3} p, q) \ge \tfrac{\pi}{3}.$$

Proof (1) First, observe that $bx - ay \ge 0$ by Remark 7.1(2). Hence,

$$E_{p,q}\big(\tfrac{2\pi}{3}\big) = \langle T_{2\pi/3} p, q\rangle = -\tfrac{1}{2}\langle p,q\rangle + \tfrac{\sqrt{3}}{2}(bx - ay) \ge -\tfrac{1}{2}\langle p,q\rangle \ge -\tfrac{1}{2}.$$

Therefore,

$$d_{\mathbb{S}^3}(T_{2\pi/3} p, q) = \arccos\big(E_{p,q}\big(\tfrac{2\pi}{3}\big)\big) \le \arccos\big(-\tfrac{1}{2}\big) = \tfrac{2\pi}{3}.$$

(2) The proof of this case is similar to the proof of (1), so we omit it. □

Proof of Proposition 1.18 It is enough to find a surjective map $\phi_{3,1} : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^1$ such that $\mathrm{dis}(\phi_{3,1}) \le \zeta_1 = \frac{2\pi}{3}$, since this map gives rise to a correspondence $R_{3,1} := \mathrm{graph}(\phi_{3,1})$ with $\mathrm{dis}(R_{3,1}) = \mathrm{dis}(\phi_{3,1}) \le \zeta_1$.

We construct the required surjective map $\phi_{3,1} : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^1$ as

$$q \mapsto \begin{cases} \phi_{2,1}(q) & \text{if } q \in \mathbb{S}^2, \\ T_{\alpha_q}\phi_{2,1}(p_q) & \text{if } q \in \mathbb{S}^3 \setminus \mathbb{S}^2 \text{ and } q = T_{\alpha_q} p_q \text{ for the \textit{unique} such} \\ & \qquad \alpha_q \in (0,\pi) \text{ and } p_q \in \mathbb{S}^2 \setminus \mathbb{S}^1. \end{cases}$$

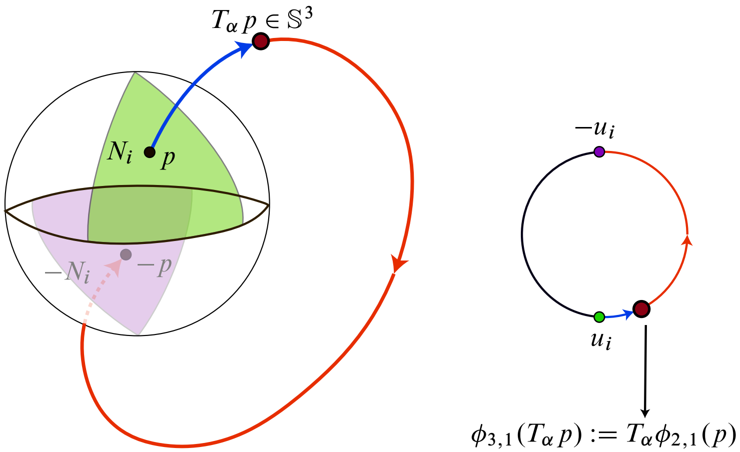


Figure 10: The definition of the map $\phi_{3,1}$ via the map $\phi_{2,1}$. The point $T_\alpha p$ on $\mathbb{S}^3$ is mapped to the point $T_\alpha \phi_{2,1}(p)$ on $\mathbb{S}^1$. The antipode-preserving map $\phi_{2,1}$ maps the whole region N_i to the point u_i.

Note that $\phi_{3,1}$ is surjective, since $\phi_{3,1}|_{\mathbb{S}^2} = \phi_{2,1}$ and $\phi_{2,1}$ is surjective.

See Figures 9 and 10 for an explanation of the construction of the map $\phi_{3,1}$.

Let us now verify that

$$|d_{\mathbb{S}^3}(q,q') - d_{\mathbb{S}^1}(\phi_{3,1}(q), \phi_{3,1}(q'))| \le \zeta_1$$

for every $q, q' \in \mathbb{S}^3$. Without loss of generality, we can assume that $q = T_\alpha p$ and $q' = T_\beta p'$ for some $p, p' \in \mathbb{S}^2$ and $0 \le \beta \le \alpha < \pi$. Then

$$\begin{aligned} |d_{\mathbb{S}^3}(q,q') - d_{\mathbb{S}^1}(\phi_{3,1}(q), \phi_{3,1}(q'))| \\ &= |d_{\mathbb{S}^3}(T_\alpha p, T_\beta p') - d_{\mathbb{S}^1}(T_\alpha \phi_{2,1}(p), T_\beta \phi_{2,1}(p'))| \\ &= |d_{\mathbb{S}^3}(T_{(\alpha-\beta)} p, p') - d_{\mathbb{S}^1}(T_{(\alpha-\beta)} \phi_{2,1}(p), \phi_{2,1}(p'))|. \end{aligned}$$

Hence, it is enough to prove

$$\left| d_{\mathbb{S}^3}(T_\alpha p, q) - d_{\mathbb{S}^1}(T_\alpha \phi_{2,1}(p), \phi_{2,1}(q)) \right| \le \zeta_1 \tag{9}$$

for any $p, q \in \mathbb{S}^2$ and $\alpha \in [0, \pi)$.

If $p \in \mathbb{S}^1$, then $\phi_{2,1}(p) = p$. Hence,

$$\begin{aligned} |d_{\mathbb{S}^3}(T_\alpha p, q) - d_{\mathbb{S}^1}(T_\alpha \phi_{2,1}(p), \phi_{2,1}(q))| &= |d_{\mathbb{S}^3}(T_\alpha p, q) - d_{\mathbb{S}^1}(T_\alpha p, \phi_{2,1}(q))| \\ &\le d_{\mathbb{S}^3}(q, \phi_{2,1}(q)) \le \zeta_1, \end{aligned}$$

where the last inequality holds by Remark 7.1(1). One can carry out a similar computation if $q \in \mathbb{S}^1$. So let's assume $p = (x, y, z, 0), q = (a, b, c, 0) \in \mathbb{S}^2 \setminus \mathbb{S}^1$. Since $\phi_{2,1}$ is antipode-preserving, it is enough to check inequality (9) only for $p, q \in \boldsymbol{H}_{>0}(\mathbb{S}^2)$. We do this by following the same idea as in the proof of Lemma 5.7.

We do a case-by-case analysis:

(1) Suppose $p \in N_i$ and $q \in N_j$ for $(i, j) = (1, 2)$, $(2, 3)$ or $(3, 1)$. By Remark 7.1(2), the two points $\phi_{2,1}(p)$ and $\phi_{2,1}(q)$ are in clockwise order. Hence,

$$d_{\mathbb{S}^1}(T_\alpha \phi_{2,1}(p), \phi_{2,1}(q)) = \begin{cases} \frac{2\pi}{3} - \alpha & \text{if } \alpha \in \left[0, \frac{2\pi}{3}\right], \\ \alpha - \frac{2\pi}{3} & \text{if } \alpha \in \left[\frac{2\pi}{3}, \pi\right). \end{cases}$$

Consider first the case when $\alpha \in \left[0, \frac{2\pi}{3}\right]$. We have to prove that

$$-\tfrac{2\pi}{3} \leq d_{\mathbb{S}^3}(T_\alpha p, q) - \left(\tfrac{2\pi}{3} - \alpha\right) \leq \tfrac{2\pi}{3}.$$

Equivalently, we have to prove

$$0 \leq G_{p,q}(\alpha) \leq \tfrac{4\pi}{3}.$$

The left-hand side inequality is obvious since $G_{p,q}(\alpha) \geq d_{\mathbb{S}^2}(p, q) \geq 0$ by Lemma 7.3(4). The right-hand side inequality is true by Lemmas 7.3(4) and 7.4(1).

Next, consider the case when $\alpha \in \left[\frac{2\pi}{3}, \pi\right)$. We have to prove

$$-\tfrac{2\pi}{3} \leq d_{\mathbb{S}^3}(T_\alpha p, q) - \left(\alpha - \tfrac{2\pi}{3}\right) \leq \tfrac{2\pi}{3}.$$

Equivalently, we have to prove

$$-\tfrac{4\pi}{3} \leq F_{p,q}(\alpha) \leq 0.$$

The left inequality is obvious since $F_{p,q}(\alpha) \geq -d_{\mathbb{S}^2}(p, q) \geq -\frac{4\pi}{3}$ by Lemma 7.3(3). The right-hand side inequality is true by Lemmas 7.3(3) and 7.4(1).

(2) Suppose $p \in N_i$ and $q \in N_j$ for $(i, j) = (2, 1)$, $(3, 2)$ or $(1, 3)$. This is almost the same as case (1) except we use Lemma 7.4(2).

(3) Suppose $p, q \in N_i$ for $i = 1, 2, 3$. In this case, $d_{\mathbb{S}^1}(T_\alpha \phi_{2,1}(p), \phi_{2,1}(q)) = \alpha$, which follows from $\phi_{2,1}(p) = \phi_{2,1}(q)$ and Lemma 7.2(5). Hence, we have to show

$$-\tfrac{2\pi}{3} \leq d_{\mathbb{S}^3}(T_\alpha p, q) - \alpha = F_{p,q}(\alpha) \leq \tfrac{2\pi}{3}.$$

But this is obvious by Remark 7.1(1) and Lemma 7.3(3).

Thus, indeed $\operatorname{dis}(\phi_{3,1}) \leq \zeta_1$. □

8 The proof of Proposition 1.19

In this section we provide a construction of an optimal correspondence, $R_{3,2}$, between $\mathbb{S}^3$ and $\mathbb{S}^2$. The structure of this correspondence is different from those described in the proofs of Propositions 1.16 and 1.20. As a matter of fact, as Remark 6.7 mentions, the distortion of the surjection $\phi_{(m+1),m}\colon \mathbb{S}^{m+1} \twoheadrightarrow \mathbb{S}^m$ constructed in Proposition 1.20 is *exactly equal* to η_m. Since $\zeta_2 < \eta_2$, this means that a different construction is required for the case $m = 2$.

Let u_1, u_2, u_3 and u_4 be the vertices of a regular tetrahedron inscribed in $\mathbb{S}^2$ (ie $\langle u_i, u_j\rangle = -\frac{1}{3} = \cos\zeta_2$ for any $i \neq j$). We consider

$$u_1 = (1,0,0), \qquad u_2 = \left(-\tfrac{1}{3}, \tfrac{2\sqrt{2}}{3}, 0\right),$$
$$u_3 = \left(-\tfrac{1}{3}, -\tfrac{\sqrt{2}}{3}, \tfrac{\sqrt{2}}{\sqrt{3}}\right), \quad u_4 = \left(-\tfrac{1}{3}, -\tfrac{\sqrt{2}}{3}, -\tfrac{\sqrt{2}}{\sqrt{3}}\right).$$

Now, let $V_1, V_2, V_3, V_4 \subset \mathbb{S}^2$ be the Voronoi partition of $\mathbb{S}^2$ induced by u_1, u_2, u_3, and u_4. Then, for each i, $\overline{V_i}$ is the spherical convex hull of the set

$$\{-u_j \in \mathbb{S}^2 \mid j \in \{1,2,3,4\} \setminus \{i\}\}.$$

Let

$$r := \arccos\left(\tfrac{2\sqrt{2}}{3}\right).$$

For $i \neq j \in \{1,2,3,4\}$, let $u_{i,j}$ be the point on the shortest geodesic between u_i and $-u_j$ such that $d_{\mathbb{S}^2}(u_i, u_{i,j}) = r$. See Figure 11 for an illustration of V_1.

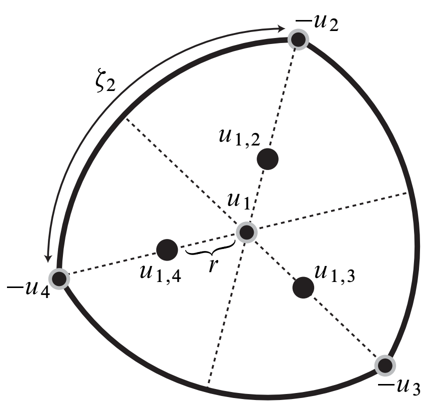


Figure 11: V_1. All the sides of the spherical triangle V_1 (determined by the three points $-u_2$, $-u_3$, and $-u_4$) have the same length ζ_2.

Remark 8.1 One can directly compute the coordinates

$$u_{1,2}=\left(\tfrac{2\sqrt{2}}{3},-\tfrac{1}{3},0\right),\quad u_{1,3}=\left(\tfrac{2\sqrt{2}}{3},\tfrac{1}{6},-\tfrac{1}{2\sqrt{3}}\right),\qquad u_{1,4}=\left(\tfrac{2\sqrt{2}}{3},\tfrac{1}{6},\tfrac{1}{2\sqrt{3}}\right),$$
$$u_{2,1}=\left(-\tfrac{4\sqrt{2}}{9},\tfrac{7}{9},0\right),\quad u_{2,3}=\left(-\tfrac{\sqrt{2}}{9},\tfrac{17}{18},-\tfrac{1}{2\sqrt{3}}\right),\quad u_{2,4}=\left(-\tfrac{\sqrt{2}}{9},\tfrac{17}{18},\tfrac{1}{2\sqrt{3}}\right).$$

Lemma 8.2 *For any $i\neq j\in\{1,2,3,4\}$, the following hold*:

(1) $\langle u_{i,k},u_{i,l}\rangle=\frac{5}{6}$ *for any* $k\neq l\in 1,2,3,4\setminus\{i\}$.

(2) $\langle u_{i,k},u_{j,k}\rangle=\frac{5}{54}$ *for any* $k\in 1,2,3,4\setminus\{i,j\}$.

(3) $\langle u_{i,k},u_{j,l}\rangle=-\frac{2}{27}$ *for any* $k\neq l\in 1,2,3,4\setminus\{i,j\}$.

(4) $\langle u_{i,k},u_{j,i}\rangle=-\frac{25}{54}$ *for any* $k\in 1,2,3,4\setminus\{i,j\}$.

(5) $\langle u_{i,j},u_{j,i}\rangle=-\frac{23}{27}$.

(6) $\langle u_{i},u_{j,k}\rangle=-\frac{\sqrt{2}}{9}$ *for any* $k\in 1,2,3,4\setminus\{i,j\}$.

(7) $\langle u_{i},u_{j,i}\rangle=-\frac{4\sqrt{2}}{9}$.

Proof By symmetry, without loss of generality one can assume $i=1$ and $j=2$. Then use the coordinate values given in Remark 8.1. □

Next, for each i, let $\{V_{i,j}\subset V_i \mid j\in\{1,2,3,4\}\setminus\{i\}\}$ be the Voronoi partition of V_i induced by $\{u_{i,j}\in V_i\mid j\in\{1,2,3,4\}\setminus\{i\}\}$.

From now on, in this section, we will identify $\mathbb{S}^2$ with $\boldsymbol{E}(\mathbb{S}^3)\subset\mathbb{S}^3$. Then obviously

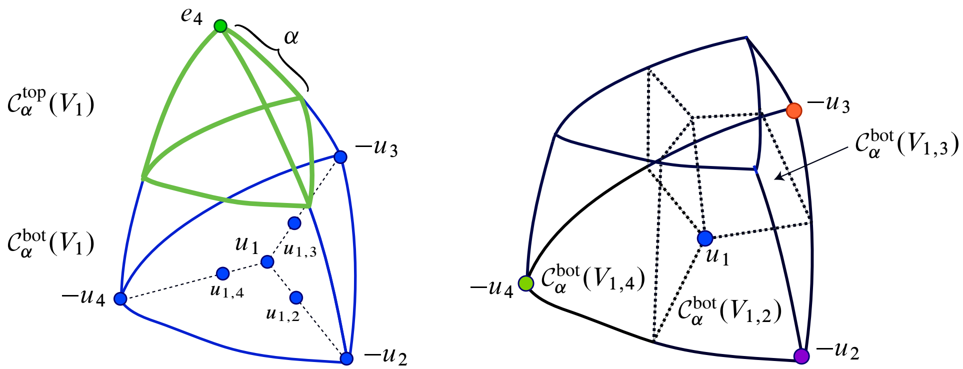
$$\boldsymbol{H}_{\geq 0}(\mathbb{S}^3)=\mathcal{C}(V_1)\cup\mathcal{C}(V_2)\cup\mathcal{C}(V_3)\cup\mathcal{C}(V_4).$$

Moreover, for any $i\in\{1,2,3,4\}$ and $\alpha\in\left[0,\frac{\pi}{2}\right]$, we divide $\mathcal{C}(V_i)$ into

- $\mathcal{C}^{\mathrm{top}}_\alpha(V_i):=\{p\in\mathcal{C}(V_i)\mid d_{\mathbb{S}^{n+1}}(e_4,p)\leq\alpha\}$,
- $\mathcal{C}^{\mathrm{bot}}_\alpha(V_i):=\{p\in\mathcal{C}(V_i)\mid d_{\mathbb{S}^{n+1}}(e_4,p)>\alpha\}$,
- $\mathcal{C}^{\mathrm{bot}}_\alpha(V_{i,j}):=\{p\in\mathcal{C}(V_i)\mid d_{\mathbb{S}^{n+1}}(e_4,p)>\alpha \text{ and } \Omega(p)\in V_{i,j}\}$ for any j in $\{1,2,3,4\}\setminus\{i\}$,

where

$$\Omega\colon \boldsymbol{H}_{\geq 0}(\mathbb{S}^3)\setminus\{e_4\}\to\boldsymbol{E}(\mathbb{S}^3)=\mathbb{S}^2,\quad (x,y,z,w)\mapsto\frac{1}{\sqrt{1-w^2}}(x,y,z,0),$$



$\mathcal{C}_\alpha^{\mathrm{top}}(V_1)$ and $\mathcal{C}_\alpha^{\mathrm{bot}}(V_1)$. $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,2})$, $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,3})$, and $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,4})$.

Figure 12: The regions into which $\mathcal{C}(V_1)$ is split.

is the orthogonal projection onto the equator. Then obviously

$$\mathcal{C}(V_i) = \mathcal{C}_\alpha^{\mathrm{top}}(V_i) \cup \bigcup_{j \in \{1,2,3,4\} \setminus \{i\}} \mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,j})$$

for each $i \in \{1,2,3,4\}$. See Figure 12 for illustrations of $\mathcal{C}_\alpha^{\mathrm{top}}(V_1)$, $\mathcal{C}_\alpha^{\mathrm{bot}}(V_1)$, $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,2})$, $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,3})$, and $\mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,4})$.

Lemma 8.3 *For $p, q \in \boldsymbol{H}_{\geq 0}(\mathbb{S}^3)$, the following inequalities hold*:

(1) *If $p, q \in \mathcal{C}_\alpha^{\mathrm{top}}(V_i)$ for some $i \in \{1,2,3,4\}$, then*

$$\langle p, q \rangle \geq \cos^2\alpha - \tfrac{1}{\sqrt{3}} \sin^2\alpha = \left(1 + \tfrac{1}{\sqrt{3}}\right)\cos^2\alpha - \tfrac{1}{\sqrt{3}}.$$

In particular, this is equivalent to

$$d_{\mathbb{S}^3}(p,q) \leq \arccos\left(\left(1 + \tfrac{1}{\sqrt{3}}\right)\cos^2\alpha - \tfrac{1}{\sqrt{3}}\right).$$

(2) *If $p \in \mathcal{C}_\alpha^{\mathrm{top}}(V_i)$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ for some $i \neq j \in \{1,2,3,4\}$, then*

$$\langle p, q \rangle \leq \sqrt{\tfrac{2}{3}\cos^2\alpha + \tfrac{1}{3}}.$$

In particular, this is equivalent to

$$d_{\mathbb{S}^3}(p,q) \geq \arccos\left(\sqrt{\tfrac{2}{3}\cos^2\alpha + \tfrac{1}{3}}\right).$$

(3) *If $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,k})$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ for distinct $i, j, k \in \{1,2,3,4\}$, then*

$$\langle p, q \rangle \leq \left(1 - \tfrac{1}{\sqrt{3}}\right)\cos^2\alpha + \tfrac{1}{\sqrt{3}}.$$

In particular, this is equivalent to the condition

$$d_{\mathbb{S}^3}(p,q) \geq \arccos\bigl(\bigl(1-\tfrac{1}{\sqrt{3}}\bigr)\cos^2\alpha + \tfrac{1}{\sqrt{3}}\bigr).$$

(4) *If* $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,j})$ *and* $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ *for some* $i \neq j \in \{1,2,3,4\}$, *then*

$$\langle p,q\rangle \leq \cos^2\alpha.$$

In particular, this is equivalent to

$$d_{\mathbb{S}^3}(p,q) \geq \arccos(\cos^2\alpha).$$

Proof We express p and q as

$$p = \cos\theta\cdot e_4 + \sin\theta\cdot\iota_2(x), \quad q = \cos\theta'\cdot e_4 + \sin\theta'\cdot\iota_2(y),$$

where $e_4 = (0,0,0,1)$ for some $\theta,\theta' \in \bigl[0,\frac{\pi}{2}\bigr]$ and $x,y \in \mathbb{S}^2$. Then

$$\langle p,q\rangle = \cos\theta\cos\theta' + \langle x,y\rangle \sin\theta\sin\theta'.$$

(1) If $p,q \in \mathcal{C}_\alpha^{\mathrm{top}}(V_i)$ for some $i \in \{1,2,3,4\}$, then we can assume $x,y \in V_i$ and $\theta,\theta' \in [0,\alpha]$. Hence,

$$\langle p,q\rangle \geq \cos\theta\cos\theta' - \tfrac{1}{\sqrt{3}}\sin\theta\sin\theta' \geq \cos^2\alpha - \tfrac{1}{\sqrt{3}}\sin^2\alpha = \bigl(1+\tfrac{1}{\sqrt{3}}\bigr)\cos^2\alpha - \tfrac{1}{\sqrt{3}},$$

where the first inequality holds because $\langle x,y\rangle \geq -\frac{1}{\sqrt{3}}$, by Remark 6.5, and the second holds since $\cos\theta\cos\theta' + \langle x,y\rangle\sin\theta\sin\theta'$ is decreasing in both θ and θ'.

(2) If $p \in \mathcal{C}_\alpha^{\mathrm{top}}(V_i)$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ for some $i \neq j \in \{1,2,3,4\}$, then we can assume $x \in V_i$, $y \in V_{j,i}$, $\theta \in [0,\alpha]$, and $\theta' \in \bigl[\alpha,\frac{\pi}{2}\bigr]$. Now, consider two cases separately.

If $\langle x,y\rangle \leq 0$, then $\cos\theta\cos\theta' + \langle x,y\rangle\sin\theta\sin\theta'$ is decreasing with respect to both θ and θ'. Hence,

$$\langle p,q\rangle \leq \cos 0\cos\alpha + \langle x,y\rangle\sin 0\sin\alpha = \cos\alpha.$$

If $\langle x,y\rangle \geq 0$, observe that

$$\langle p,q\rangle = (1-\langle x,y\rangle)\cos\theta\cos\theta' + \langle x,y\rangle\cos(\theta'-\theta).$$

If we view θ' as a variable on $\bigl[\alpha,\frac{\pi}{2}\bigr]$,

$$\frac{\partial}{\partial\theta'}\bigl((1-\langle x,y\rangle)\cos\theta\cos\theta' + \langle x,y\rangle\cos(\theta'-\theta)\bigr) = -(1-\langle x,y\rangle)\cos\theta\sin\theta' - \langle x,y\rangle\sin(\theta'-\theta) \leq 0.$$

Hence, $\langle p,q\rangle$ is maximized when $\theta'=\alpha$. So $\langle p,q\rangle \le \cos\theta\cos\alpha + \langle x,y\rangle\sin\theta\sin\alpha$. Now, if we view θ as a variable and take a derivative,

$$\frac{\partial}{\partial\theta}(\cos\theta\cos\alpha+\langle x,y\rangle\sin\theta\sin\alpha) = -\sin\theta\cos\alpha+\langle x,y\rangle\cos\theta\sin\alpha.$$

One can easily check that

$$-\sin\theta\cos\alpha+\langle x,y\rangle\cos\theta\sin\alpha \begin{cases}\ge 0 & \text{if } \theta\in[0,\theta_0],\\ \le 0 & \text{if } \theta\in[\theta_0,\alpha],\end{cases}$$

where θ_0 is the unique critical point satisfying $\tan\theta_0=\langle x,y\rangle\tan\alpha$. Hence,

$$\cos\theta\cos\alpha+\langle x,y\rangle\sin\theta\sin\alpha$$

is maximized when $\theta=\theta_0$. Hence,

$$\langle p,q\rangle \le \cos\theta\cos\alpha+\langle x,y\rangle\sin\theta\sin\alpha \le \sqrt{\cos^2\alpha+\langle x,y\rangle^2\sin^2\alpha}.$$

Note that $\langle x,y\rangle \le \frac{1}{\sqrt{3}}$ since $x\in V_i$ and $y\in V_{ji}$ (this value $\frac{1}{\sqrt{3}}$ can be achieved when x is the midpoint of $-u_k$ and $-u_l$ for $k\ne l\in\{1,2,3,4\}\setminus\{i,j\}$ and $y=u_j$). Hence, one can conclude

$$\langle p,q\rangle \le \sqrt{\cos^2\alpha+\tfrac{1}{3}\sin^2\alpha} = \sqrt{\tfrac{2}{3}\cos^2\alpha+\tfrac{1}{3}}.$$

Since obviously $\cos\alpha \le \sqrt{\cos^2\alpha+\frac{1}{3}\sin^2\alpha} = \sqrt{\frac{2}{3}\cos^2\alpha+\frac{1}{3}}$, this completes the proof of this case.

(3) If $p\in\mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,k})$ and $q\in\mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ for distinct $i,j,k\in\{1,2,3,4\}$, then one can assume $x\in V_{i,k}$, $y\in V_{j,i}$, and $\theta,\theta'\in\left[\alpha,\frac{\pi}{2}\right]$. Now, consider two cases separately.

If $\langle x,y\rangle\le 0$, then $\cos\theta\cos\theta'+\langle x,y\rangle\sin\theta\sin\theta'$ is decreasing with respect to both θ and θ'. Hence,

$$\langle p,q\rangle \le \cos^2\alpha+\langle x,y\rangle\sin^2\alpha\le\cos^2\alpha.$$

If $\langle x,y\rangle\ge 0$, without loss of generality, one can assume $\theta\ge\theta'$. Also, observe that

$$\langle p,q\rangle = (1-\langle x,y\rangle)\cos\theta\cos\theta'+\langle x,y\rangle\cos(\theta-\theta').$$

If we view θ as a variable on $\left[\theta',\frac{\pi}{2}\right]$,

$$\begin{aligned}\frac{\partial}{\partial\theta}\big((1-\langle x,y\rangle)\cos\theta\cos\theta'+\langle x,y\rangle\cos(\theta-\theta')\big)\\ &= -(1-\langle x,y\rangle)\sin\theta\cos\theta'-\langle x,y\rangle\sin(\theta-\theta')\\ &\le 0.\end{aligned}$$

Hence, $\langle p,q\rangle$ is maximized when $\theta=\theta'$. So $\langle p,q\rangle\le\cos^2\theta'+\langle x,y\rangle\sin^2\theta'$. Now, if we view θ' as a variable and take a derivative,

$$\frac{\partial}{\partial\theta'}(\cos^2\theta'+\langle x,y\rangle\sin^2\theta')=-2(1-\langle x,y\rangle)\cos\theta'\sin\theta'\le 0.$$

Therefore, $\cos^2\theta'+\langle x,y\rangle\sin^2\theta'$ is maximized when $\theta'=\alpha$. Hence,

$$\langle p,q\rangle\le\cos^2\alpha+\langle x,y\rangle\sin^2\alpha.$$

Note that $\langle x,y\rangle\le\frac{1}{\sqrt{3}}$ as in the proof of the previous case. Hence, finally we get $\langle p,q\rangle\le\cos^2\alpha+\frac{1}{\sqrt{3}}\sin^2\alpha=\bigl(1-\frac{1}{\sqrt{3}}\bigr)\cos^2\alpha+\frac{1}{\sqrt{3}}$. Since $\cos^2\alpha$ is obviously smaller than $\cos^2\alpha+\frac{1}{\sqrt{3}}\sin^2\alpha=\bigl(1-\frac{1}{\sqrt{3}}\bigr)\cos^2\alpha+\frac{1}{\sqrt{3}}$, this completes the proof of this case.

(4) If $p\in\mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,j})$ and $q\in\mathcal{C}_\alpha^{\mathrm{bot}}(V_{j,i})$ for some $i\ne j\in\{1,2,3,4\}$, then one can assume $x\in V_{i,j}$, $y\in V_{j,i}$, and $\theta,\theta'\in\bigl[\alpha,\frac{\pi}{2}\bigr]$. Since $\langle x,y\rangle\le 0$ always in this case, $\cos\theta\cos\theta'+\langle x,y\rangle\sin\theta\sin\theta'$ is decreasing with respect to both θ and θ'. Hence, $\langle p,q\rangle$ is maximized when $\theta=\theta'=\alpha$. Therefore,

$$\langle p,q\rangle\le\cos^2\alpha+\langle x,y\rangle\sin^2\alpha\le\cos^2\alpha. \qquad \square$$

Finally, we are ready to construct the map

$$\tilde{\phi}_{3,2}^{\alpha}\colon \boldsymbol{H}_{>0}(\mathbb{S}^3)\to\mathbb{S}^2,\qquad p\mapsto\begin{cases}u_i & \text{if } p\in\mathcal{C}_\alpha^{\mathrm{top}}(V_i)\text{ for some } i\in\{1,2,3,4\},\\ u_{i,j} & \text{if } p\in\mathcal{C}_\alpha^{\mathrm{bot}}(V_{i,j})\text{ for some } i\ne j\in\{1,2,3,4\}.\end{cases}$$

Proposition 8.4 *For $\alpha\in\bigl[0,\frac{\pi}{2}\bigr]$ such that $\cos^2\alpha\in\bigl[\frac{\sqrt{3}-1}{3+\sqrt{3}},\frac{7}{9}\bigr]$,*

$$\operatorname{dis}\bigl(\tilde{\phi}_{3,2}^{\alpha}\bigr)\le\zeta_2.$$

Proof We need to check

$$\bigl|d_{\mathbb{S}^3}(p,q)-d_{\mathbb{S}^2}\bigl(\tilde{\phi}_{3,2}^{\alpha}(p),\tilde{\phi}_{3,2}^{\alpha}(q)\bigr)\bigr|\le\zeta_2$$

for any $p,q\in\boldsymbol{H}_{>0}(\mathbb{S}^3)$. We carry out a case-by-case analysis.

(1) Suppose $p,q\in\mathcal{C}(V_i)$ for some $i\in\{1,2,3,4\}$. Without loss of generality, one can assume $i=1$. Then

$$d_{\mathbb{S}^2}\bigl(\tilde{\phi}_{3,2}^{\alpha}(p),\tilde{\phi}_{3,2}^{\alpha}(q)\bigr)\le\operatorname{diam}(\{u_1,u_{1,2},u_{1,3},u_{1,4}\})=\arccos\tfrac{5}{6}<\zeta_2$$

by Lemma 8.2(1). Therefore,

$$d_{\mathbb{S}^2}\bigl(\tilde{\phi}_{3,2}^{\alpha}(p),\tilde{\phi}_{3,2}^{\alpha}(q)\bigr)-d_{\mathbb{S}^3}(p,q)\le\arccos\tfrac{5}{6}<\zeta_2.$$

So it is enough to prove $d_{\mathbb{S}^3}(p,q)-d_{\mathbb{S}^2}\bigl(\tilde{\phi}_{3,2}^{\alpha}(p),\tilde{\phi}_{3,2}^{\alpha}(q)\bigr)\le\zeta_2$. But, for this direction, we need more subtle case-by-case analysis.

(a) Suppose $p, q \in \mathcal{C}_\alpha^{\text{top}}(V_1)$. Then $\tilde{\phi}^\alpha_{3,2}(p) = \tilde{\phi}^\alpha_{3,2}(q) = u_1$. Also, by Lemma 8.3(1) and the choice of α,

$$d_{\mathbb{S}^3}(p,q) \le \arccos\big(\big(1+\tfrac{1}{\sqrt{3}}\big)\cos^2\alpha - \tfrac{1}{\sqrt{3}}\big) \le \zeta_2.$$

Hence,

$$d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) = d_{\mathbb{S}^3}(p,q) \le \zeta_2.$$

(b) If $p \in \mathcal{C}_\alpha^{\text{top}}(V_1)$ and $q \in \mathcal{C}_\alpha^{\text{bot}}(V_1)$, then $\tilde{\phi}^\alpha_{3,2}(p) = u_1$ and $\tilde{\phi}^\alpha_{3,2}(q) = u_{1,j}$ for some $j \in \{2,3,4\}$. Therefore,

$$d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) \le \arccos\big(-\tfrac{1}{\sqrt{3}}\big) - \arccos\big(\tfrac{2\sqrt{2}}{3}\big) < \zeta_2.$$

(c) Suppose $p, q \in \mathcal{C}_\alpha^{\text{bot}}(V_1)$.

(i) If $p, q \in \mathcal{C}_\alpha^{\text{bot}}(V_{1,j})$ for some $j \in \{2,3,4\}$, then $\tilde{\phi}^\alpha_{3,2}(p) = \tilde{\phi}^\alpha_{3,2}(q) = u_{1,j}$. Also, it is easy to check the diameter of $\mathcal{C}_\alpha^{\text{bot}}(V_{1,j})$ is $\frac{\pi}{2}$. Hence,

$$d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) = d_{\mathbb{S}^3}(p,q) \le \tfrac{\pi}{2} < \zeta_2.$$

(ii) If $p \in \mathcal{C}_\alpha^{\text{bot}}(V_{1,k})$ and $p \in \mathcal{C}_\alpha^{\text{bot}}(V_{1,l})$ for some $k \neq l \in \{2,3,4\}$, then

$$d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) = d_{\mathbb{S}^2}(u_{1,k}, u_{1,l}) = \arccos\big(\tfrac{5}{6}\big)$$

by Lemma 8.2(1). Therefore,

$$d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) \le \arccos\big(-\tfrac{1}{\sqrt{3}}\big) - \arccos\big(\tfrac{5}{6}\big) < \zeta_2.$$

(2) Suppose $p \in \mathcal{C}(V_i)$ and $q \in \mathcal{C}(V_j)$ for some $i \neq j \in \{1,2,3,4\}$. Without loss of generality, one can assume $i = 1$ and $j = 2$. Then, by Lemma 8.2,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) \ge \arccos\big(\tfrac{5}{54}\big) > \arccos\big(\tfrac{1}{3}\big).$$

Therefore,

$$d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) < \pi - \arccos\big(\tfrac{1}{3}\big) = \zeta_2.$$

So, it is enough to prove $d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) - d_{\mathbb{S}^3}(p,q) \le \zeta_2$. Again we need more subtle case-by-case analysis.

(a) If $p \in \mathcal{C}_\alpha^{\text{top}}(V_1)$ and $q \in \mathcal{C}_\alpha^{\text{top}}(V_2)$, then $\tilde{\phi}^\alpha_{3,2}(p) = u_1$ and $\tilde{\phi}^\alpha_{3,2}(q) = u_2$, so $d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) = d_{\mathbb{S}^2}(u_1, u_2) = \zeta_2$. Thus,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}^\alpha_{3,2}(p), \tilde{\phi}^\alpha_{3,2}(q)\big) - d_{\mathbb{S}^3}(p,q) \le \zeta_2.$$

(b) Suppose $p \in \mathcal{C}_\alpha^{\mathrm{top}}(V_1)$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_2)$.

(i) If $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{2,j})$ for some $j \in \{3,4\}$, then, by Lemma 8.2(6),

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) = d_{\mathbb{S}^3}(u_1, u_{2,j}) = \arccos\big(-\tfrac{\sqrt{2}}{9}\big).$$

Hence,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) - d_{\mathbb{S}^3}(p,q) \le \arccos\big(-\tfrac{\sqrt{2}}{9}\big) < \zeta_2.$$

(ii) If $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{2,1})$ then $d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) = d_{\mathbb{S}^2}(u_1, u_{2,1}) = \arccos\big(-\frac{4\sqrt{2}}{9}\big)$ by Lemma 8.2(7). Moreover, by Lemma 8.3(2) and the choice of α,

$$d_{\mathbb{S}^3}(p,q) \ge \arccos\Big(\sqrt{\tfrac{2}{3}\cos^2\alpha + \tfrac{1}{3}}\Big) > \arccos\big(\tfrac{2\sqrt{2}}{3}\big),$$

which implies

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) - d_{\mathbb{S}^3}(p,q) < \arccos\big(-\tfrac{4\sqrt{2}}{9}\big) - \arccos\big(\tfrac{2\sqrt{2}}{3}\big) = \zeta_2.$$

(c) Suppose $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_1)$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_2)$. Considering symmetry, there are basically four subcases:

(i) If $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,3})$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{2,3})$, then, by Lemma 8.2(2),

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) = d_{\mathbb{S}^2}(u_{1,3}, u_{2,3}) = \arccos\big(\tfrac{5}{54}\big).$$

Hence,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) - d_{\mathbb{S}^3}(p,q) \le \arccos\big(\tfrac{5}{54}\big) < \zeta_2.$$

(ii) If $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,3})$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{2,4})$, then, by Lemma 8.2(3),

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) = d_{\mathbb{S}^2}(u_{1,3}, u_{2,4}) = \arccos\big(-\tfrac{2}{27}\big).$$

Hence,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) - d_{\mathbb{S}^3}(p,q) \le \arccos\big(-\tfrac{2}{27}\big) < \zeta_2.$$

(iii) If $p \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{1,3})$ and $q \in \mathcal{C}_\alpha^{\mathrm{bot}}(V_{2,1})$, then, by Lemma 8.2(4),

$$d_{\mathbb{S}^2}\big(\tilde{\phi}_{3,2}^\alpha(p), \tilde{\phi}_{3,2}^\alpha(q)\big) = d_{\mathbb{S}^2}(u_{1,3}, u_{2,1}) = \arccos\big(-\tfrac{25}{54}\big).$$

Moreover, by Lemma 8.3(3) and the choice of α,

$$d_{\mathbb{S}^3}(p,q) \ge \arccos\Big(\big(1 - \tfrac{1}{\sqrt{3}}\big)\cos^2\alpha + \tfrac{1}{\sqrt{3}}\Big) > \arccos\big(-\tfrac{25}{54}\big) - \zeta_2.$$

Hence,

$$d_{\mathbb{S}^2}(\tilde{\phi}^{\alpha}_{3,2}(p), \tilde{\phi}^{\alpha}_{3,2}(q)) - d_{\mathbb{S}^3}(p,q) < \zeta_2.$$

(iv) If $p \in \mathcal{C}^{\mathrm{bot}}_{\alpha}(V_{1,2})$ and $q \in \mathcal{C}^{\mathrm{bot}}_{\alpha}(V_{2,1})$, then, by Lemma 8.2(5),

$$d_{\mathbb{S}^2}\big(\tilde{\phi}^{\alpha}_{3,2}(p), \tilde{\phi}^{\alpha}_{3,2}(q)\big) = d_{\mathbb{S}^2}(u_{1,2}, u_{2,1}) = \arccos\big(-\tfrac{23}{27}\big).$$

Moreover, by Lemma 8.3(4) and the choice of α,

$$d_{\mathbb{S}^3}(p,q) \geq \arccos(\cos^2\alpha) \geq \arccos\big(\tfrac{7}{9}\big).$$

Hence,

$$d_{\mathbb{S}^2}\big(\tilde{\phi}^{\alpha}_{3,2}(p), \tilde{\phi}^{\alpha}_{3,2}(q)\big) - d_{\mathbb{S}^3}(p,q) \leq \arccos\big(-\tfrac{23}{27}\big) - \arccos\big(\tfrac{7}{9}\big) = \zeta_2. \qquad \square$$

Lemma 8.5 *For any* $p \in \boldsymbol{H}_{>0}(\mathbb{S}^3)$, $d_{\mathbb{S}^3}\big(p, \tilde{\phi}^{\alpha}_{3,2}(p)\big) \leq \frac{\pi}{2}$.

Proof Without loss of generality, one can assume $p \in \mathcal{C}(V_1)$. Then one can express p as $p = \cos\theta \cdot e_4 + \sin\theta \cdot \iota_2(x)$, where $e_4 = (0,0,0,1)$ for some $\theta \in \big[0, \frac{\pi}{2}\big]$ and $x \in V_1$. Moreover, since $\tilde{\phi}^{\alpha}_{3,2}(p) \in \{u_1, u_{1,2}, u_{1,3}, u_{1,4}\}$,

$$\big\langle p, \tilde{\phi}^{\alpha}_{3,2}(p)\big\rangle = \big\langle x, \tilde{\phi}^{\alpha}_{3,2}(p)\big\rangle \cdot \sin\theta.$$

Also, it is easy to check that $\big\langle x, \tilde{\phi}^{\alpha}_{3,2}(p)\big\rangle \geq 0$ (more precisely, $\langle u_1, x\rangle \geq \frac{1}{3}$ and $\langle u_{1,j}, x\rangle \geq \frac{\sqrt{2}}{9}$ for any $x \in N_1$ with $j \neq 1$). This implies $\big\langle p, \tilde{\phi}^{\alpha}_{3,2}(p)\big\rangle \geq 0$; hence we have the required inequality. $\square$

We are now ready to prove Proposition 1.19.

Proof of Proposition 1.19 It is enough to find a surjective map $\phi_{3,2}\colon \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$ such that $\operatorname{dis}(\phi_{3,2}) \leq \zeta_2$ since this map gives rise to the correspondence $R_{3,2} := \operatorname{graph}(\phi_{3,2})$ with $\operatorname{dis}(R_{3,2}) = \operatorname{dis}(\phi_{3,2}) \leq \zeta_2$.

Let

$$\hat{\phi}^{\alpha}_{3,2}\colon \boldsymbol{A}(\mathbb{S}^3) \to \mathbb{S}^2, \qquad p \mapsto \begin{cases} \tilde{\phi}^{\alpha}_{3,2}(p) & \text{if } p \in \boldsymbol{H}_{>0}(\mathbb{S}^3), \\ p & \text{if } p \in \iota_2(\boldsymbol{A}(\mathbb{S}^2)). \end{cases}$$

We claim that $\operatorname{dis}(\hat{\phi}^{\alpha}_{3,2}) = \operatorname{dis}(\tilde{\phi}^{\alpha}_{3,2})$. To check this, it is enough to show that

$$\big|d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}(\hat{\phi}^{\alpha}_{3,2}(p), \hat{\phi}^{\alpha}_{3,2}(q))\big| \leq \zeta_2$$

for any $p \in \boldsymbol{H}_{>0}(\mathbb{S}^3)$ and $q \in \iota_2(\boldsymbol{A}(\mathbb{S}^2))$. But, this is true since

$$\begin{aligned}\big|d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\hat{\phi}^{\alpha}_{3,2}(p), \hat{\phi}^{\alpha}_{3,2}(q)\big)\big| &= \big|d_{\mathbb{S}^3}(p,q) - d_{\mathbb{S}^2}\big(\hat{\phi}^{\alpha}_{3,2}(p), q\big)\big| \\ &\leq d_{\mathbb{S}^3}\big(p, \hat{\phi}^{\alpha}_{3,2}(p)\big),\end{aligned}$$

and $d_{\mathbb{S}^3}\big(p,\hat{\phi}^{\alpha}_{3,2}(p)\big)=d_{\mathbb{S}^3}\big(p,\tilde{\phi}^{\alpha}_{3,2}(p)\big)\le\frac{\pi}{2}<\zeta_2$ for any $p\in \boldsymbol{H}_{>0}(\mathbb{S}^3)$ by Lemma 8.5. Hence, $\mathrm{dis}\big(\hat{\phi}^{\alpha}_{3,2}\big)=\mathrm{dis}\big(\tilde{\phi}^{\alpha}_{3,2}\big)$. Finally, apply Lemma 5.7 to construct a surjective map $\phi_{3,2}\colon\mathbb{S}^3\twoheadrightarrow\mathbb{S}^2$. Then

$$\mathrm{dis}(\phi_{3,2})=\mathrm{dis}\big(\hat{\phi}^{\alpha}_{3,2}\big)=\mathrm{dis}\big(\tilde{\phi}^{\alpha}_{3,2}\big)\le\zeta_2$$

by Proposition 8.4. □

9 The Gromov–Hausdorff distance between spheres with Euclidean metric

For any nonempty subset $X\subseteq\mathbb{S}^n$, let X_{E} denote the metric space with the inherited Euclidean metric. In particular, $\mathbb{S}^n_{\mathrm{E}}$ will denote the unit sphere with the Euclidean metric d_{E} inherited from $\mathbb{R}^{n+1}$. A natural question is: what is the value of

$$\mathfrak{g}^{\mathrm{E}}_{m,n}:=d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}},\mathbb{S}^n_{\mathrm{E}})$$

for $0\le m<n\le\infty$? We found that, interestingly, these values do not always directly follow from those of $\mathfrak{g}_{m,n}$.

Any correspondence R between $\mathbb{S}^m$ and $\mathbb{S}^n$ can of course be regarded as a correspondence between $\mathbb{S}^m_{\mathrm{E}}$ and $\mathbb{S}^n_{\mathrm{E}}$. Throughout this section, let $\mathrm{dis}(R)$ denote the distortion with respect to the geodesic metric (as usual), and let $\mathrm{dis}_{\mathrm{E}}(R)$ denote the distortion with respect to the Euclidean metric.

The following are direct extensions of parallel results for spheres with geodesic distance:

Remark 9.1 As in Remark 1.4, for all $0\le m\le n\le\infty$,

$$d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}},\mathbb{S}^n_{\mathrm{E}})\le 1.$$

Lemma 9.2 *For any integer $m\ge1$ and any finite metric space P with cardinality at most $m+1$, we have $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}},P)\ge1$.*

Proof Fix an arbitrary correspondence R between $\mathbb{S}^m_{\mathrm{E}}$ and P. Then one can prove that $\mathrm{dis}_{\mathrm{E}}(R)\ge2$ as in the proof of Lemma 3.2 (via the aid of Lyusternik–Schnirelmann theorem). Since R is arbitrary, one can conclude $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}},P)\ge1$. □

Corollary 9.3 *Let R be any correspondence between a finite metric space P and $\mathbb{S}^\infty_{\mathrm{E}}$. Then $\mathrm{dis}_{\mathrm{E}}(R)\ge2$. In particular, $d_{\mathrm{GH}}(P,\mathbb{S}^\infty_{\mathrm{E}})\ge1$.*

Proof See the proof of Corollary 3.4. □

Proposition 9.4 *Let X be any totally bounded metric space. Then $d_{\mathrm{GH}}(X,\mathbb{S}^\infty_{\mathrm{E}})\ge 1$.*

Proof Follow the idea of the proof of Proposition 3.5. □

Proposition 9.5 *For any $n\ge 1$, $d_{\mathrm{GH}}(\mathbb{S}^0_{\mathrm{E}},\mathbb{S}^n_{\mathrm{E}})=1$.*

Proof Apply Remark 9.1 and Lemma 9.2. □

Proposition 9.6 *For any integer $m\ge 0$, $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}},\mathbb{S}^\infty_{\mathrm{E}})=1$.*

Proof Apply Remark 9.1 and Proposition 9.4. □

The following lemma permits bounding $\mathrm{dis}_{\mathrm{E}}(R)$ via $\mathrm{dis}(R)$:

Lemma 9.7 *Let $0\le m<n\le\infty$, and let R be an arbitrary nonempty relation between $\mathbb{S}^m_{\mathrm{E}}$ and $\mathbb{S}^n_{\mathrm{E}}$. Then*
$$\mathrm{dis}_{\mathrm{E}}(R)\le 2\sin\bigl(\tfrac12\,\mathrm{dis}(R)\bigr).$$

Proof First of all, note that $\mathrm{dis}(R):=\sup_{(x,y),(x',y')\in R}|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\le\pi$, since both $\mathrm{diam}(\mathbb{S}^m)$ and $\mathrm{diam}(\mathbb{S}^n)$ are at most π. Fix arbitrary $(x,y),(x',y')\in R$. Then

$$\begin{aligned}
d_{\mathrm{E}}(x,x') &= 2\sin\bigl(\tfrac12 d_{\mathbb{S}^m}(x,x')\bigr)\\
&= 2\sin\bigl(\tfrac12 d_{\mathbb{S}^m}(x,x')-\tfrac12 d_{\mathbb{S}^n}(y,y')+\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\\
&= 2\sin\bigl(\tfrac12 d_{\mathbb{S}^m}(x,x')-\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\cos\bigl(\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\\
&\qquad\qquad +2\cos\bigl(\tfrac12 d_{\mathbb{S}^m}(x,x')-\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\sin\bigl(\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\\
&\le 2\sin\bigl(\tfrac12|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\bigr)+2\sin\bigl(\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\\
&= 2\sin\bigl(\tfrac12|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\bigr)+d_{\mathrm{E}}(y,y'),
\end{aligned}$$

where the inequality follows since $\cos\bigl(\tfrac12 d_{\mathbb{S}^m}(x,x')-\tfrac12 d_{\mathbb{S}^n}(y,y')\bigr)\in[0,1]$.

Hence,
$$d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')\le 2\sin\bigl(\tfrac12|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\bigr).$$

Similarly, one can also prove
$$d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2\sin\bigl(\tfrac12|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\bigr).$$

Therefore,
$$|d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')|\le 2\sin\bigl(\tfrac12|d_{\mathbb{S}^m}(x,x')-d_{\mathbb{S}^n}(y,y')|\bigr).$$

Since $(x,y),(x',y')\in R$ were arbitrary, this leads to the required conclusion. □

Corollary 9.8 *For any* $0 \le m < n \le \infty$:

(1) $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}}, \mathbb{S}^n_{\mathrm{E}}) \le \sin(d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n))$.

(2) *In more generality, for any* $X \subseteq \mathbb{S}^m$ *and* $Y \subseteq \mathbb{S}^n$, $d_{\mathrm{GH}}(X_{\mathrm{E}}, Y_{\mathrm{E}}) \le \sin(d_{\mathrm{GH}}(X, Y))$.

Corollary 9.9 $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}}, \mathbb{S}^n_{\mathrm{E}}) < 1$ *for all* $0 < m \ne n < \infty$.

Proof Invoke Corollary 9.8 and Theorem A. □

Given the above, and that we proved $\mathfrak{g}_{1,2} = \frac{\pi}{3}$ and $\mathfrak{g}_{2,3} = \frac{1}{2}\zeta_2$, one might expect that $\mathfrak{g}^{\mathrm{E}}_{1,2} = d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, \mathbb{S}^2_{\mathrm{E}}) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and similarly that $\mathfrak{g}^{\mathrm{E}}_{2,3} = \frac{\sqrt{2}}{\sqrt{3}}$. However, rather surprisingly, we were able to construct a correspondence R_{E} between $\mathbb{S}^1_{\mathrm{E}}$ and $\boldsymbol{H}_{\ge 0}(\mathbb{S}^2_{\mathrm{E}})$ such that $\mathrm{dis}_{\mathrm{E}}(R_{\mathrm{E}}) < \sqrt{3}$ (see Proposition 9.10 and its proof in Section 9.1). This correspondence then naturally induces a function $\phi_{\mathrm{E}} \colon \boldsymbol{A}(\mathbb{S}^2_{\mathrm{E}}) \to \mathbb{S}^1_{\mathrm{E}}$ from the "helmet" on $\mathbb{S}^2_{\mathrm{E}}$ into $\mathbb{S}^1_{\mathrm{E}}$ also with $\mathrm{dis}_{\mathrm{E}}(\phi_{\mathrm{E}}) < \sqrt{3}$.

Proposition 9.10
$$d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, \boldsymbol{H}_{\ge 0}(\mathbb{S}^2_{\mathrm{E}})) < \frac{\sqrt{3}}{2}.$$

This proposition was motivated by Ilya Bogdanov's answer [2] to a MathOverflow question regarding the Gromov–Hausdorff distance between $\mathbb{S}^1_{\mathrm{E}}$ and the unit disk in $\mathbb{R}^2$.

We now discuss the possibility that the correspondence R_{E} described above permits proving that, in fact, $d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, \mathbb{S}^2_{\mathrm{E}}) < \frac{\sqrt{3}}{2}$ via extending R_{E} into a correspondence between $\mathbb{S}^2_{\mathrm{E}}$ and $\mathbb{S}^1_{\mathrm{E}}$ much in the same way that we did so in the case of spheres with their geodesic distance (see Lemma 5.7).

By the same method of proof as that of Corollary 5.5 (giving the lower bound $\mathrm{dis}(g) \ge \zeta_n$ for any antipode-preserving map $g \colon \mathbb{S}^n \to \mathbb{S}^{n-1}$), one obtains the following Euclidean analogue:

Corollary 9.11 *For each integer* $n > 0$, *any function* $g \colon \mathbb{S}^n_{\mathrm{E}} \to \mathbb{S}^{n-1}_{\mathrm{E}}$ *which maps every pair of antipodal points on* $\mathbb{S}^n_{\mathrm{E}}$ *onto antipodal points on* $\mathbb{S}^{n-1}_{\mathrm{E}}$ *satisfies*
$$\mathrm{dis}_{\mathrm{E}}(g) \ge \sqrt{2 + \frac{2}{n}}.$$

Remark 9.12 (extending Lemma 5.7 to the case of spheres with Euclidean metric) Lemma 5.7 was instrumental in our quest for lower bounds for the Gromov–Hausdorff distance between spheres with the geodesic distance. It is natural to attempt to obtain a suitable version of that result to the case of the Euclidean metric. However, there is a caveat. Indeed, one should *not* expect to be able to prove a version in which $\operatorname{dis}_{\mathrm{E}}(\phi^*)$ is *equal* to $\operatorname{dis}(\phi)$ where $\phi\colon \boldsymbol{A}(\mathbb{S}^n_{\mathrm{E}}) \to \mathbb{S}^m_{\mathrm{E}}$ and ϕ^* is its antipode-preserving extension obtained via the "helmet trick" (as described in the statement of Lemma 5.7). If this was the case, then the antipode-preserving extension ϕ^*_{E} of the function ϕ_{E} mentioned above would satisfy

$$\operatorname{dis}_{\mathrm{E}}(\phi^*_{\mathrm{E}}) < \sqrt{3}. \tag{10}$$

However, note that Corollary 9.11 implies that, in the case of spheres with Euclidean distance, any antipode-preserving map $\psi\colon \mathbb{S}^{m+1}_{\mathrm{E}} \to \mathbb{S}^m_{\mathrm{E}}$ must satisfy

$$\operatorname{dis}_{\mathrm{E}}(\psi) \geq \sqrt{2+\frac{2}{m+1}}.$$

In particular, it must be that $\operatorname{dis}_{\mathrm{E}}(\psi) \geq \sqrt{3}$ for any antipode-preserving map $\psi\colon \mathbb{S}^2_{\mathrm{E}} \to \mathbb{S}^1_{\mathrm{E}}$, and this would contradict (10).

Still, as we describe next, there is a suitable generalization of Lemma 5.7 which yields nontrivial lower bounds (see Proposition 9.16).

Lemma 9.13 *If $|a-b| =: \delta \in [0,2]$ for some $a, b \in [0,2]$, then*

$$\left|\sqrt{4-a^2}-\sqrt{4-b^2}\right| \leq \sqrt{\delta(4-\delta)},$$

and the inequality is tight.

Proof The claim is obvious if $\delta = 0$. Henceforth, we will assume that $\delta > 0$. Observe that

$$\begin{aligned}\left|\sqrt{4-a^2}-\sqrt{4-b^2}\right| &= \frac{|a^2-b^2|}{\sqrt{4-a^2}+\sqrt{4-b^2}} \\ &= |a-b|\cdot\frac{a+b}{\sqrt{4-a^2}+\sqrt{4-b^2}} \\ &\leq \delta\cdot\frac{4-\delta}{\sqrt{4\delta-\delta^2}} \\ &= \sqrt{\delta(4-\delta)}.\end{aligned}$$

Finally, the equality holds if $a = 2$ and $b = 2-\delta$, or $a = 2-\delta$ and $b = 2$. □

Lemma 9.14 *For any $m, n \geq 0$, let $\varnothing \neq C \subseteq \mathbb{S}^n_{\mathrm{E}}$ satisfy $C \cap (-C) = \varnothing$ and let $\phi\colon C \to \mathbb{S}^m_{\mathrm{E}}$ be any map. Then the extension ϕ^* of ϕ to the set $C \cup (-C)$ defined by*

$$\phi^*\colon C\cup(-C) \to \mathbb{S}^m, \qquad x \mapsto \phi(x), \quad -x \mapsto -\phi(x) \quad \text{for } x \in C,$$

is antipode-preserving and satisfies $\operatorname{dis}_{\mathrm{E}}(\phi^*) \leq \sqrt{\operatorname{dis}_{\mathrm{E}}(\phi)(4-\operatorname{dis}_{\mathrm{E}}(\phi))}$.

Proof By definition, ϕ^* is antipode-preserving. Now, fix arbitrary $x, x' \in C$. Then

$$\begin{aligned}|d_{\mathrm{E}}(x,-x') - d_{\mathrm{E}}(\phi^*(x),\phi^*(-x'))| \\ &= \left|\sqrt{4-(d_{\mathrm{E}}(x,x'))^2} - \sqrt{4-\big(d_{\mathrm{E}}(\phi(x),\phi(x'))\big)^2}\right| \\ &\le \sqrt{|d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(\phi(x),\phi(x'))|\big(4-|d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(\phi(x),\phi(x'))|\big)} \\ &\le \sqrt{\mathrm{dis}_{\mathrm{E}}(\phi)(4-\mathrm{dis}_{\mathrm{E}}(\phi))}\end{aligned}$$

and

$$|d_{\mathrm{E}}(-x,-x') - d_{\mathrm{E}}(\phi^*(-x),\phi^*(-x'))| = |d_{\mathrm{E}}(x,x') - d_{\mathrm{E}}(\phi(x),\phi(x'))| \le \mathrm{dis}_{\mathrm{E}}(\phi).$$

Hence,

$$\mathrm{dis}_{\mathrm{E}}(\phi^*) \le \max\big\{\mathrm{dis}_{\mathrm{E}}(\phi), \sqrt{\mathrm{dis}_{\mathrm{E}}(\phi)(4-\mathrm{dis}_{\mathrm{E}}(\phi))}\big\} = \sqrt{\mathrm{dis}_{\mathrm{E}}(\phi)(4-\mathrm{dis}_{\mathrm{E}}(\phi))}. \qquad \square$$

Corollary 9.15 *For each $n \in \mathbb{Z}_{>0}$ and any map $\phi\colon \mathbb{S}^n_{\mathrm{E}} \to \mathbb{S}^{n-1}_{\mathrm{E}}$, there exists an antipode-preserving map $\phi^*\colon \mathbb{S}^n_{\mathrm{E}} \to \mathbb{S}^{n-1}_{\mathrm{E}}$ such that $\mathrm{dis}_{\mathrm{E}}(\phi^*) \le \sqrt{\mathrm{dis}_{\mathrm{E}}(\phi)(4-\mathrm{dis}_{\mathrm{E}}(\phi))}$.*

Proof Consider the restriction of ϕ to the "helmet" $\boldsymbol{A}(\mathbb{S}^n)$ (see Section 5.1) and apply Lemma 9.14. $\square$

Proposition 9.16 *For all integers $0 < m < n$,*

$$d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}}, \mathbb{S}^n_{\mathrm{E}}) \ge \frac{1}{2}\bigg(2 - \sqrt{2 - \frac{2}{m+1}}\bigg) \ge \frac{1}{2}.$$

Proof Suppose to the contrary that $d_{\mathrm{GH}}(\mathbb{S}^m_{\mathrm{E}}, \mathbb{S}^n_{\mathrm{E}}) < \frac{1}{2}(2 - \sqrt{2-2/(m+1)})$. This implies that there exist a correspondence Γ between $\mathbb{S}^m_{\mathrm{E}}$ and $\mathbb{S}^n_{\mathrm{E}}$ such that $\mathrm{dis}_{\mathrm{E}}(\Gamma) < \frac{1}{2}(2 - \sqrt{2-2/(m+1)})$. Moreover, since $n \ge m+1$, $\mathbb{S}^{m+1}_{\mathrm{E}}$ can be isometrically embedded in $\mathbb{S}^n_{\mathrm{E}}$, so we are able to construct a map $g\colon \mathbb{S}^{m+1}_{\mathrm{E}} \to \mathbb{S}^m_{\mathrm{E}}$ in the following way: for each $x \in \mathbb{S}^{m+1}_{\mathrm{E}} \subseteq \mathbb{S}^n_{\mathrm{E}}$, choose $g(x) \in \mathbb{S}^m_{\mathrm{E}}$ such that $(g(x), x) \in \Gamma$. Then $\mathrm{dis}_{\mathrm{E}}(g) < (2 - \sqrt{2-2/(m+1)})$ as well. By applying Corollary 9.15, one can modify this g into an antipode-preserving map $\hat{g}\colon \mathbb{S}^{m+1}_{\mathrm{E}} \to \mathbb{S}^m_{\mathrm{E}}$ with

$$\mathrm{dis}_{\mathrm{E}}(\hat{g}) \le \sqrt{\mathrm{dis}_{\mathrm{E}}(g)(4-\mathrm{dis}_{\mathrm{E}}(g))} < \sqrt{2 + \frac{2}{m+1}},$$

which contradicts Corollary 9.11. $\square$

Note that in contrast to the case of geodesic distances (where the upper bound given by Proposition 1.16 and the lower bound given by Theorem B agree when $m = 1$ and $n = 2$), Proposition 9.16 yields $\mathfrak{g}^{\mathrm{E}}_{1,2} \ge \frac{1}{2}$, which is strictly smaller than the upper bound $\frac{\sqrt{3}}{2}$ provided by Corollary 9.8 and Proposition 1.16.

9.1 The proof of Proposition 9.10

The proof will be based on a geometric construction which is illustrated in Figures 13 and 14.

Proof To prove the claim, note that it is enough to construct a correspondence R_{E} between $\mathbb{S}^1_{\mathrm{E}}$ and $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2_{\mathrm{E}})$ such that $\mathrm{dis}_{\mathrm{E}}(R_{\mathrm{E}}) < \sqrt{3}$.

First, let $u_1, \dots, u_7$ be the vertices of a regular heptagon inscribed in $\mathbb{S}^1$. Let $v_i := -u_i$ for $i = 1, \dots, 7$. See Figure 13 for a description.

Second, divide $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2_E)$ into seven regions $A_1, \dots, A_7$ as in Figure 14. The precise "disjointification" (on the boundary) of the seven regions is not relevant to the analysis that follows, as it is easy to check.

Now, choose $a_i \in A_i$ for each $i = 1, \dots, 7$ in the following way, where α is some number which is very close to $\frac{\sqrt{3}}{2}$ but still strictly smaller than $\frac{\sqrt{3}}{2}$ (for example, choose $\alpha = 0.866$):

$$a_1 = \left(\sqrt{1-(\sqrt{1-\alpha^2}+2-\sqrt{3})^2}, \sqrt{1-\alpha^2}+2-\sqrt{3}, 0\right) \approx (0.640511, 0.767949, 0),$$

$$a_2 = \left(0, \sqrt{1-\alpha^2}+2-\sqrt{3}, \sqrt{1-(\sqrt{1-\alpha^2}+2-\sqrt{3})^2}\right) \approx (0, 0.767949, 0.640511),$$

$$a_3 = \left(0, \sqrt{1-\alpha^2}, \alpha\right) \approx (0, 0.5, 0.866),$$

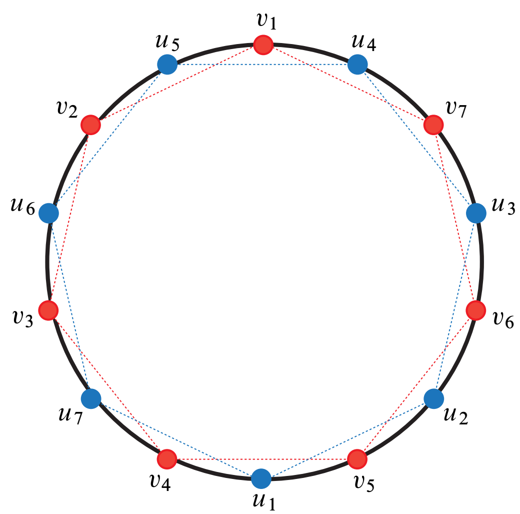


Figure 13: The points $v_1, \dots, v_7$ and $u_1, \dots, u_7$. These arise from two antipodal regular heptagons inscribed in $\mathbb{S}^1$.

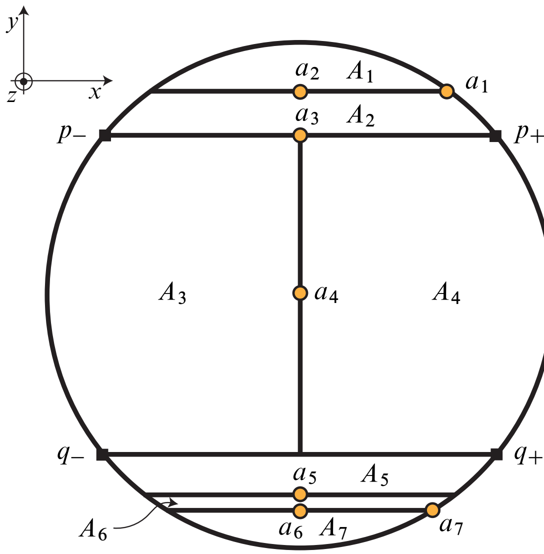


Figure 14: View from above of the seven regions $A_1, \dots, A_7$ of $\boldsymbol{H}_{\geq 0}(\mathbb{S}^2_E)$. All lines shown in the figure (which are projections of circular arcs) are aligned with the either the x or y axis. Also, $p_\pm = (\pm\alpha, \sqrt{1-\alpha^2}, 0)$ and $q_\pm = (\pm\alpha, -\sqrt{1-\alpha^2}, 0)$.

$$
\begin{aligned}
a_4 &= (0,0,1),\\
a_5 &= \left(0, -\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}\right), \sqrt{1-\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}\right)^2}\right)\\
&\approx (0, -0.717805, 0.696244),\\
a_6 &= \left(0, -\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}+\rho_5-\sqrt{3}\right), \sqrt{1-\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}+\rho_5-\sqrt{3}\right)^2}\right)\\
&\approx (0, -0.787692, 0.616069),\\
a_7 &= \left(\sqrt{1-\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}+\rho_5-\sqrt{3}\right)^2}, -\left(\sqrt{1-\alpha^2}+\rho_6-\sqrt{3}+\rho_5-\sqrt{3}\right), 0\right)\\
&\approx (0.616069, -0.787692, 0),
\end{aligned}
$$

where $\rho_k := \sqrt{2-2\cos(k\pi/7)}$ for $k \in \{1, \dots, 7\}$.

One can directly check that the following seven conditions are satisfied:

(1) $d_{\mathrm{E}}(A_i, A_j) > \rho_6 - \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i-j| = 3$.

(2) $d_{\mathrm{E}}(a_i, a_j) > \rho_6 - \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i-j| = 2$.

(3) $d_{\mathrm{E}}(a_i, a_j) > 2 - \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i-j| = 3$.

(4) $d_{\mathrm{E}}(A_i, a_j) > \rho_5 - \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i - j| = 2$.

(5) $d_{\mathrm{E}}(A_i, a_j) > 2 - \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i - j| = 3$.

(6) $\mathrm{diam}(A_i) < \sqrt{3}$ for any $i \in \{1, \dots, 7\}$.

(7) $d_{\mathrm{E}}(a_i, a_j) < \sqrt{3}$ for any $i, j \in \{1, \dots, 7\}$ with $|i - j| = 1$.

In what follows, for two points $v, w \in \mathbb{S}^1$ with $d_{\mathrm{E}}(v, w) < 2$, $\widehat{vw}$ will denote the (unique) shortest circular arc determined by these two points.

Now we define a correspondence R_{E} by

$$R_{\mathrm{E}} := \bigcup_{i=1}^{7} \{(u_i, y) : y \in A_i\} \cup \bigcup_{i=1}^{7} \{(x, a_i) : x \in \widehat{v_{i+3} v_{i+4}}\}.$$

We now prove that $\mathrm{dis}_{\mathrm{E}}(R_{\mathrm{E}}) < \sqrt{3}$:

First, let us prove that

$$\sup_{(x,y),(x',y') \in R_{\mathrm{E}}} (d_{\mathrm{E}}(x, x') - d_{\mathrm{E}}(y, y')) < \sqrt{3}.$$

For this we verify the inequality $d_{\mathrm{E}}(x, x') - d_{\mathrm{E}}(y, y') < \sqrt{3}$ for all cases induced by the structure of the correspondence R_E:

(1) If $(x, y), (x', y') \in \{u_i\} \times A_i$ for some $i \in \{1, \dots, 7\}$, then $d_{\mathrm{E}}(x, x') = d_{\mathrm{E}}(u_i, u_i) = 0$.

(2) If $(x, y) \in \{u_i\} \times A_i$ and $(x', y') \in \{u_j\} \times A_j$ for some $i, j \in \{1, \dots, 7\}$ with $|i - j| = 1$, then $d_{\mathrm{E}}(x, x') = d_{\mathrm{E}}(u_i, u_j) = \rho_2 < \sqrt{3}$.

(3) If $(x, y) \in \{u_i\} \times A_i$ and $(x', y') \in \{u_j\} \times A_j$ for some $i, j \in \{1, \dots, 7\}$ with $|i - j| = 2$, then $d_{\mathrm{E}}(x, x') = d_{\mathrm{E}}(u_i, u_j) = \rho_4 < \sqrt{3}$.

(4) If $(x, y) \in \{u_i\} \times A_i$ and $(x', y') \in \{u_j\} \times A_j$ for some $i, j \in \{1, \dots, 7\}$ with $|i - j| = 3$, then $d_{\mathrm{E}}(x, x') = d_{\mathrm{E}}(u_i, u_j) = \rho_6 > \sqrt{3}$. However, since $d_{\mathrm{E}}(A_i, A_j) > \rho_6 - \sqrt{3}$ by condition (1) above, we have $d_{\mathrm{E}}(x, x') - d_{\mathrm{E}}(y, y') < \sqrt{3}$.

(5) If $(x, y), (x', y') \in \widehat{v_{i+3} v_{i+4}} \times \{a_i\}$ for some $i \in \{1, \dots, 7\}$, then $d_{\mathrm{E}}(x, x') \le \mathrm{diam}(\widehat{v_{i+3} v_{i+4}}) = \rho_2 < \sqrt{3}$.

(6) If $(x, y) \in \widehat{v_{i+3} v_{i+4}} \times \{a_i\}$ and $(x', y') \in \widehat{v_{j+3} v_{j+4}} \times \{a_j\}$ for some $i, j \in \{1, \dots, 7\}$ with $|i - j| = 1$, then $d_{\mathrm{E}}(x, x') \le \mathrm{diam}(\widehat{v_{i+3} v_{i+4}} \cup \widehat{v_{j+3} v_{j+4}}) = \rho_4 < \sqrt{3}$.

(7) If $(x, y) \in \widehat{v_{i+3} v_{i+4}} \times \{a_i\}$ and $(x', y') \in \widehat{v_{j+3} v_{j+4}} \times \{a_j\}$ for some $i, j \in \{1, \dots, 7\}$ with $|i - j| = 2$, then $d_{\mathrm{E}}(x, x') \le \mathrm{diam}(\widehat{v_{i+3} v_{i+4}} \cup \widehat{v_{j+3} v_{j+4}}) = \rho_6 > \sqrt{3}$.

However, since $d_{\mathrm{E}}(y,y')=d_{\mathrm{E}}(a_i,a_j)>\rho_6-\sqrt{3}$ by condition (2) above, we have $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<\sqrt{3}$.

(8) If $(x,y)\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=3$, then, since $d_{\mathrm{E}}(y,y')=d_{\mathrm{E}}(a_i,a_j)>2-\sqrt{3}$ by condition (3) above, we have $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<2-(2-\sqrt{3})=\sqrt{3}$.

(9) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ for some $i\in\{1,\dots,7\}$, then $u_i\in\widehat{v_{i+3}v_{i+4}}$. Hence, $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,x')\le\operatorname{diam}(\widehat{v_{i+3}v_{i+4}})<\sqrt{3}$. So $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<\sqrt{3}$.

(10) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=1$, then $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,x')\le\operatorname{diam}(\{u_i\}\cup\widehat{v_{j+3}v_{j+4}})=\rho_3<\sqrt{3}$.

(11) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=2$, then $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,x')\le\operatorname{diam}(\{u_i\}\cup\widehat{v_{j+3}v_{j+4}})=\rho_5>\sqrt{3}$. However, since $d_{\mathrm{E}}(A_i,a_j)>\rho_5-\sqrt{3}$ by condition (4) above, $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<\sqrt{3}$.

(12) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=3$, then, since $d_{\mathrm{E}}(A_i,a_j)>2-\sqrt{3}$ by condition (5) above, we have $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<\sqrt{3}$.

Next, we prove

$$\sup_{(x,y),(x',y')\in R_{\mathrm{E}}}(d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x'))<\sqrt{3},$$

for which we verify the inequality $d_{\mathrm{E}}(x,x')-d_{\mathrm{E}}(y,y')<\sqrt{3}$ in a number of cases.

(1) If $(x,y),(x',y')\in\{u_i\}\times A_i$ for some $i\in\{1,\dots,7\}$, then, since $\operatorname{diam}(A_i)<\sqrt{3}$ by condition (6) above, $d_{\mathrm{E}}(y,y')<\sqrt{3}$, so $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')<\sqrt{3}$.

(2) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\{u_j\}\times A_j$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=1$, then $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,u_j)=\rho_2$ and $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_2<\sqrt{3}$.

(3) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\{u_j\}\times A_j$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=2$, then $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,u_j)=\rho_4$ and $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_4<\sqrt{3}$.

(4) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\{u_j\}\times A_j$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=3$, then $d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(u_i,u_j)=\rho_6$ and $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_6<\sqrt{3}$.

(5) If $(x,y),(x',y')\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ for some $i\in\{1,\dots,7\}$, then $d_{\mathrm{E}}(y,y')=d_{\mathrm{E}}(a_i,a_i)=0$.

(6) If $(x,y)\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=1$, then, since $d_{\mathrm{E}}(a_i,a_j)<\sqrt{3}$ by condition (7) above, we have $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')=d_{\mathrm{E}}(a_i,a_j)-d_{\mathrm{E}}(x,x')<\sqrt{3}$.

(7) If $(x,y)\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=2$, then $d_{\mathrm{E}}(x,x')\ge\rho_2$. Hence, $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_2<\sqrt{3}$.

(8) If $(x,y)\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=3$, then $d_{\mathrm{E}}(x,x')\ge\rho_4$. Hence, $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_4<\sqrt{3}$.

(9) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{i+3}v_{i+4}}\times\{a_i\}$ for some $i\in\{1,\dots,7\}$, then, since $a_i\in A_i$ and $\operatorname{diam}(A_i)<\sqrt{3}$ by condition (6), we have $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')<\sqrt{3}$.

(10) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=1$, then $d_{\mathrm{E}}(x,x')\ge\rho_1$. Hence, $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_1<\sqrt{3}$.

(11) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=2$, then $d_{\mathrm{E}}(x,x')\ge\rho_3$. Hence, $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_3<\sqrt{3}$.

(12) If $(x,y)\in\{u_i\}\times A_i$ and $(x',y')\in\widehat{v_{j+3}v_{j+4}}\times\{a_j\}$ for some $i,j\in\{1,\dots,7\}$ with $|i-j|=3$: Observe that $d_{\mathrm{E}}(x,x')\ge\rho_5$. Hence, $d_{\mathrm{E}}(y,y')-d_{\mathrm{E}}(x,x')\le 2-\rho_5<\sqrt{3}$.

Hence, $\operatorname{dis}_{\mathrm{E}}(R_{\mathrm{E}})<\sqrt{3}$, as required. □

Appendix A A succinct proof of Theorem G

In this appendix we provide a proof of Theorem G following a strategy suggested by Matoušek in [24, page 41] and due to Arnold Waßmer.

Lemma A.1 *If a simplex contains $0\in\mathbb{R}^n$ and all of its vertices lie on $\mathbb{S}^{n-1}$, then there are vertices u and v of the simplex such that $d_{\mathbb{S}^{n-1}}(u,v)\ge\zeta_{n-1}$.*

Proof We give the proof here for completeness — the proof is basically the same as that of [11, Lemma 1]. Let $u_1,\dots,u_{n+1}$ be (not necessarily distinct) vertices of a simplex such that their convex hull contains the origin $0\in\mathbb{R}^n$. Therefore, there are nonnegative numbers $\lambda_1,\dots,\lambda_{n+1}$ such that $\sum_{i=1}^{n+1}\lambda_n=1$ and $0=\sum_{i=1}^{n+1}\lambda_i u_i$. Then

$$0=\left\|\sum_{i=1}^{n+1}\lambda_i u_i\right\|^2=\sum_{i\neq j}\lambda_i\lambda_j\langle u_i,u_j\rangle+\sum_{i=1}^{n+1}\lambda_i^2.$$

Moreover, since $0\le\sum_{i\neq j}(\lambda_i-\lambda_j)^2=2n\sum_{i=1}^{n+1}\lambda_i^2-2\sum_{i\neq j}\lambda_i\lambda_j$,

$$\sum_{i=1}^{n+1}\lambda_i^2\ge\frac{1}{n}\sum_{i\neq j}\lambda_i\lambda_j.$$

Hence,

$$0 \geq \sum_{i \neq j} \lambda_i \lambda_j \Big(\langle u_i, u_j \rangle + \frac{1}{n} \Big).$$

Thus, there must be some distinct i and j such that $\langle u_i, u_j \rangle \leq -1/n$, so that

$$d_{\mathbb{S}^{n-1}}(u_i, u_j) \geq \arccos\Big(-\frac{1}{n}\Big) = \zeta_{n-1}. \qquad \square$$

Below, the notation $V(T)$ for a triangulation T of the cross-polytope $\widehat{\mathbb{B}}^n$ will denote its set of vertices.

Lemma A.2 *Let T be a triangulation of the cross-polytope $\widehat{\mathbb{B}}^n$ which is antipodally symmetric at the boundary (ie if $\Delta \subset \partial\widehat{\mathbb{B}}^n$ is a simplex in T, then $-\Delta \subset \partial\widehat{\mathbb{B}}^n$ is also in T), and let $g\colon V(T) \to \mathbb{S}^{n-1}$ be a mapping that satisfies $g(-v) = -g(v) \in \mathbb{S}^{n-1}$ for all vertices $v \in V(T)$ lying on the boundary of $\widehat{\mathbb{B}}^n$. Then there exist vertices $u, v \in V(T)$ with $d_{\mathbb{S}^{n-1}}(g(u), g(v)) \geq \zeta_{n-1}$.*

Proof By Lemma A.1 it is enough to show that some simplex $\{v_1, \dots, v_m\}$ of T satisfies

$$0 \in \mathrm{Conv}(g(v_1), g(v_2), \dots, g(v_m)).$$

Suppose not; then one can construct the continuous map $\phi\colon \widehat{\mathbb{B}}^n \to \mathbb{R}^n \setminus \{0\}$ such that $\phi(a_1u_1 + \cdots + a_mu_m) := a_1 g(u_1) + \cdots + a_m g(u_m)$, where $\{u_1, \dots, u_m\}$ is a simplex of T, $a_1, \dots, a_m \in [0, 1]$, and $\sum_{i=1}^m a_i = 1$. Next, one can construct the continuous map $\hat{\phi}\colon \widehat{\mathbb{B}}^n \to \mathbb{S}^{n-1}$ such that $\hat{\phi}(x) := \phi(x)/\|\phi(x)\|$ for each $x \in \widehat{\mathbb{B}}^n$. Moreover, this map $\hat{\phi}$ is antipode-preserving on the boundary since if $x \in \partial\widehat{\mathbb{B}}^n$ satisfies $x = a_1v_1 + \cdots + a_mv_m$ where $\{v_1, \dots, v_m\}$ is a simplex of $\partial\widehat{\mathbb{B}}^n$, then $\phi(x) = a_1 g(v_1) + \cdots + a_m g(v_m)$ and $\phi(-x) = a_1 g(-v_1) + \cdots + a_m g(-v_m)$, so $\phi(-x) = -\phi(x)$. This is a contradiction to the classical Borsuk–Ulam theorem since $\hat{\phi} \circ \alpha^{-1}\colon \mathbb{B}^n \to \mathbb{S}^{n-1}$ is continuous and antipode-preserving on the boundary, where (below, for a vector v we denote by $\|v\|_1$ its 1–norm)

$$\alpha\colon \widehat{\mathbb{B}}^n \to \mathbb{B}^n, \qquad x \mapsto \begin{cases} (0, \dots, 0) & \text{if } x = (0, \dots, 0), \\ x\|x\|_1/\|x\| & \text{otherwise,} \end{cases}$$

is the natural bi-Lipschitz homeomorphism between $\widehat{\mathbb{B}}^n$ and $\mathbb{B}^n$ from the unit cross-polytope to the closed unit ball. $\square$

Now we are ready to prove Theorem G.

Proof of Theorem G Let $f\colon \mathbb{B}^n \to \mathbb{S}^{n-1}$ be a map that is antipode-preserving on the boundary of $\mathbb{B}^n$. Now, fix an arbitrary $\delta \geq 0$ such that for any $x \in \mathbb{B}^n$ there exists an open neighborhood U_x of x with $\operatorname{diam}(f(U_x)) \le \delta$. Fix $\varepsilon > 0$ smaller than the Lebesgue number of the open covering $\{U_x\}_{x \in \mathbb{B}^n}$.

Let $\alpha\colon \widehat{\mathbb{B}}^n \to \mathbb{B}^n$ be the natural (fattening) homeomorphism used in the proof of Lemma A.2. One can construct a triangulation T of $\widehat{\mathbb{B}}^n$ satisfying the following two properties:

(1) T is antipodally symmetric on the boundary of $\widehat{\mathbb{B}}^n$.

(2) T is fine enough that $\|\alpha(u) - \alpha(v)\| \le \varepsilon$ for any two adjacent vertices u and v.

Then, by Lemma A.2, there exist adjacent vertices u and v such that

$$d_{\mathbb{S}^{n-1}}(f \circ \alpha(u), f \circ \alpha(v)) \ge \zeta_{n-1}.$$

Choose $x = \alpha(u)$ and $y = \alpha(v)$. Because of the choice of ε, both x and y are contained in some U_x. Hence, $\delta \ge \operatorname{diam}(f(U_x)) \ge \zeta_{n-1}$, which concludes as in the proof of Corollary 5.4. □

Appendix B The Gromov–Hausdorff distance between a sphere and an interval

To make this paper self-contained, we include a proof of the following proposition.

Proposition B.1 *Let n be any positive integer. Then $\operatorname{dis}(f) \ge \frac{2\pi}{3}$ for any function $f\colon \mathbb{S}^n \to \mathbb{R}$.*

As a consequence, $d_{\mathrm{GH}}(\mathbb{S}^n, I) \ge \frac{\pi}{3}$ for any interval $I \subseteq \mathbb{R}$.

Proof Note that it is enough to prove the claim for $n = 1$. We adapt an argument from the proof of [20, Lemma 2.3].

Fix an arbitrary $\varepsilon > 0$. Consider an antipodally symmetric triangulation of $\mathbb{S}^1$ with vertex set $V \subset \mathbb{S}^1$ such that $d_{\mathbb{S}^1}(p,q) \le \varepsilon$ for any two adjacent vertices $p, q \in V$. Then let $\tilde{f}\colon \mathbb{S}^1 \to I$ be the linear interpolation of $f|_V\colon V \to I$. Now, by the classical Borsuk–Ulam theorem, there exists $x \in \mathbb{S}^1$ such that $\tilde{f}(x) = \tilde{f}(-x)$. Let $p, q \in V$ be such that $x \in \widehat{pq}$. Then $I \cap J \ne \varnothing$ where I is the closed interval between $f(p)$ and $f(q)$, and J is the closed interval between $f(-p)$ and $f(-q)$ (since I and J both contain $\tilde{f}(x) = \tilde{f}(-x)$). Without loss of generality, we can assume that $f(-p) \in I$. Now, let

$$r := \begin{cases} p & \text{if } |f(-p) - f(p)| \le |f(-p) - f(q)|, \\ q & \text{if } |f(-p) - f(p)| > |f(-p) - f(q)|. \end{cases}$$

Then $|f(-p)-f(r)| \le \frac{1}{2}\operatorname{length}(I) \le \frac{1}{2}(\operatorname{dis}(f)+\varepsilon)$. Hence,

$$\pi-\varepsilon \le d_{\mathbb{S}^1}(-p.r) \le \operatorname{dis}(f)+|f(-p)-f(r)| \le \tfrac{3}{2}\operatorname{dis}(f)+\tfrac{1}{2}\varepsilon,$$

so $\operatorname{dis}(f) \ge \frac{2\pi}{3}-\varepsilon$. □

Appendix C Regular polygons and $\mathbb{S}^1$

In this appendix we compute the distance between regular polygons and also between the circle and a regular polygon.

The following map from metric spaces to metric spaces will be useful. For a metric space (X, d_X), consider the pseudo-ultrametric space (X, u_X) where $u_X\colon X\times X\to\mathbb{R}$ is defined by

$$(x,x')\mapsto u_X(x,x') := \inf\Big\{\max_{0\le i\le n-1} d_X(x_i,x_{i+1}) \,\Big|\, x=x_0,\dots,x_n=x' \text{ for some } n\ge 1\Big\}.$$

Now, define $\boldsymbol{U}(X)$ to be the quotient metric space induced by (X,u_X) under the equivalence $x\sim x'$ if and only if $u_X(x,x')=0$. One then has the following, whose proof we omit:

Proposition C.1 *For any path-connected metric space X it holds that $\boldsymbol{U}(X)=*$.*

We also have the following result, establishing that $\boldsymbol{U}\colon \mathcal{M}_b\to\mathcal{M}_b$ is 1–Lipschitz:

Theorem H [7] *For all bounded metric spaces X and Y,*

$$d_{\mathrm{GH}}(X,Y) \ge d_{\mathrm{GH}}(\boldsymbol{U}(X),\boldsymbol{U}(Y)).$$

For each integer $n\ge 3$, let P_n be the regular polygon with n vertices inscribed in $\mathbb{S}^1$. We also let $P_2=\mathbb{S}^0$. Furthermore, we endow P_n with the restriction of the geodesic distance on $\mathbb{S}^1$. We then have:

Proposition C.2 (d_{GH} between $\mathbb{S}^1$ and inscribed regular polygons) *For all $n\ge 2$,*

$$d_{\mathrm{GH}}(\mathbb{S}^1,P_n)=\frac{\pi}{n}.$$

Proof That $d_{\mathrm{GH}}(\mathbb{S}^1,P_n)\ge \pi/n$ can be obtained as follows: by Theorem H,

$$d_{\mathrm{GH}}(\mathbb{S}^1,P_n)\ge d_{\mathrm{GH}}(\boldsymbol{U}(\mathbb{S}^1),\boldsymbol{U}(P_n)),$$

but, since $\boldsymbol{U}(\mathbb{S}^1) = *$ by Proposition C.1, and $\boldsymbol{U}(P_n)$ is isometric to the metric space over n points with all nonzero pairwise distances equal to $2\pi/n$, from the above inequality and (7) we have $d_{\mathrm{GH}}(\mathbb{S}^1, P_n) \geq \frac{1}{2}\operatorname{diam}(\boldsymbol{U}(P_n)) = \pi/n$. The inequality $d_{\mathrm{GH}}(\mathbb{S}^1, P_n) \leq \pi/n$ follows from the fact that $d_{\mathrm{GH}}(\mathbb{S}^1, P_n) \leq d_{\mathrm{H}}(\mathbb{S}^1, P_n) = \pi/n$. □

Note that, if $\mathbb{S}^1$ and P_n are both endowed with the Euclidean distance (respectively denoted by $\mathbb{S}^1_{\mathrm{E}}$ and $(P_n)_{\mathrm{E}}$), then, in analogy with Proposition C.2, we have the following proposition which solves a question posed in [1]. The proof is slightly different from that of Proposition C.2.

Proposition C.3 *For all* $n \geq 2$, $d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, (P_n)_{\mathrm{E}}) = \sin(\pi/n)$.

Proof One can prove $d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, (P_n)_{\mathrm{E}}) \geq \sin(\pi/n)$ by invoking $\boldsymbol{U}$ as in the proof of Proposition C.2. In order to prove $d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, (P_n)_{\mathrm{E}}) \leq \sin(\pi/n)$, let us construct a specific correspondence R between $\mathbb{S}^1_{\mathrm{E}}$ and $(P_n)_{\mathrm{E}}$. Let $u_1, \dots, u_n$ be the vertices of $(P_n)_{\mathrm{E}}$, and $V_1, \dots, V_n$ be the Voronoi regions of $\mathbb{S}^1$ induced by $u_1, \dots, u_n$. Now let

$$R := \bigcup_{i=1}^{n} V_i \times \{u_i\}.$$

Then we claim $\operatorname{dis}_{\mathrm{E}}(R) \leq 2\sin(\pi/n)$. To prove this, it is enough to check the following two conditions via standard trigonometric identities:

(1) $2\sin(k\pi/n) - 2\sin((k-1)\pi/n) \leq 2\sin(\pi/n)$ for $1 \leq k \leq \lfloor \frac{1}{2}n \rfloor$.

(2) $2 - 2\sin\bigl(\lfloor \frac{1}{2}n \rfloor \pi/n\bigr) \leq 2\sin(\pi/n)$.

Hence, $d_{\mathrm{GH}}(\mathbb{S}^1_{\mathrm{E}}, (P_n)_{\mathrm{E}}) \leq \sin(\pi/n)$. □

We now pose the following question and provide partial information about it in Proposition C.5:

Question VI *Determine, for all* $m, n \in \mathbb{N}$, *the value of* $\mathfrak{p}_{m,n} := d_{\mathrm{GH}}(P_m, P_n)$.

Remark C.4 By simple arguments, which we omit, one can prove that $\mathfrak{p}_{2,3} = \frac{\pi}{3}$, $\mathfrak{p}_{2,4} = \frac{\pi}{4}$, $\mathfrak{p}_{2,5} = \frac{2\pi}{5}$ and $\mathfrak{p}_{2,6} = \frac{\pi}{3}$. Also Proposition C.2 indicates that $\mathfrak{p}_{2,n}$ tends to $\frac{\pi}{2}$ as $n \to \infty$. Then these calculations imply that $n \mapsto \mathfrak{p}_{2,n}$ is *not* monotonically increasing towards $\frac{\pi}{2}$; cf Question I.

Proposition C.5 *For any integer* $0 < m < \infty$, $\mathfrak{p}_{m,m+1} = \pi/(m+1)$.

Proof First, let us prove that $\mathfrak{p}_{m,m+1} \le \pi/(m+1)$. We construct a correspondence R between P_m and P_{m+1} such that $\operatorname{dis}(R) \le 2\pi/(m+1)$. Let $u_1,\dots,u_m$ be the vertices of P_m and $v_1,\dots,v_m,v_{m+1}$ be the vertices of P_{m+1}. Consider the correspondence

$$R := \bigcup_{i=1}^{m} \{(u_m, v_m)\} \cup \{(u_m, v_{m+1})\}.$$

Then, for any $i, j \in \{1,\dots,m\}$,

$$\begin{aligned}
&|d_{\mathbb{S}^1}(u_i,u_j) - d_{\mathbb{S}^1}(v_i,v_j)| \\
&\qquad = \left| \frac{2\pi}{m}\cdot \min\{|i-j|, m-|i-j|\} - \frac{2\pi}{m+1}\cdot\min\{|i-j|, m+1-|i-j|\} \right| \\
&\qquad = \left| \frac{2\pi k}{m} - \frac{2\pi k}{m+1} \right| \text{ or } \left| \frac{2\pi k}{m} - \frac{2\pi(k+1)}{m+1} \right| \qquad \left(\text{for some } 0 \le k \le \left\lfloor \tfrac{1}{2} m \right\rfloor\right) \\
&\qquad = \frac{2\pi k}{m(m+1)} \text{ or } \frac{2\pi}{m+1}\left(1 - \frac{k}{m}\right) \qquad \left(\text{for some } 0 \le k \le \left\lfloor \tfrac{1}{2} m \right\rfloor\right) \\
&\qquad \le \frac{2\pi}{m+1}.
\end{aligned}$$

Also, for any $i \in \{1,\dots,m\}$,

$$\begin{aligned}
&|d_{\mathbb{S}^1}(u_i,u_m) - d_{\mathbb{S}^1}(v_i,v_{m+1})| \\
&\qquad = \left| \frac{2\pi}{m}\cdot \min\{m-i, i\} - \frac{2\pi}{m+1}\cdot\min\{m+1-i, i\} \right| \\
&\qquad = \left| \frac{2\pi k}{m} - \frac{2\pi k}{m+1} \right| \text{ or } \left| \frac{2\pi k}{m} - \frac{2\pi(k+1)}{m+1} \right| \qquad \left(\text{for some } 0 \le k \le \left\lfloor \tfrac{1}{2} m \right\rfloor\right) \\
&\qquad = \frac{2\pi k}{m(m+1)} \text{ or } \frac{2\pi}{m+1}\left(1 - \frac{k}{m}\right) \qquad \left(\text{for some } 0 \le k \le \left\lfloor \tfrac{1}{2} m \right\rfloor\right) \\
&\qquad \le \frac{2\pi}{m+1}.
\end{aligned}$$

Hence, one concludes that $\operatorname{dis}(R) \le 2\pi/(m+1)$.

Next, let us prove that $\mathfrak{p}_{m,m+1} \ge \pi/(m+1)$. Fix an arbitrary correspondence R between P_m and P_{m+1}. Then there must be a vertex u_i of P_m, and two vertices v_j and v_k of P_{m+1} such that $(u_i,v_j),(u_i,v_k) \in R$. Hence,

$$\operatorname{dis}(R) \ge |d_{\mathbb{S}^1}(u_i,u_i) - d_{\mathbb{S}^1}(v_j,v_k)| = \frac{2\pi}{m+1}.$$

Since R is arbitrary, one concludes that $\mathfrak{p}_{m,m+1} \ge \pi/(m+1)$. ◻

References

[1] **H Adams**, **S Chowdhury**, **A Q Jaffe**, **B Sibanda**, *Vietoris–Rips complexes of regular polygons*, preprint (2018) arXiv 1807.10971

[2] **I Bogdanov**, *Gromov–Hausdorff distance between a disk and a circle*, MathOverflow answer (2018) Available at `https://mathoverflow.net/a/312046/160011`

[3] **A M Bronstein**, **M M Bronstein**, **R Kimmel**, *Numerical geometry of non-rigid shapes*, Springer (2008) MR Zbl

[4] **D Burago**, **Y Burago**, **S Ivanov**, *A course in metric geometry*, Graduate Studies in Mathematics 33, Amer. Math. Soc., Providence, RI (2001) MR Zbl

[5] **G Carlsson**, *Topology and data*, Bull. Amer. Math. Soc. 46 (2009) 255–308 MR Zbl

[6] **G Carlsson**, *Topological pattern recognition for point cloud data*, Acta Numer. 23 (2014) 289–368 MR Zbl

[7] **G Carlsson**, **F Mémoli**, *Characterization, stability and convergence of hierarchical clustering methods*, J. Mach. Learn. Res. 11 (2010) 1425–1470 MR Zbl

[8] **M Cho**, *On the optimal covering of equal metric balls in a sphere*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 4 (1997) 137–144 MR

[9] **S Chowdhury**, **F Mémoli**, *Explicit geodesics in Gromov–Hausdorff space*, Electron. Res. Announc. Math. Sci. 25 (2018) 48–59 MR Zbl

[10] **T H Colding**, *Large manifolds with positive Ricci curvature*, Invent. Math. 124 (1996) 193–214 MR Zbl

[11] **L E Dubins**, **G Schwarz**, *Equidiscontinuity of Borsuk–Ulam functions*, Pacific J. Math. 95 (1981) 51–59 MR Zbl

[12] **D A Edwards**, *The structure of superspace*, from "Studies in topology" (N M Stavrakas, K R Allen, editors), Academic, New York (1975) 121–133 MR Zbl

[13] **G B Folland**, *Real analysis: modern techniques and their applications*, 2nd edition, John Wiley and Sons, New York (1999) MR Zbl

[14] **K Funano**, *Estimates of Gromov's box distance*, Proc. Amer. Math. Soc. 136 (2008) 2911–2920 MR Zbl

[15] **A Gray**, *Tubes*, 2nd edition, Progr. Math. 221, Birkhäuser, Basel (2004) MR Zbl

[16] **M Gromov**, *Metric structures for Riemannian and non-Riemannian spaces*, Progr. Math. 152, Birkhäuser, Boston, MA (1999) MR Zbl

[17] **Y Ji**, **A A Tuzhilin**, *Gromov–Hausdorff distance between interval and circle*, Topology Appl. 307 (2022) art. id. 107938 MR Zbl

[18] **N J Kalton**, **M I Ostrovskii**, *Distances between Banach spaces*, Forum Math. 11 (1999) 17–48 MR Zbl

[19] **M Katz**, *The filling radius of two-point homogeneous spaces*, J. Differential Geom. 18 (1983) 505–511 MR Zbl

[20] **M G Katz**, *Torus cannot collapse to a segment*, J. Geom. 111 (2020) art. id. 13 MR Zbl

[21] **S Lim**, **F Mémoli**, **O B Okutan**, *Vietoris–Rips persistent homology, injective metric spaces, and the filling radius*, preprint (2020) arXiv 2001.07588

[22] **S Lim**, **F Mémoli**, **Z Smith**, GitHub repository Available at `https://github.com/ndag/dgh-spheres`

[23] **S Lim**, **F Mémoli**, **Z Smith**, *The Gromov–Hausdorff distance between spheres*, preprint (2021) arXiv 2105.00611

[24] **J Matoušek**, *Using the Borsuk–Ulam theorem: lectures on topological methods in combinatorics and geometry*, Springer (2003) MR Zbl

[25] **F Mémoli**, *Gromov–Wasserstein distances and the metric approach to object matching*, Found. Comput. Math. 11 (2011) 417–487 MR Zbl

[26] **F Mémoli**, *Some properties of Gromov–Hausdorff distances*, Discrete Comput. Geom. 48 (2012) 416–440 MR Zbl

[27] **F Mémoli**, **G Sapiro**, *A theoretical and computational framework for isometry invariant recognition of point cloud data*, Found. Comput. Math. 5 (2005) 313–347 MR Zbl

[28] **H J Munkholm**, *A Borsuk–Ulam theorem for maps from a sphere to a compact topological manifold*, Illinois J. Math. 13 (1969) 116–124 MR Zbl

[29] **G Peano**, *Sur une courbe, qui remplit toute une aire plane*, Math. Ann. 36 (1890) 157–160 MR Zbl

[30] **P Petersen**, *Riemannian geometry*, Graduate Texts in Math. 171, Springer (1998) MR Zbl

[31] **L A Santaló**, *Convex regions on the n–dimensional spherical surface*, Ann. of Math. 47 (1946) 448–459 MR Zbl

Max Planck Institute for Mathematics in the Sciences
Leipzig, Germany

Department of Mathematics, The Ohio State University
Columbus, OH, United States

Department of Computer Science, University of Minnesota
Minneapolis, MN, United States

`sulim@mis.mpg.de`, `facundo.memoli@gmail.com`, `smit9474@umn.edu`

`https://sites.google.com/view/sunhyuklim`, `http://facundo-memoli.org`

Proposed: Dmitri Burago
Seconded: Dan Abramovich, David Fisher

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