Part III

2-Knots

Chapter 14

Knots and links

In this chapter we introduce the basic notions and constructions of knot theory. Many of these apply equally well in all dimensions, and for the most part we have framed our definitions in such generality, although our main concern is with 2-knots (embeddings of S^2 in S^4). In particular, we show how the classification of higher dimensional knots may be reduced (essentially) to the classification of certain closed manifolds, and we give Kervaire's characterization of high dimensional knot groups. In the final sections we comment briefly on links and the groups of links, homology spheres and their groups.

14.1 Knots

The standard orientation of R^n induces an orientation on the unit n-disc $D^n = \{(x_1, \dots x_n) \in R^n \mid \Sigma x_i^2 \leq 1\}$ and hence on its boundary $S^{n-1} = \partial D^n$, by the convention "outward normal first". We shall assume that standard discs and spheres have such orientations. Qualifications shall usually be omitted when there is no risk of ambiguity. In particular, we shall often abbreviate $X(K)$, $M(K)$ and πK (defined below) as X, M and π, respectively.

An *n-knot* is a locally flat embedding $K : S^n \to S^{n+2}$. (We shall also use the terms "classical knot" when $n = 1$, "higher dimensional knot" when $n \geq 2$ and "high dimensional knot" when $n \geq 3$.) It is determined up to (ambient) isotopy by its image $K(S^n)$, considered as an oriented codimension 2 submanifold of S^{n+2}, and so we may let K also denote this submanifold. Let r_n be an orientation reversing self homeomorphism of S^n. Then K is *invertible*, *$+$amphicheiral* or *$-$amphicheiral* if it is isotopic to $K\rho = Kr_n$, $rK = r_{n+2}K$ or $-K = rK\rho$, respectively. An n-knot is *trivial* if it is isotopic to the composite of equatorial inclusions $S^n \subset S^{n+1} \subset S^{n+2}$.

Every knot has a product neighbourhood: there is an embedding $j : S^n \times D^2$ onto a closed neighbourhood N of K, such that $j(S^n \times \{0\}) = K$ and ∂N is bicollared in S^{n+2} [KS75, FQ]. We may assume that j is orientation preserving. If $n \geq 2$ it is then unique up to isotopy *rel* $S^n \times \{0\}$. The *exterior* of K is the compact $(n+2)$-manifold $X(K) = S^{n+2} \setminus intN$ with boundary $\partial X(K) \cong S^n \times S^1$, and is well defined up to homeomorphism. It inherits an orientation from S^{n+2}. An n-knot K is trivial if and only if $X(K) \simeq S^1$; this follows from Dehn's Lemma if $n = 1$, is due to Freedman if $n = 2$ ([FQ] – see Corollary 17.1.1 below) and is an easy consequence of the s-cobordism theorem if $n \geq 3$.

The *knot group* is $\pi K = \pi_1(X(K))$. An oriented simple closed curve isotopic to the oriented boundary of a transverse disc $\{j\} \times S^1$ is called a *meridian* for K, and we shall also use this term to denote the corresponding elements of π. If μ is a meridian for K, represented by a simple closed curve on ∂X then $X \cup_\mu D^2$ is a deformation retract of $S^{n+2} - \{*\}$ and so is contractible. Hence π is generated by the conjugacy class of its meridians.

Assume for the remainder of this section that $n \geq 2$. The group of pseudoisotopy classes of self homeomorphisms of $S^n \times S^1$ is $(Z/2Z)^3$, generated by reflections in either factor and by the map τ given by $\tau(x, y) = (\rho(y)(x), y)$ for all x in S^n and y in S^1, where $\rho : S^1 \to SO(n + 1)$ is an essential map [Gl62, Br67, Kt69]. As any self homeomorphism of $S^n \times S^1$ extends across $D^{n+1} \times S^1$ the *knot manifold* $M(K) = X(K) \cup (D^{n+1} \times S^1)$ obtained from S^{n+2} by surgery on K is well defined, and it inherits an orientation from S^{n+2} via X. Moreover $\pi_1(M(K)) \cong \pi K$ and $\chi(M(K)) = 0$. Conversely, suppose that M is a closed orientable 4-manifold with $\chi(M) = 0$ and $\pi_1(M)$ is generated by the conjugacy class of a single element. (Note that each conjugacy class in π corresponds to an unique isotopy class of oriented simple closed curves in M.) Surgery on a loop in M representing such an element gives a 1-connected 4-manifold Σ with $\chi(\Sigma) = 2$ which is thus homeomorphic to S^4 and which contains an embedded 2-sphere as the cocore of the surgery. We shall in fact study 2-knots through such 4-manifolds, as it is simpler to consider closed manifolds rather than pairs.

There is however an ambiguity when we attempt to recover K from $M = M(K)$. The cocore $\gamma = \{0\} \times S^1 \subset D^{n+1} \times S^1 \subset M$ of the original surgery is well defined up to isotopy by the conjugacy class of a meridian in $\pi K = \pi_1(M)$. (In fact the orientation of γ is irrelevant for what follows.) Its normal bundle is trivial, so γ has a product neighbourhood, P say, and we may assume that $M \setminus intP = X(K)$. But there are two essentially distinct ways of identifying ∂X with $S^n \times S^1 = \partial(S^n \times D^2)$, modulo self homeomorphisms of $S^n \times S^1$ that extend across $S^n \times D^2$. If we reverse the original construction of M we recover $(S^{n+2}, K) = (X \cup_j S^n \times D^2, S^n \times \{0\})$. If however we identify $S^n \times S^1$ with ∂X by means of $j\tau$ we obtain a new pair

$$(\Sigma, K^*) = (X \cup_{j\tau} S^n \times D^2, S^n \times \{0\}).$$

It is easily seen that $\Sigma \simeq S^{n+2}$, and hence $\Sigma \cong S^{n+2}$. We may assume that the homeomorphism is orientation preserving. Thus we obtain a new n-knot

K^*, which we shall call the *Gluck reconstruction* of K. The knot K is *reflexive* if it is determined as an unoriented submanifold by its exterior, i.e., if K^* is isotopic to K, rK, $K\rho$ or $-K$.

If there is an orientation preserving homeomorphism from $X(K_1)$ to $X(K)$ then K_1 is isotopic to K, K^*, $K\rho$ or $K^*\rho$. If the homeomorphism also preserves the homology class of the meridians then K_1 is isotopic to K or to K^*. Thus K is determined up to an ambiguity of order at most 2 by $M(K)$ together with the conjugacy class of a meridian.

A *Seifert hypersurface* for K is a locally flat, oriented codimension 1 submanifold V of S^{n+2} with (oriented) boundary K. By a standard argument these always exist. (Using obstruction theory it may be shown that the projection $pr_2 j^{-1} : \partial X \to S^n \times S^1 \to S^1$ extends to a map $p : X \to S^1$ [Ke65]. By topological transversality we may assume that $p^{-1}(1)$ is a bicollared, proper codimension 1 submanifold of X. The union $p^{-1}(1) \cup j(S^n \times [0,1])$ is then a Seifert hypersurface for K.) We shall say that V is *minimal* if the natural homomorphism from $\pi_1(V)$ to πK is a monomorphism.

In general there is no canonical choice of Seifert surface. However there is one important special case. An n-knot K is *fibred* if there is such a map $p : X \to S^1$ which is the projection of a fibre bundle. (Clearly K^* is then fibred also.) The exterior is then the mapping torus of a self homeomorphism θ of the fibre F of p, called the *(geometric) monodromy* of the bundle. Such a map p extends to a fibre bundle projection $q : M(K) \to S^1$, with fibre $\widehat{F} = F \cup D^{n+1}$, called the *closed fibre* of K. Conversely, if $M(K)$ fibres over S^1 then the cocore γ is homotopic (and thus isotopic) to a cross-section of the bundle projection, and so K is fibred. If the monodromy is represented by a self-homeomorphism of finite order then it has nonempty fixed point set, and the closed monodromy $\widehat{\theta}$ has finite order. However the results of [Hn] and [La] may be used to show that the closed monodromy of the spun trefoil knot $\sigma 3_1$ has finite order, but as $\pi_1(F) \cong F(2)$ has no automorphism of order 6 [Me74] there is no representative of finite order with nonempty fixed point set.

14.2 Covering spaces

Let K be an n-knot. Then $H_1(X(K);\mathbb{Z}) \cong Z$ and $H_i(X(K);\mathbb{Z}) = 0$ if $i > 1$, by Alexander duality. The meridians are all homologous and generate $\pi/\pi' = H_1(X;\mathbb{Z})$, and so determine a canonical isomorphism with Z. Moreover $H_2(\pi;\mathbb{Z}) = 0$, since it is a quotient of $H_2(X;\mathbb{Z}) = 0$.

We shall let $X'(K)$ and $M'(K)$ denote the covering spaces corresponding to the commutator subgroup. (The cover X'/X is also known as the *infinite cyclic cover* of the knot.) Since $\pi/\pi' = Z$ the (co)homology groups of X' are modules over the group ring $\mathbb{Z}[Z]$, which may be identified with the ring of integral Laurent polynomials $\Lambda = \mathbb{Z}[t, t^{-1}]$. If A is a Λ-module, let zA be the $\mathbb{Z}$-torsion submodule, and let $e^i A = Ext^i_\Lambda(A, \Lambda)$.

Since Λ is noetherian the (co)homology of a finitely generated free Λ-chain complex is finitely generated. The Wang sequence for the projection of X' onto X may be identified with the long exact sequence of homology corresponding to the exact sequence of coefficients

$$0 \to \Lambda \to \Lambda \to \mathbb{Z} \to 0.$$

Since X has the homology of a circle it follows easily that multiplication by $t-1$ induces automorphisms of the modules $H_i(X; \Lambda)$ for $i > 0$. Hence these homology modules are all finitely generated torsion Λ-modules. It follows that $Hom_\Lambda(H_i(X; \Lambda), \Lambda)$ is 0 for all i, and the UCSS collapses to a collection of short exact sequences

$$0 \to e^2 H_{i-2}(X; \Lambda) \to H^i(X; \Lambda) \to e^1 H_{i-1}(X; \Lambda) \to 0.$$

The infinite cyclic covering spaces X' and M' behave homologically much like $(n+1)$-manifolds, at least if we use field coefficients [Mi68, Ba80]. If $H_i(X; \Lambda) = 0$ for $1 \le i \le (n+1)/2$ then X' is acyclic; thus if also $\pi = Z$ then $X \simeq S^1$ and so K is trivial. All the classifications of high dimensional knots to date assume that $\pi = Z$ and that X' is highly connected.

When $n = 1$ or 2 knots with $\pi = Z$ are trivial, and it is more profitable to work with the universal cover $\widetilde{X}$ (or $\widetilde{M}$). In the classical case $\widetilde{X}$ is contractible [Pa57]. In higher dimensions X is aspherical only when the knot is trivial [DV73]. Nevertheless the closed 4-manifolds $M(K)$ obtained by surgery on 2-knots are often aspherical. (This asphericity is an additional reason for choosing to work with $M(K)$ rather than $X(K)$.)

14.3 Sums, factorization and satellites

The *sum* of two knots K_1 and K_2 may be defined (up to isotopy) as the n-knot $K_1 \sharp K_2$ obtained as follows. Let $D^n(\pm)$ denote the upper and lower hemispheres of S^n. We may isotope K_1 and K_2 so that each $K_i(D^n(\pm))$ contained in $D^{n+2}(\pm)$, $K_1(D^n(+))$ is a trivial n-disc in $D^{n+2}(+)$, $K_2(D^n(-))$ is a trivial n-disc in $D^{n+2}(-)$ and $K_1|_{S^{n-1}} = K_2|_{S^{n-1}}$ (as the oriented boundaries of the images of $D^n(-)$). Then we let $K_1 \sharp K_2 = K_1|_{D^n(-)} \cup K_2|_{D^n(+)}$. By van

Kampen's theorem $\pi(K_1\sharp K_2) = \pi K_1 *_Z \pi K_2$ where the amalgamating subgroup is generated by a meridian in each knot group. It is not hard to see that $X'(K_1\sharp K_2) \simeq X'(K_1) \vee X'(K_2)$, and so $\pi(K_1\sharp K_2)' \cong \pi(K_1)' * \pi(K_2)'$.

The knot K is *irreducible* if it is not the sum of two nontrivial knots. Every knot has a finite factorization into irreducible knots [DF87]. (For 1- and 2-knots whose groups have finitely generated commutator subgroups this follows easily from the Grushko-Neumann theorem on factorizations of groups as free products.) In the classical case the factorization is essentially unique, but if $n \geq 3$ there are n-knots with several distinct such factorizations [BHK81]. Almost nothing is known about the factorization of 2-knots.

If K_1 and K_2 are fibred then so is their sum, and the closed fibre of $K_1\sharp K_2$ is the connected sum of the closed fibres of K_1 and K_2. However in the absence of an adequate criterion for a 2-knot to fibre, we do not know whether every summand of a fibred 2-knot is fibred. In view of the unique factorization theorem for oriented 3-manifolds we might hope that there would be a similar theorem for fibred 2-knots. However the closed fibre of an irreducible 2-knot need not be an irreducible 3-manifold. (For instance, the Artin spin of a trefoil knot is an irreducible fibred 2-knot, but its closed fibre is $(S^2 \times S^1)\sharp(S^2 \times S^1)$.)

A more general method of combining two knots is the process of forming satellites. Although this process arose in the classical case, where it is intimately connected with the notion of torus decomposition, we shall describe only the higher-dimensional version of [Kn83]. Let K_1 and K_2 be n-knots (with $n \geq 2$) and let γ be a simple closed curve in $X(K_1)$, with a product neighbourhood U. Then there is a homeomomorphism h which carries $S^{n+2} \setminus intU \cong S^n \times D^2$ onto a product neighbourhood of K_2. The knot $\Sigma(K_2; K_1, \gamma) = hK_1$ is called the *satellite* of K_1 about K_2 relative to γ. We also call K_2 a *companion* of hK_1. If either $\gamma = 1$ or K_2 is trivial then $\Sigma(K_2; K_1, \gamma) = K_1$. If γ is a meridian for K_1 then $\Sigma(K_2; K_1, \gamma) = K_1\sharp K_2$. If γ has finite order in πK_1 let q be that order; otherwise let $q = 0$. Let w be a meridian in πK_2. Then $\pi\Sigma(K_2; K_1, \gamma) \cong (\pi K_2/\langle\langle w^q\rangle\rangle) *_{Z/qZ} \pi K_1$, where w is identified with γ in πK_1, by Van Kampen's theorem.

14.4 Spinning, twist spinning and deform spinning

The first nontrivial examples of higher dimensional knots were given by Artin [Ar25]. We may paraphrase his original idea as follows. As the half space $R^3_+ = \{(w, x, y, z) \in R^4 \mid w = 0, z \geq 0\}$ is spun about the axis $A = \{(0, x, y, 0)\}$ it sweeps out R^4, and any arc in R^3_+ with endpoints on A sweeps out a 2-sphere.

This construction has been extended, first to twist-spinning and roll-spinning [Fo66, Ze65], and then more generally to deform spinning [Li79]. Let g an orientation preserving self-homeomorphism of S^{n+2} which is the identity on the n-knot K. If g does not twist the normal bundle of K in S^{n+2} then the sections of $M(g)$ determined by points of K have canonical "constant" framings, and surgery on such a section in the pair $(M(g), K\times S^1)$ gives an $(n+1)$-knot, called the *deform spin* of K determined by g. The deform spin is *untwisted* if g preserves a Seifert hypersurface for K. If g is the identity this gives the *Artin spin* σK, and $\pi\sigma K=\pi K$.

Twist spins are defined by maps supported in a collar of $\partial X=K\times S^1$. (If $n=1$ we use the 0-framing.) Let r be an integer. The self-map t_r of S^{n+2} defined by $t_r(k,z,x)=(k,e^{2\pi irx}z,x)$ on $\partial X\times[0,1]$ and the identity elsewhere gives the r-*twist spin* τ_rK. Clearly $\tau_0K=\sigma K$. The group of τ_rK is obtained from πK by adjoining the relation making the r^{th} power of (any) meridian central. Zeeman discovered the remarkable fact that if $r\neq 0$ then τ_rK is fibred, with closed fibre the r-fold cyclic branched cover of S^{n+2}, branched over K, and monodromy the canonical generator of the group of covering transformations [Ze65]. Hence τ_1K is always trivial. More generally, if g is an *un*twisted deformation of K and $r\neq 0$ then the knot determined by t_rg is fibred [Li79]. (See also [GK78, Mo83, Mo84] and [Pl84'].) Twist spins of $-$amphicheiral knots are $-$amphicheiral, while twist spinning interchanges invertibility and $+$amphicheirality [Li85].

If K is a classical knot the factors of the closed fibre of τ_rK are the cyclic branched covers of the prime factors of K, and are Haken, hyperbolic or Seifert fibred. With some exceptions for small values of r, the factors are aspherical, and $S^2\times S^1$ is never a factor [Pl84]. If $r>1$ and K is nontrivial then τ_rK is nontrivial, by the Smith Conjecture. If K is a deform spun 2-knot then the order ideal of $H_1(\pi K;\mathbb{Q}\Lambda)$ is invariant under the involution $t\mapsto t^{-1}$ [BM09].

14.5 Ribbon and slice knots

Two n-knots K_0 and K_1 are *concordant* if there is a locally flat embedding $\mathcal{K}: S^n\times[0,1]\to S^{n+2}\times[0,1]$ such that $K_i=\mathcal{K}\cap S^n\times\{i\}$ for $i=0,1$. They are s-*concordant* if there is a concordance whose exterior is an s-cobordism (*rel* ∂) from $X(K_0)$ to $X(K_1)$. (If $n>2$ this is equivalent to ambient isotopy, by the s-cobordism theorem.)

An n-knot K is a *slice* knot if it is concordant to the unknot; equivalently, if it bounds a properly embedded $(n+1)$-disc Δ in D^{n+3}. Such a disc is called a *slice disc* for K. Doubling the pair (D^{n+3}, Δ) gives an $(n+1)$-knot which meets the equatorial S^{n+2} of S^{n+3} transversally in K; if the $(n+1)$-knot can be chosen to be trivial then K is *doubly slice*. All even-dimensional knots are slice [Ke65], but not all slice knots are doubly slice, and no adequate criterion is yet known. The sum $K\sharp -K$ is a slice of $\tau_1 K$ and so is doubly slice [Su71]. Twist spins of doubly slice knots are doubly slice.

An n-knot K is a *ribbon* knot if it is the boundary of an immersed $(n+1)$-disc Δ in S^{n+2} whose only singularities are transverse double points, the double point sets being a disjoint union of discs. Given such a "ribbon" $(n+1)$-disc Δ in S^{n+2} the cartesian product $\Delta \times D^p \subset S^{n+2} \times D^p \subset S^{n+2+p}$ determines a ribbon $(n+1+p)$-disc in S^{n+2+p}. All higher dimensional ribbon knots derive from ribbon 1-knots by this process [Yn77]. As the p-disc has an orientation reversing involution, this easily imples that all ribbon n-knots with $n \geq 2$ are $-$amphicheiral. The Artin spin of a 1-knot is a ribbon 2-knot. Each ribbon 2-knot has a Seifert hypersurface which is a once-punctured connected sum of copies of $S^1 \times S^2$ [Yn69]. Hence such knots are reflexive. (See [Su76] for more on geometric properties of such knots.)

An n-knot K is a *homotopy ribbon* knot if it is a slice knot with a slice disc whose exterior W has a handlebody decomposition consisting of 0-, 1- and 2-handles. The dual decomposition of W relative to $\partial W = M(K)$ has only handles of index $\geq n+1$, and so (W, M) is n-connected. (The definition of "homotopically ribbon" for 1-knots used in Problem 4.22 of [GK] requires only that this latter condition be satisfied.) More generally, we shall say that K is π_1-*slice* if the inclusion of $X(K)$ into the exterior of some slice disc induces an isomorphism on fundamental groups.

Every ribbon knot is homotopy ribbon and hence slice [Hi79], while if $n \geq 2$ every homotopy ribbon n-knot is π_1-slice. Nontrivial classical knots are never π_1-slice, since the longitude of a slice knot is nullhomotopic in the exterior of a slice disc. It is an open question whether every classical slice knot is ribbon. However in higher dimensions "slice" does not even imply "homotopy ribbon". (The simplest example is $\tau_2 3_1$ - see below.)

Most of the conditions considered here depend only on the h-cobordism class of $M(K)$. An n-knot K is slice if and only if $M = \partial W$, where W is an homology $S^1 \times D^{n+2}$ and the image of πK normally generates $\pi_1(W)$, and it is π_1-slice if and only if we may assume also that the inclusion of M into ∂W induces an isomorphism on π_1. The knot K is doubly slice if and only if M embeds in $S^1 \times S^{n+2}$ via a map which induces an isomorphism on first homology.

14.6 The Kervaire conditions

A group G has *weight* 1 if it has an element whose conjugates generate G. Such an element is called a *weight element* for G, and its conjugacy class is called a *weight class* for G. If G is solvable then it has weight 1 if and only if G/G' is cyclic, for a solvable group with trivial abelianization must be trivial.

If π is the group of an n-knot K then

(1) *π is finitely presentable;*

(2) *π is of weight 1;*

(3) *$H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong Z$; and*

(4) *$H_2(\pi;\mathbb{Z}) = 0$.*

Kervaire showed that any group satisfying these conditions is an n-knot group, for every $n \geq 3$ [Ke65]. These conditions are also necessary when $n = 1$ or 2, but are then no longer sufficient, and there are as yet no corresponding characterizations for 1- and 2-knot groups. If (4) is replaced by the stronger condition that $\mathrm{def}(\pi) = 1$ then π is a 2-knot group, but this condition is not necessary [Ke65]. (See §9 of this chapter, §4 of Chapter 15 and §4 of Chapter 16 for examples with deficiency ≤ 0.) Gonzalez-Acuña has given a characterization of 2-knot groups as groups admitting certain presentations [GA94]. (Note also that if π is a high dimensional knot group then $q(\pi) \geq 0$, and $q(\pi) = 0$ if and only if π is a 2-knot group.)

Every knot group has a Wirtinger presentation, i.e., one in which the relations are all of the form $x_j = w_j x_0 w_j^{-1}$, where $\{x_i, 0 \leq i \leq n\}$ is the generating set [Yj70]. If K is a nontrivial 1-knot then πK has a Wirtinger presentation of deficiency 1. A group has such a presentation if and only if it has weight 1 and has a deficiency 1 presentation P such that the presentation of the trivial group obtained by adjoining the relation killing a weight element is AC-equivalent to the empty presentation [Yo82']. Any such group is the group of a 2-knot which is a smooth embedding in the standard smooth structure on S^4 [Le78]. The group of a nontrivial 1-knot K has one end [Pa57], so $X(K)$ is aspherical, and $X(K)$ collapses to a finite 2-complex, so $g.d.\pi K = 2$. If π is an n-knot group then $g.d.\pi = 2$ if and only if $c.d.\pi = 2$ and $\mathrm{def}(\pi) = 1$, by Theorem 2.8.

Since the group of a homotopy ribbon n-knot (with $n \geq 2$) is the fundamental group of a $(n+3)$-manifold W with $\chi(W) = 0$ and which can be built with 0-, 1- and 2-handles only, such groups also have deficiency 1. Conversely, if a finitely presentable group π has weight 1 and deficiency 1 then we may use such a presentation to construct a 5-dimensional handlebody $W = D^5 \cup \{h_i^1\} \cup \{h_j^2\}$

with $\pi_1(\partial W) = \pi_1(W) \cong \pi$ and $\chi(W) = 0$. Adjoining another 2-handle h along a loop representing a weight class for $\pi_1(\partial W)$ gives a homotopy 5-ball B with 1-connected boundary. Thus $\partial B \cong S^4$, and the boundary of the cocore of the 2-handle h is clearly a homotopy ribbon 2-knot with group π. (In fact any group of weight 1 with a Wirtinger presentation of deficiency 1 is the group of a ribbon n-knot, for each $n \geq 2$ [Yj69] – see [H3].)

The deficiency may be estimated in terms of the minimum number of generators of the Λ-module $e^2(\pi'/\pi'')$. Using this observation, it may be shown that if K is the sum of $m+1$ copies of $\tau_2 3_1$ then $\mathrm{def}(\pi K) = -m$ [Le78]. There are irreducible 2-knots whose groups have deficiency $-m$, for each $m \geq 0$ [Kn83].

A knot group π has two ends if and only if π' is finite. We shall determine all such 2-knots in §4 of Chapter 15. Nontrivial torsion-free knot groups have one end [Kl93]. There are also many 2-knot groups with infinitely many ends [GM78]. The simplest is perhaps the group with presentation

$$\langle a, b, t \mid a^3 = 1,\ aba^{-1} = b^2,\ tat^{-1} = a^2 \rangle.$$

The first two relations imply that $b^7 = 1$, and so this group is an HNN extension of $\langle a, b\rangle \cong Z/7Z \rtimes_2 Z/3Z$, with associated subgroups both $\langle a \rangle \cong Z/3Z$. It is also the group of a satellite of $\tau_2 3_1$ with companion Fox's Example 10.

14.7 Weight elements, classes and orbits

Two 2-knots K and K_1 have homeomorphic exteriors if and only if there is a homeomorphism from $M(K_1)$ to $M(K)$ which carries the conjugacy class of a meridian of K_1 to that of K (up to inversion). In fact if M is any closed orientable 4-manifold with $\chi(M) = 0$ and with $\pi = \pi_1(M)$ of weight 1 then surgery on a weight class gives a 2-knot with group π. Moreover, if t and u are two weight elements and f is a self homeomorphism of M such that u is conjugate to $f_*(t^{\pm 1})$ then surgeries on t and u lead to knots whose exteriors are homeomorphic (via the restriction of a self homeomorphism of M isotopic to f). Thus the natural invariant to distinguish between knots with isomorphic groups is not the weight class, but rather the orbit of the weight class under the action of self homeomorphisms of M. In particular, the orbit of a weight element under $Aut(\pi)$ is a well defined invariant, which we shall call the *weight orbit*. If every automorphism of π is realized by a self homeomorphism of M then the homeomorphism class of M and the weight orbit together form a complete invariant for the (unoriented) knot, up to Gluck reconstruction. (This is the case if M is an infrasolvmanifold.)

For oriented knots we need a refinement of this notion. If w is a weight element for π then we shall call the set $\{\alpha(w) \mid \alpha \in Aut(\pi),\ \alpha(w) \equiv w \ mod\ \pi'\}$ a *strict* weight orbit for π. A strict weight orbit determines a transverse orientation for the corresponding knot (and its Gluck reconstruction). An orientation for the ambient sphere is determined by an orientation for $M(K)$. If K is invertible or $+$amphicheiral then there is a self homeomorphism of M which is orientation preserving or reversing (respectively) and which reverses the transverse orientation of the knot, i.e., carries the strict weight orbit to its inverse. Similarly, if K is $-$amphicheiral there is an orientation reversing self homeomorphism of M which preserves the strict weight orbit.

Theorem 14.1 *Let G be a group of weight 1 and with $G/G' \cong Z$. Let t be an element of G whose image generates G/G' and let c_t be the automorphism of G' induced by conjugation by t. Then*

(1) *t is a weight element if and only if c_t is meridianal;*

(2) *two weight elements t, u are in the same weight class if and only if there is an inner automorphism c_g of G' such that $c_u = c_g c_t c_g^{-1}$;*

(3) *two weight elements t, u are in the same strict weight orbit if and only if there is an automorphism d of G' such that $c_u = dc_t d^{-1}$ and $dc_t d^{-1} c_t^{-1}$ is an inner automorphism;*

(4) *if t and u are weight elements then u is conjugate to $(g''t)^{\pm 1}$ for some g'' in G''.*

Proof The verification of (1-3) is routine. If t and u are weight elements then, up to inversion, u must equal $g't$ for some g' in G'. Since multiplication by $t-1$ is invertible on G'/G'' we have $g' = khth^{-1}t^{-1}$ for some h in G' and k in G''. Let $g'' = h^{-1}kh$. Then $u = g't = hg''th^{-1}$. □

An immediate consequence of this theorem is that if t and u are in the same strict weight orbit then c_t and c_u have the same order. Moreover if C is the centralizer of c_t in $Aut(G')$ then the strict weight orbit of t contains at most $[Aut(G') : C.Inn(G')] \leq |Out(G')|$ weight classes. In general there may be infinitely many weight orbits [Pl83']. However if π is metabelian the weight class (and hence the weight orbit) is unique up to inversion, by part (4) of the theorem.

14.8 The commutator subgroup

It shall be useful to reformulate the Kervaire conditions in terms of the automorphism of the commutator subgroup induced by conjugation by a meridian. An automorphism ϕ of a group G is *meridianal* if $\langle\langle g^{-1}\phi(g) \mid g \in G\rangle\rangle_G = G$. If H is a characteristic subgroup of G and ϕ is meridianal the induced automorphism of G/H is then also meridianal. In particular, $H_1(\phi) - 1$ maps $H_1(G;\mathbb{Z}) = G/G'$ onto itself. If G is solvable an automorphism satisfying the latter condition is meridianal, for a solvable perfect group is trivial.

It is easy to see that no group G with $G/G' \cong Z$ can have $G' \cong Z$ or D. It follows that the commutator subgroup of a knot group never has two ends.

Theorem 14.2 [HK78],[Le78] *A finitely presentable group π is a high dimensional knot group if and only if $\pi \cong \pi' \rtimes_\theta Z$ for some meridianal automorphism θ of π' such that $H_2(\theta) - 1$ is an automorphism of $H_2(\pi';\mathbb{Z})$.* □

If π is a knot group then π'/π'' is a finitely generated Λ-module. Levine and Weber have made explicit the conditions under which a finitely generated Λ-module may be the commutator subgroup of a metabelian high dimensional knot group [LW78]. Leaving aside the Λ-module structure, Hausmann and Kervaire have characterized the finitely generated abelian groups A that may be commutator subgroups of high dimensional knot groups [HK78]. "Most" can occur; there are mild restrictions on 2- and 3-torsion, and if A is infinite it must have rank at least 3. We shall show that the abelian groups which are commutator subgroups of 2-knot groups are Z^3, $Z[\frac{1}{2}]$ (the additive group of dyadic rationals) and the cyclic groups of odd order. (See Theorems 15.7 and 15.12.) The commutator subgroup of a nontrivial classical knot group is never abelian.

Hausmann and Kervaire also showed that any finitely generated abelian group could be the centre of a high dimensional knot group [HK78']. We shall show that the centre of a 2-knot group is either Z^2, torsion-free of rank 1, finitely generated of rank 1 or is a torsion group. (See Theorems 15.7 and 16.3. In all known cases the centre is Z^2, $Z \oplus (Z/2Z)$, Z, $Z/2Z$ or 1.) A classical knot group has nontrivial centre if and only if the knot is a torus knot [BZ]; the centre is then Z.

Silver has given examples of high dimensional knot groups π with π' finitely generated but not finitely presentable [Si91]. He has also shown that there are embeddings $j : T \to S^4$ such that $\pi_1(S^4 \setminus j(T))'$ is finitely generated but not finitely presentable [Si97]. However no such 2-knot groups are known. If the commutator subgroup is finitely generated then it is the unique HNN base [Si96]. Thus knots with such groups have no minimal Seifert hypersurfaces.

The first examples of high dimensional knot groups which are not 2-knot groups made use of Poincaré duality with coefficients Λ. Farber [Fa77] and Levine [Le77] independently found the following theorem.

Theorem 14.3 (Farber, Levine) *Let K be a 2-knot and $A = H_1(M(K);\Lambda)$. Then $H_2(M(K);\Lambda) \cong \overline{e^1 A}$, and there is a nondegenerate $\mathbb{Z}$-bilinear pairing $[\ ,\ ] : zA \times zA \to \mathbb{Q}/\mathbb{Z}$ such that $[t\alpha, t\beta] = [\alpha, \beta]$ for all α and β in zA.* $\square$

Most of this theorem follows easily from Poincaré duality with coefficients Λ, but some care is needed in order to establish the symmetry of the pairing. When K is a fibred 2-knot, with closed fibre $\widehat{F}$, the Farber-Levine pairing is just the standard linking pairing on the torsion subgroup of $H_1(\widehat{F};\mathbb{Z})$, together with the automorphism induced by the monodromy. In particular, Farber observed that although the group π with presentation

$$\langle a, t \mid tat^{-1} = a^2,\ a^5 = 1\rangle$$

is a high dimensional knot group, if ℓ is any nondegenerate $\mathbb{Z}$-bilinear pairing on $\pi' \cong Z/5Z$ with values in $\mathbb{Q}/\mathbb{Z}$ then $\ell(t\alpha, t\beta) = -\ell(\alpha, \beta)$ for all α, β in π', and so π is not a 2-knot group.

Corollary 14.3.1 [Le78] *$H_2(\pi';\mathbb{Z})$ is a quotient of $\overline{Hom_\Lambda(\pi'/\pi'', \mathbb{Q}(t)/\Lambda)}$.* $\square$

Every orientation preserving meridianal automorphism of a torsion-free 3-manifold group is realizable by a fibred 2-knot.

Theorem 14.4 *Let N be a closed orientable 3-manifold such that $\nu = \pi_1(N)$ is torsion-free. If K is a 2-knot such that $(\pi K)' \cong \nu$ then $M(K)$ is homotopy equivalent to the mapping torus of a self homeomorphism of N. If θ is a meridianal automorphism of ν then $\pi = \nu \rtimes_\theta Z$ is the group of a fibred 2-knot with fibre N if and only if $\theta_*(c_{N*}[N]) = c_{N*}[N]$.*

Proof The first assertion follows from Corollary 4.5.4.

The classifying maps for the fundamental groups induce a commuting diagram involving the Wang sequences of $M(K)$ and π from which the necessity of the orientation condition follows easily. (It is vacuous if ν is a free group.)

Let $N = P \sharp R$ where P is a connected sum of r copies of $S^1 \times S^2$ and the summands of R are aspherical. If $\theta_*(c_{N*}[N]) = c_{N*}[N]$ then θ may be realized by an orientation preserving self homotopy equivalence g of N [Sw74]. We may assume that g is a connected sum of homotopy equivalences between the irreducible factors of R and a self homotopy equivalence of P, by the Splitting Theorem of [HL74]. The factors of R are either Haken, hyperbolic or Seifert-fibred, by the Geometrization Conjecture (see [B-P]), and homotopy equivalences between such manifolds are homotopic to homeomorphisms, by [Hm], Mostow rigidity and [Sc83], respectively. A similar result holds for $P = \sharp^r(S^1 \times S^2)$, by [La]. Thus we may assume that g is a self homeomorphism of N. Surgery on a weight class in the mapping torus of g gives a fibred 2-knot with closed fibre N and group π. □

If N is hyperbolic, Seifert fibred or if its prime factors are Haken or $S^1 \times S^2$ then the mapping torus is determined up to homeomorphism among fibred 4-manifolds by its homotopy type, since homotopy implies isotopy in each case, by Mostow rigidity, [Sc85, BO91] and [HL74], respectively.

Yoshikawa has shown that a finitely generated abelian group is the base of some HNN extension which is a high dimensional knot group if and only if it satisfies the restrictions on torsion of [HK78], while if a knot group has a non-finitely generated abelian base then it is metabelian. Moreover a 2-knot group π which is an HNN extension with abelian base is either metabelian or has base $Z \oplus (Z/\beta Z)$ for some odd $\beta \geq 1$ [Yo86, Yo92]. We shall show that in the latter case β must be 1, and so π has a deficiency 1 presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1} \rangle$. (See Theorem 15.14.) No nontrivial classical knot group is an HNN extension with abelian base. (This is implicit in Yoshikawa's work, and can also be deduced from the facts that classical knot groups have cohomological dimension ≤ 2 and symmetric Alexander polynomial.)

14.9 Deficiency and geometric dimension

J.H.C.Whitehead raised the question "is every subcomplex of an aspherical 2-complex also aspherical?" This is so if the fundamental group of the subcomplex is a 1-relator group [Go81] or is locally indicable [Ho82] or has no nontrivial superperfect normal subgroup [Dy87]. Whitehead's question has interesting connections with knot theory. (For instance, the exterior of a ribbon n-knot or of a ribbon concordance between classical knots is homotopy equivalent to such a 2-complex. The asphericity of such ribbon exteriors has been raised in [Co83] and [Go81].)

If the answer to Whitehead's question is YES, then a high dimensional knot group has geometric dimension at most 2 if and only if it has deficiency 1 (in which case it is a 2-knot group). For let G be a group of weight 1 and with $G/G' \cong Z$. If $C(P)$ is the 2-complex corresponding to a presentation of deficiency 1 then the 2-complex obtained by adjoining a 2-cell to $C(P)$ along a loop representing a weight element for G is 1-connected and has Euler characteristic 1, and so is contractible. The converse follows from Theorem 2.8. On the other hand a positive answer in general implies that there is a group G such that $c.d.G = 2$ and $g.d.G = 3$ [BB97].

If the answer is NO then either there is a finite nonaspherical 2-complex X such that $X \cup_f D^2$ is contractible for some $f : S^1 \to X$ or there is an infinite ascending chain of nonaspherical 2-complexes whose union is contractible [Ho83]. In the finite case $\chi(X) = 0$ and so $\pi = \pi_1(X)$ has deficiency 1; moreover, π has weight 1 since it is normally generated by the conjugacy class represented by f. Such groups are 2-knot groups. Since X is not aspherical $\beta_1^{(2)}(\pi) \neq 0$, by Theorem 2.4, and so π' cannot be finitely generated, by Lemma 2.1.

A group is called *knot-like* if it has abelianization Z and deficiency 1. If the commutator subgroup of a classical knot group is finitely generated then it is free. Using the result of Corollary 2.5.1 above and the fact that the Novikov completions of $\mathbb{Z}[G]$ with respect to epimorphisms from G onto Z are weakly finite Kochloukova has shown that this holds more generally for all knot-like groups [Ko06]. (See Corollary 4.3.1 above.) This answers an old question of Rapaport, who established this in the 2-generator, 1-relator case [Rp60].

In particular, if the group of a fibred 2-knot has a presentation of deficiency 1 then its commutator subgroup is free. Any 2-knot with such a group is s-concordant to a fibred homotopy ribbon knot. (See §6 of Chapter 17.) As $S^2 \times S^1$ is never a factor of the closed fibre of a nontrivial twist spin $\tau_r K$ [Pl84], it follows that if $r > 1$ and K is nontrivial then $\mathrm{def}(\pi\tau_r K) \leq 0$ and $\tau_r K$ is not a homotopy ribbon 2-knot.

If a knot group has a 2-generator 1-relator Wirtinger presentation it is an HNN extension with free base and associated subgroups [Yo88]. This paper also gives an example π with $g.d.\pi = 2$ and a deficiency 1 Wirtinger presentation which also has a 2-generator 1-relator presentation but which is not such an HNN extension (and so has no 2-generator 1-relator *Wirtinger* presentation).

Lemma 14.5 *If G is a group with* $\mathrm{def}(G) = 1$ *and* $e(G) = 2$ *then* $G \cong Z$.

Proof The group G has an infinite cyclic subgroup A of finite index, since $e(G) = 2$. Let C be the finite 2-complex corresponding to a presentation of deficiency 1 for G, and let D be the covering space corresponding to A. Then

D is a finite 2-complex with $\pi_1(D) = A \cong Z$ and $\chi(D) = [\pi : A]\chi(C) = 0$. Since $H_2(D; \mathbb{Z}[A]) = H_2(\widetilde{D}; \mathbb{Z})$ is a submodule of a free $\mathbb{Z}[A]$-module and is of rank $\chi(D) = 0$ it is 0. Hence $\widetilde{D}$ is contractible, and so G must be torsion-free and hence abelian. $\square$

This lemma is also a consequence of Theorem 2.5. It follows immediately that $\mathrm{def}(\pi\tau_2 3_1) = 0$, since $\pi\tau_2 3_1 \cong (Z/3Z) \rtimes_{-1} Z$. Moreover, if K is a classical knot such that π' finitely generated but nontrivial then $H^1(\pi; \mathbb{Z}[\pi]) = 0$, and so $X(K)$ is aspherical, by Poincaré duality.

Theorem 14.6 *Let K be a 2-knot. Then $\pi = \pi K \cong Z$ if and only if $\mathrm{def}(\pi) = 1$ and $\pi_2(M(K)) = 0$.*

Proof The conditions are necessary, by Theorem 11.1. If they hold then $\beta_j^{(2)}(M) = \beta_j^{(2)}(\pi)$ for $j \leq 2$, by Theorem 6.54 of [Lü], and so $0 = \chi(M) = \beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi)$. Now $\beta_1^{(2)}(\pi) - \beta_2^{(2)}(\pi) \geq \mathrm{def}(\pi) - 1 = 0$, by Corollary 2.4.1. Therefore $\beta_1^{(2)}(\pi) = \beta_2^{(2)}(\pi) = 0$ and so $g.d.\pi \leq 2$, by the same Corollary. In particular, the manifold M is not aspherical. Hence $H^1(\pi; \mathbb{Z}[\pi]) \cong H_3(M; \mathbb{Z}[\pi]) \neq 0$. Since π is torsion-free it is indecomposable as a free product [Kl93]. Therefore $e(\pi) = 2$ and so $\pi \cong Z$, by Lemma 14.5. $\square$

In fact K must be trivial ([FQ] - see Corollary 17.1.1). A simpler argument is used in [H1] to show that if $\mathrm{def}(\pi) = 1$ then $\pi_2(M)$ maps onto $H_2(M; \Lambda)$, which is nonzero if $\pi' \neq \pi''$.

14.10 Asphericity

The outstanding property of the exterior of a classical knot is that it is aspherical. Swarup extended the classical Dehn's lemma criterion for unknotting to show that if K is an n-knot such that the natural inclusion of S^n (as a factor of $\partial X(K)$) into $X(K)$ is null homotopic then $X(K) \simeq S^1$, provided πK is accessible [Sw75]. Since it is now known that finitely presentable groups are accessible [DD], it follows that the exterior of a higher dimensional knot is aspherical if and only if the knot is trivial. Nevertheless, we shall see that the closed 4-manifolds $M(K)$ obtained by surgery on 2-knots are often aspherical.

Theorem 14.7 *Let K be a 2-knot. Then $M(K)$ is aspherical if and only if $\pi = \pi K$ is a PD_4-group, which must then be orientable.*

Proof The condition is clearly necessary. Suppose that it holds. Let M^+ be the covering space associated to $\pi^+ = \mathrm{Ker}(w_1(\pi))$. Then $[\pi : \pi^+] \leq 2$, so $\pi' < \pi^+$. Since $\pi/\pi' \cong Z$ and $t-1$ acts invertibly on $H_1(\pi';\mathbb{Z})$ it follows that $\beta_1(\pi^+) = 1$. Hence $\beta_2(M^+) = 0$, since M^+ is orientable and $\chi(M^+) = 0$. Hence $\beta_2(\pi^+)$ is also 0, so $\chi(\pi^+) = 0$, by Poincaré duality for π^+. Therefore $\chi(\pi) = 0$ and so M must be aspherical, by Corollary 3.5.1. □

We may use this theorem to give more examples of high dimensional knot groups which are not 2-knot groups. Let $A \in GL(3,\mathbb{Z})$ be such that $det(A) = -1$, $det(A-I) = \pm 1$ and $det(A+I) = \pm 1$. The characteristic polynomial of A must be either $f_1(X) = X^3 - X^2 - 2X + 1$, $f_2(X) = X^3 - X^2 + 1$, $f_3(X) = X^3 f_1(X^{-1})$ or $f_4(X) = X^3 f_2(X^{-1})$. (There are only two conjugacy classes of such matrices, up to inversion, for it may be shown that the rings $\mathbb{Z}[X]/(f_i(X))$ are principal ideal domains.) The group $Z^3 \rtimes_A Z$ satifies the Kervaire conditions, and is a PD_4-group. However it cannot be a 2-knot group, since it is nonorientable. (Such matrices have been used to construct fake RP^4s [CS76'].)

Is every (torsion-free) 2-knot group π with $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$ a PD_4-group? Is every 3-knot group which is also a PD_4^+-group a 2-knot group? (Note that by Theorem 3.6 such a group cannot have deficiency 1.)

We show next that knots with such groups cannot be a nontrivial satellite.

Theorem 14.8 *Let $K = \Sigma(K_2; K_1, \gamma)$ be a satellite 2-knot. If $\pi = \pi K$ is a PD_4-group then $K = K_1$ or K_2. In particular, K is irreducible.*

Proof Let q be the order of γ in πK_1. Then $\pi \cong \pi K_1 *_C B$, where $B = \pi K_2/\langle\langle w^q \rangle\rangle$, and C is cyclic. Since π is torsion-free $q = 0$ or 1. Suppose that $K \neq K_1$. Then $q = 0$, so $C \cong Z$, while $B \neq C$. If $\pi K_1 \neq C$ then πK_1 and B have infinite index in π, and so $c.d.\pi K_1 \leq 3$ and $c.d.B \leq 3$, by Strebel's Theorem. A Mayer-Vietoris argument then gives $4 = c.d.\pi \leq 3$, which is impossible. Therefore K_1 is trivial and so $K = K_2$. □

14.11 Links

A *μ-component n-link* is a locally flat embedding $L : \mu S^n \to S^{n+2}$. The *exterior* of L is $X(L) = S^{n+2} \setminus intN(L)$, where $N(L) \cong \mu S^n \times D^2$ is a regular neighbourhood of the image of L, and the *group* of L is $\pi L = \pi_1(X(L))$. Let $M(L) = X(L) \cup \mu D^{n+1} \times S^1$ be the closed manifold obtained by surgery on L.

An n-link L is *trivial* if it bounds a collection of μ disjoint locally flat 2-discs in S^n. It is *split* if it is isotopic to one which is the union of nonempty sublinks L_1 and L_2 whose images lie in disjoint discs in S^{n+2}, in which case we write $L = L_1 \amalg L_2$, and it is a *boundary* link if it bounds a collection of μ disjoint hypersurfaces in S^{n+2}. Clearly a trivial link is split, and a split link is a boundary link; neither implication can be reversed if $\mu > 1$. Knots are boundary links, and many arguments about knots that depend on Seifert hypersurfaces extend readily to boundary links. The definitions of slice and ribbon knots and s-concordance extend naturally to links.

A 1-link is trivial if and only if its group is free, and is split if and only if its group is a nontrivial free product, by the Loop Theorem and Sphere Theorem, respectively. (See Chapter 1 of [H3].) Gutiérrez has shown that if $n \geq 4$ an n-link L is trivial if and only if πL is freely generated by meridians and the homotopy groups $\pi_j(X(L))$ are all 0, for $2 \leq j \leq (n+1)/2$ [Gu72]. His argument applies also when $n = 3$. While the fundamental group condition is necessary when $n = 2$, we cannot yet use surgery to show that it is a complete criterion for triviality of 2-links with more than one component. We shall settle for a weaker result.

Theorem 14.9 *Let M be a closed 4-manifold with $\pi_1(M)$ free of rank r and $\chi(M) = 2(1-r)$. If M is orientable it is s-cobordant to $\sharp^r(S^1 \times S^3)$, while if it is nonorientable it is s-cobordant to $(S^1 \tilde{\times} S^3)\sharp(\sharp^{r-1}(S^1 \times S^3))$.*

Proof We may assume without loss of generality that $\pi_1(M)$ has a free basis $\{x_1, ...x_r\}$ such that x_i is an orientation preserving loop for all $i > 1$, and we shall use c_{M*} to identify $\pi_1(M)$ with $F(r)$. Let $N = \sharp^r(S^1 \times S^3)$ if M is orientable and let $N = (S^1 \tilde{\times} S^3)\sharp(\sharp^{r-1}(S^1 \times S^3))$ otherwise. (Note that $w_1(N) = w_1(M)$ as homomorphisms from $F(r)$ to $\{\pm 1\}$.) Since $c.d.\pi_1(M) \leq 2$ and $\chi(M) = 2\chi(\pi_1(M))$ we have $\pi_2(M) \cong \overline{H^2(F(r); \mathbb{Z}[F(r)])}$, by Theorem 3.12. Hence $\pi_2(M) = 0$ and so $\pi_3(M) \cong H_3(\widetilde{M}; \mathbb{Z}) \cong D = \overline{H^1(F(r); \mathbb{Z}[F(r)])}$, by the Hurewicz theorem and Poincaré duality. Similarly, we have $\pi_2(N) = 0$ and $\pi_3(N) \cong D$.

Let $c_M = g_M h_M$ be the factorization of c_M through $P_3(M)$, the third stage of the Postnikov tower for M. Thus $\pi_i(h_M)$ is an isomorphism if $i \leq 3$ and $\pi_j(P_3(M)) = 0$ if $j > 3$. As $K(F(r), 1) = \vee^r S^1$ each of the fibrations g_M and g_N clearly have cross-sections and so there is a homotopy equivalence $k : P_3(M) \to P_3(N)$ such that $g_M = g_N k$. (See Section 5.2 of [Ba].) We may assume that k is cellular. Since $P_3(M) = M \cup \{\mathit{cells\ of\ dimension} \geq 5\}$ it follows that $kh_M = h_N f$ for some map $f : M \to N$. Clearly $\pi_i(f)$ is an isomorphism for $i \leq 3$. Since the universal covers $\widetilde{M}$ and $\widetilde{N}$ are 2-connected open 4-manifolds the induced map $\tilde{f} : \widetilde{M} \to \widetilde{N}$ is an homology isomorphism, and so is a homotopy equivalence. Hence f is itself a homotopy equivalence. As $Wh(F(r)) = 0$ any such homotopy equivalence is simple.

If M is orientable $[M, G/TOP] \cong Z$, since $H^2(M; \mathbb{Z}/2\mathbb{Z}) = 0$. As the surgery obstruction in $L_4(F(r)) \cong Z$ is given by a signature difference, it is a bijection, and so the normal invariant of f is trivial. Hence there is a normal cobordism $F : P \to N \times I$ with $F|\partial_- P = f$ and $F|\partial_+ P = id_N$. There is another normal cobordism $F' : P' \to N \times I$ from id_N to itself with surgery obstruction $\sigma_5(P', F') = -\sigma_5(P, F)$ in $L_5(F(r))$, by Theorem 6.7 and Lemma 6.9. The union of these two normal cobordisms along $\partial_+ P = \partial_- P'$ is a normal cobordism from f to id_N with surgery obstruction 0, and so we may obtain an s-cobordism W by 5-dimensional surgery (rel ∂).

A similar argument applies in the nonorientable case. The surgery obstruction is then a bijection from $[N; G/TOP]$ to $L_4(F(r), -) = Z/2Z$, so f is normally cobordant to id_N, while $L_5(Z, -) = 0$, so $L_5(F(r), -) \cong L_5(F(r-1))$ and the argument of [FQ] still applies. □

Corollary 14.9.1 *Let L be a μ-component 2-link such that πL is freely generated by μ meridians. Then L is s-concordant to the trivial μ-component link.*

Proof Since $M(L)$ is orientable, $\chi(M(L)) = 2(1-\mu)$ and $\pi_1(M(L)) \cong \pi L = F(\mu)$, there is an s-cobordism W with $\partial W = M(L) \cup M(\mu)$, by Theorem 14.9. Moreover it is clear from the proof of that theorem that we may assume that the elements of the meridianal basis for πL are freely homotopic to loops representing the standard basis for $\pi_1(M(\mu))$. We may realise such homotopies by μ disjoint embeddings of annuli running from meridians for L to such standard loops in $M(\mu)$. Surgery on these annuli (i.e., replacing $D^3 \times S^1 \times [0,1]$ by $S^2 \times D^2 \times [0,1]$) then gives an s-concordance from L to the trivial μ-component link. □

A similar strategy may be used to give an alternative proof of the higher dimensional unlinking theorem of [Gu72] which applies uniformly for $n \geq 3$. The hypothesis that πL be freely generated by meridians cannot be dropped entirely [Po71]. On the other hand, if L is a 2-link whose longitudes are all null homotopic then the pair $(X(L), \partial X(L))$ is homotopy equivalent to the pair $(\sharp^\mu S^1 \times D^3, \partial(\sharp^\mu S^1 \times D^3))$ [Sw77], and hence the Corollary applies.

There is as yet no satisfactory splitting criterion for higher-dimensional links. However we can give a stable version for 2-links.

Theorem 14.10 *Let M be a closed 4-manifold such that $\pi = \pi_1(M)$ is isomorphic to a nontrivial free product $G*H$. Then M is stably homeomorphic to a connected sum $M_G\sharp M_H$ with $\pi_1(M_G)\cong G$ and $\pi_1(M_H)\cong H$.*

Proof Let $K = K_G\cup[-1,1]\cup K_H/(*_G\sim -1, +1\sim *_H)$, where K_G and K_H are $K(G,1)$- and $K(H,1)$-spaces with basepoints $*_G$ and $*_H$ (respectively). Then K is a $K(\pi,1)$-space and so there is a map $f: M\to K$ which induces an isomorphism of fundamental groups. We may assume that f is transverse to $0\in[-1,1]$, so $V = f^{-1}(0)$ is a submanifold of M with a product neighbourhood $V\times[-\epsilon,\epsilon]$. We may also assume that V is connected, by the arc-chasing argument of Stallings' proof of Kneser's conjecture. (See page 67 of [Hm].) Let $j: V\to M$ be the inclusion. Since fj is a constant map and $\pi_1(f)$ is an isomorphism $\pi_1(j)$ is the trivial homomorphism, and so $j^*w_1(M) = 0$. Hence V is orientable and so there is a framed link $L\subset V$ such that surgery on L in V gives S^3 [Li62]. The framings of the components of L in V extend to framings in M. Let $W = M\times[0,1]\cup_{L\times D^2\times[-\epsilon,\epsilon]\times\{1\}}(\mu D^2\times D^2\times[-\epsilon,\epsilon])$, where μ is the number of components of L. Note that if $w_2(M) = 0$ then we may choose the framed link L so that $w_2(W) = 0$ also [Kp79]. Then $\partial W = M\cup\widehat{M}$, where $\widehat{M}$ is the result of surgery on L in M. The map f extends to a map $F: W\to K$ such that $\pi_1(F|_{\widehat{M}})$ is an isomorphism and $(F|_{\widehat{M}})^{-1}(0)\cong S^3$. Hence $\widehat{M}$ is a connected sum as in the statement. Since the components of L are null-homotopic in M they may be isotoped into disjoint discs, and so $\widehat{M}\cong M\sharp(\sharp^\mu S^2\times S^2)$. This proves the theorem. □

Note that if V is a homotopy 3-sphere then M is a connected sum, for $V\times R$ is then homeomorphic to $S^3\times R$, by 1-connected surgery.

Theorem 14.11 *Let L be a μ-component 2-link with sublinks L_1 and $L_2 = L\setminus L_1$ such that there is an isomorphism from πL to $\pi L_1 * \pi L_2$ which is compatible with the homomorphisms determined by the inclusions of $X(L)$ into $X(L_1)$ and $X(L_2)$. Then $X(L)$ is stably homeomorphic to $X(L_1\amalg L_2)$.*

Proof By Theorem 14.10, $M(L)\sharp(\sharp^a S^2\times S^2)\cong N\sharp P$, where $\pi_1(N)\cong\pi L_1$ and $\pi_1(P)\cong\pi L_2$. On undoing the surgeries on the components of L_1 and L_2, respectively, we see that $M(L_2)\sharp(\sharp^a S^2\times S^2)\cong N\sharp\bar{P}$, and $M(L_1)\sharp(\sharp^a S^2\times S^2)\cong\bar{N}\sharp P$, where $\bar{N}$ and $\bar{P}$ are simply connected. Since undoing the surgeries on all the components of L gives $\sharp^a S^2\times S^2\cong\bar{N}\sharp\bar{P}$, $\bar{N}$ and $\bar{P}$ are each connected sums of copies of $S^2\times S^2$, so N and P are stably homeomorphic to $M(L_1)$ and $M(L_2)$, respectively. The result now follows easily. □

Similar arguments may be used to show that, firstly, if L is a 2-link such that $c.d.\pi L \leq 2$ and there is an isomorphism $\theta : \pi L \to \pi L_1 * \pi L_2$ which is compatible with the natural maps to the factors then there is a map $f_o : M(L)_o = M(L)\setminus intD^4 \to M(L_1)\sharp M(L_2)$ such that $\pi_1(f_o) = \theta$ and $\pi_2(f_o)$ is an isomorphism; and secondly, if moreover f_o extends to a homotopy equivalence $f : M(L) \to M(L_1)\sharp M(L_2)$ and the factors of πL are either classical link groups or are square root closed accessible then L is s-concordant to the split link $L_1 \amalg L_2$. (The surgery arguments rely on [AFR97] and [Ca73], respectively.) However we do not know how to bridge the gap between the algebraic hypothesis and obtaining a homotopy equivalence.

14.12 Link groups

If π is the group of a μ-component n-link L then

(1) *π is finitely presentable;*

(2) *π is of weight μ;*

(3) *$H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong Z^\mu$; and*

(4) *(if $n > 1$) $H_2(\pi;\mathbb{Z}) = 0$.*

Conversely, any group satisfying these conditions is the group of an n-link, for every $n \geq 3$ [Ke65']. (Note that $q(\pi) \geq 2(1-\mu)$, with equality if and only if π is the group of a 2-link.) If (4) is replaced by the stronger condition that $\mathrm{def}(\pi) = \mu$ (and π has a deficiency μ Wirtinger presentation) then π is the group of a (ribbon) 2-link which is a sublink of a (ribbon) link whose group is a free group. (See Chapter 1 of [H3].) The group of a classical link satisfies (4) if and only if the link splits completely as a union of knots in disjoint balls. If subcomplexes of aspherical 2-complexes are aspherical then a higher-dimensional link group group has geometric dimension at most 2 if and only if it has deficiency μ (in which case it is a 2-link group).

A link L is a boundary link if and only if there is an epimorphism from $\pi(L)$ to the free group $F(\mu)$ which carries a set of meridians to a free basis. If the latter condition is dropped L is said to be an homology boundary link. Although sublinks of boundary links are clearly boundary links, the corresponding result is not true for homology boundary links. It is an attractive conjecture that every even-dimensional link is a slice link. This has been verified under additional hypotheses on the link group. For a 2-link L it suffices that there be a homomorphism $\phi : \pi L \to G$ where G is a high-dimensional link group such that $H_3(G;\mathbb{F}_2) = H_4(G;\mathbb{Z}) = 0$ and where the normal closure of the image of

ϕ is G [Co84]. In particular, sublinks of homology boundary 2-links are slice links.

A choice of (based) meridians for the components of a link L determines a homomorphism $f : F(\mu) \to \pi L$ which induces an isomorphism on abelianization. If L is a higher dimensional link $H_2(\pi L;\mathbb{Z}) = H_2(F(\mu);\mathbb{Z}) = 0$ and hence f induces isomorphisms on all the nilpotent quotients $F(\mu)/F(\mu)_{[n]} \cong \pi L/(\pi L)_{[n]}$, and a monomorphism $F(\mu) \to \pi L/(\pi L)_{[\omega]} = \pi L/\cap_{n\geq 1}(\pi L)_{[n]}$ [St65]. (In particular, if $\mu \geq 2$ then πL contains a nonabelian free subgroup.) The latter map is an isomorphism if and only if L is a homology boundary link. In that case the homology groups of the covering space $X(L)^\omega$ corresponding to $\pi L/(\pi L)_{[\omega]}$ are modules over $\mathbb{Z}[\pi L/(\pi L)_{[\omega]}] \cong \mathbb{Z}[F(\mu)]$, which is a coherent ring of global dimension 2. Poincaré duality and the UCSS then give rise to an isomorphism $e^2e^2(\pi L/(\pi L)_{[\omega]}) \cong \overline{e^2(\pi L/(\pi L)_{[\omega]})}$, where $e^i(M) = Ext^i_{\mathbb{Z}[F(\mu)]}(M,\mathbb{Z}[F(\mu)])$, which is the analogue of the Farber-Levine pairing for 2-knots.

The argument of [HK78'] may be adapted to show that every finitely generated abelian group is the centre of the group of some μ-component boundary n-link, for any $\mu \geq 1$ and $n \geq 3$. However the centre of the group of a 2-link with more than one component must be finite. (All known examples have trivial centre.)

Theorem 14.12 *Let L be a μ-component 2-link. If $\mu > 1$ then*

(1) *πL has no infinite amenable normal subgroup;*

(2) *πL is not an ascending HNN extension over a finitely generated base.*

Proof Since $\chi(M(L)) = 2(1-\mu)$ the L^2-Euler characteristic formula gives $\beta_1^{(2)}(\pi L) \geq \mu - 1$. Therefore $\beta_1^{(2)}(\pi L) \neq 0$ if $\mu > 1$, and so the result follows from Lemma 2.1 and Corollary 2.3.1. ∎

In particular, the exterior of a 2-link with more than one component never fibres over S^1. (This is true of all higher dimensional links: see Theorem 5.12 of [H3].) Moreover a 2-link group has finite centre and is never amenable. In contrast, we shall see that there are many 2-knot groups which have infinite centre or are solvable.

The exterior of a classical link is aspherical if and only the link is unsplittable, while the exterior of a higher dimensional link with more than one component is never aspherical [Ec76]. Is $M(L)$ ever aspherical?

14.13 Homology spheres

A closed connected n-manifold M is an *homology n-sphere* if $H_q(M;\mathbb{Z})=0$ for $0<q<n$. In particular, it is orientable and so $H_n(M;\mathbb{Z})\cong Z$. If π is the group of an homology n-sphere then

(1) *π is finitely presentable;*

(2) *π is perfect, i.e., $\pi=\pi'$; and*

(3) $H_2(\pi;\mathbb{Z})=0$.

A group satisfying the latter two conditions is said to be *superperfect*. Every finitely presentable superperfect group is the group of an homology n-sphere, for every $n\geq 5$ [Ke69], but in low dimensions more stringent conditions hold. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. Every perfect group with a presentation of deficiency 0 is superperfect, and is an homology 4-sphere group [Ke69]. However none of the implications "G is an homology 3-sphere group" $\Rightarrow$ "G is finitely presentable, perfect and $\mathrm{def}(G)=0$" $\Rightarrow$ "G is an homology 4-sphere group" $\Rightarrow$ "G is finitely presentable and superperfect" can be reversed, as we shall now show.

Although the finite groups $SL(2,\mathbb{F}_p)$ are perfect and have deficiency 0 for each prime $p\geq 5$ [CR80], the binary icosahedral group $I^*=SL(2,\mathbb{F}_5)$ is the only nontrivial finite perfect group with cohomological period 4, and thus is the only finite homology 3-sphere group.

Let $G=\langle x,s\mid x^3=1, sxs^{-1}=x^{-1}\rangle$ be the group of $\tau_2 3_1$ and let

$$H=\langle a,b,c,d\mid bab^{-1}=a^2, cbc^{-1}=b^2, dcd^{-1}=c^2, ada^{-1}=d^2\rangle$$

be the Higman group [Hg51]. Then H is perfect and $\mathrm{def}(H)=0$, so there is an homology 4-sphere Σ with group H. Surgery on a loop representing sa^{-1} in $\Sigma\sharp M(\tau_2 3_1)$ gives an homology 4-sphere with group $\pi=(G*H)/\langle\langle sa^{-1}\rangle\rangle$. Then π is the semidirect product $\rho\rtimes H$, where $\rho=\langle\langle G'\rangle\rangle_\pi$ is the normal closure of the image of G' in π. The obvious presentation for this group has deficiency -1. We shall show that this is best possible.

Let $\Gamma=\mathbb{Z}[H]$. Since H has cohomological dimension 2 [DV73'] the augmentation ideal $I=\mathrm{Ker}(\varepsilon:\Gamma\to\mathbb{Z})$ has a short free resolution

$$C_*: 0\to\Gamma^4\to\Gamma^4\to I\to 0.$$

Let $B=H_1(\pi;\Gamma)\cong\rho/\rho'$. Then $B\cong\Gamma/\Gamma(3,a+1)$ as a left Γ-module and there is an exact sequence

$$0\to B\to A\to I\to 0,$$

in which $A = H_1(\pi, 1; \Gamma)$ is a relative homology group [Cr61]. Since $B \cong \Gamma \otimes_\Lambda (\Lambda/\Lambda(3, a+1))$, where $\Lambda = \mathbb{Z}[a, a^{-1}]$, there is a free resolution

$$0 \to \Gamma \xrightarrow{(3,a+1)} \Gamma^2 \xrightarrow{\binom{a+1}{-3}} \Gamma \to B \to 0.$$

Suppose that π has deficiency 0. Evaluating the Jacobian matrix associated to an optimal presentation for π via the natural epimorphism from $\mathbb{Z}[\pi]$ to Γ gives a presentation matrix for A as a module [Cr61, Fo62]. Thus there is an exact sequence

$$D_* : \dots \to \Gamma^n \to \Gamma^n \to A \to 0.$$

A mapping cone construction leads to an exact sequence of the form

$$D_1 \to C_1 \oplus D_0 \to B \oplus C_0 \to 0$$

and hence to a presentation of deficiency 0 for B of the form

$$D_1 \oplus C_0 \to C_1 \oplus D_0 \to B.$$

Hence there is a free resolution

$$0 \to L \to \Gamma^p \to \Gamma^p \to B \to 0.$$

Schanuel's Lemma gives an isomorphism $\Gamma^{1+p+1} \cong L \oplus \Gamma^{p+2}$, on comparing these two resolutions of B. Since Γ is weakly finite the endomorphism of Γ^{p+2} given by projection onto the second summand is an automorphism. Hence $L = 0$ and so B has a short free resolution. In particular, $Tor_2^\Gamma(R, B) = 0$ for any right Γ-module R. But it is easily verified that if $\overline{B} \cong \Gamma/(3, a+1)\Gamma$ is the conjugate right Γ-module then $Tor_2^\Gamma(\overline{B}, B) \neq 0$. Thus our assumption was wrong, and $\mathrm{def}(\pi) = -1 < 0$.

Our example of an homology 4-sphere group with negative deficiency is "very infinite" in the sense that the Higman group H has no finite quotients, and therefore no finite-dimensional representations over any field [Hg51]. Livingston has constructed examples with arbitrarily large negative deficiency, which are extensions of I^* by groups which map onto Z. His argument uses only homological algebra for infinite cyclic covers [Li05].

Let $I^* = SL(2, \mathbb{F}_5)$ act diagonally on $(\mathbb{F}_5^2)^k$, and let G_k be the universal central extension of the perfect group $(\mathbb{F}_5^2)^k \rtimes I^*$. In [HW85] it is shown that for k large, the superperfect group is not the group of an homology 4-sphere. In particular, it has negative deficiency. It seems unlikely that deficiency 0 is a necessary condition for a finite perfect group to be an homology 4-sphere group.

Kervaire's criteria may be extended further to the groups of links in homology spheres. Unfortunately, the condition $\chi(M) = 0$ is central to most of our arguments, and is satisfied only when the link has one component.

We may also use the Higman group H to construct fibred knots with group of cohomological dimension d, for any finite $d \geq 1$. If Σ is an homology 4-sphere with $\pi_1(\Sigma) \cong H$ then surgery on a loop in $\Sigma \times S^1$ representing $(a, 1) \in H \times Z$ gives a homotopy 5-sphere, and so $H \times Z$ is the group of a fibred 3-knot. Every superperfect group is the group of an homology 5-sphere, and a similar construction shows that $H^k \times Z$ is the group of a fibred 4-knot, for all $k \geq 0$. Similarly, $\pi = H^k \times \pi 3_1$ is a high-dimensional knot group with π' finitely presentable and $c.d.\pi = 2k + 2$, for all $k \geq 0$.

On the other hand, if K is a 2-knot with group $\pi = \pi K$ such that π' is finitely generated then $M(K)'$ is a PD_3-space, by Theorem 4.5. Hence π' has a factorization as a free product of PD_3-groups and indecomposable virtually free groups, by the theorems of Turaev and Crisp. In particular, $v.c.d.\pi' = 0$, 1 or 3, and so $v.c.d.\pi = 1$, 2 or 4. Thus $H^k \times Z$ is not a 2-knot group, if $k \geq 1$.

These observations suggest several questions:

(1) are there any 2-knot groups π with $c.d.\pi = 3$?

(2) what are the groups of fibred n-knots?

(3) in particular, is $H^k \times \pi 3_1$ realized by a fibred 3-knot, if $k \geq 2$?

Chapter 15

Restrained normal subgroups

It is plausible that if K is a 2-knot whose group $\pi = \pi K$ has an infinite restrained normal subgroup N then either π' is finite or $\pi \cong \Phi$ (the group of Fox's Example 10) or $M(K)$ is aspherical and $\sqrt{\pi} \neq 1$ or N is virtually Z and π/N has infinitely many ends. In this chapter we shall give some evidence in this direction. In order to clarify the statements and arguments in later sections, we begin with several characterizations of Φ, which plays a somewhat exceptional role. In §2 we assume that N is almost coherent and locally virtually indicable, but not locally finite. In §3 we establish the above tetrachotomy for the cases with N nilpotent and $h(N) \geq 2$ or abelian of rank 1. In §4 we determine all such π with π' finite, and in §5 we give a version of the Tits alternative for 2-knot groups. In §6 we shall complete Yoshikawa's determination of the 2-knot groups which are HNN extensions over abelian bases. We conclude with some observations on 2-knot groups with infinite locally finite normal subgroups.

15.1 The group Φ

Let $\Phi \cong Z*_2$ be the group with presentation $\langle a, t \mid tat^{-1} = a^2 \rangle$. This group is an ascending HNN extension with base Z, is metabelian, and has commutator subgroup isomorphic to the dyadic rationals. The 2-complex corresponding to this presentation is aspherical and so $g.d.\Phi = 2$.

The group Φ is the group of Example 10 of Fox, which is the boundary of the ribbon D^3 in S^4 obtained by "thickening" a suitable immersed ribbon D^2 in S^3 for the stevedore's knot 6_2 [Fo62]. Such a ribbon disc may be constructed by applying the method of §7 of Chapter 1 of [H3] to the equivalent presentation $\langle t, u, v \mid vuv^{-1} = t,\ tut^{-1} = v \rangle$ for Φ (where $u = ta$ and $v = t^2at^{-1}$).

Theorem 15.1 *Let π be a group such that $\pi/\pi' \cong Z$, $c.d.\pi = 2$ and π has a nontrivial normal subgroup E which is either elementary amenable or almost coherent, locally virtually indicable and restrained. Then either $\pi \cong \Phi$ or π is an iterated free product of (one or more) torus knot groups, amalgamated over central subgroups, $E \leq \zeta\pi$ and $\zeta\pi \cap \pi' = 1$. In either case* $\mathrm{def}(\pi) = 1$.

Proof If π is solvable then $\pi \cong Z*_m$, for some $m \neq 0$, by Corollary 2.6.1. Since $\pi/\pi' \cong Z$ we must have $m = 2$ and so $\pi \cong \Phi$.

Otherwise $E \cong Z$, by Theorem 2.7. Then $[\pi : C_\pi(E)] \leq 2$ and $C_\pi(E)'$ is free, by Bieri's Theorem. This free subgroup must be nonabelian for otherwise π would be solvable. Hence $E \cap C_\pi(E)' = 1$ and so E maps injectively to $H = \pi/C_\pi(E)'$. As H has an abelian normal subgroup of index at most 2 and $H/H' \cong Z$ we must in fact have $H \cong Z$. It follows easily that $C_\pi(E) = \pi$, and so π' is free. The further structure of π is then due to Strebel [St76]. The final observation follows readily. □

The following alternative characterizations of Φ shall be useful.

Theorem 15.2 *Let π be a 2-knot group with maximal locally finite normal subgroup T. Then $\pi/T \cong \Phi$ if and only if π is elementary amenable and $h(\pi) = 2$. Moreover the following are equivalent:*

(1) *π has an abelian normal subgroup A of rank 1 such that π/A has two ends;*
(2) *π is elementary amenable, $h(\pi) = 2$ and π has an abelian normal subgroup A of rank 1;*
(3) *π is almost coherent, elementary amenable and $h(\pi) = 2$;*
(4) *$\pi \cong \Phi$.*

Proof Since π is finitely presentable and has infinite cyclic abelianization it is an HNN extension $\pi \cong B*_\phi$ with base B a finitely generated subgroup of π', by Theorem 1.13. Since π is elementary amenable the extension must be ascending. Since $h(\pi'/T) = 1$ and π'/T has no nontrivial locally-finite normal subgroup $[\pi'/T : \sqrt{\pi'/T}] \leq 2$. The meridianal automorphism of π' induces a meridianal automorphism on $(\pi'/T)/\sqrt{\pi'/T}$ and so $\pi'/T = \sqrt{\pi'/T}$. Hence π'/T is a torsion-free rank 1 abelian group. Let $J = B/B \cap T$. Then $h(J) = 1$ and $J \leq \pi'/T$ so $J \cong Z$. Now ϕ induces a monomorphism $\psi : J \to J$ and $\pi/T \cong J*_\psi$. Since $\pi/\pi' \cong Z$ we must have $J*_\psi \cong \Phi$.

If (1) holds then π is elementary amenable and $h(\pi) = 2$. Suppose (2) holds. We may assume without loss of generality that A is the normal closure of an element of infinite order, and so π/A is finitely presentable. Since π/A is elementary amenable and $h(\pi/A) = 1$ it is virtually Z. Therefore π is virtually an HNN extension with base a finitely generated subgroup of A, and so is coherent. If (3) holds then $\pi \cong \Phi$, by Corollary 3.17.1. Since Φ clearly satisfies conditions (1-3) this proves the theorem. □

Corollary 15.2.1 *If T is finite and $\pi/T \cong \Phi$ then $T = 1$ and $\pi \cong \Phi$.* □

15.2 Almost coherent, restrained and locally virtually indicable

We shall show that the basic tetrachotomy of the introduction is essentially correct, under mild coherence hypotheses on πK or N. Recall that a restrained group has no noncyclic free subgroups. Thus if N is a countable restrained group either it is elementary amenable and $h(N) \leq 1$ or it is an increasing union of finitely generated one-ended groups.

Theorem 15.3 *Let K be a 2-knot whose group $\pi = \pi K$ is an ascending HNN extension over an FP_2 base B with finitely many ends. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical.*

Proof This follows from Theorem 3.17, since a group with abelianization Z cannot be virtually Z^2. □

Is $M(K)$ still aspherical if we assume only that B is finitely generated and one-ended?

Corollary 15.3.1 *If π' is locally finite then it is finite.* □

Corollary 15.3.2 *If B is FP_3 and has one end then $\pi' = B$ and is a PD_3^+-group.*

Proof This follows from Lemma 3.4 of [BG85], as in Theorem 2.13. □

Does this remain true if we assume only that B is FP_2 and has one end?

Corollary 15.3.3 *If π is an ascending HNN extension over an FP_2 base B and has an infinite restrained normal subgroup A then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $\pi' \cap A = 1$ and π/A has infinitely many ends.*

Proof If B is finite or $A \cap B$ is infinite then B has finitely many ends (cf. Corollary 1.15.1) and Theorem 15.3 applies. Therefore we may assume that B has infinitely many ends and $A \cap B$ is finite. But then $A \not\leq \pi'$, so π is virtually $\pi' \times Z$. Hence $\pi' = B$ and $M(K)'$ is a PD_3-complex. In particular, $\pi' \cap A = 1$ and π/A has infinitely many ends. □

In §4 we shall determine all 2-knot groups with π' finite. If K is the r-twist spin of an irreducible 1-knot then the r^{th} power of a meridian is central in π and either π' is finite or $M(K)$ is aspherical. (See §3 of Chapter 16.) The final possibility is realized by Artin spins of nontrivial torus knots.

Theorem 15.4 *Let K be a 2-knot whose group $\pi = \pi K$ is an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$. If π has a restrained normal subgroup N which is not locally finite and $\beta_1^{(2)}(\pi) = 0$ then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or N is locally virtually Z and π/N has infinitely many ends.*

Proof If $\pi' \cap N$ is locally finite then it follows from Britton's lemma (on normal forms in HNN extensions) that either $B \cap N = I \cap N$ or $B \cap N = J \cap N$. Moreover $N \not\leq \pi'$ (since N is not locally finite), and so $\pi'/\pi' \cap N$ is finitely generated. Hence $B/B \cap N \cong I/I \cap N \cong J/J \cap N$. Thus either $B = I$ or $B = J$ and so the HNN extension is ascending. If B has finitely many ends we may apply Theorem 15.3. Otherwise $B \cap N$ is finite, so $\pi' \cap N = B \cap N$ and N is virtually Z. Hence π/N is commensurable with $B/B \cap N$, and $e(\pi/N) = \infty$.

If $\pi' \cap N$ is locally virtually Z and $\pi/\pi' \cap N$ has two ends then π is elementary amenable and $h(\pi) = 2$, so $\pi \cong \Phi$. Otherwise we may assume that either $\pi/\pi' \cap N$ has one end or $\pi' \cap N$ has a finitely generated, one-ended subgroup. In either case $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.18, and so $M(K)$ is aspherical, by Theorem 3.5. □

Note that $\beta_1^{(2)}(\pi) = 0$ if N is amenable. Every knot group is an HNN extension with finitely generated base and associated subgroups, by Theorem 1.13, and in all known cases these subgroups are FP_2.

Theorem 15.5 *Let K be a 2-knot such that $\pi = \pi K$ has an almost coherent, locally virtually indicable, restrained normal subgroup E which is not locally finite. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or E is abelian of rank 1 and π/E has infinitely many ends or E is locally virtually Z and π/E has one or infinitely many ends.*

Proof Let F be a finitely generated subgroup of E. Since F is FP_2 and virtually indicable it has a subgroup of finite index which is an HNN extension over a finitely generated base, by Theorem 1.13. Since F is restrained the HNN extension is ascending, and so $\beta_1^{(2)}(F) = 0$, by Lemma 2.1. Hence $\beta_1^{(2)}(E) = 0$ and so $\beta_1^{(2)}(\pi) = 0$, by part (3) of Theorem 7.2 of [Lü].

If every finitely generated infinite subgroup of E has two ends, then E is elementary amenable and $h(E) = 1$. If π/E is finite then π' is finite. If π/E has two ends then π is almost coherent, elementary amenable and $h(\pi) = 2$, and so $\pi \cong \Phi$, by Theorem 15.2. If E is abelian and π/E has one end, or if E

has a finitely generated, one-ended subgroup and π is not elementary amenable of Hirsch length 2 then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17. Hence $M(K)$ is aspherical, by Theorem 3.5.

The remaining possibility is that E is locally virtually Z and π/E has one or infinitely many ends. $\square$

Does this theorem hold without any coherence hypothesis? Note that the other hypotheses hold if E is elementary amenable and $h(E) \geq 2$. If E is elementary amenable, $h(E) = 1$ and π/E has one end is $H^2(\pi;\mathbb{Z}[\pi]) = 0$?

Corollary 15.5.1 *Let K be a 2-knot with group $\pi = \pi K$. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical and $\sqrt{\pi}$ is nilpotent or $h(\sqrt{\pi}) = 1$ and $\pi/\sqrt{\pi}$ has one or infinitely many ends or $\sqrt{\pi}$ is locally finite.*

Proof Finitely generated nilpotent groups are polycyclic. If $\pi/\sqrt{\pi}$ has two ends we may apply Theorem 15.3. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$, by Theorem 9.2, while if $h > 2$ then π is virtually poly-Z, by Theorem 8.1. $\square$

If $M(K)$ is aspherical then $c.d.E \leq 4$ and $c.d.E \cap \pi' \leq 3$. Slightly stronger hypotheses on E then imply that π has a nontrivial torsion-free abelian normal subgroup.

Theorem 15.6 *Let N be a group which is either elementary amenable or locally FP_3, virtually indicable and restrained. If $c.d.N \leq 3$ then N is virtually solvable.*

Proof Suppose first that N is locally FP_3 and virtually indicable, and let E be a finitely generated subgroup of N which maps onto Z. Then E is an ascending HNN extension $B*_\phi$ with FP_3 base B and associated subgroups. If $c.d.B = 3$ then $H^3(B;\mathbb{Z}[E]) \cong H^3(B;\mathbb{Z}[B]) \otimes_{\mathbb{Z}[B]} \mathbb{Z}[E] \neq 0$ and the homomorphism $H^3(B;\mathbb{Z}[E]) \to H^3(B;\mathbb{Z}[E])$ in the Mayer-Vietoris sequence for the HNN extension is not onto, by Lemma 3.4 and the subsequent Remark 3.5 of [BG85]. But then $H^4(E;\mathbb{Z}[E]) \neq 0$, contrary to $c.d.N \leq 3$. Therefore $c.d.B \leq 2$, and so B is elementary amenable, by Theorem 2.7. Hence N is elementary amenable, and so is virtually solvable by Theorem 1.11. $\square$

In particular, $\zeta\sqrt{N}$ is a nontrivial, torsion-free abelian characteristic subgroup of N. A similar argument shows that if N is locally FP_n, virtually indicable, restrained and $c.d.N \leq n$ then N is virtually solvable.

15.3 Nilpotent normal subgroups

In this section we shall consider 2-knot groups with infinite nilpotent normal subgroups. The class of groups with such subgroups includes the groups of torus knots and twist spins, Φ, and all 2-knot groups with finite commutator subgroup. If there is such a subgroup of rank > 1 the knot manifold is aspherical; this case is considered further in Chapter 16. (Little is known about infinite locally-finite normal subgroups: see §15.7 below.)

Lemma 15.7 *Let G be a group with normal subgroups $T < N$ such that T is a torsion group, N is nilpotent, $N/T \cong Z$ and G/N is finitely generated. Then G has an infinite cyclic normal subgroup $C \leq N$.*

Proof Let $z \in N$ have infinite order, and let $P = \langle\langle z \rangle\rangle$. Then P is nilpotent and $P/P \cap T \cong Z$. We may choose $g_0, \dots, g_n \in G$ such that their images generate G/N, and such that $g_0 z g_0^{-1} = z^\varepsilon t_0$ and $g_i z g_i^{-1} = z t_i$, where $\varepsilon = \pm 1$ and t_i has finite order, for all $0 \leq i \leq n$. The conjugates of $t_0, \dots, t_n$ generate the torsion subgroup of P/P', and so it has bounded exponent, e say.

Let $Q = \langle\langle z^e \rangle\rangle$. If P is abelian then we may take $C = Q$. Otherwise, we may replace P by Q, which has strictly smaller nilpotency class, and the result follows by induction on the nilpotency class. □

Theorem 15.8 *Let K be a 2-knot whose group $\pi = \pi K$ has a nilpotent normal subgroup N with $h = h(N) \geq 1$. Then $h \leq 4$ and*

(1) *if π' is finitely generated then so is N;*
(2) *if $h = 1$ then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $e(\pi/N) = 1$ and $h(\zeta N) = 0$ or $e(\pi/N) = \infty$;*
(3) *if $h = 1$ and $N \not\leq \pi'$ then N is finitely generated and $M(K)'$ is a PD_3^+-complex, and $M(K)'$ is aspherical if and only if $e(\pi') = 1$;*
(4) *if $h = 1$, $N \leq \pi'$ and $e(\pi/N) = \infty$ then N is finitely generated;*
(5) *if $h = 2$ then $N \cong Z^2$ and $M(K)$ is aspherical;*
(6) *if $h = 3$ or 4 then $M(K)$ is homeomorphic to an infrasolvmanifold.*

In all cases, π has a nontrivial torsion-free abelian normal subgroup $A \leq N$.

Proof If π' is finitely generated and $N \cap \pi'$ is infinite then M is aspherical and π' is the fundamental group of a Seifert fibred 3-manifold or is virtually poly-Z, by Theorems 4.5 and 2.14. Thus N must be finitely generated.

The four possibilities in case (2) correspond to whether π/N is finite or has one, two or infinitely many ends, by Theorem 15.5. These possibilities are mutually exclusive; if $e(\pi/N)=\infty$ then a Mayer-Vietoris argument as in Lemma 14.8 implies that π cannot be a PD_4-group.

If $h=1$ and $N\not\leq\pi'$ then $T=\pi'\cap N$ is the torsion subgroup of N and $N/T\cong Z$. There is an infinite cyclic subgroup $S\leq N$ which is normal in π, by Lemma 15.6. Since $S\cap\pi'=1$ and $Aut(S)$ is abelian S is in fact central, and $\pi'S$ has finite index in π. Hence $\pi'S\cong\pi'\times S$, and so π' is finitely presentable. Therefore $M(K)'$ is a PD_3^+-complex, and is aspherical if and only if $e(\pi')=1$, by Theorem 4.5. (Note also that T is finite, and so $e(\pi')=e(\pi/N)$.)

Suppose that $h=1$, $N\leq\pi'$ and N is not finitely generated. We may write π' as an increasing union $\pi'=\cup_{n\geq1}G_n$ of subgroups G_n which contain N, and such that G_n/N is finitely generated. If G_n/N is elementary amenable for all n then so are π'/N and π. Hence $e(\pi/N)<\infty$. Thus we may also assume that G_n/N is not elementary amenable, for all n.

Since G_n/N is finitely generated, G_n has an infinite cyclic normal subgroup C_n, by Lemma 15.6. Let $J_n=G_n/C_n$. Then G_n is an extension of J_n by Z, and J_n is in turn an extension of G_n/N by N/C_n. Since G_n/N is finitely generated J_n is an increasing union $J_n=\cup_{m\geq1}J_{n,m}$ of finitely generated infinite subgroups which are extensions of G_n/N by torsion groups. If these torsion subgroups are all finite then J_n is not finitely generated, since N/C_n is infinite, and $[J_{n,m+1}:J_{n,m}]<\infty$ for all m. Moreover, $H^0(J_{n,m};\mathbb{Z}[J_{n,m}])=0$, for all m, since $J_{n,m}$ is infinite. Therefore $H^s(J_n;F)=0$ for $s\leq1$ and any free $\mathbb{Z}[J_n]$-module F, by the Gildenhuys-Strebel Theorem (with $r=1$). Otherwise, $J_{n,m}$ has one end for m large, and we may apply Theorem 1.15 to obtain the same conclusion. An LHSSS argument now shows that $H^s(G_n;F)=0$ for $s\leq2$ and any free $\mathbb{Z}[G_n]$-module F. Another application of Theorem 1.15 shows that $H^s(\pi';F)=0$ for $s\leq2$ and any free $\mathbb{Z}[\pi']$-module F, and then another LHSSS argument gives $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq2$. Thus $M(K)$ is aspherical. Thus if $e(\pi/N)=\infty$ then N must be finitely generated.

Since $h\leq h(\sqrt{\pi})$ parts (5) and (6) follow immediately from Theorems 9.2 and 8.1, respectively.

The final assertion is clear. □

If $h(N)\geq1$ and N is not finitely generated then N is torsion-free abelian of rank 1, and $\pi\cong\Phi$ or $M(K)$ is aspherical. (It is not known whether abelian normal subgroups of PD_n groups must be finitely generated.)

If π has a nilpotent normal subgroup N with $h(N) \geq 1$ and $\sqrt{\pi}/N$ is infinite then $e(\pi/N) \leq 2$. It then follows from the theorem that $\sqrt{\pi}$ is nilpotent. Is $\sqrt{\pi}$ always nilpotent?

Corollary 15.8.1 *If π' is infinite and $\pi'/\pi' \cap N$ is finitely generated then N is torsion-free.* $\square$

Does this corollary hold without assuming that $\pi'/\pi' \cap N$ is finitely generated?

Corollary 15.8.2 *If π' finitely generated and infinite then N is torsion-free poly-Z and either $\pi' \cap N = 1$ or $M(K)'$ is homotopy equivalent to an aspherical Seifert fibred 3-manifold or $M(K)$ is homeomorphic to an infrasolvmanifold. If moreover $h(\pi' \cap N) = 1$ then $\zeta\pi' \neq 1$.*

Proof If $\pi' \cap N$ is torsion-free and $h(\pi' \cap N) = 1$ then $Aut(\pi' \cap N)$ is abelian. Hence $\pi' \cap N \leq \zeta\pi'$. The rest of the corollary is clear from the theorem, with Theorems 8.1 and 9.2. $\square$

Twists spins of classical knots give examples with N abelian, of rank 1, 2 or 4, and π' finite, M aspherical or $e(\pi/N) = \infty$. If $\pi' \cap N = 1$ and π' is torsion free then π' is a free product of PD_3^+-groups and free groups. Is every 2-knot K such that $\zeta\pi \not\leq \pi'$ and π is torsion-free s-concordant to a fibred knot?

We may construct examples of 2-knot groups π such that $\zeta\pi' \neq 1$ as follows. Let N be a closed 3-manifold such that $\nu = \pi_1(N)$ has weight 1 and $\nu/\nu' \cong Z$, and let $w = w_1(N)$. Then $H^2(N; \mathbb{Z}^w) \cong Z$. Let M_e be the total space of the S^1-bundle over N with Euler class $e \in H^2(N; \mathbb{Z}^w)$. Then M_e is orientable, and $\pi_1(M_e)$ has weight 1 if $e = \pm 1$ or if $w \neq 0$ and e is odd. In such cases surgery on a weight class in M_e gives S^4, so $M_e \cong M(K)$ for some 2-knot K.

In particular, we may take N to be the result of 0-framed surgery on a 1-knot. If the 1-knot is 3_1 or 4_1 (i.e., is fibred of genus 1) then the resulting 2-knot group has commutator subgroup Γ_1. If instead we assume that the 1-knot is *not* fibred then N is not fibred [Ga87] and so we get a 2-knot group π with $\zeta\pi \cong Z$ but π' not finitely generated. For examples with $w \neq 0$ we may take one of the nonorientable surface bundles with group

$$\langle t, a_i, b_i \ (1 \leq i \leq n) \mid \Pi[a_i, b_i] = 1,\ ta_it^{-1} = b_i,\ tb_it^{-1} = a_ib_i \ (1 \leq i \leq n)\rangle,$$

where n is odd. (When $n = 1$ we get the third of the three 2-knot groups with commutator subgroup Γ_1. See Theorem 16.14.)

Theorem 15.9 *Let K be a 2-knot with a minimal Seifert hypersurface, and such that $\pi = \pi K$ has an abelian normal subgroup A. Then $A \cap \pi'$ is finite cyclic or is torsion-free, and $\zeta\pi$ is finitely generated.*

Proof By assumption, $\pi = HNN(B; \phi : I \cong J)$ for some finitely presentable group B and isomorphism ϕ of subgroups I and J, where $I \cong J \cong \pi_1(V)$ for some Seifert hypersurface V. Let $t \in \pi$ be the stable letter. Either $B \cap A = I \cap A$ or $B \cap A = J \cap A$ (by Britton's Lemma). Hence $\pi' \cap A = \cup_{n \in Z} t^n (I \cap A) t^{-n}$ is a monotone union. Since $I \cap A$ is an abelian normal subgroup of a 3-manifold group it is finitely generated, by Theorem 2.14, and since V is orientable $I \cap A$ is torsion-free or finite. If $A \cap I$ is finite cyclic or is central in π then $A \cap I = t^n (A \cap I) t^{-n}$, for all n, and so $A \cap \pi' = A \cap I$. (In particular, $\zeta\pi$ is finitely generated.) Otherwise $A \cap \pi'$ is torsion-free. □

This argument derives from [Yo92, Yo97], where it was shown that if A is a finitely generated abelian normal subgroup then $\pi' \cap A \leq I \cap J$.

Corollary 15.9.1 *Let K be a 2-knot with a minimal Seifert hypersurface. If $\pi = \pi K$ has a nontrivial abelian normal subgroup A then $\pi' \cap A$ is finite cyclic or is torsion-free. Moreover $\zeta\pi \cong 1$, $Z/2Z$, Z, $Z \oplus (Z/2Z)$ or Z^2.* □

The knots $\tau_0 4_1$, the trivial knot, $\tau_3 3_1$ and $\tau_6 3_1$ are fibred and their groups have centres 1, Z, $Z \oplus (Z/2Z)$ and Z^2, respectively. A 2-knot with a minimal Seifert hypersurface and such that $\zeta\pi = Z/2Z$ is constructed in [Yo82]. This paper also gives an example with $\zeta\pi \cong Z$, $\zeta\pi < \pi'$ and such that $\pi/\zeta\pi$ has infinitely many ends. In all known cases the centre of a 2-knot group is cyclic, $Z \oplus (Z/2Z)$ or Z^2.

15.4 Finite commutator subgroup

It is a well known consequence of the asphericity of the exteriors of classical knots that classical knot groups are torsion-free. The first examples of higher dimensional knots whose groups have nontrivial torsion were given by Mazur [Ma62] and Fox [Fo62]. These examples are 2-knots whose groups have finite commutator subgroup. We shall show that if π is such a group π' must be a CK group, and that the images of meridianal automorphisms in $Out(\pi')$ are conjugate, up to inversion. In each case there is just one 2-knot group with given finite commutator subgroup. Many of these groups can be realized by twist spinning classical knots. Zeeman introduced twist spinning in order to study Mazur's example; Fox used hyperplane cross sections, but his examples (with $\pi' \cong Z/3Z$) were later shown to be twist spins [Kn83'].

Lemma 15.10 *An automorphism of $Q(8)$ is meridianal if and only if it is conjugate to σ.*

Proof Since $Q(8)$ is solvable an automorphism is meridianal if and only if the induced automorphism of $Q(8)/Q(8)'$ is meridianal. It is easily verified that all such elements of $Aut(Q(8)) \cong (Z/2Z)^2 \rtimes SL(2, \mathbb{F}_2)$ are conjugate to σ. □

Lemma 15.11 *All nontrivial automorphisms of I^* are meridianal. Moreover each automorphism is conjugate to its inverse. The nontrivial outer automorphism class of I^* cannot be realised by a 2-knot group.*

Proof The first assertion is immediate, since ζI^* is the only nontrivial proper normal subgroup of I^*. Each automorphism is conjugate to its inverse, since $Aut(I^*) \cong S_5$. Considering the Wang sequence for the map $: M(K)' \to M(K)$ shows that the meridianal automorphism induces the identity on $H_3(\pi'; \mathbb{Z})$, and so the nontrivial outer automorphism class cannot occur, by Lemma 11.4. □

The elements of order 2 in $A_5 \cong Inn(I^*)$ are all conjugate, as are the elements of order 3. There are two conjugacy classes of elements of order 5.

Lemma 15.12 *An automorphism of T_k^* is meridianal if and only if it is conjugate to $\rho^{3^{k-1}}$ or $\rho^{3^{k-1}}\eta$. These have the same image in $Out(T_k^*)$.*

Proof Since T_k^* is solvable an automorphism is meridianal if and only if the induced automorphism of $T_k^*/(T_k^*)'$ is meridianal. Any such automorphism is conjugate to either ρ^{2j+1} or to $\rho^{2j+1}\eta$ for some $0 \leq j < 3^{k-1}$. (Note that 3 divides $2^{2j}-1$ but does not divide $2^{2j+1}-1$.) However among them only those with $2j+1 = 3^{k-1}$ satisfy the isometry condition of Theorem 14.3. □

Theorem 15.13 *Let K be a 2-knot with group $\pi = \pi K$. If π' is finite then $\pi' \cong P \times (Z/nZ)$ where $P = 1$, $Q(8)$, I^* or T_k^*, and $(n, 2|P|) = 1$, and the meridianal automorphism sends x and y in $Q(8)$ to y and xy, is conjugation by a noncentral element on I^*, sends x, y and z in T_k^* to y^{-1}, x^{-1} and z^{-1}, and is -1 on the cyclic factor.*

Proof Since $\chi(M(K)) = 0$ and π has two ends π' has cohomological period dividing 4, by Theorem 11.1, and so is among the groups listed in §2 of Chapter 11. As the meridianal automorphism of π' induces a meridianal automorphism on the quotient by any characteristic subgroup, we may eliminate immediately the groups $O^*(k)$, $A(m,e)$ and $Z/2mZ$ and their products with Z/nZ since these all have abelianization cyclic of even order. If $k>1$ the subgroup generated by x in $Q(8k)$ is a characteristic subgroup of index 2. Since $Q(2^n a)$ is a quotient of $Q(2^n a,b,c)$ by a characteristic subgroup (of order bc) this eliminates this class also. Thus there remain only the above groups.

Automorphisms of a group $G=H\times J$ such that $(|H|,|J|)=1$ correspond to pairs of automorphisms ϕ_H and ϕ_J of H and J, respectively, and ϕ is meridianal if and only if ϕ_H and ϕ_J are. Multiplication by s induces a meridianal automorphism of Z/mZ if and only if $(s-1,m)=(s,m)=1$. If Z/mZ is a direct factor of π' then it is a direct summand of $\pi'/\pi''=H_1(M(K);\Lambda)$ and so $s^2\equiv 1$ modulo (m), by Theorem 14.3. Hence we must have $s\equiv -1$ modulo (m). The theorem now follows from Lemmas 15.10–15.12. □

Finite cyclic groups are realized by the 2-twist spins of 2-bridge knots, while $Q(8)$, T_1^* and I^* are realized by $\tau_3 3_1$, $\tau_4 3_1$ and $\tau_5 3_1$, respectively. As the groups of 2-bridge knots have 2 generator 1 relator presentations the groups of these twist spins have 2 generator presentations of deficiency 0. In particular, $\pi\tau_r 3_1$ has the presentation $\langle a,t \mid tat^{-1}=at^2at^{-2},\ t^r a=at^r\rangle$.

The groups with $\pi'\cong Q(8)\times(Z/nZ)$ also have such optimal presentations:

$$\langle t,a \mid ta^2t^{-1}=a^{-2},\ ta^nt^{-1}=a^nt^2a^nt^{-2}\rangle.$$

For let $x=a^n$, $y=ta^nt^{-1}$ and $z=a^4$. Then $xz=zx$, $y=txt^{-1}$ and the relations imply $tyt^{-1}=x^{-1}y$, $tzt^{-1}=z^{-1}$ and $y^2=x^{-2}$. Hence $yz=zy$ and $x^2=(x^{-1}y)^2$. It follows easily that $x^4=1$, so $z^n=1$, and $\pi'\cong Q(8)\times(Z/nZ)$ (since n is odd). Conjugation by tx induces the meridianal automorphism of Theorem 15.13. When $n=1$ the generators correspond to those of the above presentation for $\pi\tau_3 3_1$. These groups are realized by fibred 2-knots [Yo82], but if $n>1$ no such group can be realized by a twist spin. (See §3 of Chapter 16.) An extension of the twist spin construction may be used to realize such groups by smooth fibred knots in the standard S^4, if $n=3$, 5, 11, 13, 19, 21 or 27 [Kn88, Tr90]. Is this so in general?

The other groups are realized by the 2-twist spins of certain pretzel knots [Yo82]. If $\pi'\cong T_k^*\times(Z/nZ)$ then π has a presentation

$$\langle s,x,y,z \mid x^2=(xy)^2=y^2,\ z^\alpha=1,\ zxz^{-1}=y,\ zyz^{-1}=xy,$$
$$sxs^{-1}=y^{-1},\ sys^{-1}=x^{-1},\ szs^{-1}=z^{-1}\rangle,$$

where $\alpha=3^k n$. This is equivalent to the presentation

$$\langle s,x,y,z \mid z^\alpha=1, zxz^{-1}=y,\ zyz^{-1}=xy,\ sxs^{-1}=y^{-1},\ szs^{-1}=z^{-1}\rangle.$$

For conjugating $zxz^{-1} = y$ by s gives $z^{-1}y^{-1}z = sys^{-1}$, so $sys^{-1} = x^{-1}$, while conjugating $zyz^{-1} = xy$ by s gives $x = yxy$, so $x^2 = (xy)^2$, and conjugating this by s gives $y^2 = (xy)^2$. On replacing s by $t = xzs$ we obtain the presentation

$$\langle t, x, y, z \mid z^\alpha = 1,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ txt^{-1} = xy,\ tzt^{-1} = yz^{-1}\rangle.$$

We may use the second and final relations to eliminate the generators x and y to obtain a 2-generator presentation of deficiency -1. (When $n = k = 1$ we may relate this to the above presentation for $\pi\tau_4 3_1$ by setting $a = zx^2$.) Are there presentations of deficiency 0?

If $\pi' \cong I^* \times (Z/nZ)$ then $\pi \cong I^* \times (\pi/I^*)$ and π has a presentation

$$\langle t, w \mid tw^n t^{-1} = w^n t^2 w^n t^{-2},\ t^5 w^n = w^n t^5,\ tw^{10}t^{-1} = w^{-10}\rangle.$$

For if G is the group with this presentation $t \mapsto t$ and $a \mapsto w^n$ defines a homomorphism from $\pi\tau_5 3_1$ to G, and so the first two relations imply that $w^{10n} = 1$, since $a^{10} = 1$ in $(\pi\tau_5 3_1)'$. (In particular, if $n = 1$ the third relation is redundant.) Since G' is generated by the conjugates of w the final relation implies that w^{10} is central in G'. We assume that $(n, 30) = 1$ and so there is an integer p such that $np \equiv 1\ mod\ (10)$. Then $t \mapsto t$ and $w \mapsto a^p$ defines an epimorphism from G to $\pi\tau_5 3_1$. Since the image of w in $G/\langle\langle w^n\rangle\rangle$ clearly has order n it follows that $G' \cong \pi'$, and conjugation by t induces the meridianal automorphism of Theorem 15.13. Thus π has a 2-generator presentation of deficiency -1. Are there presentations of deficiency 0?

If $P = 1$ or $Q(8)$ the weight class is unique up to inversion, while T_k^* and I^* have 2 and 4 weight orbits, respectively, by Theorem 14.1. If $\pi' = T_1^*$ or I^* each weight orbit is realized by a branched twist spun torus knot [PS87].

The group $\pi\tau_5 3_1 \cong Z \times I^* = Z \times SL(2, \mathbb{F}_5)$ is the common member of two families of high dimensional knot groups which are not otherwise 2-knot groups. If p is a prime greater than 3 then $SL(2, \mathbb{F}_p)$ is a finite superperfect group. Let $e_p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $(1, e_p)$ is a weight element for $Z \times SL(2, \mathbb{F}_p)$. Similarly, $(I^*)^m$ is superperfect and $(1, e_5, \dots, e_5)$ is a weight element for $G = Z \times (I^*)^m$, for any $m \geq 0$. However $SL(2, \mathbb{F}_p)$ has cohomological period $p - 1$ (see Corollary 1.27 of [DM85]), while $\zeta(I^*)^m \cong (Z/2Z)^m$ and so $(I^*)^m$ does not have periodic cohomology if $m > 1$.

Let $G(n) \cong Q(8n)*_{Q(8)}$ be the HNN extension with presentation

$$\langle t, x, y \mid x^{2n} = y^2,\ yxy^{-1} = x^{-1},\ tyt^{-1} = yx^{-1},\ ty^{-1}x^n t^{-1} = x^n\rangle.$$

Then $G(n)$ is a 2-knot group with an element of order $4n$, for all $n \geq 1$ [Kn89].

15.5 The Tits alternative

An HNN extension (such as a knot group) is restrained if and only if it is ascending and the base is restrained. The class of groups considered in the next result probably includes all restrained 2-knot groups.

Theorem 15.14 *Let π be a 2-knot group. Then the following are equivalent:*

(1) *π is restrained, locally FP_3 and locally virtually indicable;*

(2) *π is an ascending HNN extension $B*_\phi$ where B is FP_3, restrained and virtually indicable;*

(3) *π is elementary amenable and has a nontrivial torsion-free normal subgroup N;*

(4) *π is elementary amenable and has an abelian normal subgroup of rank >0;*

(5) *π is elementary amenable and is an ascending HNN extension $B*_\phi$ where B is FP_2;*

(6) *π' is finite or $\pi \cong \Phi$ or π is torsion-free virtually poly-Z and $h(\pi) = 4$.*

Proof Condition (1) implies (2) by Theorem 1.13. If (2) holds then either π' is finite or $\pi \cong \Phi$ or $\pi' = B$ and is a PD_3-group, by Theorem 15.3 and Corollary 15.3.2. In the latter case B has a subgroup of finite index which maps onto Z^2, since it is virtually indicable and admits a meridianal automorphism. Hence B is virtually poly-Z, by Corollary 2.13.1 (together with the remark following it). Hence (2) implies (6). If (3) holds then either $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for all $s \geq 0$ or N is virtually solvable, by Proposition 3 of [Kr93']. Hence either π is torsion-free virtually poly-Z and $h(\pi) = 4$, by Theorem 8.1, or (4) holds. Conditions (4) and (5) imply (6) by Theorems 1.17 and 15.2, and by Theorem 15.3, respectively. On the other hand (6) implies (1-5). □

In particular, if K is a 2-knot with a minimal Seifert hypersurface, πK is restrained and the Alexander polynomial of K is nontrivial then either $\pi \cong \Phi$ or π is torsion-free virtually poly-Z and $h(\pi) = 4$.

15.6 Abelian HNN bases

We shall complete Yoshikawa's study of 2-knot groups which are HNN extensions with abelian base. The first four paragraphs of the following proof outline the arguments of [Yo86, Yo92]. (Our contribution is the argument in the final paragraph, eliminating possible torsion when the base has rank 1.)

Theorem 15.15 *Let π be a 2-knot group which is an HNN extension with abelian base. Then either π is metabelian or it has a deficiency 1 presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1} \rangle$ for some $n > 1$.*

Proof Suppose that $\pi = A*_\phi = HNN(A; \phi : B \to C)$ where A is abelian. Let j and j_C be the inclusions of B and C into A, and let $\tilde{\phi} = j_C\phi$. Then $\tilde{\phi} - j : B \to A$ is an isomorphism, by the Mayer-Vietoris sequence for homology with coefficients $\mathbb{Z}$ for the HNN extension. Hence $rank(A) = rank(B) = r$, say, and the torsion subgroups TA, TB and TC of A, B and C coincide.

Suppose first that A is not finitely generated. Since π is finitely presentable and $\pi/\pi' \cong Z$ it is also an HNN extension with finitely generated base and associated subgroups, by the Bieri-Strebel Theorem (1.13). Moreover we may assume the base is a subgroup of A. Considerations of normal forms with respect to the latter HNN structure imply that it must be ascending, and so π is metabelian [Yo92].

Assume now that A is finitely generated. Then the image of TA in π is a finite normal subgroup N, and π/N is a torsion-free HNN extension with base $A/TA \cong Z^r$. Let j_F and ϕ_F be the induced inclusions of B/TB into A/TA, and let $M_j = |det(j_F)|$ and $M_\phi = |det(\phi_F)|$. Applying the Mayer-Vietoris sequence for homology with coefficients Λ, we find that $t\tilde{\phi} - j$ is injective and $\pi'/\pi'' \cong H_1(\pi; \Lambda)$ has rank r as an abelian group. Now $H_2(A; \mathbb{Z}) \cong A \wedge A$ (see page 334 of [Ro]) and so $H_2(\pi; \Lambda) \cong \mathrm{Cok}(t \wedge_2 \tilde{\phi} - \wedge_2 j)$ has rank $\binom{r}{2}$. Let $\delta_i(t) = \Delta_0(H_i(\pi; \Lambda))$, for $i = 1$ and 2. Then $\delta_1(t) = det(t\phi_F - j_F)$ and $\delta_2(t) = det(t\phi_F \wedge \phi_F - j_F \wedge j_F)$. Moreover $\delta_2(t^{-1})$ divides $\delta_1(t)$, by Theorem 14.3. In particular, $\binom{r}{2} \leq r$, and so $r \leq 3$.

If $r = 0$ then clearly $B = A$ and so π is metabelian. If $r = 2$ then $\binom{r}{2} = 1$ and $\delta_2(t) = \pm(tM_\phi - M_j)$. Comparing coefficients of the terms of highest and lowest degree in $\delta_1(t)$ and $\delta_2(t^{-1})$, we see that $M_j = M_\phi$, so $\delta_2(1) \equiv 0 \mod (2)$, which is impossible since $|\delta_1(1)| = 1$. If $r = 3$ a similar comparison of coefficients in $\delta_1(t)$ and $\delta_2(t^{-1})$ shows that M_j^3 divides M_ϕ and M_ϕ^3 divides M_j, so $M_j = M_\phi = 1$. Hence ϕ is an isomorphism, and so π is metabelian.

There remains the case $r = 1$. Yoshikawa used similar arguments involving coefficients $\mathbb{F}_p\Lambda$ instead to show that in this case $N \cong Z/\beta Z$ for some odd $\beta \geq 1$. The group π/N then has a presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1} \rangle$ (with $n \geq 1$). Let p be a prime. There is an isomorphism of the subfields $\mathbb{F}_p(X^n)$ and $\mathbb{F}_p(X^{n+1})$ of the rational function field $\mathbb{F}_p(X)$ which carries X^n to X^{n+1}. Therefore $\mathbb{F}_p(X)$ embeds in a skew field L containing an element t such that $tX^n t^{-1} = X^{n+1}$, by Theorem 5.5.1 of [Cn]. It is clear from the argument of this theorem that the group ring $\mathbb{F}_p[\pi/N]$ embeds as a subring of L, and so this group ring is weakly finite. Therefore so is the subring $\mathbb{F}_p[C_\pi(N)/N]$. It now follows from Lemma 3.16 that N must be trivial. Since π is metabelian if $n=1$ this completes the proof. $\square$

15.7 Locally finite normal subgroups

Let K be a 2-knot such that $\pi=\pi K$ has an infinite locally finite normal subgroup T, which we may assume maximal. As π has one end and $\beta_1^{(2)}(\pi)=0$, by Gromov's Theorem (2.3), $H^2(\pi;\mathbb{Z}[\pi])\neq 0$. For otherwise $M(K)$ would be aspherical and so π would be torsion-free, by Theorem 3.5. Moreover $T<\pi'$ and π/T is not virtually Z, so $e(\pi/T)=1$ or ∞. (No examples of such 2-knot groups are known, and we expect that there are none with $e(\pi/T)=1$.)

Suppose that $\mathbb{Z}[\pi/T]$ has a safe extension S with an involution extending that of the group ring. If $e(\pi/T)=1$ then either $H^2(\pi/T;\mathbb{Q}[\pi/T])\neq 0$ or π/T is a PD_4^+-group over $\mathbb{Q}$, by the Addendum to Theorem 2.6 of [H2].

In particular, if π/T has a nontrivial locally nilpotent normal subgroup U/T, then U/T is torsion-free, by Proposition 5.2.7 of [Ro], and so $\mathbb{Z}[\pi/T]$ has such a safe extension, by Theorem 1.7 of [H2]. Moreover $e(\pi/T)=1$. An iterated LHSSS argument shows that if $h(U/T)>1$ or if $U/T\cong Z$ and $e(\pi/U)=1$ then $H^2(\pi/T;\mathbb{Q}[\pi/T])=0$. (This is also the case if $h(U/T)=1$, $e(\pi/U)=1$ and π/T is finitely presentable, by Theorem 1 of [Mi87] with [GM86].) Thus if $H^2(\pi/T;\mathbb{Q}[\pi/T])\neq 0$ then U/T is abelian of rank 1 and either $e(\pi/U)=2$ (in which case $\pi/T\cong\Phi$, by Theorem 15.2), $e(\pi/U)=1$ (and U/T not finitely generated and π/U not finitely presentable) or $e(\pi/U)=\infty$. As $Aut(U/T)$ is then abelian U/T is central in π'/T. Moreover π/U can have no nontrivial locally finite normal subgroups, for otherwise T would not be maximal in π, by an easy extension of Schur's Theorem (Proposition 10.1.4 of [Ro]).

Hence if π has an ascending series whose factors are either locally finite or locally nilpotent then either $\pi/T\cong\Phi$ or $h(\sqrt{\pi/T})\geq 2$. In the latter case π/T is an elementary amenable PD_4^+-group over $\mathbb{Q}$. Since it has no nontrivial locally finite normal subgroup, it is virtually solvable, by the argument of Theorem 1.11. It can be shown that π/T is virtually poly-Z and $(\pi/T)'\cap\sqrt{\pi/T}\cong Z^3$ or Γ_q for some $q\geq 1$. (See Theorem VI.2 of [H1].) The possibilities for $(\pi/T)'$ are examined in Theorems VI.3-5 and VI.9 of [H1]. We shall not repeat this discussion here as we expect that if G is finitely presentable and T is an infinite locally finite normal subgroup such that $e(G/T)=1$ then $H^2(G;\mathbb{Z}[G])=0$.

The following lemma suggests that there may be a homological route to showing that solvable 2-knot groups are virtually torsion-free.

Lemma 15.16 *Let G be an FP_2 group with a torsion normal subgroup T such that $\mathbb{Z}[G/T]$ is coherent. Then T/T' has finite exponent as an abelian group. In particular, if G is solvable then $T=1$ if and only if $H_1(T;\mathbb{F}_p)=0$ for all primes p.*

Proof Let C_* be a free $\mathbb{Z}[G]$-resolution of the augmentation module Z which is finitely generated in degrees ≤ 2. Then $T/T' = H_1(\mathbb{Z}[G/T]\otimes_G C_*)$, and so it is finitely presentable as a $\mathbb{Z}[G/T]$-module, since $\mathbb{Z}[G/T]$ is coherent. If T/T' is generated by elements t_i of order e_i then Πe_i is a finite exponent for T/T'.

If G is solvable then so is T, and $T=1$ if and only if $T/T'=1$. Since T/T' has finite exponent $T/T'=1$ if and only if $H_1(T;\mathbb{F}_p)=0$ for all primes p. □

If G/T is virtually $Z*_m$ or virtually poly-Z then $\mathbb{Z}[G/T]$ is coherent [BS79]. Note also that $\mathbb{F}_p[Z*_m]$ is a coherent Ore domain of global dimension 2, while if J is a torsion-free virtually poly-Z group then $\mathbb{F}_p[J]$ is a noetherian Ore domain of global dimension $h(J)$. (See §4.4 and §13.3 of [Pa].)

Chapter 16

Abelian normal subgroups of rank ≥ 2

If K is a 2-knot such that $h(\sqrt{\pi K}) = 2$ then $\sqrt{\pi K} \cong Z^2$, by Corollary 15.5.1. The most familiar examples of such knots are the branched twist spins of torus knots, but there are others. In all cases, the knot manifolds are s-cobordant to $\mathbb{SL} \times \mathbb{E}^1$-manifolds. The groups of the branched twist spins of torus knots are the 3-knot groups which are PD_4^+-groups and have centre of rank 2, with some power of a weight element being central. There are related characterizations of the groups of branched twist spins of other prime 1-knots.

If $h(\sqrt{\pi K}) > 2$ then $M(K)$ is homeomorphic to an infrasolvmanifold, so K is determined up to Gluck reconstruction and change of orientations by πK and a weight orbit. We find the groups and their strict weight orbits, and give optimal presentations for most such groups. Two are virtually Z^4; otherwise, $h(\sqrt{\pi K}) = 3$. Whether K is amphicheiral or invertible is known, and in most cases whether it is reflexive is also known. (See Chapter 18 and [Hi11'].)

16.1 The Brieskorn manifolds $M(p, q, r)$

Let $M(p, q, r) = \{(u, v, w) \in \mathbb{C}^3 \mid u^p + v^q + w^r = 0\} \cap S^5$. Thus $M(p, q, r)$ is a *Brieskorn* 3*-manifold* (the link of an isolated singularity of the intersection of n algebraic hypersurfaces in $\mathbb{C}^{n+2}$, for some $n \geq 1$). It is clear that $M(p, q, r)$ is unchanged by a permutation of $\{p,\ q,\ r\}$.

Let $s = hcf\{pq, pr, qr\}$. Then $M(p, q, r)$ admits an effective S^1-action given by $z(u, v, w) = (z^{qr/s}u, z^{pr/s}v, z^{pq/s}w)$ for $z \in S^1$ and $(u, v, w) \in M(p, q, r)$, and so is Seifert fibred. More precisely, let $\ell = lcm\{p, q, r\}$, $p' = lcm\{p, r\}$, $q' = lcm\{q, r\}$ and $r' = lcm\{p, q\}$, $s_1 = qr/q'$, $s_2 = pr/p'$ and $s_3 = pq/r'$ and $t_1 = \ell/q'$, $t_2 = \ell/p'$ and $t_3 = \ell/r'$. Let $g = (2 + (pqr/\ell) - s_1 - s_2 - s_3)/2$. Then $M(p, q, r) = M(g; s_1(t_1, \beta_1), s_2(t_2, \beta_2), s_3(t_3, \beta_3))$, in the notation of [NR78], where the coefficients β_i are determined *mod* t_i by the equation

$$e = -(qr\beta_1 + pr\beta_2 + pq\beta_3)/\ell) = -pqr/\ell^2$$

for the generalized Euler number. (See [NR78].) If $p^{-1} + q^{-1} + r^{-1} \leq 1$ the Seifert fibration is essentially unique. (See Theorem 3.8 of [Sc83'].) In most cases the triple $\{p,q,r\}$ is determined by the Seifert structure of $M(p,q,r)$. (Note however that, for example, $M(2,9,18) \cong M(3,5,15)$ [Mi75].)

The map $f : M(p,q,r) \to \mathbb{CP}^1$ given by $f(u,v,w) = [u^p : v^q]$ is constant on the orbits of the S^1-action, and the exceptional fibres are those above 0, -1 and ∞ in $\mathbb{CP}^1$. In particular, if p, q and r are pairwise relatively prime f is the orbit map and $M(p,q,r)$ is Seifert fibred over the orbifold $S^2(p,q,r)$. The involution c of $M(p,q,r)$ induced by complex conjugation in $\mathbb{C}^3$ is orientation preserving and is compatible with f and complex conjugation in $\mathbb{CP}^1$. Let $A(u,v,w) = (u,v,e^{2\pi i/r}w)$ and $h(u,v,w) = (u,v)/(|u|^2+|v|^2)$, for $(u,v,w) \in M(p,q,r)$. The Z/rZ-action generated by A commutes with the S^1-action, and these actions agree on their subgroups of order r/s. The projection to the orbit space $M(p,q,r)/\langle A\rangle$ may be identified with the map $h : M(p,q,r) \to S^3$, which is an r-fold cyclic branched covering, branched over the (p,q)-torus link. (See Lemma 1.1 of [Mi75].)

The 3-manifold $M(p,q,r)$ is a $\mathbb{S}^3$-manifold if and only if $p^{-1}+q^{-1}+r^{-1} > 1$. The triples $(2,2,r)$ give lens spaces. The other triples with $p^{-1}+q^{-1}+r^{-1} > 1$ are permutations of $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, and give the three CK 3-manifolds with fundamental groups $Q(8)$, T_1^* and I^*. The manifolds $M(2,3,6)$, $M(3,3,3)$ and $M(2,4,4)$ are $\mathbb{N}il^3$-manifolds; in all other cases $M(p,q,r)$ is a $\widetilde{\mathbb{SL}}$-manifold (in fact, a coset space of $\widetilde{SL}$ [Mi75]), and $\sqrt{\pi_1(M(p,q,r))} \cong Z$. If p, q and r are pairwise relatively prime $M(p,q,r)$ is a $\mathbb{Z}$-homology 3-sphere.

16.2 Rank 2 subgroups

In this section we shall show that an abelian normal subgroup of rank 2 in a 2-knot group is free abelian and not contained in the commutator subgroup.

Lemma 16.1 *Let ν be the fundamental group of a closed $\mathbb{H}^2\times\mathbb{E}^1$-, $\mathbb{S}ol^3$- or $\mathbb{S}^2\times\mathbb{E}^1$-manifold. Then ν admits no meridianal automorphism.*

Proof The fundamental group of a closed $\mathbb{S}ol^3$- or $\mathbb{S}^2\times\mathbb{E}^1$-manifold has a characteristic subgroup with quotient having two ends. If ν is a lattice in $Isom^+(\mathbb{H}^2\times\mathbb{E}^1)$ then $\sqrt{\nu} \cong Z$, and either $\sqrt{\nu} = \zeta\nu$ and is not contained in ν' or $C_\nu(\sqrt{\nu})$ is a characteristic subgroup of index 2 in ν. In none of these cases can ν admit a meridianal automorphism. □

Theorem 16.2 *Let π be a finitely presentable group. Then the following are equivalent:*

(1) *π is a PD_4^+-group of type FF and weight 1, and with an abelian normal subgroup A of rank 2;*

(2) *$\pi = \pi K$ where K is a 2-knot and π has an abelian normal subgroup A of rank 2;*

(3) *$\pi = \pi K$ where K is a 2-knot such that $M(K)$ is an aspherical Seifert fibred 4-manifold.*

If so, then $A \cong Z^2$, $\pi' \cap A \cong Z$, $\pi' \cap A \leq \zeta\pi' \cap I(\pi')$, $[\pi : C_\pi(A)] \leq 2$ and $\pi' = \pi_1(N)$, where N is a $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold. If π is solvable then $M(K)$ is homeomorphic to a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold. If π is not solvable then $M(K)$ is s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold which fibres over S^1 with fibre N, and closed monodromy of finite order.

Proof If (1) holds then $\chi(\pi) = 0$ [Ro84], and so $\beta_1(\pi) > 0$, by Lemma 3.14. Hence $H_1(\pi;\mathbb{Z}) \cong \mathbb{Z}$, since π has weight 1. If $h(\sqrt{\pi}) = 2$ then $A \cong Z^2$, by Theorem 9.2. Otherwise, π is virtually poly-Z, and so $A \cong Z^2$ again. We may assume that A is maximal among such subgroups. Then $\pi/A \cong \pi_1^{orb}(B)$, for some aspherical 2-orbifold B, and $\pi = \pi_1(M)$, where M is an orientable 4-manifold which is Seifert fibred over B, by Corollary 7.3.1. Surgery on a loop in M representing a normal generator of π gives a 2-knot K with $M(K) \cong M$ and $\pi K \cong \pi$. Thus (2) and (3) hold. Conversely, these each imply (1).

Since no 2-orbifold group has infinite cyclic abelianization, $A \not\leq \pi'$, and so $\pi' \cap A \cong Z$. If τ is the meridianal automorphism of $\pi'/I(\pi')$ then $\tau - 1$ is invertible, and so cannot have ± 1 as an eigenvalue. Hence $\pi' \cap A \leq I(\pi')$. (In particular, π' is not abelian.) The image of $\pi/C_\pi(A)$ in $Aut(A) \cong GL(2,\mathbb{Z})$ is triangular, since $\pi' \cap A \cong Z$ is normal in π. Therefore as $\pi/C_\pi(A)$ has finite cyclic abelianization it must have order at most 2. Thus $[\pi : C_\pi(A)] \leq 2$, so $\pi' < C_\pi(A)$ and $\pi' \cap A < \zeta\pi'$. The subgroup H generated by $\pi' \cup A$ has finite index in π and so is also a PD_4^+-group. Since A is central in H and maps onto H/π' we have $H \cong \pi' \times Z$. Hence π' is a PD_3^+-group with nontrivial centre. As the nonabelian flat 3-manifold groups either admit no meridianal automorphism or have trivial centre, $\pi' = \pi_1(N)$ for some $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold N, by Theorem 2.14 and Lemma 16.1.

The manifold $M(K)$ is s-cobordant to the mapping torus $M(\Theta)$ of a self homeomorphism of N, by Theorem 13.2. If N is a $\mathbb{N}il^3$-manifold $M(K)$ is homeomorphic to $M(\Theta)$, by Theorem 8.1, and $M(K)$ must be a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold, since $\mathbb{S}ol_1^4$-lattices do not have rank 2 abelian normal subgroups, while $\mathbb{N}il^4$-lattices cannot have abelianization Z, as they have characteristic rank 2 subgroups contained in their commutator subgroups. Since $[\pi : C_\pi(A)] \leq 2$ and

$A \not\leq \pi'$ the meridianal outer automorphism class has finite order. Therefore if N is a $\widetilde{\mathbb{SL}}$-manifold then $M(\Theta)$ is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold, by Theorem 9.8. $\square$

Since $\beta = \pi_1^{orb}(B)$ has weight 1 the space underlying B is S^2, RP^2 or D^2. However the possible bases are not yet known. If $A = \zeta\pi$ then B is orientable, and so $B = S^2(\alpha_1, \dots, \alpha_r)$, for some $r \geq 3$. Hence β is a one-relator product of r cyclic groups. Must $r = 3$ or 4? (See also Lemma 16.7 below.)

If $(p,q,r) = (2,3,5)$ or $(2,3,7)$ then $\pi_1(M(p,q,r))$ is perfect, and has the presentation

$$\langle a_1, a_2, a_3, h \mid a_1^p = a_2^q = a_3^r = a_1a_2a_3 = h\rangle.$$

(See [Mi75].) The involution c of $M(p,q,r)$ induces the automorphism c_* of ν determined by $c_*(a_1) = a_1^{-1}$, $c_*(a_2) = a_2^{-1}$ and $c_*(h) = h^{-1}$. (Hence $c_*(a_3) = a_2a_3^{-1}a_2^{-1}$.) Surgery on the mapping torus of c gives rise to a 2-knot whose group $\nu \rtimes_{c_*} Z$ has an abelian normal subgroup $A = \langle t^2, h\rangle$. If $r = 5$ then $h^2 = 1$ and A is central, but if $r = 7$ then $A \cong Z^2$ and is not central. The groups of 2-twist spins of Montesinos knots have noncentral rank 2 abelian normal subgroups. See Theorem 16.15 below for the virtually poly-Z case.

Theorem 16.3 *Let π be a 2-knot group such that $\zeta\pi$ has rank $r > 0$. If $\zeta\pi$ has nontrivial torsion it is finitely generated and $r = 1$. If $r > 1$ then $\zeta\pi \cong Z^2$, $\zeta\pi' = \pi' \cap \zeta\pi \cong Z$, and $\zeta\pi' \leq \pi''$.*

Proof The first assertion follows from Theorem 15.7. If $\zeta\pi$ had rank greater than 2 then $\pi' \cap \zeta\pi$ would contain an abelian normal subgroup of rank 2, contrary to Theorem 16.2. Therefore $\zeta\pi \cong Z^2$ and $\pi' \cap \zeta\pi \cong Z$. Moreover $\pi' \cap \zeta\pi \leq \pi''$, since $\pi/\pi' \cong Z$. In particular π' is nonabelian and π'' has nontrivial centre. Hence π' is the fundamental group of a $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2, and so $\zeta\pi' \cong Z$. It follows easily that $\pi' \cap \zeta\pi = \zeta\pi'$. $\square$

16.3 Twist spins of torus knots

The commutator subgroup of the group of the r-twist spin of a classical knot K is the fundamental group of the r-fold cyclic branched cover of S^3, branched over K [Ze65]. The r-fold cyclic branched cover of a sum of knots is the connected sum of the r-fold cyclic branched covers of the factors, and is irreducible if and only if the knot is prime. Moreover the cyclic branched covers of a prime knot are either aspherical or finitely covered by S^3; in particular no summand has free fundamental group [Pl84]. The cyclic branched covers of prime knots with nontrivial companions are Haken 3-manifolds [GL84]. The r-fold cyclic branched cover of the (p,q)-torus knot $k_{p,q}$ is the Brieskorn manifold $M(p,q,r)$ [Mi75]. The r-fold cyclic branched cover of a simple nontorus knot is a hyperbolic 3-manifold if $r \geq 3$, excepting only the 3-fold cyclic branched cover of the figure-eight knot, which is the Hanztsche-Wendt flat 3-manifold [Du83]. (In particular, there are only four r-fold cyclic branched covers of nontrivial knots for any $r > 2$ which have finite fundamental group.)

Theorem 16.4 *Let M be the r-fold cyclic branched cover of S^3, branched over a knot K, and suppose that $r > 2$ and that $\sqrt{\pi_1(M)}$ is infinite. Then K is uniquely determined by M and r, and either K is a torus knot or $K \cong 4_1$ and $r = 3$.*

Proof As the connected summands of M are the cyclic branched covers of the factors of K, any homotopy sphere summand must be standard, by the proof of the Smith conjecture. Therefore M is aspherical, and is either Seifert fibred or is a $\mathbb{S}ol^3$-manifold, by Theorem 2.14. It must in fact be a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold, by Lemma 16.1. If there is a Seifert fibration which is preserved by the automorphisms of the branched cover the fixed circle (the branch set of M) must be a fibre of the fibration (since $r > 2$) which therefore passes to a Seifert fibration of $X(K)$. Thus K must be a (p,q)-torus knot, for some relatively prime integers p and q [BZ]. These integers may be determined arithmetically from r and the formulae for the Seifert invariants of $M(p,q,r)$ given in §1. Otherwise M is flat [MS86] and so $K \cong 4_1$ and $r = 3$, by [Du83]. $\square$

All the knots whose 2-fold branched covers are Seifert fibred are torus knots or Montesinos knots. (This class includes the 2-bridge knots and pretzel knots, and was first described in [Mo73].) The number of distinct knots whose 2-fold branched cover is a given Seifert fibred 3-manifold can be arbitrarily large [Be84]. Moreover for each $r \geq 2$ there are distinct simple 1-knots whose r-fold cyclic branched covers are homeomorphic [Sa81, Ko86].

If K is a fibred 2-knot with monodromy of finite order r and if $(r,s) = 1$ then the s-fold cyclic branched cover of S^4, branched over K is again a 4-sphere and so the branch set gives a new 2-knot, which we shall call the s-fold cyclic branched cover of K. This new knot is again fibred, with the same fibre and monodromy the s^{th} power of that of K [Pa78, Pl86]. If K is a classical knot we shall let $\tau_{r,s}K$ denote the s-fold cyclic branched cover of the r-twist spin of K. We shall call such knots *branched twist spins*, for brevity.

Using properties of S^1-actions on smooth homotopy 4-spheres, Plotnick obtains the following result [Pl86].

Theorem (Plotnick) *A 2-knot is fibred with periodic monodromy if and only if it is a branched twist spin of a knot in a homotopy 3-sphere.* ◻

Here "periodic monodromy" means that the fibration of the exterior of the knot has a characteristic map of finite order. It is not in general sufficient that the *closed* monodromy be represented by a map of finite order. (For instance, if K is a fibred 2-knot with $\pi' \cong Q(8)\times(Z/nZ)$ for some $n>1$ then the meridianal automorphism of π' has order 6, and so it follows from the observations above that K is not a twist spin.)

The homotopy 3-sphere must be standard, by Perelman's work (see [B-P]). In our application in the next theorem we are able to show this directly.

Theorem 16.5 *A group G which is not virtually solvable is the group of a branched twist spin of a torus knot if and only if it is a 3-knot group and a PD_4^+-group with centre of rank 2, some nonzero power of a weight element being central.*

Proof If K is a cyclic branched cover of $\tau_r k_{p,q}$ then $M(K)$ fibres over S^1 with fibre $M(p,q,r)$ and monodromy of order r, and so the r^{th} power of a meridian is central. Moreover the monodromy commutes with the natural S^1-action on $M(p,q,r)$ (see Lemma 1.1 of [Mi75]) and hence preserves a Seifert fibration. Hence the meridian commutes with $\zeta\pi_1(M(p,q,r))$, which is therefore also central in G. Since $\pi_1(M(p,q,r))$ is either a PD_3^+-group with infinite centre or is finite, the necessity of the conditions is evident.

Conversely, if G is such a group then G' is the fundamental group of a Seifert fibred 3-manifold, N say, by Theorem 2.14. Moreover N is "sufficiently complicated" in the sense of [Zn79], since G' is not virtually solvable. Let t be an element of G whose normal closure is the whole group, and such that t^n is central for some $n>0$. Let θ be the automorphism of G' determined by t, and let m be the order of the outer automorphism class $[\theta]\in Out(G')$. By Corollary 3.3 of [Zn79] there is a fibre preserving self homeomorphism τ of N inducing $[\theta]$ such that the group of homeomorphisms of $\widetilde{N}\cong R^3$ generated by the covering group G' together with the lifts of τ is an extension of Z/mZ by G', and which is a quotient of the semidirect product $\hat{G}=G/\langle\langle t^n\rangle\rangle\cong G'\rtimes_\theta(Z/nZ)$. Since the self homeomorphism of $\widetilde{N}$ corresponding to the image of t has finite order it has a connected 1-dimensional fixed point set, by Smith theory. The image P of a fixed point in N determines a cross-section $\gamma=\{P\}\times S^1$ of the mapping torus $M(\tau)$. Surgery on γ in $M(\tau)$ gives a 2-knot with group G which is fibred with monodromy (of the fibration of the exterior X) of finite order. We may then apply Plotnick's Theorem to conclude that the 2-knot is a branched twist spin of a knot in a homotopy 3-sphere. Since the monodromy respects the Seifert fibration and leaves the centre of G' invariant, the branch set must be a fibre, and the orbit manifold a Seifert fibred homotopy 3-sphere. Therefore the orbit knot is a torus knot in S^3, and the 2-knot is a branched twist spin of a torus knot. □

If K is a 2-knot with group as in Theorem 16.5 then $M(K)$ is aspherical, and so is homotopy equivalent to some such knot manifold. (See also Theorem 16.2.)

Can we avoid the appeal to Plotnick's Theorem in the above argument? There is a stronger result for fibred 2-knots. The next theorem is a version of Proposition 6.1 of [Pl86], starting from more algebraic hypotheses.

Theorem 16.6 *Let K be a fibred 2-knot such that πK has centre of rank 2, some power of a weight element being central. Then $M(K)$ is homeomorphic to $M(K_1)$, where K_1 is some branched twist spin of a torus knot.*

Proof Let F be the closed fibre and $\phi : F \to F$ the characteristic map. Then F is a Seifert fibred manifold, as above. Now the automorphism of F constructed as in Theorem 16.5 induces the same outer automorphism of $\pi_1(F)$ as ϕ, and so these maps must be homotopic. Therefore they are in fact isotopic [Sc85, BO91]. The theorem now follows. □

If $(p,q) = (r,s) = 1$ then $M(\tau_{r,s}k_{p,q})$ is Seifert fibred over $S^2(p,q,r)$. (This can be derived from §16.1.) More generally, we have the following lemma.

Lemma 16.7 *Let B be an aspherical orientable 2-orbifold. If $\beta = \pi_1^{orb}(B)$ is normally generated by an element of finite order r then $B = S^2(m,n,r)$, where $(m,n) = 1$.*

Proof Since β is infinite and β/β' is finite cyclic $B = S^2(\alpha_1,\dots,\alpha_k)$, for some $\alpha_i > 1$ and $k \geq 3$. Hence β has the presentation

$$\langle x_1,\dots,x_k \mid x_i^{\alpha_i} = 1 \; \forall\; 1 \leq i \leq k,\; \Pi x_i = 1\rangle.$$

Every such group is nontrivial, and elements of finite order are conjugate to powers of a cone point generator x_i, by Theorem 4.8.1 of [ZVC]. If x_k^s is a weight element then so is x_k, and so $\langle x_1,\dots,x_{k-1} \mid x_i^{\alpha_i} = 1 \;\forall\; 1 \leq i \leq k-1,\; \Pi x_i = 1\rangle$ is trivial. Hence $k-1 = 2$ and $(\alpha_1,\alpha_2) = 1$. Similarly, $(s,\alpha_3) = 1$, so $r = \alpha_3$. Setting $m = \alpha_1$ and $n = \alpha_2$, we see that $B = S^2(m,n,r)$ and $(m,n) = 1$. □

The central extension of $\pi_1^{orb}(S^2(6,10,15))$ by Z^2 with presentation

$$\langle u,v,x,y,z \mid u,v\ central,\ x^6=u^5,\ y^{10}=uv,\ z^{15}=uv^{-1},\ xyz=u\rangle$$

is torsion-free, has abelianization Z, and is normally generated by yz^{-1}. The corresponding Seifert 4-manifold with base $S^2(6,10,15)$ is a knot manifold.

The manifold obtained by 0-framed surgery on the reef knot $3_1\#-3_1$ is the Seifert fibred 3-manifold $N=M(0;(2,1),(2,-1),(3,1),(3,-1))$, and the total space of the S^1-bundle over N with Euler class a generator of $H^2(N;\mathbb{Z})$ is a knot manifold which is Seifert fibred over $S^2(2,2,3,3)$. (Using $k_{p,q}$ instead of 3_1 gives a knot manifold with Seifert base $S^2(p,p,q,q)$.)

In each of these cases, it follows from Lemma 16.7 that no power of any weight element is central. Thus no such knot is a branched twist spin, and the final hypothesis of Theorem 16.5 is not a consequence of the others.

If p, q and r are pairwise relatively prime then $M(p,q,r)$ is an homology sphere and $\pi=\pi\tau_r k_{p,q}$ has a central element which maps to a generator of π/π'. Hence $\pi\cong\pi'\times Z$ and π' has weight 1. Moreover if t is a generator for the Z summand then an element h of π' is a weight element for π' if and only if ht is a weight element for π. This correspondance also gives a bijection between conjugacy classes of such weight elements. If we exclude the case $(2,3,5)$ then π' has infinitely many distinct weight orbits, and moreover there are weight elements such that no power is central [Pl83]. Therefore we may obtain many 2-knots whose groups are as in Theorem 16.5 but which are not themselves branched twist spins by surgery on weight elements in $M(p,q,r)\times S^1$.

We may apply arguments similar to those of Theorems 16.5 and 16.6 in attempting to understand twist spins of other knots. As only the existence of homeomorphisms of finite order and "homotopy implies isotopy" require different justifications, we shall not give proofs for the following assertions.

Let G be a 3-knot group such that G' is the fundamental group of an aspherical 3-manifold N and in which some nonzero power of a weight element is central. Then N is Seifert fibred, hyperbolic or Haken, by Perelman's work [B-P]. If N is hyperbolic we may use Mostow rigidity to show that G is the group of some branched twist spin K of a simple non-torus knot. Moreover, if K_1 is another fibred 2-knot with group G and hyperbolic fibre then $M(K_1)$ is homeomorphic to $M(K)$. In particular the simple knot and the order of the twist are determined by G. Similarly if N is Haken, but neither hyperbolic nor Seifert fibred, then we may use [Zn82] to show that G is the group of some branched twist spin of a prime non-torus knot. Moreover, the prime knot and the order of the twist are determined by G [Zn86], since all finite group actions on N are geometric [B-P].

16.4 Solvable PD_4-groups

If K is a 2-knot such that $h(\sqrt{\pi K} > 2$ then πK is virtually poly-Z and $h(\pi) = 4$, by Theorem 8.1. In this section we shall determine all such groups and their weight orbits.

Lemma 16.8 *Let G be torsion-free and virtually poly-Z with $h(G) = 4$ and $G/G' \cong Z$. Then $G' \cong Z^3$ or G_6 or $\sqrt{G'} \cong \Gamma_q$ (for some $q > 0$) and $G'/\sqrt{G'} \cong Z/3Z$ or 1.*

Proof Let $H = G/\sqrt{G'}$. Then $H/H' \cong Z$ and $h(H') \leq 1$, since $\sqrt{G'} = G' \cap \sqrt{G}$ and $h(G' \cap \sqrt{G}) \geq h(\sqrt{G}) - 1 \geq 2$, by Theorem 1.4. Now H' cannot have two ends, since $H/H' \cong Z$, and so $H' = G'/\sqrt{G'}$ is finite.

If $\sqrt{G'} \cong Z^3$ then $G' \cong Z^3$ or G_6, since these are the only flat 3-manifold groups which admit meridianal automorphisms.

If $\sqrt{G'} \cong \Gamma_q$ for some $q > 0$ then $\zeta\sqrt{G'} \cong Z$ is normal in G and so is central in G'. Using the known structure of automorphisms of Γ_q, it follows that the finite group $G'/\sqrt{G'}$ must act on $\sqrt{G'}/\zeta\sqrt{G'} \cong Z^2$ via $SL(2,\mathbb{Z})$ and so must be cyclic. Moreover it must have odd order, and hence be 1 or $Z/3Z$, since $G/\sqrt{G'}$ has infinite cyclic abelianization. □

There is a fibred 2-knot K with $\pi K \cong G$ if and only if G is orientable, by Theorems 14.4 and 14.7.

Theorem 16.9 *Let π be a 2-knot group with $\pi' \cong Z^3$, and let C be the image of the meridianal automorphism in $SL(3,\mathbb{Z})$. Then $\Delta_C(t) = \det(tI - C)$ is irreducible, $|\Delta_C(1)| = 1$ and π' is isomorphic to an ideal in the domain $R = \Lambda/(\Delta_C(t))$. Two such groups are isomorphic if and only if the polynomials are equal (after inverting t, if necessary) and the ideal classes then agree. There are finitely many ideal classes for each such polynomial and each class (equivalently, each such matrix) is realized by some 2-knot group. Moreover $\sqrt{\pi} = \pi'$ and $\zeta\pi = 1$. Each such group π has two strict weight orbits.*

Proof Let t be a weight element for π and let C be the matrix of the action of t by conjugation on π', with respect to some basis. Then $\det(C - I) = \pm 1$, since $t-1$ acts invertibly. Moreover if K is a 2-knot with group π then $M(K)$ is orientable and aspherical, so $\det(C) = +1$. Conversely, surgery on the mapping torus of the self homeomorphism of $S^1 \times S^1 \times S^1$ determined by such a matrix C gives a 2-knot with group $Z^3 \rtimes_C Z$.

The Alexander polynomial of K is the characteristic polynomial $\Delta_K(t) = \det(tI - C)$ which has the form $t^3 - at^2 + bt - 1$, for some a and $b = a \pm 1$. It is irreducible, since it does not vanish at ± 1. Since π' is annihilated by $\Delta_K(t)$ it is an R-module; moreover as it is torsion-free it embeds in $\mathbb{Q} \otimes \pi'$, which is a vector space over the field of fractions $\mathbb{Q} \otimes R$. Since π' is finitely generated and π' and R each have rank 3 as abelian groups it follows that π' is isomorphic to an ideal in R. Moreover the characteristic polynomial of C cannot be cyclotomic and so no power of t can commute with any nontrivial element of π'. Hence $\sqrt{\pi} = \pi'$ and $\zeta\pi = 1$.

By Lemma 1.1 two such semidirect products are isomorphic if and only if the matrices are conjugate up to inversion. The conjugacy classes of matrices in $SL(3, \mathbb{Z})$ with given irreducible characteristic polynomial $\Delta(t)$ correspond to the ideal classes of $\Lambda/(\Delta(t))$, by Theorem 1.4. Therefore π is determined by the ideal class of π', and there are finitely many such 2-knot groups with given Alexander polynomial.

Since $\pi'' = 1$ there are just two strict weight orbits, by Theorem 14.1. □

We shall call 2-knots with such groups *Cappell-Shaneson* 2-knots.

In [AR84] matrix calculations are used to show that any matrix C as in Theorem 16.9 is conjugate to one with first row $(0, 0, 1)$. Given this, it is easy to see that the corresponding Cappell-Shaneson 2-knot group has a presentation

$$\langle t, x, y, z \mid xy = yx,\ xz = zx,\ txt^{-1} = z,\ tyt^{-1} = x^m y^n z^p,\ tzt^{-1} = x^q y^r z^s \rangle.$$

Since p and s must be relatively prime these relations imply $yz = zy$. We may reduce the number of generators and relations on setting $z = txt^{-1}$. The next lemma gives a more conceptual exposition of part of this result.

Lemma 16.10 *Let $\Delta_a(t) = t^3 - at^2 + (a-1)t - 1$ for some $a \in \mathbb{Z}$, and let M be an ideal in the domain $R = \Lambda/(\Delta_a(t))$. Then M can be generated by 2 elements as an R-module.*

Proof Let $D = a(a-2)(a-3)(a-5) - 23$ be the discriminant of $\Delta_a(t)$. If a prime p does not divide D then $\Delta_a(t)$ has no repeated roots *mod* (p). If p divides D then $p > 5$, and there are integers α_p, β_p such that $\Delta_a(t) \equiv (t - \alpha_p)^2(t - \beta_p)$ *mod* (p). Let $K_p = \{m \in M \mid (t - \beta_p)m \in pM\}$.

If $\beta_p \equiv \alpha_p$ *mod* (p) then $\alpha_p^3 \equiv 1$ and $(1 - \alpha_p)^3 \equiv -1$ *mod* (p). Together these congruences imply that $\alpha_p \equiv 2$ *mod* (p), and hence that $p = 7$. Assume that this is so, and A be the matrix of multiplication by t on $M \cong \mathbb{Z}^3$, with respect to some basis. If $M/7M \cong (R/(7,t-2))^3$ then $A = 2I+7B$ for some $\mathbb{Z}$-matrix B. On expanding $\det(A) = \det(2I+7B)$ and $\det(A-I) = \det(I+7B)$ *mod* (49), we find that $\det(A) = 1$ and $\det(A-I_3) = \pm 1$ cannot both hold. (Compare Lemma A2 of [AR84].) Thus $M/7M \cong R/(7,(t-2)^3)$ or $R/(7,(t-2)^2) \oplus R/(7,t-2)$, and K_7 has index at least 7 in M. For all primes $p \neq 7$ dividing D the index of K_p is p^2. Therefore $M - \cup_{p|D} K_p$ is nonempty, since

$$\frac{1}{7} + \Sigma_{p|D,p\neq 7}\frac{1}{p^2} < \frac{1}{7} + \int_2^\infty \frac{1}{t^2}dt < 1.$$

Let x be an element of $M - \cup_{p|D} K_p$ which is not $\mathbb{Z}$-divisible in M. Then $N = M/Rx$ is finite, and is generated by at most two elements as an abelian group, since $M \cong Z^3$ as an abelian group. For each prime p the $\Lambda/p\Lambda$-module M/pM is an extension of N/pN by the submodule X_p generated by the image of x and its order ideal is generated by the image of $\Delta_a(t)$ in the P.I.D. $\Lambda/p\Lambda \cong \mathbb{F}_p[t,t^{-1}]$. If p does not divide D the image of $\Delta_a(t)$ in $\Lambda/p\Lambda$ is square free. If $p|D$ the order ideal of X_p is divisible by $t-\alpha_p$. In all cases the order ideal of N/pN is square free and so N/pN is cyclic as a Λ-module. By the Chinese Remainder Theorem there is an element $y \in M$ whose image is a generator of N/pN, for each prime p dividing the order of N. The image of y in N generates N, by Nakayama's Lemma. □

The cited result of [AR84] is equivalent to showing that M has an element x such that the image of tx in $M/\mathbb{Z}x$ is indivisible, from which it follows that M is generated as an abelian group by x, tx and some third element y.

Lemma 16.11 *Let π be a finitely presentable group such that $\pi/\pi' \cong Z$, and let $R = \Lambda$ or $\Lambda/p\Lambda$ for some prime $p \geq 2$. Then*

(1) *if π can be generated by 2 elements $H_1(\pi;R)$ is cyclic as an R-module;*
(2) *if $\mathrm{def}(\pi) = 0$ then $H_2(\pi;R)$ is cyclic as an R-module.*

Proof If π is generated by two elements t and x, say, we may assume that the image of t generates π/π' and that $x \in \pi'$. Then π' is generated by the conjugates of x under powers of t, and so $H_1(\pi;R) = R\otimes_\Lambda(\pi'/\pi'')$ is generated by the image of x.

If X is the finite 2-complex determined by a deficiency 0 presentation for π then $H_0(X;R) = R/(t-1)$ and $H_1(X;R)$ are R-torsion modules, and $H_2(X;R)$ is a submodule of a finitely generated free R-module. Hence $H_2(X;R) \cong R$, as it has rank 1 and R is an UFD. Therefore $H_2(\pi;R)$ is cyclic as an R-module, since it is a quotient of $H_2(X;R)$, by Hopf's Theorem. □

If $M(K)$ is aspherical $H_2(\pi;R) \cong \overline{Ext^1_R(H_1(\pi;R),R)}$, by Poincaré duality and the UCSS. In particular, if $R = \Lambda/p\Lambda$ for some prime $p \geq 2$ then $H_1(\pi;R)$ and $\overline{H_2(\pi;R)}$ are non-canonically isomorphic.

Theorem 16.12 *Let $\pi = Z^3 \rtimes_C Z$ be the group of a Cappell-Shaneson 2-knot, and let $\Delta(t) = \det(tI - C)$. Then π has a 3 generator presentation of deficiency -2. Moreover the following are equivalent.*

(1) *π has a 2 generator presentation of deficiency 0;*

(2) *π is generated by 2 elements;*

(3) $\mathrm{def}(\pi) = 0$*;*

(4) *π' is cyclic as a Λ-module.*

Proof The first assertion follows immediately from Lemma 16.10. Condition (1) implies (2) and (3), since $\mathrm{def}(\pi) \leq 0$, by Theorem 2.5, while (2) implies (4), by Lemma 16.11. If $\mathrm{def}(\pi) = 0$ then $H_2(\pi;\Lambda)$ is cyclic as a Λ-module, by Lemma 16.11. Since $\pi' = H_1(\pi;\Lambda) \cong \overline{H^3(\pi;\Lambda)} \cong \overline{Ext^1_\Lambda(H_2(\pi;\Lambda),\Lambda)}$, by Poincaré duality and the UCSS, it is also cyclic and so (3) also implies (4). If x generates π' as a Λ-module it is easy to see that π has a presentation

$$\langle t, x \mid xtxt^{-1} = txt^{-1}x,\ t^3xt^{-3} = t^2x^at^{-2}tx^bt^{-1}x\rangle,$$

for some integers a, b, and so (1) holds. $\square$

In fact Theorem A.3 of [AR84] implies that any such group has a 3 generator presentation of deficiency -1, as remarked before Lemma 16.10. Since $\Delta(t)$ is irreducible the Λ-module π' is determined by the Steinitz-Fox-Smythe row ideal $(t-n, m+np)$ in the domain $\Lambda/(\Delta(t))$. (See Chapter 3 of [H3].) Thus π' is cyclic if and only if this ideal is principal. In particular, this is not so for the concluding example of [AR84], which gives rise to the group with presentation

$$\langle t, x, y, z \mid xz = zx,\ yz = zy,\ txt^{-1} = y^{-5}z^{-8},\ tyt^{-1} = y^2z^3,\ tzt^{-1} = xz^{-7}\rangle.$$

Since $(\Delta_a(t)) \neq (\Delta_a(t^{-1}))$ (as ideals of $\mathbb{Q}\Lambda$), for all a, no Cappell-Shaneson 2-knot is a deform spin, by the criterion of [BM09].

Let $G(+)$ and $G(-)$ be the extensions of Z by G_6 with presentations

$$\langle t, x, y \mid xy^2x^{-1}y^2 = 1, txt^{-1} = (xy)^{\mp 1}, tyt^{-1} = x^{\pm 1}\rangle.$$

In each case, using the final relation to eliminate the generator x gives a 2-generator presentation of deficiency 0, which is optimal, by Theorem 2.5.

Theorem 16.13 *Let π be a 2-knot group with $\pi' \cong G_6$. Then $\pi \cong G(+)$ or $G(-)$. In each case π is virtually Z^4, $\pi' \cap \zeta\pi = 1$ and $\zeta\pi \cong Z$. Every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $x^{2n}t$, and every such element is a weight element.*

Proof Since $Out(G_6)$ is finite π is virtually $G_6 \times Z$ and hence is virtually Z^4. The groups $G(+)$ and $G(-)$ are the only orientable flat 4-manifold groups with $\pi/\pi' \cong Z$. The next assertion ($\pi' \cap \zeta\pi = 1$) follows as $\zeta G_6 = 1$. It is easily seen that $\zeta G(+)$ and $\zeta G(-)$ are generated by the images of t^3 and $t^6x^{-2}y^2(xy)^{-2}$, respectively, and so in each case $\zeta\pi \cong Z$.

If $t \in \pi$ represents a generator of π/π' it is a weight element, since π is solvable. We shall use the notation of §2 of Chapter 8 for automorphisms of G_6.

Suppose first that $\pi = G(+)$ and $c_t = ja$. If u is another weight element with the same image in π/π' then we may assume that $u = gt$ for some $g \in \pi'' = G_6'$, by Theorem 14.1. Suppose that $g = x^{2m}y^{2n}z^{2p}$. Let $\lambda(g) = m + n - p$ and $w = x^{2n}y^{2p}$. Then $w^{-1}gtw = x^{2\lambda(g)}t$. On the other hand, if $\psi \in Aut(G_6)$ then $\psi c_{gt}\psi^{-1} = c_{ht}$ for some $h \in G_6'$ if and only if the images of ψ and ja in $Aut(G_6)/G_6'$ commute. If so, ψ is in the subgroup generated by $\{def^{-1}, jb, ce\}$. It is easily verified that $\lambda(h) = \lambda(g)$ for any such ψ. (It suffices to check this for the generators of $Aut(G_6)$.) Thus $x^{2n}t$ is a weight element representing $[ja]$, for all $n \in \mathbb{Z}$, and $x^{2m}t$ and $x^{2n}t$ are in the same strict weight orbit if and only if $m = n$.

The proof is very similar when $\pi = G(-)$ and $c_t = jb$. The main change is that we should define the homomorphism λ by $\lambda(x^{2m}y^{2n}z^{2p}) = m - n + p$. □

The group $G(+)$ is the group of the 3-twist spin of the figure eight knot ($G(+) \cong \pi\tau_3 4_1$.) Although $G(-)$ is the group of a fibred 2-knot, by Theorem 14.4, it is not the group of any twist spin, by Theorem 16.4. We can also see this directly.

Corollary 16.13.1 *No 2-knot with group $G(-)$ is a twist-spin.*

Proof If $G(-) = \pi\tau_r K$ for some 1-knot K then the rth power of a meridian is central in $G(-)$. The power $(x^{2n}t)^r$ is central if and only if $(d^{2n}jb)^r = 1$ in $Aut(G_6)$. But $(d^{2n}jb)^3 = d^{2n}f^{2n}e^{-2n}(jb)^3 = (de^{-1}f)^{2n+1}$. Therefore $d^{2n}jb$ has infinite order, and so $G(-)$ is not the group of a twist-spin. □

Theorem 16.14 *Let π be a 2-knot group with $\pi' \cong \Gamma_q$ for some $q > 0$, and let θ be the image of the meridianal automorphism in $Out(\Gamma_q)$. Then either*

$q = 1$ and θ is conjugate to $[\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), 0]$ or $[\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), 0]$, or q is odd and θ is conjugate to $[\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), 0]$ or its inverse. Each such group π has two strict weight orbits.

Proof If (A, μ) is a meridianal automorphism of Γ_q the induced automorphisms of $\Gamma_q/\zeta\Gamma_q \cong Z^2$ and $tors(\Gamma_q/\Gamma_q') \cong Z/qZ$ are also meridianal, and so $\det(A-I) = \pm 1$ and $\det(A)-1$ is a unit $mod\ (q)$. Therefore q must be odd and $\det(A) = -1$ if $q > 1$, and the characteristic polynomial $\Delta_A(X)$ of A must be $X^2 - X + 1$, $X^2 - 3X + 1$, $X^2 - X - 1$ or $X^2 + X - 1$. Since the corresponding rings $\mathbb{Z}[X]/(\Delta_A(X))$ are isomorphic to $\mathbb{Z}[(1+\sqrt{-3})/2]$ or $\mathbb{Z}[(1+\sqrt{5})/2])$, which are PIDs, A is conjugate to one of $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, by Theorem 1.4. Now $[A, \mu][A, 0][A, \mu]^{-1} = [A, \mu(I - \det(A)A)^{-1}]$ in $Out(\Gamma_q)$. (See §7 of Chapter 8.) As in each case $I - \det(A)A$ is invertible, it follows that θ is conjugate to $[A, 0]$ or to $[A^{-1}, 0] = [A, 0]^{-1}$. Since $\pi'' \leq \zeta\pi'$ the final observation follows from Theorem 14.1. □

The groups Γ_q are discrete cocompact subgroups of the Lie group Nil^3 and the coset spaces are S^1-bundles over the torus. Every automorphism of Γ_q is orientation preserving and each of the groups allowed by Theorem 16.14 is the group of some fibred 2-knot, by Theorem 14.4. The automorphism $[\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), 0]$ is realised by $\tau_6 3_1$, and $M(\tau_6 3_1)$ is Seifert fibred over $S^2(2, 3, 6)$. In all the other cases θ has infinite order and the group is not the group of any twist spin. If $q > 1$ no such 2-knot is deform spun, by the criterion of [BM09], since $(t^2 - t - 1) \neq (t^{-2} - t^{-1} - 1)$ (as ideals of $\mathbb{Q}\Lambda$).

The 2-knot groups with commutator subgroup Γ_1 have presentations

$$\langle t, x, y \mid xyxy^{-1} = yxy^{-1}x,\ txt^{-1} = xy,\ tyt^{-1} = w \rangle,$$

where $w = x^{-1}$, xy^2 or x (respectively), while those with commutator subgroup Γ_q with $q > 1$ have presentations

$$\langle t, u, v, z \mid uvu^{-1}v^{-1} = z^q,\ tut^{-1} = v,\ tvt^{-1} = zuv,\ tzt^{-1} = z^{-1} \rangle.$$

(Note that as $[v, u] = t[u, v]t^{-1} = [v, zuv] = [v, z]z[v, u]z^{-1} = [v, z][v, u]$, we have $vz = zv$ and hence $uz = zu$ also.) These are easily seen to have 2 generator presentations of deficiency 0 also.

The other $\mathbb{N}il^3$-manifolds which arise as the closed fibres of fibred 2-knots are Seifert fibred over $S^2(3, 3, 3)$. (In all other cases the fundamental group has no meridianal automorphism.) These are 2-fold branched covers of $(S^3, k(e, \eta))$, where $k(e, \eta) = \mathfrak{m}(e; (3, 1), (3, 1), (3, \eta))$ is a Montesinos link, for some $e \in \mathbb{Z}$ and $\eta = \pm 1$ [Mo73, BZ]. If e is even this link is a knot, and is invertible, but not amphicheiral. (See §12E of [BZ].) Let $\pi(e, \eta) = \pi\tau_2 k(e, \eta)$.

Theorem 16.15 *Let π be a 2-knot group such that $\sqrt{\pi'} \cong \Gamma_q$ (for some $q \geq 1$) and $\pi'/\sqrt{\pi'} \cong Z/3Z$. Then q is odd and either*

(1) *$\pi = \pi(e, \eta)$, where e is even and $\eta = 1$ or -1, and has the presentation $\langle t, x, z \mid x^3 = (x^{3e-1}z^{-1})^3 = z^{3\eta},\ txt^{-1} = x^{-1},\ tzt^{-1} = z^{-1}\rangle$; or*

(2) *π has the presentation $\langle t, x \mid x^3 = t^{-1}x^{-3}t,\ x^{3e-1}t^{-1}xt = xtx^{-1}t^{-1}x^{-1}\rangle$, where e is even.*

Every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $u^n t$, where $u = z^{-1}x$ in case (1) and $u = t^{-1}xtx$ in case (2), and every such element is a weight element.

Proof It follows easily from Lemma 16.8 that $\zeta\sqrt{\pi'} = \zeta\pi'$ and $G = \pi'/\zeta\pi'$ is isomorphic to $\pi_1^{orb}(S^2(3,3,3)) \cong Z^2 \rtimes_{-B} (Z/3Z)$, where $-B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. As π' is a torsion-free central extension of G by Z it has a presentation

$$\langle h, x, y, z \mid x^3 = y^3 = z^{3\eta} = h,\ xyz = h^e\rangle$$

for some $\eta = \pm 1$ and $e \in \mathbb{Z}$. The image of h in π' generates $\zeta\pi'$, and the images of $u = z^{-1}x$ and $v = zuz^{-1}$ in $G = \pi'/\langle h\rangle$ form a basis for the translation subgroup $T(G) \cong Z^2$. Hence $\sqrt{\pi'}$ is generated by u, v and h. Since

$$vuv^{-1}u^{-1} = xz^{-2}xzx^{-2}z = xz^{-3}.zxzx.x^{-3}z = x.x^{-3}z^{-3}(y^{-1}h^e)^3x^{-1} = h^{3e-2-\eta},$$

we must have $q = |3e - 2 - \eta|$. Since π' admits a meridianal automorphism and $\pi'/(\pi')^2 \cong Z/(2, e-1)$, e must be even, and so q is odd. Moreover, $\eta = 1$ if and only if 3 divides q. In terms of the new generators, π' has the presentation

$$\langle u, v, z \mid zuz^{-1} = v,\ zvz^{-1} = u^{-1}v^{-1}z^{9e-6\eta-3},\ vuv^{-1}u^{-1} = z^{9e-6-3\eta}\rangle.$$

Since π'/π'' is finite, $Hom(\pi', \zeta\pi') = 0$, and so the natural homomorphism from $Aut(\pi')$ to $Aut(G)$ is injective. An automorphism ϕ of π' must preserve characteristic subgroups such as $\zeta\pi' = \langle z^3\rangle$ and $\sqrt{\pi'} = \langle u, v, z^3\rangle$. Let K be the subgroup of $Aut(\pi')$ consisting of automorphisms which induce the identity on the subquotients $\pi'/\sqrt{\pi'} \cong Z/3Z$ and $\sqrt{\pi'}/\zeta\pi' \cong Z^2$. Automorphisms in K also fix the centre, and are of the form $k_{m,n}$, where

$$k_{m,n}(u) = uz^{3\eta s},\ k_{m,n}(v) = vz^{3\eta t} \quad \text{and} \quad k_{m,n}(z) = z^{3\eta p+1}u^m v^n,$$

for $(m, n) \in \mathbb{Z}^2$. These formulae define an automorphism if and only if

$$s - t = -n(3e - 2 - \eta), \quad s + 2t = m(3e - 2 - \eta) \quad \text{and}$$

$$6p = (m + n)((m + n - 1)(3e - 2 - \eta) + 2(\eta - 1)).$$

In particular, the parameters m and n determine p, s and t. Conjugation by u and v give $c_u = k_{-2,-1}$ and $c_v = k_{1,-1}$, respectively. Let $k = k_{1,0}$. If $\eta = 1$

these equations have solutions $p, s, t \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$, and $K = \langle k, c_u \rangle$. If $\eta = -1$ we must have $m + n \equiv 0 \ mod\ (3)$, and $K = \langle c_u, c_v \rangle$. In each case, $K \cong \mathbb{Z}^2$. We may define further automorphisms b and r by setting

$$b(u) = v^{-1} z^{3\eta e - 3}, \quad b(v) = uvz^{3\eta(e-1)} \quad \text{and} \quad b(z) = z; \quad \text{and}$$

$$r(u) = v^{-1}, \quad r(v) = u^{-1} \quad \text{and} \quad r(z) = z^{-1}.$$

Then $b^6 = r^2 = (br)^2 = 1$ and $rk = kr$, while $b^4 = c_z$ is conjugation by z. If $\eta = -1$ then $Out(\pi') \cong (Z/2Z)^2$ is generated by the images of b and r, while if $\eta = 1$ then $Out(\pi') \cong S_3 \times Z/2Z$ is generated by the images of b, k and r. Since $bk^n r(u) \equiv u \ mod\ \pi''$, such automorphisms are not meridianal. Thus if $\eta = +1$ there are two classes of meridianal automorphisms (up to conjugacy and inversion in $Out(\pi')$), represented by r and kr, while if $\eta = -1$ there is only one, represented by r. It is easily seen that $\pi(e, \eta) \cong \pi' \rtimes_r Z$. If $\eta = 1$ and the meridianal automorphism is kr then $kr(x) = x^{-1} z x^{2-3e}$ and $kr(z) = x^{-1}$, which leads to the presentation (2).

In each case, π has an automorphism which is the identity on π' and sends t to th, and the argument of Theorem 16.13 may be adapted to prove the final assertion. $\square$

If $\pi \cong \pi(e, \eta)$ then $H_1(\pi; \Lambda/3\Lambda) \cong H_2(\pi; \Lambda/3\Lambda) \cong (\Lambda/(3, t+1))^2$, and so the presentation in (1) is optimal, by Lemma 16.11. The subgroup $A = \langle t^2, x^3 \rangle \cong \mathbb{Z}^2$ is normal but not central. The quotient $\pi(e, \eta)/A$ is $\pi_1^{orb}(\mathbb{D}^2(\overline{3}, \overline{3}, \overline{3}))$, and $M(\tau_2 k(e, \eta))$ is Seifert fibred over $\mathbb{D}^2(\overline{3}, \overline{3}, \overline{3})$.

The presentation in (2) is also optimal, by Theorem 3.6. Since $k^3 = c_u^{-1} c_v$ and $(kr)^6 = k^6$ in $Aut(\pi')$, $t^6 u^2 v^{-2}$ is central in π. The subgroup $A = \langle t^6 u^2 v^{-2}, x^3 \rangle \cong \mathbb{Z}^2$ is again normal but not central. The quotient π/A is $\pi_1^{orb}(\mathbb{D}^2(3, \overline{3}))$, and the knot manifold is Seifert fibred over $\mathbb{D}^2(3, \overline{3})$.

The knot $k(0, -1) = 9_{46}$ is doubly slice, and hence so are all of its twist spins. Since $M(\tau_2 k(0, -1))$ is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold it is determined up to homeomorphism by its group. Hence every knot with group $\pi(0, -1)$ must be doubly slice. No other non-trivial 2-knot with torsion-free, elementary amenable group is doubly slice. (The Alexander polynomials of Fox's knot, the Cappell-Shaneson 2-knots and the knots of Theorem 16.14 are irreducible and not constant, while the Farber-Levine pairings of the others are not hyperbolic.)

Chapter 17

Knot manifolds and geometries

In this chapter we shall attempt to characterize certain 2-knots in terms of algebraic invariants. As every 2-knot K may be recovered (up to orientations and Gluck reconstruction) from $M(K)$ together with the orbit of a weight class in $\pi = \pi K$ under the action of self homeomorphisms of M, we need to characterize $M(K)$ up to homeomorphism. After some general remarks on the algebraic 2-type in §1, and on surgery in §2, we shall concentrate on three special cases: when $M(K)$ is aspherical, when π' is finite and when $g.d.\pi = 2$.

When π is torsion-free and virtually poly-Z the surgery obstructions vanish, and when it is poly-Z the weight class is unique. The surgery obstruction groups are notoriously difficult to compute if π has torsion. However we can show that there are infinitely many distinct 2-knots K such that $M(K)$ is simple homotopy equivalent to $M(\tau_2 3_1)$; among these knots only $\tau_2 3_1$ has a minimal Seifert hypersurface. If $\pi = \Phi$ the homotopy type of $M(K)$ determines the exterior of the knot; the difficulty here is in finding a homotopy equivalence from $M(K)$ to a standard model.

In the final sections we shall consider which knot manifolds are homeomorphic to geometric 4-manifolds or complex surfaces. If $M(K)$ is geometric then either K is a Cappell-Shaneson knot or the geometry must be one of $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_1^4$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\mathbb{S}^3 \times \mathbb{E}^1$. If $M(K)$ is homeomorphic to a complex surface then either K is a branched twist spin of a torus knot or $M(K)$ admits one of the geometries $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.

17.1 Homotopy classification of $M(K)$

Let K and K_1 be 2-knots and suppose that $\alpha : \pi = \pi K \to \pi K_1$ and $\beta : \pi_2(M) \to \pi_2(M_1)$ determine an isomorphism of the algebraic 2-types of $M = M(K)$ and $M_1 = M(K_1)$. Since the infinite cyclic covers M' and M_1' are homotopy equivalent to 3-complexes there is a map $h : M' \to M_1'$ such that $\pi_1(h) = \alpha|_\pi$ and $\pi_2(h) = \beta$. If $\pi = \pi K$ has one end then $\pi_3(M) \cong \Gamma(\pi_2(M))$ and so h is a homotopy equivalence. Let t and $t_1 = \alpha(t)$ be corresponding generators of $Aut(M'/M)$ and $Aut(M_1'/M_1)$, respectively. Then $h^{-1}t_1^{-1}ht$ is a self homotopy equivalence of M' which fixes the algebraic 2-type. If this is homotopic to $id_{M'}$ then M and M_1 are homotopy equivalent, since up to homotopy they are the mapping tori of t and t_1, respectively. Thus the homotopy classification of such knot manifolds may be largely reduced to determining the obstructions to homotoping a self-map of a 3-complex to the identity.

We may use a similar idea to approach this problem in another way. Under the same hypotheses on K and K_1 there is a map $f_o : M \setminus intD^4 \to M_1$ inducing isomorphisms of the algebraic 2-types. If π has one end $\pi_3(f_o)$ is an epimorphism, and so f_o is 3-connected. If there is an extension $f : M \to M_1$ then it is a homotopy equivalence, as it induces isomorphisms on the homology of the universal covering spaces.

If $c.d.\pi \leq 2$ the algebraic 2-type is determined by π, for then $\pi_2(M) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Theorem 3.12, and the k-invariant is 0. In particular, if π' is free of rank r then $M(K)$ is homotopy equivalent to the mapping torus of a self-homeomorphism of $\sharp^r S^1 \times S^2$, by Corollary 4.5.1. If $\pi = \Phi$ then K is one of Fox's examples [Hi09].

The related problem of determining the homotopy type of the exterior of a 2-knot has been considered in [Lo81, Pl83] and [PS85]. The examples considered in [Pl83] do not test the adequacy of the algebraic 2-type for the present problem, as in each case either π' is finite or $M(K)$ is aspherical. The examples of [PS85] probably show that in general $M(K)$ is not determined by π and $\pi_2(M(K))$ alone.

17.2 Surgery

The natural transformations $I_G : G \to L_5^s(G)$ defined in Chapter 6 clearly factor through G/G'. If $\alpha : G \to Z$ induces an isomorphism on abelianization the homomorphism $\hat{I}_G = I_G\alpha^{-1}I_Z^{-1}$ is a canonical splitting for $L_5(\alpha)$.

Theorem 17.1 *Let K be a 2-knot. If $L_5^s(\pi K) \cong Z$ and N is simple homotopy equivalent to $M(K)$ then N is s-cobordant to $M(K)$.*

Proof Since $M = M(K)$ is orientable and $[M, G/TOP] \cong H^4(M;\mathbb{Z}) \cong Z$ the surgery obstruction map $\sigma_4 : [M(K), G/TOP] \to L_4^s(\pi K)$ is injective, by Theorem 6.6. The image of $L_5(Z)$ under $\hat{I}_{\pi K}$ acts trivially on $S_{TOP}^s(M(K))$, by Theorem 6.7. Hence there is a normal cobordism with obstruction 0 from any simple homotopy equivalence $f : N \to M$ to id_M. ∎

Corollary 17.1.1 (Freedman) *A 2-knot K is trivial if and only if $\pi K \cong Z$.*

Proof The condition is clearly necessary. Conversely, if $\pi K \cong Z$ then $M(K)$ is homeomorphic to $S^3 \times S^1$, by Theorem 6.11. Since the meridian is unique up to inversion and the unknot is clearly reflexive the result follows. □

Theorem 17.1 applies if π is a classical knot group [AFR97], or if $c.d.\pi = 2$ and π is square root closed accessible [Ca73].

Surgery on an s-concordance $\mathcal{K}$ from K_0 to K_1 gives an s-cobordism from $M(K_0)$ to $M(K_1)$ in which the meridians are conjugate. Conversely, if $M(K)$ and $M(K_1)$ are s-cobordant via such an s-cobordism then K_1 is s-concordant to K or K^*. In particular, if K is reflexive then K and K_1 are s-concordant.

The next lemma follows immediately from Perelman's work (see [B-P]). In the original version of this book, we used a 4-dimensional surgery argument. We retain the statement so that the numbering of the other results not be changed.

Lemma 17.2 *Let K be a 2-knot. Then K has a Seifert hypersurface which contains no fake 3-cells.* □

17.3 The aspherical cases

Whenever the group of a 2-knot K contains a sufficiently large abelian normal subgroup $M(K)$ is aspherical. This holds for most twist spins of prime knots.

Theorem 17.3 *Let K be a 2-knot with group $\pi = \pi K$. If $\sqrt{\pi}$ is abelian of rank 1 and $e(\pi/\sqrt{\pi}) = 1$ or if $h(\sqrt{\pi}) \geq 2$ then $\widetilde{M}(K)$ is homeomorphic to R^4.*

Proof If $\sqrt{\pi}$ is abelian of rank 1 and $\pi/\sqrt{\pi}$ has one end M is aspherical, by Theorem 15.5, and π is 1-connected at ∞, by Theorem 1 of [Mi87]. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$ and M is s-cobordant to the mapping torus of a self homeomorphism of a $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2. If $h(\sqrt{\pi}) \geq 3$ then M is homeomorphic to an infrasolvmanifold, by Theorem 8.1. In all cases, $\widetilde{M}$ is contractible and 1-connected at ∞, and so is homeomorphic to R^4 [FQ]. □

Is there a 2-knot K with $\widetilde{M}(K)$ contractible but not 1-connected at ∞?

Theorem 17.4 *Let K be a 2-knot such that $\pi = \pi K$ is torsion-free and virtually poly-Z. Then K is determined up to Gluck reconstruction by π together with a generator of $H_4(\pi; \mathbb{Z})$ and the strict weight orbit of a meridian.*

Proof If $\pi \cong Z$ then K must be trivial, and so we may assume that π is torsion-free and virtually poly-Z of Hirsch length 4. Hence $M(K)$ is aspherical and is determined up to homeomorphism by π, and every automorphism of π may be realized by a self homeomorphism of $M(K)$, by Theorem 6.11. Since $M(K)$ is aspherical its orientation is determined by a generator of $H_4(\pi;\mathbb{Z})$. $\square$

This theorem applies in particular to the Cappell-Shaneson 2-knots, which have an unique strict weight orbit, up to inversion. (A similar argument applies to Cappell-Shaneson n-knots with $n > 2$, provided we assume also that $\pi_i(X(K)) = 0$ for $2 \leq i \leq (n+1)/2$.)

Theorem 17.5 *Let K be a 2-knot with group $\pi = \pi K$. Then K is s-concordant to a fibred knot with closed fibre a $\widetilde{\mathbb{SL}}$-manifold if and only if π' is finitely generated, $\zeta\pi' \cong Z$ and π is not virtually solvable. The fibred knot is determined up to Gluck reconstruction by π together with a generator of $H_4(\pi;\mathbb{Z})$ and the strict weight orbit of a meridian.*

Proof The conditions are clearly necessary. If they hold then $M(K)$ is aspherical, by Theorem 15.8.(2), so every automorphism of π is induced by a self homotopy equivalence of $M(K)$. Moreover π' is a PD_3-group, by Theorem 4.5.(3). As $\zeta\pi' \cong Z$ and π is not virtually solvable, π' is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold, by Lemma 16.1. Therefore $M(K)$ is determined up to s-cobordism by π, by Theorem 13.2. The rest is standard. $\square$

Branched twist spins of torus knots are perhaps the most important examples of such knots, but there are others. (See Chapter 16.) Is every 2-knot K such that $\pi = \pi K$ is a PD_4^+-group determined up to s-concordance and Gluck reconstruction by π together with a generator of $H_4(\pi;\mathbb{Z})$ and a strict weight orbit? Is K s-concordant to a fibred knot with aspherical closed fibre if and only if π' is finitely generated and has one end? (This follows from [Ro11] and Theorem 6.8 if $Wh(\pi) = 0$.)

17.4 Quasifibres and minimal Seifert hypersurfaces

Let M be a closed 4-manifold with fundamental group π. If $f : M \to S^1$ is a map which is transverse to $p \in S^1$ then $\widehat{V} = f^{-1}(p)$ is a codimension 1 submanifold with a product neighbourhood $N \cong \widehat{V} \times [-1,1]$. If moreover the induced homomorphism $f_* : \pi \to Z$ is an epimorphism and each of the inclusions $j_\pm : \widehat{V} \cong \widehat{V} \times \{\pm 1\} \subset W = M \setminus V \times (-1,1)$ induces monomorphisms on fundamental groups then we shall say that $\widehat{V}$ is a *quasifibre* for f. The group π is then an HNN extension with base $\pi_1(W)$ and associated subgroups $j_{\pm*}(\pi_1(\widehat{V})$, by Van Kampen's Theorem. Every fibre of a bundle projection is a quasifibre. We may use the notion of quasifibre to interpolate between the homotopy fibration theorem of Chapter 4 and a TOP fibration theorem. (See also Theorem 6.12 and Theorem 17.7.)

Theorem 17.6 *Let M be a closed 4-manifold with $\chi(M)=0$ and such that $\pi=\pi_1(M)$ is an extension of Z by a finitely generated normal subgroup ν. If there is a map $f:M\to S^1$ inducing an epimorphism with kernel ν and which has a quasifibre $\widehat{V}$ then the infinite cyclic covering space M_ν associated with ν is homotopy equivalent to $\widehat{V}$.*

Proof As ν is finitely generated the monomorphisms $j_{\pm*}$ must be isomorphisms. Therefore ν is finitely presentable, and so M_ν is a PD_3-complex, by Theorem 4.5. Now $M_\nu\cong W\times Z/\sim$, where $(j_+(v),n)\sim(j_-(v),n+1)$ for all $v\in\widehat{V}$ and $n\in Z$. Let $\tilde{j}(v)$ be the image of $(j_+(v),0)$ in M_ν. Then $\pi_1(\tilde{j})$ is an isomorphism. A Mayer-Vietoris argument shows that $\tilde{j}$ has degree 1, and so $\tilde{j}$ is a homotopy equivalence. □

One could use duality instead to show that $H_s=H_s(W,\partial_\pm W;\mathbb{Z}[\pi])=0$ for $s\neq 2$, while H_2 is a stably free $\mathbb{Z}[\pi]$-module, of rank $\chi(W,\partial_\pm W)=0$. Since $\mathbb{Z}[\pi]$ is weakly finite this module is 0, and so W is an h-cobordism.

Corollary 17.6.1 *Let K be a 2-knot with group $\pi=\pi K$. If π' is finitely generated and K has a minimal Seifert hypersurface V such that every self homotopy equivalence of $\widehat{V}$ is homotopic to a homeomorphism then $M(K)$ is homotopy equivalent to $M(K_1)$, where $M(K_1)$ is a fibred 2-knot with fibre V.*

Proof Let $j_+^{-1}:M(K)'\to\widehat{V}$ be a homotopy inverse to the homotopy equivalence j_+, and let θ be a self homeomorphism of $\widehat{V}$ homotopic to $j_+^{-1}j_-$. Then $j_+\theta j_+^{-1}$ is homotopic to a generator of $Aut(M(K)'/M(K))$, and so the mapping torus of θ is homotopy equivalent to $M(K)$. Surgery on this mapping torus gives such a knot K_1. □

If a Seifert hypersurface V for a 2-knot has fundamental group Z then the Mayer-Vietoris sequence for $H_*(M(K);\Lambda)$ gives $H_1(X')\cong\Lambda/(ta_+-a_-)$, where $a_\pm:H_1(V)\to H_1(S^4\setminus V)$. Since $H_1(X)=Z$ we must have $a_+-a_-=\pm1$. If $a_+a_-\neq 0$ then V is minimal. However one of a_+ or a_- could be 0, in which case V may not be minimal. The group Φ is realized by ribbon knots with such minimal Seifert hypersurfaces (homeomorphic to $S^2 \times S^1 \setminus intD^3$) [Fo62]. Thus minimality does not imply that π' is finitely generated.

If a 2-knot K has a minimal Seifert hypersurface then πK is an HNN extension with finitely presentable base and associated subgroups. It remains an open question whether every 2-knot group is of this type. (There are high dimensional knot groups which are not so [Si91, Si96].) Yoshikawa has shown that there are ribbon 2-knots whose groups are HNN extensions with base a torus knot group and associated subgroups Z but which cannot be expressed as HNN extensions with base a free group [Yo88].

An argument of Trace implies that if V is a Seifert hypersurface for a fibred n-knot K then there is a degree-1 map from $\widehat{V} = V \cup D^{n+1}$ to the closed fibre $\widehat{F}$ [Tr86]. For the embedding of V in X extends to an embedding of $\widehat{V}$ in M, which lifts to an embedding in M'. Since the image of $[\widehat{V}]$ in $H_{n+1}(M;\mathbb{Z})$ is Poincaré dual to a generator of $H^1(M;\mathbb{Z}) = Hom(\pi,\mathbb{Z}) = [M,S^1]$ its image in $H_{n+1}(M';\mathbb{Z}) \cong Z$ is a generator. In particular, if K is a fibred 2-knot and $\widehat{F}$ has a summand which is aspherical or whose fundamental group is a nontrivial finite group then $\pi_1(V)$ cannot be free. Similarly, as the Gromov norm of a 3-manifold does not increase under degree 1 maps, if $\widehat{F}$ is a $\mathbb{H}^3$-manifold then $\widehat{V}$ cannot be a graph manifold [Ru90]. Rubermann observes also that the "Seifert volume" of [BG84] may be used instead to show that if $\widehat{F}$ is a $\widetilde{\mathbb{SL}}$-manifold then $\widehat{V}$ must have nonzero Seifert volume. (Connected sums of $\mathbb{E}^3$-, $\mathbb{S}^3$-, $\mathbb{N}il^3$-, $\mathbb{S}ol^3$-, $\mathbb{S}^2 \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds all have Seifert volume 0 [BG84].)

17.5 The spherical cases

Let π be a 2-knot group with commutator subgroup $\pi' \cong P \times (Z/(2r+1)Z)$, where $P = 1$, $Q(8)$, T_k^* or I^*. The meridianal automorphism induces the identity on the set of irreducible real representations of π', except when $P = Q(8)$. (It permutes the three nontrivial 1-dimensional representations when $\pi' \cong Q(8)$, and similarly when $\pi' \cong Q(8) \times (Z/nZ)$.) It then follows as in Chapter 11 that $L_5^s(\pi)$ has rank $r+1$, $3(r+1)$, $3^{k-1}(5+7r)$ or $9(r+1)$, respectively. Hence if $\pi' \neq 1$ then there are infinitely many distinct 2-knots with group π, since the group of self homotopy equivalences of $M(K)$ is finite.

The simplest nontrivial such group is $\pi = (Z/3Z) \rtimes_{-1} Z$. If K is any 2-knot with this group then $M(K)$ is homotopy equivalent to $M(\tau_2 3_1)$. Since $Wh(Z/3Z) = 0$ [Hg40] and $L_5(Z/3Z) = 0$ [Ba75] we have $L_5^s(\pi) \cong L_4(\pi') \cong Z^2$, but we do not know whether $Wh(\pi) = 0$.

Theorem 17.7 *Let K be a 2-knot with group $\pi = \pi K$ such that $\pi' \cong Z/3Z$, and which has a minimal Seifert hypersurface. Then $K = \tau_2 3_1$.*

Proof Let V be a minimal Seifert hypersurface for K. Let $\widehat{V} = V \cup D^3$ and $W = M(K) \backslash V \times (-1, 1)$. Then W is an h-cobordism from $\widehat{V}$ to itself. (See the remark following Theorem 6.) Therefore $W \cong \widehat{V} \times I$, by surgery over $Z/3Z$. (Note that $Wh(Z/3Z) = L_5(Z/3Z) = 0$.) Hence M fibres over S^1. The closed fibre must be the lens space $L(3, 1)$, by Perelman's work (see [B-P]), and so K must be $\tau_2 3_1$. □

As none of the other 2-knots with this group has a minimal Seifert surface, they are all further counter-examples to the most natural 4-dimensional analogue of Farrell's fibration theorem. We do not know whether any of these knots (other than $\tau_2 3_1$) is PL in some PL structure on S^4.

Let F be an $\mathbb{S}^3$-group, and let $W = (W; j_\pm)$ be an h-cobordism with homeomorphisms $j_\pm : N \to \partial_\pm W$, where $N = S^3/F$. Then W is an s-cobordism [KS92]. The set of such s-cobordisms from N to itself is a finite abelian group with respect to stacking of cobordisms. All such s-cobordisms are products if F is cyclic, but there are nontrivial examples if $F \cong Q(8) \times (Z/pZ)$, for any odd prime p [KS95]. If ϕ is a self-homeomorphism of N the closed 4-manifold Z_ϕ obtained by identifying the ends of W via $j_+ \phi j_-^{-1}$ is homotopy equivalent to $M(\phi)$. However if Z_ϕ is a mapping torus of a self-homeomorphism of N then W is trivial. In particular, if ϕ induces a meridianal automorphism of F then $Z_\phi \cong M(K)$ for an exotic 2-knot K with $\pi' \cong F$ and which has a minimal Seifert hypersurface, but which is not fibred with geometric fibre.

17.6 Finite geometric dimension 2

Knot groups with finite 2-dimensional Eilenberg-Mac Lane complexes have deficiency 1, by Theorem 2.8, and so are 2-knot groups. This class includes all classical knot groups, all knot groups with free commutator subgroup and all knot groups in the class $\mathcal{X}$ (such as those of Theorems 15.1 and 15.14).

Theorem 17.8 *Let K be a 2-knot. If πK is a 1-knot group or an $\mathcal{X}$-group then $M(K)$ is determined up to s-cobordism by its homotopy type.*

Proof This is an immediate consequence of Lemma 6.9, if π is an $\mathcal{X}$-group. If π is a nontrivial classical knot group it follows from Theorem 17.1, since $Wh(\pi K) = 0$ [Wd78] and $L_5^s(\pi K) \cong Z$ [AFR97]. □

Corollary 17.8.1 *Any 2-knot K with group Φ is ambient isotopic to one of Fox's examples.*

Proof In [Hi09] it is shown that the homotopy type of a closed orientable 4-manifold M with $\pi_1(M)=\Phi$ and $\chi(M)=0$ is unique. Since Φ is metabelian s-cobordism implies homeomorphism and there is an unique weight class up to inversion, so $X(K)$ is determined by the homotopy type of $M(K)$. Since Examples 10 and 11 of [Fo62] are ribbon knots they are $-$amphicheiral and determined by their exterior. □

Fox's examples are mirror images. Since their Alexander polynomials are asymmetric, they are not invertible. (Thus they are not isotopic.) Nor are they deform spins, by the criterion of [BM09].

Theorem 17.9 *A 2-knot K with group $\pi=\pi K$ is s-concordant to a fibred knot with closed fibre $\sharp^r(S^1\times S^2)$ if and only if $\mathrm{def}(\pi)=1$ and π' is finitely generated. Moreover any such fibred 2-knot is reflexive and homotopy ribbon.*

Proof The conditions are clearly necessary. If they hold then $\pi'\cong F(r)$, for some $r\geq 0$, by Corollary 4.3.1. Then $M=M(K)$ is homotopy equivalent to a PL 4-manifold N which fibres over S^1 with fibre $\sharp^r(S^1\times S^2)$, by Corollary 4.5.1. Moreover $Wh(\pi)=0$, by Lemma 6.3, and π is square root closed accessible, so I_π is an isomorphism, by Lemma 6.9, so there is an s-cobordism W from M to N, by Theorem 17.1. We may embed an annulus $A=S^1\times[0,1]$ in W so that $M\cap A=S^1\times\{0\}$ is a meridian for K and $N\cap A=S^1\times\{1\}$. Surgery on A in W then gives an s-concordance from K to such a fibred knot K_1, which is reflexive [Gl62] and homotopy ribbon [Co83]. □

Based self homeomorphisms of $N=\sharp^r(S^1\times S^2)$ which induce the same automorphism of $\pi_1(N)=F(r)$ are conjugate up to isotopy if their images in $E_0(N)$ are conjugate [HL74]. Their mapping tori are then orientation-preserving homeomorphic. (Compare Corollary 4.5.) The group $E_0(N)$ is a semidirect product $(Z/2Z)^r\rtimes Aut(F(r))$, with the natural action [Hn]. It follows easily that every fibred 2-knot with π' free is determined (among such knots) by its group together with the weight orbit of a meridian. (However, $\pi 3_1$ has infinitely many weight orbits [Su85].) Is every such group the group of a ribbon knot?

If $K=\sigma k$ is the Artin spin of a fibred 1-knot then $M(K)$ fibres over S^1 with fibre $\sharp^r(S^2\times S^1)$. However not all such fibred 2-knots arise in this way. It follows easily from Lemma 1.1 and the fact that $Out(F(2))\cong GL(2,\mathbb{Z})$ that there are just three knot groups $G \cong F(2) \rtimes Z$, namely $\pi 3_1$ (the trefoil knot group), $\pi 4_1$ (the figure eight knot group) and the group with presentation

$$\langle x, y, t \mid txt^{-1} = y,\ tyt^{-1} = xy \rangle.$$

(Two of the four presentations given in [Rp60] present isomorphic groups.) The Alexander polynomial of the latter example is $t^2 - t - 1$, which is not symmetric, and so this is not a classical knot group. (See also [AY81, Rt83].)

Theorem 17.10 *Let K be a 2-knot with group $\pi = \pi K$. Then*

(1) *if $c.d.\pi \leq 2$ then K is π_1-slice;*

(2) *if K is π_1-slice then $H_3(c_{M'(K)}; \mathbb{Z}) = 0$;*

(3) *if π' is finitely generated then K is π_1-slice if and only if π' is free;*

Proof Let $M = M(K)$. If $c.d.\pi \leq 2$ then $H_p(\pi; \Omega_{4-p}^{Spin}) = 0$ for $p > 0$, and so signature gives an isomorphism $\Omega_4^{Spin}(K(\pi,1)) \cong \mathbb{Z}$. Hence $M = \partial W$ for some *Spin* 5-manifold W with a map f to $K(\pi,1)$ such that $f|_M = c_M$. We may modify W by elementary surgeries to make $\pi_1(f)$ an isomorphism and $H_2(W; \mathbb{Z}) = 0$. Attaching a 2-handle to W along a meridian for K then gives a contractible 5-manifold, and so K is π_1-slice.

Let R be an open regular neighbourhood in D^5 of a π_1-slice disc Δ. Since c_M factors through $D^5 \setminus R$ the first assertion follows from the exact sequence of homology (with coefficients Λ) for the pair $(D^5 \setminus R, M)$.

If π' is finitely generated then M' is a PD_3-space, by Theorem 4.5. The image of $[M']$ in $H_3(\pi'; \mathbb{Z})$ determines a projective homotopy equivalence of modules $C^2/\partial^1(C^1) \simeq A(\pi')$, by the argument of Theorem 4 of [Tu90]. (This does not need Turaev's assumption that π' be finitely presentable.) If this image is 0 then $id_{A(\pi')} \sim 0$, so $A(\pi')$ is projective and $c.d.\pi' \leq 1$. Therefore π' is free. The converse follows from Theorem 17.9, or from part (1) of this theorem. □

If π' is free must K be homotopy ribbon? This would follow from "homotopy connectivity implies geometric connectivity", but our situation is just beyond the range of known results. Is $M(K)$ determined up to s-cobordism by its group whenever $g.d.\pi \leq 2$? If $g.d.\pi \leq 2$ then $\mathrm{def}(\pi) = 1$, and so π is the group of a homotopy ribbon 2-knot. (See §6 of Chapter 14.) The conditions "$g.d.\pi \leq 2$" and "$\mathrm{def}(\pi) = 1$" are equivalent if the Whitehead Conjecture holds. (See §9 of Chapter 14.) Are these conditions also equivalent to "homotopy ribbon"? (See [Hi08'] for further connections between these properties.)

17.7 Geometric 2-knot manifolds

The 2-knots K for which $M(K)$ is homeomorphic to an infrasolvmanifold are essentially known. There are three other geometries which may be realized by such knot manifolds. All known examples are fibred, and most are derived from twist spins of classical knots. However there are examples (for instance, those with $\pi' \cong Q(8) \times (Z/nZ)$ for some $n > 1$) which cannot be constructed from twist spins. The remaining geometries may be eliminated very easily; only $\mathbb{H}^2 \times \mathbb{E}^2$ and $\mathbb{S}^2 \times \mathbb{E}^2$ require a little argument.

Theorem 17.11 *Let K be a 2-knot with group $\pi = \pi K$. If $M(K)$ admits a geometry then the geometry is one of $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\mathbb{S}ol_{m,n}^4$ (for certain $m \neq n$ only), $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. All these geometries occur.*

Proof The knot manifold $M(K)$ is homeomorphic to an infrasolvmanifold if and only if $h(\sqrt{\pi}) \geq 3$, by Theorem 8.1. It is then determined up to homeomorphism by π. We may then use the observations of §10 of Chapter 8 to show that $M(K)$ admits a geometry of solvable Lie type. By Lemma 16.7 and Theorems 16.12 and 16.14 π must be either $G(+)$ or $G(-)$, $\pi(e,\eta)$ for some even b and $\epsilon = \pm 1$ or $\pi' \cong Z^3$ or Γ_q for some odd q. We may identify the geometry on looking more closely at the meridianal automorphism.

If $\pi \cong G(+)$ or $G(-)$ then $M(K)$ admits the geometry $\mathbb{E}^4$. If $\pi \cong \pi(e,\eta)$ then $M(K)$ is the mapping torus of an involution of a $\mathbb{N}il^3$-manifold, and so admits the geometry $\mathbb{N}il^3 \times \mathbb{E}^1$. If $\pi' \cong Z^3$ then $M(K)$ is homeomorphic to a $\mathbb{S}ol_{m,n}^4$- or $\mathbb{S}ol_0^4$-manifold. More precisely, we may assume (up to change of orientations) that the Alexander polynomial of K is $t^3 - (m-1)t^2 + mt - 1$ for some integer m. If $m \geq 6$ all the roots of this cubic are positive and the geometry is $\mathbb{S}ol_{m-1,m}^4$. If $0 \leq m \leq 5$ two of the roots are complex conjugates and the geometry is $\mathbb{S}ol_0^4$. If $m < 0$ two of the roots are negative and π has a subgroup of index 2 which is a discrete cocompact subgroup of $Sol_{m',n'}^4$, where $m' = m^2 - 2m + 2$ and $n' = m^2 - 4m + 1$, so the geometry is $\mathbb{S}ol_{m',n'}^4$.

If $\pi' \cong \Gamma_q$ and the image of the meridianal automorphism in $Out(\Gamma_q)$ has finite order then $q = 1$ and $K = \tau_6 3_1$ or $(\tau_6 3_1)^* = \tau_{6,5} 3_1$. In this case $M(K)$ admits the geometry $\mathbb{N}il^3 \times \mathbb{E}^1$. Otherwise (if $\pi' \cong \Gamma_q$ and the image of the meridianal automorphism in $Out(\Gamma_q)$ has infinite order) $M(K)$ admits the geometry $\mathbb{S}ol_1^4$.

If K is a branched r-twist spin of the (p,q)-torus knot then $M(K)$ is a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold if $p^{-1} + q^{-1} + r^{-1} > 1$, and is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold if $p^{-1} + q^{-1} + r^{-1} < 1$.

(The case $p^{-1}+q^{-1}+r^{-1}=1$ gives the $\mathbb{N}il^3\times\mathbb{E}^1$-manifold $M(\tau_6 3_1)$.) The manifolds obtained from 2-twist spins of 2-bridge knots and certain other "small" simple knots also have geometry $\mathbb{S}^3\times\mathbb{E}^1$. Branched r-twist spins of simple (nontorus) knots with $r>2$ give $\mathbb{H}^3\times\mathbb{E}^1$-manifolds, excepting $M(\tau_3 4_1)\cong M(\tau_{3,2}4_1)$, which is the $\mathbb{E}^4$-manifold with group $G(+)$.

Every orientable $\mathbb{H}^2\times\mathbb{E}^2$-manifold is double covered by a Kähler surface [Wl86]. Since the unique double cover of a 2-knot manifold $M(K)$ has first Betti number 1 no such manifold can be an $\mathbb{H}^2\times\mathbb{E}^2$-manifold. (If K is fibred we could instead exclude this geometry by Lemma 16.1.) Since π is infinite and $\chi(M(K))=0$ we may exclude the geometries $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2\times\mathbb{S}^2$, and $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^2\times\mathbb{H}^2$ and $\mathbb{S}^2\times\mathbb{H}^2$, respectively. The geometry $\mathbb{S}^2\times\mathbb{E}^2$ may be excluded by Theorem 10.10 or Lemma 16.1 (no group with two ends admits a meridianal automorphism), while $\mathbb{F}^4$ is not realized by any closed 4-manifold. $\square$

In particular, no knot manifold is a $\mathbb{N}il^4$-manifold or a $\mathbb{S}ol^3\times\mathbb{E}^1$-manifold, and many $\mathbb{S}ol^4_{m,n}$-geometries do not arise in this way. The knot manifolds which are infrasolvmanifolds or have geometry $\mathbb{S}^3\times\mathbb{E}^1$ are essentially known, by Theorems 8.1, 11.1, 15.12 and §4 of Chapter 16. The knot is uniquely determined up to Gluck reconstruction and change of orientations if $\pi'\cong Z^3$ (see Theorem 17.4 and the subsequent remarks above), Γ_q (see §3 of Chapter 18) or $Q(8)\times(Z/nZ)$ (since the weight class is then unique up to inversion). There are only six knots whose knot manifold admits the geometry $\mathbb{S}ol^4_0$, for $Z[X]/(\Delta_a(X))$ is a PID if $0\leq a\leq 5$. (See the tables of [AR84].) If the knot is fibred with closed fibre a lens space it is a 2-twist spin of a 2-bridge knot [Tr89]. The other knot groups corresponding to infrasolvmanifolds have infinitely many weight orbits.

Corollary 17.11.1 *If $M(K)$ admits a geometry then it fibres over S^1, and if π is not solvable the closed monodromy has finite order.*

Proof This is clear if $M(K)$ is an $\mathbb{S}^3\times\mathbb{E}^1$-manifold or an infrasolvmanifold, and follows from Corollary 13.1.1 and Theorem 16.2 if the geometry is $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$.

If $M(K)$ is a $\mathbb{H}^3\times\mathbb{E}^1$-manifold we refine the argument of Theorem 9.3. Since $\pi/\pi'\cong Z$ and $\sqrt{\pi}=\pi\cap(\{1\}\times R)\neq 0$ we may assume $\pi\leq Isom(\mathbb{H}^3)\times R$, and so $\pi'\leq Isom(\mathbb{H}^3)\times\{1\}$. Hence π' is the fundamental group of a closed $\mathbb{H}^3$-manifold, N say, and $M(K)$ is the quotient of $N\times\mathbb{R}$ by the action induced by a meridian. Thus $M(K)$ is a mapping torus, and so fibres over S^1. $\square$

If the geometry is $\mathbb{H}^3\times\mathbb{E}^1$ is $M(K)\cong M(K_1)$ for some branched twist spin of a simple non-torus knot? (See §3 of Chapter 16.)

Corollary 17.11.2 *If $M(K)$ is Seifert fibred it is a $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$-, $\mathbb{N}il^3\times\mathbb{E}^1$- or $\mathbb{S}^3\times\mathbb{E}^1$-manifold.*

Proof This follows from Ue's Theorem, Theorem 16.2 and Theorem 17.11. □

Are any 2-knot manifolds M total spaces of orbifold bundles with hyperbolic general fibre? (The base B must be flat, since $\chi(M)=0$, and $\pi_1^{orb}(B)$ must have cyclic abelianization. Hence $B=S^2(2,3,6)$, $\mathbb{D}^2(\overline{3},\overline{3},\overline{3})$, or $\mathbb{D}^2(3,\overline{3})$.)

It may be shown that if k is a nontrivial 1-knot and $r\geq 2$ then $M(\tau_r k)$ is geometric if and only if k is simple, and has a proper geometric decomposition if and only if k is prime but not simple. (The geometries of the pieces are then $\mathbb{H}^2\times\mathbb{E}^2$, $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$ or $\mathbb{H}^3\times\mathbb{E}^1$.) This follows from the fact that the r-fold cyclic branched cover of (S^3,k) admits an equivariant JSJ decomposition, and has finitely generated π_2 if and only if k is prime.

17.8 Complex surfaces and 2-knot manifolds

If a complex surface S is homeomorphic to a 2-knot manifold $M(K)$ then S is minimal, since $\beta_2(S)=0$, and has Kodaira dimension $\kappa(S)=1$, 0 or $-\infty$, since $\beta_1(S)=1$ is odd. If $\kappa(S)=1$ or 0 then S is elliptic and admits a compatible geometric structure, of type $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$ or $\mathbb{N}il^3\times\mathbb{E}^1$, respectively [Ue90,91, Wl86]. The only complex surfaces with $\kappa(S)=-\infty$, $\beta_1(S)=1$ and $\beta_2(S)=0$ are Inoue surfaces, which are not elliptic, but admit compatible geometries of type $\mathbb{S}ol_0^4$ or $\mathbb{S}ol_1^4$, and Hopf surfaces [Tl94]. An elliptic surface with Euler characteristic 0 has no exceptional fibres other than multiple tori.

If $M(K)$ has a complex structure compatible with a geometry then the geometry is one of $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}^3\times\mathbb{E}^1$ or $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$, by Theorem 4.5 of [Wl86]. Conversely, if $M(K)$ admits one of the first three of these geometries then it is homeomorphic to an Inoue surface of type S_M, an Inoue surface of type $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$, or an elliptic surface of Kodaira dimension 0, respectively. (See [In74, EO94] and Chapter V of [BHPV].)

Lemma 17.12 *Let K be a branched r-twist spin of the (p,q)-torus knot. Then $M(K)$ is homeomorphic to an elliptic surface.*

Proof We shall adapt the argument of Lemma 1.1 of [Mi75]. (See also [Ne83].) Let $V_0=\{(z_1,z_2,z_3)\in C^3\setminus\{0\}|z_1^p+z_2^q+z_3^r=0\}$, and define an action of

$\mathbb{C}^\times$ on V_0 by $u.v = (u^{qr}z_1, u^{pr}z_2, u^{pq}z_3)$ for all u in $\mathbb{C}^\times$ and $v = (z_1, z_2, z_3)$ in V_0. Define functions $m : V_0 \to R^+$ and $n : V_0 \to m^{-1}(1)$ by $m(v) = (|z_1|^p + |z_2|^q + |z_3|^r)^{1/pqr}$ and $n(v) = m(v)^{-1}.v$ for all v in V_0. Then the map $(m, n) : V_0 \to m^{-1}(1) \times R^+$ is an R^+-equivariant homeomorphism, and so $m^{-1}(1)$ is homeomorphic to V_0/R^+. Therefore there is a homeomorphism from $m^{-1}(1)$ to the Brieskorn manifold $M(p, q, r)$, under which the action of the group of r^{th} roots of unity on $m^{-1}(1) = V_0/R^+$ corresponds to the group of covering homeomorphisms of $M(p, q, r)$ as the branched cyclic cover of S^3, branched over the (p, q)-torus knot [Mi75]. The manifold $M(K)$ is the mapping torus of some generator of this group of self homeomorphisms of $M(p, q, r)$. Let ω be the corresponding primitive r^{th} root of unity. If $t > 1$ then $t\omega$ generates a subgroup Ω of $\mathbb{C}^\times$ which acts freely and holomorphically on V_0, and the quotient V_0/Ω is an elliptic surface over the curve V_0/Ω. Moreover V_0/Ω is homeomorphic to the mapping torus of the self homeomorphism of $m^{-1}(1)$ which maps v to $m(t\omega.v)^{-1}.t\omega.v = \omega m(t.v)^{-1}t.v$. Since this map is isotopic to the map sending v to $\omega.v$ this mapping torus is homeomorphic to $M(K)$. □

The Kodaira dimension of the elliptic surface in the above lemma is 1, 0 or $-\infty$ according as $p^{-1} + q^{-1} + r^{-1}$ is < 1, 1 or > 1. In the next theorem we shall settle the case of elliptic surfaces with $\kappa = -\infty$.

Theorem 17.13 *Let K be a 2-knot. Then $M(K)$ is homeomorphic to a Hopf surface if and only if K or its Gluck reconstruction is a branched r-twist spin of the (p, q)-torus knot for some p, q and r such that $p^{-1} + q^{-1} + r^{-1} > 1$.*

Proof If $K = \tau_{r,s}k_{p,q}$ then $M(K)$ is homeomorphic to an elliptic surface, by Lemma 17.13, and the surface must be a Hopf surface if $p^{-1} + q^{-1} + r^{-1} > 1$.

If $M(K)$ is homeomorphic to a Hopf surface then π has two ends, and there is a monomorphism $h : \pi = \pi K \to GL(2, \mathbb{C})$ onto a subgroup which contains a contraction c (Kodaira - see [Kt75]). Hence π' is finite and $h(\pi') = h(\pi) \cap SL(2, \mathbb{C})$, since $det(c) \neq 1$ and $\pi/\pi' \cong Z$. Finite subgroups of $SL(2, \mathbb{C})$ are conjugate to subgroups of $SU(2) = S^3$, and so are cyclic, binary dihedral or isomorphic to T_1^*, O_1^* or I^*. Therefore $\pi \cong \pi\tau_2 k_{2,n}$, $\pi\tau_3 3_1$, $\pi\tau_4 3_1$ or $\pi\tau_5 3_1$, by Theorem 15.12 and the subsequent remarks. Hopf surfaces with $\pi \cong Z$ or π nonabelian are determined up to diffeomorphism by their fundamental groups, by Theorem 12 of [Kt75]. Therefore $M(K)$ is homeomorphic to the manifold of the corresponding torus knot. If π' is cyclic there is an unique weight orbit. The weight orbits of $\tau_4 3_1$ are realized by $\tau_2 k_{3,4}$ and $\tau_4 3_1$, while the weight orbits of T_1^* are realized by $\tau_2 k_{3,5}$, $\tau_3 k_{2,5}$, $\tau_5 3_1$ and $\tau_{5,2} 3_1$ [PS87]. Therefore K agrees up to Gluck reconstruction with a branched twist spin of a torus knot. □

The Gluck reconstruction of a branched twist spin of a classical knot is another branched twist spin of that knot, by §6 of [Pl84'].

Elliptic surfaces with $\beta_1 = 1$ and $\kappa = 0$ are $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds, and so a knot manifold $M(K)$ is homeomorphic to such an elliptic surface if and only if πK is virtually poly-Z and $\zeta\pi K \cong Z^2$. For minimal properly elliptic surfaces (those with $\kappa = 1$) we must settle for a characterization up to s-cobordism.

Theorem 17.14 *Let K be a 2-knot with group $\pi = \pi K$. Then $M(K)$ is s-cobordant to a minimal properly elliptic surface if and only if $\zeta\pi \cong Z^2$ and π' is not virtually poly-Z.*

Proof If $M(K)$ is a minimal properly elliptic surface then it admits a compatible geometry of type $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ and π is isomorphic to a discrete cocompact subgroup of $Isom_o(\widetilde{\mathbb{SL}}) \times R$, the maximal connected subgroup of $Isom_o(\widetilde{\mathbb{SL}} \times \mathbb{E}^1)$, for the other components consist of orientation reversing or antiholomorphic isometries. (See Theorem 3.3 of [Wl86].) Since π meets $\zeta(Isom_o(\widetilde{\mathbb{SL}}) \times R)) \cong R^2$ in a lattice subgroup $\zeta\pi \cong Z^2$ and projects nontrivially onto the second factor $\pi' = \pi \cap Isom_o(\widetilde{\mathbb{SL}})$ and is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold. Thus the conditions are necessary.

Suppose that they hold. Then $M(K)$ is s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold which is the mapping torus $M(\Theta)$ of a self homeomorphism of a $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2. As Θ must be orientation preserving and induce the identity on $\zeta\pi' \cong Z$ the group π is contained in $Isom_o(\widetilde{\mathbb{SL}}) \times R$. Hence $M(\Theta)$ has a compatible structure as an elliptic surface, by Theorem 3.3 of [Wl86]. □

An elliptic surface with Euler characteristic 0 is a Seifert fibred 4-manifold, and so is determined up to diffeomorphism by its fundamental group if the base orbifold is euclidean or hyperbolic [Ue90, Ue91]. Using this result (instead of [Kt75]) together with Theorem 16.6 and Lemma 17.12, it may be shown that if $M(K)$ is homeomorphic to a minimal properly elliptic surface and some power of a weight element is central in πK then $M(K)$ is homeomorphic to $M(K_1)$, where K_1 is some branched twist spin of a torus knot. However in general there may be infinitely many algebraically distinct weight classes in πK and we cannot conclude that K is itself such a branched twist spin.

Chapter 18

Reflexivity

The most familiar invariants of knots are derived from the knot complements, and so it is natural to ask whether every knot is determined by its complement. This has been confirmed for classical knots [GL89]. Given a higher dimensional knot there is at most one other knot (up to change of orientations) with homeomorphic exterior. The first examples of non-reflexive 2-knots were given by Cappell and Shaneson [CS76]; these are fibred with closed fibre R^3/Z^3. Gordon gave a different family of examples [Go76], and Plotnick extended his work to show that no fibred 2-knot with monodromy of odd order is reflexive. It is plausible that this may be so whenever the order is greater than 2, but this is at present unknown.

We shall decide the questions of amphicheirality, invertibility and reflexivity for most fibred 2-knots with closed fibre an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold, excepting only the knots with groups as in part (ii) of Theorem 16.15, for which reflexivity remains open. The other geometrically fibred 2-knots have closed fibre an $\mathbb{S}^3$-, $\mathbb{H}^3$- or $\widetilde{\mathbb{SL}}$-manifold. Branched twist spins $\tau_{r,s}k$ of simple 1-knots k form an important subclass. We shall show that such branched twist spins are reflexive if and only if $r = 2$. If, moreover, k is a torus knot then $\tau_{r,s}k$ is $+$amphicheiral but is not invertible.

This chapter is partly based on joint work with Plotnick and Wilson (in [HP88] and [HW89], respectively).

18.1 Sections of the mapping torus

Let θ be a self-homeomorphism of a closed 3-manifold F, with mapping torus $M(\theta) = F\times_\theta S^1$, and canonical projection $p_\theta : M(\theta) \to S^1$, given by $p_\theta([x, s]) = e^{2\pi is}$ for all $[x, s] \in M(\theta)$. Then $M(\theta)$ is orientable if and only if θ is orientation-preserving. If $\theta' = h\theta h^{-1}$ for some self-homeomorphism h of F then $[x, s] \mapsto [h(x), s]$ defines a homeomorphism $m(h) : M(\theta) \to M(\theta')$ such that $p_{\theta'} m(h) = p_\theta$. Similarly, if θ' is isotopic to θ then $M(\theta') \cong M(\theta)$.

If $P \in F$ is fixed by θ then the image of $P \times [0, 1]$ in $M(\theta)$ is a section of p_θ. In particular, if the fixed point set of θ is connected there is a canonical isotopy class of sections. (Two sections are isotopic if and only if they represent conjugate elements of π.) In general, we may isotope θ to have a fixed point P. Let $t \in \pi_1(M(\theta))$ correspond to the constant section of $M(\theta)$, and let $u = gt$ with $g \in \pi_1(F)$. Let $\gamma : [0,1] \to F$ be a loop representing g. There is an isotopy h_s from $h_0 = id_F$ to $h = h_1$ which drags P around γ, so that $h_s(P) = \gamma(s)$ for all $0 \leq s \leq 1$. Then $H([f,s]) = [(h_s)^{-1}(f), s]$ defines a homeomorphism $M(\theta) \cong M(h^{-1}\theta)$. Under this homeomorphism the constant section of $p_{h^{-1}\theta}$ corresponds to the section of p_θ given by $m_u(t) = [\gamma(t), t]$, which represents u. If F is a geometric 3-manifold we may assume that γ is a geodesic path.

Suppose henceforth that F is orientable, θ is orientation-preserving and fixes a basepoint P, and induces a meridianal automorphism θ_* of $\nu = \pi_1(F)$. The loop sending $[u] = e^{2\pi i u}$ to $[P, u]$, for all $0 \leq u \leq 1$, is the *canonical* cross-section of the mapping torus, and the corresponding element $t \in \pi = \pi_1(M) = \nu \rtimes_{\theta_*} Z$ is a weight element for π. Surgery on the image $C = \{P\} \times S^1$ of this section gives a 2-knot, with exterior the complement of an open regular neighbourhood R of C. Choose an embedding $J : D^3 \times S^1 \to M$ onto R. Let $M_o = M \setminus intR$ and let $j = J|_{\partial D^3 \times S^1}$. Then $\Sigma = M_o \cup_j S^2 \times D^2$ and $\Sigma_\tau = M_o \cup_{j\tau} S^2 \times D^2$ are homotopy 4-spheres and the images of $S^2 \times \{0\}$ represent 2-knots K and K^* with group π and exterior M_o.

Let $\widetilde{F}$ be the universal covering space of F, and let $\tilde{\theta}$ be the lift of θ which fixes some chosen basepoint. Let $\widehat{M} = \widetilde{N} \times_{\tilde{\theta}} S^1$ be the (irregular) covering space corresponding to the subgroup of π generated by t. This covering space shall serve as a natural model for a regular neighbourhood of C in our geometric arguments below.

18.2 Reflexivity for fibred 2-knots

Let K be an n-knot with exterior X and group π. If it is reflexive there is such a self-homeomorphism which changes the framing of the normal bundle. This restricts to a self-homeomorphism of X which (up to changes of orientations) is the nontrivial twist τ on $\partial X \cong S^n \times S^1$. (See §1 of Chapter 14).

If K is invertible or $\pm$amphicheiral there is a self-homeomorphism h of (S^{n+2}, K) which changes the orientations appropriately, but does not twist the normal bundle of $K(S^n) \subset S^{n+2}$. Thus if K is $-$amphicheiral there is such an h which reverses the orientation of $M(K)$ and for which the induced automorphism h'_* of π' commutes with the meridianal automorphism c_t, while if K is invertible or $+$ampicheiral there is a self-homeomorphism h such that $h'_* c_t h'_* = c_t^{-1}$ and which preserves or reverses the orientation. We shall say that K is *strongly* $\pm$amphicheiral or invertible if there is such an h which is an involution.

Suppose now that $n = 2$. Every self-homeomorphism f of X extends "radially" to a self-homeomorphism h of $M(K) = X \cup D^3 \times S^1$ which maps the cocore $C = \{0\} \times S^1$ to itself. If f preserves both orientations or reverses both orientations then it fixes the meridian, and we may assume that $h|_C = id_C$. If f reverses the meridian, we may still assume that it fixes a point on C. We take such a fixed point as the basepoint for $M(K)$.

Now $\partial X \cong S^2 \times_A S^1$, where A is the restriction of the monodromy to $\partial(F \setminus intD^3) \cong S^2$. (There is an unique isotopy class of homeomorphisms which are compatible with the orientations of the spheres S^1, S^2 and $X \subset S^4$.) Roughly speaking, the local situation – the behaviour of f and A on $D^3 \times S^1$ – determines the global situation. Assume that f is a fibre preserving self homeomorphism of $D^3 \times_A S^1$ which induces a linear map B on each fibre D^3. If A has infinite order, the question as to when f "changes the framing", i.e., induces τ on $\partial D^3 \times_A S^1$ is delicate. (See §2 and §3 below). But if A has finite order we have the following easy result.

Lemma 18.1 *Let A in $SO(3)$ be a rotation of order $r \geq 2$ and let B in $O(3)$ be such that $BAB^{-1} = A^{\pm 1}$, so that B induces a diffeomorphism f_B of $D^3 \times_A S^1$. If f_B changes the framing then $r = 2$.*

Proof We may choose coordinates for R^3 so that $A = \rho_{s/r}$, where ρ_u is the matrix of rotation through $2\pi u$ radians about the z-axis in R^3, and $0 < s < r$. Let $\rho : D^3 \times_A S^1 \to D^3 \times S^1$ be the diffeomorphism given by $\rho([x, u]) = (\rho_{-su/r}, e^{2\pi i u})$, for all $x \in D^3$ and $0 \leq u \leq 1$.

If $BA = AB$ then $f_B([x, u]) = [Bx, u]$. If moreover $r \geq 3$ then $B = \rho_v$ for some v, and so $\rho f_B \rho^{-1}(x, e^{2\pi i u}) = (Bx, e^{2\pi i u})$ does not change the framing. But if $r = 2$ then $A = diag[-1, -1, 1]$ and there is more choice for B. In particular, $B = diag[1, -1, 1]$ acts dihedrally: $\rho_{-u} B \rho_u = \rho_{-2u} B$, and so $\rho_{-u} f_B \rho_u(x, e^{2\pi i u}) = (\rho_{-u} x, e^{2\pi i u})$, i.e., $\rho_{-u} f_B \rho_u$ is the twist τ.

If $BAB^{-1} = B^{-1}$ then $f_B([x, u]) = [Bx, 1-u]$. In this case $\rho f_B \rho^{-1}(x, e^{2\pi i u}) = (\rho_{-s(1-u)/r} B \rho_{su/r} x, e^{-2\pi i u})$. If $r \geq 3$ then B must act as a reflection in the first two coordinates, so $\rho f_B \rho^{-1}(x, e^{2\pi i u}) = (\rho_{-s/r} Bx, e^{-2\pi i u})$ does not change the framing. But if $r = 2$ we may take $B = I$, and then $\rho f_B \rho^{-1}(x, e^{2\pi i u}) = (\rho_{(u-1)/2} \rho_{u/2} x, e^{-2\pi i u}) = (\rho_{(u-\frac{1}{2})} x, e^{-2\pi i u})$, which after reversing the S^1 factor is just τ. ∎

We can sometimes show that reflexivity depends only on the knot manifold, and not the weight orbit.

Lemma 18.2 *Let K be a fibred 2-knot. If there is a self homeomorphism h of $X(K)$ which is the identity on one fibre and such that $h|_{\partial X} = \tau$ then all knots $\tilde{K}$ with $M(\tilde{K}) \cong M(K)$ are reflexive. In particular, this is so if the monodromy of K has order 2.*

Proof We may extend h to a self-homeomorphism $\hat{h}$ of $M(K)$ which fixes the surgery cocore $C \cong S^1$. After an isotopy of h, we may assume that it is the identity on a product neighbourhood $N = \hat{F} \times [-\epsilon, \epsilon]$ of the closed fibre. Since any weight element for π may be represented by a section γ of the bundle which coincides with C outside N, we may use h to change the framing of the normal bundle of γ for any such knot. Hence every such knot is reflexive.

If the monodromy θ has order 2 the diffeomorphism h of $\hat{F} \times_\theta S^1$ given by $h([x,s]) = [x, 1-s]$ which "turns the bundle upside down" changes the framing of the normal bundle and fixes one fibre. □

This explains why $r = 2$ is special. The reflexivity of 2-twist spins is due to Litherland. See the footnote to [Go76] and also [Mo83, Pl84'].

The hypotheses in the next lemma seem very stringent, but are satisfied in our applications below, where we shall seek a homeomorphism $\widetilde{F} \cong \mathbb{R}^3$ which gives convenient representations of the maps in question, and then use an isotopy from the identity to $\widetilde{\theta}$ to identify $M(\widetilde{\theta})$ with $\mathbb{R}^3 \times S^1$.

Lemma 18.3 *Suppose that $\widetilde{F} \cong R^3$ and that h is an orientation preserving self-homeomorphism of M which which fixes C pointwise. If $h_*|_{\pi'}$ is induced by a basepoint preserving self-homeomorphism ω of F which commutes with θ and if there is an isotopy γ from $id_{\widetilde{F}}$ to $\widetilde{\theta}$ which commutes with the basepoint preserving lift $\tilde{\omega}$ then h does not change the framing.*

Proof Let h be an orientation preserving self homeomorphism of M which fixes C pointwise. Suppose that h changes the framing. We may assume that $h|_R$ is a bundle automorphism and hence that it agrees with the radial extension of τ from $\partial R = S^2 \times S^1$ to R. Since h fixes the meridian, $h_*\theta_* = \theta_* h_*$. Let ω be a basepoint preserving self diffeomorphism of F which induces $h_*|_\nu$ and commutes with θ. Then we may define a self diffeomorphism h_ω of M by $h_\omega([x,s]) = [\omega(x), s]$ for all $[x,s]$ in $M = F \times_\theta S^1$.

Since $h_{\omega *} = h_*$ and M is aspherical, h and h_ω are homotopic. Therefore the lifts $\hat{h}$ and $\hat{h}_\omega$ to basepoint preserving maps of $\widehat{M}$ are properly homotopic. Let $\tilde{\omega}$ be the lift of ω to a basepoint preserving map of $\widetilde{F}$. Note that $\tilde{\omega}$ is orientation preserving, and so is isotopic to $id_{\widetilde{F}}$.

Given an isotopy γ from $\gamma(0) = id_{\widetilde{F}}$ to $\gamma(1) = \tilde{\theta}$ we may define a diffeomorphism $\rho_\gamma : \widetilde{F} \times S^1 \to \widehat{M}$ by $\rho_\gamma(x, e^{2\pi it}) = [\gamma(t)(x), t]$. Now $\rho_\gamma^{-1}\hat{h}_\omega\rho_\gamma(l, [u]) = (\gamma(u)^{-1}\tilde{\omega}\gamma(u)(l), [u])$. Thus if $\gamma(t)\tilde{\omega} = \tilde{\omega}\gamma(t)$ for all t then $\rho_\gamma^{-1}\hat{h}_\omega\rho_\gamma = \tilde{\omega} \times id_{S^1}$, and so $\hat{h}$ is properly homotopic to $id_{\widehat{M}}$.

Since the radial extension of τ and $\rho_\gamma^{-1}\hat{h}\rho_\gamma$ agree on $D^3 \times S^1$ they are properly homotopic on $R^3 \times S^1$ and so τ is properly homotopic to the identity. Now τ extends uniquely to a self diffeomorphism τ of $S^3 \times S^1$, and any such proper homotopy extends to a homotopy from τ to the identity. Let p be the projection of $S^3 \times S^1$ onto S^3. The suspension of $p\tau$, restricted to the top cell of $\Sigma(S^3 \times S^1) = S^2 \vee S^4 \vee S^5$ is the nontrivial element of $\pi_5(S^4)$, whereas the corresponding restriction of the suspension of p is trivial. (See [CS76, Go76]). This is contradicts $p\tau \sim p$. Therefore h cannot change the framing. □

Note that in general there is no isotopy from id_F to θ.

We may use a similar argument to give a sufficient condition for knots constructed from mapping tori to be $-$amphicheiral. As we shall not use this result below we shall only sketch a proof.

Lemma 18.4 *Let F be a closed orientable 3-manifold with universal cover $\widetilde{F} \cong R^3$. Suppose now that there is an orientation reversing self diffeomorphism $\psi : F \to F$ which commutes with θ and which fixes P. If there is a path γ from I to $\Theta = D\theta(P)$ which commutes with $\Psi = D\psi(P)$ then each of K and K^* is $-$amphicheiral.*

Proof The map ψ induces an orientation reversing self diffeomorphism of M which fixes C pointwise. We may use such a path γ to define a diffeomorphism $\rho_\gamma : \widetilde{F} \times S^1 \to \widetilde{M}$. We may then verify that $\rho_\gamma^{-1}\hat{h}\rho_\gamma$ is isotopic to $\Psi \times id_{S^1}$, and so $\rho_\gamma^{-1}\hat{h}\rho_\gamma|_{\partial D^3 \times S^1}$ extends across $S^2 \times D^2$. □

18.3 Cappell-Shaneson knots

Let $A \in SL(3, \mathbb{Z})$ be such that $\det(A - I) = \pm 1$, and let K be the Cappell-Shaneson knot determined by A. Inversion in each fibre of $M(K)$ fixes a circle, and passes to an orientation reversing involution of (S^4, K). Hence K is strongly $-$amphicheiral. However, it is not invertible, since $\Delta_K(t) = \det(tI - A)$ is not symmetric.

Cappell and Shaneson showed that if none of the eigenvalues of A are negative then the knot is not reflexive. In a footnote they observed that if A has negative eigenvalues the two knots obtained from A are equivalent if and only if there is a matrix B in $GL(3,\mathbb{Z})$ such that $AB = BA$ and the restriction of B to the negative eigenspace of A has negative determinant. We shall translate this matrix criterion into one involving algebraic numbers, and settle the issue by showing that up to change of orientations there is just one reflexive Cappell-Shaneson 2-knot.

We note first that on replacing A by A^{-1} (which corresponds to changing the orientation of the knot), if necessary, we may assume that $\det(A-I) = +1$.

Theorem 18.5 *Let $A \in SL(3,\mathbb{Z})$ satisfy $\det(A-I) = 1$. If A has trace -1 then the corresponding Cappell-Shaneson knot is reflexive, and is determined up to change of orientations among all 2-knots with metabelian group by its Alexander polynomial t^3+t^2-2t-1. If the trace of A is not -1 then the corresponding Cappell-Shaneson knots are not reflexive.*

Proof Let a be the trace of A. Then the characteristic polynomial of A is $\Delta_a(t) = t^3 - at^2 + (a-1)t - 1 = t(t-1)(t-a+1) - 1$. It is easy to see that Δ_a is irreducible; indeed, it is irreducible modulo (2). Since the leading coefficient of Δ_a is positive and $\Delta_a(1) < 0$ there is at least one positive eigenvalue. If $a > 5$ all three eigenvalues are positive (since $\Delta_a(0) = -1$, $\Delta_a(\frac{1}{2}) = (2a-11)/8 > 0$ and $\Delta_a(1) = -1$). If $0 \leq a \leq 5$ there is a pair of complex eigenvalues.

Thus if $a \geq 0$ there are no negative eigenvalues, and so $\gamma(s) = sA + (1-s)I$ (for $0 \leq s \leq 1$) defines an isotopy from I to A in $GL(3,\mathbb{R})$. Let h be a self homeomorphism of (M,C) such that $h(*) = *$. We may assume that h is orientation preserving and preserves the meridian. Since M is aspherical h is homotopic to a map h_B, where $B \in SL(3,\mathbb{Z})$ commutes with A. Applying Lemma 18.3 with $\tilde{\theta} = A$ and $\tilde{\omega} = B$, we see that h must preserve the framing and so K is not reflexive.

We may assume henceforth that $a < 0$. There are then three real roots λ_i, for $1 \leq i \leq 3$, such that $a-1 < \lambda_3 < a < \lambda_2 < 0 < 1 < \lambda_1 < 2$. Note that the products $\lambda_i(\lambda_i - 1)$ are all positive, for $1 \leq i \leq 3$.

Since the eigenvalues of A are real and distinct there is a matrix P in $GL(3,\mathbb{R})$ such that $\tilde{A} = PAP^{-1}$ is the diagonal matrix $diag[\lambda_1, \lambda_2, \lambda_3]$. If B in $GL(3,\mathbb{Z})$ commutes with A then $\tilde{B} = PBP^{-1}$ commutes with $\tilde{A}$ and hence is also diagonal (as the λ_i are distinct). On replacing B by $-B$ if necessary we may assume that $\det(B) = +1$. Suppose that $\tilde{B} = diag[\beta_1, \beta_2, \beta_3]$. We may isotope PAP^{-1} linearly to $diag[1,-1,-1]$. If $\beta_2\beta_3 > 0$ for all such B then PBP^{-1} is isotopic to I through block diagonal matrices and we may again conclude that the knot is not reflexive. On the other hand if there is such a B with $\beta_2\beta_3 < 0$ then the knot is reflexive. Since $\det(B) = +1$ an equivalent criterion for reflexivity is that $\beta_1 < 0$.

The roots of $\Delta_{-1}(t) = t^3+t^2-2t-1$ are the Galois conjugates of $\zeta_7+\zeta_7^{-1}$, where ζ_7 is a primitive 7th root of unity. The discriminant of $\Delta_{-1}(t)$ is 49, which is not properly divisible by a perfect square, and so $\mathbb{Z}[t]/(\Delta_{-1}(t))$ is the full ring of integers in the maximal totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_7)$. As this ring has class number 1 it is a PID. Hence any two matrices in $SL(3,\mathbb{Z})$ with this characteristic polynomial are conjugate, by Theorem 1.4. Therefore the knot group is unique and determines K up to Gluck reconstruction and change of orientations, by Theorem 17.5. Since $B = -A - I$ has determinant 1 and $\beta_1 = -\lambda_1 - 1 < 0$, the corresponding knot is reflexive.

Suppose now that $a < -1$. Let F be the field $\mathbb{Q}[t]/(\Delta_a(t))$ and let λ be the image of X in F. We may view Q^3 as a $\mathbb{Q}[t]$-module and hence as a 1-dimensional F-vector space via the action of A. If B commutes with A then it induces an automorphism of this vector space which preserves a lattice and so determines a unit $u(B)$ in O_F, the ring of integers in F. Moreover $\det(B) = N_{F/\mathbb{Q}}u(B)$. If σ is the embedding of F in R which sends λ to λ_1 and P and B are as above we must have $\sigma(u(B)) = \beta_1$.

Let $U = O_F^\times$ be the group of all units in O_F, and let U^ν, U^σ, U^+ and U^2 be the subgroups of units of norm 1, units whose image under σ is positive, totally positive units and squares, respectively. Then $U \cong Z^2 \times \{\pm 1\}$, since F is a totally real cubic number field, and so $[U : U^2] = 8$. The unit -1 has norm -1, and λ is a unit of norm 1 in U^σ which is not totally positive. Hence $[U : U^\nu] = [U^\nu \cap U^\sigma : U^+] = 2$. It is now easy to see that there is a unit of norm 1 that is not in U^σ (i.e., $U^\nu \neq U^\nu \cap U^\sigma$) if and only if every totally positive unit is a square (i.e., $U^+ = U^2$).

The image of $t(t-1)$ in F is $\lambda(\lambda-1)$, which is totally positive and is a unit (since $t(t-1)(t-a+1) = 1 + \Delta_a(t)$). Suppose that it is a square in F. Then $\phi = \lambda - (a-1)$ is a square (since $\lambda(\lambda-1)(\lambda-(a-1)) = 1$). The minimal polynomial of ϕ is $g(Y) = Y^3 + (2a-3)Y^2 + (a^2-3a+2)Y - 1$. If $\phi = \psi^2$ for some ψ in F then ψ is a root of $h(Z) = g(Z^2)$ and so the minimal polynomial of ψ divides h. This polynomial has degree 3 also, since $Q(\psi) = F$, and so $h(Z) = p(Z)q(Z)$ for some polynomials $p(Z) = Z^3 + rZ^2 + sZ + 1$ and $q(Z) = Z^3 + r'Z^2 + s'Z - 1$ with integer coefficients. Since the coefficients of Z and Z^5 in h are 0 we must have $r' = -r$ and $s' = -s$. Comparing the coefficients of Z^2 and Z^4 then gives the equations $2s - r^2 = 2a - 3$ and $s^2 - 2r = a^2 - 3a + 2$. Eliminating s we find that $r(r^3 + (4a-6)r - 8) = -1$

and so $1/r$ is an integer. Hence $r = \pm 1$ and so $a = -1$ or 3, contrary to hypothesis. Thus there is no such matrix B and so the Cappell-Shaneson knots corresponding to A are not reflexive. $\square$

18.4 The Hantzsche-Wendt flat 3-manifold

In §2 of Chapter 8 we determined the automorphisms of the flat 3-manifold groups from a purely algebraic point of view. In this chapter we use instead the geometric approach, to study fibred 2-knots with closed fibre the *Hantzsche-Wendt* flat 3-manifold, $HW = G_6\backslash\mathbb{R}^3$.

The group of affine motions of 3-space is $Aff(3) = \mathbb{R}^3 \rtimes GL(3,\mathbb{R})$. The action is given by $(v,A)(x) = Ax + v$, for all $x \in \mathbb{R}^3$. Therefore $(v,A)(w,B) = (v + Aw, AB)$. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, and let $X, Y, Z \in GL(3,\mathbb{Z})$ be the diagonal matrices $X = diag[1,-1,-1]$, $Y = diag[-1,1,-1]$ and $Z = diag[-1,-1,1]$. Let $x = (\frac{1}{2}e_1, X)$, $y = (\frac{1}{2}(e_2 - e_3), Y)$ and $z = (\frac{1}{2}(e_1 - e_2 + e_3), Z)$. Then the subgroup of $Aff(3)$ generated by x and y is G_6. The translation subgroup $T = G_6 \cap \mathbb{R}^3$ is free abelian, with basis $\{x^2, y^2, z^2\}$. The holonomy group $H = \{I, X, Y, Z\} \cong (Z/2Z)^2$ is the image of G_6 in $GL(3,\mathbb{R})$. (We may clearly take $\{1, x, y, z\}$ as coset representatives for H in G_6. The commutator subgroup G_6' is free abelian, with basis $\{x^4, y^4, x^2y^2z^{-2}\}$. Thus $2T < G_6' < T$, $T/G_6' \cong (Z/2Z)^2$ and $G_6'/2T \cong Z/2Z$.

Every automorphism of the fundamental group of a flat n-manifold is induced by conjugation in $Aff(n)$, by a theorem of Bieberbach. In particular, $Aut(G_6) \cong N/C$ and $Out(G_6) \cong N/CG_6$, where $C = C_{Aff(3)}(G_6)$ and $N = N_{Aff(3)}(G_6)$.

If $(v,A) \in Aff(3)$ commutes with all elements of G_6 then $AB = BA$ for all $B \in H$, so A is diagonal, and $v + Aw = w + Bv$ for all $(w,B) \in G_6$. Taking $B = I$, we see that $Aw = w$ for all $w \in \mathbb{Z}^3$, so $A = I$, and then $v = Bv$ for all $B \in H$, so $v = 0$. Thus $C = 1$, and so $Aut(G_6) \cong N$.

If $(v,A) \in N$ then $A \in N_{GL(3,\mathbb{R})}(H)$ and A preserves $T = \mathbb{Z}^3$, so $A \in N_{GL(3,\mathbb{Z})}(H)$. Therefore $W = AXA^{-1}$ is in H. Hence $WA = AX$ and so $WAe_1 = Ae_1$ is up to sign the unique basis vector fixed by W. Applying the same argument to AYA^{-1} and AZA^{-1}, we see that $N_{GL(3,\mathbb{R})}(H)$ is the group of "signed permutation matrices", generated by the diagonal matrices and permutation matrices. Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If A is a diagonal matrix in $GL(3,\mathbb{Z})$ then $(0,A)\in N$. Thus $\tilde{a}=(0,-X)$, $\tilde{b}=(0,-Y)$ and $\tilde{c}=(0,-Z)$ are in N. It is easily seen that $N\cap\mathbb{R}^3=\frac{1}{2}\mathbb{Z}^3$, with basis $\tilde{d}=(\frac{1}{2}e_1,I)$, $\tilde{e}=(\frac{1}{2}e_2,I)$ and $\tilde{f}=(\frac{1}{2}e_3,I)$. It is also easily verified that $\tilde{i}=(-\frac{1}{4}e_3,P)$ and $\tilde{j}=(\frac{1}{4}(e_1-e_2),J)$ are in N, and that N is generated by $\{\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f},\tilde{i},\tilde{j}\}$. These generators correspond to the generators given in §2 of Chapter 8, and we shall henceforth drop the tildes.

The natural action of N on $\mathbb{R}^3$ is isometric, since $N_{GL(3,\mathbb{R})}(H)<O(3)$, and so N/G_6 acts isometrically on the orbit space HW. In fact every isometry of HW lifts to an affine transformation of $\mathbb{R}^3$ which normalizes G_6, and so $Isom(HW)\cong Out(G_6)$. The isometries which preserve the orientation are represented by pairs (v,A) with $\det(A)=1$.

Since G_6 is solvable and $H_1(G_6)\cong(Z/4Z)^2$, an automorphism (v,A) of G_6 is meridianal if and only if its image in $Aut(G_6/T)\cong GL(2,2)$ has order 3. Thus its image in $Out(G_6)$ is conjugate to $[j]$, $[j]^{-1}$, $[ja]$ or $[jb]$. The latter pair are orientation-preserving and each is conjugate to its inverse (via $[i]$). However $(ja)^3=1$ while $(jb)^3=de^{-1}f$, so $[jb]^3=[ab]\neq 1$. Thus $[ja]$ is not conjugate to $[jb]^{\pm}$, and the knot groups $G(+)=G_6\rtimes_{[ja]}Z$ and $G(-)=G_6\rtimes_{[jb]}Z$ are distinct. The corresponding knot manifolds are the mapping tori of the isometries of HW determined by $[ja]$ and $[jb]$, and are flat 4-manifolds.

18.5 2-knots with group $G(\pm)$

We shall use the geometry of flat 3-manifolds to show that 2-knots with group $G(+)$ or $G(-)$ are not reflexive, and that among these knots only $\tau_3 4_1,\tau_3 4_1^*$ and the knots obtained by surgery on the section of $M([jb])$ defined by $\gamma|_{[0,1]}$ admit orientation-changing symmetries.

Theorem 18.6 *Let K be a 2-knot with group $\pi\cong G(+)$ or $G(-)$. Then K is not reflexive.*

Proof The knot manifold $M=M(K)$ is homeomorphic to the flat 4-manifold $\mathbb{R}^4/\pi$, by Theorem 8.1 and the discussion in Chapter 16. The weight orbit of K may be represented by a geodesic simple closed curve C through the basepoint P of M. Let γ be the image of C in π. Let h be a self-homeomorphism of M which fixes C pointwise. Then h is based-homotopic to an affine diffeomorphism α, and then $\alpha_*(\gamma)=h_*(\gamma)=\gamma$. Let $\widehat{M}\cong\mathbb{R}^3\times S^1$ be the covering space corresponding to the subgroup $\langle\gamma\rangle\cong Z$, and fix a lift $\widehat{C}$. A homotopy from h to α lifts to a proper homotopy between the lifts $\widehat{h}$ and $\widehat{\alpha}$

to self-homeomorphisms fixing $\widehat{C}$. Now the behaviour at ∞ of these maps is determined by the behaviour near the fixed point sets, as in Lemma 18.3. Since the affine diffeomorphism $\widehat{\alpha}$ does not change the framing of the normal to $\widehat{C}$ it follows that $\widehat{h}$ and h do not change the normal framings either. $\square$

The orthogonal matrix $-JX$ is a rotation though $\frac{2\pi}{3}$ about the axis in the direction $e_1 + e_2 - e_3$. The fixed point set of the isometry $[ja]$ of HW is the image of the line $\lambda(s) = s(e_1 + e_2 - e_3) - \frac{1}{4}e_2$. The knots corresponding to the canonical section are $\tau_3 4_1$ and its Gluck reconstruction $\tau_3 4_1^*$.

Lemma 18.7 *If $n = 0$ then $C_{Aut(G_6)}(ja)$ and $N_{Aut(G_6)}(\langle ja \rangle)$ are generated by $\{ja, def^{-1}, abce\}$ and $\{ja, ice, abce\}$, respectively. The subgroup which preserves the orientation of $\mathbb{R}^3$ is generated by $\{ja, ice\}$.*

If $n \neq 0$ then $N_{Aut(G_6)}(\langle d^{2n} ja \rangle) = C_{Aut(G_6)}(d^{2n} ja)$, and is generated by $\{d^{2n} ja, def^{-1}\}$. This subgroup acts orientably on $\mathbb{R}^3$.

Proof This is straightforward. (Note that $abce = j^3$ and $def^{-1} = (ice)^2$.) $\square$

Lemma 18.8 *The mapping torus $M([ja])$ has an orientation reversing involution which fixes a canonical section pointwise, and an orientation reversing involution which fixes a canonical section setwise but reverses its orientation. There is no orientation preserving involution of M which reverses the orientation of any section.*

Proof Let $\omega = abcd^{-1}f = abce(ice)^{-2}$, and let $p = \lambda(\frac{1}{4}) = \frac{1}{4}(e_1 - e_3)$. Then $\omega = (2p, -I_3)$, $\omega^2 = 1$, $\omega ja = ja\omega$ and $\omega(p) = ja(p) = p$. Hence $\Omega = m([\omega])$ is an orientation reversing involution of $M([ja])$ which fixes the canonical section determined by the image of p in HW.

Let $\Psi([f, s]) = [[iab](f), 1 - s]$ for all $[f, s] \in M([ja])$. This is well-defined, since $(iab)ja(iab)^{-1} = (ja)^{-1}$, and is an involution, since $(iab)^2 = 1$. It is clearly orientation reversing, and since $iab(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ it reverses the section determined by the image of $\lambda(\frac{1}{8})$ in HW.

On the other hand, $\langle ja, ice \rangle \cong Z/3Z \rtimes_{-1} Z$, and the elements of finite order in this group do not invert ja. $\square$

Theorem 18.9 *Let K be a 2-knot with group $G(+)$ and weight element $u = x^{2n}t$, where t is the canonical section. If $n = 0$ then K is strongly $\pm$amphicheiral, but is not strongly invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof Suppose first that $n = 0$. Since $-JX$ has order 3 it is conjugate in $GL(3,\mathbb{R})$ to a block diagonal matrix $\Lambda(-JX)\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{2\pi}{3}) \end{pmatrix}$, where $R(\theta) \in GL(2,\mathbb{R})$ is rotation through θ. Let $R_s = R(\frac{2\pi}{3}s)$ and $\xi(s) = ((I_3 - A_s)p, A_s)$, where $A_s = \Lambda^{-1}\begin{pmatrix} 1 & 0 \\ 0 & R_s \end{pmatrix}\Lambda$, for $s \in \mathbb{R}$. Then ξ is a 1-parameter subgroup of $Aff(3)$, such that $\xi(s)(p) = p$ and $\xi(s)\omega = \omega\xi(s)$ for all s. In particular, $\xi|_{[0,1]}$ is a path from $\xi(0) = 1$ to $\xi(1) = ja$ in $Aff(3)$. Let $\Xi : \mathbb{R}^3 \times S^1 \to M(ja)$ be the homeomorphism given by $\Xi(v, e^{2\pi is}) = [\xi(v), s]$ for all $(v,s) \in \mathbb{R}^3 \times [0,1]$. Then $\Xi^{-1}\Omega\Xi = \omega \times id_{S^1}$ and so Ω does not change the framing. Therefore K is strongly $-$amphicheiral.

Similarly, if we let $\zeta(s) = ((I_3 - A_s)\lambda(\frac{1}{8}), A_s)$ then $\zeta(s)(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ and $iab\zeta(s)iab = \zeta(s)^{-1}$, for all $s \in \mathbb{R}$, and $\zeta|_{[0,1]}$ is a path from 1 to ja in $Aff(3)$. Let $Z : \mathbb{R}^3 \times S^1 \to M(ja)$ be the homeomorphism given by $Z(v, e^{2\pi is}) = [\zeta(s)(v), s]$ for all $(v,s) \in \mathbb{R}^3 \times S^1$. Then

$$Z^{-1}\Psi Z(v,z) = (\zeta(1-s)^{-1}iab\zeta(s)(v), z^{-1}) = (\zeta(1-s)^{-1}\zeta(s)^{-1}iab(v), z^{-1})$$
$$= ((ja)^{-1}iab(v), z^{-1})$$

for all $(v,z) \in \mathbb{R}^3 \times S^1$. Hence Ψ does not change the framing, and so K is strongly $+$amphicheiral. However it is not strongly invertible, by Lemma 18.15.

If $n \neq 0$ every self-homeomorphism h of $M(K)$ preserves the orientation and fixes the meridian, by Lemma 18.7, and so K is neither amphicheiral nor invertible. $\square$

A similar analysis applies when the knot group is $G(-)$, i.e., when the meridianal automorphism is $jb = (\frac{1}{4}(e_1 - e_2), -JY)$. (The orthogonal matrix $-JY$ is now a rotation though $\frac{2\pi}{3}$ about the axis in the direction $e_1 - e_2 + e_3$.) All 2-knots with group $G(-)$ are fibred, and the characteristic map $[jb]$ has finite order, but none of these knots are twist-spins, by Corollary 16.12.1.

Lemma 18.10 *If $n = 0$ then $C_{Aut(G_6)}(jb)$ and $N_{Aut(G_6)}(\langle jb\rangle)$ are generated by $\{jb\}$ and $\{jb, i\}$, respectively. If $n \neq 0$ then $N_{Aut(G_6)}(\langle d^{2n}jb\rangle) = C_{Aut(G_6)}(d^{2n}jb)$ and is generated by $\{d^{2n}jb, de^{-1}f\}$. These subgroups act orientably on $\mathbb{R}^3$.* $\square$

The isometry $[jb]$ has no fixed points in $G_6\backslash\mathbb{R}^3$. We shall defined a preferred section as follows. Let $\gamma(s) = \frac{2s-1}{8}(e_1 - e_2) - \frac{1}{8}e_3$, for $s \in \mathbb{R}$. Then $\gamma(1) = jb(\gamma(0))$, and so $\gamma|_{[0,1]}$ defines a section of $p_{[jb]}$. We shall let the image of $(\gamma(0), 0)$ be the basepoint for $M([jb])$.

Theorem 18.11 *Let K be a 2-knot with group $G(-)$ and weight element $u = x^{2n}t$, where t is the canonical section. If $n = 0$ then K is strongly $+$amphicheiral but not invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof Suppose first that $n = 0$. Since $i(\gamma(s)) = \gamma(1-s)$ for all $s \in \mathbb{R}$ the section defined by $\gamma|_{[0,1]}$ is fixed setwise and reversed by the orientation reversing involution $[f,s] \mapsto [[i](f), 1-s]$. Let B_s be a 1-parameter subgroup of $O(3)$ such that $B_1 = jb$. Then we may define a path from 1 to jb in $Aff(3)$ by setting $\zeta(s) = ((I_3 - B_s)\gamma(s), B_s)$ for $s \in \mathbb{R}$. We see that $\zeta(0) = 1$, $\zeta(1) = jb$, $\zeta(s)(\gamma(s)) = \gamma(s)$ and $i\zeta(s)i = \zeta(s)^{-1}$ for all $0 \leq s \leq 1$. As in Theorem 18.9 it follows that the involution does not change the framing and so K is strongly $+$amphicheiral.

The other assertions follow from Lemma 18.10, as in Theorem 18.9. □

18.6 $\mathbb{N}il^3$-fibred knots

The 2-knot groups π with π' a $\mathbb{N}il^3$-lattice were determined in Theorems 14 and 15 of Chapter 16. The group $Nil = Nil^3$ is a subgroup of $SL(3,\mathbb{R})$ and is diffeomorphic to R^3, with multiplication given by

$$[r,s,t][r',s',t'] = [r+r', s+s', rs'+t+t'].$$

(See Chapter 7). The kernel of the natural homomorphism from $Aut_{Lie}(Nil)$ to $Aut_{Lie}(R^2) = GL(2,\mathbb{R})$ induced by abelianization ($Nil/Nil' \cong R^2$) is isomorphic to $Hom_{Lie}(Nil, \zeta Nil) \cong R^2$. The set underlying the group $Aut_{Lie}(Nil)$ is the cartesian product $GL(2,\mathbb{R}) \times R^2$, with $(A,\mu) = ((\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}), (m_1, m_2))$ acting via $(A,\mu)([r,s,t]) =$

$$[ar+cs, br+ds, m_1 r + m_2 s + (ad-bc)t + bcrs + \frac{ab}{2}r(r-1) + \frac{cd}{2}s(s-1)].$$

The Jacobian of such an automorphism is $(ad-bc)^2$, and so it is orientation preserving. The product of (A,μ) with $(B,\nu) = ((\begin{smallmatrix} g & j \\ h & k \end{smallmatrix}), (n_1, n_2))$ is

$$(A,\mu) \circ (B,\nu) = (AB, \mu B + \det(A)\nu + \frac{1}{2}\eta(A,B)),$$

where

$$\eta(A,B) = (abg(1-g) + cdh(1-h) - 2bcgh, abj(1-j) + cdk(1-k) - 2bcjk).$$

In particular, $Aut_{Lie}(Nil)$ is *not* a semidirect product of $GL(2,\mathbb{R})$ with R^2. For each $q > 0$ in Z the stabilizer of Γ_q in $Aut_{Lie}(Nil)$ is $GL(2,\mathbb{Z}) \times (q^{-1}Z^2)$, which is easily verified to be $Aut(\Gamma_q)$. (See §7 of Chapter 8). Thus every automorphism of Γ_q extends to an automorphism of Nil. (This is a special case of a theorem of Malcev on embeddings of torsion-free nilpotent groups in 1-connected nilpotent Lie groups - see [Rg]).

Let the identity element $[0,0,0]$ and its images in $N_q = Nil/\Gamma_q$ be the basepoints for Nil and for these coset spaces. The extension of each automorphism of Γ_q to Nil induces a basepoint and orientation preserving self homeomorphism of N_q.

If K is a 2-knot with group $\pi = \pi K$ and $\pi' \cong \Gamma_q$ then $M = M(K)$ is homeomorphic to the mapping torus of such a self homeomorphism of N_q. (In fact, such mapping tori are determined up to diffeomorphism by their fundamental groups). Up to conjugacy and involution there are just three classes of meridianal automorphisms of Γ_1 and one of Γ_q, for each odd $q > 1$. (See Theorem 16.13). Since $\pi'' \leq \zeta\pi'$ it is easily seen that π has just two strict weight orbits. Hence K is determined up to Gluck reconstruction and changes of orientation by π alone, by Theorem 17.4. (Instead of appealing to 4-dimensional surgery to realize automorphisms of π by basepoint and orientation preserving self homeomorphisms of M we may use the S^1-action on N_q to construct such a self homeomorphism which in addition preserves the fibration over S^1). We shall show that the knots with $\pi' \cong \Gamma_1$ and whose characteristic polynomials are $X^2 - X + 1$ and $X^2 - 3X + 1$ are not reflexive, while all other 2-knots with $\pi' \cong \Gamma_q$ for some $q > 1$ are reflexive.

Lemma 18.12 *Let K be a fibred 2-knot with closed fibre N_1 and Alexander polynomial $X^2 - 3X + 1$. Then K is +amphicheiral.*

Proof Let $\Theta = (A, (0,0))$ be the automorphism of Γ_1 with $A = (\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix})$. Then Θ induces a basepoint and orientation preserving self diffeomorphism θ of N_1. Let $M = N_1 \times_\theta S^1$ and let C be the canonical section. A basepoint and orientation preserving self diffeomorphism ψ of N_1 such that $\psi\theta\psi^{-1} = \theta^{-1}$ induces a self diffeomorphism of M which reverses the orientations of M and C. If moreover it does not twist the normal bundle of C then each of the 2-knots K and K^* obtained by surgery on C is +amphicheiral. We may check the normal bundle condition by using an isotopy from Θ to id_{Nil} to identify $\widehat{M}$ with $Nil \times S^1$. Thus we seek an automorphism $\Psi = (B, \mu)$ of Γ_1 such that $\Psi\Theta_t\Psi^{-1} = \Theta_t^{-1}$, or equivalently $\Theta_t\Psi\Theta_t = \Psi$, for some isotopy Θ_t from $\Theta_0 = id_{Nil}$ to $\Theta_1 = \Theta$.

Let $P = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $PAP^{-1} = A^{-1}$, or $APA = P$. It may be checked that the equation $\Theta(P,\mu)\Theta = (P,\mu)$ reduces to a linear equation for μ with unique solution $\mu = -(2,3)$. Let $\Psi = (P, -(2,3))$ and let h be the induced diffeomorphism of M.

As the eigenvalues of A are both positive it lies on a 1-parameter subgroup, determined by $L = \ln(A) = m\left(\begin{smallmatrix} 1 & -2 \\ -2 & -1 \end{smallmatrix}\right)$, where $m = (\ln((3+\sqrt{5})/2))/\sqrt{5}$. Now $PLP^{-1} = -L$ and so $P\exp(tL)P^{-1} = \exp(-tL) = (\exp(tL)^{-1}$, for all t. We seek an isotopy $\Theta_t = (\exp(tL), v_t)$ from id_{Nil} to Θ such that $\Theta_t\Psi\Theta_t = \Psi$ for all t. It is easily seen that this imposes a linear condition on v_t which has an unique solution, and moreover $v_0 = v_1 = (0,0)$.

Now $\rho^{-1}h\rho(x,u) = (\Theta_{1-u}\Psi\Theta_u(x), 1-u) = (\Psi\Theta_{1-u}\Theta_u, 1-u)$. The loop $u \mapsto \Theta_{1-u}\Theta_u$ is freely contractible in $Aut_{Lie}(Nil)$, since $\exp((1-u)L)\exp(uL) = \exp(L)$. It follows easily that h does not change the framing of C. □

Instead of using the one-parameter subgroup determined by $L = \ln(A)$ we may use the polynomial isotopy given by $A_t = \left(\begin{smallmatrix} 1 & t \\ t & 1+t^2 \end{smallmatrix}\right)$, for $0 \leq t \leq 1$. A similar argument could be used for the polynomial $X^2 - X + 1$, which is realized by $\tau_6 3_1$ and its Gluck reconstruction. On the other hand, the polynomial $X^2 + X - 1$ is not symmetric, and so the corresponding knots are not $+$amphicheiral.

Theorem 18.13 *Let K be a fibred 2-knot with closed fibre N_q.*

(1) *If the fibre is N_1 and the monodromy has characteristic polynomial $X^2 - X + 1$ or $X^2 - 3X + 1$ then K is not reflexive;*

(2) *If the fibre is N_q (q odd) and the monodromy has characteristic polynomial $X^2 \pm X - 1$ then K is reflexive.*

Proof As $\tau_6 3_1$ is shown to be not reflexive in §7 below, we shall concentrate on the knots with polynomial $X^2 - 3X + 1$, and then comment on how our argument may be modified to handle the other cases.

Let Θ, θ and $M = N_1 \times_\theta S^1$ be as in Lemma 18.12, and let $\widehat{M} = Nil \times_\Theta S^1$ be as in §1. We shall take $[0,0,0,0]$ as the basepoint of $\widehat{M}$ and its image in M as the basepoint there.

Suppose that $\Omega = (B, \nu)$ is an automorphism of Γ_1 which commutes with Θ. Since the eigenvalues of A are both positive the matrix $A(u) = uA + (1-u)I$ is invertible and $A(u)B = BA(u)$, for all $0 \leq u \leq 1$. We seek a path of the form $\gamma(u) = (A(u), \mu(u))$ with commutes with Ω. Equating the second elements of the ordered pairs $\gamma(u)\Omega$ and $\Omega\gamma(u)$, we find that $\mu(u)(B - \det(B)I)$ is uniquely determined. If $\det(B)$ is an eigenvalue of B then there is a corresponding eigenvector ξ in Z^2. Then $BA\xi = AB\xi = \det(B)A\xi$, so $A\xi$

is also an eigenvector of B. Since the eigenvalues of A are irrational we must have $B = \det(B)I$ and so $B = I$. But then $\Omega\Theta = (A, \nu A)$ and $\Theta\Omega = (A, \nu)$, so $\nu(A - I) = 0$ and hence $\nu = 0$. Therefore $\Omega = id_{Nil}$ and there is no difficulty in finding such a path. Thus we may assume that $B - \det(B)I$ is invertible, and then $\mu(u)$ is uniquely determined. Moreover, by the uniqueness, when $A(u) = A$ or I we must have $\mu(u) = (0, 0)$. Thus γ is an isotopy from $\gamma(0) = id_{Nil}$ to $\gamma(1) = \Theta$ (through diffeomorphisms of Nil) and so determines a diffeomorphism ρ_γ from $R^3 \times S^1$ to $\widehat{M}$ via $\rho_\gamma(r, s, t, [u]) = [\gamma(u)([r, s, t]), u]$.

A homeomorphism f from Σ to Σ_τ carrying K to K_τ (as unoriented submanifolds) extends to a self homeomorphism h of M which leaves C invariant, but changes the framing. We may assume that h preserves the orientations of M and C, by Lemma 18.12. But then h must preserve the framing, by Lemma 18.3. Hence there is no such homeomorphism and such knots are not reflexive.

If $\pi \cong \pi\tau_6 3_1$ then we may assume that the meridianal automorphism is $\Theta = \left(\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), (0, 0)\right)$. As an automorphism of Nil, Θ fixes the centre pointwise, and it has order 6. Moreover $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), (0, 0)\right.$ is an involution of Nil which conjugates Θ to its inverse, and so M admits an orientation reversing involution. It can easily be seen that any automorphism of Γ_1 which commutes with Θ is a power of Θ, and the rest of the argument is similar.

If the monodromy has characteristic polynomial $X^2 \pm X - 1$ we may assume that the meridianal automorphism is $\Theta = (D, (0, 0))$, where $D = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or its inverse. As $\Omega = (-I, (-1, 1))$ commutes with Θ (in either case) it determines a self homeomorphism h_ω of $M = N_q \times_\theta S^1$ which leaves the meridianal circle $\{0\} \times S^1$ pointwise fixed. The action of h_ω on the normal bundle may be detected by the induced action on $\widehat{M}$. In each case there is an isotopy from Θ to $\Upsilon = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), (1, 0)\right)$ which commutes with Ω, and so we may replace $\widehat{M}$ by the mapping torus $Nil \times_\Upsilon S^1$. (Note also that Υ and Ω act linearly under the standard identification of Nil with R^3).

Let $R(u) \in SO(2)$ be rotation through πu radians, and let $v(u) = \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right)$, for $0 \le u \le 1$. Then $\gamma(u) = \left(\begin{smallmatrix} 1 & 0 \\ v(u) & R(u) \end{smallmatrix}\right)$ defines a path γ in $SL(3, \mathbb{R})$ from $\gamma(0) = id_{Nil}$ to $\gamma(1) = \Upsilon$ which we may use to identify the mapping torus of Υ with $R^3 \times S^1$. In the "new coordinates" h_ω acts by sending $(r, s, t, e^{2\pi iu})$ to $(\gamma(u)^{-1}\Omega\gamma(u)(r, s, t), e^{2\pi iu})$. The loop sending $e^{2\pi iu}$ in S^1 to $\gamma(u)^{-1}\Omega\gamma(u)$ in $SL(3, \mathbb{R})$ is freely homotopic to the loop $\gamma_1(u)^{-1}\Omega_1\gamma_1(u)$, where $\gamma_1(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R(u) \end{smallmatrix}\right)$ and $\Omega_1 = diag[-1, -1, 1]$. These loops are essential in $SL(3, \mathbb{R})$, since $\Omega_1\gamma_1(u)^{-1}\Omega_1\gamma_1(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R(2u) \end{smallmatrix}\right)$. Thus h_ω induces the twist τ on the normal bundle of the meridian, and so the knot is equivalent to its Gluck reconstruction. □

We may refine one aspect of Litherland's observations on symmetries of twist spins [Li85] as follows. Let K be a strongly invertible 1-knot. Then there is an involution h which rotates S^3 about an unknotted axis A and such that $h(K) = K$ and $A \cap K = S^0$. Let $P \in A \cap K$ and let B be a small ball about P such that $h(B) = B$. Define an involution H of $S^4 = ((S^3 \setminus B) \times S^1) \cup (S^2 \times D^2)$ by $H(s, \theta) = (\rho_{-r\theta} h \rho_{-r\theta}(s), -\theta)$ for $(S, \theta) \in (S^3 \setminus B) \times S^1$ and $H(s, z) = (h(s), \bar{z})$ for $(s, z) \in S^2 \times D^2$. Then $H(\tau_r K) = \tau_r K$ and so $\tau_r K$ is strongly $+$amphicheiral.

Since the trefoil knot 3_1 is strongly invertible $\tau_6 3_1$ is strongly $+$amphicheiral. The involution of $X(\tau_6 3_1)$ extends to an involution of $M(\tau_6 3_1)$ which fixes the canonical section C pointwise and does not change the framing of the normal bundle, and hence $(\tau_6 3_1)^*$ is also $+$amphicheiral.

The other 2-knot groups π with π' a $\mathbb{N}il^3$-lattice are as in Theorem 16.15.

Theorem 18.14 *Let K be a 2-knot with group $\pi(e, \eta)$. Then K is reflexive. If $K = \tau_2 k(e, \eta)$ it is strongly $+$amphicheiral, but no other 2-knot with this group is $+$amphicheiral.*

Proof The first assertion follows from Lemma 18.2, while $\tau_2 k(e, \eta)$ is strongly $+$amphicheiral since the Montesinos knot $k(e, \eta)$ is strongly invertible. (See the paragraph just above.)

Let t be the image of the canonical section in π. Every strict weight orbit representing the preferred meridian in π/π' contains an unique element of the form $u^n t$, by Theorem 16.15. If $n \neq 0$ then $u^n t$ is not conjugate to its inverse. Hence K is not $+$amphicheiral. □

The geometric argument used in [Hi11'] for the knots with group $\pi(e, \eta)$ appears to break down for the 2-knots with groups as in part (2) of Theorem 16.15, and the question of reflexivity is open for these.

Automorphisms of $\mathbb{N}il^3$-lattices preserve orientation, and so no fibred 2-knot with closed fibre a $\mathbb{N}il^3$-manifold is $-$amphicheiral or invertible.

It has been shown that for many of the Cappell-Shaneson knots at least one of the (possibly two) corresponding smooth homotopy 4-spheres is the standard S^4 [AR84]. Can a similar study be made in the Nil cases?

18.7 Other geometrically fibred knots

We shall assume henceforth throughout this section that k is a prime simple 1-knot, i.e., that k is either a torus knot or a hyperbolic knot.

Lemma 18.15 *Let A and B be automorphisms of a group π such that $AB = BA$, $A(h) = h$ for all h in $\zeta\pi$ and the images of A^i and B in $Aut(\pi/\zeta\pi)$ are equal. Let $[A]$ denote the induced automorphism of π/π'. If $I-[A]$ is invertible in $End(\pi/\pi')$ then $B = A^i$ in $Aut(\pi)$.*

Proof There is a homomorphism $\epsilon : \pi \to \zeta\pi$ such that $BA^{-i}(x) = x\epsilon(x)$ for all x in π. Moreover $\epsilon A = \epsilon$, since $BA = AB$. Equivalently, $[\epsilon](I - [A]) = 0$, where $[\epsilon] : \pi/\pi' \to \zeta\pi$ is induced by ϵ. If $I - [A]$ is invertible in $End(\pi/\pi')$ then $[\epsilon] = 0$ and so $B = A^i$. $\square$

Let $p = ap'$, $q = bq'$ and $r = p'q'c$, where $(p, q) = (a, r) = (b, r) = 1$. Let A denote both the canonical generator of the Z/rZ action on the Brieskorn manifold $M(p, q, r)$ given by $A(u, v, w) = (u, v, e^{2\pi i/r}w)$ and its effect on $\pi_1(M(p, q, r))$. The image of the Seifert fibration of $M(p, q, r)$ under the projection to the orbit space $M(p, q, r)/\langle A\rangle \cong S^3$ is the Seifert fibration of S^3 with one fibre of multiplicity p and one of multiplicity q. The quotient of $M(p, q, r)$ by the subgroup generated by $A^{p'q'}$ may be identified with $M(p, q, p'q')$. We may display the factorization of these actions as follows:

$$\begin{array}{ccc} M(p,q,r) & \xrightarrow{/S^1} & P^2(p,q,r) \\ \downarrow & & \downarrow \\ M(p,q,p'q') & \xrightarrow{/S^1} & P^2(p,q,p'q') \\ \downarrow & & \downarrow \\ (S^3,(p,q)) & \xrightarrow{/S^1} & S^2 \end{array}$$

Sitting above the fibre in S^3 of multiplicity p in both M's we find q' fibres of multiplicity a, and above the fibre of multiplicity q we find p' fibres of multiplicity b. But above the branch set, a principal fibre in S^3, we have one fibre of multiplicity c in $M(p, q, r)$, but a principal fibre in $M(p, q, p'q')$. (Note that the orbit spaces $P^2(p, q, r)$ and $P^2(p, q, p'q')$ are in fact homeomorphic.)

We have the following characterization of the centralizer of A in $Aut(\pi)$.

Theorem 18.16 *Assume that $p^{-1}+q^{-1}+r^{-1}\leq 1$, and let A be the automorphism of $\pi=\pi_1(M(p,q,r))$ of order r induced by the canonical generator of the branched covering transformations. If B in $Aut(\pi)$ commutes with A then $B=A^i$ for some $0\leq i<r$.*

Proof The 3-manifold $M=M(p,q,r)$ is aspherical, with universal cover R^3, and π is a central extension of $Q(p,q,r)$ by Z. Here $Q=Q(p,q,r)$ is a discrete planar group with signature $((1-p')(1-q')/2; a\dots a, b\dots b, c)$ (where there are q' entries a and p' entries b). Note that Q is Fuchsian except for $Q(2,3,6)\cong Z^2$. (In general, $Q(p,q,pq)$ is a PD_2^+-group of genus $(1-p)(1-q)/2$).

There is a natural homomorphism from $Aut(\pi)$ to $Aut(Q)=Aut(\pi/\zeta\pi)$. The strategy shall be to show first that $B=A^i$ in $Aut(Q)$ and then lift to $Aut(\pi)$. The proof in $Aut(Q)$ falls naturally into three cases.

Case 1. $r=c$. In this case M is a homology 3-sphere, fibred over S^2 with three exceptional fibres of multiplicity p, q and r. Thus $Q\cong\Delta(p,q,r)=\langle q_1,q_2,q_3\mid q_1^p=q_2^q=q_3^r=q_1q_2q_3=1\rangle$, the group of orientation preserving symmetries of a tesselation of H^2 by triangles with angles π/p, π/q and π/r. Since Z_r is contained in S^1, A is inner. (In fact it is not hard to see that the image of A in $Aut(Q)$ is conjugation by q_3^{-1}. See §3 of [Pl83]).

It is well known that the automorphisms of a triangle group correspond to symmetries of the tessellation (see Chapters V and VI of [ZVC]). Since p, q and r are pairwise relatively prime there are no self symmetries of the (p,q,r) triangle. So, fixing a triangle T, all symmetries take T to another triangle. Those that preserve orientation correspond to elements of Q acting by inner automorphisms, and there is one nontrivial outerautomorphism, R say, given by reflection in one of the sides of T. We can assume $R(q_3)=q_3^{-1}$.

Let B in $Aut(Q)$ commute with A. If B is conjugation by b in Q then $BA=AB$ is equivalent to $bq_3=q_3b$, since Q is centreless. If B is R followed by conjugation by b then $bq_3=q_3^{-1}b$. But since $\langle q_3\rangle=Z_r$ in Q is generated by an elliptic element the normalizer of $\langle q_3\rangle$ in $PSL(2,\mathbb{R})$ consists of elliptic elements with the same fixed point as q_3. Hence the normalizer of $\langle q_3\rangle$ in Q is just $\langle q_3\rangle$. Since $r>2$, $q_3\neq q_3^{-1}$ and so we must have $bq_3=q_3b$, $b=q_3^i$ and $B=A^i$. (Note that if $r=2$ then R commutes with A in $Aut(Q)$).

Case 2. $r=p'q'$ so that $Z_r\cap S^1=1$. The map from $S^2(p,q,p'q')$ to S^2 is branched over three points in S^2. Over the point corresponding to the fibre of multiplicity p in S^3 the map is p'-fold branched; it is q'-fold branched over the point corresponding to the fibre of multiplicity q in S^3, and it is $p'q'$-fold branched over the point $*$ corresponding to the branching locus of M over S^3.

Represent S^2 as a hyperbolic orbifold $H^2/\Delta(p,q,p'q')$. (If $(p,q,r)=(2,3,6)$ we use instead the flat orbifold $E^2/\Delta(2,3,6)$). Lift this to an orbifold structure on $S^2(p,q,p'q')$, thereby representing $Q=Q(p,q,p'q')$ into $PSL(2,\mathbb{R})$. Lifting the $Z_{p'q'}$-action to H^2 gives an action of the semidirect product $Q\rtimes Z_{p'q'}$ on H^2, with $Z_{p'q'}$ acting as rotations about a point $\tilde{*}$ of H^2 lying above $*$. Since the map from H^2 to $S^2(p,q,p'q')$ is unbranched at $\tilde{*}$ (equivalently, $Z_r\cap S^1=1$), $Q\cap Z_{p'q'}=1$. Thus $Q\rtimes Z_{p'q'}$ acts effectively on H^2, with quotient S^2 and three branch points, of orders p, q and $p'q'$.

In other words, $Q\rtimes Z_{p'q'}$ is isomorphic to $\Delta(p,q,p'q')$. The automorphism A extends naturally to an automorphism of Δ, namely conjugation by an element of order $p'q'$, and B also extends to $Aut(\Delta)$, since $BA=AB$.

We claim $B=A^i$ in $Aut(\Delta)$. We cannot directly apply the argument in Case 1, since $p'q'$ is not prime to pq. We argue as follows. In the notation of Case 1, A is conjugation by q_3^{-1}. Since $BA=AB$, $B(q_3)=q_3^{-1}B(q_3)q_3$, which forces $B(q_3)=q_3^j$. Now $q_3^{-1}B(q_2)q_3=AB(q_2)=B(q_3^{-1})B(q_2)B(q_3)=q_3^{-j}B(q_2)q_3^j$, or $B(q_2)=q_3^{1-j}B(q_3)q_3^{j-1}$. But $B(q_2)$ is not a power of q_3, so $q_3^{1-j}=1$, or $j\equiv 1$ modulo (r). Thus $B(q_3)=q_3$. This means that the symmetry of the tessellation that realizes B shares the same fixed point as A, so B is in the dihedral group fixing that point, and now the proof is as before.

Case 3. $r=p'q'c$ (the general case). We have $Z_{p'q'c}$ contained in $Aut(\pi)$, but $Z_{p'q'c}\cap S^1=Z_c$, so that Z_c is the kernel of the composition

$$Z_r\to Out(\pi)\to Out(Q).$$

Let $\bar{Q}$ be the extension corresponding to the abstract kernel $Z_{p'q'}\to Out(Q)$. (The extension is unique since $\zeta Q=1$). Then $\bar{Q}$ is a quotient of the semidirect product $Q(p,q,r)\rtimes(Z/rZ)$ by a cyclic normal subgroup of order c.

Geometrically, this corresponds to the following. The map from $S^2(p,q,r)$ to S^2 is branched as in Case 2, over three points with branching indices p, q and $p'q'$. This time, represent S^2 as $H^2/\Delta(p,q,p'q')$. Lift to an orbifold structure on $S^2(p,q,r)$ with one cone point of order c. Lifting an elliptic element of order r in $\Delta(p,q,r)$ to the universal orbifold cover of $S^2(p,q,r)$ gives Z_r contained in $Aut(Q(p,q,r))$ defining the semidirect product. But $Q(p,q,r)\cap Z_r=Z_c$, so the action is ineffective. Projecting to $Z_{p'q'}$ and taking the extension $\bar{Q}$ kills the ineffective part of the action. Note that $Q(p,q,r)$ and Z_r inject into $\bar{Q}$.

As in Case 2, $\bar{Q}\cong\Delta(p,q,r)$, A extends to conjugation by an element of order r in $\bar{Q}$, and B extends to an automorphism of $Q(p,q,r)\rtimes Z_r$, since $BA=AB$. Now $(q_3,p'q')$ in $Q(p,q,r)\rtimes Z_r$ normally generates the kernel of

$Q(p,q,r) \rtimes Z_r \to \bar{Q}$, where q_3 is a rotation of order c with the same fixed point as the generator of Z_r. In other words, A in $Aut(Q(p,q,r))$ is such that $A^{p'q'}$ is conjugation by q_3. Since $BA^{p'q'} = A^{p'q'}B$ the argument in Case 2 shows that $B(q_3) = q_3$. So B also gives an automorphism of $\bar{Q}$, and now the argument of Case 2 finishes the proof.

We have shown that $B = A^i$ in $Aut(Q)$. Since A in $Aut(\pi)$ is the monodromy of a fibred knot in S^4 (or, more directly, since A is induced by a branched cover of a knot in a homology sphere), $I - [A]$ is invertible. Thus the Theorem now follows from Lemma 18.15. $\square$

Theorem 18.17 *Let k be a prime simple knot in S^3. Let $0 < s < r$, $(r,s) = 1$ and $r > 2$. Then $\tau_{r,s}k$ is not reflexive.*

Proof We shall consider separately the three cases (a) k a torus knot and the branched cover aspherical; (b) k a torus knot and the branched cover spherical; and (c) k a hyperbolic knot.

Aspherical branched covers of torus knots. Let $K = \tau_{r,s}(k_{p,q})$ where $r > 2$ and $M(p,q,r)$ is aspherical. Then $X(K) = (M(p,q,r) \setminus intD^3) \times_{A^s} S^1$, $M = M(K) = M(p,q,r) \times_{A^s} S^1$ and $\pi = \pi K \cong \pi_1(M(p,q,r)) \rtimes_{A^s} Z$. If K is reflexive there is a homeomorphism f of X which changes the framing on ∂X. Now $k_{p,q}$ is strongly invertible - there is an involution of $(S^3, k_{p,q})$ fixing two points of the knot and reversing the meridian. This lifts to an involution of $M(p,q,r)$ fixing two points of the branch set and conjugating A^s to A^{-s}, thus inducing a diffeomorphism of $X(K)$ which reverses the meridian. By Lemma 18.1 this preserves the framing, so we can assume that f preserves the meridian of K. It must also preserve the orientation, by the remark following Theorem 2.14. Since $M(p,q,r)$ is a $\widetilde{\mathbb{SL}}$-manifold $\widetilde{M(p,q,r)} \cong R^3$ and automorphisms of $\pi_1(M(p,q,r))$ are induced by fibre-preserving self-diffeomorphisms [Sc83']. The remaining hypothesis of Lemma 18.3 is satisfied, by Theorem 18.16. Therefore there is no such self homeomorphism f, and K is not reflexive.

Spherical branched covers of torus knots. We now adapt the previous argument to the spherical cases. The analogue of Theorem 18.16 is valid, except for $(2,5,3)$. We sketch the proofs.

$(2,3,3)$: $M(2,3,3 = S^3/Q(8)$. The image in $Aut(Q(8)/\zeta Q(8)) \cong S_3$ of the automorphism A induced by the 3-fold cover of the trefoil knot has order 3 and so generates its own centralizer.

$(2,3,4)$: $M(2,3,4) = S^3/T_1^*$. In this case the image of A in $Aut(T_1^*) \cong S_4$ must be a 4-cycle, and generates its own centralizer.

$(2,3,5)$: $M(2,3,5) = S^3/I^*$. In this case the image of A in $Aut(I^*) \cong S_5$ must be a 5-cycle, and generates its own centralizer.

$(2,5,3)$: We again have I^*, but in this case $A^3 = I$, say $A = (123)(4)(5)$. Suppose $BA = AB$. If B fixes 4 and 5 then it is a power of A. But B may transpose 4 and 5, and then $B = A^iC$, where $C = (1)(2)(3)(45)$ represents the nontrivial outer automorphism class of I^*.

Now let $K = \tau_{r,s}(k_{p,q})$ as usual, with (p,q,r) one of the above four triples, and let $M = M(p,q,r) \times_{A^s} S^1$. As earlier, if K is reflexive we have a homeomorphism f which preserves the meridian t and changes the framing on $D^3 \times_{A^s} S^1$. Let $\widehat{M} = S^3 \times_{\hat{A}^s} S^1$ be the cover of M corresponding to the meridian subgroup, where $\hat{A}$ is a rotation about an axis. Let f be a basepoint preserving self homotopy equivalence of M such that $f_*(t) = t$ in π. Let B in $Aut(\pi_1(M(p,q,r))$ be induced by f_*, so $BA^s = A^sB$. The discussion above shows that $B = A^{si}$ except possibly for $(2,5,3)$. But if B represented the outer automorphism of I^* then after lifting to infinite cyclic covers we would have a homotopy equivalence of S^3/I^* inducing C, contradicting Lemma 11.4. So we have an obvious fibre preserving diffeomorphism f_B of M.

The proof that $\hat{f}_B$ is homotopic to $id_{\widehat{M}}$ is exactly as in the aspherical case. To see that $\hat{f}_B$ is homotopic to $\hat{f}$ (the lift of f to a basepoint preserving proper self homotopy equivalence of $\widehat{M}$) we investigate whether f_B is homotopic to f. Since $\pi_2(M) = 0$ we can homotope f_B to f on the 2-skeleton of M. On the 3-skeleton we meet an obstruction in $H^3(M;\pi_3) \cong H^3(M;\mathbb{Z}) = Z$, since M has the homology of $S^3 \times S^1$. But this obstruction is detected on the top cell of $M(p,q,r)$ and just measures the difference of the degrees of f and f_B on the infinite cyclic covers [Ol53]. Since both f and f_B are orientation preserving homotopy equivalences this obstruction vanishes. On the 4-skeleton we have an obstruction in $H^4(M;\pi_4) = Z/2Z$, which may not vanish. But this obstruction is killed when we lift to $\widehat{M}$, since the map from $\widehat{M}$ to M has even degree, proving that $\hat{f}_B \simeq \hat{f}$.

We now use radial homotopies on $S^3 \times S^1$ to finish, as before.

Branched covers of hyperbolic knots. Let k be hyperbolic. Excluding $N_3(4_1)$ (the 3-fold cyclic branched cover of the figure eight knot), $N = N_r(k)$ is a closed hyperbolic 3-manifold, with $\langle\alpha\rangle \cong Z/rZ$ acting by isometries. As usual, we assume there is a homeomorphism f of $M = M(\tau_{r,s}(k))$ which changes the framing on $D^3 \times_{A^s} S^1$. As in the aspherical torus knot case, it shall suffice to show that the lift $\hat{f}$ on $\widehat{M}$ is properly homotopic to a map of $(R^3 \times S^1, D^3 \times S^1)$ that does not change the framing on $D^3 \times S^1$.

Letting $B = f_*$ on $\nu = \pi_1(N)$, we have $BA^sB^{-1} = A^{\pm s}$, depending on whether $f_*(t) = t^{\pm 1}$ in $\pi = \nu \rtimes_{A^s} Z$. There is an unique isometry β of N realizing the class of B in $Out(\nu)$, by Mostow rigidity, and $\beta\alpha^s\beta^{-1} = \alpha^{\pm s}$. Hence there is an induced self diffeomorphism f_β of $M = N \times_{\alpha^s} S^1$. Note that $f_* = (f_\beta)_*$ in $Out(\pi)$, so f is homotopic to f_β. We cannot claim that β fixes the basepoint of N, but β preserves the closed geodesic fixed by α^s.

Now $\widehat{M} = H^3 \times_{\hat{\alpha}^s} S^1$ where $\hat{\alpha}^s$ is an elliptic rotation about an axis L, and $\hat{f}_\beta$ is fibrewise an isometry $\hat{\beta}$ preserving L. We can write $H^3 = R^2 \times L$ (non-metrically!) by considering the family of hyperplanes perpendicular to L, and then $\hat{\beta}$ is just an element of $O(2) \times E(1)$ and $\hat{\alpha}^s$ is an element of $SO(2) \times \{1\}$. The proof of Lemma 18.1, with trivial modifications, shows that, after picking coordinates and ignoring orientations, $\hat{f}_\beta$ is the identity. This completes the proof of the theorem. □

The manifolds $M(p, q, r)$ with $p^{-1} + q^{-1} + r^{-1} < 1$ are coset spaces of $\widetilde{SL}$ [Mi75]. Conversely, let K be a 2-knot obtained by surgery on the canonical cross-section of $N \times_\theta S^1$, where N is such a coset space. If θ is induced by an automorphism of $\widetilde{SL}$ which normalizes $\nu = \pi_1(N)$ then it has finite order, since $N_{\widetilde{SL}}(\nu)/\nu \cong N_{PSL(2,\mathbb{R})}(\nu/\zeta\nu)/(\nu/\zeta\nu)$. Thus if θ has infinite order we cannot expect to use such geometric arguments to analyze the question of reflexivity.

We note finally that since torus knots are strongly invertible, their twist spins are strongly +amphicheiral. (See the paragraph after Theorem 18.13.) However, since automorphisms of $\widetilde{\mathbb{SL}}$-lattices preserve orientation, no such knot is −amphicheiral or invertible.

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$\Gamma(-)$ (Whitehead quadratic functor), 241
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ζG (centre of a group), 3
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σK (Artin spin of K), 276
τ (the twist of $S^2 \times S^1$), 83
$\tau_r K$ (r-twist spin of a knot K), 276
$\tau_{r,s} K$ (branched twist spin of a knot K), 315
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$\chi(\pi)$ (Euler characteristic of vFP group π), 14

Non-alphabetic symbols

boundary $\partial : \pi_2(B) \to \pi_1(F)$ (connecting homomorphism), 89
dagger †: $L^\dagger = \overline{Hom_{\mathbb{Z}[\pi]}(L, \mathbb{Z}[\pi])}$ the conjugate dual module, 67
double angle brackets $\langle\langle\ \rangle\rangle$: $\langle\langle S \rangle\rangle_G$ (normal closure of S in G), 3
overbar $^-$: anti-involution $\bar{g} = w(g)g^{-1}$, conjugate module $\overline{M}$, 13
prime $'$: commutator subgroup G', maximal abelian cover X', 3, 273
semidirect product: $G \rtimes_\theta Z$, 4
sharp ♯: sum of knots $K_1 \sharp K_2$, 274
star *: K^* (Gluck reconstruction of a knot K), 272
surd $\sqrt{\ }$: $\sqrt{G}$ (Hirsch-Plotkin radical of a group G), 7
tilde $\widetilde{\ }$: $\widetilde{X}$ (universal cover), 25