Homologically arc-homogeneous ENRs

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We prove that an arc-homogeneous Euclidean neighborhood retract is a homology manifold.

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1 Introduction

The so-called Modified Bing–Borsuk Conjecture, which grew out of a question in [1], asserts that a homogeneous Euclidean neighborhood retract is a homology manifold. At this mini-workshop on exotic homology manifolds, Frank Quinn asked whether a space that satisfies a similar property, which he calls homological arc-homogeneity, is a homology manifold. The purpose of this note is to show that the answer to this question is yes.

2 Statement and proof of the main result

Theorem 2.1 Suppose that $X$ is an $n$-dimensional homologically arc-homogeneous ENR. Then $X$ is a homology $n$-manifold.

Definitions A homology $n$-manifold is a space $X$ having the property that, for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A Euclidean neighborhood retract (ENR) is a space homeomorphic to a closed subset of Euclidean space that is a retract of some neighborhood of itself. A space $X$ is homologically arc-homogeneous provided that for every path $\alpha: [0, 1] \to X$, the inclusion induced map

$$H_* (X \times 0, X \times 0 - (\alpha(0), 0)) \to H_* (X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of $\alpha$. The local homology sheaf $\mathcal{H}_k$ in dimension $k$ on a space $X$ is the sheaf with stalks $H_k(X, X - x), x \in X$. 

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By a result of Bredon [2, Theorem 15.2], if an \( n \)-dimensional space \( X \) is cohomologically locally connected (over \( \mathbb{Z} \)), has finitely generated local homology groups \( H_k(X, X - x) \) for each \( k \), and if each \( H_k \) is locally constant, then \( X \) is a homology manifold. We shall show that an \( n \)-dimensional, homologically arc-connected ENR satisfies the hypotheses of Bredon’s theorem.

Assume from now on that \( X \) represents an \( n \)-dimensional, homologically arc-homogeneous ENR. Unless otherwise specified, all homology groups are assumed to have integer coefficients. The following lemma is a straightforward application of the definition and the Mayer–Vietoris theorem.

**Lemma 2.2** Given a path \( \alpha \colon [0, 1] \to X \) and \( t \in [0, 1] \), the inclusion induced map

\[
H_\ast(X \times t, X \times t - (\alpha(t), t)) \to H_\ast(X \times I, X \times I - \Gamma(\alpha))
\]

is an isomorphism.

Given points \( x, y \in X \), an arc \( \alpha \colon I \to X \) from \( x \) to \( y \), and an integer \( k \geq 0 \), let \( \alpha_* \colon H_k(X, X - x) \to H_k(X, X - y) \) be defined by the composition

\[
H_k(X, X - x) \xrightarrow{\times 0} H_\ast(X \times I, X \times I - \Gamma(\alpha)) \xrightarrow{\times 1} H_k(X, X - y).
\]

Clearly \( (\alpha^{-1})_* = \alpha_*^{-1} \) and \( (\alpha \beta)_* = \beta_* \alpha_* \), whenever \( \alpha \beta \) is defined.

**Lemma 2.3** Given \( x \in X \) and \( \eta \in H_k(X, X - x) \), there is a neighborhood \( U \) of \( x \) in \( X \) such that if \( \alpha \) and \( \beta \) are paths in \( U \) from \( x \) to \( y \), then \( \alpha_* (\eta) = \beta_* (\eta) \in H_k(X, X - y) \).

**Proof** We will prove the equivalent statement: for each \( x \in X \) and \( \eta \in H_k(X, X - x) \) there is a neighborhood \( U \) of \( x \) with \( \alpha_* (\eta) = \eta \) for any loop \( \alpha \) in \( U \) based at \( x \).

Suppose \( x \in X \) and \( \eta \in H_k(X, X - x) \). Since \( H_k(X, X - x) \) is the direct limit of the groups \( H_k(X, X - W) \), where \( W \) ranges over the (open) neighborhoods of \( x \) in \( X \), there is a neighborhood \( U \) of \( x \) and an \( \eta_U \in H_k(X, X - U) \) that goes to \( \eta \) under the inclusion \( H_k(X, X - U) \to H_k(X, X - x) \).

Suppose \( \alpha \) is a loop in \( U \) based at \( x \). Let \( \eta_0 \in H_k(X \times I, X \times I - \Gamma(\alpha)) \) correspond to \( \eta \) under the isomorphism \( H_k(X, X - x) \xrightarrow{\times 0} H_k(X \times I, X \times I - \Gamma(\alpha)) \) guaranteed by homological arc-homogeneity.

Let

\[
\eta_{U \times I} = \eta_U \times 0 \in H_k(X \times I, X \times I - U \times I).
\]
Then the image of $\eta_{U \times I}$ in $H_k(X \times I, X \times I - \Gamma(\alpha))$ is $\eta_\alpha$, as can be seen by chasing the following diagram around the lower square.

\[
\begin{array}{ccc}
H_k(X, X - U) & \xrightarrow{\times 1} & H_k(X, X - x) \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_k(X \times I, X \times I - U \times I) & \xrightarrow{\times 0} & H_k(X \times I, X \times I - \Gamma(\alpha)) \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_k(X, X - U) & \xrightarrow{\times 0} & H_k(X, X - x)
\end{array}
\]

But from the upper square we see that $\eta_\alpha$ must also come from $\eta$ after including into $X \times 1$. That is, $\alpha_*(\eta) = \eta$. \qed

**Corollary 2.4** Suppose the neighborhood $U$ above is path connected and $F$ is the cyclic subgroup of $H_k(X, X - U)$ generated by $\eta_U$. Then, for every $y \in U$, the inclusion $H_k(X, X - U) \to H_k(X, X - y)$ takes $F$ one-to-one onto the subgroup $F_y$ generated by $\alpha_*(\eta)$, where $\alpha$ is any path in $U$ from $x$ to $y$.

**Lemma 2.5** Suppose $x, y \in X$ and $\alpha$ and $\beta$ are path-homotopic paths in $X$ from $x$ to $y$. Then $\alpha_* = \beta_*: H_k(X, X - x) \to H_k(X, X - y)$.

**Proof** By a standard compactness argument it suffices to show that, for a given path $\alpha$ from $x$ to $y$ and element $\eta \in H_k(X, X - x)$, there is an $\epsilon > 0$ such that $\alpha_*(\eta) = \beta_*(\eta)$ for any path $\beta$ from $x$ to $y$ $\epsilon$-homotopic (rel $\{x, y\}$) to $\alpha$.

Given a path $\alpha$ from $x$ to $y$, $\eta \in H_k(X, X - x)$, and $t \in I$, let $U_t$ be a path-connected neighborhood of $\alpha(t)$ associated with $(\alpha_t)_*(\eta) \in H_k(X, X - \alpha(t))$ given by Lemma 2.3, where $\alpha_t$ is the path $\alpha|[0, t]$. There is a subdivision

$$\{0 = t_0 < t_1 < \cdots < t_m = 1\}$$

of $I$ such that $\alpha([t_{i-1}, t_i]) \subseteq U_t$ for each $i = 1, \ldots, m$, where $U_t = U_t$ for some $t$. There is an $\epsilon > 0$ so that if $H: I \times I \to X$ is an $\epsilon$-path-homotopy from $\alpha$ to a path $\beta$, then $H([t_{i-1}, t_i] \times I) \subseteq U_t$.

For each $i = 1, \ldots, m$, let $\alpha_i = \alpha|[t_{i-1}, t_i]$ and $\beta_i = \beta|[t_{i-1}, t_i]$, and for $i = 0, \ldots, m$, let $\gamma_i = H|t_i \times I$ and $\eta_i = (\alpha_{ti})_*(\eta)$. By Corollary 2.4,

$$(\alpha_i)_*(\eta_{i-1}) = (\gamma_{i-1}\beta_i\gamma_i^{-1})_*(\eta_{i-1}) = \eta_i$$

where $\eta_0 = \eta$. Since $\gamma_0$ and $\gamma_n$ are the constant paths, it follows easily that

$$\alpha_*(\eta) = (\alpha_n)_* \cdots (\alpha_1)_*(\eta) = (\beta_n)_* \cdots (\beta_1)_*(\eta) = \beta_*(\eta).$$

\qed

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Proof of Theorem 2.1  As indicated at the beginning of this note, we need only show that the hypotheses of [2, Theorem 15.2] are satisfied.

Since $X$ is an ENR, it is locally contractible, and hence cohomologically locally connected over $\mathbb{Z}$.

Given $x \in X$, let $W$ be a path-connected neighborhood of $x$ such that $W$ is contractible in $X$. If $\alpha$ and $\beta$ are two paths in $W$ from $x$ to a point $y \in W$, then $\alpha$ and $\beta$ are path-homotopic in $X$. Hence, by Lemma 2.5, $\alpha_\#: H_k(X, X - x) \to H_k(X, X - y)$ is a well-defined isomorphism that is independent of $\alpha$ for every $k \geq 0$. Hence, $\mathcal{H}_k|W$ is the constant sheaf, and so $\mathcal{H}_k$ is locally constant.

Finally, we need to show that the local homology groups of $X$ are finitely generated. This can be seen by working with a mapping cylinder neighborhood of $X$. Assume $X$ is nicely embedded in $\mathbb{R}^{n+m}$, for some $m \geq 3$, so that $X$ has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi: \partial N \to X$, with mapping cylinder projection $\pi: N \to X$ (see [3]). Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\hat{A} = \phi^{-1}(A)$.

Lemma 2.6  If $A$ is a closed subset of $X$, then $H_k(X, X - A) \cong \overline{H}_c^{n+m-k}(A^*, \hat{A})$.

Proof  Suppose $A$ is closed in $X$. Since $\pi: N \to X$ is a proper homotopy equivalence,

$H_k(X, X - A) \cong H_k(N, N - A^*)$.

Since $\partial N$ is collared in $N$,

$H_k(N, N - A^*) \cong H_k(\text{int } N, \text{int } N - A^*)$,

and by Alexander duality,

$H_k(\text{int } N, \text{int } N - A^*) \cong \overline{H}_c^{n+m-k}(A^* - \hat{A})$

$\cong \overline{H}_c^{n+m-k}(A^*, \hat{A})$

(since $\hat{A}$ is also collared in $A^*$).

Corollary 2.7  If $A$ is a closed subset of $X$, then $\overline{H}^q(A^*, \hat{A}) = 0$, if $q < m$ or $q > n + m$.

Thus, the local homology sheaf $\mathcal{H}_k$ of $X$ is isomorphic to the Leray sheaf $\mathcal{H}^{n+m-k}$ of the map $\pi: N \to X$ whose stalks are $\overline{H}_c^{n+m-k}(x^*, \hat{x})$. For each $k \geq 0$, this sheaf is also locally constant, so there is a path-connected neighborhood $U$ of $x$ such that
\( \mathcal{H}^q \big| U \) is constant for all \( q \geq 0 \). Given such a \( U \), there is a path-connected neighborhood \( V \) of \( x \) lying in \( U \) such that the inclusion of \( V \) into \( U \) is null-homotopic. Thus, for any coefficient group \( G \), the inclusion \( H^p(U, G) \to H^p(V, G) \) is zero if \( p \neq 0 \) and is an isomorphism for \( p = 0 \).

The Leray spectral sequences of \( \pi \big| \pi^{-1}(U) \) and \( \pi \big| \pi^{-1}(V) \) have \( E_2 \) terms

\[
E_2^{p,q}(U) \cong H^p(U; \mathcal{H}^q), \quad E_2^{p,q}(V) \cong H^p(V; \mathcal{H}^q)
\]

and converge to

\[
E_\infty^{p,q}(U) \subseteq H^{p+q}(U^*, \bar{U}; \mathbb{Z}), \quad E_\infty^{p,q}(V) \subseteq H^{p+q}(V^*, \bar{V}; \mathbb{Z}),
\]

respectively (see [2, Theorem 6.1]). Since the sheaf \( \mathcal{H}^q \) is constant on \( U \) and \( V \), \( H^p(U; \mathcal{H}^q) \) and \( H^p(V; \mathcal{H}^q) \) represent ordinary cohomology groups with coefficients in \( G_q \cong \check{H}^q(x^*, \check{x}) \).

By naturality, we have the commutative diagram

\[
\begin{array}{ccc}
E_2^{0,q}(U) & \longrightarrow & E_2^{2,q-1}(U) \\
\downarrow & & \downarrow 0 \\
E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V)
\end{array}
\]

which implies that the differential \( d_2: E_2^{0,q}(V) \to E_2^{2,q-1}(V) \) is the zero map. Hence,

\[
E_3^{0,q}(V) = \ker(E_2^{0,q}(V) \to E_2^{2,q-1}(V)) / \operatorname{im}(E_2^{-2,q+1}(V) \to E_2^{0,q}(V)) = E_2^{0,q}(V),
\]

and, similarly, \( E_3^{0,q}(V) = E_4^{0,q}(V) = \cdots = E_\infty^{0,q}(V) \). Thus,

\[
H^q(V^*, \check{V}; \mathbb{Z}) \supseteq E_\infty^{0,q}(V) \cong E_2^{0,q}(V) \cong H^0(V; \mathcal{H}^q) \cong H^0(V; G_q) \cong G_q.
\]

Applying the same argument to the inclusion \( (x^*, \check{x}) \subseteq (V^*, \check{V}) \) yields the commutative diagram

\[
\begin{array}{ccc}
E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \\
\downarrow & & \downarrow 0 \\
E_2^{0,q}(x) & \longrightarrow & E_2^{2,q-1}(x)
\end{array}
\]
which, in turn, gives

\[
\begin{align*}
G_q &\cong H^0(V; G_q) \xrightarrow{\cong} H^q(V^*; \hat{V}; \mathbb{Z}) \\
&\cong H^0(x; G_q) \xrightarrow{\cong} H^q(x^*; \hat{x}; \mathbb{Z}) \cong G_q
\end{align*}
\]

from which it follows that the inclusion \( H^q(V^*; \hat{V}; \mathbb{Z}) \to H^q(x^*; \hat{x}; \mathbb{Z}) \cong G_q \) is an isomorphism of \( G_q \). Since \((x^*, \hat{x})\) is a compact pair in the manifold pair \((V^*, \hat{V})\), it has a compact manifold pair neighborhood \((W, \partial W)\). Since the inclusion \( H^q(V^*, \hat{V}) \to \tilde{H}^q(x^*, \hat{x}) \) factors through \( H^q(W, \partial W) \), its image is finitely generated for each \( q \).

Hence, \( H_k(X, X - x) \cong \tilde{H}^{n+m-k}(x^*, \hat{x}) \) is finitely generated for each \( k \). \( \square \)

The following theorem, which may be of independent interest, emerges from the proof of Theorem 2.1.

**Theorem 2.8** Suppose \( X \) is an \( n \)-dimensional ENR whose local homology sheaf \( \mathcal{H}_k \) is locally constant for each \( k \geq 0 \). Then \( X \) is a homology \( n \)-manifold.

**References**


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