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Algebraic uniqueness of Kähler–Ricci flow limits and optimal degenerations of Fano varieties

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We prove that for any Fano manifold X, the special $\mathbb{R}$–test configuration that minimizes the $\boldsymbol{H}^{\mathrm{NA}}$–functional is unique and has a K–semistable $\mathbb{Q}$–Fano central fiber (W, ξ). Moreover there is a unique K–polystable degeneration of (W, ξ). As an application, we confirm the conjecture of Chen, Sun and Wang about the algebraic uniqueness for Kähler–Ricci flow limits on Fano manifolds, which implies that the Gromov–Hausdorff limit of the flow does not depend on the choice of initial Kähler metrics. The results are achieved by studying algebraic optimal degeneration problems via new functionals for real valuations over $\mathbb{Q}$–Fano varieties, which are analogous to the minimization problem for normalized volumes.

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1 Introduction

Let X be a smooth Fano manifold. It is now known that X admits a Kähler–Einstein metric if and only if X is K–polystable; see Berman [5], Chen, Donaldson and Sun [25; 26; 27] and Tian [67; 68]. In this paper, we are interested in the case when X is not K–polystable. If X is strictly K–semistable, then X admits a unique K–polystable degeneration by Li, Wang and Xu [55]. If X is K–unstable (ie not K–semistable), several kinds of optimal degenerations have been studied which are related to continuity methods or geometric flows in the analytic study of canonical metrics. For example, related to Aubin's continuity method, there is a (not necessarily unique) special degeneration whose associated valuation minimizes the δ invariant; see Blum, Liu and Zhou [16] and Székelyhidi [65]. There is also a unique destabilizing geodesic ray which arises in the study of inverse Monge–Ampère flow (resp. Calabi flow) and whose associated non-Archimedean metric minimizes an L^2–normalized non-Archimedean Ding invariant (resp. L^2–normalized radial Calabi functional); see Donaldson [33], Hisamoto [42] and Xia [77]. In this paper we are interested in optimal degenerations that arise in the study of Hamilton–Tian conjecture about the long time behavior of Kähler–Ricci flows. The latter conjecture states that starting from any Kähler metric $\omega \in c_1(X)$, the normalized Kähler–Ricci converges in the Gromov–Hausdorff sense to a Kähler–Ricci soliton on a $\mathbb{Q}$–Fano variety X_∞. The Hamilton–Tian conjecture has been solved (see Bamler [4], Chen and Wang [29] and Tian and Zhang [70]) and applied to give a proof of the Yau–Tian–Donaldson conjecture in Chen, Sun and Wang [28].

It is known that X_∞ coincides with X if and only if there is already a Kähler–Ricci soliton on X; see Dervan and Székelyhidi [31] and Tian and Zhu [72]. In general, Chen, Sun and Wang [28] proved the following phenomenon. The metric degeneration from X to X_∞ induces a finitely generated filtration $\mathcal{F}$ on $R = \bigoplus_m H^0(X, -mK_X)$, and there is a two-step degeneration:

(i) The filtration $\mathcal{F}$ as an $\mathbb{R}$–test configuration (see Definition 2.8) degenerates X to a normal Fano variety W with a torus $\mathbb{T}$–action generated by a holomorphic vector field ξ. For simplicity, we call this step the semistable degeneration.

(ii) There is a $\mathbb{T}$–equivariant test configuration of (W, ξ) to (X_∞, ξ). We call this step the polystable degeneration.

As explained in Chen, Sun and Wang [28], this picture is a global analogue of the picture in Donaldson and Sun's study [34] of metric tangent cones on Gromov–Hausdorff limits of Fano Kähler–Einstein manifolds. In [34], Donaldson and Sun conjectured that metric tangent cones depend only on the algebraic structure near the singularity. This conjecture has been confirmed in a series of works of the second author with his collaborators (see Li [51], Li and Xu [58; 57] and Li, Wang and Xu [55]), which depends on the study of the minimization problem of a normalized volume functional over the space of valuations centered at the singularity; see Li, Liu and Xu [54] for a survey. Analogous to this conjecture on metric tangent cones, the following conjecture was proposed in [28]:

Conjecture 1.1 *The data $\mathcal{F}$, W and X_∞ depend only on the algebraic structure of X but not on the initial metric for the Kähler–Ricci flow.*

In this paper we will confirm Conjecture 1.1. The idea and method to prove this conjecture are in some sense parallel to the study of minimizing normalized volumes. However, the correct framework for achieving this goal has not been established until now. So the second purpose of this paper is to study an analogous minimization problem in the global setting, which can be studied for all $\mathbb{Q}$–Fano varieties possibly singular, and prove various results about it.

The functional we want to minimize is called the $\boldsymbol{H}^{\mathrm{NA}}$–functional of $\mathbb{R}$–test configurations.[1] Tian, Zhang, Zhang and Zhu [69, Proposition 5.1] first introduced the $\boldsymbol{H}^{\mathrm{NA}}$–functional for holomorphic vector fields in their study of Kähler–Ricci flow on Fano manifolds. This invariant was generalized to any *special* $\mathbb{R}$–test configuration by Dervan and Székelyhidi [31], who then used the results of Chen and Wang [29], Chen, Sun and Wang [28] and He [40] to prove that the semistable degeneration mentioned above minimizes the $\boldsymbol{H}^{\mathrm{NA}}$–functional among all special $\mathbb{R}$–test configurations; see Remark 2.44. For general test configurations, such an $\boldsymbol{H}^{\mathrm{NA}}$–functional is a nonlinear version of the non-Archimedean Berman–Ding functional, and was first explicitly used by Hisamoto in [43] to reprove Dervan and Székelyhidi's result using pluripotential theory. Note that in this paper, for the convenience of our argument and comparison with the case of the δ invariant (or with the β invariant, see equation (107)), we will use the negative of the sign convention in these previous works.

Conjecture 1.1 follows from two purely algebrogeometric statements for each step of the semistable and polystable degenerations.

Theorem 1.2 *For any $\mathbb{Q}$–Fano variety, the special $\mathbb{R}$–test configuration that minimizes $\boldsymbol{H}^{\mathrm{NA}}$ is unique and its central fiber (W, ξ) is K–semistable* (Definition 2.49).

Theorem 1.3 *If (X, ξ) is K–semistable, then there exists a unique K–polystable degeneration.*

Corollary 1.4 *Conjecture 1.1 is true for any smooth Fano manifold. In particular, the Gromov–Hausdorff limit X_∞ for the Kähler–Ricci flow does not depend on the initial metric of the flow.*

To prepare for the proof of such results, we will first carry out an algebraic study of the $\boldsymbol{H}^{\mathrm{NA}}$–functional, which is analogous to the study of the minimization problem for normalized volume or the δ invariant. We will prove a new interesting fact in Theorem 3.5, that the MMP process devised in [56] decreases the $\boldsymbol{H}^{\mathrm{NA}}$ invariant of test configurations. This requires us to derive new intersection formulas (see (121)) and derivative formulas for the $\boldsymbol{H}^{\mathrm{NA}}$ invariant. The proof of such formulas depends on a fibration technique in the study of equivariant cohomology. This technique is partly motivated by some construction from our previous work [39], although there are key differences which require more concrete calculations; see Remark 3.2.

We will then introduce the following $\tilde{\beta}$–functional on $\mathrm{Val}(X)$ the space of valuations on X: for any $v \in \mathrm{Val}(X)$ with $A_X(v) < +\infty$, we define

$$\tilde{\beta}(v) = A_X(v) + \log\left(\frac{1}{(-K_X)^{\cdot n}} \int_{\mathbb{R}} e^{-\lambda}(-d\,\mathrm{vol}(\mathcal{F}_v^{(\lambda)}))\right). \tag{1}$$

[1]We will mostly use the notation of non-Archimedean functionals, as advocated in Boucksom, Hisamoto and Jonsson [19]. However, note that $\boldsymbol{H}^{\mathrm{NA}}$ here is not the non-Archimedean entropy functional used in [19]. We will not use the non-Archimedean entropy in this article.

See Section 4 for details. If $A_X(v)=+\infty$, then we define $\widetilde{\beta}(v)=+\infty$. The transition from $\boldsymbol{H}^{\mathrm{NA}}$ to the $\widetilde{\beta}$–functional is similar to the transition from Ding–Tian's generalized Futaki invariant to the β–functional in the literature of K–stability (as first appeared in Li [50], where it was called Θ, and in Fujita [37]). In other words, $\widetilde{\beta}$ is a nonlinear version of β and could also be considered as a global analogue of the normalized volume. Unlike the case of normalized volume functional, the $\widetilde{\beta}$ invariant is not invariant under rescaling of valuations. Indeed, we find the following new phenomenon: when restricted to the ray of multiples of a fixed valuation $v\in\mathrm{Val}(X)$ with $A(v)<+\infty$, it is strictly convex and proper and its derivative at the origin is exactly $\beta(v)$. As a consequence there is a unique minimizer along the ray, which is nontrivial if and only if $\beta(v)<0$ (Proposition 4.6). The above MMP result implies that the minimum can be approached by a sequence of special divisorial valuations. As a consequence, one can adapt the method developed in [14] to show that there is minimizing valuation which is quasimonomial; see Theorem 4.10. On the other hand, the $\boldsymbol{H}^{\mathrm{NA}}$ invariant for special test configurations is expressed as the $\widetilde{\beta}$ invariant; see Lemma 4.2. Combining these discussions, we will prove (see Sections 2.2 and 2.5 for relevant notation):

Theorem 1.5 *For any $\mathbb{Q}$–Fano variety X, we have the identity*

$$\inf_{\mathcal{F}\text{ filtration}} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F})=\inf_{(\mathcal{X},\mathcal{L},a\eta)\text{ special}}(\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a\eta))=\inf_{v\in\mathrm{Val}(X)}\widetilde{\beta}(v). \tag{2}$$

Moreover, the last infimum is achieved by a quasimonomial valuation.

As in the cases of normalized volume, we conjecture that the minimizer is unique and induces a special $\mathbb{R}$–test configuration (see Conjecture 4.11)[2] whose central fiber (with the induced vector field) must then be K–semistable by the following result. When X is smooth, by the result of Dervan and Székelyhidi [31] the existence of such special minimizing valuation is implied by the work of Chen and Wang [29] and Chen, Sun and Wang [28]. We also note that optimal degenerations (of various kinds) in the toric case are well studied; see Wang and Zhu [75] for the toric result for Kähler–Ricci flow.

Theorem 1.6 (Theorem 5.2) *A special $\mathbb{R}$–test configuration minimizes $\boldsymbol{H}^{\mathrm{NA}}$ if and only if its central fiber is K–semistable.*

The uniqueness in Theorem 1.2 about the semistable degeneration is nothing but the result on the uniqueness of the minimizer of $\widetilde{\beta}$ among all quasimonomial valuations associated to special $\mathbb{R}$–test configurations. The proof of this fact uses the technique of initial term degeneration, again motivated by study of normalized volumes; see Li [50] and Li and Xu [58; 57]. This process essentially reduces the question to the uniqueness of minimizer of $\boldsymbol{H}^{\mathrm{NA}}$ (actually a variant of $\boldsymbol{H}^{\mathrm{NA}}$ after the work of Xu and Zhuang [79]) along an interpolation between a fixed filtration and a weight filtration (induced by a holomorphic vector field) on the central fiber. The interpolation is constructed by using the rescaling of

[2] This has recently been confirmed by Blum, Liu, Xu and Zhuang [15].

twist of the fixed filtration (the twist of filtration is in the sense of Li [52] generalizing Hisamoto [41]), which we can deal with using the technique of Newton–Okounkov bodies and Boucksom and Jonsson's work on the characterization of asymptotically equivalent filtrations. Our valuative formulation is useful because filtrations associated to valuations are asymptotically equivalent if and only if they are the same (Corollary 2.28 and Lemma 2.29). Again unlike the case of normalized volumes or the case of the δ invariant in Blum, Liu and Zhou [16], the minimizing valuation in the current global setting is expected to be absolutely unique, not just up to rescaling or twisting. This is because of a strict convex property of the $\boldsymbol{H}^{\mathrm{NA}}$–functional, which goes back to Tian and Zhu's work in [71] on the uniqueness of Kähler–Ricci vector fields from the Lie algebra of a torus.

To deal with the polystable step, we first introduce the equivariant version of normalized volumes. Most results about normalized volumes can be generalized for the equivariant version. Finally we complete the proof of Theorem 1.3 by adapting the argument in Li, Wang and Xu [55] about uniqueness of K–polystable degeneration of K–semistable Fano varieties.

To end this introduction, the following table summarizes the quantities used in each of the two steps:

degenerations	semistable	polystable
valuations	$\mathrm{Val}(X)$	$\mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}}$
antiderivative	$\boldsymbol{H}^{\mathrm{NA}}$, $\widetilde{\beta}$	$\widehat{\mathrm{vol}}_g$
derivative	$\boldsymbol{D}_\xi^{\mathrm{NA}}$, Fut_ξ	$\boldsymbol{D}_\xi^{\mathrm{NA}}$, β_g
derivative formula	(172)	(191)

Postscript After we finished the paper, we were informed by F Wang and X Zhu that they use analytic methods to prove related uniqueness results for Kähler–Ricci flow limits based on their recent work on the Hamilton–Tian conjecture; see [73; 74].

After this paper appeared, there have been several important developments. In Li and Li [59], based on the minimization setup and uniqueness results in this paper, the authors calculated nontrivial examples of limits of Kähler–Ricci flows on some unstable Fano varieties which are compactifications of homogeneous varieties under some complex reductive group. Very recently, the paper of Blum, Liu, Xu and Zhuang [15] continues and completes the algebraic study of minimization problem proposed in this paper, based on the recent breakthrough on the high-rank finite generation conjecture. In particular, Conjecture 4.11 in our paper is now confirmed in [15].

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2 Preliminaries

2.1 Some notation

Let X be a $\mathbb{Q}$–Fano variety. In this paper for the simplicity of notation, we assume that $-K_X$ is Cartier. The modification to the general $\mathbb{Q}$–Cartier case is straightforward; see eg [52]. For any $m, \ell \in \mathbb{N}$, set

$$R_m := H^0(X, -mK_X), \qquad R = \bigoplus_{m=0}^{+\infty} R_m, \tag{3}$$

$$N_m = \dim R_m, \qquad V = (-K_X)^{\cdot n} = \lim_{m\to+\infty} \frac{N_m}{m^n/n!}, \tag{4}$$

$$R_m^{(\ell)} := H^0(X, -m\ell K_X), \quad R^{(\ell)} = \bigoplus_{m=0}^{+\infty} R_m^{(\ell)}. \tag{5}$$

We will denote by $\mathrm{Val}(X)$ the space of real valuations on $\mathbb{C}(X)$, by $\mathring{\mathrm{Val}}(X)$ the set of real valuations v with $A_X(v) < +\infty$, and by $X_{\mathbb{Q}}^{\mathrm{div}}$ the set of divisorial valuations, ie the valuations of the form $a \cdot \mathrm{ord}_E$ with $a \geq 0$ and E a prime divisor over X. A valuation $v \in \mathrm{Val}(X)$ is quasimonomial if there exist a birational morphism $Y \to X$ and a simple normal crossing divisors $E = \bigcup_{i=1}^d E_i \subset Y$ such that v is a monomial valuation on Y with respect to the local coordinates defining E_i, whose center of v over Y is an irreducible component of $\bigcap_{i\in J} E_i$, where $J \subseteq \{1, \dots, d\}$ is a subset. We denote by $\mathrm{QM}(Y, E)$ the set of such quasimonomial valuations. We refer to [35; 44] for more details about such quasimonomial (or equivalently the Abhyankar) valuations.

In this paper, $\mathbb{T}$ denotes a complex torus $(\mathbb{C}^*)^r = ((S^1)^r)_{\mathbb{C}}$ that acts effectively on a $\mathbb{Q}$–Fano variety X. There is a canonical action of $\mathbb{T}$ on (any multiple of) $-K_X$. Set

$$N_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}), \quad N_{\mathbb{R}} = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*), \quad M_{\mathbb{R}} = M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{6}$$

For any $\xi \in N_{\mathbb{R}}$, we have a valuation $\mathrm{wt}_\xi \in \mathrm{Val}(X)$ as follows. For any $f \in \mathbb{C}(X) = \bigoplus_{\alpha\in M_{\mathbb{Z}}} \mathbb{C}(X)_\alpha$,

$$\mathrm{wt}_\xi(f) = \min\left\{ \langle \alpha, \xi \rangle \;\middle|\; f = \sum_\alpha f_\alpha, f_\alpha \neq 0 \right\}. \tag{7}$$

Moreover, for any $m \in \mathbb{N}$, we have a weight decomposition induced by the canonical $\mathbb{T}$–action on $(X, -mK_X)$:

$$R_m = \bigoplus_{\alpha\in M_{\mathbb{Z}}} (R_m)_\alpha = (R_m)_{\alpha_1^{(m)}} \oplus \cdots \oplus (R_m)_{\alpha_{N_m}^{(m)}}. \tag{8}$$

Moreover, we will use the following notation for any $\mathbb{Q}$–Fano variety. Let $e^{-\tilde{\varphi}}$ be an $(S^1)^r$–invariant smooth positively curved Hermitian metric on $-K_X$ (eg as the restriction of a Fubini–Study metric under an equivariant embedding of X into projective space). We identify any $\eta \in N_{\mathbb{R}}$ with the corresponding holomorphic vector field on X. Because $\mathbb{T}$–action canonically lifts to an action on $-K_X$, we can set

$$\theta_{\widetilde{\varphi}}(\eta) = -\frac{\mathfrak{L}_\eta e^{-\widetilde{\varphi}}}{e^{-\widetilde{\varphi}}}. \tag{9}$$

Then $\theta_{\widetilde{\varphi}}(\eta)$ is a Hamiltonian function of η with respect to $\mathrm{dd^c}\widetilde{\varphi} = (\sqrt{-1}/2\pi)\partial\bar{\partial}\widetilde{\varphi} \geq 0$:

$$\iota_\eta \mathrm{dd^c}\widetilde{\varphi} = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_{\widetilde{\varphi}}(\eta). \tag{10}$$

Moreover, $(z,\eta) \mapsto \theta_{\widetilde{\varphi}}(\eta)(z)$ is equivalent to the moment map $\boldsymbol{m}_{\widetilde{\varphi}} \colon X \to M_{\mathbb{R}}$ whose image is the moment polytope P of $\mathbb{T}$–action on $(X, -K_X)$ which does not depend on the choice of $\widetilde{\varphi}$. It is known that the measure

$$\frac{n!}{m^n}\sum_i \dim(R_m)_{\alpha_i^{(m)}} \cdot \delta_{\alpha_i^{(m)}/m} \tag{11}$$

converges weakly to the Duistermaat–Heckman measure $(\boldsymbol{m}_{\widetilde{\varphi}})_*(\mathrm{dd^c}\widetilde{\varphi})^n$; see [23] or [8, Proposition 4.1].

For any subset $S \subseteq \mathbb{R}^n$, we will use dy_S or just dy to denote the Lebesgue measure of S.

2.2 $\mathbb{R}$–test configuration and filtrations

We will use extensively the language of filtrations:

Definition 2.1 [17] A filtration $\mathcal{F} := \mathcal{F}R_\bullet$ of the graded $\mathbb{C}$–algebra $R = \bigoplus_{m=0}^{+\infty} R_m$ consists of a family of subspaces $\{\mathcal{F}^\lambda R_m\}_x$ of R_m for each $m \geq 0$ with the following properties:

- **Decreasing** $\mathcal{F}^\lambda R_m \subseteq \mathcal{F}^{\lambda'} R_m$ if $\lambda \geq \lambda'$.
- **Left continuous** $\mathcal{F}^\lambda R_m = \bigcap_{\lambda'<\lambda} \mathcal{F}^{\lambda'} R_m$.
- **Multiplicative** $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subseteq \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$ for any $\lambda, \lambda' \in \mathbb{R}$ and $m, m' \in \mathbb{Z}_{\geq 0}$.
- **Linearly bounded** There exist $e_-, e_+ \in \mathbb{Z}$ such that $\mathcal{F}^{me_-} R_m = R_m$ and $\mathcal{F}^{me_+} R_m = 0$ for all $m \in \mathbb{Z}_{\geq 0}$.

Similarly one defines filtration on $R^{(\ell)}$ for any $\ell \geq 1 \in \mathbb{N}$.

Example 2.2 Given any valuation $v \in \mathring{\mathrm{Val}}(X)$, we have an associated filtration $\mathcal{F} = \mathcal{F}_v$:

$$\mathcal{F}_v^\lambda R_m := \{s \in R_m \mid v(s) \geq \lambda\}. \tag{12}$$

In particular, if there is a $\mathbb{T}$–action on X, for any $\xi \in N_{\mathbb{R}}$, we have a filtration $\mathcal{F}_{\mathrm{wt}_\xi}$ associated to the valuation wt_ξ in (7).

The trivial filtration $\mathcal{F}_{\mathrm{triv}}$ is the filtration associated to the trivial valuation: $\mathcal{F}_{\mathrm{triv}}^x R_m$ is equal to R_m if $x \leq 0$, and is equal to 0 if $x > 0$.

Example 2.3 For any filtration $\mathcal{F}$, we will denote by $\mathcal{F}_{\mathbb{Z}}$ the filtration defined by $\mathcal{F}_{\mathbb{Z}}^{\lambda} R_m = \mathcal{F}^{\lceil \lambda \rceil} R_m$.

Definition 2.4 [45; 46; 49] We say a valuation $\mathfrak{v}\colon \mathbb{C}(X) \to \mathbb{Z}^n$ (where $\mathbb{Z}^n$ is ordered lexicographically) is a faithful valuation if $\mathfrak{v}(\mathbb{C}(X)) \cong \mathbb{Z}^n$. Note that such a valuation always has at most one-dimensional leaves (in the sense of [45]): if $\mathfrak{v}(f) = \mathfrak{v}(g)$ for $f, g \in \mathbb{C}(X)$, then there exists $c \in \mathbb{C}^*$ satisfying $\mathfrak{v}(f + cg) > \mathfrak{v}(f)$.

Fix such a faithful valuation $\mathfrak{v}$. For any $t \in \mathbb{R}$, define the Newton–Okounkov body of the graded linear series

$$\mathcal{F}^{(t)} := \mathcal{F}^{(t)} R_\bullet := \{\mathcal{F}^{tm} R_m\} \tag{13}$$

as the closed convex hull of unions of rescaled values of elements from $\mathcal{F}^{(t)}$:

$$\Delta(\mathcal{F}^{(t)}) = \overline{\bigcup_{m=1}^{+\infty} \frac{1}{m} \mathfrak{v}(\mathcal{F}^{tm} R_m)}. \tag{14}$$

By the theory of Newton–Okounkov bodies [62; 49; 46], we know that

$$n! \cdot \mathrm{vol}(\Delta(\mathcal{F}^{(t)})) = \mathrm{vol}(\mathcal{F}^{(t)} R_\bullet) = \lim_{m \to +\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} R_m}{m^n / n!}. \tag{15}$$

When $t \ll 0$,

$$\Delta(\mathcal{F}^{(t)}) =: \Delta_{\mathfrak{v}}(X, -K_X) = \Delta(X) \tag{16}$$

is associated to the complete graded linear series $\{R_m\}_m$. Following [17], define the concave transform

$$G^{\mathcal{F}}\colon \Delta(X) \to \mathbb{R}, \quad G^{\mathcal{F}}(y) = \sup\{t \mid y \in \Delta(\mathcal{F}^{(t)})\}. \tag{17}$$

Given any filtration $\mathcal{F} = \{\mathcal{F}^{\lambda} R_m\}_{\lambda \in \mathbb{R}}$ and $m \in \mathbb{Z}_{\geq 0}$, the successive minima on R_m is the decreasing sequence

$$\lambda_{\max}^{(m)} = \lambda_1^{(m)} \geq \cdots \geq \lambda_{N_m}^{(m)} = \lambda_{\min}^{(m)}$$

defined by

$$\lambda_j^{(m)} = \max\{\lambda \in \mathbb{R} \mid \dim_{\mathbb{C}} \mathcal{F}^{\lambda} R_m \geq j\}.$$

Theorem 2.5 [17] (i) *The function $x \mapsto \mathrm{vol}(\mathcal{F}^{(x)} R_\bullet)^{1/n}$ is concave on $(-\infty, \lambda_{\max})$ and vanishes on $(\lambda_{\max}, +\infty)$.*

(ii) *As $m \to +\infty$, the Dirac-type measure*

$$\nu_m = \frac{n!}{m^n} \sum_i \delta_{\lambda_i^{(m)}/m} = -\frac{d}{dt} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} H^0(Z, m\ell_0 L)}{m^n / n!} \tag{18}$$

converges weakly to a measure with total mass $V = (-K_X)^{\cdot n}$:

$$\mathrm{DH}(\mathcal{F}) := n! \cdot (G^{\mathcal{F}})_* \, dy = -d\,\mathrm{vol}(\mathcal{F}^{(t)}), \tag{19}$$

where dy is the Lebesgue measure on $\Delta(X)$.

(iii) *The support of the measure* $\mathrm{DH}(\mathcal{F})$ *is given by* $\mathrm{supp}(\mathrm{DH}(\mathcal{F})) = [\lambda_{\min}, \lambda_{\max}]$, *with*

$$\lambda_{\min} := \lambda_{\min}(\mathcal{F}) := \inf\{t \in \mathbb{R} \mid \mathrm{vol}(\mathcal{F}^{(t)}) < V\}, \tag{20}$$

$$\lambda_{\max} := \lambda_{\max}(\mathcal{F}) := \lim_{m\to+\infty} \frac{\lambda_{\max}^{(m)}}{m} = \sup_{m\geq 1} \frac{\lambda_{\max}^{(m)}}{m}. \tag{21}$$

Moreover, $\mathrm{DH}(\mathcal{F})$ *is absolutely continuous with respect to the Lebesgue measure*, *except perhaps for a point mass at* $\lambda_{\max}$.

Example 2.6 If $v \in \mathrm{Val}(X)$ is quasimonomial, it is shown in [22] that DH is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, ie there is no Dirac mass at $\lambda_{\max}(\mathcal{F}_v)$.

Definition 2.7 Let $\mathcal{F}$ be any filtration. For any $a > 0$ the a–rescaling of $\mathcal{F}$ is given by

$$(a\mathcal{F})^\lambda R_m = \mathcal{F}^{\lambda/a} R_m. \tag{22}$$

For any $b \in \mathbb{R}$, the b–shift is given by

$$\mathcal{F}(b)^\lambda R_m = \mathcal{F}^{\lambda - bm} R_m. \tag{23}$$

Set

$$a\mathcal{F}(b) = (a\mathcal{F})(b) = a(\mathcal{F}(b/a)), \quad \text{ie } a\mathcal{F}(b)^x R_m = \mathcal{F}^{(x-bm)/a} R_m. \tag{24}$$

We have the easy identities

$$\Delta(a\mathcal{F}(b)^{(t)}) = \Delta(\mathcal{F}^{((t-b)/a)}), \quad G_{a\mathcal{F}(b)} = aG_{\mathcal{F}} + b, \quad \mathrm{vol}(a\mathcal{F}(b)^{(t)}) = \mathrm{vol}(\mathcal{F}^{((t-b)/a)}). \tag{25}$$

For any $f_m \in R_m$, set

$$\overline{v}_{\mathcal{F}}(f_m) = \sup\{\lambda \mid f_m \in \mathcal{F}^\lambda R_m\} = \max\{\lambda;\, f_m \in \mathcal{F}^\lambda R_m\}, \tag{26}$$

and for any $f = \sum_m f_m \in R = \bigoplus_m R_m$ with $f_m \in R_m$, set

$$\overline{v}_{\mathcal{F}}\Big(\sum_m f_m\Big) = \min\{\overline{v}_{\mathcal{F}}(f_m) \mid f_m \neq 0 \in R_m\}. \tag{27}$$

Then $\overline{v}_{\mathcal{F}}$ is a semivaluation on $R = \bigoplus_m R_m$, satisfying

$$\overline{v}_{\mathcal{F}}(f+g) \geq \min\{\overline{v}_{\mathcal{F}}(f), \overline{v}_{\mathcal{F}}(g)\} \quad \text{and} \quad \overline{v}_{\mathcal{F}}(fg) \geq \overline{v}_{\mathcal{F}}(f)\cdot\overline{v}_{\mathcal{F}}(g). \tag{28}$$

Set

$$\Gamma^+(\mathcal{F}) := \{\lambda_i^{(m)} \mid m \geq 0,\, 1 \leq i \leq N_m\}. \tag{29}$$

Denote by $\Gamma(\mathcal{F})$ the group of $\mathbb{R}$ generated by $\Gamma^+(\mathcal{F})$.

Definition 2.8 • The extended Rees algebra and associated graded algebra of a filtration $\mathcal{F}$ are defined as

$$\mathcal{R}(\mathcal{F}) = \bigoplus_{m\geq 0} \bigoplus_{\lambda\in\Gamma(\mathcal{F})} t^{-\lambda}\mathcal{F}^{\lambda}R_m, \tag{30}$$

$$\mathrm{Gr}(\mathcal{F}) = \bigoplus_{m\geq 0} \bigoplus_{\lambda\in\Gamma(\mathcal{F})} t^{-\lambda}\mathcal{F}^{\lambda}R_m/\mathcal{F}^{>\lambda}R_m, \tag{31}$$

where $\mathcal{F}^{>\lambda}R_m = \{f \in R_m \mid v_{\mathcal{F}}(f) > \lambda\}$.

- If $\mathcal{R}(\mathcal{F})$ is finitely generated, we say that $\mathcal{F}$ is finitely generated and call $\mathcal{F}$ an $\mathbb{R}$–test configuration. In this case, $\Gamma(\mathcal{F})$ is a finitely generated free Abelian group: $\Gamma(\mathcal{F}) \cong \mathbb{Z}^{\mathrm{rk}(\mathcal{F})}$ for some positive integer $\mathrm{rk}(\mathcal{F}) \in \mathbb{Z}_{>0}$, and we will call $\mathrm{rk}(\mathcal{F})$ the rank of $\mathcal{F}$. Moreover, $\mathrm{Gr}(\mathcal{F})$ is also finitely generated, and we call the projective scheme $\mathrm{Proj}(\mathrm{Gr}(\mathcal{F})) =: X_{\mathcal{F},0}$ the central fiber of $\mathcal{F}$.

  There is an induced filtration $\mathcal{F}|_{X_{\mathcal{F},0}} := \mathcal{F}'R' = \{\mathcal{F}'R'_m\}$ on $R' := \mathrm{Gr}(\mathcal{F})$, the homogeneous coordinate ring of the central fiber:

$$\mathcal{F}'^{\lambda}R'_m = \bigoplus_{\lambda_i^{(m)}\geq\lambda} \mathcal{F}^{\lambda_i^{(m)}}R_m/\mathcal{F}^{>\lambda_i^{(m)}}R_m. \tag{32}$$

  The $\Gamma(\mathcal{F})$ grading of $\mathrm{Gr}(\mathcal{F})$ corresponds to a holomorphic vector field $\eta = \eta_{\mathcal{F}}$ on the central fiber, which generates an action by a complex torus of dimension $\mathrm{rk}(\mathcal{F})$.

- We say an $\mathbb{R}$–test configuration $\mathcal{F}$ is special if its central fiber $X_{\mathcal{F},0}$ is a $\mathbb{Q}$–Fano variety and there is an isomorphism $\mathrm{Gr}(\mathcal{F}) \cong R(X_{\mathcal{F},0}, -K_{X_{\mathcal{F},0}}) =: R'$. In this case, there is a $\sigma \in \mathbb{R}$ such that

$$\mathcal{F}'R' = \mathcal{F}'_{\mathrm{wt}_\eta}R'(-\sigma). \tag{33}$$

Remark 2.9 We can naturally extend the above definition to filtrations on $R^{(\ell)}$ for any $\ell \in \mathbb{N}_{\geq 1}$. Indeed we will actually identify two filtrations if they induce the same non-Archimedean metric on $(X^{\mathrm{NA}}, L^{\mathrm{NA}})$ with $L = -K_X$. See Definition 2.17.

There are two equivalent geometric descriptions of $\mathbb{R}$–test configurations, which we now explain.

(I) **Geometric $\mathbb{R}$–TC I** Let $\iota\colon X \to \mathbb{P}^{N_\ell}$ be a Kodaira embedding by a basis of $R_\ell = H^0(X, \ell(-K_X))$ for some $\ell > 0$, and let η be a holomorphic vector field on $\mathbb{P}^{N_\ell-1} = \mathbb{P}(H^0(X, \ell(-K_X)^*)$ that generates an effective holomorphic action on $\mathbb{P}^{N_\ell-1}$ by a torus $\mathbb{T}$ of rank r. Then we get a weight decomposition $R_\ell = \bigoplus_{\alpha\in\mathbb{Z}^r} R_{\ell,\alpha}$ and a filtration on R_ℓ by setting

$$\mathcal{F}^{\lambda}R_\ell = \bigoplus_{\langle\alpha,\eta\rangle\geq\lambda} R_{\ell,\alpha}. \tag{34}$$

The filtration $\mathcal{F}R_\ell$ generates a filtration on $\mathcal{F}R^{(\ell)}$, which is an $\mathbb{R}$–test configuration $\mathcal{F}$. The following lemma generalizes the well-known fact for test configurations; see [33; 76; 64].

Lemma 2.10 *Any $\mathbb{R}$–test configuration, which by definition is a finitely generated filtration, is obtained in this way.*

Proof To see this, we assume again that $\mathcal{F}$ is generated by $\mathcal{F}R_\ell$. For simplicity of notation, set $V = R_\ell$ and $\lambda_i = \lambda_i^{(\ell)}$. By shifting the filtration, we can normalize $\lambda_{N_\ell} = 0$ and assume that we have the relation

$$\begin{aligned}\lambda_1 = \cdots = \lambda_{i_1} =: w_1 &> \lambda_{i_1+1} = \cdots = \lambda_{i_2} =: w_2 \\ &\ \vdots \\ &> \lambda_{i_{k-2}+1} = \cdots = \lambda_{i_{k-1}} =: w_{k-1} \\ &> \lambda_{i_{k-1}+1} = \cdots = \lambda_{N_\ell} =: w_k = 0.\end{aligned}$$

In other words, $\{w_1, \dots, w_k\}$ is the set of distinct values of successive minima and we have a usual filtration,

$$\{0\} \subsetneq \mathcal{F}^{w_1} V \subsetneq \mathcal{F}^{w_2} V \subsetneq \cdots \subsetneq \mathcal{F}^{w_k} = V. \tag{35}$$

In other words, we can equivalently describe an $\mathbb{R}$–filtration by the language of weighted flags. Fixing a reference Hermitian inner product H_0 on $V = R_\ell$, we can assign to the flag (35) a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \tag{36}$$

where $V_1 = \mathcal{F}^{w_1} V$ and V_j is the H_0–orthogonal complement of $\mathcal{F}^{w_{j-1}} V$ inside $\mathcal{F}^{w_j} V$, which has dimension $i_j - i_{j-1} =: d_j$.

Fix a maximal $\mathbb{Q}$–linearly independent subset of $\{w_1, \dots, w_k\}$ to be

$$0 > w_2 =: \zeta_1 > \cdots > w_{p_r} =: \zeta_r. \tag{37}$$

So for each w_j we can find a vector of rational numbers $\vec{r}_j = (r_{j1}, \dots, r_{jr}) \in \mathbb{Q}$ such that $w_j = \sum_{p=1}^r r_{jp}\zeta_p$. Finding a common multiple D of the denominators of $\{r_{jp} \mid 1 \le j \le k, 1 \le p \le r\}$, we set $\eta = \zeta/D$ and $\alpha_j = D\vec{r}_j$, so that

$$w_j = \sum_{p=1}^r \alpha_{jp}\zeta_p = \langle \alpha_j, \eta\rangle. \tag{38}$$

In this way we get a $(\mathbb{C}^*)^r$ representation V, whose weight decomposition is given by (36), where V_j consists of elements of weight α_j, and

$$\mathcal{F}^\lambda V = \bigoplus_{\langle \alpha_j, \eta\rangle \ge a} V_j = \left\{ v = \sum_{j=1}^k v_j \;\middle|\; \min\{\langle \alpha_j, \eta\rangle \mid v_j \ne 0\} \ge \lambda \right\}. \qquad \square$$

From another point of view, let $\mathcal{I}_X \subset \mathbb{C}[Z_1, \dots, Z_{N_\ell}] = S$ be the homogeneous ideal of X. For each $d \in \mathbb{N}$, the $\mathbb{T}$–action induces a representation of $\mathbb{T}$ on S_d, the set of degree-d homogeneous polynomials. The holomorphic vector field η induces an order on the weights of these $\mathbb{T}$–representations. Choosing a set of homogeneous generators of $\mathcal{I}_X$, the initial term with respect to this order generates the ideal of $X_{\mathcal{F},0}$. If σ_η denotes the one-parameter $\mathbb{R}$–group generated by η, we have the convergence of algebraic cycles (or schemes)

$$\lim_{s\to+\infty}\sigma_\eta(s)\circ[X]=[X_{\mathcal{F},0}]. \tag{39}$$

So we say that the $\mathbb{R}$–action generated by η degenerates X into a projective scheme $X_{\mathcal{F},0}$.

By perturbing $\eta\in N_{\mathbb{R}}$, we can find a sequence of rational vector $\eta_k\in N_{\mathbb{Q}}$ converging to η. For $k\gg 1$, η_k induces an $\mathbb{R}$–test configuration of rank one with the same central fiber $X_{\mathcal{F},0}$.

(II) **Geometric $\mathbb{R}$–TC II** This description is essentially contained in [66, Section 2]. For any $\mathbb{R}$–test configuration, we set $B=\mathrm{Spec}(\mathbb{C}(\Gamma^+(\mathcal{F}))\cong\mathbb{C}^r$. Then there is a flat family

$$\mathcal{X}=\mathrm{Proj}_{\mathbb{C}^r}(\mathcal{R}(\mathcal{F}))\to B \tag{40}$$

such that the generic fiber is isomorphic to X and a special fiber isomorphic to $X_{\mathcal{F},0}$. Set $\mathcal{L}$ to be the relative ample line bundle $\mathcal{O}_{\mathcal{X}/\mathbb{C}^r}(1)$. Fix $m\ge 0$. For any $\lambda\in\mathbb{R}$, we set $\lceil\lambda\rceil=\min\{\lambda_i^{(m)}\mid\lambda_i^{(m)}\ge\lambda\}=\langle\alpha,\eta_{\mathcal{F}}\rangle$ for $\alpha\in M_{\mathbb{Z}}$. Then for any $\tau=(\tau_1,\dots,\tau_r)\in\mathbb{C}^r$, we set $\tau^{-\lceil\lambda\rceil}=\prod_{i=1}^r\tau_i^{\alpha_i}$, to get

$$\mathcal{F}^\lambda R_m=\{s\in R_m\mid\tau^{-\lceil\lambda\rceil}\bar{s}\text{ extends to a holomorphic section of }m\mathcal{L}\to\mathcal{X}\}, \tag{41}$$

where $\bar{s}$ is the meromorphic section of $m\mathcal{L}$ defined as the pullback of s via the projection $(\mathcal{X},\mathcal{L})\times_B(\mathbb{C}^*)^r\cong(X,-K_X)\times(\mathbb{C}^*)^r\to X$.

Lemma 2.11 *If $\mathrm{Gr}(\mathcal{F})$ is an integral domain, then the semivaluation $\bar{v}_{\mathcal{F}}$ in (26) defines a valuation on the quotient field of R. Denote by $v_{\mathcal{F}}$ the restriction of $\bar{v}_{\mathcal{F}}$ to $\mathbb{C}(X)$: for $f=s_1/s_2\in\mathbb{C}(X)$ with $s_1,s_2\in R_m$, set*

$$v_{\mathcal{F}}(f)=\bar{v}_{\mathcal{F}}(s_1)-\bar{v}_{\mathcal{F}}(s_2). \tag{42}$$

Then there exists $\sigma>0$ such that $\mathcal{F}=\mathcal{F}_{v_{\mathcal{F}}}(-\sigma)$. In particular, this statement applies to any special $\mathbb{R}$–test configuration.

Proof Fix any two homogeneous elements $s_i\in R_{m_i}$ for $i=1,2$. Assume that $\bar{v}_{\mathcal{F}}(f_i)=s_i$. Then $s_i'\in R'_{m_i,x_i}$. Because $\mathrm{Gr}(\mathcal{F})$ is integral, $s_1's_2'\neq 0\in R'_{m_1+m_2,x_1+x_2}$, which implies that $\bar{v}_{\mathcal{F}}(s_1s_2)=x_1+x_2=\bar{v}_{\mathcal{F}}(s_1)+\bar{v}_{\mathcal{F}}(s_2)$. From this, we easily see that $\bar{v}_{\mathcal{F}}$ is a real valuation.

Assume $f=s_1/s_2=\tilde{s}_1/\tilde{s}_2$. Then $s_1\cdot\tilde{s}_2=s_2\cdot\tilde{s}_1$ and hence $\bar{v}_{\mathcal{F}}(s_1)-\bar{v}_{\mathcal{F}}(s_2)=\bar{v}_{\mathcal{F}}(\tilde{s}_1)-\bar{v}_{\mathcal{F}}(\tilde{s}_2)$. So $v_{\mathcal{F}}$ in (42) is well defined.

For any $s_i\neq 0\in R_m$ with $i=1,2$, by construction $\bar{v}_{\mathcal{F}}(s_1)-v_{\mathcal{F}}(s_1)=\bar{v}_{\mathcal{F}}(s_2)-\bar{v}(s_2)$. This means $b_m:=v_{\mathcal{F}}-\bar{v}_{\mathcal{F}}$ is constant on $R_m\setminus\{0\}$. It is easy to see that $\sigma_{m_1}\sigma_{m_2}=\sigma_{m_1+m_2}$. So we can set $\sigma=\sigma_m/m$ to get the conclusion. □

An $\mathbb{R}$–test configuration with $\mathrm{rk}(\mathcal{F})=1$ is, up to rescaling, associated to the usual test configuration, a notion that plays a basic role in the subject of K–stability.

Definition 2.12 [67; 32; 56] A test configuration of (X, L) is a triple $(\mathcal{X}, \mathcal{L}, \eta)$, sometimes just denoted by $(\mathcal{X}, \mathcal{L})$, that consists of

- a variety $\mathcal{X}$ admitting a $\mathbb{C}^*$–action generated by a holomorphic vector field η and a $\mathbb{C}^*$–equivariant morphism $\pi : \mathcal{X} \to \mathbb{C}$, where the action of $\mathbb{C}^*$ on $\mathbb{C}$ is given by the standard multiplication generated by $-t\partial_t$, and
- a $\mathbb{C}^*$–equivariant π–semiample $\mathbb{Q}$–Cartier $\mathbb{Q}$–divisor $\mathcal{L}$ on $\mathcal{X}$ such that there is an $\mathbb{C}^*$–equivariant isomorphism $\mathrm{i}_\eta : (\mathcal{X}, \mathcal{L})|_{\pi^{-1}(\mathbb{C}\setminus\{0\})} \cong (X, L) \times \mathbb{C}^*$.

We denote by $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ the natural compactification of $(\mathcal{X}, \mathcal{L})$ obtained by adding a trivial fiber at infinity using the isomorphism i_η.

$(X_{\mathbb{C}}, (-K_X)_{\mathbb{C}}, \eta_{\mathrm{triv}}) := (X \times \mathbb{C}, -K_X \times \mathbb{C}, -t\partial_t)$ is called the trivial test configuration. $(\mathcal{X}, \mathcal{L}, \eta)$ is a normal test configuration if $\mathcal{X}$ is a normal variety.

A normal test configuration $(\mathcal{X}, \mathcal{L}, \eta)$ is a special test configuration (resp. weakly special) if $(\mathcal{X}, \mathcal{X}_0)$ is plt (resp. if $(\mathcal{X}, \mathcal{X}_0)$ is log canonical) and $\mathcal{L} = -K_{\mathcal{X}} + c\mathcal{X}_0$ for some $c \in \mathbb{Q}$. By inversion of adjunction, $(\mathcal{X}, \mathcal{L}, \eta)$ being special is equivalent to the condition that $(\mathcal{X}_0, -K_{\mathcal{X}_0})$ is $\mathbb{Q}$–Fano.

Two test configurations $(\mathcal{X}_i, \mathcal{L}_i)$ for $i = 1, 2$ are equivalent if there exists a test configuration $(\mathcal{X}', \mathcal{L}')$ and two $\mathbb{C}^*$–equivariant birational morphisms $\rho_i : \mathcal{X}' \to \mathcal{X}_i$ such that $\rho_1^*\mathcal{L}_1 = \mathcal{L}' = \rho_2^*\mathcal{L}_2$.

Assume that $\mathbb{G}$ is a reductive complex Lie group acting on (X, L). A $\mathbb{G}$–equivariant test configuration of (X, L) is a test configuration $(\mathcal{X}, \mathcal{L}, \eta)$ with the following property:

- There is a $\mathbb{G}$–action on $(\mathcal{X}, \mathcal{L})$ that commutes with the $\mathbb{C}^*$–action generated by η and the action of $\mathbb{G}$ on $(\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \overset{\mathrm{i}_\eta}{\cong} (X, L) \times \mathbb{C}^*$ coincides with the fiberwise action of $\mathbb{G}$ on (the first factor of) $(X, L) \times \mathbb{C}^*$.

As mentioned above, by the work of [76; 65; 19], for any $\mathbb{R}$–test configuration $\mathcal{F}$ with $\mathrm{rk}(\mathcal{F}) = 1$, there exists a test configuration $(\mathcal{X}, \mathcal{L}, \eta)$ and $a > 0$, such that $\Gamma(\mathcal{F}) \cong a\mathbb{Z}$ and $\mathcal{F} = a\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)}$. In this case, we will also denote the $\mathbb{R}$–test configuration $\mathcal{F}$ by $(\mathcal{X}, \mathcal{L}, a\eta)$ and set

$$\mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)} := a\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)}. \tag{43}$$

The identity (40) becomes

$$\mathcal{X} = \mathrm{Proj}_{\mathbb{C}[t]}\Big(\bigoplus_{m\geq 0}\bigoplus_{j\in\mathbb{Z}} t^{-aj}\mathcal{F}^j R_m\Big). \tag{44}$$

Conversely, assume $(\mathcal{X}, \mathcal{L})$ is a test configuration of $(X, L := -K_X)$. Then we associate to it a filtration $\mathcal{F} = \mathcal{F}_{(\mathcal{X},\mathcal{L})}$ as in (41); so $s \in \mathcal{F}^\lambda R_m$ if and only if $t^{-\lceil\lambda\rceil}\bar{s}$ extends to a holomorphic section of $m\mathcal{L}$. In particular, such a construction sets up a one-to-one correspondence between test configurations $(\mathcal{X}, \mathcal{L})$ with ample $\mathcal{L}$, and $\mathbb{R}$–test configurations $\mathcal{F}$ with $\Gamma(\mathcal{F}) \subseteq \mathbb{Z}$; see [19, Proposition 2.15].

Now assume that $(\mathcal{X},\mathcal{L})$ is normal and there is a $\mathbb{C}^*$–equivariant birational morphism $\rho\colon\mathcal{X}\to X_{\mathbb{C}}:=X\times\mathbb{C}$. Write $\mathcal{L}=\rho^*L_{\mathbb{C}}+D$, where $L_{\mathbb{C}}=p_1^*L$. Then by [19, Lemma 5.17], the filtration $\mathcal{F}$ has the more explicit description

$$\mathcal{F}^{\lambda}R_m=\bigcap_E\{s\in H^0(X,mL)\mid r(\mathrm{ord}_E)(s)+m\ell_0\,\mathrm{ord}_E(D)\geq xb_E\},\tag{45}$$

where E runs over the irreducible components of the central fiber $\mathcal{X}_0$, and $b_E=\mathrm{ord}_E(\mathcal{X}_0)=\mathrm{ord}_E(t)$ while $r(\mathrm{ord}_E)$ denotes the restriction of ord_E to $\mathbb{C}(Z)$ under the inclusion $\mathbb{C}(Z)\subset\mathbb{C}(X\times\mathbb{C}^*)=\mathbb{C}(\mathcal{X})$.

When $\mathcal{F}=\mathcal{F}_{(\mathcal{X},-K_{\mathcal{X}},\eta)}$ is associated to a special test configuration, Lemma 2.11 applies. In fact, by [19], $v_{\mathcal{F}}=v_{\mathcal{X}_0}=r(\mathrm{ord}_{\mathcal{X}_0})$ and by [50], $\sigma=A_X(v_{\mathcal{X}_0})$, so $\mathcal{F}_{(\mathcal{X},-K_{\mathcal{X}},\eta)}=\mathcal{F}_{v_{\mathcal{X}_0}}(-A(v_{\mathcal{X}_0}))$. As a consequence, for any $a>0$, by (24) we have the identity

$$\mathcal{F}_{(\mathcal{X},-K_{\mathcal{X}},a\eta)}=\mathcal{F}_{av_{\mathcal{X}_0}}(-A(av_{\mathcal{X}_0})).\tag{46}$$

Note that following Definition 2.8, for any $a>0$ we say that $(\mathcal{X},\mathcal{L},a\eta)$ is a special (resp. normal) $\mathbb{R}$–test configuration if $(\mathcal{X},\mathcal{L},\eta)$ is a special (resp. normal) test configuration.

Note that we use the negative sign $-t\partial_t$ in our Definition 2.12. This sign convention will be convenient for our subsequent computations, as illustrated in the following simple example.

Example 2.13 Consider the product test configuration $(\mathcal{X},\mathcal{L})$ of $(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ induced by the $\mathbb{C}^*$–action

$$t\circ[Z_0,Z_1]=[Z_0,tZ_1].$$

Let s_i for $i=0,1$ be two holomorphic sections of $H^0(\mathbb{P}^1,\mathcal{O}(1))$ corresponding to the homogeneous coordinates Z_i for $i=0,1$. Then t acts on the holomorphic sections by $t\cdot s_0=s_0$ and $t\circ s_1=t^{-1}s_1$. The corresponding filtration is given by

$$\mathcal{F}^{\lambda}R_m=\mathrm{Span}\{s_0^{m-i}s_1^i\mid 0\geq -i\geq\lambda\};\tag{47}$$

cf (34). The natural compactification $\overline{\mathcal{X}}$ can be identified with the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1})$, and $\overline{\mathcal{L}}$ is given by $\mathcal{O}_{\overline{\mathcal{X}}}(D_\infty)$, where D_∞ is the divisor at infinity; see [56, Example 3]. The successive minima are given by $\{\lambda_i^{(m)}\}=\{-m,-m+1,\dots,0\}$. In particular, we have

$$\sum_i\lambda_i^{(m)}=-\tfrac12m^2-\tfrac12m=\tfrac12\overline{\mathcal{L}}^2m^2+\bigl(\tfrac12K_{\overline{\mathcal{X}}}^{-1}\cdot\overline{\mathcal{L}}-1\bigr)m.\tag{48}$$

Moreover, $\eta=-z\,\partial/\partial z$, whose Hamiltonian function is given by $\theta(\eta)=-|Z_1|^2/(|Z_1|^2+|Z_2|^2)$. Note that $\theta(\eta)_*\omega_{\mathrm{FS}}=dy_{[-1,0]}=\mathrm{DH}(\mathcal{F})$.

Example 2.14 If $\mathcal{F}$ is an $\mathbb{R}$–test configuration, then $a\mathcal{F}(b)$ is an $\mathbb{R}$–test configurations for any $(a,b)\in\mathbb{R}_{>0}\times\mathbb{R}$.

Assume $\mathcal{F}=\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)}$ for a test configuration $(\mathcal{X},\mathcal{L},\eta)$. Then as mentioned above, for simplicity of notation we will identify $a\mathcal{F}(b)$ with the data $(\mathcal{X},\mathcal{L}+b\mathcal{X}_0,a\eta)$.

For any $d > 0 \in \mathbb{N}$, we can consider the normalization of the base change,

$$(\mathcal{X}, \mathcal{L}, \eta)^{(d)} := ((\mathcal{X}, \mathcal{L}, \eta) \times_{\mathbb{C}, t \to t^d} \mathbb{C})^{\mathrm{norm}} =: (\mathcal{X}^{(d)}, \mathcal{L}^{(d)}, \eta^{(d)}). \tag{49}$$

On the other hand, $\mathbb{Z}_d = \langle e^{2\pi\sqrt{-1}/d} \rangle \hookrightarrow \mathbb{C}^*$ naturally acts on the $(\mathcal{X}, \mathcal{L})$ and we can take a quotient

$$(\mathcal{X}, \mathcal{L}, \eta)/_{\mathbb{Z}_d} = (\mathcal{X}^{(1/d)}, \mathcal{L}^{(1/d)}, \eta^{(1/d)}) \tag{50}$$

to get a test configuration with a nonreduced central fiber in general.

With this notation, for any $a > 0 \in \mathbb{Q}$ we then have the natural identification

$$\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)^{(a)}} = a \cdot \mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)} = \mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)}. \tag{51}$$

For a filtration $\mathcal{F}R_\bullet$, choose e_- and e_+ as in Definition 2.1. For convenience, we can choose $e_+ = \lceil \lambda_{\max}(\mathcal{F}R) \rceil \in \mathbb{Z}$. Set $e = e_+ - e_-$ and define (fractional) ideals

$$I_{m,\lambda} := I^{\mathcal{F}}_{m,\lambda} := \mathrm{Image}(\mathcal{F}^\lambda R_m \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X), \tag{52}$$

$$\tilde{\mathcal{I}}_m := \tilde{\mathcal{I}}^{\mathcal{F}}_m := I^{\mathcal{F}}_{(m,me_+)} t^{-me_+} + I^{\mathcal{F}}_{(m,me_+-1)} t^{1-me_+} + \cdots + I^{\mathcal{F}}_{(m,me_-+1)} t^{-me_- -1} + \mathcal{O}_X \cdot t^{-me_-}, \tag{53}$$

$$\mathcal{I}_m := \mathcal{I}^{\mathcal{F}(e_+)}_m = \tilde{\mathcal{I}}^{\mathcal{F}}_m \cdot t^{me_+} = I^{\mathcal{F}}_{(m,me_+)} + I^{\mathcal{F}}_{(m,me_+-1)} t^1 + \cdots + I^{\mathcal{F}}_{(m,me_-+1)} t^{me-1} + (t^{me}) \subseteq \mathcal{O}_{X_{\mathbb{C}}}. \tag{54}$$

Definition–Proposition 2.15 [36, Lemma 4.6] *With the above notation, for m sufficiently divisible, define the m^{th} approximating test configuration $(\check{\mathcal{X}}^{\mathcal{F}}_m, \check{\mathcal{L}}^{\mathcal{F}}_m)$ as follows:*

(i) *$\check{\mathcal{X}}^{\mathcal{F}}_m$ is the normalization of blowup of $X \times \mathbb{C}$ along the ideal sheaf $\mathcal{I}^{\mathcal{F}(e_+)}_m$.*

(ii) *The semiample $\mathbb{Q}$–divisor is given by*

$$\check{\mathcal{L}}^{\mathcal{F}}_m = \pi^*((-K_X) \times \mathbb{C}) - \frac{1}{m} E_m + e_+ \check{\mathcal{X}}_0, \tag{55}$$

where E_m is the exceptional divisor of the normalized blowup.

For simplicity of notation, we also denote the data by $(\check{\mathcal{X}}_m, \check{\mathcal{L}}_m)$ if the filtration is clear.

It is easy to see that the filtration $\mathcal{F}_{(\check{\mathcal{X}}_m, \check{\mathcal{L}}_m)}$ on $R^{(m)}$ is induced by $\mathcal{F}_{\mathbb{Z}} R_m$ under the canonical map $S^k R_m \to R_{km}$. By [20, Proof of Theorem 4.13], we have the following approximation result.

Proposition 2.16 [20, Proof of Theorem 4.13] *With notation as in Definition–Proposition 2.15, the Duistermaat–Heckmann measures $\mathrm{DH}(\check{\mathcal{X}}_m, \check{\mathcal{L}}_m)$ converge weakly to $\mathrm{DH}(\mathcal{F})$ as $m \to +\infty$.*

Following Boucksom and Jonsson, it is very convenient to use the non-Archimedean metric defined by filtrations. Any filtration (in the sense of Definition 2.1) defines a non-Archimedean metric on $L^{\mathrm{NA}} \to X^{\mathrm{NA}}$. If we denote by ϕ_{triv} the non-Archimedean metric associated to the trivial filtration, then any non-Archimedean metric ϕ on L^{NA} is represented by the real valued function $\phi - \phi_{\mathrm{triv}}$ on $X^{\mathrm{div}}_{\mathbb{Q}}$.

Definition 2.17 Let $\mathcal{F}=\mathcal{F}R_\bullet$ be a filtration. For any $w\in \mathring{\mathrm{Val}}(X)$, define the non-Archimedean metric associated to $\mathcal{F}$ by

$$(56)\qquad (\phi^{\mathcal{F}}_m-\phi_{\mathrm{triv}})(w)=-\frac{1}{m}G(w)(\widetilde{\mathcal{I}}^{\mathcal{F}}_m)=-\frac{1}{m}G(w)(\mathcal{I}_m^{\mathcal{F}(e_+)}t^{-me_+})=-\frac{1}{m}G(w)(\mathcal{I}_m^{\mathcal{F}(e_+)})+e_+,$$

$$(57)\qquad (\phi^{\mathcal{F}}-\phi_{\mathrm{triv}})(w)=-G(w)(\widetilde{\mathcal{I}}^{\mathcal{F}}_\bullet)=\lim_{m\to+\infty}\phi^{\mathcal{F}}_m(w).$$

In particular, if $v\in\mathring{\mathrm{Val}}(Z)$ and $\mathcal{F}=\mathcal{F}_v$, then we write $\phi_v=\phi^{\mathcal{F}_v}$.

Note that $\phi^{\mathcal{F}}_m=\phi_{\mathcal{F}(\check{\mathcal{X}}_m,\check{\mathcal{L}}_m)}$ converges to ϕ as $m\to+\infty$. Moreover, if (for simplicity) we assume that $S^kR_m\to R_{km}$ is surjective for all $k,m\ge 1$, then it is an increasing sequence in the sense that if $m_1\,|\,m_2$, then $\phi^{\mathcal{F}}_{m_1}\le\phi^{\mathcal{F}}_{m_2}$. If $\phi^{\mathcal{F}}$ is continuous, then ϕ_m converges to ϕ uniformly by Dini's theorem.

The following transformation rule can be easily verified.

Lemma 2.18 *For any filtration $\mathcal{F}$ and any $(a,b)\in\mathbb{R}_{>0}\times\mathbb{R}$ and $v\in X^{\mathrm{div}}_{\mathbb{Q}}$,*

$$(58)\qquad (\phi_{a\mathcal{F}(b)}-\phi_{\mathrm{triv}})(v)=a(\phi_{\mathcal{F}}-\phi_{\mathrm{triv}})\Big(\frac{v}{a}\Big)+b.$$

2.3 Twist of filtrations

Let $\mathcal{F}=\mathcal{F}R_\bullet$ be a $\mathbb{T}$–equivariant filtration, which means that $\mathcal{F}^\lambda R_m$ is a $\mathbb{T}$–invariant subspace of R_m for any $x\in\mathbb{R}$. For $\alpha\in M_{\mathbb{Z}}=N^\vee_{\mathbb{Z}}$, denote the weight space by

$$(59)\qquad (R_m)_\alpha=\{s\in R_m\mid \tau\circ s=\tau^\alpha s\text{ for all }\tau\in(\mathbb{C}^*)^r\}.$$

Then we have

$$(60)\qquad (\mathcal{F}^\lambda R_m)_\alpha:=\{s\in\mathcal{F}^\lambda R_m\mid\tau\circ s=\tau^\alpha s\}=\mathcal{F}^\lambda R_m\cap(R_m)_\alpha,$$

and the decomposition

$$(61)\qquad \mathcal{F}^\lambda R_m=\bigoplus_{\alpha\in M_{\mathbb{Z}}}(\mathcal{F}^\lambda R_m)_\alpha.$$

Definition 2.19 [52] For any $\xi\in N_{\mathbb{R}}$, the ξ–twist of $\mathcal{F}$ is the filtration $\mathcal{F}_\xi R_\bullet$ defined by

$$(62)\qquad \mathcal{F}^\lambda_\xi R_m=\bigoplus_{\alpha\in M_{\mathbb{Z}}}(\mathcal{F}^\lambda_\xi R_m)_\alpha,\quad\text{where }(\mathcal{F}^\lambda_\xi R_m)_\alpha:=(\mathcal{F}^{\lambda-\langle\alpha,\xi\rangle}R_m)_\alpha.$$

Example 2.20 If $\mathcal{F}$ is a $\mathbb{T}$–equivariant $\mathbb{R}$–test configuration, then $\mathcal{F}_\xi$ is also an $\mathbb{R}$–test configuration.

If $\mathcal{F}=\mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)}$ for a test configuration, then we can identify the data $\mathcal{F}_\xi$ with the data $(\mathcal{X},\mathcal{L},a\eta+\xi)$; see [41]. If $\xi\in N_{\mathbb{Z}}$, then $(\mathcal{X},\mathcal{L},a\eta+\xi)$ is equivalent to the birational image of the $(\mathcal{X},\mathcal{L})$ via the birational transform $\sigma_\xi\colon\mathcal{X}\dashrightarrow\mathcal{X}$, $(z,t)\to(\sigma_\xi(t)\cdot z,t)$; see [52].

Moreover, if we start with the trivial filtration $\mathcal{F}_{\mathrm{triv}}=\mathcal{F}_{(X_{\mathbb{C}},(-K_X)_{\mathbb{C}},-t\partial_t)}$, then $(\mathcal{F}_{\mathrm{triv}})_\xi$ is equal to $\mathcal{F}_{\mathrm{wt}_\xi}$.

Definition 2.21 We say that a faithful valuation $\mathfrak{v}$ in the sense of Definition 2.4 is adapted to the torus action if for any $f \in \mathbb{C}(X)_\alpha$ we have $\mathfrak{v}(f) = (\alpha, \mathfrak{v}^{r+1}(f), \dots, \mathfrak{v}^n(f)) \in \mathbb{Z}^r \times \mathbb{Z}^{n-r}$.

There always exists a faithful valuation that is adapted to the torus action. This can be constructed as follows. First we choose a $\mathbb{T}$–invariant Zariski-open set U of X as in [3]. Then by the theory of affine T–varieties as developed in [2], there exists a variety Y of dimension $n-r$ and a polyhedral divisor $\mathfrak{D}$ such that

$$U = \operatorname{Spec}_{\alpha \in M_{\mathbb{Z}}} H^0(Y, \mathcal{O}(\mathfrak{D}(\alpha))). \tag{63}$$

We can choose a faithful valuation $\mathfrak{v}_Y$ on Y (for example via a flag of varieties as in [49]) and define, for any $f \in H^0(Y, \mathcal{O}(\mathfrak{D}(\alpha))$,

$$\mathfrak{v}(f) = (\alpha, \mathfrak{v}_Y(f)). \tag{64}$$

Let $\mathfrak{v}$ be such a valuation and $\Delta = \Delta_{\mathfrak{v}}(X, -K_X) \subset \mathbb{R}^n$ be the associated Newton–Okounkov body. If $p\colon \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^r$ denotes the natural projection, then we have

$$p(\Delta) = P = \text{moment map of the } \mathbb{T}\text{–action on } (X, -K_X). \tag{65}$$

The following lemma was already observed in [81], in which a faithful valuation adapted to the torus action was constructed using equivariant infinitesimal flags in the sense of [49]. Here we give a different and direct proof for the reader's convenience.

For simplicity of notation, we write $y = (y_1, \dots, y_n) = (y', y'') \in \mathbb{R}^r \times \mathbb{R}^{n-r}$ and set

$$\langle y', \xi \rangle = \sum_{i=1}^r y_i' \xi^i =: \langle y, \xi \rangle. \tag{66}$$

In the last identity, we identify $\xi \in N_{\mathbb{R}} = \mathbb{R}^r$ with $(\xi, 0) \in \mathbb{R}^n$.

Lemma 2.22 [81] *If $\mathfrak{v}$ is a $\mathbb{Z}^n$–valued valuation adapted to the torus action, then for any $y \in \Delta(-K_X)$,*

$$G_{\mathcal{F}_\xi}(y) = G_{\mathcal{F}}(y) + \langle y', \xi \rangle. \tag{67}$$

Proof For any $t > G^{\mathcal{F}}(y) = \lambda$, there exists $\epsilon > 0$ such that $y \notin \Delta(\mathcal{F}^{(t-\epsilon)})$. Let $\delta_1 = \operatorname{dist}(y, \Delta(\mathcal{F}^{(t-\epsilon)}))$. Choose any $f \in \mathcal{F}_\xi^{(t+\langle y', \xi\rangle)m} R_{m,\alpha} = \mathcal{F}^{(t+\langle y',\xi\rangle)m - \langle \alpha, \xi\rangle} R_{m,\alpha}$. Consider two cases:

(i) $\langle \alpha/m, \xi\rangle - \langle y', \xi\rangle < \epsilon$. Then $\mathfrak{v}(f) \in \Delta^{(t-\epsilon)}$, so $|\mathfrak{v}(f)/m - y| \geq \delta_1$.

(ii) $\langle \alpha/m, \xi\rangle - \langle y', \xi\rangle \geq \epsilon$. Then $|\mathfrak{v}(f)/m - y| \geq |\alpha/m - y'| \geq \epsilon/|\xi|$.

The two cases together imply that $y \notin \Delta(\mathcal{F}_\xi^{(t+\langle y',\xi\rangle)})$. So we get the inequality $G_{\mathcal{F}_\xi} \leq G_{\mathcal{F}} + \langle y', \xi\rangle$. On the other hand, since $\mathcal{F} = (\mathcal{F}_\xi)_{-\xi}$, we also get $G_{\mathcal{F}} \leq G_{\mathcal{F}_\xi} - \langle y', \xi\rangle$. So we get the desired identity. □

2.4 Asymptotically equivalent filtrations

In this section we recall Boucksom and Jonsson's characterization in [20; 21] of asymptotically equivalent filtrations; see also [1].

For a filtration $\mathcal{F}R_m$ of R_m, we say that a basis $\mathcal{B}=\{s_1,\dots,s_{N_m}\}$ of R_m is compatible with $\mathcal{F}R_m$ if for any $\lambda\in\mathbb{R}$ there exists a subset of $\mathcal{B}$ that spans $\mathcal{F}^\lambda R_m$.

Let $\mathcal{F}_i=\{\mathcal{F}_i R_m\}$ for $i=0,1$ be two filtrations. For each m, we can find a basis $\mathcal{B}:=\{s_1,\dots,s_{N_m}\}$ of R_m that is compatible with both $\mathcal{F}_i R_m$ with $i=0,1$. We refer to [1; 18] and the discussion in Section 5 for more details. Assume that for each $i=0,1$, we have $s_k\in\mathcal{F}_i^{\mu_{k,i}}\setminus\mathcal{F}^{>\mu_{k,i}}$. Then $\mathcal{B}$ is an orthogonal basis for the non-Archimedean norm $\|\cdot\|_{m,i}$ corresponding to $\mathcal{F}_i$: for any $s=\sum_k a_k s_k\in R_m$,

$$\|s\|_{m,i}=e^{-\max\{\lambda\mid s\in\mathcal{F}^\lambda R_m\}}=\max_k |a_k|_0 e^{-\mu_{k,i}}, \tag{68}$$

where $|\cdot|_0$ is the trivial norm on $\mathbb{C}$.

Following [24; 20], we define the set of successive minima of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ to be the set $\{\mu_{k,1}-\mu_{k,0}\}$. The following result was proved in [18; 24].

Theorem 2.23 [18; 24] *As $m\to+\infty$, the measures*

$$\frac{n!}{m^n}\sum_{k=1}^{N_m}\delta_{(\mu_{k,1}-\mu_{k,0})/m} \tag{69}$$

converge weakly as $m\to+\infty$ to a relative limit measure, denoted by $d\nu:=d\nu(\mathcal{F}_0,\mathcal{F}_1)$.

Corollary 2.24 *For any $p\in[1,\infty)$, the limit*

$$d_p(\mathcal{F}_0,\mathcal{F}_1):=\lim_{m\to+\infty}\left(\frac{n!}{m^n}\sum_{k=1}^{N_m} m^{-1}|\mu_{k,1}-\mu_{k,0}|^p\right)^{1/p} \tag{70}$$

exists and is given by

$$d_p(\mathcal{F}_0,\mathcal{F}_1)=\left(\int_{\mathbb{R}}|\lambda|^p\,d\nu(\lambda)\right)^{1/p}. \tag{71}$$

Definition 2.25 [20, Section 3.6] $\mathcal{F}_0$ and $\mathcal{F}_1$ are asymptotically equivalent if $d_2(\mathcal{F}_0,\mathcal{F}_1)=0$.

In fact, by [20] the d_p are comparable to each other for all $p\in[1,\infty)$, and the above equivalence can be defined by using any $p\in[1,+\infty)$.

Theorem 2.26 [20, Theorem 4.16] *Assume that X is smooth. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two filtrations on R. Then $\mathcal{F}_0$ and $\mathcal{F}_1$ are asymptotically equivalent if and only if $\phi_{\mathcal{F}_1}=\phi_{\mathcal{F}_2}$.*

We also need:

Proposition 2.27 *If $\mathcal{F}_{v_i}$ for $i=0,1$ are two $\mathbb{R}$–test configurations associated to two valuations $v_i\in\mathring{\mathrm{Val}}(X)$ for $i=0,1$, then $\phi_{\mathcal{F}_{v_1}}=\phi_{\mathcal{F}_{v_2}}+c$ for a constant $c\in\mathbb{R}$ if and only if $v_1=v_2$ (and hence $\mathcal{F}_{v_1}=\mathcal{F}_{v_2}$).*

Proof Recall that $\phi_{\mathcal{F}_{v_i}} = \lim_{m\to+\infty} \phi_m^{\mathcal{F}_{v_i}}$ is an increasing limit along the subsequence $m = 2^k$, where for any $w \in \mathrm{Val}(X)$,

$$\phi_m^{\mathcal{F}_{v_i}}(w) = -\frac{1}{m} G(w)\biggl(\sum_{\lambda\in\mathbb{N}} I_{m,\lambda}^{\mathcal{F}_{v_i}} t^{-\lambda}\biggr), \tag{72}$$

where $I_{m,\lambda}^{\mathcal{F}_{v_i}}$ is the base ideal of the sublinear system $\mathcal{F}_{v_i}^{\lambda} R_m$. Note that it is easy to see that $v_1 = v_2$ if and only if $\mathfrak{a}_\lambda(v_1) = \mathfrak{a}_\lambda(v_2)$ for any $\lambda \in \mathbb{N}$, where $\mathfrak{a}_\lambda(v_i) = \{f \in \mathcal{O}_X \mid v_i(f) \geq \lambda\}$.

For any $d \in \mathbb{N}$, by choosing $m \gg 1$ we can assume that $mL \otimes \mathfrak{a}_d(v_1)$ is globally generated. Then we get $I_{m,d}^{\mathcal{F}_{v_1}} = \mathfrak{a}_d(v_1)$. From this it is also clear that $\phi_{\mathcal{F}_{v_i}}(v_i) = 0$. So we get

$$-c = -\phi_{\mathcal{F}_{v_1}}(v_2) \leq -\phi_{2^k}^{\mathcal{F}_{v_1}}(v_2) = \frac{1}{2^k} G(v_2)\biggl(\sum_{\lambda} I_{2^k,\lambda}^{\mathcal{F}_{v_1}} t^{-\lambda}\biggr) \leq \frac{1}{2^k}\bigl(v_2(I_{2^k,d}^{\mathcal{F}_{v_1}}) - d\bigr) = \frac{1}{2^k}\bigl(v_2(\mathfrak{a}_d(v_1)) - d\bigr).$$

Since k can be arbitrarily large, we get $-c \leq 0$, ie $c \geq 0$. Switching v_1 and v_2 in the above argument, we get $c \leq 0$. So $c = 0$. We then have the inequality $v_2(\mathfrak{a}_d(v_1)) \geq d$ for any $d \in \mathbb{N}$. This easily implies $v_2 \geq v_1$. Switching v_1 and v_2, we get $v_1 \leq v_2$. Hence $v_1 = v_2$, as required. $\square$

Corollary 2.28 *Assume that X is smooth. With the same notation as above, if $\mathcal{F}_{v_1}$ is asymptotically equivalent to $\mathcal{F}_{v_2}$, then $v_1 = v_2$.*

More recently, this result has been proved for any $\mathbb{Q}$–Fano variety:

Lemma 2.29 [15, Lemma 3.16; 21, Theorem C] *For any $\mathbb{Q}$–Fano variety, if v_i for $i = 1, 2$ are two valuations in $\mathring{\mathrm{Val}}(X)$ such that $\mathcal{F}_{v_1}$ is asymptotically equivalent to $\mathcal{F}_{v_2}$, then $v_1 = v_2$.*

Remark 2.30 In the first version of this paper, Corollary 2.28 was stated for any $\mathbb{Q}$–Fano variety. However, it has been pointed out by experts that the validity of Theorem 2.26 from [20] for singular $\mathbb{Q}$–Fano varieties depends on a still conjectural property called *continuity of envelopes*. Fortunately, recently, in [15; 21], the result in Lemma 2.29 has been given a direct proof without using the continuity of envelopes.

2.5 Non-Archimedean invariants of filtrations

For any filtration $\mathcal{F}$ on $R = R(X, -K_X)$, we set

$$\boldsymbol{L}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{L}_X^{\mathrm{NA}}(\mathcal{F}) = \inf_{v\in X_{\mathbb{Q}}^{\mathrm{div}}} \bigl(A_X(v) + (\phi_{\mathcal{F}} - \phi_{\mathrm{triv}})(v)\bigr), \tag{73}$$

$$\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}) = \widetilde{\boldsymbol{S}}_X^{\mathrm{NA}}(\mathcal{F}) = -\log\biggl(\frac{1}{V}\int_{\mathbb{R}} e^{-\lambda}\,\mathrm{DH}(\mathcal{F})\biggr) = -\log\biggl(\frac{n!}{V}\int_{\Delta} e^{-G_{\mathcal{F}}(y)}\,dy\biggr), \tag{74}$$

$$\boldsymbol{E}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \boldsymbol{E}^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{E}_X^{\mathrm{NA}}(\mathcal{F}) = \frac{1}{V}\int_{\mathbb{R}} \lambda\cdot\mathrm{DH}(\mathcal{F}) = \frac{n!}{V}\int_{\Delta} G_{\mathcal{F}}(y)\,dy, \tag{75}$$

$$\boldsymbol{H}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{H}_X^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}), \tag{76}$$

$$\boldsymbol{D}^{\mathrm{NA}}(\phi^{\mathcal{F}}) = \boldsymbol{D}^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{D}_X^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) - \boldsymbol{E}^{\mathrm{NA}}(\mathcal{F}). \tag{77}$$

The above functionals are by now well known, and we use notation following that in [19; 43]. The formula involving $G_{\mathcal{F}}$ follows from Theorem 2.5(ii).

Proposition 2.31 (see [37; 20; 52]) *For a filtration $\mathcal{F}$, with the notation from Definition 2.17, we have the following convergence: the sequence from Definition–Proposition 2.15 satisfies, for any $\boldsymbol{F} \in \{\widetilde{\boldsymbol{S}}, \boldsymbol{E}\}$,*

$$\lim_{m\to+\infty} \boldsymbol{F}^{\mathrm{NA}}(\phi_m^{\mathcal{F}}) = \boldsymbol{F}^{\mathrm{NA}}(\phi^{\mathcal{F}}). \tag{78}$$

Moreover, we have

$$\lim_{m\to+\infty} \boldsymbol{L}^{\mathrm{NA}}(\phi_m^{\mathcal{F}}) \le \boldsymbol{L}^{\mathrm{NA}}(\phi^{\mathcal{F}}). \tag{79}$$

Proof By Proposition 2.16 we know that $\mathrm{DH}(\mathcal{F}_{(\mathcal{X}_m,\mathcal{L}_m,\eta_m)})$ converges weakly to $\mathrm{DH}(\mathcal{F})$ as $m\to+\infty$, from this we easily get the convergence of $\widetilde{\boldsymbol{S}}^{\mathrm{NA}}$ and $\boldsymbol{E}^{\mathrm{NA}}$.

The inequality (79) follows easily from the inequality $\phi_m^{\mathcal{F}} \le \phi^{\mathcal{F}}$. □

For our later argument, we will use a different formulation of the $\boldsymbol{L}^{\mathrm{NA}}$–functional studied in [80; 15]. For any filtration $\mathcal{F}$, denote by $I_\bullet^{\mathcal{F}^{(x)}} = \{I_{m,mx}^{\mathcal{F}}\}$ the graded sequence of base ideals defined in (52). In [79], Xu and Zhuang introduced the functional

$$\widehat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}) = \sup\{x \in \mathbb{R} \mid \mathrm{lct}(X; I_\bullet^{\mathcal{F}^{(x)}}) \ge 1\}, \tag{80}$$

and proved that $\widehat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}) \ge \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F})$. More recently it has been shown that in fact the two functionals are identical to each other. More specifically, we will need the following comparison results.

Proposition 2.32 [79, Proposition 4.2 and Theorem 4.3; 15, Lemma 3.8] *For any filtration $\mathcal{F}$, we have:*

(i) *$A_X(E) \ge \widehat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}_{\mathrm{ord}_E})$ for any prime divisor E over X, with equality holding if ord_E induces a weakly special test configuration.*

(ii) *$\widehat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F})$ for any filtration $\mathcal{F}$.*

For later purposes, we also introduce, for any $a>0$,

$$\boldsymbol{E}_k^{\mathrm{NA}}(\mathcal{F}) = \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} x^k \,\mathrm{DH}(\mathcal{F}) = \lim_{m\to+\infty}\frac{1}{N_m}\sum_i \left(\frac{\lambda_i^{(m)}}{m}\right)^k, \tag{81}$$

$$\boldsymbol{Q}^{(a)}(\mathcal{F}) = \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} e^{-ax}\,\mathrm{DH}(\mathcal{F}) = \frac{1}{\boldsymbol{V}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}a^k \boldsymbol{E}_k^{\mathrm{NA}}(\mathcal{F}), \tag{82}$$

$$\boldsymbol{Q}(\mathcal{F}) := \boldsymbol{Q}^{(1)}(\mathcal{F}). \tag{83}$$

Note that $\boldsymbol{E}_1^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{E}^{\mathrm{NA}}(\mathcal{F})$ and $\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}) = -\log \boldsymbol{Q}(\mathcal{F})$.

For any $v \in \mathring{\mathrm{Val}}(X)$ (resp. test configuration $(\mathcal{X},\mathcal{L},a\eta)$), we will often write $\boldsymbol{F}^{\mathrm{NA}}(v)$ (resp. $\boldsymbol{F}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a\eta)$) for the above various functionals $\boldsymbol{F}^{\mathrm{NA}}(\mathcal{F})$ with $\mathcal{F}$ being the corresponding filtration.

Example 2.33 If $(\mathcal{X}, \mathcal{L}, a\eta)$ is a normal $\mathbb{R}$–test configuration, then we have

$$\boldsymbol{E}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = a \cdot \frac{\bar{\mathcal{L}}^{\cdot n+1}}{(n+1)\boldsymbol{V}}, \tag{84}$$

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = a \cdot (\mathrm{lct}(\mathcal{X}, -K_{\mathcal{X}} - \mathcal{L}; \mathcal{X}_0) - 1). \tag{85}$$

If $(\mathcal{X}, \mathcal{X}_0)$ has log canonical singularities and $K_{\mathcal{X}} + \mathcal{L} = \sum_i e_i E_i$ (which is centered at $\mathcal{X}_0$), then

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = a \cdot \min_i e_i. \tag{86}$$

Example 2.34 Let $\mathcal{F}$ be a special $\mathbb{R}$–test configuration and let $(X_0, \eta) = (X_{\mathcal{F},0}, \eta_{\mathcal{F}})$ be the corresponding central fiber. Assume that $\mathcal{F}|_{X_0} = \mathcal{F}'_{\mathrm{wt}_\eta} R'(-\sigma)$; see (33). Let $\tilde{\varphi}$ be any $(S^1)^r$–invariant smooth positively curved Hermitian metric on $-K_X$. Then with the notation as in the paragraph containing (9), we have

$$\boldsymbol{L}_X^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{L}_{X_0}^{\mathrm{NA}}(\mathcal{F}|_{X_0}) = \frac{\int_{X_0} \theta_{\tilde{\varphi}}(\eta) e^{-\tilde{\varphi}}}{\int_{X_0} e^{-\tilde{\varphi}}} - \sigma = -\sigma, \tag{87}$$

$$\boldsymbol{E}_X^{\mathrm{NA}}(\mathcal{F}) = \boldsymbol{E}_{X_0}^{\mathrm{NA}}(\mathcal{F}|_{X_0}) = \frac{1}{\boldsymbol{V}} \int_{X_0} \theta_{\tilde{\varphi}}(\eta) (\mathrm{dd^c}\tilde{\varphi})^n - \sigma, \tag{88}$$

$$\tilde{\boldsymbol{S}}_X^{\mathrm{NA}}(\mathcal{F}) = \tilde{\boldsymbol{S}}_{X_0}^{\mathrm{NA}}(\mathcal{F}|_{X_0}) = -\log\left(\frac{1}{\boldsymbol{V}} \int_{X_0} e^{-\theta_\eta} (\mathrm{dd^c}\tilde{\varphi})^n\right) - \sigma. \tag{89}$$

The above identity is well known if $\mathcal{F}$ comes from a special test configuration. For more general $\mathcal{F}$, one can use a sequence of special test configuration to approximate and get the above formula.

Corresponding to (58), we have the following simple transformation rule, which can be checked easily from the defining expressions of the functionals.

Lemma 2.35 *For any $(a, b) \in \mathbb{R}_{>0} \times \mathbb{R}$, we have*

$$\boldsymbol{L}^{\mathrm{NA}}(a\mathcal{F}(b)) = \hat{\boldsymbol{L}}^{\mathrm{NA}}(a\mathcal{F}(b)) = a\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) + b, \tag{90}$$

$$\tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}(b)) = \tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}) + b, \tag{91}$$

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}(b)) = \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}). \tag{92}$$

We also note:

Lemma 2.36 *The function $a \mapsto \boldsymbol{H}^{\mathrm{NA}}(a\mathcal{F})$ is a convex function on $\mathbb{R}_{\geq 0}$.*

Proof Since $\boldsymbol{L}^{\mathrm{NA}}(a\mathcal{F})$ is linear in a by (90), we just need to show that $f(a) := -\tilde{\boldsymbol{S}}^{\mathrm{NA}}(a\mathcal{F})$ is convex in $a \in \mathbb{R}_{\geq 0}$. By (25) and (74), we get

$$f(a) = \log\left(\frac{n!}{\boldsymbol{V}} \int_\Delta e^{-aG(y)}\, dy\right),$$

where $G = G(y) = G_{\mathcal{F}}(y)$. So we can calculate

$$f''(a) = \frac{\int_\Delta G^2 e^{-aG}\,dy}{\int_\Delta e^{-aG}\,dy} - \frac{\big(\int_\Delta G e^{-aG}\,dy\big)^2}{\big(\int_\Delta e^{-aG}\,dy\big)^2} \geq 0$$

by Hölder's inequality. □

Note the first identity in (90) comes from Proposition 2.32. Moreover, combining [52, Lemma 3.10] and Proposition 2.32, we have the following invariance property under the twisting.

Lemma 2.37 *Let $\mathcal{F}$ be a $\mathbb{T}$–equivariant filtration. For any $\xi \in N_{\mathbb{R}}$, we have*

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_\xi) = \hat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}_\xi) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}). \tag{93}$$

As a consequence, we have

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_{\mathrm{wt}_\xi}) = \hat{\boldsymbol{L}}^{\mathrm{NA}}(\mathcal{F}_{\mathrm{wt}_\xi}) = 0. \tag{94}$$

The following lemma is a prototype uniqueness result in this paper, and can be seen as a generalization of the uniqueness of Kähler–Ricci soliton vector fields shown by Tian and Zhu [71] (the case when $\mathcal{F} = \mathcal{F}_{\mathrm{triv}}$). See Section 2.6 for more discussion.

Lemma 2.38 *Let $\mathcal{F}$ be a $\mathbb{T}$–equivariant filtration. Then the function $\xi \mapsto \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_\xi)$ on $N_{\mathbb{R}}$ admits a unique minimizer.*

Proof By (93), $\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_\xi)$ is constant in ξ. Using the identity (67) and (74),

$$-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_\xi) = \log\bigg(\frac{n!}{V}\int_\Delta e^{-G_{\mathcal{F}_\xi}(y)}\,dy\bigg) = \log\bigg(\frac{n!}{V}\int_\Delta e^{-G_{\mathcal{F}}(y)-\langle y,\xi\rangle}\,dy\bigg).$$

It is easy to use this expression to show that $f(\xi) := -\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_\xi)$ is strictly convex in $\xi \in N_{\mathbb{R}}$, which implies the uniqueness of minimizer. To prove the existence of minimizer, we need to show that $f(\xi)$ is proper, ie $\lim_{|\xi|\to+\infty} f(\xi) = +\infty$. To see this, recall that we have the vanishing

$$\int_X \theta_{\widetilde{\varphi}}(\xi) e^{-\widetilde{\varphi}} = -\int_X \mathfrak{L}_\eta e^{-\widetilde{\varphi}} = 0. \tag{95}$$

This implies that $0 > \inf_X \theta_{\widetilde{\varphi}}(\xi) = \inf_\Delta \langle y, \xi\rangle$ if $\xi \neq 0$, which indeed implies the properness. □

Definition 2.39 We say that a filtration $\mathcal{F}$ is normalized if

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) = 0. \tag{96}$$

A test configuration $(\mathcal{X}, \mathcal{L}, a\eta)$ is normalized if $\mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)}$ is normalized.

With the above discussion, the following lemma is easy to prove.

Lemma 2.40 (i) *Any special test configuration $(\mathcal{X}, -K_{\mathcal{X}})$ is normalized. More generally, a special $\mathbb{R}$–test configuration $\mathcal{F}$ (see Definition 2.8) if and only if $\sigma = 0$ in* (33).

(ii) *For any filtration $\mathcal{F}$, the shift $\mathcal{F}(-\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}))$ is normalized. If $\mathcal{F}$ is normalized, then so are $a\mathcal{F}$ for any $a > 0$, and any twist $\mathcal{F}_\xi$.*

As a consequence of this approximation result in Proposition 2.31, it is convenient for us to introduce:

Definition–Proposition 2.41 *For any $\mathbb{Q}$–Fano variety X, we define*

$$h(X) = \inf_{(\mathcal{X},\mathcal{L},a\eta)} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = \inf_{\mathcal{F}} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}), \tag{97}$$

where $(\mathcal{X}, \mathcal{L}, a\eta)$ ranges over all test configurations, and $\mathcal{F}$ ranges over all filtrations or $\mathbb{R}$–test configurations.

The following lemma is similar to [31, Lemma 2.5].

Lemma 2.42 *For any filtration $\mathcal{F}$, we have*

$$\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}) \le \boldsymbol{E}^{\mathrm{NA}}(\mathcal{F}) \quad \text{and} \quad \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}) \ge \boldsymbol{D}^{\mathrm{NA}}(\mathcal{F}). \tag{98}$$

The identities hold true if and only if $\mathcal{F}(c)$ is asymptotically equivalent to the trivial filtration for some $c \in \mathbb{R}$; see Definition 2.25.

Proof The first inequality, which implies the second, follows from the concavity of the logarithmic function. When the identity holds, the DH measure $\mathrm{DH}(\mathcal{F})$ is a Dirac measure $V \cdot \delta_c$. Then $d_2(\mathcal{F}(c), \mathcal{F}_{\mathrm{triv}}) = 0$, which by Definition 2.25 means that $\mathcal{F}(c)$ is asymptotically equivalent to the trivial filtration. □

Based on the work in [29; 28; 40], Dervan and Székelyhidi proved:

Theorem 2.43 [31] *Assume that X is a smooth Fano manifold. There is an identity*

$$h(X) := \inf_{(\mathcal{X},\mathcal{L},a\eta)\ \text{special}} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = -\inf_{\omega \in c_1(X)} \int_X h_\omega e^{h_\omega} \omega^n, \tag{99}$$

where ω ranges over smooth Kähler metrics from $c_1(X)$ and h_ω is the normalized Ricci potential of ω. Moreover, the infimum is achieved by a special test configuration constructed via the Gromov–Hausdorff limit Kähler–Ricci soliton from [29; 28].

More recently, Hisamoto [43] gave a different proof of (99) based on the destabilizing geodesic rays constructed from [30].

Remark 2.44 Our sign convention differs from that of Dervan–Székelyhidi and Hisamoto by a minus. Dervan and Székelyhidi defined a non-Archimedean functional for general $\mathbb{R}$–test configuration by mimicking Tian's CM weight (or the so-called Donaldson–Futaki invariant). But in such generality, their normalization seems imprecise. Differently from their definition, for any test configuration $(\mathcal{X}, \mathcal{L}, a\eta)$ one could define

$$\widetilde{\boldsymbol{H}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = \frac{a}{V}(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} + \bar{\mathcal{L}}^{\cdot n+1}) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta). \tag{100}$$

By the same argument as [19, Proposition 7.32], we have

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta) \le \widetilde{\boldsymbol{H}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta), \tag{101}$$

with strict inequality if $(\mathcal{X}, \mathcal{L}, \eta)$ is anticanonical. Moreover, by (98) we also get

$$\widetilde{\boldsymbol{H}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta) \ge \mathrm{CM}(\mathcal{X}, \mathcal{L}, \eta) = \frac{1}{\boldsymbol{V}}\Big(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{n} + \frac{n}{n+1}\bar{\mathcal{L}}^{\cdot n+1}\Big), \tag{102}$$

with the identity being true only if $(\mathcal{X}, \mathcal{L}, \eta)$ is trivial. One advantage of $\boldsymbol{H}^{\mathrm{NA}}$ over $\widetilde{\boldsymbol{H}}^{\mathrm{NA}}$ is that the former can be defined for any filtration, not necessarily finitely generated. Due to this reason, we will not use $\widetilde{\boldsymbol{H}}^{\mathrm{NA}}$ in this paper.

2.6 g–Ding-stability and Kähler–Ricci solitons

Let $\mathcal{F}$ be a $\mathbb{T}$–equivariant filtration. For any $\lambda \in \mathbb{R}$, we have a (finite) decomposition

$$\mathcal{F}^{\lambda} R_m = \bigoplus_{\alpha \in M_{\mathbb{Z}}} \mathcal{F}^{\lambda} R_{m,\alpha}. \tag{103}$$

Let P be the moment polytope of $(X, -K_X)$ with respect to the $\mathbb{T}$–action. Let g be a smooth positive function on P. Fix a faithful $\mathbb{Z}^n$–valuation that is adapted to the torus action (see Definition 2.21) and let $\Delta \subset \mathbb{R}^n$ be the Okounkov body that satisfies (65): $p(\Delta) = P$, where $p: \mathbb{R}^n \to \mathbb{R}^r$ is the natural projection. Still denote by g the function $p^* g$ on Δ. Define the g–volume of graded linear series $\{\mathcal{F}^{(t)} R_m\}$ as

$$\mathrm{vol}_g(\mathcal{F}^{(t)}) := \lim_{m\to+\infty} \sum_{\alpha} g\Big(\frac{\alpha}{m}\Big) \frac{\dim \mathcal{F}^{mt} R_{m,\alpha}}{m^n/n!} = n! \cdot \int_{\Delta(\mathcal{F}^{(t)})} g(y)\, dy_{\mathrm{Leb}} =: n! \cdot \mathrm{vol}_g(\Delta(\mathcal{F}^{(t)})).$$

Then, as in the $g \equiv 1$ case, we have the convergence

$$\mathrm{DH}_g(\mathcal{F}) := \lim_{m\to+\infty} \sum_{\alpha} g\Big(\frac{\alpha}{m}\Big) \delta_{\lambda_i^{(m,\alpha)}/m} = -d\,\mathrm{vol}_g(\mathcal{F}^{(t)}) = n! \cdot (G_{\mathcal{F}})_*(g(y)\, dy_{\mathrm{Leb}}).$$

We also set

$$\boldsymbol{V}_g := n! \cdot \mathrm{vol}_g(\Delta) = n! \cdot \int_{\Delta} g(y)\, dy_{\mathrm{Leb}} = \int_{\mathbb{R}} \mathrm{DH}_g(\mathcal{F}), \tag{104}$$

$$\boldsymbol{E}_g^{\mathrm{NA}}(\mathcal{F}) := \frac{n!}{\boldsymbol{V}_g} \int_{\Delta} G_{\mathcal{F}}(y) g(y)\, dy_{\mathrm{Leb}} = \frac{1}{\boldsymbol{V}_g} \int_{\mathbb{R}} \lambda \cdot \mathrm{DH}_g(\mathcal{F}), \tag{105}$$

$$\boldsymbol{D}_g^{\mathrm{NA}}(\mathcal{F}) := \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) - \boldsymbol{E}_g^{\mathrm{NA}}(\mathcal{F}). \tag{106}$$

If $(\mathcal{X}, \mathcal{L}, \eta)$ is a test configuration, then we set $\boldsymbol{D}_g^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta) = \boldsymbol{D}_g^{\mathrm{NA}}(\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)})$.

Definition 2.45 (X,ξ) is g–Ding-semistable if $\boldsymbol{D}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L},\eta)\geq 0$ for any $\mathbb{T}$–equivariant test configuration $(\mathcal{X},\mathcal{L},\eta)$ of $(X,-K_X)$.

(X,ξ) is g–Ding-polystable if it is g–Ding-semistable, and $\boldsymbol{D}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L},\eta)=0$ for a $\mathbb{T}$–equivariant weakly special test configuration (see Definition 2.12) only if $(\mathcal{X},\mathcal{L},\eta)$ is a product test configuration.

The following result was proved by adapting the techniques of MMP from [56; 37; 7].

Theorem 2.46 (see [39]) *To test the g–Ding-semistability, or the g–Ding-polystability, of (X,ξ), it suffices to test over all special test configurations.*

We have the following valuative criterion:

Theorem 2.47 [39] *X is g–Ding-semistable if and only if for any $v\in(X_{\mathbb{Q}}^{\mathrm{div}})^{\mathbb{T}}$, we have*

$$\beta_g(v):=A_X(v)-\frac{1}{\boldsymbol{V}_g}\int_0^{+\infty}\mathrm{vol}_g(\mathcal{F}_v^{(t)})\,dt\geq 0. \tag{107}$$

Now we use our notation to reformulate the holomorphic invariants of Tian and Zhu [71] in the study of Kähler–Ricci solitons. We refer to [71; 8; 39] for more details and references. Let X be a $\mathbb{Q}$–Fano variety with an effective $\mathbb{T}$–action. We use the same notation, such as an $(S^1)^r$–invariant smooth Hermitian metric $\widetilde{\varphi}$ on $-K_X$, a moment polytope $P\subset M_{\mathbb{R}}$, a function $\theta_{\widetilde{\varphi}}(\eta)=\mathfrak{L}_\eta e^{-\widetilde{\varphi}}/e^{-\widetilde{\varphi}}$, etc. We identify any $\eta\in N_{\mathbb{R}}$ with the corresponding holomorphic vector field on X.

A Kähler–Ricci soliton on (X,ξ) is a positively curved bounded Hermitian metric $e^{-\varphi}$ on $-K_X$ that satisfies the equation

$$e^{\varphi}(\mathrm{dd^c}\varphi)^n=e^{\theta_\varphi(\xi)}, \tag{108}$$

where $\theta_\varphi(\xi)=\theta_{\widetilde{\varphi}}(\xi)+\xi(\varphi-\widetilde{\varphi})$. Over X^{reg}, φ is smooth [6; 39] and satisfies the identity

$$\mathrm{Ric}(\mathrm{dd^c}\varphi)-\mathrm{dd^c}\varphi=-\mathrm{dd^c}\theta_\varphi(\xi). \tag{109}$$

As a consequence, the family of metrics $\varphi(s):=\sigma_\xi(s)^*\varphi$ satisfies the normalized Kähler–Ricci flow,

$$\frac{d}{ds}\mathrm{dd^c}\varphi(s)=-\mathrm{Ric}(\mathrm{dd^c}\varphi(s))+\mathrm{dd^c}\varphi(s). \tag{110}$$

For any $\xi\in N_{\mathbb{R}}$, we set $g_\xi(x)=e^{-\langle x,\xi\rangle}=e^{-\sum_{i=1}^r\xi^i x_i}$, which is a smooth positive function on P, and write $\boldsymbol{F}_{g_\xi}$ as $\boldsymbol{F}_\xi$ for $\boldsymbol{F}\in\{\boldsymbol{L},\boldsymbol{D}\}$ etc, and $\boldsymbol{V}_\xi:=V_{g_\xi}$. Tian and Zhu [71] defined a modified Futaki invariant as an obstruction to the existence of Kähler–Ricci solitons on (X,ξ): for any $\eta\in N_{\mathbb{R}}$,

$$\mathrm{Fut}_\xi(\eta):=-\frac{1}{\boldsymbol{V}_\xi}\int_X\theta_{\widetilde{\varphi}}(\eta)e^{-\theta_{\widetilde{\varphi}}(\xi)}(\mathrm{dd^c}\widetilde{\varphi})^n=\boldsymbol{D}_\xi^{\mathrm{NA}}(\mathrm{wt}_\eta), \tag{111}$$

where $\boldsymbol{V}_\xi=\int_X e^{-\theta_{\widetilde{\varphi}}(\xi)}(\mathrm{dd^c}\widetilde{\varphi})^n$. The second identity follows by noting that $\boldsymbol{D}_\xi^{\mathrm{NA}}(\mathrm{wt}_\eta)=-\boldsymbol{E}_\xi^{\mathrm{NA}}(\mathrm{wt}_\eta)$ because of the vanishing $\boldsymbol{L}^{\mathrm{NA}}(\mathrm{wt}_\eta)=0$.

Remark 2.48 Again, here we have used the negative sign convention compared to [71].

Fut_ξ does not depend on the choice of $\widetilde{\varphi}$ and (X,ξ) admits a KR soliton only if $\mathrm{Fut}_\xi \equiv 0$ on $N_{\mathbb{R}}$. Moreover, by [71, Lemma 2.2] the soliton vector field is a priori uniquely determined by minimizing the strictly convex functional (Tian and Zhu didn't use the logarithm) on $N_{\mathbb{R}}$ (see Lemma 2.38), which is the antiderivative of $\eta \mapsto \mathrm{Fut}_\xi(\eta)$,

$$\xi \mapsto \log\left(\frac{1}{V}\int_X e^{-\theta_{\widetilde{\varphi}}(\xi)}(\mathrm{dd^c}\widetilde{\varphi})^n\right) = \log\left(\frac{1}{V}\int_{\mathbb{R}} e^{-\lambda}\,\mathrm{DH}(\mathcal{F}_{\mathrm{wt}_\xi})\right) = -\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathrm{wt}_\xi). \tag{112}$$

Recall also that $\boldsymbol{L}^{\mathrm{NA}}(\mathrm{wt}_\eta) = \widehat{\boldsymbol{L}}^{\mathrm{NA}}(\mathrm{wt}_\eta) \equiv 0$ on $N_{\mathbb{R}}$; see (94). Combining these discussions we get the derivative identity

$$\frac{d}{ds}\boldsymbol{H}^{\mathrm{NA}}(\mathrm{wt}_{\xi+s\eta}) = \frac{d}{ds}\boldsymbol{H}^{\mathrm{NA}}(\mathrm{wt}_{\xi+s\eta}) = \boldsymbol{D}^{\mathrm{NA}}_{\xi}(\mathrm{wt}_\eta) = \mathrm{Fut}_\xi(\eta). \tag{113}$$

For simplicity of notation, we introduce:

Definition 2.49 We say that (X,ξ) is K–semistable (resp. K–polystable) if X is g_ξ–Ding-semistable (resp. g_ξ–Ding-polystable).

Remark 2.50 Since by Theorem 2.46 it is enough to test the stability on special test configurations, this definition coincides with the original modified K–(poly)stability adopted by Tian as well as Berman, Witt and Nyström, and others. To respect the original notation, we will just call (X,ξ) K–(poly)stable, although we will also freely use the notion of Ding-(poly)stability.

By [8; 31], when X is smooth, the Yau–Tian–Donaldson conjecture is true, ie K–polystability is equivalent to the existence of Kähler–Ricci solitons. For singular X, we proved in [39] a version of the Yau–Tian–Donaldson conjecture involving $\mathrm{Aut}(X,\xi)_0$–uniform Ding-stability.

3 H^{NA} invariant and MMP

3.1 An intersection formula for higher moments

Let $(\mathcal{X},\mathcal{L},\eta)$ be any normal ample test configuration. Choose a smooth (semipositive) curvature form ω in $c_1(\mathcal{L}|_{\mathcal{X}_0})$. Let θ be the Hamiltonian function for η with respect to ω, so $\iota_\eta\omega = (\sqrt{-1}/2\pi)\,\bar\partial\theta$. By the equivariant Riemann–Roch formula, we get

$$\boldsymbol{E}^{\mathrm{NA}}_k(\mathcal{X},\mathcal{L}) := \boldsymbol{E}^{\mathrm{NA}}_k(\mathcal{F}_{(\mathcal{X},\mathcal{L})}) = \lim_{m\to+\infty}\frac{1}{N_m}\sum_i\left(\frac{\lambda_i^{(m)}}{m}\right)^k = \frac{1}{V}\int_{\mathcal{X}_0}\theta^k\omega^n.$$

To motivate our calculations, we will first give a direct proof of two identities which can already be derived from the above discussion.

Lemma 3.1 *We have*

$$\boldsymbol{E}^{\mathrm{NA}}_k(\mathcal{X},\mathcal{L}) = \frac{1}{V}\int_{\mathbb{R}} x^k\,\mathrm{DH}(\mathcal{F}^{(x)}), \tag{114}$$

$$\boldsymbol{E}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \boldsymbol{E}^{\mathrm{NA}}_1(\mathcal{X},\mathcal{L}) = \frac{1}{V}\frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1}. \tag{115}$$

Proof When we change $\mathcal{L}$ to $\mathcal{L}+d\mathcal{X}_0$, $\mathcal{F}$ is changed to $\mathcal{F}(d)$, and both sides of the above identities have d added to them. So we can assume that $\overline{\mathcal{L}}$ is very ample over $\overline{\mathcal{X}}$. Then we have

$$\overline{\mathcal{X}} = \mathrm{Proj}\Big(\bigoplus_{m\geq 0}\bigoplus_{j=0}^{+\infty} t^{-j}\mathcal{F}^j R_m\Big) \tag{116}$$

and $\overline{\mathcal{L}}_d = \mathcal{O}_{\overline{\mathcal{X}}}(1)$.

For simplicity of notation, we write

$$\begin{aligned} f_k(m) = \sum_{i=1}^{N_m}(\lambda_i^{(m)})^k &= \sum_{j=0} j^k(\dim \mathcal{F}^j R_m - \dim \mathcal{F}^{j+1} R_m) \\ &= \sum_{j=1}(j^k-(j-1)^k)\dim \mathcal{F}^j R_m \ = \sum_{j=1}(kj^{k-1}+O(j^{k-2}))\dim \mathcal{F}^j R_m. \end{aligned} \tag{117}$$

We easily get the identity

$$\boldsymbol{E}_k^{\mathrm{NA}} = \frac{1}{\boldsymbol{V}}\lim_{m\to+\infty}\frac{n!}{m^{n+k}} f_k(m) = \frac{1}{\boldsymbol{V}}\int_0^\infty kx^{k-1}\,\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\,dx = \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} x^k(-d\,\mathrm{vol}(\mathcal{F}^{(x)})). \tag{118}$$

Moreover, we have the dimension formula

$$\mathcal{N}_m := h^0(\overline{\mathcal{X}}, m\overline{\mathcal{L}}) = \sum_{j=0}^{+\infty}\dim \mathcal{F}^j R_m = \frac{m^{n+1}}{n!}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\,dx + O(m^n),$$

which, by the Riemann–Roch formula, gives the identity

$$\frac{1}{\boldsymbol{V}}\frac{\overline{\mathcal{L}}^{\cdot n+1}}{n+1} = \frac{1}{\boldsymbol{V}}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\,dx = \frac{1}{\boldsymbol{V}}\int_0^{+\infty} x\,\mathrm{DH}(\mathcal{F}). \tag{119}$$ □

The formula (115) goes back to Mumford's study of GIT [61], and has also been used in the study of K–stability. The following result is a generalization of it to higher moments. We will use the following notation as in [39]. Let $\mathbb{C}^*\to\mathbb{C}^{k+1}\setminus\{0\}\to\mathbb{P}^k$ be the principal $\mathbb{C}^*$–bundle and set

$$(\overline{\mathcal{X}}^{[k]}, \overline{\mathcal{L}}^{[k]}) := ((\overline{\mathcal{X}}, \overline{\mathcal{L}})\times(\mathbb{C}^{k+1}\setminus\{0\}))/\mathbb{C}^*. \tag{120}$$

Remark 3.2 Since the $\mathbb{C}^*$–action on $\overline{\mathcal{X}}$ moves the fiber $\overline{\mathcal{X}}\to\mathbb{P}^1$, the situation here is different from the situation in [19, Corollary 3.4] or [39], where a similar fiber construction with respect to a vertical torus action is used.

Proposition 3.3 *Let $(\mathcal{X},\mathcal{L})$ be a normal ample test configuration. For any $k\geq 1$, we have the intersection formula*

$$\boldsymbol{E}_k^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{1}{\boldsymbol{V}}\frac{k!\,n!}{(n+1)!}(\overline{\mathcal{L}}^{[k-1]})^{\cdot n+k}. \tag{121}$$

Proof We use the notation from the above proof and without loss of generality assume that $\bar{\mathcal{L}}$ is very ample over $\bar{\mathcal{X}}$.

The weights $\{\mu_\alpha \mid \alpha = 1, \dots, \mathcal{N}_m\}$ and multiplicities of $\mathbb{C}^*$–action on $H^0(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ are given according to the isomorphism (44). By the identity (117), the weight of $\mathbb{C}^*$ on $\det H^0(\bar{\mathcal{X}}, m\mathcal{L})$ is given by

$$\sum_{\alpha=1}^{\mathcal{N}_m} \mu_\alpha^{k-1} = \sum_{j=0}^{+\infty} j^{k-1} \dim \mathcal{F}^j R_m = k^{-1} f_k(m) + O(m^{n+k-1}). \tag{122}$$

Choose a smooth Kähler metric $\Omega \in c_1(\bar{\mathcal{L}})$ on $\bar{\mathcal{X}}$ and let Θ be the Hamiltonian function for η. Then by the equivariant Riemann–Roch formula, we get

$$\begin{aligned}\lim_{m\to+\infty} \frac{(n+1)!}{m^{n+1}} \sum_\alpha \left(\frac{\mu_\alpha}{m}\right)^{k-1} = \int_{\bar{\mathcal{X}}} \Theta^{k-1}\Omega^{n+1} &= \frac{(k-1)!\,(n+1)!}{(k+n)!} \int_{\bar{\mathcal{X}}^{[k-1]}} (\Omega + \Theta t)^{n+k} \\ &= \frac{(k-1)!\,(n+1)!}{(k+n)!} (\bar{\mathcal{L}}^{[k-1]})^{\cdot n+k}.\end{aligned} \tag{123}$$

Combining (118), (122) and (123), we get

$$\begin{aligned}\boldsymbol{E}_k^{\mathrm{NA}} = \frac{1}{\boldsymbol{V}} \lim_{k\to} \frac{n!}{m^{n+k}} k \sum_\alpha \mu_\alpha^{k-1} &= \frac{1}{\boldsymbol{V}} \frac{k}{n+1} \frac{(k-1)!\,(n+1)!}{(k+n)!} (\bar{\mathcal{L}}^{[k-1]})^{\cdot n+k} \\ &= \frac{1}{\boldsymbol{V}} \frac{k!\,n!}{(k+n)!} (\bar{\mathcal{L}}^{[k-1]})^{\cdot n+k}.\end{aligned}$$

□

Recall from (82) that $\tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = -\log \boldsymbol{Q}^{(a)}$, where

$$\boldsymbol{Q}^{(a)} = \frac{1}{\boldsymbol{V}} \int_{\mathcal{X}_0} e^{-a\theta}\omega^n = \sum_{k=0}^{\infty} (-1)^k a^k \frac{1}{\boldsymbol{V}} \int_{\mathcal{X}_0} \frac{\theta^k}{k!}\omega^n = \sum_k (-1)^k \frac{a^k}{k!} \boldsymbol{E}_k^{\mathrm{NA}}.$$

Proposition 3.4 *Let $(\mathcal{X}, \mathcal{L}_\lambda, a\eta)_{\lambda\in(-\epsilon,\epsilon)}$ be a family of normal test configurations of $(X, -K_X)$, with a fixed total space and varying polarization. Assume that $\mathcal{X}_0 = \sum_i b_i E_i$ for irreducible components E_i, and that $\mathcal{L}_\lambda$ is differentiable with respect to λ. Then we have the derivative formula*

$$\frac{d}{d\lambda} \tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) = a \frac{\sum_i e_i \boldsymbol{Q}_i^{(a)}}{\boldsymbol{Q}^{(a)}}, \tag{124}$$

where $\boldsymbol{Q}_i^{(a)} = (1/\boldsymbol{V}) \int_{E_i} e^{-a\theta}\omega^n$.

Proof We use the intersection formula (121) to get

$$\begin{aligned}V \cdot \frac{d}{d\lambda} \boldsymbol{E}_k^{\mathrm{NA}} = \frac{d}{d\lambda} \frac{k!\,n!}{(k+n)!} (\bar{\mathcal{L}}^{[k-1]})^{\cdot n+k} &= \frac{k!\,n!}{(k+n-1)!} (\bar{\mathcal{L}}^{[k-1]})^{\cdot n+k-1} \cdot \dot{\bar{\mathcal{L}}}^{[k-1]} \\ &= \frac{k!\,n!}{(k+n-1)!} \sum_i e_i \int_{E_i^{[k-1]}} (\Omega + \Theta t)^{n+k-1} = k \sum_i e_i \int_{E_i} \theta^{k-1}\omega^n,\end{aligned}$$

where $E_i^{[k-1]} = (E_i \times \mathbb{C}^{k-1} \setminus \{0\})/\mathbb{C}^*$. So we get the desired formula

$$\begin{aligned}\frac{d}{d\lambda}\boldsymbol{Q}^{(a)} &= \sum_k (-1)^k \frac{a^k}{k!}\frac{d}{d\lambda}\boldsymbol{E}_k^{\mathrm{NA}} = \sum_{k=1}(-1)^k \frac{a^k}{(k-1)!}\sum_i e_i \frac{1}{\boldsymbol{V}}\int_{E_i}\theta^{k-1}\omega^n \\ &= -a\sum_i e_i \frac{1}{\boldsymbol{V}}\int_{E_i}\sum_{j=0}\frac{(-1)^j}{j!}(a\theta)^j\omega^n = -a\sum_i e_i\frac{1}{\boldsymbol{V}}\int_{E_i} e^{-a\theta}\omega^n = -a\sum_i \boldsymbol{Q}_i^{(a)}.\end{aligned}$$

The termwise differentiation and the change of summation are valid because of absolute convergence. □

3.2 Decreasing of H^{NA} along MMP

Theorem 3.5 *Let $\mathbb{G}$ be a reductive group and $(\mathcal{X}, \mathcal{L}, a\eta)$ be a $\mathbb{G}$–equivariant normal test configuration. There exists a $\mathbb{G}$–equivariant special test configuration $(\mathcal{X}^s, \mathcal{L}^s, a^s\eta^s)$ such that*

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) \ge \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{L}^s, a^s\eta^s). \tag{125}$$

Moreover, if $\mathcal{X}_0$ is reduced, then the identity holds true if $\mathcal{X}$ is already a special test configuration.

Proof For simplicity of notation, we assume $\mathbb{G}$ is trivial. The general case is obtained by running the $\mathbb{G}$–equivariant MMP in the following arguments.

Step 1 Choose a semistable reduction of $\mathcal{X} \to \mathbb{C}$. By this, we mean that there is an integer d and a $\mathbb{G}$–equivariant log resolution of singularities $\widetilde{\mathcal{X}} \to \mathcal{X}^{(d_1)} := (\mathcal{X} \times_{\mathbb{C}, t\to t^{d_1}} \mathbb{C})^{\mathrm{norm}}$ (see (49)) such that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}}_0)$ is simple normal crossing. In particular, $\mathcal{X}_0^{(d_1)}$ is reduced. By using the identity (51) and Lemma 2.35 we easily get

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^{(d_1)}, \mathcal{L}^{(d_1)}, a\eta^{(d_1)}/d_1) = \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta). \tag{126}$$

Step 2 In this step, we show that there exist $d_1 \in \mathbb{Z}_{>0}$, a projective birational $\mathbb{C}^*$–equivariant morphism $\pi: \mathcal{X}^{\mathrm{lc}} \to \mathcal{X}^{(d_1)}$ and a normal, ample test configuration $(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}})/\mathbb{C}$ for (X, L), such that

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^{(d_1)}, \mathcal{L}^{(d_1)}, a\eta^{(d_1)}/d_1) \ge \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}}, a\eta^{\mathrm{lc}}/d_1). \tag{127}$$

Moreover, if the equality holds, then $(\mathcal{X}^{(d_1)}, \mathcal{L}^{(d_1)})$ is isomorphic to $(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}})$, and hence $(\mathcal{X}, \mathcal{X}_0)$ is already log canonical.

We run a $\mathbb{C}^*$–equivariant MMP to get a log canonical modification $\pi^{\mathrm{lc}}: \mathcal{X}^{\mathrm{lc}} \to \mathcal{X}^{(d_1)}$ such that $(\mathcal{X}^{\mathrm{lc}}, \mathcal{X}_0^{\mathrm{lc}})$ is log canonical and $K_{\mathcal{X}^{\mathrm{lc}}}$ is relatively ample over $\mathcal{X}^{(d_1)}$. Set $E = K_{\mathcal{X}^{\mathrm{lc}}} + (\pi^{\mathrm{lc}})^*\mathcal{L} = \sum_{i=1}^k e_i \mathcal{X}_{0,i}$ with $e_1 \le e_2 \le \cdots \le e_k$ and $\mathcal{L}_\lambda^{\mathrm{lc}} = (\pi^{\mathrm{lc}})^*\mathcal{L}^{(d_1)} + \lambda E$. Then since E is relatively ample over $\mathcal{X}^{(d_1)}$, $\mathcal{L}_\lambda$ is ample over $\mathcal{X}^{\mathrm{lc}}$ for $0 < \lambda \ll 1$. So

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_\lambda^{\mathrm{lc}}, a\eta^{\mathrm{lc}}/d_1) = \frac{a}{d_1}\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_\lambda^{\mathrm{lc}}, \eta^{\mathrm{lc}}) = \frac{a}{d_1}(1+\lambda)e_1.$$

By definition (76), we have

$$\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_\lambda^{\mathrm{lc}}, a\eta^{\mathrm{lc}}/d_1) = -\log \boldsymbol{Q}^{(ad_1^{-1})}, \quad \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_\lambda^{\mathrm{lc}}, a\eta^{\mathrm{lc}}/d_1) = \frac{a(1+\lambda)e_1}{d_1} + \log \boldsymbol{Q}^{(ad_1^{-1})}.$$

We then use (124) to calculate

$$\frac{d}{d\lambda}\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^{\mathrm{lc}},\mathcal{L}^{\mathrm{lc}}_{\lambda},a\eta^{\mathrm{lc}}/d_1)=\frac{ae_1}{d_1}-\frac{a}{d_1}\frac{\sum_i e_i\boldsymbol{Q}_i^{(ad_1^{-1})}}{\sum_i\boldsymbol{Q}_i^{(ad_1^{-1})}}=-\frac{a}{d_1}\frac{\sum_i(e_i-e_1)\boldsymbol{Q}_i^{(ad_1^{-1})}}{\boldsymbol{Q}^{(ad_1^{-1})}}\le 0.$$

The last identity holds if and only if $e_i\equiv e_1$, and hence $(\mathcal{X}^{(d_1)},\mathcal{L}^{(d_1)})\cong(\mathcal{X}^{\mathrm{lc}},\mathcal{L}^{\mathrm{lc}})$. In this case, $(\mathcal{X}^{(d_1)},\mathcal{X}_0^{(d_1)})$ is log canonical, which implies that $(\mathcal{X},\mathcal{X}_0)$ is already log canonical, by the pullback formula for the log differential; see [56, page 210].

Step 3 With the $(\mathcal{X}^{\mathrm{lc}},\mathcal{L}^{\mathrm{lc}})$ obtained from the first step, we run a relative MMP with scaling to get a normal, ample test configuration $(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})/\mathbb{P}^1$ for $(X,-K_X)$ with $(\mathcal{X}^{\mathrm{ac}},\mathcal{X}^{\mathrm{ac}}_0)$ log canonical such that $-K_{\mathcal{X}^{\mathrm{ac}}}\sim_{\mathbb{Q},\mathbb{C}}\mathcal{L}^{\mathrm{ac}}$. More concretely, we take $q\gg 1$ such that $\mathcal{H}^{\mathrm{lc}}=\mathcal{L}^{\mathrm{lc}}-(q+1)^{-1}(\mathcal{L}^{\mathrm{lc}}+K_{\mathcal{X}^{\mathrm{lc}}})$ is relatively ample. Set $\mathcal{X}^0=\mathcal{X}^{\mathrm{lc}}$, $\mathcal{L}^0=\mathcal{L}^{\mathrm{lc}}$, $\mathcal{H}^0=\mathcal{H}^{\mathrm{lc}}$ and $\lambda_0=q+1$. Then $K_{\mathcal{X}^0}+\lambda_0\mathcal{H}^0=q\mathcal{L}^0$. We run a sequence of $K_{\mathcal{X}^0}$–MMP over $\mathbb{C}$ with scaling of $\mathcal{H}^0$. Then we obtain a sequence of models

$$\mathcal{X}^0\dashrightarrow\mathcal{X}^1\dashrightarrow\cdots\dashrightarrow\mathcal{X}^k$$

and a sequence of critical values

$$\lambda_{i+1}=\min\{\lambda\mid K_{\mathcal{X}^i}+\lambda\mathcal{H}^i\text{ is nef over }\mathbb{C}\}$$

with $q+1=\lambda_0\ge\lambda_1\ge\cdots\ge\lambda_k>\lambda_{k+1}=1$. For any $\lambda_i\ge\lambda\ge\lambda_{i+1}$, we let $\mathcal{H}^i$ be the pushforward of $\mathcal{H}$ to $\mathcal{X}^i$ and set

$$\mathcal{L}^i_\lambda=\frac{1}{\lambda-1}(K_{\mathcal{X}^i}+\lambda\mathcal{H}^i)=\frac{1}{\lambda-1}(K_{\mathcal{X}^i}+\mathcal{H}^i)+\mathcal{H}^i=:\frac{1}{\lambda-1}E+\mathcal{H}^i.\tag{128}$$

Write $E=\sum_{j=1}^k e_j\mathcal{X}^i_{0,j}$ with $e_1\le e_2\le\cdots\le e_k$. Then we have $(d/d\lambda)\mathcal{L}^i_\lambda=-(1/(\lambda-1)^2)E$ and

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}^i,\mathcal{L}^i,a\eta^i/d_1)=\frac{a\lambda}{\lambda-1}e_1.$$

So we can again use (124) to calculate

$$\begin{aligned}\frac{d}{d\lambda}\boldsymbol{H}^{\mathrm{NA}}\left(\mathcal{X}^i,\mathcal{L}^i_\lambda,\frac{a\eta^i}{d_1}\right)&=-\frac{a}{d_1(\lambda-1)^2}e_1+\frac{a}{d_1(\lambda-1)^2}\frac{\sum_i e_i\boldsymbol{Q}_i^{(ad_1^{-1})}}{\boldsymbol{Q}^{(ad_1^{-1})}}\\&=\frac{a}{d_1(\lambda-1)^2}\frac{\sum_i(e_i-e_1)\boldsymbol{Q}_i^{(a)}}{\boldsymbol{Q}^{(a)}}\ge 0.\end{aligned}$$

The last identity holds only if $e_i\equiv e_1$, which implies $(\mathcal{X}^{\mathrm{lc}},\mathcal{L}^{\mathrm{lc}})\cong(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}}+e_1\mathcal{X}^{\mathrm{ac}}_0)$.

Step 4 With the test configuration $(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})$ obtained from Step 2, there exists a $d_2\in\mathbb{Z}_{>0}$ and a projective birational $T_{\mathbb{C}}\times\mathbb{C}^*$–equivariant birational map $(\mathcal{X}^{\mathrm{ac}})^{(d_2)}\dashrightarrow\mathcal{X}^s$ over $\mathbb{P}^1$ such that $(\mathcal{X}^s,-K_{\mathcal{X}^s})$ is a special test configuration and

$$\boldsymbol{H}^{\mathrm{NA}}\left(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}},\frac{a\eta}{d_1d_2}\right)\ge\boldsymbol{H}^{\mathrm{NA}}\left(\mathcal{X}^s,\mathcal{L}^s,\frac{a\eta^s}{d_1d_2}\right).\tag{129}$$

As in [56], this is achieved by doing a base change and running an MMP. Let $E=-K_{\mathcal{X}^s/\mathbb{P}^1}-(-K_{\mathcal{X}'/\mathbb{P}^1})$. Then $E\geq 0$ by the negativity lemma. So $\mathcal{L}'_\lambda=-K_{\mathcal{X}'/\mathbb{P}^1}+\lambda E$, and

$$\operatorname{lct}\left(\mathcal{X}',\mathcal{L}'_\lambda,\frac{a\eta'}{d_1d_2}\right)=\frac{a}{d_1d_2}\lambda e_1. \tag{130}$$

So, as before, we get

$$\frac{d}{d\lambda}\boldsymbol{H}^{\mathrm{NA}}\left(\mathcal{X}',\mathcal{L}'_\lambda,\frac{a\eta'}{d_1d_2}\right)=\frac{a}{d_1d_2}e_1-\frac{a}{d_1d_2}\frac{\sum_i e_i\boldsymbol{Q}_i^{(ad_1^{-1})}}{\boldsymbol{Q}^{(ad_1^{-1})}}=-\frac{a}{d_1d_2}\frac{\sum_i(e_i-e_1)\boldsymbol{Q}_i^{(ad_1^{-1})}}{\boldsymbol{Q}^{(ad_1^{-1})}}\leq 0.$$

The last identity holds only if $e_i\equiv e_1$ which implies $(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})\cong(\mathcal{X}^s,\mathcal{L}^s)$. □

Corollary 3.6 *We have the identity*

$$h(X)=\inf_{(\mathcal{X},\mathcal{L},a\eta)\text{ special}}\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a\eta). \tag{131}$$

Lemma 3.7 *For any normal test configuration $(\mathcal{X},\mathcal{L},\eta)$, there exists a unique $a_*>0$ such that*

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a_*\eta)=\inf_{c>0}\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a\eta)=:\boldsymbol{H}_*^{\mathrm{NA}}(\mathcal{X},\mathcal{L}). \tag{132}$$

As a consequence, we have

$$h(X)=\inf_{(\mathcal{X},-K_{\mathcal{X}})\text{ special}}\boldsymbol{H}_*^{\mathrm{NA}}(\mathcal{X},\mathcal{L}). \tag{133}$$

Proof By taking normalization of a fiber product, without loss of generality we can assume that $\mathcal{X}$ dominates $X_{\mathbb{C}}:=X\times\mathbb{C}$ by a $\mathbb{C}^*$–equivariant birational morphism $\rho\colon\mathcal{X}\to X_{\mathbb{C}}$.

Choose a $\mathbb{C}^*$–equivariant resolution of singularities $\mu\colon\widetilde{\mathcal{X}}\to\mathcal{X}$ such that the pair $(\widetilde{\mathcal{X}},\widetilde{\mathcal{X}}_0^{\mathrm{red}})$ has simple normal crossing singularities. Set $\widetilde{\rho}=\rho\circ\mu$. Then we can write

$$K_{\widetilde{\mathcal{X}}}=\widetilde{\rho}^*K_{X_{\mathbb{C}}}+\sum_i a_iE_i+\sum_j a'_jE'_j,\quad \pi^*\mathcal{X}_0=\sum_i b_iE_i,\quad \mu^*\mathcal{L}=\widetilde{\rho}^*(-K_{X_{\mathbb{C}}})+\sum_i c_iE_i,$$

where $\{E_i\}$ are irreducible components of $\widetilde{\mathcal{X}}_0$ and $\{E'_j\}$ are horizontal exceptional divisors. Then we have the identity (see [19, Proposition 7.29])

$$\begin{aligned}\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X},\mathcal{L})&=\operatorname{lct}(\mathcal{X},-(K_{\mathcal{X}}+\mathcal{L});\mathcal{X}_0)-1\\&=\min_i\left(b_i^{-1}A_{(X\times\mathbb{C},X\times\{0\})}(E_i)+b_i^{-1}\operatorname{ord}_{E_i}\left(\sum_k c_kE_k\right)\right)=\min_i\frac{1+a_i+c_i}{b_i}-1.\end{aligned}$$

Because $\boldsymbol{H}^{\mathrm{NA}}$ is translation-invariant, by adding a multiple of $\mathcal{X}_0$ to $\mathcal{L}$ we can normalize $\phi=\phi_{\mathcal{F}}$ to satisfy $\boldsymbol{L}^{\mathrm{NA}}(\phi)=0$. So we get

$$c_i\geq b_i-1-a_i \tag{134}$$

and, without loss of generality, $c_1=b_1-1-a_1$. So

$$\lambda_{\min}=\min_i\frac{c_i}{b_i}\leq\frac{c_1}{b_1}=1-\frac{a_1+1}{b_1}=1-A_{X_{\mathbb{C}}}(b_1^{-1}\operatorname{ord}_{E_1})=-A_X(v_{E_1})\leq 0, \tag{135}$$

where $v_{E_1} := r(b_1^{-1}\mathrm{ord}_{E_1})$ is the restriction of the valuation ord_{E_1} to the function field $\mathbb{C}(X)$. Here we used the identity between log discrepancies from [19, Proposition 4.11] and the assumption that X has log terminal singularities.

Set $\mathcal{F} = \mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)}$. Then according to (43), we have $\mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)} = a\mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)}$. Moreover, by (74), (76) and (90), we get the expression

$$f(a) := \boldsymbol{H}^{\mathrm{NA}}(a\mathcal{F}) = a\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}) + \log\biggl(\frac{1}{\boldsymbol{V}}\int_{\lambda_{\min}}^{+\infty} e^{-a\lambda}\,\mathrm{DH}(\mathcal{F})\biggr).$$

By Lemma 2.36, $f(a)$ is convex in $a \in [0,+\infty)$. If $\lambda_{\min} < 0$, then $f(a)$ diverges to $+\infty$ as $a \to +\infty$ by the above expression. So $f(a)$ admits a unique minimum over $[0,+\infty)$.

If $\lambda_{\min} = 0$, then by (135) $A_X(v_{E_1}) = 0$, which implies that v_{E_1} is trivial so that

$$\lambda_{\max} = \mathrm{ord}_{E_1}\biggl(\sum_k c_k E_k\biggr) = c_1 = 0 = \lambda_{\min}.$$

See [19, Theorem 5.16]. This implies that the normal test configuration $(\mathcal{X},\mathcal{L})$ is equivalent to a trivial test configuration and hence $\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L},a\eta) \equiv 0$. □

4 A minimization problem for real valuations

In this section, we will introduce a minimization problem for valuations analogous to the normalized volume functional in the local setting [51].

Definition 4.1 For any $v \in \mathrm{Val}(X)$, define

$$\tilde{\beta}(v) = \begin{cases} A_X(v) - \tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_v) & \text{if } A_X(v) < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{136}$$

Note that by integration by parts we have

$$\begin{aligned} e^{-\tilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_v)} &= \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} e^{-x}\,\mathrm{DH}(\mathcal{F}_v) = \frac{1}{\boldsymbol{V}}\int_0^{+\infty} e^{-x}(-d\mathrm{vol}(\mathcal{F}_v^{(x)})) \\ &= 1 - \frac{1}{\boldsymbol{V}}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}_v^{(x)}R_\bullet)e^{-x}\,dx \le 1, \end{aligned} \tag{137}$$

with identity if and only if v is trivial. So we can rewrite $\tilde{\beta}(v)$ as

$$\tilde{\beta}(v) = A_X(v) + \log\biggl(1 - \frac{1}{\boldsymbol{V}}\int_0^{+\infty} e^{-x}\,\mathrm{vol}(\mathcal{F}_v^{(x)}R_\bullet)\,dx\biggr). \tag{138}$$

Lemma 4.2 *For any $v \in X_{\mathbb{Q}}^{\mathrm{div}}$, we have the inequality*

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_v) \le \tilde{\beta}(v). \tag{139}$$

Moreover, if $(\mathcal{X},-K_{\mathcal{X}},a\eta)$ is a special test configuration, then equality holds for $v = av_{\mathcal{X}_0} = a\cdot r(\mathrm{ord}_{\mathcal{X}_0})$. (See Definition 2.12.)

Proof The inequality follows immediately from

$$\inf_w (A(w)+\phi_v(w)) \le A(v)+\phi_v(v) = A(v). \tag{140}$$

When $(\mathcal{X}, -K_{\mathcal{X}}, a\eta)$ is a special test configuration and $v = a v_{\mathcal{X}_0}$, then

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}, -K_{\mathcal{X}}, a\eta) = a\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}, -K_{\mathcal{X}}, \eta) = a(\mathrm{lct}(\mathcal{X};\mathcal{X}_0)-1) = 0. \tag{141}$$

On the other hand, by (46),

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{X}, -K_{\mathcal{X}}, a\eta) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_v(-A(v)) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_v) - A(v). \tag{142}$$

So we get

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_v) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_v) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_v) = A(v) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(v) = \widetilde{\beta}(v). \tag{143}$$ □

Lemma 4.3 *For any $\phi = \phi_{\mathcal{F}}$ and $v \in X^{\mathrm{div}}_{\mathbb{Q}}$, we have the inequality*

$$\widetilde{\boldsymbol{S}}(v) + \phi(v) \ge \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\phi). \tag{144}$$

Proof We use the same argument as in [52, Section 4.1]. Set $\gamma = \phi(v)$. Then by the argument there, we have $\lambda_{\min} = \lambda_{\min}(\mathcal{F}) \le \gamma$ and we can then estimate

$$\begin{aligned} e^{-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\phi)} &= \boldsymbol{Q}(\phi) = \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} e^{-x}(-d\,\mathrm{vol}(\mathcal{F}^{(x)})) = e^{-\lambda_{\min}} - \frac{1}{\boldsymbol{V}}\int_{\lambda_{\min}} e^{-x}\,\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\,dx \\ &\ge e^{-\gamma} - \frac{1}{\boldsymbol{V}}\int_{\gamma}^{+\infty} e^{-x}\,\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\,dx \ge e^{-\gamma} - \frac{1}{\boldsymbol{V}}\int_{\gamma}^{+\infty} e^{-x}\,\mathrm{vol}(\mathcal{F}_v^{(x-a)})\,dx \\ &= e^{-\gamma} - e^{-\gamma}\frac{1}{\boldsymbol{V}}\int_0^{+\infty} e^{-x}\,\mathrm{vol}(\mathcal{F}_v^{(t)})\,dt = e^{-\gamma}\frac{1}{\boldsymbol{V}}\int_0^{+\infty} e^{-x}(-d\,\mathrm{vol}(\mathcal{F}_v^{(t)})) \\ &= e^{-\phi(v)}e^{-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(v)}. \end{aligned}$$ □

Proposition 4.4 *For any $\mathbb{Q}$–Fano variety, we have the identity*

$$h(X) = \inf_{v\in X^{\mathrm{div}}_{\mathbb{Q}}} \widetilde{\beta}(v). \tag{145}$$

Proof For any test configuration $(\mathcal{X}, \mathcal{L}, a\eta)$, by Theorem 3.5 there exists a special test configuration $(\mathcal{X}^s, \mathcal{L}^s, a^s\eta^s)$ such that

$$\boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, a\eta) \ge \boldsymbol{H}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{L}^s, a^s\eta^s) = \widetilde{\beta}(a^s v_{\mathcal{X}^s_0}). \tag{146}$$

The last identity is from Lemma 4.2. This together with Corollary 3.6 implies identity (145).

Alternatively, recall that $\boldsymbol{L}^{\mathrm{NA}}(\phi) = \inf_{v\in X^{\mathrm{div}}_{\mathbb{Q}}}(A_X(v)+\phi(v))$. So for any $\epsilon > 0$ we can choose v such that $A_X(v)+\phi(v) < \boldsymbol{L}^{\mathrm{NA}}(\phi)+\epsilon$. We can then use v in (144) to get

$$\boldsymbol{L}^{\mathrm{NA}}(\phi) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\phi) \ge A_X(v)+\phi(v)-\epsilon-(\phi(v)+\widetilde{\boldsymbol{S}}(v)) = \widetilde{\beta}(v)-\epsilon.$$

Since ϵ is arbitrary, we can use (97) to get the identity (145). □

With the identity (145) and Proposition 2.32, we get:

Corollary 4.5 *For any* $\mathbb{Q}$*–Fano variety, we have the equality*

$$h(X) = \inf_{\mathcal{F}} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}). \tag{147}$$

The next result should be compared to Lemma 3.7.

Proposition 4.6 *For any* $v \in \mathring{\mathrm{Val}}(X)$, *there exists a unique* $a_* = a_*(v) \geq 0$ *such that*

$$\tilde{\beta}(a_* v) = \inf_{a>0} \tilde{\beta}(av) =: \tilde{\beta}_*(v). \tag{148}$$

When $\beta(v) \geq 0$, *then* $a_* = 0$, *so that* $a_* v$ *is the trivial valuation and* $\tilde{\beta}_*(v) = 0$. *Otherwise,* $a_*(v) > 0$ *and* $\tilde{\beta}_*(v) < 0$.

Proof Consider the function defined on $\mathbb{R}_{\geq 0}$ by

$$\begin{aligned} f(a) = A(av) - \tilde{\boldsymbol{S}}^{\mathrm{NA}}(av) &= aA(v) + \log\left(\frac{1}{V}\int_0^{+\infty} e^{-x}\,\mathrm{DH}(\mathcal{F}_{av})\right) \\ &= aA(v) + \log\left(\frac{1}{V}\int_0^{+\infty} e^{-ax}\,\mathrm{DH}(\mathcal{F}_{v})\right). \end{aligned} \tag{149}$$

We will show that $a \mapsto f(a)$ is convex and goes to $+\infty$ as $a \to +\infty$. Now

$$f'(a) = A(v) - \frac{\int_0^{+\infty} x e^{-ax}\,\mathrm{DH}(\mathcal{F}_v)}{\int_0^{+\infty} e^{-ax}\,\mathrm{DH}(\mathcal{F}_v)},$$

$$f''(a) = \frac{\int x^2 e^{-ax}\,\mathrm{DH}}{\int e^{-ax}\,\mathrm{DH}} - \frac{\left(\int x e^{-ax}\,\mathrm{DH}\right)^2}{\left(\int e^{-ax}\,\mathrm{DH}\right)^2} = \|x-\bar{x}\|^2_{L^2(d\nu)} \geq 0,$$

where

$$d\nu = \frac{e^{-ax}\,\mathrm{DH}}{\int e^{-ax}\,\mathrm{DH}} \quad \text{and} \quad \bar{x} = \int x\,d\nu. \tag{150}$$

So $f''(a) = 0$ if and only if av is trivial. Moreover, $f'(0) = A(v) - (1/V)\int_0^{+\infty} x\,\mathrm{DH}(\mathcal{F}_v) = \beta(v)$.

On the other hand, $f(0) = 0$ and we claim that $\lim_{a\to+\infty} f(a) = +\infty$, which then implies the statement. To prove this divergence, we set $g(x) = V^{-1/n}\,\mathrm{vol}(\mathcal{F}^{(x)} R_\bullet)^{1/n}$. Then $g(x)$ is decreasing, and concave on $[0, \lambda_{\max}]$ by Theorem 2.5. As a consequence, the subset $\{x \in \mathbb{R}_{\geq 0} \mid g'(x) \text{ exists}\}$ is dense in $\mathbb{R}_{\geq 0}$, by Aleksandrov's differentiability theorem for concave functions. Fix $0 < \epsilon \ll \lambda_{\max}$ such that $g'(\epsilon)$ exists and $g(\epsilon) < g(0) = 1$. Setting $C = -g'(\epsilon) > 0$ and $T = (1 + C\epsilon)/C$, define a function

$$\hat{g}(x) = \begin{cases} 1 & \text{if } x \in [0,\epsilon], \\ 1 + C\epsilon - Cx & \text{if } x \in (\epsilon, T], \\ 0 & \text{if } x \in (T, +\infty). \end{cases} \tag{151}$$

Then $\hat{g}(x) \geq g(x)$ over $[0, +\infty)$, by concavity. Then we calculate to get

$$a\int_0^{+\infty} \hat{g}^n(x) e^{-ax}\,dx = 1 - nCm_{n-1}, \tag{152}$$

where $m_k = \int_\epsilon^T (1+C\epsilon - Cx)^k e^{-ax}\,dx$ satisfies

$$m_k = \frac{1}{a}e^{-a\epsilon} - \frac{kC}{a}m_{k-1} = \frac{1}{a}e^{-a\epsilon} - \frac{kC}{a}\Big(\frac{1}{a}e^{-a\epsilon} - \frac{(k-1)C}{a}m_{k-2}\Big).$$

Using induction we get $m_{n-1} = a^{-1}e^{-a\epsilon}(1+O(a^{-1}))$. So

$$\begin{aligned} e^{-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}}(\mathcal{F}_{av}) = 1 - a\int_0^{+\infty} g^n(x)e^{-ax}\,dx &\geq 1 - a\int_0^{+\infty} \hat{g}^n(x)e^{-ax}\,dx \\ &= nCm_{n-1} = nCa^{-1}e^{-a\epsilon}(1+O(a^{-1})). \end{aligned}$$

So we get $-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_{av}) \geq -\log a - a\epsilon + O(1)$, giving

$$f(a) = \tilde{\beta}(av) \geq (A(v)-\epsilon)a - \log a + O(1), \tag{153}$$

which approaches $+\infty$ as $a \to +\infty$ if we choose $0 < \epsilon < A(v)$. □

Remark 4.7 By the above proof, we get an estimate: for any $C_1 > 0$, there exists a $C_2 = C_2(C_1, v) > 0$ such that for any $w \in \mathrm{Val}(X)$ with $w \leq C_1 v$, we have

$$a_*(w) \leq \frac{C_2}{A(v)}. \tag{154}$$

Corollary 4.8 *We always have $h(X) \leq 0$, with equality holding if and only if $h(X) = 0$.*

Proof By [37; 50], X is K–semistable if and only if $\beta(v) \geq 0$, which implies $\tilde{\beta}_*(v) = 0$. If X is not K–semistable then there exists v' such that $\beta(v') < 0$. By Proposition 4.6, we then have $\tilde{\beta}_*(v') < 0$, which implies $h(X) < 0$. □

Lemma 4.9 *If v computes $h(X)$, then v is the unique valuation, up to rescaling, that computes $\mathrm{lct}(\mathfrak{a}_\bullet(v))$.*

Proof Recall that $\mathrm{lct}(\mathfrak{a}_\bullet) = \inf_w A(w)/w(\mathfrak{a}_\bullet(w))$. For any $w \in \mathrm{Val}(X)$, assume that $w(\mathfrak{a}_\bullet(v)) = a > 0$. Then $a^{-1}w \geq v$. By Proposition A.1, the function

$$w \mapsto \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(w) = -\log \frac{1}{\boldsymbol{V}}\int_{\mathbb{R}} e^{-\lambda}\,\mathrm{DH}(\mathcal{F}_w)$$

is strictly increasing on $\mathrm{Val}(X)$. So we use the assumption to get

$$\begin{aligned} \frac{A(w)}{w(\mathfrak{a}_\bullet(v))} = A(a^{-1}w) &= A(a^{-1}w) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(a^{-1}w) + \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(a^{-1}w) \\ &\geq A(v) - \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(v) + \widetilde{\boldsymbol{S}}^{\mathrm{NA}}(v) = A(v) \\ &= \frac{A(v)}{v(\mathfrak{a}_\bullet(v))}. \end{aligned}$$

When the equality holds, then $a^{-1}w = v$. □

We now observe that the method developed in [14] can be used to prove:

Theorem 4.10 *For any $\mathbb{Q}$–Fano variety, there exists a minimizing valuation of $\tilde{\beta}$ which is quasi-monomial.*

Since the argument is almost verbatim to [14] except for the continuity property of $\widetilde{\beta}$, we just give a sketch of key points and explain the required continuity of $\widetilde{\beta}$ in Section 8. Without the properties of $\widetilde{\beta}(S)$ explained in Section 8, the existence of a valuation calculating $h(X)$ (but without the quasimonomial property) can also be obtained using the argument in [11, Section 6].

Proof By Corollary 3.6, $h(X) = \inf_E \widetilde{\beta}_*(E)$, where E ranges over prime divisors over X that induce special test configurations of $(X, -K_X)$. By [14, Theorem A.2], we know that such an E is an lc place of an N–complement D of X, where N depends only on the dimension n (this depends on the deep result of Birkar about the boundedness of $\mathbb{Q}$–complements). So we have

$$h(X) = \inf_v \widetilde{\beta}_*(v), \tag{155}$$

where v ranges over all divisorial valuations that are lc places of an N–complement. For such a valuation v, there exists a $D \in (1/N)|-NK_X|$ such that (X, D) is lc and $A_{(X,D)}(v) = 0$. We then parametrize such $\mathbb{Q}$–divisors as in [14, Proof of Theorem 4.5]. Set $W = \mathbb{P}(H^0(X, \mathcal{O}_X(-NK_X))$ and denote by H the universal divisor on $X \times W$ parametrizing divisors in $|-NK_X|$ and set $D := (1/N)H$. By the lower semicontinuity of log canonical thresholds, the locus $Z = \{w \in W \mid \mathrm{lct}(X_w; D_w) = 1\}$ is locally closed in W. For each $z \in Z$, set $b_z := \inf_v \widetilde{\beta}(v)$, where v ranges over all $v \in \mathrm{Val}(X)$ with $A_{(X,D_z)}(v) = 0$.

Let $g\colon Y_z \to X$ be a log resolution of (X, D_z). Write $K_Y + D_{Y_z} = g^*(K_X + D_z)$. Consider the section of the simplicial cone, $\mathcal{S} := \mathrm{QM}(Y_z, D_{Y_z}) \bigcap \{v \in \mathrm{Val}(X) \mid A(v) = 1\}$. By Proposition 4.6, we know that for each $v \in \mathcal{S}$ there exists $a_*(v)$ such that $\inf_{a>0} \widetilde{\beta}(av) = \widetilde{\beta}(a_*(v)v) =: \widetilde{\beta}_*(v)$. By Izumi's estimate (see [48, Example 11.3.9; 44, Proposition 5.10] for the smooth case, and [51, Section 3] in the klt case), we know that there exists $C_1 > 0$ such that for any $v \in \mathcal{S}$ we have $v \le C_1 \cdot \mathrm{ord}_F$, where $F = \bigcap_i D_{Y_z,i}$. Now by the proof of Proposition 4.6 (see Remark 4.7), we know that $a_*(v)$ is uniformly bounded for any $v \in \mathcal{S}$. By Proposition A.2, we know that $v \mapsto \widetilde{\beta}(v)$ is continuous on $\mathrm{QM}(Y_z, D_{Y_z})$ and hence is uniformly continuous over compact subsets. We then get the continuity of $v \mapsto \widetilde{\beta}_*(v)$ over the compact set $\mathcal{S}$. So we know that there exists $v_z^* \in \mathcal{S}$ such that $\widetilde{\beta}_*(v_z^*) = \min_{v \in \mathcal{S}} \widetilde{\beta}_*(v)$ and $a_*(v_z^*) \cdot v_z^*$ is then a minimizer of $\widetilde{\beta}$ over $\mathrm{QM}(Y_z, D_{Y_z})$.

Then as [14, Proof of Theorem 4.5], choose a locally closed decomposition $Z = \bigcup_{i=1}^r Z_i$ so that Z_i is smooth and there is an étale map $Z_i' \to Z_i$ such that $(X_{Z_i'}, D_{Z_i'})$ admits fiberwise log resolutions. By the same arguments as [14, Proof of Propositions 4.1 and 4.2], which depend on the deformation invariance of log plurigenera in the work of Hacon, McKernan and Xu, we know that b_z is independent of $z \in Z_i$. So b_z takes finitely many values and there is a $z_0 \in Z$ such that $h(X) = \min_{z \in Z} b_z = b_{z_0}$ is computed by $v_{z_0}^*$. □

As in the case of normalized volume, we expect the following:

Conjecture 4.11 *The minimizer v_* is unique, and is special, which means that $\mathcal{F}_{v_*}$ is a special $\mathbb{R}$–test configuration.*

Remark 4.12 As [11, Proposition 4.11], using Lemma 4.9 one can show that any divisorial (ie rational rank one) minimizing valuation is primitive and plt.

Besides the case of stability threshold treated in [14], in the local setting of normalized volumes, the existence of quasimonomial minimizers is also known thanks to the work of Blum [10] and Xu [78]. Moreover, one might also be able to adapt the techniques in Xu and Zhuang [80] to the current global setting to prove the uniqueness of minimizing valuations.[3] We will prove in Section 6 the uniqueness of special minimizers (in a similar spirit to the work in [58; 57; 55]).

5 Initial term degeneration of filtrations

Let $\mathcal{F}_0$ be a special $\mathbb{R}$–test configuration of $(X, -K_X)$ with central fiber $(W := \mathrm{Proj}(\mathrm{Gr}(\mathcal{F}_0)), \xi_0 := \xi_{\mathcal{F}_0})$. Let $\mathcal{F}_1$ be another filtration of R. We define a filtration on

$$R' := R(W, -K_W) = \bigoplus_{m\geq 0} \bigoplus_{\lambda\in\Gamma(\mathcal{F}_0)} t^{-\lambda} \mathcal{F}_0^{\lambda} R_m / \mathcal{F}_0^{>\lambda} R_m =: \bigoplus_{m\geq 0} R'_m \tag{156}$$

in the following way. Recall that we can write

$$R'_m = \bigoplus_{\alpha\in M_{\mathbb{Z}}} t^{-\langle\alpha,\xi_0\rangle} \mathcal{F}_0^{\langle\alpha,\xi_0\rangle} R_m / \mathcal{F}_0^{>\langle\alpha,\xi_0\rangle} R_m. \tag{157}$$

For any $f \in R_m$, set

$$\mathbf{in}_{\mathcal{F}_0}(f) = (t^{-\langle\alpha,\xi\rangle}\bar{f})(0) =: f' \in \mathcal{F}_0^{\langle\alpha,\xi_0\rangle} R_m / \mathcal{F}_0^{>\langle\alpha,\xi_0\rangle} R_m, \quad \text{where } \langle\alpha,\xi_0\rangle = v_{\mathcal{F}_0}(f). \tag{158}$$

For any $\lambda \in \mathbb{R}$, take the Gröbner base-type degeneration

$$\mathcal{F}_1'^{\lambda} R'_m = \mathrm{Span}_{\mathbb{C}}\{\mathbf{in}_{\mathcal{F}_0}(f) \mid f \in \mathcal{F}_1^{\lambda} R_m\} \subseteq R'_m. \tag{159}$$

Note that because R' is integral, $\mathbf{in}_{\mathcal{F}_0}(fg) = \mathbf{in}_{\mathcal{F}_0}(f)\cdot\mathbf{in}_{\mathcal{F}_0}(g)$ if $f \in R_{m_1}$ and $g \in R_{m_2}$. So in this way, we get a $\mathbb{T}_0$–equivariant filtration

$$\mathcal{F}_1'^{\lambda} R'_m = \bigoplus_{\alpha\in\mathbb{Z}^{r_0}} \mathcal{F}_1'^{\lambda} R'_{m,\alpha}. \tag{160}$$

There is an equivalent way to describe $\mathcal{F}_1'^{\lambda} R'_m$, as follows. For any $f' \in R'_{m,\alpha}$, we choose $f \in R_m$ such that $f' = t^{-\langle\alpha,\xi_0\rangle}\bar{f}(0)$. Then we have

$$\mathcal{F}_1'^{\lambda} R'_{m,\alpha} = \{f' \in R'_{m,\alpha} \mid f + h \in \mathcal{F}_1^{\lambda} R_m \text{ for some } h \in \mathcal{F}_0^{>\langle\alpha,\xi_0\rangle} R_m\}. \tag{161}$$

This is well defined since f is determined up to addition by elements from $\mathcal{F}_0^{>\langle\alpha,\xi_0\rangle} R_m$.

Note that this construction allows us to find a basis $\mathcal{B} = \{s_1, \dots, s_{N_m}\}$ of R_m that is compatible with both $\mathcal{F}_0 R_m$ and $\mathcal{F}_1 R_m$. Recall that this means that for any $\lambda \in \mathbb{R}$ and $i = 0, 1$, there exists a subset of $\mathcal{B}$ which depends on λ and i and is a basis of $\mathcal{F}_i^{\lambda} R_m$. To find such a basis, we can first find a basis $\mathcal{B}'_\alpha$ of $R'_{m,\alpha}$ which is compatible with $\mathcal{F}'_1 R_{m,\alpha}$. Then $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha =: \{f'_1, \dots, f'_{N_m}\}$ is a basis compatible with

[3] This has indeed been recently achieved in [15].

both $\mathcal{F}'_1 R_m$ and $\mathcal{F}'_{\mathrm{wt}_{\xi_0}} R'_m$. For each $f'_k \in R'_{m,\alpha_k}$, there exists $\lambda_k \in \mathbb{R}$ such that $f'_k \in \mathcal{F}'^{\lambda_k}_1 R'_m \setminus \mathcal{F}'^{>\lambda_k}_1 R'_m$. Then by (161), there exists $h_k \in \mathcal{F}_0^{>\langle\alpha_k,\xi\rangle} R_m$ such that $s_k := f_k + h_k \in \mathcal{F}_1^{\lambda_k} R_m$. Moreover, we have $s_k \notin \mathcal{F}_1^{>\lambda_k} R_m$ since otherwise $\mathbf{in}(s_k) = \mathbf{in}(f_k) = f'_k \in \mathcal{F}'^{>\lambda_k} R'_m$. It is easy to verify that $\{s_k\}$ is the desired basis. So the relative successive minima of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ (see [20]) is given by the set $\{\lambda_k - \langle\alpha_k, \xi_0\rangle\}$, which is the same as the relative successive minima of $\mathcal{F}'_1 := \mathcal{F}'_1 R'$ with respect to $\mathcal{F}'_0 := \mathcal{F}'_{\mathrm{wt}_{\xi_0}} R'$. This immediately proves a useful fact:

Lemma 5.1 *With the above constructions and notation, we have the identity*

$$d_2^X(\mathcal{F}_0, \mathcal{F}_1) = d_2^W(\mathcal{F}'_0, \mathcal{F}'_1). \tag{162}$$

Since the initial term degeneration does not change the dimension of vector spaces, it is clear that the successive minima of $\mathcal{F}_1$ and $\mathcal{F}'_1$ coincide. As a consequence, we get

$$\widetilde{\boldsymbol{S}}^{\mathrm{NA}}_X(\mathcal{F}_1) = \widetilde{\boldsymbol{S}}^{\mathrm{NA}}_W(\mathcal{F}'_1). \tag{163}$$

On the other hand, consider the $\mathbb{T}_0$–equivariant graded filtration of the Rees algebra $\mathcal{R}' := \mathcal{R}(\mathcal{F}_0)$ (see (30)) given by

$$\mathcal{F}'^{\lambda}\mathcal{R}'_{m,\alpha} = \{s = t^{-\langle\alpha,\xi\rangle}\,\bar{f} \in \mathcal{R}'_{m,\alpha} \mid t^{-\lambda}\,\bar{f} \in \mathcal{R}(\mathcal{F}_1)\}. \tag{164}$$

Then $\mathcal{F}'\mathcal{R}'$ coincides with $\mathcal{F}R$ on the generic fiber and coincides with $\mathcal{F}'R'$ on the central fiber. By the lower semicontinuity of lct for a family, it is easy to see that $\widehat{\boldsymbol{L}}^{\mathrm{NA}}$ in (80) is also lower semicontinuous for a family. This is standard if $\mathcal{F}_0$ has rank one, which corresponds to a special test configuration; see [47, Lemma 8.1; 13, Proof of Lemma 6.5]. In general, one can restrict to a generic curve passing through 0 in the family in Teissier's construction in the paragraph above Lemma 2.11; alternatively, see Remark 6.2. So we can get

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_1) = \widehat{\boldsymbol{L}}^{\mathrm{NA}}_X(\mathcal{F}_1) \geq \widehat{\boldsymbol{L}}^{\mathrm{NA}}_W(\mathcal{F}'_1) = \boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}'_1), \tag{165}$$

where the first and the last identity come from Proposition 2.32. Combining the above discussion, we get the inequality

$$\boldsymbol{H}^{\mathrm{NA}}_X(\mathcal{F}_1) \geq \boldsymbol{H}^{\mathrm{NA}}_W(\mathcal{F}'_1). \tag{166}$$

Theorem 5.2 *Assume v induces a special $\mathbb{R}$–test configuration $\mathcal{F}_v$ of X. Then v is a minimizer of $\widetilde{\beta}$ over* $\mathrm{Val}(X)$ *if and only if v is Ding-semistable (or equivalently K–semistable).*

Proof For simplicity of notation, set $\mathcal{F}_0 = \mathcal{F}_v$ and $(W, \xi_0) := (X_{\mathcal{F}_v,0}, \xi_{\mathcal{F}_0})$ and let $\mathbb{T}_0$ be the torus generated by ξ_0.

We first prove that minimizer is Ding-semistable. Suppose (W, ξ_0) is not Ding-semistable. Then by Theorem 2.46 from [39], there exists a $\mathbb{T}$–equivariant special test configuration $(\mathcal{W}, -K_{\mathcal{W}})$ of $(W, -K_W)$ with central fiber $Y := \mathcal{W}_0$ such that

$$\boldsymbol{D}^{\mathrm{NA}}_g(\mathcal{W}, -K_{\mathcal{W}}) = \mathrm{Fut}_{Y,\xi}(\eta) < 0. \tag{167}$$

We can now construct a family of valuations $\{v_\epsilon\}$ such that v_ϵ induces special test configurations with central fiber Y and corresponds to a vector field $\xi_\epsilon = \xi_0 + \epsilon\eta$ on Y. This can be done by using the cone construction to reduce to the situation in [58, Section 6] or [57, Proof of Theorem 2.64]. Alternatively, one can use an argument involving the Hilbert scheme as in [55, Proof of Lemma 3.1].

Here we will use the Chow variety to explain this construction. Recall that the Chow point of a cycle $Z \subset \mathbb{P}^{N-1}$ of degree d and dimension n corresponds to a divisor in the Grassmannian $\mathrm{Gr}(N-n-1, \mathbb{C}^N)$ which is the zero scheme of a section:

$$\mathrm{CH}(Z) \in H^0(\mathrm{Gr}(N-n-1, \mathbb{C}^N), \mathcal{O}(d)) =: \mathbb{M}.$$

$\mathrm{CH}(Z)$ is determined up to rescaling and we call it the Chow coordinate of Z. Let $\mathrm{CH}(X)$, $\mathrm{CH}(W)$ and $\mathrm{CH}(Y)$ be the Chow coordinates of X, W and Y, respectively. Denote by $[\mathrm{CH}(X)]$ the Chow point of X in the projectivization $\mathbb{P}(\mathbb{M})$, and similarly for Y and W. Since the $\mathbb{T}$–action on $\mathbb{P}^{N-1}$ induces a weight decomposition $\mathbb{M} = \bigoplus_\alpha \mathbb{M}_\alpha$, we have

$$\lim_{s\to+\infty} \sigma_\xi(s) \circ [\mathrm{CH}(X)] = [\mathrm{CH}(W)] \quad \text{and} \quad \lim_{s\to+\infty} \sigma_\eta(s) \circ [\mathrm{CH}(W)] = [\mathrm{CH}(Y)]. \tag{168}$$

If we set

$$\mathrm{CW}_\xi(X) = \min\{\langle\alpha,\xi\rangle \mid \mathrm{CH}(X)_\alpha \neq 0\} \quad \text{and} \quad \mathrm{CW}_\eta(W) = \min\{\langle\alpha,\xi\rangle \mid \mathrm{CH}(W)_\alpha \neq 0\}, \tag{169}$$

then

$$[\mathrm{CH}(W)] = \Bigl[\sum_{\alpha\in I_W} \mathrm{CH}(X)_\alpha\Bigr], \quad \text{where } I_W = \{\alpha \mid \mathrm{CH}(X)_\alpha \neq 0, \langle\alpha,\xi\rangle = \mathrm{CW}_\xi(X)\},$$

$$[\mathrm{CH}(Y)] = \Bigl[\sum_{\alpha\in I_Y} \mathrm{CH}(W)_\alpha\Bigr], \quad \text{where } I_Y = \{\alpha \mid \mathrm{CH}(W)_\alpha \neq 0, \langle\alpha,\eta\rangle = \mathrm{CW}_\eta(W)\}.$$

Note that $I_Y \subseteq I_W$. For any $\alpha \in M_{\mathbb{Z}}$ with $\mathrm{CH}(X)_\alpha \neq 0$, we have that $\langle\alpha,\xi\rangle \ge \mathrm{CW}_\xi(X)$, with equality if and only if $\alpha \in I_W$. Similarly, for any $\alpha \in M_{\mathbb{Z}}$ with $\mathrm{CH}(W)_\alpha \neq 0$ (and hence $\mathrm{CH}(X)_\alpha \neq 0$), we have that $\langle\alpha,\eta\rangle \ge \mathrm{CW}_\eta(W)$, with equality if and only if $\alpha \in I_Y$. So when $0 < \epsilon \ll 1$ and for any $\mathrm{CH}(X)_\alpha \neq 0$, we have that $\langle\alpha,\xi+\epsilon\eta\rangle \ge \mathrm{CW}_\xi(X) + \epsilon\mathrm{CW}_\eta(W)$, with equality if and only if $\alpha \in I_Y$. So we get

$$\lim_{t\to 0} \sigma_{\xi+\epsilon\eta}(t) \circ [\mathrm{CH}(X)] = \lim_{t\to 0}\Bigl[\sum_\alpha t^{\langle\alpha,\xi+\epsilon\eta\rangle}\mathrm{CH}(X)_\alpha\Bigr] = [\mathrm{CH}(Y)].$$

So for $0 < \epsilon \ll 1$, $\xi + \epsilon\eta$ induces an $\mathbb{R}$–test configuration that degenerates X to Y. By Lemma 2.11, we get the corresponding valuations v_ϵ.

Now we use the identity (113) to get

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{\beta}(v_\epsilon) = \frac{d}{d\epsilon} \boldsymbol{H}_Y^{\mathrm{NA}}(\mathcal{F}_{\mathrm{wt}_{\xi+\epsilon\eta}}) = \mathrm{Fut}_{Y,\xi}(\eta) < 0. \tag{170}$$

But this contradicts the assumption that $v_0 = v$ is the minimizer of $\tilde{\beta}$.

Conversely, we need to show that a Ding-semistable valuation is a minimizer. Let $(\mathcal{X},\mathcal{L},a\eta)$ be any special test configuration of $(X,-K_X)$ and $\mathcal{F}_1=\mathcal{F}_{(\mathcal{X},\mathcal{L},a\eta)}$ be the associated filtration. We consider the initial term degeneration of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ defined as above. Then we can use (166) to get

$$\boldsymbol{H}_X^{\mathrm{NA}}(\mathcal{F}_1)\geq \boldsymbol{H}_W^{\mathrm{NA}}(\mathcal{F}_1')\geq \boldsymbol{H}_W^{\mathrm{NA}}(\mathcal{F}_{\mathrm{wt}_{\xi_0}})=\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_0)=\widetilde{\beta}(v),$$

where the second inequality follows from the results in Lemma 6.1 in the next section and the assumption that (W,ξ_0) is Ding-semistable. □

6 Uniqueness of minimizing special $\mathbb{R}$–test configurations

We prove Theorem 1.2 in this section. We first generalize the formula (113). Let $(X,-K_X,\mathbb{T},\xi)$ be the data as before and $\mathcal{F}$ be a $\mathbb{T}$–equivariant filtration. We consider a family of $\mathbb{T}$–equivariant filtrations

$$\mathcal{F}_s=s\mathcal{F}_{((1-s)/s)\xi}\quad\text{for } s\in(0,1],\ \text{with } \mathcal{F}_0=\mathcal{F}_{\mathrm{wt}_\xi} \text{ and } \mathcal{F}_1=\mathcal{F},\tag{171}$$

which interpolates $\mathcal{F}_{\mathrm{wt}_\xi}$ and $\mathcal{F}$.

Lemma 6.1 *For the family of filtrations* (171), *the following statements hold true*:

(i) *The map $s\mapsto \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_s)$ is smooth and convex. It is affine if and only if $G_{\mathcal{F}}$ is a multiple of $\langle x,\xi\rangle$.*

(ii) *We have the derivative formula*

$$\frac{d}{ds}\Big|_{s=0}\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_s)=\beta_\xi(\mathcal{F}_{-\xi}).\tag{172}$$

To get (113) from (172), we just need to set $\mathcal{F}=\mathcal{F}_{\mathrm{wt}_{\xi+\eta}}$ so that $\mathcal{F}_s=\mathcal{F}_{\xi+s\eta}$. Moreover, we fix a faithful valuation that is adapted to the torus action (see Definition 2.21) and will freely use the associated Newton–Okounkov body $\Delta=\Delta(-K_X)$ of $(X,-K_X)$.

Proof By Lemma 2.22 and (25), as functions on $\Delta=\Delta(-K_X)$, we have

$$G(s,y):=G_{\mathcal{F}_s}(y)=(1-s)\langle y,\xi\rangle+sG_{\mathcal{F}}(y).\tag{173}$$

So, by using Lemma 2.35, we get

$$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_s)=s\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}),\tag{174}$$

$$-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_s)=\log\biggl(\frac{n!}{\boldsymbol{V}}\int_\Delta e^{-G(s,y)}\,dy\biggr).\tag{175}$$

$\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_s)$ is linear in s and $-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_s)$ is smooth in s. By Hölder's inequality, $-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_s)$ is strictly convex in s unless $G_{\mathcal{F}}$ is a multiple of $\langle x,\xi\rangle$. This implies that $\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_s)=\boldsymbol{L}^{\mathrm{NA}}-\boldsymbol{S}^{\mathrm{NA}}$ is convex in $s\in[0,1]$.

To see (172), we calculate

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}_s)&=\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F})+\frac{\int_\Delta(\langle y,\xi\rangle-G_{\mathcal{F}}(y))e^{-G(0,y)}\,dx}{\int_\Delta e^{-G(0,y)}\,dy}\\&=\boldsymbol{L}^{\mathrm{NA}}(\mathcal{F}_{-\xi})-\frac{n!}{\boldsymbol{V}_\xi}\int_\Delta G_{\mathcal{F}_{-\xi}}(y)e^{-\langle y,\xi\rangle}\,dy=\beta_\xi(\mathcal{F}_{-\xi}).\end{aligned}$$

□

Assume that there are two special $\mathbb{R}$–test configurations $\mathcal{F}_i = \{\mathcal{F}_i R_m\}$ for $i = 0, 1$ of $(X, -K_X)$ that minimize $\boldsymbol{H}^{\mathrm{NA}}$. By Theorem 5.2, the central fibers $(W^{(i)} := \mathrm{Proj}(\mathrm{Gr}_{\mathcal{F}_i}), \xi_i = \xi_{\mathcal{F}_i})$ are both Ding-semistable. Now consider the initial term degeneration of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ as in the above section. We get a $\mathbb{T}_0$–equivariant filtration $\mathcal{F}'_1$ on $R' = R(W^{(0)}, -K_{W^{(0)}})$ and by (166), $\boldsymbol{H}^{\mathrm{NA}}_X(\mathcal{F}_1) \geq \boldsymbol{H}^{\mathrm{NA}}_{W^{(0)}}(\mathcal{F}'_1)$.

Now, as at the beginning of this section, consider the family of filtrations that interpolates $\mathcal{F}'_1$ and $\mathcal{F}_{\mathrm{wt}_{\xi_0}} R' =: \mathcal{F}'_{\mathrm{wt}_{\xi_0}}$,

$$\mathcal{F}'_s := s\mathcal{F}'_{((1-s)/s)\xi_0} R'. \tag{176}$$

Applying Lemma 6.1 to $(W^{(0)}, \xi_0, \mathcal{F}'_s)$, we know that $\boldsymbol{D}(s) := \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}'_s)$ is convex in $s \in [0, 1]$. Moreover we have the relation

$$\boldsymbol{D}(0) = \boldsymbol{H}^{\mathrm{NA}}_{W^{(0)}}(\mathcal{F}_{\mathrm{wt}_{\xi_0}}) = \boldsymbol{H}^{\mathrm{NA}}_X(\mathcal{F}_0) = \boldsymbol{H}^{\mathrm{NA}}_X(\mathcal{F}_1) \geq \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}'_1) = \boldsymbol{D}(1). \tag{177}$$

The 3rd identity is by Theorem 5.2, that the $\mathcal{F}_i$ for $i = 0, 1$ both obtain the minimum of $\boldsymbol{H}^{\mathrm{NA}}$.

On the other hand, by (172),

$$\frac{d}{ds}\Big|_{s=0} \boldsymbol{H}^{\mathrm{NA}}(\mathcal{F}'_s) = \beta_{\xi_0}(\mathcal{F}'_{-\xi_0}) \geq 0.$$

The last inequality is because $(W^{(0)}, \xi_0)$ is Ding-semistable.

By convexity of $\boldsymbol{D}(s)$, we conclude that $\boldsymbol{D}(s)$ is constant in s and by Lemma 6.1 that $G_{\mathcal{F}'_1}(y) \equiv \langle y, \xi_0\rangle$ for any $y \in \Delta' = \Delta(W^{(0)}, -K_{W^{(0)}})$ (the Okounkov body of $(W^{(0)}, -K_{W^{(0)}})$).

By the discussion in previous section, we know that the relative successive minima of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ is the same as the relative successive minima of $\mathcal{F}'_1$ with respect to $\mathcal{F}'_{\mathrm{wt}_{\xi_0}}$, which is the same as the successive minima of $\mathcal{F}'_{-\xi_0}$ and is given by the difference $\lambda_k - \langle \alpha_k, \xi_0\rangle$ with the notation there. So we get by Lemma 5.1 that

$$\begin{aligned} d_2(\mathcal{F}_0, \mathcal{F}_1)^2 = d_2(\mathcal{F}'_0, \mathcal{F}'_1) &= \lim_{m\to+\infty} \sum_k \frac{(\lambda_k - \langle \alpha_k, \xi_0\rangle)^2}{m^2} = \lim_{m\to+\infty} \sum_i \frac{\lambda_i^{(m)}(\mathcal{F}'_{-\xi_0})^2}{m^2} \\ &= \int_{\mathbb{R}} \lambda^2\, \mathrm{DH}(\mathcal{F}'_{-\xi_0})^2 = \int_{\Delta'} G^2_{\mathcal{F}'_{-\xi_0}}\, dy = \int_{\Delta'} (G_{\mathcal{F}'} - \langle y, \xi_0\rangle)^2\, dy = 0. \end{aligned}$$

By [20], we know that $\mathcal{F}_0$ is asymptotically equivalent to $\mathcal{F}_1$. By Lemmas 2.11 and 2.29 (see also Proposition 2.27), we get $\mathcal{F}_0 = \mathcal{F}_1$.

Remark 6.2 Although here we are dealing with filtration of arbitrary ranks, the unique result in this section (and minimization result in previous section) can also be proved by using $r := \mathrm{rk}(\mathcal{F}_0)$–step degenerations to reduce to the rank-one case. To see this, we first choose $\{\eta_1, \dots, \eta_r\} \in N_{\mathbb{Q}} \cong \mathbb{Q}^r$ (where $N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}_0)$ as before) such that:

- $\mathrm{Span}_{\mathbb{R}}\{\eta_1, \dots, \eta_r\} = N_{\mathbb{R}}$.
- For any $1 \leq k \leq r$, η_k induces a special test configuration whose central fiber is the same as $W^{(0)}$. This is achieved by choosing η_k satisfying $|\eta_k - \xi_0| \ll 1$.

By abuse of notation, we denote by $\mathcal{F}'_{\xi_0}$ (resp. $\mathcal{F}'_{\eta_1}$) the filtration on $R = R(X, -K_X)$ corresponding to the $\mathbb{R}$–test configuration induced by ξ_0 (resp. η_1), and also the filtration on $R' = R(W^{(0)}, -K_{W_0^{(0)}})$ corresponding to the weight filtration induced by ξ_0 (resp. η_k for $2 \le k \le r$). Set $\mathcal{F}_1'^{(0)} = \mathcal{F}_1$ and inductively define $\mathcal{F}_1'^{(k)}$ to be the initial term degeneration of $\mathcal{F}_1'^{(k-1)}$ with respect to $\mathcal{F}'_{\eta_k}$ for $1 \le k \le r$. By (166) for the rank-one case, we have

$$\boldsymbol{H}_X^{\mathrm{NA}}(\mathcal{F}_1'^{(0)}) \ge \boldsymbol{H}_{W^{(0)}}^{\mathrm{NA}}(\mathcal{F}_1'^{(1)}) \quad \text{and} \quad \boldsymbol{H}_{W^{(0)}}^{\mathrm{NA}}(\mathcal{F}_1'^{(k-1)}) \ge \boldsymbol{H}_{W^{(0)}}^{\mathrm{NA}}(\mathcal{F}_1'^{(k)}) \quad \text{for } 2 \le k \le r. \tag{178}$$

So if $\mathcal{F}_1 = \mathcal{F}_1'^{(0)}$ obtains the minimum of $\boldsymbol{H}_X^{\mathrm{NA}}$, then $\mathcal{F}'^{(k)}$ for any $1 \le k \le r$ also obtains the minimum of $\boldsymbol{H}_{W^{(0)}}^{\mathrm{NA}}$. Now because $\mathcal{F}'^{(r)}$ is $\mathbb{T}_0$–invariant and $\mathcal{F}'_{\xi_0} = \mathcal{F}'_{\mathrm{wt}_{\xi_0}}$ also obtains the minimum of $\boldsymbol{H}_{W^{(0)}}^{\mathrm{NA}}$, we can use Lemma 2.38 to conclude that $\mathcal{F}'^{(r)} = \mathcal{F}'_{\xi_0}$.

On the other hand, by Lemma 5.1, we get for $2 \le k \le r$ that

$$d_2^X(\mathcal{F}_1'^{(0)}, \mathcal{F}'_{\eta_1}) = d_2^{W^{(0)}}(\mathcal{F}_1'^{(1)}, \mathcal{F}'_{\eta_1}) \quad \text{and} \quad d_2^{W^{(0)}}(\mathcal{F}_1'^{(k-1)}, \mathcal{F}'_{\eta_k}) = d_2^{W^{(0)}}(\mathcal{F}_1'^{(k)}, \mathcal{F}'_{\eta_k}).$$

So for any $1 \le k \le r$, we get, by omitting the superscripts and using the triangle inequality,

$$\begin{aligned} d_2(\mathcal{F}_1'^{(k-1)}, \mathcal{F}'_{\xi_0}) &\le d_2(\mathcal{F}_1'^{(k-1)}, \mathcal{F}'_{\eta_k}) + d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0}) \\ &= d_2(\mathcal{F}_1'^{(k)}, \mathcal{F}'_{\eta_k}) + d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0}) \\ &\le d_2(\mathcal{F}_1'^{(k)}, \mathcal{F}'_{\xi_0}) + 2d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0}). \end{aligned}$$

So we can inductively estimate

$$\begin{aligned} d_2(\mathcal{F}_1, \mathcal{F}_0) = d_2(\mathcal{F}_1'^{(0)}, \mathcal{F}'_{\xi_0}) &\le d_2(\mathcal{F}_1'^{(1)}, \mathcal{F}'_{\xi_0}) + 2d_2(\mathcal{F}'_{\eta_1}, \mathcal{F}'_{\xi_0}) \\ &\le d_2(\mathcal{F}_1'^{(2)}, \mathcal{F}'_{\xi_0}) + 2(d_2(\mathcal{F}'_{\eta_2}, \mathcal{F}'_{\xi_0}) + d_2(\mathcal{F}'_{\eta_1}, \mathcal{F}'_{\xi_0})) \\ &\ \ \vdots \\ &\le d_2(\mathcal{F}_1'^{(r)}, \mathcal{F}'_{\xi_0}) + 2\sum_{k=1}^{r} d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0}) \\ &= 2\sum_{k=1}^{r} d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0}). \end{aligned}$$

Now we can choose η_k so that $d_2(\mathcal{F}'_{\eta_k}, \mathcal{F}'_{\xi_0})$ is arbitrarily small for all $1 \le k \le r$. So we indeed get $d_2(\mathcal{F}_1, \mathcal{F}_0) = 0$, as desired.

7 Cone construction and g–normalized volume

Let X be an n–dimensional $\mathbb{Q}$–Fano variety and for simplicity of notation, assume that $-K_X$ is Cartier. Recall that $R = \bigoplus_m R_m = \bigoplus H^0(X, m(-K_X))$. We define the cone

$$C = C(X, -K_X) = \mathrm{Spec}_{\mathbb{C}} R, \quad o = \mathfrak{m} = \bigoplus_{m>0} R_m. \tag{179}$$

Then (C, o) is a klt cone singularity.

Since X admits a $\mathbb{C}^* \times \mathbb{T}$–action, we have a decomposition of the coordinate ring of R,

$$R = \bigoplus_{m\geq 0} \bigoplus_{\alpha\in\mathbb{Z}^r} R_{m,\alpha}. \tag{180}$$

For any $\mathbb{T}$–invariant homogeneous primary ideal $\mathfrak{a} = \bigoplus_m \bigoplus_\alpha \mathfrak{a}_{m,\alpha} \subset R$, define the g–colength and g–multiplicity of $\mathfrak{a}$ by

$$\mathrm{colen}_g(\mathfrak{a}) = \sum_{m\geq 0}\sum_{\alpha} g\Big(\frac{\alpha}{m}\Big) \dim R_{m,\alpha}/\mathfrak{a}_{m,\alpha}, \tag{181}$$

$$\mathrm{mult}_g(\mathfrak{a}) = \lim_{k\to+\infty} \frac{\mathrm{colen}_g(\mathfrak{a}^k)}{k^{n+1}/(n+1)!}. \tag{182}$$

See [63] for the study of such equivariant multiplicity. More generally, let $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ be a graded sequence of $\mathbb{C}^*\times\mathbb{T}$–invariant ideals. We define

$$\mathrm{mult}_g(\mathfrak{a}_\bullet) = \lim_{k\to+\infty} \frac{\mathrm{colen}_g(\mathfrak{a}_k)}{k^{n+1}/(n+1)!}. \tag{183}$$

One can use the techniques of Newton–Okounkov bodies to show that the limit exists. To see this, we can adapt the argument in the work in [46] as follows. First choose a valuation $\mathfrak{v}$ adapted to the $\mathbb{T}$–action on X (in the sense of Definition 2.21). We can construct a $\mathbb{C}^*\times\mathbb{T}$–invariant $\mathbb{Z}^{n+1}$–valuation on C by

$$\mathfrak{V}(f) = (m, \mathfrak{v}(f)) \quad \text{for any } f \in R_m. \tag{184}$$

Denote by $\mathfrak{C}$ the strongly convex cone which is the closure of the convex hull of the value semigroup $\mathfrak{V}(R)$. To each graded sequence of $\mathbb{C}^*\times\mathbb{T}$–invariant ideals $\mathfrak{a}_\bullet$, one can associate a convex region $\overline{P} := \overline{P}(\mathfrak{a}_\bullet) \subset \mathfrak{C}$ such that $\overline{P}^c := \mathfrak{C} \setminus \overline{P}$ is bounded. If we still denote by $g(y)$ the pullback of function g by the projection $\mathbb{R}^{n+1} = \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, then mult_g is given by the weighted volume of the co-convex set $\overline{P}^c$,

$$\mathrm{mult}_g(\mathfrak{a}_\bullet) = (n+1)! \int_{\overline{P}^c} g(y)\, dy. \tag{185}$$

Let $\mathrm{Val}_{C,o}$ be the space of real valuations that are centered at o, and by $\mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}}$ the subset of $\mathbb{C}^*\times\mathbb{T}$–invariant real valuations in $\mathrm{Val}_{C,o}$. If $\widetilde{C} \to C$ is the blowup of the vertex $o \in X$, then the exceptional divisor on $\widetilde{C}$ is isomorphic to X, and we will denote by ord_X the associated divisorial valuation contained in $\mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}}$.

Let $\overline{v} \in \mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}}$ be any $\mathbb{C}^*\times\mathbb{T}$–invariant valuation. Then for any $\lambda \in \mathbb{R}$, $\mathfrak{a}_\lambda(\overline{v}) = \{f \in R \mid \overline{v}(f) > m\}$ is a $\mathbb{T}$–invariant homogeneous primary ideal. Set $\mathfrak{a}_\bullet(\overline{v}) = \mathfrak{a}_m(\overline{v})$ and define (see [35] for the $g = 1$ case)

$$\mathrm{vol}_g(\overline{v}) := \mathrm{mult}_g(\mathfrak{a}_\bullet(v)) = \lim_{m\to+\infty} \frac{\mathrm{colen}_g(\mathfrak{a}_\lambda(\overline{v}))}{\lambda^{n+1}/(n+1)!}.$$

We define the following equivariant version of normalized volume [51]:

$$\widehat{\mathrm{vol}}_g : \mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}} \to \mathbb{R}_{>0} \cup \{+\infty\}, \qquad \widehat{\mathrm{vol}}_g(\overline{v}) = \begin{cases} A_C(\overline{v})^{n+1} \cdot \mathrm{vol}_g(\overline{v}) & \text{when } A_C(\overline{v}) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

By using the same argument as in the study of normalized volumes, one can generalize almost all the results about normalized volume to work for the g–normalized volume functional. Here we just write down a few results that we need in the next section. We have the following equivariant version of an identity from [60].

Lemma 7.1 *With the above notation, we have the identity*

$$\inf_{\overline{v}} \widehat{\mathrm{vol}}_g(\overline{v}) = \inf_{\mathfrak{a}} \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}_g(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}_g(\mathfrak{a}_\bullet), \tag{186}$$

where $\overline{v}$ ranges over $\mathbb{C}^*\times\mathbb{T}$–invariant valuations, and $\mathfrak{a}$ (resp. $\mathfrak{a}_\bullet$) ranges over $\mathbb{C}^*\times\mathbb{T}$–invariant ideals (resp. graded sequences of $\mathbb{C}^*\times\mathbb{T}$–invariant ideals).

This is proved by using exactly the same argument. For the reader's convenience, we give the short proof.

Proof For any $\overline{v} \in \mathring{\mathrm{Val}}_{C,o}$, we have

$$\mathrm{lct}(\mathfrak{a}_\bullet(\overline{v}))^n \cdot \mathrm{mult}_g(\mathfrak{a}_\bullet(\overline{v})) \le \left(\frac{A_C(\overline{v})}{\overline{v}(\mathfrak{a}_\bullet)}\right)^n \mathrm{vol}_g(\overline{v}) = A_C(\overline{v})^n \,\mathrm{vol}_g(\overline{v}). \tag{187}$$

Conversely, for any graded sequence of ideals $\mathfrak{a}_\bullet$, let $\overline{w} \in \mathring{\mathrm{Val}}_{C,o}$ be the valuation that calculates $\mathrm{lct}(\mathfrak{a}_\bullet)$, which exists by [44]. By multiplying by a constant, we can assume $1 = \overline{w}(\mathfrak{a}_\bullet) = \inf_m \overline{w}(\mathfrak{a}_m)/m$. So $\mathfrak{a}_m \subseteq \mathfrak{a}_m(\overline{w})$, which implies $\mathrm{mult}_g(\mathfrak{a}_\bullet) \ge \mathrm{mult}_g(\mathfrak{a}_\bullet(\overline{w})) = \mathrm{vol}_g(\overline{w})$. Then we get

$$\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}_g(\mathfrak{a}_\bullet) = \left(\frac{A_C(\overline{w})}{\overline{w}(\mathfrak{a}_\bullet)}\right)^n \cdot \mathrm{mult}_g(\mathfrak{a}_\bullet) \ge A_C(\overline{w})^n \cdot \mathrm{vol}_g(\overline{w}) = \widehat{\mathrm{vol}}_g(\overline{w}). \tag{188}$$ □

For any $v \in X^{\mathrm{div}}_{\mathbb{Q}}$ and $\tau > 0$, we denote by $\overline{v}_\tau$ the $\mathbb{C}^*$–invariant valuation on C given by

$$\overline{v}_\tau\left(\sum_i f_i t^i\right) = \min_i (v(f_i) + \tau i). \tag{189}$$

By using the same calculation as in [50], we get:

Theorem 7.2 *The g–volume of $\overline{v}_\tau$ is given by the formula*

$$\mathrm{vol}_g(\overline{v}_\tau) = \frac{1}{\tau^{n+1}} V_g - (n+1)\int_0^{+\infty} \mathrm{vol}_g(\mathcal{F}_v R^{(x)}) \frac{dx}{(x+\tau)^{n+2}}. \tag{190}$$

We have the following criterion for g–Ding-semistability, which generalizes the results in [50; 53; 58] about normalized volumes.

Theorem 7.3 *The pair (X, η) is g–Ding-semistable if and only if ord_X obtains the minimum of $\widehat{\mathrm{vol}}_g$ over $\mathrm{Val}^{\mathbb{C}^*\times\mathbb{T}}_{C,o}$.*

Proof For any $v \in (X^{\mathrm{div}}_{\mathbb{Q}})^{\mathbb{T}}$, consider $w_s := \overline{(sv)}_{(1-s)A_X(v)} \in \mathrm{Val}^{\mathbb{C}^*\times\mathbb{T}}_{C,o}$. Then $w_0 = A_X(v)\overline{v}_0$ and $w_1 = v$. We also have $A_C(w_s) \equiv A_X(v)$. Set

$$\begin{aligned} f(s) &= \widehat{\mathrm{vol}}(w_s) = A_C(w_s)^{n+1}\,\mathrm{vol}_g(w_s) \\ &= A_X(v)^{n+1}\left(\frac{V_g}{(1-s)^{n+1}A_X(v)^{n+1}} - (n+1)\int_0^{+\infty} \mathrm{vol}_g(\mathcal{F}_v R^{(x)})\frac{s\,dx}{(sx+(1-s)A_X(v))^{n+2}}\right) \\ &= A_X(v)^{n+1}\int_0^{+\infty}\frac{-d\,\mathrm{vol}_g(\mathcal{F}_v R^{(x)})}{(sx+(1-s)A_X(v))^{n+1}}. \end{aligned}$$

Then $f(s)$ is a convex function in $s \in [0,1]$. Its derivative at $s=0$ is given by

$$\begin{aligned} f'(0) &= A_X(v)^{n+1}\left((n+1)\frac{V_g}{A_X(v)^{n+1}} - (n+1)\int_0^{+\infty}\mathrm{vol}_g(\mathcal{F}_v R^{(x)})\,dx\,\frac{1}{A_X(v)^{n+2}}\right) \\ &= \frac{n+1}{A_X(v)V_g}\left(A_X(v) - \frac{1}{V_g}\int_0^{+\infty}\mathrm{vol}_g(\mathcal{F}_v R^{(x)})\,dx\right) \\ &= \frac{n+1}{A_X(v)V_g}\cdot\beta_g(v). \end{aligned} \tag{191}$$

With this and Theorem 2.47, we can easily derive the conclusion as in [50]. □

Remark 7.4 By the same argument as in the case of normalized volume [12; 78], one shows that g–Ding-semistability is Zariski-open for a $\mathbb{T}$–equivariant family of Fano varieties.

8 Uniqueness of polystable degeneration

In this section, we prove Theorem 1.3. The proof is verbatim the same as the proof of the existence and uniqueness of K–polystable degenerations for any K–semistable $\mathbb{Q}$–Fano varieties, as proved in [55]; see also [16]. Indeed, we just need to carry out the same argument by using the equivariant version of normalized volume and the modified Futaki invariant Fut_ξ, etc. To avoid redundancy, we only sketch the key steps and refer to [55; 16] for more details.

Assume that (X,ξ) is semistable and admits two polystable degenerations via two special test configurations $(\mathcal{X}^{(i)}, -K_{\mathcal{X}^{(i)}})$ for $i=0,1$. Take cones fiberwise to get a special test configuration of Fano cones $(\mathcal{C}^{(i)}), \zeta^{(i)})$, where $\zeta^{(i)}$ is the radial vector field.

Let E_k be the Kollár component (see [58] for the definition) obtained by blowing up the vertex of $\mathcal{C}^{(0)}$ with weight $(k,1)$. Then we have

$$\widehat{\mathrm{vol}}_g(E_k) = \widehat{\mathrm{vol}}_g(\mathrm{ord}_X) + O(k^{-2}).$$

Set $\mathfrak{a}_\bullet = \{\mathfrak{a}_\ell(E_k)\}$. Then

$$\mathrm{lct}(\mathfrak{a}_\bullet) = \frac{A(E_k)}{\mathrm{ord}_{E_k}(\mathfrak{a}_\bullet)} = A(E_k) =: c_k = O(k), \quad \mathrm{lct}(X,\mathfrak{a}_\bullet)^n\cdot\mathrm{mult}_g(\mathfrak{a}_\bullet) = \widehat{\mathrm{vol}}_g(E_k).$$

Consider the initial degeneration of $\mathfrak{a}_\bullet$ with respect to $\mathcal{C}^{(1)}$,

$$\mathbf{in}(\mathfrak{a}_\ell) = \mathrm{span}_{\mathbb{C}}\{\mathbf{in}(f), f \in \mathfrak{a}_\ell(E_k)\}. \tag{192}$$

Using the preservation of co-length under initial term degeneration, we get

$$\begin{aligned}\mathrm{lct}^n(\mathcal{C}_0^{(1)}, \mathbf{in}(\mathfrak{a}_\bullet)) &\geq \frac{\widehat{\mathrm{vol}}_g(\mathrm{ord}_{\mathcal{X}_0^{(1)}})}{\mathrm{mult}_g(\mathbf{in}(\mathfrak{a}_\bullet))} = \frac{V_g}{\mathrm{mult}_g(\mathfrak{a}_\bullet)} \\ &= \frac{V_g}{\widehat{\mathrm{vol}}_g(E_k)}\mathrm{lct}(\mathfrak{a}_\bullet)^n = \frac{V_g}{V_g + O(k^{-2})}\mathrm{lct}(\mathfrak{a}_\bullet)^n \\ &= (1 + O(k^{-2}))c_k = c_k + O(k^{-1}).\end{aligned}$$

Let $Z_k \to \mathcal{C}^{(0)}$ be the extraction of E_k, and let $Z_k \times \mathbb{C}^*$ be the product along $\mathcal{C}^{(1)} \setminus \mathcal{C}_0^{(1)} \cong C \times \mathbb{C}^*$ with exceptional divisor $\mathcal{E}_k$. Let $\mathfrak{B}_\bullet = \{\mathfrak{B}_\ell\}$ be ideal on the total space $\mathcal{C}^{(1)}$ obtained by the above degenerating $\mathfrak{a}_\ell$. Then we have

$$A(\mathcal{C}^{(1)}, c_k(1-\epsilon k^{-1})\mathfrak{B}_\bullet, \mathcal{E}_k) = A(C, c_k(1-\epsilon k^{-1})\mathfrak{a}_\bullet, E_k) = \epsilon k^{-1} c_k = \epsilon O(1), \tag{193}$$

$$\mathrm{lct}(\mathcal{C}_0^{(1)}, c_k(1-\epsilon k^{-1})\mathbf{in}(\mathfrak{a}_\bullet)) \geq c_k^{-1}(1-\epsilon k^{-1})(c_k + O(k^{-1})) = 1 - \epsilon k^{-1} + O(k^{-2}). \tag{194}$$

By inversion of adjunction,

$$\mathrm{lct}(\mathcal{C}^{(1)}, c_k(1-\epsilon k^{-1})\mathfrak{B}_\bullet) \geq 1 - \epsilon k^{-1} + O(k^{-2}). \tag{195}$$

When $0 < \epsilon \ll 1$, by [9], we can extract the divisor $\mathcal{E}_k$ over $\mathcal{C}^{(1)}$. By the same argument as [55], we get the commutative diagram

$$\begin{tikzcd}
C_0^{(1)} \arrow[dr, dashed] \arrow[ddd, squiggly, "\mathcal{C}'^{(1)}"'] & & & C \arrow[lll, squiggly, "\mathcal{C}^{(1)}\leftarrow\mathcal{Z}_k^{(1)}\leftarrow\mathcal{E}_k^{(1)}"'] \arrow[dl, dashed] \arrow[ddd, squiggly, "\mathcal{C}^{(0)}\leftarrow\mathcal{Z}_k\leftarrow\mathcal{E}_k=E_k\times\mathbb{A}^1"] & Z_k \arrow[l] & E_k \arrow[l] \\
& X_0^{(1)} \arrow[d, squiggly, "\mathcal{X}'^{(1)}"'] & X \arrow[l, squiggly, "\mathcal{X}^{(1)}"'] \arrow[d, squiggly, "\mathcal{X}^{(0)}"] & & & \\
& X_0' & X_0^{(0)} \arrow[l, squiggly, "\mathcal{X}'^{(0)}"] & & & \\
C_0' \arrow[ur, dashed] & & & C_0^{(0)} \arrow[lll, squiggly, "\mathcal{C}'^{(0)}"] \arrow[ul, dashed] & Z_{k,0} \arrow[l] & E_k \arrow[l]
\end{tikzcd} \tag{196}$$

By the same argument as in [55], we know that both test configurations $\mathcal{X}'^{(i)}$ for $i = 0, 1$ are weakly special and have vanishing Fut_ξ invariant. By [39], we know that both of them are special and hence $X_0^{(1)} \cong X_0' \cong X_0^{(0)}$ by the polystability of $X_0^{(i)}$.

The existence part can again be proved by the similar arguments as in [55], which deals with the case when $\xi = 0$. We just sketch the arguments. If (X, ξ) is K–polystable, then we are done. Otherwise, we can find a nontrivial $\mathbb{T}$–equivariant special test configuration such that the central fiber (with the vector field ξ) has a vanishing Fut_ξ invariant. By [55, Proof of Lemma 3.1], we know that the central fiber is K–semistable, and has an effective action by a larger torus. If the central fiber is K–polystable, then we are done again. Otherwise, we can continue this process, which must stop since the dimension of the torus is bounded by the dimension of X.

Proof of Corollary 1.4 By the work of Chen, Sun and Wang in [28], which is based on the resolution of the Hamilton–Tian conjecture [29], we get a special $\mathbb{R}$–test configuration $\mathcal{F}^{\mathrm{ss}}$ with central fiber (W,ξ), and a special test configuration of (W,ξ) with central fiber (X_∞,ξ), which admits a Kähler–Ricci soliton and hence is K–polystable. By the work of Dervan and Székelyhidi [31], $\mathcal{F}^{\mathrm{ss}}$ obtains the minimum $h(X)$. The statement follows directly from Theorems 1.2 and 1.3. □

Remark 8.1 The fact that $\mathcal{F}^{\mathrm{ss}}$ obtains the minimum also follows from the K–semistability of (W,ξ) and Theorem 5.2. The K–semistability of (W,ξ) follows from the same degeneration argument as used in [58], or the Zariski-openness of K–semistability as pointed out in Remark 7.4.

Remark 8.2 As in the more general setting of [55] or [52], the algebraic results in this paper can be generalized to the log Fano case in a straightforward way.

Appendix Properties of $\widetilde{S}(v)$

Recall that by (137), for any valuation $v \in \mathring{\mathrm{Val}}(X)$ we have

$$\boldsymbol{Q}(v) := \boldsymbol{Q}(\mathcal{F}_v) = e^{-\widetilde{\boldsymbol{S}}^{\mathrm{NA}}(\mathcal{F}_v)} = 1 - \frac{1}{\boldsymbol{V}}\int_0^{T(v)} e^{-x}\,\mathrm{vol}(\mathcal{F}_v^{(x)}R_\bullet)\,dx =: 1-\Psi(v), \tag{197}$$

where, for simplicity of notation, we have written

$$T(v) = \lambda_{\max}(\mathcal{F}_v) \quad \text{and} \quad \Psi(v) = \frac{1}{\boldsymbol{V}}\int_0^{T(v)} e^{-x}\,\mathrm{vol}(\mathcal{F}_v^{(x)}R_\bullet)\,dx. \tag{198}$$

Proposition A.1 *The function $v \mapsto \Psi(v)$ is strictly increasing on $\mathring{\mathrm{Val}}(X)$. In other words, if $v \le w$, then $\Psi(v) \le \Psi(w)$, with the identity true only if $v = w$. As a consequence, $v \mapsto \widetilde{\boldsymbol{S}}(v)$ is strictly increasing on* $\mathrm{Val}(X)$.

This is proved as in [11, Proof of Proposition 3.15] (which is based on an argument in the local case from [58]). We sketch the argument for the reader's convenience.

Proof First, by using Theorem 2.5, we can show that

$$\Psi(v) = \lim_{m\to+\infty}\frac{1}{mN_m}\sum_{j\ge1} e^{-j/m}\dim\mathcal{F}_v^j R_m. \tag{199}$$

Suppose that $v \le w$ but $v \ne w$. Then by rescaling v, w and $L = -K_X$, we can assume that there exists $s \in H^0(X,L)$ with $w(s) = p \in \mathbb{N}^*$ and $v(s) \le p-1$. Then, arguing as in [11, Proof of Proposition 3.15], we have

$$\dim(\mathcal{F}_w^j R_m/\mathcal{F}_v^j R_m) \geq \sum_{1\leq i\leq \min\{j/p,m\}} \dim(\mathcal{F}_v^{j-ip} R_m/\mathcal{F}_v^{j-ip+1} R_{m-i}). \tag{200}$$

One the other hand, with $C=\max\{T(v),T(w)\}$, we get

$$\begin{aligned}\sum_{j\geq 1}\dim e^{-j/m}(\mathcal{F}_w^j R_m-\mathcal{F}_v^j R_m) &\geq e^{-C}\sum_{j\geq 1}(\mathcal{F}_w^j R_m-\mathcal{F}_v^j R_m)\\ &\geq e^{-C}\sum_{1\leq i\leq m}\sum_{j\geq pi}(\dim \mathcal{F}_v^{j-ip} R_{m-i}-\mathcal{F}_v^{j-ip+1} R_{m-i})\\ &= e^{-C}\sum_{1\leq i\leq m}\dim R_{m-i}.\end{aligned}$$

So we conclude

$$\Psi(v)-\Psi(w)\geq e^{-C}\lim_{m\to+\infty}\frac{1}{mN_m}\sum_{1\leq i\leq m}\dim R_{m-i}>0.$$ □

Let $\pi\colon Y\to X$ be a proper birational morphism with Y a regular and $E=\sum_i E_i$ a reduced simple normal crossing divisor.

Proposition A.2 *The function $v\mapsto \boldsymbol{Q}(v)$ is continuous on* $\mathrm{QM}(Y,E)$.

We use the same strategy as [14, Proposition 2.4]. As noted in [38], for any $v\in \mathrm{Val}(X)$, we have $A(v)/T(v)\geq \alpha(X)>0$, which implies, with $C=\alpha(X)^{-1}$,

$$T(v)\leq CA(v). \tag{201}$$

Lemma A.3 *For any $v\in\mathrm{Val}(X)$, we have the inequality*

$$\Psi(v)\leq CA(v). \tag{202}$$

Proof Since $\mathrm{vol}(\mathcal{F}^{(x)}R_\bullet)\leq \boldsymbol{V}$, we immediately get

$$\Psi(v)\leq\int_0^{T(v)} e^{-x}\,dx = 1-e^{-T(v)}\leq T(v)\leq CA(v),$$

where we used the inequality $1-e^{-x}\leq x$ for any $x\in\mathbb{R}_{\geq 0}$, and the inequality (201). □

Similarly to [38; 11], we introduce the approximation

$$\boldsymbol{Q}_m(\mathcal{F})=\frac{1}{N_m}\sum_i e^{-\lambda_i^{(m)}/m}=\frac{1}{N_m}\int_0^{+\infty} e^{-x/m}d(-\dim\mathcal{F}^x R_m) \tag{203}$$

$$=1-\frac{1}{N_m}\int_0^{\lambda_{\max}^{(m)}(\mathcal{F})/m} e^{-x}\dim\mathcal{F}^{xm}R_m\,dx =: 1-\Psi_m(\mathcal{F}), \tag{204}$$

where we set

$$\Psi_m(v)=\frac{1}{N_m}\int_0^{\lambda_{\max}^{(m)}} e^{-x}\dim\mathcal{F}^{xm}R_m\,dx=\frac{1}{N_m}\int_0^{T(v)} e^{-x}\dim\mathcal{F}^{xm}R_m\,dx. \tag{205}$$

Similarly to [38; 11], for any valuation $v \in \mathrm{Val}(X)$ we have the identity

$$\boldsymbol{Q}_m(v) = \boldsymbol{Q}_m(\mathcal{F}_v) = \min_{\{s_j\}} \frac{1}{N_m} \sum_{j=1}^{N_m} e^{-v(s_j)/m}, \tag{206}$$

where the minimum is taken over all bases $s_1, \dots, s_{N_m}$ of $H^0(X, -mK_X)$.

For any $\mathfrak{s} := \{s_1, \dots, s_{N_m}\} \in H^0(X, -mK_X)^{N_m}$, define a function

$$\varphi_{\mathfrak{s}}(v) := \sum_{j=1}^{N_m} e^{-v(s_j)/m}. \tag{207}$$

By the same argument as in [14, Proof of Lemma 2.5], the set of functions $\{\varphi_{\mathfrak{s}}(v) | \mathfrak{s} \in R_m^{N_m}\}$ is finite. So $\boldsymbol{Q}_m$ is continuous on $\mathrm{QM}(Y, E)$.

As in [14, Proof of Proposition 2.4], the continuity of Ψ and hence $\boldsymbol{Q}$ follows easily from the following proposition, which we prove by using the techniques developed in [11; 13].

Lemma A.4 (i) *For any $v \in \mathrm{Val}(X)$ with $A(v) < +\infty$, we have the convergence*

$$\lim_{m \to +\infty} \Psi_m(v) = \Psi(v). \tag{208}$$

(ii) *For any $\epsilon > 0$ and any $C_1 > 0$, there exists $C_2 > 0$ and $m_0 > 0$ such that if $v \in \mathrm{Val}(X)$ satisfies $A(v) < C_1$, we have*

$$|\Psi_m(\mathcal{F}_v) - \Psi(\mathcal{F}_v)| \le \epsilon \tag{209}$$

for all m divisible by m_0.

Proof The first statement follows from Theorem 2.5(ii). We focus on the second statement.

Note that e^{-G} is convex and $0 \le e^{-G} \le 1$. By [11, Lemma 2.2], for any $\epsilon' > 0$ there exists $m_0(\epsilon')$ such that for any $m \ge m_0$,

$$\int_\Delta e^{-G}\, d\rho_m \ge \int_\Delta e^{-G}\, dy - \epsilon'. \tag{210}$$

By the same argument as [11, Proof of Lemma 2.9], we get

$$\boldsymbol{Q}_m(\mathcal{F}_v) \ge \frac{m^n}{N_m} \int_\Delta e^{-G}\, d\rho_m. \tag{211}$$

Note that $\lim_{m \to +\infty} m^n/N_m = V$. So for any $\epsilon > 0$ there exists m_0 such that for any $m \ge m_0$,

$$\boldsymbol{Q}_m(\mathcal{F}_v) \ge \frac{n!}{V} \int_\Delta e^{-G}\, dy - \epsilon = \boldsymbol{Q}(\mathcal{F}_v) - \epsilon. \tag{212}$$

We need to prove the other direction of inequality. Following [11], define a graded linear series

$$\widetilde{\mathcal{F}}^{(t)}_{m,p} R_{mp} := H^0\big(X, mpL \otimes \overline{\mathfrak{b}(|\mathcal{F}^{mt} R_m|)^p}\big), \tag{213}$$

where $\mathfrak{b}(|\mathcal{F}^{mt}R_m|)$ is the base ideal of the sublinear system $\mathcal{F}^{tm}R_m$. Set

$$\widetilde{\Psi}_m(\mathcal{F}) = \int_0^{T(v)} e^{-t}\,\mathrm{vol}(\widetilde{\mathcal{F}}^{(t)}_{m,\bullet})\,dt. \tag{214}$$

By [11, Proposition 5.13], there exists $a = a(X, -K_X) > 0$ such that for all $t \in \mathbb{Q}_{>0}$ with $mt > A(v)$, we have, with $t' = t - m^{-1}(at + A(v))$,

$$\Big(\frac{m-a}{m}\Big)^{n+1}\mathrm{vol}(\mathcal{F}^{(t)}_{\bullet}) \le \frac{1}{m^n}\,\mathrm{vol}(\widetilde{\mathcal{F}}^{(t')}_{m,\bullet}). \tag{215}$$

So we can estimate as in [11, Proof of Proposition 5.15] to get

$$\begin{aligned}\widetilde{\Psi}_m(v) &\ge \Big(\frac{m-a}{m}\Big)^{n+1}\Big(\Psi(v) - e^{(aT(v)+A(v))/m}\int_0^{A(v)/(m-a)}\frac{\mathrm{vol}(\mathcal{F}^{(t)})}{V}e^{-t}\,dt\Big)\\ &\ge \Big(\frac{m-a}{m}\Big)^{n+1}\Big(\Psi(v) - e^{CA(v)/m}\frac{A(v)}{m-a}\Big).\end{aligned}$$

From this it is easy to get

$$\Psi(v) - \widetilde{\Psi}_m(v) \le C\frac{A(v)}{m}.$$

To compare with Ψ_m, we further set

$$\mathcal{F}^{(x)}_{m,p} = \mathrm{Im}\big(S^p\mathcal{F}^{mx}R_m \to H^0(X, pmL)\big). \tag{216}$$

By [13, Propositions 5.14 and 3.2], there exists a positive constant $C > 0$ independent of v such that for all $x \le T(v) - CA(v)/m$, we have $\mathrm{vol}(\mathcal{F}^{(x)}_{m,\bullet}) = \mathrm{vol}(\widetilde{\mathcal{F}}^{(x)}_{m,\bullet})$. So as in [13, Proof of Proposition 5.15], we get

$$\begin{aligned}\Psi(v) \le \widetilde{\Psi}_m(v) + C\frac{A(v)}{m} &= \frac{1}{V}\int_0^{T(v)}\frac{\mathrm{vol}(\widetilde{\mathcal{F}}^{(x)}_{m,\bullet})}{m^n}e^{-x}\,dx + \frac{CA(v)}{m}\\ &\le \frac{1}{V}\int_0^{T(v)-CA(v)/m}\frac{\mathrm{vol}(\mathcal{F}^{(x)}_{m,\bullet})}{m^n}e^{-x}\,dx + \frac{CA(v)}{m}\\ &\le \frac{1}{V}\int_0^{T(v)}\frac{\mathrm{vol}(\mathcal{F}^{(x)}_{m,\bullet})}{m^n}e^{-x}\,dx + \frac{CA(v)}{m}.\end{aligned}$$

For the second inequality we used the estimate that, as $m \to +\infty$,

$$\int_{T(v)-CA(v)/m}^{T(v)} e^{-x}\,dx = e^{-(T(v)-CA(v)/m)} - e^{-T(v)} \le e^{CA(v)/m} - 1 = O\Big(\frac{A(v)}{m}\Big).$$

Fixing any $\epsilon > 0$, by choosing $m \gg 1$ and $p \gg 1$ we have (see [13, equation (5.6)]):

$$\left|\frac{\mathrm{vol}(\mathcal{F}^{(x)}_{m,\bullet})}{m^nV} - \frac{\dim\mathcal{F}^{(x)}_{m,p}}{N_{mp}}\right| < \epsilon. \tag{217}$$

Finally we can estimate as in [13, Proof of Theorem 5.13]: for $m \gg 1$,

$$\begin{aligned}\Psi(v) &\le \frac{1}{V}\int_0^{T(v)}\frac{\mathrm{vol}(\mathcal{F}^{(x)}_{m,\bullet})}{m^n}e^{-x}\,dx + \frac{CA(v)}{m} \le \int_0^{+\infty}\frac{\dim\mathcal{F}^{(x)}_{m,p}}{N_{mp}}e^{-x}\,dx + \epsilon T(v) + \frac{CA(v)}{m}\\ &\le \int_0^{+\infty}\frac{\dim\mathcal{F}^{pmx}R_m}{N_{mp}}e^{-x}\,dx + 2\epsilon A(v) = \Psi(v) + 2\epsilon A(v).\end{aligned}$$

In the third inequality, we used again the inequality $1 - e^{-T(v)} \leq T(v)$. Since $\epsilon > 0$ is arbitrary, we get the conclusion. □

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