On the homotopy groups of $E(n)$–local spectra with unusual invariant ideals

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Let $E(n)$ and $T(m)$ for nonnegative integers n and m denote the Johnson–Wilson and the Ravenel spectra, respectively. Given a spectrum whose $E(n)_*$–homology is $E(n)_*(T(m))/(v_1,\dots,v_{n-1})$, then each homotopy group of it estimates the order of each homotopy group of $L_nT(m)$. We here study the $E(n)$–based Adams E_2–term of it and present that the determination of the E_2–term is unexpectedly complex for odd prime case. At the prime two, we determine the E_∞–term for $\pi_*(L_2T(1)/(v_1))$, whose computation is easier than that of $\pi_*(L_2T(1))$ as we expect.

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1 Introduction

In [4], Ravenel has constructed the homotopy associative commutative ring spectrum $T(m)$ as a summand of p–component of the Thom spectrum associated with the map $\Omega SU(p^m)\to BU$. It is extensively used in [4, Section 7] to compute the homotopy groups of spheres in terms of "the method of infinite descent". The Adams–Novikov E_2–term converging to the stable homotopy groups $\pi_*(T(m))$ is described by use of the Hopf algebroid $(BP_*,\Gamma(m+1))$ (cf [4, Definition 7.1.1]). In particular, the 0–th line is

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*,BP_*)=\mathbb{Z}_{(p)}[v_1,\dots,v_m]\subset BP_*=\mathbb{Z}_{(p)}[v_1,\dots],$$

and the more the value of m, the more primitives we obtain. Since v_k for $1\le k\le m$ is a permanent cycle of the spectral sequence, we obtain spectra $T(m)/(v_k)$ and $T(m)/(v_k,v_l)$ for $1\le k,l\le m$ (see Lemma 3.7.) Here $T(m)/J$ for an ideal J of BP_* denotes a spectrum such that $BP_*(T(m)/J)=BP_*/J$.

Let $BP\langle n\rangle$ denote the Johnson–Wilson ring spectrum with $BP\langle n\rangle_*=\mathbb{Z}_{(p)}[v_1,\dots,\ v_n]$ and put $E(n)=v_n^{-1}BP\langle n\rangle$ as usual. Then we have the $E(n)$–based Adams spectral sequence $E_r^{s,t}(X)\Rightarrow\pi_*(L_nX)$ for a spectrum X, whose E_2–term is $E_2^*(X)=\mathrm{Ext}^*_{E(n)_*(E(n))}(E(n)_*,E(n)_*(X))$. Here L_n denotes the Bousfield localization functor with respect to $E(n)$. Note that $BP_*(T(m))=BP_*[t_1,\dots,t_m]\subset BP_*[t_1,\dots]=$

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$BP_*(BP)$. In order to study the E_2–term for a spectrum X with $E(n)_*(X) = E(n)_*/J[t_1, \dots, t_m]$ for an ideal J of $E(n)_*$, we introduce the generalized Johnson–Wilson spectrum $E_m(n) = v_n^{-1} BP\langle n+m\rangle$. Then

$$\Sigma(n,m) = E_m(n)_* \otimes_{BP_*} BP_*[t_{m+1}, t_{m+2}, \dots] \otimes_{BP_*} E_m(n)_*$$

is a Hopf algebroid over $E_m(n)_*$, and the $E(n)$–based Adams E_2–term $E_2^*(X)$ is isomorphic to $\mathrm{Ext}^*_{\Sigma(n,m+1)}(E_m(n)_*, E_m(n)_*/J)$, which we denote $\mathrm{Ext}^*(E_m(n)_*/J)$, by a similar change-of-rings theorem of Hovey and Sadofsky [1].

Consider J_n be the sequence $v_1, v_2, \dots, v_{n-1}$. Then $T(m)/(J_n)$ exists if $n \le 2$ as commented above. Besides, if $L_n T(m)/J$ exists, then the $E(n)$–based Adams E_2–term for $\pi_*(L_n T(m)/J)$ is isomorphic to an Ext group $\mathrm{Ext}^*(E(n)_*/J)$. Consider the long exact sequence of Ext groups associated to the short exact sequence

$$0 \longrightarrow E_m(n)_*/(J_n) \longrightarrow p^{-1}E_m(n)_*/(J_n) \longrightarrow E_m(n)_*/(p^\infty, J_n) \longrightarrow 0.$$

Since $\mathrm{Ext}^*(p^{-1}E_m(n)_*/(J_n)) = \mathbb{Q}$, Corollary 4.5 implies our first theorem:

Theorem 1.1 *The* Ext *group* $\mathrm{Ext}^0(E_m(n)_*/(J_n))$ *is isomorphic to* $\mathbb{Z}_{(p)}$, *and the group* $E_2^1(E_m(n)_*/(J_n))$ *is isomorphic to the direct sum of the cyclic module over the ring* $\mathbb{Z}_{(p)}[v_n^{\pm 1}, v_{n+1}, \dots, v_m]$ *generated by*

$$\frac{v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}$$

of order $p^{1+\nu(e_k)}$ *with* $\nu(e_k) = \min\{\nu(e_1), \dots, \nu(e_n)\}$, *where the integer* $\nu(\ell)$ *denotes the maximal power of* p *that divides* ℓ.

For the case where $n > m$, we have an example which has a similar result to the above theorem (cf Proposition 4.7):

Proposition 1.2 *The* $E(2)$*–based Adams* E_2*–term* $E_2^0(T(1)/(v_1))$ *is isomorphic to* $\mathbb{Z}_{(p)}$ *and* $E_2^1(T(1)/(v_1))$ *is the direct sum of the cyclic module over* $\mathbb{Z}_{(p)}$ *generated by* $v_2^{sp^i} v_3^{tp^j}/p^{1+\min(i,j)}$ *of order* $p^{1+\min(i,j)}$.

In these cases, we did not determine E_2^s for $s > 1$ since there is an obstruction, which comes from the generators known as $b_{i,j}$ (see (3–2)). This is what we did not expect. For $p = 2$, we have the relation $b_{i,j} = h_{i,j}^2$, which makes possible to compute for $s > 1$. Since the $E(2)$–based Adams differentials are read off from Mahowald and Shimomura [2], we obtain the E_∞–term.

Theorem 1.3 *Let $p = 2$. The $E(2)$–based Adams E_∞–term for $\pi_*(L_2T(1)/(v_1))$ is isomorphic to $\mathbb{Z}_{(2)}$ if $s = 0$ and is isomorphic to the tensor product of $\Lambda(\rho_2)$ and the direct sum of*

(1) $\widetilde{v_2A[h_{20}]}$, $v_3B[h_{30}]/(h_{30}^3)$ *and* $v_3Bh_{30}h_{31}$ *whose elements are of order two,*

(2) M^0 *and* M^1.

Here the modules are given in Section 5.

In Section 2, we consider the Hopf algebroid $(E_m(n)_*, \Sigma(n, m+1))$ and show a variation of the change-of-rings theorem given in Hovey and Sadofsky [1]. In Section 3, we exhibit the formulas for the structure maps (the right unit η_R and the diagonal maps Δ). We then observe the existence of spectra of the form $T(m)/J$. Section 4 is devoted to prove Theorem 1.1 and Proposition 1.2. In Section 5, we determine the E_∞–term for $\pi_*(L_2T(1)/(2^\infty, v_1))$. The homotopy groups $\pi_*(L_2T(1))$ is determined easily if p is odd, and stays undetermined if $p = 2$. The result of this section is the first step to understand $\pi_*(L_2T(1))$ at the prime two.

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2 A generalized Johnson–Wilson theory

Let BP and $BP\langle n\rangle$ denote the Brown–Peterson and the Johnson–Wilson spectra characterized by $\pi_*(BP) = BP_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, \dots]$ and $\pi_*(BP\langle n\rangle) = BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP_*$ with $|v_n| = |t_n| = 2(p^n - 1)$. Then the BP_*–homology of BP is $BP_*(BP) = BP_*[t_1, \dots, t_n, \dots]$, We put

$$E_m(n) = v_n^{-1}BP\langle n+m\rangle$$

for nonnegative integers n and m. Then

$$E_m(n)_* = E(n)_*[v_{n+1}, \dots, v_{n+m}] \subset v_n^{-1}BP_*.$$

We notice that $E_0(n)$ is the localized Johnson–Wilson spectrum $E(n)$.

Let $\Gamma(m+1)$ (cf Ravenel [4, 7.1.1]) be the $BP_*(BP)$–comodule defined by

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots].$$

Then the pair $(BP_*, \Gamma(m+1))$ has the structure of the Hopf algebroid inherited from $(BP_*, BP_*(BP))$. Put

$$\Sigma_m(n,i) = E_m(n)_* \otimes_{BP_*} \Gamma(i) \otimes_{BP_*} E_m(n)_*.$$

In particular, we write

$$\Sigma(n,m+1) = \Sigma_m(n,m+1) = E_m(n)_* \otimes_{BP_*} \Gamma(m+1) \otimes_{BP_*} E_m(n)_*.$$

The pair $(E_m(n)_*, \Sigma_m(n,i))$ is a Hopf algebroid with the structure maps inherited from those of the Hopf algebroid $(BP_*, \Gamma(i))$ for all $i > 0$. Consider the map between Hopf algebroids $(E_m(n)_*, \Sigma_m(n,1)) \longrightarrow (E_m(n)_*, \Sigma(n,m+1))$ induced from the projection from $BP_*(BP)$ to $\Gamma(m+1)$. The map is normal and that

$$E_m(n)_*(T(m)) = E_m(n)_* \square_{\Sigma(n,m+1)} \Sigma_m(n,1) \tag{2–1}$$

if $m > 0$. Here, $T(m)$ denotes the Ravenel spectrum [4, 6.5.1], which is an associative commutative ring spectrum characterized by $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$. Since $\Sigma_m(n,1)$ is $E_m(n)_*(E_m(n))$, the change-of-rings theorem [4, A1.3.12] shows the following:

Lemma 2.2 *There is an isomorphism*

$$\mathrm{Ext}_{E_m(n)_*(E_m(n))}(E_m(n)_*, E_m(n)_*(T(m))) = \mathrm{Ext}_{\Sigma(n,m+1)}(E_m(n)_*, E_m(n)_*).$$

Remark 2.3 In general, equation (2–1) does not hold if we work on $E(n)_*E(n)$–comodules. For example, if we set $(n,i) = (2,3)$, then

$$\Sigma_0(2,3) = E(2)_*[t_3, t_4, \dots]/(\eta_R(v_k) : k > 2).$$

In the right hand side we have the relation $v_2 t_1^{p^2} \equiv v_2^p t_1 \mod (p)$ since $\eta_R(v_3) = 0$. On the other hand, we do not have any relation on t_1 in $E(2)_*T(2) = E(2)_*[t_1, t_2]$.

Since $E_m(n)_*$ is a free $E(n)_*$–module over the bases $v^E = v_{n+1}^{e_1} \dots v_{n+m}^{e_m}$ for $E = (e_1, \dots, e_m)$ with $e_k \geq 0$, there is a homotopy equivalence $E_m(n) = \bigvee_E \Sigma^{|E|} E(n)$. This shows that the $E(n)$–based and the $E_m(n)$–based Adams spectral sequences agrees from the E_2–term (cf Hovey and Sadofsky [1]).

3 Existence of some spectra

An ideal $I = (a_0, a_1, \dots, a_{n-1})$ of BP_* is called *invariant* if $\eta_R(a_i) \equiv a_i \mod (a_0, a_1, \dots, a_{i-1})$ for each $0 \leq i < n$ as a BP_*BP–comodule. It is well known that if there is a spectrum X such that $BP_*(X) = BP_*/I$, then I is invariant. Consider now the Ravenel spectrum $T(m)$. Then the E_2–term of the Adams–Novikov spectral sequence for $\pi_*(W\wedge T(m))$ for a spectrum W is isomorphic to an Ext group over the Hopf algebroid $(BP_*,\Gamma(m+1))$. We call an ideal $J=(w_0,w_1,\dots,w_{n-1})$ of BP_* *unusual* if it is not invariant and $\eta_R(w_i)\equiv w_i \mod (w_0,w_1,\dots,w_{i-1})$ for each $0\le i<n$ as a $\Gamma(m+1)$–comodule. In the same manner as above, if there is a spectrum X such that $BP_*(X)=BP_*/J[t_1,\dots,t_m]$ for $m>0$, then J is invariant or unusual. In this section, we study the existence of a spectrum X with BP_*–homology (resp. $E(n)_*$–homology)

$$BP_*(X)=BP_*/J[t_1,\dots,t_m]\quad (\text{resp. } E(n)_*(X)=E(n)_*/J[t_1,\dots,t_m])$$

for an unusual ideal J. We write $T(m)/J$ (resp. $L_nT(m)/J$) for such X.

The next lemma is verified by Hazewinkel's and Quillen's formulas (see Miller, Ravenel and Wilson [3, (1.1)–(1.3)]):

Lemma 3.1 *Assume that $n\le m$. Let J_n denote the ideal $(v_1,\dots,v_{n-1})$ of BP_*. Then the structure maps in $(BP_*,\Gamma(m+1))$ act as*

$$\begin{aligned}
\eta_R(v_k) &\equiv v_k && \text{for } n\le k\le m,\\
\eta_R(v_{m+k}) &\equiv v_{m+k}+pt_{m+k} && \text{for } 0<k\le n,\\
\Delta(t_{m+k}) &\equiv t_{m+k}\otimes 1+1\otimes t_{m+k} && \text{for } 0\le k\le n,\\
\Delta(t_{m+n+1}) &\equiv t_{m+n+1}\otimes 1+1\otimes t_{m+n+1}+v_nb_{m+1,n-1}
\end{aligned}$$

mod J_n, *where*

$$b_{i,j}=\left(t_i^{p^{j+1}}\otimes 1+1\otimes t_i^{p^{j+1}}-(t_i\otimes 1+1\otimes t_i)^{p^{j+1}}\right)/p. \tag{3–2}$$

By this lemma, we read off the behavior of the structure maps η_R and Δ mod J_n of the Hopf algebroid $(E_m(n)_*,\Sigma(n,m+1))$. For $n>m$, we only consider the case where $n=2$ and $m=1$.

Lemma 3.3 *The structure maps in $(BP_*,\Gamma(2))$ acts as*

$$\begin{aligned}
\eta_R(v_i) &\equiv v_i+pt_i \quad \text{for } i=2 \text{ and } 3,\\
\eta_R(v_4) &\equiv v_4+v_2t_2^{p^2}+pt_4+v_2c_{21}-\eta_R(v_2)^{p^2}t_2,\\
\eta_R(v_5) &\equiv v_5+v_3t_2^{p^3}+v_2t_3^{p^2}+pt_5+v_2c_{31}+v_3c_{22}-\eta_R(v_3)^{p^2}t_2-\eta_R(v_2)^{p^3}t_3,\\
\Delta(t_i) &\equiv t_i\otimes 1+1\otimes t_i \quad \text{for } i=2 \text{ and } 3,\\
\Delta(t_4) &\equiv t_4\otimes 1+1\otimes t_4+t_2\otimes t_2^{p^2}+v_2b_{21},\\
\Delta(t_5) &\equiv t_5\otimes 1+1\otimes t_5+t_3\otimes t_2^{p^3}+t_2\otimes t_3^{p^2}+v_2b_{31}+v_3b_{22}
\end{aligned}$$

mod (v_1), *where* $c_{i,j} = p^{-1}(v_i^{p^{j+1}} - \eta_R(v_i^{p^{j+1}}))$. *In particular,* $b_{i,j} \equiv t_i^{2^j} \otimes t_i^{2^j}$ mod (2) *for* $p = 2$.

We consider the Adams–Novikov spectral sequence

$$E_2^{*,*}(X) = \mathrm{Ext}^{*,*}_{BP_*(BP)}(BP_*, BP_*(X)) \quad \Longrightarrow \quad \pi_*(X). \tag{3–4}$$

By the change-of-rings theorem [4, A1.3.12], we have an isomorphism

$$E_2^*(T(m)/I_n) = \mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n). \tag{3–5}$$

Hereafter we use the abbreviation:

$$\mathrm{Ext}_\Gamma(A, -) = \mathrm{Ext}_\Gamma(-) \quad \text{for a Hopf algebroid } (A, \Gamma).$$

Lemma 3.1 implies the following:

Lemma 3.6 *For* $0 \le k \le m$,

$$v_{n+k} \in E_2^0(T(m)/I_n) = \mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n),$$

where $I_n = (p) + J_n$.

Lemma 3.7 *Let* M *be a* $T(m)$*–module spectrum. If* α *and* $\beta \in E_2(T(m))$ *are permanent cycles in the spectral sequence* (3–4)*, then there exist spectra of the form* $M/(\alpha^a)$ *and* $M/(\alpha^a, \beta^b)$ *for positive integers* a *and* b. *In particular, we have* $T(m)/(v_k^a)$ *and* $T(m)/(v_i^a, v_j^b)$ *for* $i, j, k < m+2$.

Proof Since M is a $T(m)$–module spectrum, the elements α and β yield the self maps on M, which we also denote by α and β. Now $M/(\alpha^a)$ is a cofiber of the self map α^a, and the $M/(\alpha^a, \beta^b)$ is obtained by use of Verdier's axiom on the equation $\alpha^a\beta^b = \beta^b\alpha^a$ in $[M, M]_*$.

Since the reduced comodule $\overline{\Gamma(m+1)}$ is $(2p^{m+1}-3)$–connected, we have the vanishing line $E_2^{s,t}(T(m)) = 0$ for $t < 2s(p^{m+1}-1)$ by (3–5). It follows that $v_k \in E_2^*(T(m))$ in Lemma 3.6 is permanent if $k < m+2$. □

The existence of a spectrum with BP_*–homology BP_*/I_n is problematic and we still have little information for such a spectrum, which we usually call the $((n-1)$st) Smith–Toda spectrum and is denoted by $V(n-1)$ (eg Smith [6], Toda [7] and Ravenel [4]). For $n \le 3$, it is shown that $V(n)$ exists if and only if $p > 2n$. On the other hand, $L_nV(n-1)$ exists if $n^2+n < 2p$ [5]. The smash products $T(m)$ and these Smith–Toda spectra show the following:

Proposition 3.8 *If* $p > 2n$, $T(m)/I_n$ *exists, and if* $n^2+n < 2p$, $L_nT(m)/I_n$ *exists.*

4 $\mathrm{Ext}^s_{\Sigma(n,m+1)}(E_m(n)_*/J_n)$ for small s

In this section, let J_n denote the sequence $v_1, \ldots, v_{n-1}$ of elements of $E_m(n)_*$. Applying Ext to the short exact sequence

$$0 \longrightarrow E_m(n)_*/(p, J_n) \xrightarrow{1/p} E_m(n)_*/(p^\infty, J_n) \xrightarrow{p} E_m(n)_*/(p^\infty, J_n) \longrightarrow 0,$$

we have the long exact sequence of Ext groups with connecting homomorphism

$$\delta\colon \mathrm{Ext}^r(E_m(n)_*/(p^\infty, J_n)) \longrightarrow \mathrm{Ext}^{r+1}(E_m(n)_*/(p, J_n)). \tag{4–1}$$

By [4, Theorem 6.5.6], we know the structure of $\mathrm{Ext}(E_m(n)_*/(p, J_n))$, which means that $\mathrm{Ext}(E_m(n)_*/(J_n))$ is a computable object.

To compute $\mathrm{Ext}(E_m(n)_*/(p^\infty, J_n))$, we redefine the class $h_{m+k,0}$ $(0 < k \le n)$ by

$$h_{i,0} = \left[\frac{\log(1 + pv_i^{-1}t_i)}{p}\right] = \left[\sum_{n>0}(-1)^{n-1}\frac{(pv_i^{-1}t_i)^n}{pn}\right]. \tag{4–2}$$

Lemma 4.3 *For $0 < k \le n$, the connecting homomorphism δ in (4–1) acts for all ℓ as $\delta(h_{m+k,0}/p^\ell) = 0$.*

Proof It suffices to show that $ph_{m+k,0} = d(\log(v_{m+k}))$. By Lemma 3.1, we have $\eta_R(v_{m+k}) = v_{m+k} + pt_{m+k}$ for $0 < k \le n$, so the equation

$$\log(1 + pv_{m+k}^{-1}t_{m+k}) = \log(\eta_R(v_{m+k})) - \log(v_{m+k}) = d(\log(v_{m+k}))$$

holds. □

The element $v_{m+k}^{k+1}x$ is well-defined in $\Sigma(n, m+1)/(p^k)$, although the representative $x = \log(1 + pv_{m+k}^{-1}t_{m+k})/p$ of $h_{m+k,0}$ has negative exponents of v_{m+k} in the coefficient.

An easy computation with Lemma 3.1 shows the following:

Lemma 4.4 *Put $\nu(e_k) = \min\{\nu(e_1), \ldots, \nu(e_n)\}$. Then we have*

$$\delta\left(\frac{v_{m+1}^{e_1}\cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}\right) = v_{m+1}^{e_1}\cdots v_{m+n}^{e_n}h_{m+k,0} + \cdots$$

in $\mathrm{Ext}^1(E_m(n)_*/(p, J_n))$ up to unit. For ν, see Theorem 1.1.

Corollary 4.5 $\mathrm{Ext}^0(E_m(n)_*/(p^\infty, J_n))$ *is the direct sum of*

(1) *the cyclic* $\mathbb{Z}_{(p)}[v_n^{\pm 1}, v_{n+1}, \dots, v_m]$*–module generated by*

$$\frac{v_{m+1}^{e_1} \dots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}$$

of order $p^{1+\nu(e_k)}$ *with* $\nu(e_k) = \min\{\nu(e_1), \dots, \nu(e_n)\}$ *and*

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}[v_n^{\pm 1}, v_{n+1}, \dots, v_m]$.

Example 4.6 For $m = n = 2$, we have

$$\delta\left(\frac{v_3^{sp^i} v_4^{tp^j}}{p^{1+\min(i,j)}}\right) = \begin{cases} v_3^{sp^i} v_4^{tp^j} h_{40} & \text{for } i > j \\ v_3^{sp^i} v_4^{tp^j} h_{30} & \text{for } i < j \\ v_3^{sp^i} v_4^{tp^j} (h_{30} + a h_{40}) & \text{for } i = j \end{cases}$$

in $\mathrm{Ext}^1(E_2(2)_*/(p, v_1))$ up to unit (where $a \in (\mathbb{Z}/(p))^\times$), and $\mathrm{Ext}^0(E_2(2)_*/(p^\infty, v_1))$ is the direct sum of

(1) the cyclic module over $\mathbb{Z}_{(p)}[v_2^{\pm}]$ generated by $v_3^{sp^i} v_4^{tp^j} / p^{1+\min(i,j)}$ of order $p^{1+\min(i,j)}$ and

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}[v_2^{\pm 1}]$.

In the computations for $\delta(h_{31})$ and $\delta(h_{41})$, the elements $b_{i,j}$ (cf Lemma 3.1) occur, which are hard to express in terms of generators appearing in [4, Theorem 6.5.6]. We observe that the specific property $b_{i,j} = h_{i,j}^2$ at $p = 2$ makes the computations easy.

We consider the spectrum $L_n T(m)/(J_n)$ for $(n, m) = (2, 1)$, which is the simplest case satisfying $n > m$, and compute $\mathrm{Ext}^s_{\Sigma(2,2)}(E_1(2)_*/(v_1))$ for $s < 2$ for an odd prime. We consider the case for $p = 2$ in the next Section 5. Since p is odd, the condition of [4, Theorem 6.5.6] is always satisfied and $\mathrm{Ext}_{\Sigma(2,2)}(E_1(2)_*/(p, v_1))$ is obtained as

$$K(2)_*[v_3] \otimes \Lambda(h_{i,j} : 2 \le i \le 3,\ j \in \mathbb{Z}/2).$$

Starting from this, $\mathrm{Ext}^0_{\Sigma(2,2)}(E_1(2)_*/(p^\infty, v_1))$ is determined by computing the connecting homomorphism (4–1) for $(m, n) = (1, 2)$ as follows Corollary 4.5:

Proposition 4.7 *For* $sp^i \in \mathbb{Z}$ *and* $tp^j \ge 0$*, we have*

$$\delta(v_2^{sp^i} v_3^{tp^j} / p^{1+\min(i,j)}) = \begin{cases} s v_2^{sp^i - 1} v_3^{tp^j} h_{20} & \text{if } i < j, \\ t v_2^{sp^i} v_3^{tp^j - 1} h_{30} & \text{if } i > j, \\ v_2^{sp^i - 1} v_3^{tp^j - 1} (s v_3 h_{20} + t v_2 h_{30}) & \text{if } i = j, \end{cases}$$

and $\mathrm{Ext}^0_{\Sigma(2,2)}(E_1(2)_*/(p^\infty, v_1))$ *is the direct sum of*

(1) *the cyclic* $\mathbb{Z}_{(p)}$*–module generated by* $v_2^{sp^i}v_3^{tp^j}/p^{1+\min(i,j)}$ *of order* $p^{1+\min(i,j)}$ *and*

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}$.

5 The homotopy groups $\pi_*(L_2T(1)/(v_1))$ at the prime two

We begin with recalling the result of Mahowald and Shimomura [2]:

(5–1) $$\mathrm{Ext}(E_1(2)_*/(2,v_1)) = K(2)_*[v_3,h_{20}]\otimes\Lambda(h_{21},h_{30},h_{31},\rho_2)$$

where ρ_2 is the generator of degree 0 represented by the cocycle $v_2^{-5}t_4+v_2^{-10}t_4^2$. We see that (4–2) for $p=2$ is also a cocycle with leading term $v_i^{-2}t_i^2$, and replace the representative cocycles by

$$h_{i,0}=[t_i] \quad\text{and}\quad h_{i,1}=\Big[\sum_{n>0}(-1)^{n-1}\frac{(2v_i^{-1}t_i)^n}{2n}\Big].$$

Setting $B=\mathbb{Z}/2[v_2^{\pm2},v_3^2]$, we rewrite the right hand side of (5–1) as

$$B\otimes\Lambda(v_3)\otimes\Lambda(h_{21},h_{30},h_{31})\otimes\Lambda(v_2)\otimes\mathbb{Z}/2[h_{20}]\otimes\Lambda(\rho_2).$$

Since $$h_{21}h_{31}=v_2^{-1}v_3^2h_{20}h_{21}+v_2^2h_{30}^2+v_2h_{20}h_{31}$$

by [2, p 243 (1)], we replace $h_{21}h_{31}$ (resp. $h_{21}h_{30}h_{31}$) with h_{30}^2 (resp. h_{30}^3).

Lemma 5.2 *As the* $\mathbb{Z}/2$*–module,* $\mathrm{Ext}(E_1(2)_*/(2,v_1))$ *is isomorphic to*

$$A\otimes\Lambda(v_2)\otimes\mathbb{Z}/2[h_{20}]\otimes\Lambda(\rho_2)$$

where $$A=B\otimes\Lambda(v_3)\otimes\big(\mathbb{Z}/2[h_{30}]/(h_{30}^4)\oplus\mathbb{Z}/2\{h_{21},h_{31}\}\otimes\Lambda(h_{30})\big)$$

and $$B=\mathbb{Z}/2[a_2^{\pm1},a_3] \qquad \text{with } a_i=v_i^2.$$

Lemma 5.3 *The connecting homomorphism* (4–1) *for* $(m,n)=(1,2)$ *acts as*

$$\delta(v_i^s/2)=v_i^{s-1}h_{i,0} \quad\text{and}\quad \delta(a_i^{2^ns}/2^{n+2})=a_i^{2^ns}h_{i,1} \qquad (i=2,3)$$

for odd s *and* $n\ge 0$.

Proof It follows from

$$\begin{aligned} (v_i^s) &\equiv 2v_i^{s-1}t_i && \bmod (4),\\ d(a_i^{2^ns}) &\equiv 2^{n+2}v_i^{2^{n+1}s}(v_i^{-1}t_i+v_i^{-2}t_i^2) && \bmod (2^{n+3}). \end{aligned}$$ □

$\mathrm{Ext}(E_1(2)_*/(2, v_1))$ is decomposed into the following four summands tensoring with $\Lambda(\rho_2)$:

$$\begin{aligned}
&v_2 A \oplus \Lambda(v_2) \otimes A \otimes \mathbb{Z}/p(h_{20})h_{20} \\
&v_3 B \oplus \Lambda(v_3) \otimes B\{h_{30}, h_{30}^2, h_{30}^3\} \oplus v_3 h_{30} h_{31} B \\
&B \oplus B\,\{h_{21}, h_{31}\} \oplus v_3 h_{21} h_{30} B \\
&B h_{30}\,\{h_{21}, h_{31}\} \oplus v_3 B\,\{h_{21}, h_{31}\}
\end{aligned}$$

With respect to each summand, we construct a long exact sequence in Lemma 5.4, Lemma 5.5 and Lemma 5.6. We often use the replacement

$$h_{31} = [v_3^{-1} t_3 + v_3^{-2} t_3^2] = v_3^{-1} h_{30} + \cdots .$$

If we define P_i $(i \geq 0)$ and Q_j $(j > 0)$ by

$$\begin{aligned}
P_i &= \mathbb{Z}_{(2)}\bigl\{a_2^{2^i s} a_3^{2^j t} : 0 \leq j \leq i, 0 \neq s \in \mathbb{Z}, t \geq 0\bigr\}, \\
Q_j &= \mathbb{Z}_{(2)}\bigl\{a_2^{2^i s} a_3^{2^j t} : 0 \leq i < j, s \in \mathbb{Z}, t > 0\bigr\},
\end{aligned}$$

then we decompose B into

$$B = \Bigl(\bigoplus_{i \geq 0} P_i\Bigr) \oplus \Bigl(\bigoplus_{j > 0} Q_j\Bigr).$$

Define M^0 and M^1 by

$$\begin{aligned}
M^0 &= \Bigl(\bigoplus_{i \geq 0} P_i\Bigl\{\frac{1}{2^{i+2}}\Bigr\}\Bigr) \oplus \Bigl(\bigoplus_{j > 0} Q_j\Bigl\{\frac{1}{2^{j+2}}\Bigr\}\Bigr) \oplus \mathbb{Q}/\mathbb{Z}_{(2)}, \\
M^1 &= \Bigl(\bigoplus_{i \geq 0} P_i\Bigl\{\frac{h_{21}}{2^{i+2}}\Bigr\}\Bigr) \oplus \Bigl(\bigoplus_{j > 0} Q_j\Bigl\{\frac{h_{31}}{2^{j+2}}\Bigr\}\Bigr).
\end{aligned}$$

Then we have the following results:

Lemma 5.4 *We have two long exact sequences*

$$\begin{tikzcd}
B \arrow[r, tail] & M^0 \arrow[r, "2"] & M^0 \arrow[dll, "\delta"'] \\
B\{h_{21}, h_{31}\} \arrow[r] & M^1 \arrow[r, "2"] & M^1 \arrow[dll, two heads, "\delta"] \\
v_3 h_{21} h_{30} B & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
v_3B\{h_{21},h_{31}\} \arrow[r, tail] & (v_3/2)B\{h_{21},h_{31}\} \arrow[r, "2"] & (v_3/2)B\{h_{21},h_{31}\} \arrow[dll, two heads, "\delta"] \\
Bh_{30}\{h_{21},h_{31}\}. & &
\end{tikzcd}$$

Proof In the first sequence the connecting homomorphism acts as:

$$\delta(a_2^{2^is}a_3^{2^jt}/2^{2+\min(i,j)}) = \begin{cases} a_2^{2^is}a_3^{2^jt}h_{21} & (i<j) \\ a_2^{2^is}a_3^{2^jt}h_{31} & (i>j) \\ a_2^{2^is}a_3^{2^jt}(h_{21}+h_{31}) & (i=j) \end{cases}$$

We also see that $\delta(a_2^{2^is}a_3^{2^jt}h_{31}/2^{i+2})$ for $i<j$, $\delta(a_2^{2^is}a_3^{2^jt}h_{21}/2^{j+2})$ for $i>j$, and $\delta(a_2^{2^is}a_3^{2^it}h_{21}/2^{i+2})$ are equal to $a_2^{2^is}a_3^{2^jt}h_{21}h_{31}$. Replacing h_{31} with $v_3^{-1}h_{30}+\cdots$, we have the first sequence. The second sequence is obvious. □

Lemma 5.5 *We have a long exact sequence*

$$\begin{tikzcd}
v_3B \arrow[r, tail] & (v_3/2)B \arrow[r, "2"] & (v_3/2)B \arrow[dll, "\delta"'] \\
h_{30}B\otimes\Lambda(v_3) \arrow[r] & (v_3h_{30}/2)B \arrow[r, "2"] & (v_3h_{30}/2)B \arrow[dll, "\delta"'] \\
\begin{matrix} h_{30}^2B\otimes\Lambda(v_3) \\ \oplus(v_3h_{30}h_{31})B \end{matrix} \arrow[r] & \begin{matrix} (v_3h_{30}^2/2)B \\ \oplus(v_3h_{30}h_{31}/2)B \end{matrix} \arrow[r, "2"] & \begin{matrix} (v_3h_{30}^2/2)B \\ \oplus(v_3h_{30}h_{31}/2)B \end{matrix} \arrow[dll, two heads, "\delta"] \\
h_{30}^3B\otimes\Lambda(v_3). & &
\end{tikzcd}$$

Proof It follows from

$$\delta(a_2^{2^is}a_3^{2^jt}v_3h_{30}^k/2) = a_2^{2^is}a_3^{2^jt}h_{30}^{k+1} \qquad \text{for } 0\le k\le 2,$$
$$\delta(a_2^{2^is}a_3^{2^jt}v_3h_{30}h_{31}/2) = a_2^{2^is}a_3^{2^jt}h_{30}^2h_{31} = a_2^{2^is}a_3^{2^jt-1}v_3h_{30}^3+\cdots$$ □

Lemma 5.6 *We have a long exact sequence*

$$\begin{tikzcd}
v_2A \arrow[r, tail] & (v_2/2)A \arrow[r, "2"] & (v_2/2)A \arrow[dll, "\delta"'] \\
h_{20}A\otimes\Lambda(v_2) \arrow[r] & (v_2h_{20}/2)A \arrow[r, "2"] & (v_2h_{20}/2)A \arrow[dll, "\delta"'] \\
h_{20}^2A\otimes\Lambda(v_2) \arrow[r] & (v_2h_{20}^2/2)A \arrow[r, "2"] & \cdots .
\end{tikzcd}$$

Proof Notice that each exponent of v_2 in $(v_2h_{20}^k/2)A$ is odd. Since we have $d(x)=0$ for $x \in A$ in the cobar complex, we have

$$d(v_2^{2s+1}v_3^tx) = d(v_2^{2s+1}v_3^t) \otimes x.$$

We see that

$$d(v_2^{2s+1}v_3^t) = \begin{cases} 2v_2^{2s}v_3^{2n}t_2 + \cdots & \text{for } t = 2n, \\ d(v_2^{2s+1}v_3^t) = 2v_2^{2s}v_3^{2n}(v_3t_2 + v_2t_3) + \cdots & \text{for } t = 2n+1. \end{cases}$$

In both cases we obtain

$$\delta\left(\frac{v_2^{2s+1}v_3^tx}{2}\right) = v_2^{2s}v_3^th_{20}x$$

replacing v_3h_{20} by $v_3h_{20} = [v_3t_2 + v_2t_3]$ only for the case $t = 2n+1$. □

By the above three lemmas, we obtain the chart of differentials

$$\boxed{v_3B} \longrightarrow h_{30}B$$

$$\boxed{v_3h_{30}B} \longrightarrow h_{30}^2B$$

$$\boxed{v_3h_{30}^2B} \longrightarrow h_{30}^3B$$

$$\boxed{v_3h_{30}h_{31}B} \longrightarrow v_3h_{30}^3B$$

$$\begin{tikzcd}
\boxed{B} \arrow[r] \arrow[dr] & h_{21}B \arrow[dr] & \\
& \boxed{h_{31}B} \arrow[r] & v_3h_{21}h_{30}B
\end{tikzcd}$$

$$\boxed{v_3B\{h_{21}, h_{31}\}} \longrightarrow h_{30}B\{h_{21}, h_{31}\}$$

$$\boxed{v_2A} \longrightarrow h_{20}A$$

$$\boxed{v_2h_{20}A} \longrightarrow h_{20}^2A$$

$$\boxed{v_2h_{20}^2A} \longrightarrow h_{20}^3A$$

$$\vdots$$

Thus we conclude the following:

Lemma 5.7 $\mathrm{Ext}_{\Sigma(2,2)}(E_1(2)_*, E_1(2)_*/(2^\infty, v_1))$ *is the tensor product of* $\Lambda(\rho_2)$ *and the direct sum of*

(1) $v_2A[h_{20}]$, $v_3B[h_{30}]/(h_{30}^3)$ *and* $v_3Bh_{30}h_{31}$ *whose elements are of order two,*

(2) M^0 *and* M^1.

Let $E_\infty^*(X)$ for a spectrum X denote the E_∞–term of the $E(2)$–based Adams spectral sequence converging to the homotopy groups $\pi_*(L_2X)$.

Theorem 5.8 *The* E_∞*–term* $E_\infty^*(L_2T(1)/(2^\infty, v_1))$ *is the tensor product of* $\Lambda(\rho_2)$ *and the direct sum of*

(1) $\widetilde{v_2A[h_{20}]}$, $v_3B[h_{30}]/(h_{30}^3)$ *and* $v_3Bh_{30}h_{31}$ *whose elements are of order two,*

(2) M^0 *and* M^1,

where $\widetilde{v_2A[h_{20}]}$ *denotes the module*

$$\big(\mathbb{Z}/2[v_2^{\pm 2}, v_3^4] \otimes \Lambda(v_3) \otimes \big(\mathbb{Z}/2[h_{30}]/(h_{30}^4) \oplus \mathbb{Z}/2\{h_{21}, h_{31}\} \otimes \Lambda(h_{30})\big)\big)[h_{20}]/(h_{20}^3).$$

Proof In [2], the differentials of $E(2)$–based Adams spectral sequence for $L_2T(1)/I_2$ (written as D in [2]) are determined as

$$d_3(v_3) = 0 \quad \text{and} \quad d_3(v_3^k) = v_2^2 v_3^{k-2} h_{20}^3 \quad \text{for } 2 \le k \le 3,$$

and $d_3(v_3^k x) = d_3(v_3^k)x$ for $x = h_{20}$, h_{21}, h_{30} and h_{31}. Note that for each element $wa_3^{2t+1} \in v_2A[h_{20}]$, we see that

$$d_3(wa_3^{2t+1}/2) = wa_3^{2t} h_{20}^3/2 \in v_2A[h_{20}].$$

This shows the structure of $\pi_*(L_2T(1)/(2^\infty, v_1))$, since it has a horizontal vanishing line. □

Proof of Theorem 1.3 Consider the cofiber sequence

$$T(1)/(v_1) \longrightarrow T(1)/(v_1) \wedge S\mathbb{Q} \longrightarrow T(1)/(2^\infty, v_1).$$

Then the homotopy groups of $T(m)/(v_1) \wedge S\mathbb{Q}$ and $T(1)/(2^\infty, v_1)$ are determined in [4, Corollary 6.5.6] and Theorem 5.8, respectively. □

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