The Heegaard structure of Dehn filled manifolds

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We expect manifolds obtained by Dehn filling to inherit properties from the knot manifold. To what extent does that hold true for the Heegaard structure? We study four changes to the Heegaard structure that may occur after filling: (1) Heegaard genus decreases, (2) a new Heegaard surface is created, (3) a non-stabilized Heegaard surface destabilizes, and (4) two or more non-isotopic Heegaard surfaces become isotopic. We survey general results that give quite satisfactory restrictions to phenomena (1) and (2) and, in a parallel thread, give a complete classification of when all four phenomena occur when filling most torus knot exteriors. This latter thread yields sufficient (and perhaps necessary) conditions for the occurrence of phenomena (3) and (4).

57N10; 57M27

1 Introduction

Let X be a *knot manifold*, that is a compact, orientable and irreducible 3–manifold with a single torus boundary component. There are many results demonstrating that most of the manifolds obtained by filling inherit properties from the knot manifold. We would also expect the Heegaard structure of filled manifolds to be closely related to the Heegaard structure of the knot manifold. For example, it is easy to see that every Heegaard surface for the knot manifold is a Heegaard surface for each filled manifold. In particular, this implies that the Heegaard genus of X is an upper bound on the genus of each filled manifold. However, the Heegaard structure of a filled manifold can differ from that of the knot manifold. Here are four ways that this could occur:

(1) Heegaard genus decreases.

(2) A new Heegaard surface is created.

(3) A non-stabilized Heegaard surface destabilizes.

(4) Two or more non-isotopic Heegaard surfaces become isotopic.

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By a *new* Heegaard surface, we mean that a filled manifold contains a Heegaard surface that is not isotopic (in the filled manifold) to a Heegaard surface for the knot manifold X. When the genus decreases (1), the filled manifold has a Heegaard surface of lower genus than every Heegaard surface for X. Indeed, it is a new Heegaard surface. So, restricting (2) also restricts (1).

In each of these cases, we would like to either demonstrate that the set of fillings for which the phenomenon occurs is special, for example finite, a line of slopes, and/or conclude that the Heegaard surface(s) in question are special in some regard, for example γ–primitive, padded, or boundary stabilized.

In Section 5 we survey known work that gives quite satisfactory restrictions to phenomena (1) and (2). We also give an extended example: Dehn filling on a torus knot exterior, for which we have almost complete knowledge. We are able to completely specify the fillings for which each of these four phenomena occur. This also illustrates sufficient conditions for (3) and (4) to occur.

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2 Background

2.1 Dehn filling and slopes

For simplicity assume that X is a *knot manifold*, an orientable, irreducible 3–manifold with boundary consisting of a single incompressible torus. Much of the discussion, and some of the results quoted, also pertain to manifolds with multiple torus boundary components but it will simplify our discussion not to consider them here.

A *slope* α is the isotopy class of a simple closed curve α in the boundary torus ∂X. With a choice of basis, for example the meridian longitude pair (μ, λ) for a knot exterior in S^3, we can naturally identify the set of slopes with $\mathbb{Q} \cup \{\infty = 1/0\}$. It will be important to be able to identify L_α, the *line of slopes associated with a given slope* α. These are precisely the slopes that meet α exactly once: $L_\alpha = \{\beta \mid \Delta(\alpha, \beta) = 1\}$, where $\Delta(\alpha, \beta)$ indicates the geometric intersection number. We will also construct a *line of lines associated with* α, LL_α, which is the set of slopes $\gamma \in L_\beta$ for some $\beta \in L_\alpha$, that is, $LL_\alpha = \{\gamma \mid \exists \beta \ s.t. \ \Delta(\alpha, \beta) = \Delta(\beta, \gamma) = 1\}$.

2.2 Triviality conditions

We assume that the reader is familiar with the notions of stabilization, (ir)reducibility, and weak reducibility for Heegaard splittings, see Scharlemann [26] for the basics. For knot manifolds, it is also worth identifying splittings that are γ*–primitive*. They possess a "strong" (A_γ, D) pair: A_γ is a vertical annulus in the compression body with slope γ on ∂X, and D a disk in the handlebody so that $|\partial A_\gamma \cap \partial D| = 1$. A step down from this are splittings that are weakly γ–primitive. They possess a "weak" (A_γ, D) pair: a vertical annulus in the compression body with slope γ and an essential disk in the handlebody that are disjoint. We will introduce several other triviality conditions in later sections, identifying splittings that are *padded, parallel stabilizations*, and *boundary stabilizations*.

2.3 Heegaard structure – the Heegaard tree and canopy

What is meant by the Heegaard structure of a manifold M? Of course, we include the Heegaard genus of M as well as the set of irreducible Heegaard surfaces for M. A bit more general is the *Heegaard tree for* M, $\mathcal{HT}_M$. A vertex in the tree, v_Σ, is the class of surfaces in M isotopic to a Heegaard surface Σ. A directed edge will point from v_Σ to v'_Σ if Σ' can be obtained from Σ by a single stabilization.

The word tree is a bit of a misnomer. For a manifold with fewer than two boundary components $\mathcal{HT}_M$ is indeed a tree. But, a Heegaard splitting $M = V \cup_\Sigma W$ induces a partition $\{\partial M \cap V \parallel \partial M \cap W\}$ of the boundary components. And isotopy and stabilization, hence destabilization, cannot change this partition. So $\mathcal{HT}_M$ actually consists of $2^{|\partial M|-1}$ (infinite) components, each a tree as proved by the the Reidemeister-Singer Theorem. The leaves of the tree are precisely the non-stabilized splittings.

We also define a somewhat finer variation, *the oriented Heegaard tree for* M, $\mathcal{HT}^{\pm}_M$, where the vertices are now defined to be isotopy classes of *oriented* Heegaard surfaces. In this case we will assume that the compression bodies are ordered 1 and 2 and the orientation on the Heegaard surface points towards the second compression body. In other words, we differentiate between the Heegaard splittings $V \cup_\Sigma W$ and $W \cup_\Sigma V$. Note that flipping the orientation of the Heegaard surface swaps the partition of boundary components. It follows that $\mathcal{HT}^{\pm}_M$ will consist of $2^{|\partial M|}$ connected trees, double that of $\mathcal{HT}_M$. For a knot manifold, we can unambiguously define the *sign* $(\pm)$ of an oriented Heegaard surface: A positive $(+)$ Heegaard surface will have its orientation pointing into the handlebody, and a negative $(-)$ Heegaard surface will have its orientation pointing into the compression body.

While this may seem somewhat obvious, it does underline an important difference between the Heegaard structure of a knot manifold and those obtained by Dehn filling.

A knot manifold X has connected boundary, so $\mathcal{HT}^{\pm}_{X}$ consists of two components, $\mathcal{HT}^{+}_{X}$ and $\mathcal{HT}^{-}_{X}$, each homeomorphic to the tree $\mathcal{HT}_{X}$. But, for a filled manifold $X(\alpha)$, the tree $\mathcal{HT}^{\pm}_{X(\alpha)}$ is connected. We will say that a surface *flips* if it is isotopic to itself with reverse orientation. This is equivalent to an isotopy that takes the handlebody V to the handlebody W for the splitting $V \cup_{\Sigma} W$. In a closed manifold, there is always a Heegaard surface that flips: Since the oriented Heegaard tree is connected, an oriented surface and its reverse have a common stabilization; this surface flips. It is an easy exercise to show that the common stabilization of a Heegaard surface of genus g and its reverse has genus at most $2g$; hence the smallest genus of surfaces that flip is at most twice the genus of the manifold. But a knot manifold never has a surface that flips, because a handlebody is never isotopic to a compression body. In other words, for *every* filled manifold $X(\alpha)$ there are Heegaard surfaces for X that *flip* in $X(\alpha)$ but not in X.

Since we can stabilize a given splitting any number of times, each tree defined above is infinite. Instead of drawing $\mathcal{HT}^{\pm}_{X}$ (upside down!), we will instead draw its *canopy*, that is the smallest subset of $\mathcal{HT}^{\pm}_{X}$ that has the same number of components as $\mathcal{HT}^{\pm}_{X}$ and that contains all of its leaves (non-stabilized splittings). A result of Li [13] shows that the stabilization tree has an infinite canopy only if the manifold contains a closed essential surface. Examples of canopies for $\mathcal{HT}^{\pm}_{M}$ are drawn in Section 3.

3 Dehn Filling on Torus Knots

In this section we will review the Heegaard structure of torus knot exteriors and the manifolds that can be obtained by Dehn filling on them. Fortunately there has been a lot of work done in this area, and we know the Heegaard tree $\mathcal{HT}^{\pm}_{M}$ up to isotopy for all torus knot exteriors and almost all manifolds that can be obtained by Dehn filling on a torus knot exterior. The sole exception is a restricted class of connected sums of lens spaces whose Heegaard structure is known up to homeomorphism but not up to isotopy. These are discussed in Section 3.7.

First, we fix notation. Let T be a Heegaard torus in S^3, it separates S^3 into two solid tori that we will denote by V_i and V_o. Let μ and λ denote the meridians of V_i and V_o, respectively. Then the curve $T_{p,q} = p\lambda + q\mu$ is a (p,q)–torus knot in S^3. The exterior, $X = S^3 - N(T_{p,q})$ is a Seifert fibered space over the disk with two exceptional fibers f_i and f_o. A regular neighborhood of f_i is the solid torus V_i with a (p,q)–fibering by regular fibers and a regular neighborhood of f_o is the solid torus V_o with a (q,p) fibering by regular fibers (see Scott [32] and Jaco [11]).

3.1 Our examples:

We will restrict our attention to (p,q)–torus knot exteriors that satisfy two conditions:

(1) $p \not\equiv \pm 1 \pmod q$ and $q \not\equiv \pm 1 \pmod p$

(2) $q^2 \not\equiv \pm 1 \pmod p$ and $p^2 \not\equiv \pm 1 \pmod q$

The first condition rules out torus knot exteriors with fewer than three non-isotopic tunnels. This restriction keeps our listing of $\mathcal{HT}^{\pm}_{M}$'s a bit shorter, but the excluded knots can be analyzed in the same manner. The second rules out fillings that produce a connected sum of lens spaces whose Heegaard structure is known only up to homeomorphism. As mentioned above, this will be discussed further in Section 3.7.

3.2 Heegaard structure of {pair of pants} × S^1

Heegaard splittings of Seifert fibered spaces with boundary are vertical (see Schultens [28]). A *vertical splitting* is a Heegaard splitting for the Seifert fibered space that is also a Heegaard surface for the product manifold obtained by drilling out all of the exceptional fibers. To understand splittings of the torus knot exterior, we look to the corresponding product manifold $P \times S^1$, where P is a pair of pants. This manifold is also homeomorphic to the exterior of the three component chain in S^3, pictured in Figure 1. We have already noted that Heegaard splittings of a manifold with boundary induce partitions of the boundary components. Heegaard splittings of $P \times S^1$ are special, because any partition identifies, up to isotopy, a unique irreducible splitting (see [28]).

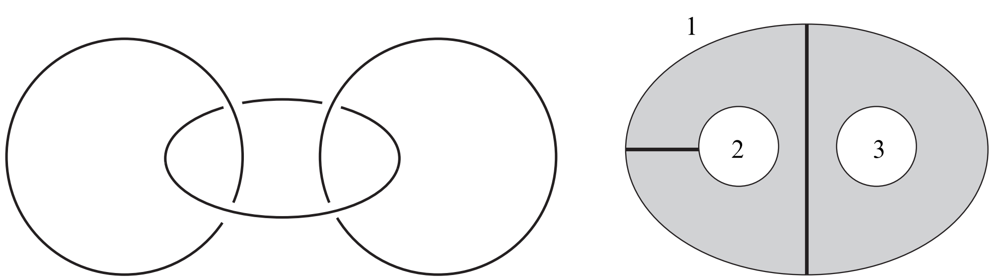

Figure 1: The link manifold $X = S^3 - N$(chain with 3 components) is homeomorphic to {pair of pants} × S^1. The arcs a_{12} and a_{11} in the pair of pants P.

While the product structure on $P \times S^1$ is not unique, the Seifert fibering is unique (see Jaco [11]). Let A_{ij} denote the unique essential annulus joining ∂X_i and ∂X_j, and $a_{ij} = A_{ij} \cap P$ a spanning arc for A_{ij}. A different choice of P will yield a different, but handle slide equivalent, spanning arc a_{ij}. Any curve in ∂X_i which meets a regular fiber once will be referred to as a *dual curve*. Dual curves determine the possible boundary slopes for different choices for P.

There are three genus two splittings (six when oriented), each identified by a partition of boundary components: $\Sigma^+_{12} \leftrightarrow \{\partial X_1, \partial X_2 \parallel \partial X_3\}$, $\Sigma^+_{13} \leftrightarrow \{\partial X_1, \partial X_3 \parallel \partial X_2\}$ and $\Sigma^+_{23} \leftrightarrow \{\partial X_2, \partial X_3 \parallel \partial X_1\}$. The compression bodies corresponding to $\Sigma_{ij} \leftrightarrow \{\partial X_i, \partial X_j \parallel \partial X_k\}$ are isotopic to $N(\partial X_i \cup a_{ij} \cup \partial X_j)$ and $N(\partial X_k \cup a_{kk})$. There is also an irreducible splitting of genus three identified by the partition $\Sigma^+_{123} \leftrightarrow \{\partial X_1, \partial X_2, \partial X_3 \parallel \varnothing\}$. In that case the compression body is given by $N(\partial X_1 \cup a_{12} \cup \partial X_2 \cup a_{23} \cup \partial X_3)$.

The canopy of $\mathcal{H}^{\pm}_{P \times S^1}$ is indicated in Figure 2. It has exactly one non-stabilized oriented Heegaard surface for each oriented partition of boundary components.

$$\begin{array}{lccccccccc}
g=3 & \Sigma^+_{123} & & & & & & & & \Sigma^-_{123} \\
g=2 & & & \Sigma^+_{12} & \Sigma^+_{13} & \Sigma^+_{23} & \Sigma^-_{23} & \Sigma^-_{13} & \Sigma^-_{12} &
\end{array}$$

Figure 2: Canopy of $\mathcal{H}^+_{\mathcal{X}}$ for $\{\text{pair of pants}\} \times S^1$. Up to isotopy, there is a single non-stabilized oriented Heegaard surface for each ordered partition of boundary components, for example $\Sigma^+_{12} \leftrightarrow \{\partial X_1, \partial X_2 \parallel \partial X_3\}$.

Of course, each of these splitting will also be a splitting after we fill in any or all of the exceptional fibers. In that case the splitting will be identified by partitions of boundary components *and* exceptional fibers f_i, for example $\{f_i, \partial X_j \parallel \partial X_k\}$. After filling, it is possible that splittings corresponding to distinct partitions now become isotopic. Fill ∂X_2 and consider the Heegaard surface Σ^+_{12} inducing the partition $\{\partial X_1, f_2 \parallel \partial X_3\}$. In this case the first compression body is a regular neighborhood of $\partial X_1 \cup a_{12} \cup f_2$. Suppose that the Seifert invariants of the fiber are (p,q) and that $q \equiv \pm 1 \pmod p$. This implies that we can find some *longitude*, a curve meeting the meridional disk of the attached solid torus once, that is also a dual curve meeting a regular fiber exactly once. In other words, we can isotope $a_{12} \cup f_2$ to appear as an eyehook on P, as in the central picture of Figure 3. Sliding the foot of the circle to ∂X_1 does not change the isotopy class of the Heegaard surface and demonstrates that this splitting is equivalent to the splitting induced by a_{11}. We have changed the partition from $\{\partial X_1, f_2 \parallel \partial X_3\}$ to $\{\partial X_1 \parallel f_2, \partial X_3\}$, demonstrating an isotopy between Σ^+_{12} and Σ^-_{23}. The isotopy *flips* f_2, moving it from one side of the partition to the other. Any fiber with Seifert invariants (p,q) so that $q \equiv \pm 1 \pmod p$ can be flipped. In essence, such fibers should be left out of the partition altogether. In fact, this is the only way that vertical splittings of a Seifert fibered spaces become isotopic (see Schultens [28, Theorem 5.1]).

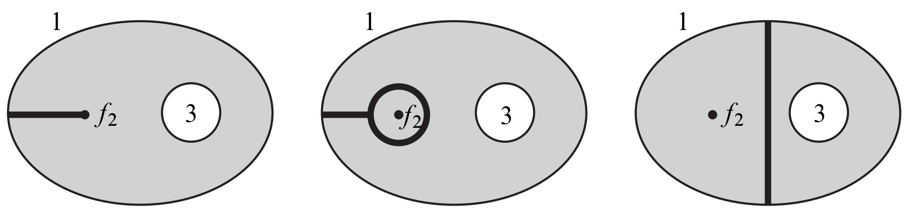

Figure 3: "Flipping" the fiber f_2: from $a_{12} \cup f_2$ to *eyehook* to a_{11}. Requires that $b \equiv \pm 1 \pmod a$ where (a,b) are the Seifert invariants of the fiber being flipped.

The genus three splitting is very fragile. While irreducible, it is boundary stabilized (see Moriah [15]). The notion of boundary stabilization will be discussed in greater detail in Section 6.1. In fact, it can be viewed as a boundary stabilization of each of the genus two splittings. This implies that it will destabilize after any filling on any one of the three boundary components.

3.3 Heegaard structure of the (p, q)–torus knot exterior

Genus two Heegaard splittings of torus knot exteriors were originally classified by the first author [14] (see also Boileau, Rost and Zieschang [3]). Since the torus knot exterior is a Seifert fibered space with boundary, any irreducible Heegaard splitting is vertical, hence isotopic to one of the three (unoriented) irreducible genus two Heegaard splittings of $\{\textit{pair of pants}\} \times S^1$, discussed in the previous section. We have pictured two of these as tunnels for the torus knot in Figure 4, they are:

(1) The inner tunnel – the inner exceptional fiber f_i joined to the knot via a vertical arc (a spanning arc for the annulus between ∂X and f_i) - $\Sigma_i^+ \leftrightarrow \{f_i, \partial X \parallel f_o\}$.

(2) The outer tunnel – the outer exceptional fiber f_o joined to the knot via a vertical arc (a spanning arc for the annulus running between ∂X and f_o) - $\Sigma_o^+ \leftrightarrow \{f_o, \partial X \parallel f_i\}$.

(3) The middle tunnel – a spanning arc for the *cabling annulus* $A = T - N(T_{p,q})$ - $\Sigma_m^+ \leftrightarrow \{\partial X \parallel f_i, f_o\}$.

These three splittings are distinct up to isotopy (and homeomorphism), unless $|p-q| = 1$, in which case all three are isotopic; or $|p-q| \neq 1$, but $p \equiv \pm 1 \mod q$ or $q \equiv \pm 1$

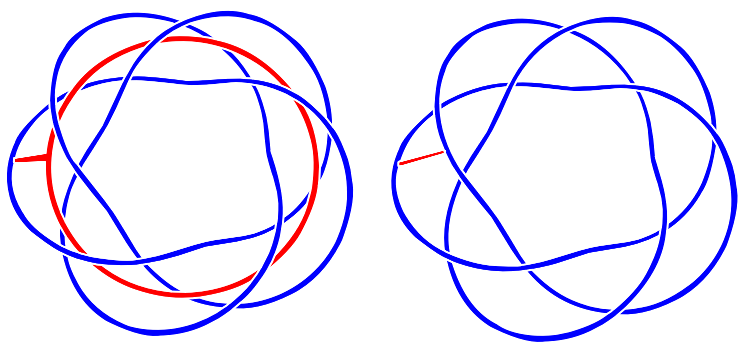

Figure 4: The inner and middle tunnels for a torus knot

mod p, in which case the middle splitting is isotopic to the inner or outer splitting, respectively (see Boileau, Rost and Zieschang [3] or Moriah [14], and the previous section). All (oriented) Heegaard surfaces of closed Seifert fibered spaces are equivalent after one stabilization (see Schultens [29]), therefore the canopy of the Heegaard tree is:

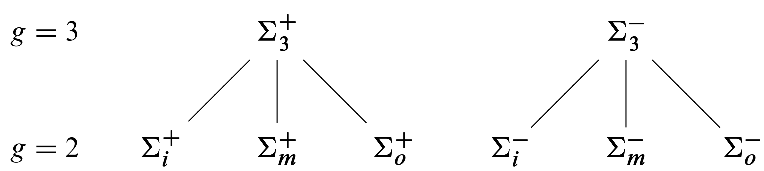

Figure 5: Canopy of $\mathcal{H}_X^+$ for a (p,q)–torus knot exterior X. (Unless $|p-q|=1$, in which case $\Sigma_i^+ = \Sigma_m^+ = \Sigma_o^+$; or $|p-q| \neq 1$, but $p \equiv \pm 1 \mod q$ or $q \equiv \pm 1 \mod p$, in which case $\Sigma_i^+ = \Sigma_m^+$ or $\Sigma_o^+ = \Sigma_m^+$, respectively).

3.4 The manifolds obtained by filling

The manifolds obtained by a Dehn filling on a torus knot exterior were classified by Moser [21], whose theorem is rephrased slightly here:

Theorem 3.1 (Moser) *Suppose that $\frac{r}{s}$–Dehn filling is performed on a non-trivial (p,q)–torus knot and let $a = \Delta_A(\frac{pq}{1}, \frac{r}{s}) = pqs - r$ be the algebraic intersection number between the slope of a regular fiber and the meridian of the attached solid torus. The type of the filled manifold $X(\frac{r}{s})$ depends on a:*

(1) $a = 0 \implies X(\frac{r}{s}) = L(p,q)\#L(q,p)$, *a connected sum of lens spaces.*

(2) $|a| = 1 \implies X(\frac{r}{s}) = L(|r|, sq^2)$, *a lens space.*

(3) $|a| > 1 \implies X(\frac{r}{s}) = SFS\{S^2|(p,q),(q,p),(a,b)\}$, *a Seifert fibered space over* S^2 *with three exceptional fibers with Seifert invariants* (p,q), (q,p) *and* (a,b).

The Seifert invariants are not normalized and $b = \Delta_A(\frac{pq}{1}, \frac{t}{u}) = pqu - t$, *where* $\frac{t}{u}$ *is the slope of any longitude for the attached solid torus.*

Remark Note that $a = 0 \Leftrightarrow \frac{r}{s} = \frac{pq}{1}$, $|a| = 1 \Leftrightarrow \frac{r}{s} \in L_{pq/1}$, and $|a| > 1$ otherwise.

3.5 Heegaard structure of $\mathbf{S^3}$

Of course, S^3 appears as a special case of Theorem 3.1 (2), the $\frac{1}{0}$–filling produces $L(1,0)$, that is, S^3. Waldhausen [36] showed S^3 has a unique non-stabilized Heegaard surface, an S^2 splitting S^3 into two balls. It is also easy to see that this surface flips. Each ball is a regular neighborhood of a point in its interior, and any two neighborhoods of points are isotopic in S^3. This defines an isotopy that reverses the orientation of the Heegaard sphere. Therefore the canopy of the oriented Heegaard tree for S^3 is as simple as possible: It is a single point.

S^2

Figure 6: Canopy of $\mathcal{HT}^{\pm}_{S^3}$ is a single point, $\alpha = \frac{1}{0}$.

3.6 Heegaard structure of $\mathbf{L(|r|, sq^2)}$

A lens space also has a unique non–stabilized Heegaard surface, a torus T (see Bonahon and Otal [4]). The splitting does not flip unless the cores of the solid tori are isotopic, hence equivalent as generators of the fundamental group. This will occur precisely for those lens spaces homeomorphic to $L(k,1)$ for some $k \in \mathbb{N}$. A check will reveal that such a manifold can not be obtained from surgery on a torus knot, except for the $\frac{5}{1}$–filling on the $(3,2)$–torus knot that which produces $L(5,4)$. So for any other lens space, including all of those we are considering, we have:

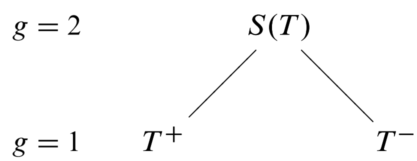

Figure 7: Canopy of $\mathcal{HT}^{\pm}_{L(p,q)}, q \not\equiv \pm 1 \pmod{p}$, $\alpha \in L_{pq/1} - \{1/0\}$.

3.7 Heegaard structure of L(p, q)#L(q, p)

The Haken Lemma implies that any Heegaard splitting of $L(p,q)\#L(q,p)$ is the connected sum of a Heegaard torus T_1 for $L(p,q)$ and a Heegaard torus T_2 for $L(q,p)$. For the connected sum, we can form two non–oriented Heegaard surfaces, $T_1^+\#T_2^+$ and $T_1^+\#T_2^-$, or four oriented Heegaard surfaces, since $(T_1^+\#T_2^+)^- = (T_1^-\#T_2^-)$ and $(T_1^+\#T_2^-)^- = T_1^-\#T_2^+$. Of course, it is possible that two or more of these surfaces are isotopic, and that will definitely be the case when a Heegaard torus for one of the summand flips, that is, when $q \equiv \pm 1 \pmod{p}$ or $p \equiv \pm 1 \pmod{q}$. However, Engmann [8] has shown that the connected sum of lens spaces $L(p_1,q_1)\#L(p_2,q_2)$ has four distinct oriented Heegaard surfaces up to homeomorphism, hence isotopy, unless: a) $q_i^2 \equiv \pm 1 \pmod{p_i}$ for some i, or b) $p_1 = p_2$ and $q_1 q_2^{-1} \equiv \pm 1 \pmod{p_1}$. Since $p \neq q$, we are concerned only with a). In that case, it is clear that the the Heegaard surfaces T_i^+ and T_i^- for $L(p_i,q_i)$ are homeomorphic, reducing the number of non-homeomorphic splittings for the connected sum. However, unless some $q_i \equiv \pm 1 \pmod{p_i}$, it is not clear that these splittings are isotopic and we conjecture that they are not. For torus knots satisfying the conditions 1) and 2) in Section 3.1, $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha = \frac{pq}{1}$ is indicated in the following figure. We also conjecture that this represents $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha = \frac{pq}{1}$ assuming only condition (1).

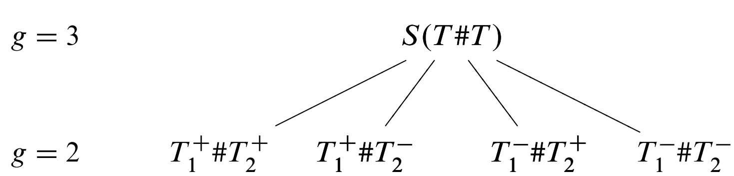

Figure 8: Canopy for $\mathcal{HT}^{\pm}_{L(p,q)\#L(q,p)}$, obtained by filling along $\alpha = \frac{pq}{1}$. This holds when $q^2 \not\equiv \pm 1 \pmod{p}$, $p^2 \not\equiv \pm 1 \pmod{q}$ and we conjecture that it holds with the weaker assumption $q \not\equiv \pm 1 \pmod{p}$, $p \not\equiv \pm 1 \pmod{q}$.

3.8 Heegaard structure of $SFS\{S^2 \mid (p,q),(q,p),(a,b)\}$

Heegaard splittings of Seifert fibered spaces are either vertical or horizontal (see Moriah–Schultens [17] and Schultens [30]). The vertical splittings of a Seifert fibered spaces over S^2 with three exceptional fibers are inherited from the torus knot exterior: Σ_i, Σ_o and Σ_m. All three are genus two, therefore minimal genus and irreducible. There may or may not be an irreducible horizontal splitting, but if it exists it is unique up to isotopy (see Moriah and Schultens [17], Sedgwick [33] and Bachman–Derby-Talbot [2]).

As already noted, the three splittings will be distinct up to isotopy unless $b \equiv \pm 1 \pmod{|a|}$ (see Moriah [14, Theorem 1] and Schultens [28, Theorem 5.1]). In that case, the inner and outer splittings are isotopic. More specifically, Σ_i^+ is isotopic to Σ_o^- (see Section 6.2). This occurs when there is a curve with slope $\frac{t}{u}$ which is a longitude for the filling solid torus and that meets the regular fiber of the Seifert fibration once, that is, $\Delta(\frac{t}{u},\frac{r}{s}) = 1$. This is precisely the set of slopes on the line of lines $LL_{pq/1}$. Equivalently, it is the set of slopes with distance 2 from the slope of the regular fiber $\frac{pq}{1}$ in the Farey graph.

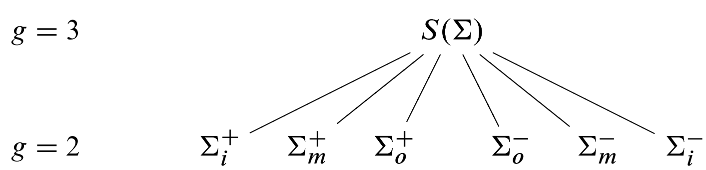

Figure 9: Canopy for the "generic" Seifert fibered space obtained by filling a torus knot exterior. There are three non-isotopic vertical splitting. $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha \notin L_{0/1} \cup LL_{pq/1}$.

There is only one line, $L_{0/1} = \{\frac{1}{n} | n \in \mathbb{Z}\}$, of fillings which produce a manifold with a strongly irreducible *horizontal* splitting. Recall that the torus knot exterior fibers (uniquely) over the circle, $X = F \tilde{\times} S^1$. In X a neighborhood of F is a handlebody, as is its complement. If the meridional slope $\frac{r}{s}$ for the filling solid torus meets the slope of $\partial F = \frac{0}{1}$ once, then the solid torus can be glued to either handlebody and in both cases the result will still be a handlebody, hence we get a Heegaard splitting of $X(\frac{1}{n})$. It has genus $g = 2g(F) = 2(p-1)(q-1)$. The splitting is (strongly) irreducible if and only if $|a| = |pqn - 1| > \mathrm{lcm}(p,q) = pq$ [33]. That is, the horizontal splitting is strongly irreducible unless $\frac{1}{n}$ is one of $\frac{1}{0}$ or $\frac{1}{1}$. (Since $X(\frac{1}{0}) = S^3$, we already know that the horizontal splitting reduces in that case.)

Although there are Seifert fibered spaces which have infinitely many non-isotopic horizontal splittings, they are always obtained by Dehn twisting in non-separating tori

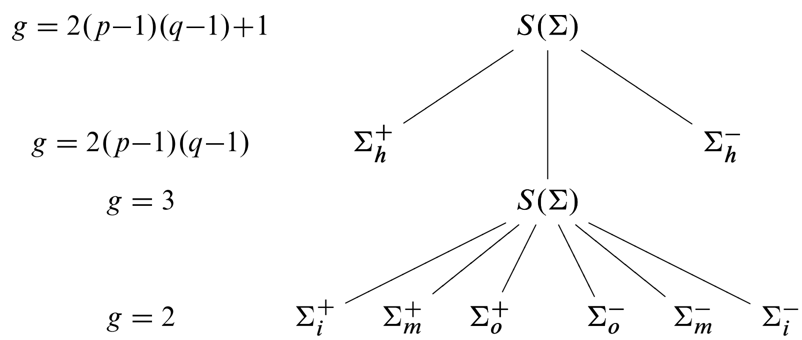

Figure 10: Canopy for Seifert fibered spaces possessing three vertical splittings and an irreducible horizontal splitting. $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha \in L_{0/1} - \{\frac{1}{0}, \frac{1}{1}, \frac{-1}{1}\}$.

[2]. However, the Seifert fibered spaces in question here have no essential tori, so the fiber F, hence the horizontal splitting, are both unique up to isotopy.

As we have already noted, Schultens has shown that Heegaard splittings of Seifert fibered spaces are equivalent after one stabilization [29]. Her argument applies to oriented Heegaard surfaces of closed Seifert fibered spaces.

It follows that we have four possibilities for the stabilization tree $\mathcal{H}^{\pm}_{X(\alpha)}$ when the filled manifold is a Seifert fibered space that is not a lens space or S^3. In the "generic" case, there will be three non-isotopic vertical splittings (six oriented). Exceptions will occur when the filled manifold has only two vertical surfaces up to isotopy ($\frac{r}{s} \in LL_{pq/1}$), the filled manifold contains an irreducible horizontal splitting of higher genus ($\frac{r}{s} \in L_{0/1}$), or both ($\frac{r}{s} \in LL_{pq/1} \cap L_{0/1}$). We leave the fact that $LL_{pq/1} \cap L_{0/1} = \{\frac{-1}{1}, \frac{1}{0}, \frac{1}{1}\}$ as an exercise. Note however, that the $\frac{1}{0}$–filling produces S^3 and the horizontal splitting is reducible after the $\frac{1}{1}$–filling, so the $\frac{-1}{1}$–filling is the unique (!) slope yielding two vertical splitting and a horizontal splitting. The four possible canopies are indicated in Figures 9–12.

4 New Heegaard surfaces – the framework

Suppose that we are given a surface $\Sigma \subset X$ and are asked for the set of filled manifolds for which it is a Heegaard surface. First, if Σ is a Heegaard surface for any $X(\alpha)$, then Σ must separate X into two components V and W, where W is a handlebody and V is a *punctured handlebody*, a handlebody with the neighborhood of a knot removed. Then Σ is a Heegaard surface for $X(\alpha)$ if and only if $V(\alpha)$ is a handlebody.

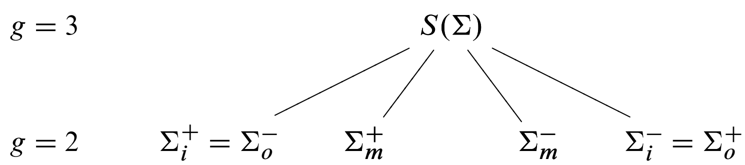

Figure 11: Canopy for the Seifert fibered space with two non-isotopic vertical splittings and no horizontal splitting, $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha \in LL_{pq/1} - \{\frac{-1}{1}, \frac{1}{0}, \frac{1}{1}\}$.

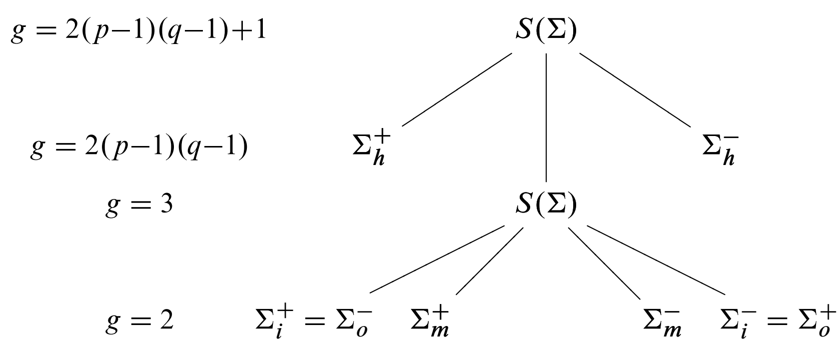

Figure 12: Canopy for Seifert fibered space with two non-isotopic vertical splittings and an a horizontal splitting. $\mathcal{HT}^{\pm}_{X(\alpha)}, \alpha = \frac{-1}{1}$.

We can use the results of Culler-Gordon-Luecke-Shalen [7] and Wu [37] to address this situation. To implement their results it is required that $\partial V - \partial X$ is incompressible, and we have not assumed that. We can reduce to that situation by first maximally compressing Σ in V to obtain $\Sigma' \subset V$ an incompressible surface bounding a punctured handlebody $V' \subset V$. If Σ' is peripheral, then Σ was itself a Heegaard surface for the knot exterior and thus for every filling on X.

Suppose that Σ' is not peripheral and that V' contains an incompressible annulus A_σ with one boundary component in Σ and the other a curve of slope σ in ∂X. If Σ bounds a handlebody in $X(\alpha)$ for $\alpha \neq \sigma$, then Σ' compresses in $X(\alpha)$ and by [7, Theorem 2.4.3], $\alpha \in L_\sigma$ and Σ is a Heegaard surface for every $\alpha \in L_\sigma$. In this case we will call Σ a *horizontal Heegaard surface*. It is a Heegaard surface for every filling $X(\alpha), \alpha \in L_\sigma$. By sliding the annulus off the scars from the disk compressions and reattaching the disks, we can assume the annulus runs between Σ and ∂X. For any slope $\alpha \in L_\sigma$, this annulus defines an isotopy of the core of the attached solid torus into the surface Σ.

If there is no incompressible annulus joining Σ and ∂X and Σ is a Heegaard surface for $X(\alpha_1)$ and $X(\alpha_2)$, then the results of Wu [37] imply that $\Delta(\alpha_1, \alpha_2) = 1$. There are therefore at most three fillings for which Σ is a Heegaard surface.

This analysis does not solve our problem because we do not have a nice short list of candidate Σ's to become Heegaard surfaces. It does, however, provide a useful framework that is used in various papers on this subject. Let Σ be a Heegaard surface for a manifold $X(\alpha)$ that is obtained by performing a Dehn filling on a knot manifold X. We regard the core curve γ of the attached solid torus as a knot in $X(\alpha)$. The core of the attached solid torus γ and the Heegaard surface Σ can have one of three possible relationships after performing isotopies in $X(\alpha)$:

(C)ore: γ is isotopic into Σ and can be further isotoped so that Σ is a Heegaard surface for the knot exterior X.

(H)orizontal but not a core: γ is isotopic into Σ but cannot be isotoped so that Σ is a Heegaard surface for X.

(N)ot level: γ cannot be isotoped into Σ.

Case C describes an old splitting not a new one. For a new splitting Σ, the issue is then whether or not the core of the attached solid torus γ is isotopic into Σ, case H, or not, case N. We would like to limit the set of slopes for which condition H or N occurs. Condition N is also referred to as a "bad" filling in Rieck–Sedgwick [25].

Which of the Heegaard surfaces from the previous section are *new*, that is, not isotopic in the filled manifold to a Heegaard surface for the torus knot exterior? By definition, the vertical splittings are not new as they are also splittings of the knot exterior. Every other non-stabilized splitting is new: the Heegaard S^2 in S^3, the Heegaard tori in the lens spaces, and the horizontal splittings. Each of these has genus different than two and therefore cannot be isotopic to a Heegaard surface for the knot exterior, all of which are genus two.

Note that in the filled manifolds, the core of the attached solid torus is not isotopic into the Heegaard surface S^2 for S^3, but is isotopic into both the Heegaard tori for the lens spaces, and the horizontal splittings, when they occur.

4.1 N – The non-level case

Note that when condition N occurs, the core curve γ has some bridge number $b > 0$ with respect to the Heegaard surface Σ. The only filling on a torus knot exterior producing condition N is the $\frac{1}{0}$–filling which produces S^3. The core γ is isotopic into every other Heegaard surface (H or C) for every other manifold obtained by filling on the torus knot. Of course, γ is a torus knot in S^3 which has bridge number $b = \min\{p,q\}$ with respect to a sweepout by S^2s (see Schubert [27] and Schultens [31]). Tubing Σ along the b upper (or lower) bridges, one builds a Heegaard surface for the knot exterior of genus $g(\Sigma)+b$. Tubing corresponds to stabilization in the filled manifold, but the resulting surface may or may not be a stabilized Heegaard surface for the knot manifold. For torus knots, this process yields a Heegaard surface of genus b for the knot exterior, which is irreducible only when its genus is minimal, that is, when $b=2$. Such torus knots have $p=2$ or $q=2$ and are excluded for consideration here by the first condition in Section 3.1.

4.2 H – The horizontal case, parallel and boundary stabilization

In case H, we can assume that $\gamma \subset \Sigma$ in the filled manifold $X(\alpha)$. The surface $\Sigma^* = \Sigma - N(\gamma)$ is a surface with boundary properly embedded in the knot exterior X. Moreover, since γ is isotopic into the surface, the meridional slope α intersects the slope σ of the surface Σ^* precisely once. The surface Σ^* has two boundary components and splits X into two handlebodies. Such a surface is referred to as an *almost Heegaard surface* in Rieck–Sedgwick [25]. As noted, Σ is a Heegaard surface for every manifold $X(\alpha')$ where $\alpha' \in L_\sigma$, where σ is the slope of Σ^*. These manifolds differ by Dehn twists in the curve γ in the Heegaard surface Σ .

The lens space fillings on the torus knot exterior are a simple example: Σ is the Heegaard torus, Σ^* is the cabling annulus and its slope is $\sigma = \frac{pq}{1}$. Moreover, Σ is a Heegaard surface for the entire line $L_{pq/1}$ of lens space fillings. The collection of Seifert fibered manifolds possessing horizontal splittings are another example, this time occurring with filling coefficients $L_{0/1}$. Note that while most of these splittings are irreducible, there are a few cases where they reduce ($\frac{1}{0} \in L_{pq/1}$ and $\frac{1}{0}, \frac{1}{1} \in L_{0/1}$). It is a theorem of Casson and Gordon [5] that if Σ^* is incompressible then the distance between weakly reducible fillings on the line is at most 6. See the appendix of Moriah–Schultens [17] for a proof. For horizontal splittings of Seifert fibered spaces, there are at most two fillings on the line that result in weakly reducible Heegaard splittings (see Sedgwick [33]).

We can also use Σ to form a Heegaard surface for the knot exterior of genus $g(\Sigma)+1$ by a process we will call *parallel stabilization*: Push the surface Σ to one side of the knot, say below, see Figure 13. Now, Σ and ∂X cobound an annulus A. Surgering Σ along A yields Σ^*, hence A and Σ^* have the same boundary slope on ∂X. Let a be a spanning arc for A. We can perform *parallel stabilization* on Σ by tubing around γ and then attaching the tube to Σ by a tube around a. To be more formal, note that the surface $\partial N(\Sigma \cup a \cup \gamma)$, (neighborhood taken in $X(\alpha)$) has two components. The parallel stabilization is the component that has one higher genus than Σ. It is a Heegaard surface for the knot exterior X. Note that we can parallel stabilize Σ in two ways, by starting with Σ either above or below γ. These surfaces are not in general isotopic in X, but they are in the filled manifold $X(\alpha)$, because there they are both stabilizations of the same surface Σ. It is not hard to check that parallel stabilization does not depend on the choice of annulus, even for annuli with different slopes.

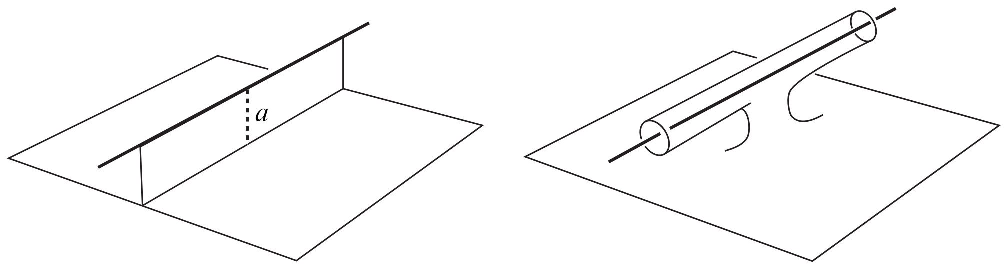

Figure 13: Parallel stabilization of a horizonal surface

Let A_α be the meridional annulus, A_σ be the longitudinal annulus, and D the disk that appear in Figure 14. The annuli A_σ and A_α meet in a single arc, $A_\sigma \cap D = \varnothing$ and $A_\alpha \cap D = \{pt\}$. Together, A_α and D demonstrate that the parallel stabilization is γ–primitive, for $\gamma = \alpha$. But, α is not unique in this regard. By twisting in A_σ we can construct an $A_{\alpha'}$ meeting D once for any $\alpha' \in L_\sigma$. The existence of the triple (A_α, A_σ, D) with the specified intersections is equivalent to the Heegaard splitting being a parallel stabilization.

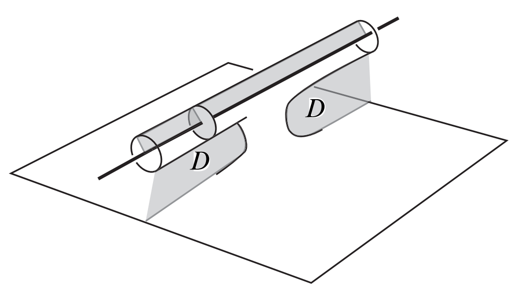

Figure 14: The disk D and annuli A_α and A_σ in a parallel stabilization

There is an isotopic picture of the parallel stabilization that will be very useful in Section 6.2. See Figure 15. Shrink the disk D so that it is small, and then flatten the top surface. This makes the knot boundary ∂X appear to be below the surface, and the hole defined by D now appears to be a tube below the surface. It is also easy to see A_α as a once punctured disk that meets the disk D once. The annulus A_σ runs between the knot and the surface above it.

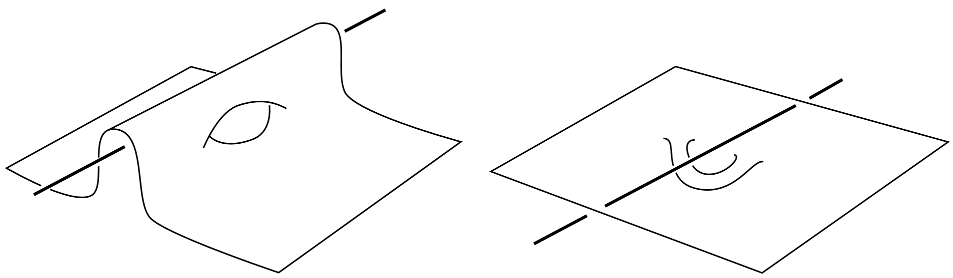

Figure 15: Surfaces that are isotopic to the parallel stabilization. First make the disk D small, and then flatten the top. This forces the hole to appear as a tube below the surface.

If Σ was a Heegaard surface for the knot exterior, and not just a horizontal surface, then we will call the parallel stabilization of Σ a *boundary stabilization*. In this case it can also be viewed as an amalgamation of Σ with a type (ii) splitting of product neighborhood of ∂X, see Moriah [15]. Since Σ cobounds annuli of every slope with ∂X, and parallel stabilization doesn't depend on this choice, a boundary stabilization is γ–primitive for every slope on L_σ for every σ. In other words, a boundary stabilization is γ–primitive for every slope γ.

An examination of Figure 4 makes it clear that the inner and outer tunnel systems of the torus knot exterior are parallel stabilizations of parallel copies of, just above and below, the standard torus in which $T_{p,q}$ is embedded. However, the middle splitting is not a parallel stabilization because it is not γ–primitive for any γ when it is not isotopic to either the inner or outer splitting, that is, for $p \not\equiv \pm 1 \pmod{q}$ and $q \not\equiv \pm 1 \pmod{p}$ [19].

When a parallel stabilization is minimal genus then it is clearly irreducible. This is the case with the inner and outer splittings of the torus knot exterior. But this need not be the case. In fact, when filling a torus knot exterior, if the filled manifold has an irreducible horizontal Heegaard splitting Σ, then its parallel stabilization is reducible. Its parallel stabilization Σ' is a splitting of the knot manifold, a Seifert fibered space with boundary. As already noted it is proved by Schultens [28] that such splittings are vertical and irreducible only when they are minimal genus. But if $p > 3$ then the genus of Σ' is $2(p-1)(q-1)+1>3$, strictly greater than two, the genus of the knot exterior.

5 New Heegaard surfaces – the results

We now survey results that restrict (N) and (H).

5.1 Moriah and Rubinstein

In order to prove the existence of knots with super-additive tunnel number, Moriah and Rubinstein [16] needed to show that there are fillings on a knot exterior for which the Heegaard genus does not drop at all. If the genus does drop, then the filled manifold has a Heegaard surface of genus lower than that of every Heegaard surface for the knot exterior, that is, the filled manifold possesses a new Heegaard surface.

In fact, they conclude that there *is* a finite list of candidates of bounded genus for all but a finite number of fillings, and the genus does not degenerate for "most" fillings, as is demonstrated by the following theorem and corollary. Their theorem applies to manifolds with multiple torus boundary components. Here we have restated it only for knot–manifolds and in terms consistent with our discussion:

Theorem 5.1 (Moriah and Rubinstein – rephrased) *Let X be hyperbolic knot–manifold and g a positive integer. Then there is finite set of slopes $\mathcal{N}_X$ and a collection of Heegaard and horizontal surfaces $\Sigma_1,\dots,\Sigma_k \subset X$, so that if $\alpha \notin \mathcal{N}_X$ then any Heegaard splitting of $X(\alpha)$ of genus less than or equal to g is isotopic to one of the Σ_i.*

Their theorem implies several things for Heegaard surfaces of bounded genus, in particular minimal genus, in hyperbolic knot–manifolds. Condition N occurs for at most finitely many slopes, those in $\mathcal{N}_X$. Away from this finite number of slopes, the core is isotopic into *every* Heegaard surface of bounded genus. And, H occurs for all slopes on a finite (possibly empty) set of lines $L_{\beta_1},\dots,L_{\beta_k}, k \geq 0$.

By avoiding a finite set of slopes for N and a finite set of lines for H, *every* Heegaard surface of bounded genus for the filled manifold is isotopic to a Heegaard surface for the knot exterior. In particular, the Heegaard genus of these manifolds is the same as that of the knot exterior. Of course, if the genus decreases and we are not in situation N, then the discussion on the horizontal case H (Section 4.2) shows that the genus decreases by at most one.

A corollary of their theorem is:

Corollary 5.2 (Moriah and Rubinstein) *Let X be a hyperbolic knot–manifold. Then there exists a finite set of slopes $\mathcal{N}_X$ and a finite set of lines $\mathcal{H}_X$ so that:*

(1) $\alpha \notin \mathcal{N}_X \cup \mathcal{H}_X \implies g(X(\alpha)) = g(X)$

(2) $\alpha \notin \mathcal{N}_X \implies g(X(\alpha)) \geq g(X) - 1$.

where g denotes the Heegaard genus of the manifold.

This theorem forbids new Heegaard splittings of bounded genus for most filled manifolds, but leaves open that possibility for new Heegaard surfaces when we do not bound genus. That is, as we increase the bound g the sets of slopes for conditions N and H could grow in an unbounded fashion. In fact, as discussed in Section 5.3, this does not happen.

5.2 Rieck

Rieck [22] took a topological approach to the same problem and computed numeric bounds on the distance between "bad", that is, type N, fillings. Suppose that N occurs with respect to Heegaard surfaces $\Sigma_1 \subset X(\alpha_1)$ and $\Sigma_2 \subset X(\alpha_2)$. Since the core is not isotopic into either $\Sigma_1 \subset X(\alpha_1)$ or $\Sigma_2 \subset X(\alpha_2)$, it can be put into non-trivial thin position with respect to sweepouts by each surface. Furthermore, thick level surfaces for each sweepout, when regarded as punctured surfaces in the knot exterior, will intersect essentially. This approach yields a bound on the distance between the slopes α_1 and α_2:

Theorem 5.3 (Rieck – rephrased) *Let X be an anannular knot manifold. Suppose that the core of the attached solid torus is not isotopic into Heegaard surfaces $\Sigma_1 \subset X(\alpha_1)$ and $\Sigma_2 \subset X(\alpha_2)$. Then $\Delta(\alpha_1, \alpha_2) < 18g_1g_2 + 18g_1 + 18g_2 + 18$, where g_1 and g_2 are the genera of Σ_1 and Σ_2, respectively.*

In [23], Rieck examines the relationship between Dehn filling and Heegaard structure from a different viewpoint. He asks which manifolds possess a *genus reducing knot*, a knot for which infinitely many surgeries decrease Heegaard genus. He answers this question for all totally orientable Seifert fibered spaces other than those with base space S^2 and three or fewer exceptional fibers. Almost all of the considered Seifert fibered spaces do contain contain genus reducing knots, the exception being Seifert fibered spaces possessing a horizontal Heegaard surface of one of two special types.

5.3 Rieck and Sedgwick

Condition H is explored further in Rieck and Segwick [25]. As we have already noted, in that case we can form the surface $\Sigma^* = \Sigma - N(\gamma)$, a properly embedded surface in the knot–manifold X.

Theorem 5.4 (Rieck and Sedgwick) *Suppose that the core of the attached solid torus is isotopic into Σ a Heegaard surface for a filled manifold $X(\alpha)$. Then one of the following holds:*

(1) *Σ is a Heegaard surface for X (perhaps after an isotopy in $X(\alpha)$, or,*

(2) *the slope of the almost Heegaard surface Σ^* is the boundary slope of a separating essential surface of genus less than or equal to that of Σ^*.*

If the second conclusion occurs, then the slope α is one that intersects the slope of an essential surface exactly once. Hatcher has shown such slopes to be finite in number [10], so if H occurs we know that the slope α belongs to one of a finite number of lines defined by slopes of essential surfaces. This improves the earlier work of Moriah and Rubinstein because the knot–manifold is not required to be hyperbolic, it applies to all surfaces without a bound on genus, and a connection is made between these slopes and the slopes of essential surfaces.

It would be nice if the surface Σ^* were itself an essential surface. However, the method of proof is similar to that of Casson and Gordon [6] and may require that Σ^* is modified by compressions and annulus swaps to obtain an essential surface.

In their second paper [24] Rieck and Sedgwick continued their investigation into the Heegaard structure of filled manifolds. We can assume that the knot–manifold X is given via a one-vertex triangulation, see Jaco and Sedgwick [12, Theorem 3.2]. They then prove (slightly rephrased):

Theorem 5.5 (Rieck and Sedgwick) *Let $\mathcal{T}$ be a one-vertex triangulation of the knot–manifold X. If γ, the core of the attached solid torus, is not isotopic in $X(\alpha)$ into a Heegaard surface Σ, then the slope α is either the slope of a boundary edge of the triangulation or the slope of a normal or almost normal slope in $(X, \mathcal{T})$.*

The proof of the theorem follows Thompson's proof [35] that a triangulation of S^3 contains an almost normal S^2. The 1–skeleton of the triangulation is put in thin position with respect to a sweepout given by the Heegaard surface Σ. If a boundary edge is isotopic into the Heegaard surface, we either have an edge with slope α or the core γ is isotopic into Σ. Otherwise, we are able to find a non-trivial thick level, yielding a normal or almost normal surface with slope α in Σ. Bachman [1] has a similar result.

They then apply a theorem of Jaco and Sedgwick [12] stating that there are only finitely many slopes bounding normal and almost normal surfaces in such a triangulation.

Theorem 5.6 (Jaco and Sedgwick) *Let X be a knot-manifold with a triangulation $\mathcal{T}$ that restricts to a one–vertex triangulation on ∂X. Then there are only a finite number of slopes realized as the slopes of embedded normal and almost normal surfaces in $(X, \mathcal{T})$.*

The proof is an analog of Hatcher's proof [10] that there are a finite number of slopes bounding essential surfaces in a knot–manifold. It is shown that normal or almost normal surfaces that are compatible, meaning that their normal sum is well defined, must have the same slope or their sum produces trivial curves in the boundary. Whereas Hatcher's theorem relies on Floyd and Oertel's work with branched surfaces [9], this proof appeals to similar properties of normal and almost normal surfaces in a triangulation.

Corollary 5.7 (Rieck and Sedgwick, rephrased) *Let X be a knot–manifold. Then there exists a finite set of slopes $\mathcal{N}_X$ in ∂X so that if $\alpha \notin \mathcal{N}_X$, then the core of the attached solid torus γ is isotopic into every Heegaard surface for $X(\alpha)$.*

Note that this theorem does not require a bound on genus and applies to non-hyperbolic as well as hyperbolic knot exteriors.

5.4 Summary of known results

For clarity, we offer a summary of the known results. Recall the trichotomy offered at the start of this section. The core of the attached solid torus γ is either (N)ot Level, (H)orizontal but not a core, or a (C)ore of a given Heegaard surface Σ.

Theorem 5.8 *Let X be a knot–manifold. Then there is a finite set of slopes $\mathcal{N}_X$ and a finite set of lines $\mathcal{H}_X$ so that:*

(1) *If $\alpha \notin \mathcal{N}_X$, then the core of the attached solid torus is isotopic into every Heegaard surface for $X(\alpha)$, and in particular $g(X) - 1 \leq g(X(\alpha)) \leq g(X)$, and,*

(2) *If $\alpha \notin \mathcal{N}_X \cup \mathcal{H}_X$ then $X(\alpha)$ does not contain a new Heegaard surface, that is, every Heegaard surface for $X(\alpha)$ is isotopic (in $X(\alpha)$) to a Heegaard for X, and in particular $g(X(\alpha)) = g(X)$.*

Remark For fillings on any torus knot exterior we can take $\mathcal{N}_X = \{\frac{1}{0}\}$ and $\mathcal{H}_X = \{L_{pq/1} \cup L_{0/1}\}$.

6 What is not known?

While we have answered many of the questions regarding the Heegaard structure of filled manifolds, there are at least several that remain.

6.1 Destabilization

Question 6.1 *Let Σ be a non-stabilized Heegaard surface for X. What can be said about the set of the fillings for which Σ destabilizes?*

When filling a torus knot exterior the inner, outer and middle splittings necessarily destabilize when the obtained manifold is a lens space or S^3, that is, for the fillings on the line $L_{pq/1}$.

This is evident in the fact that each of these splitting is padded. A Heegaard surface Σ is said to be *padded* if there exists a triple (P, A_σ, D) where:

(1) P is a punctured disk in the compression body, (a *punctured disk* is a planar surface in a compression body that has one boundary component, its "boundary", in the Heegaard surface and all others, the "punctures", in ∂X)

(2) A_σ is a vertical annulus in the compression body with slope σ on ∂X.

(3) D is a disk in the handlebody,

(4) $|\partial P \cap \partial D| = 1$

(5) $|\partial A_\sigma \cap \partial D| = 0$.

Condition (5) clearly implies that padded splittings are weakly σ–primitive. But in fact, it is much stronger: For a strongly irreducible Heegaard surface the results of Culler–Gordon–Luecke–Shalen [7] and Wu [37] imply that σ meets the slope of the punctures once. Twisting P in the annulus A_σ yields a destabilizing pair (P_α, D) for every slope $\alpha \in L_\sigma$.

We have observed that the inner and outer splittings Σ_i and Σ_o are parallel stabilizations. This is a very strong form of padded where the punctured disk P is actually an annulus (take $P = A_\alpha, A_\sigma = A_\sigma, D = D$). The middle splitting Σ_m is also padded where $\sigma = \frac{pq}{1}$, but it is slightly harder to see. The handlebody component of $X - \Sigma_m$ is a regular neighborhood of the union of the critical fibers joined by a spanning arc a_{io} for the annulus A_{io} running between them, that is, $N(f_i \cup a_{io} \cup f_o)$. The arc a_{io} punctures the cabling annulus A once so we may think of the compression body as a product $T' \times [-1, 1] - N(T_{p,q})$, where T' is a copy of T with a single puncture. Take D to be the meridional disk for the inner solid torus, $A \subset V$ as the vertical annulus $T_{p,q} \times [0,1]$ with slope $\frac{pq}{1}$ and boundary towards the outside solid torus, and $P = \{a\} \times I - N(T_{p,q})$, where a is any arc properly embedded in T' that meets ∂D exactly once.

It may be surprising to realize that there is a non-stabilized Heegaard splitting that destabilizes for every filling. As noted in Section 4, a boundary stabilization is γ–primitive for every γ so it necessarily destabilizes for every filling. Hence the question becomes: Can a boundary stabilized Heegaard splitting be non-stabilized in the knot-manifold? Yes, as is shown in Sedgwick [34] and Moriah–Segwick [18]. Let Σ_{ij} be one of the genus two splittings for $\{$*pair of pants*$\} \times S^1$ discussed in Section 3.2. Then the annuli A_{ik} and A_{jk} are annuli that we can use to parallel stabilize (boundary stabilize, since we start with a Heegaard surface). In each case, we obtain the genus three splitting Σ_{123} which is identified by the partition of boundary components $\{\partial X_1, \partial X_2, \partial X_3 \parallel \varnothing\}$. It is induced by the pair of arcs $a_{12} \cup a_{23}$ (or $a_{12} \cup a_{13}$ or $a_{13} \cup a_{23}$). There are also examples of boundary stabilized but non-stabilized Heegaard splittings for manifolds with two boundary components [18]. The question remains open for knot-manifolds.

In general, we expect to be able to say little about a Heegaard splitting that destabilizes for a finite number of fillings. But what if it destabilizes for infinite number of fillings? The above examples demonstrate that padded splittings destabilize for infinitely many fillings and a boundary stabilization destabilizes for all fillings. We conjecture the converses:

Conjecture 6.2 *Suppose that a non-stabilized Heegaard surface destabilizes for infinitely many fillings. Then it is padded.*

Conjecture 6.3 *Suppose that an irreducible Heegaard surface destabilizes for all fillings. Then it is a boundary stabilization.*

It might appear that Theorem 5.8 would answer the questions raised in this section. But Theorem 5.8 only restricts the set of slopes for which a given Heegaard surface destabilizes to a *new* Heegaard surface. But, it is possible for a non-stabilized Heegaard surface to destabilize in the filled manifold where it becomes isotopic to another Heegaard surface for the knot exterior. For example, a boundary stabilization of a Heegaard surface Σ destabilizes in every filling, and the destabilized surface is isotopic to Σ.

But, more importantly, Theorem 5.8 doesn't say anything about the structure of the surface destabilizing, even if it is of minimal genus: Let Σ be a minimal genus Heegaard surface for X that destabilizes in infinitely many fillings, in particular in some filled manifold $X(\alpha)$ where $\alpha \notin \mathcal{N}_X$. Then Σ destabilizes to a Heegaard surface Σ' for

$X(\alpha)$ and the core of the attached solid torus is isotopic into Σ'. We can construct Σ'', a parallel stabilization of Σ', that is a Heegaard surface for X of the same genus as that of Σ. The parallel stabilization Σ'' destabilizes in $X(\alpha)$, as well as in an infinite number of other filled manifolds. But we don't know that Σ and Σ'' are the same surface in X, all we know is that they become isotopic in $X(\alpha)$.

In fact, the above situation is not vacuous. The middle splitting of the (p,q)–torus knot exterior is padded but not a parallel stabilization. (A parallel stabilization is γ–primitive for all slopes meeting the slope of the annulus A once, but the middle splitting is not γ–primitive for any γ as proved in Moriah–Sedgwick [19].) Not only does it destabilize for fillings on the line $L_{pq/1}$, it also becomes isotopic to the inner and outer splittings for fillings on the same line.

6.2 Isotopic Surfaces

The above discussion leads us to our next question:

Question 6.4 *Suppose that Σ_1 and Σ_2 are non-isotopic Heegaard surfaces for X. In which fillings do Σ_1 and Σ_2 become isotopic?*

This question is also of interest if we wish to understand what happens to the oriented Heegaard tree $\mathcal{H}_X^{\pm}$ after filling. We know that some vertices are identified in every filled manifold.

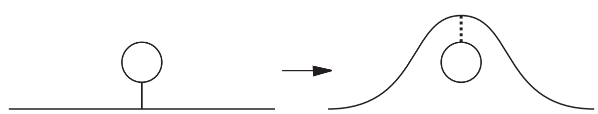

Figure 16: An annulus swap on a horizontal surface

An annulus swap is a procedure that allows us to find, in the knot-manifold, surfaces that become isotopic after filling. See Figure 16. Suppose that Σ is a horizontal surface in X, so that there is an annulus A_σ with one boundary component in Σ and another a non-trivial curve of slope σ in ∂X. Then $\partial N(\Sigma \cup A_\sigma \cup \partial X)$, where the neighborhood is taken in X, consists of two surfaces, one isotopic in the neighborhood to Σ and the other not, call it Σ'. Then we say that Σ' is obtained from Σ by an *annulus swap* along A_σ. There is an annulus A'_σ running from Σ' to ∂X, also with slope σ, that we can use to reverse the operation, obtaining Σ from Σ' by swapping Σ' along A'_σ. More importantly, after filling along any slope $\alpha \in L_\sigma$ the surfaces Σ and Σ' co–bound a product, $\Sigma \times I$, and are therefore isotopic in $X(\alpha)$. We will say that Σ and Σ' are

σ*–swap equivalent* if Σ' can be obtained from Σ by a swap along an annulus A_σ with slope σ. They are *swap equivalent* if they are σ–swap equivalent for some σ. And, we will say that Σ and Σ' are *weakly swap equivalent* if there is a sequence of swaps (with no restriction on slopes) taking Σ to Σ'.

Distinct Heegaard surfaces can be swap equivalent. For example, the dual tunnels of Morimoto and Sakuma [20] are $\frac{1}{0}$–swap equivalent. The following lemma shows that this only happens under specific circumstances:

Lemma 6.5 *Suppose that Σ is a Heegaard surface for X. Then Σ is γ–swap equivalent to a Heegaard surface Σ' for X if and only if Σ is γ–primitive.*

Proof

Let $V \cup_\Sigma W$ and $V' \cup_{\Sigma'} W'$ be the decompositions induced by Σ and Σ', where W and W' are handlebodies and V and V' are the components containing ∂X. We know that V is a compression body and must decide when V' is as well.

Let N be a regular neighborhood, $N = N(A_\gamma \cup \partial X)$, taken in V. Its boundary ∂N consists of two 2–tori $T_1 \cup T_2$, one of which, say T_2, is ∂X. The torus T_1 is composed of two annuli $T_1 = A_V \cup A_\Sigma$ where $A_\Sigma \subset \Sigma$ is a regular neighborhood in Σ of the curve component of $\partial A_\gamma \setminus \partial X$ and $A_V = T_1 \setminus A_\Sigma$ is properly embedded in V.

We obtain V' by gluing N to W along A_Σ, that is, $V' = W \cup_{A_\Sigma} N$. Since, N is a homeomorphic to a product, *2–torus* $\times I$, the component V' is a compression body if and only if there is an essential disk $D \subset W$ that meets A_Σ in a single essential arc. This occurs if and only if D meets the boundary of the annulus A_γ in a single point, that is, Σ is γ–primitive. □

It follows that if the Heegaard surfaces Σ and Σ' are swap equivalent but one of them is not γ–primitive for some γ, then the sequence of swaps must have length at least two. This still places restrictions on Σ and Σ' as detailed in the following lemma. Note that Σ is not assumed to be a Heegaard surface, although if it is swap equivalent to a Heegaard surface, it must be a horizonal surface.

Lemma 6.6 *Suppose that we can perform two distinct annulus swaps on Σ. Swapping along A_{σ_1} yields Σ_1, and swapping along A_{σ_2} produces Σ_2, where A_{σ_1} and A_{σ_2} are not isotopic. If either Σ_1 or Σ_2 is a Heegaard surface, then Σ_1 is weakly σ_1–primitive and Σ_2 is weakly σ_2–primitive.*

Proof Isotope A_{σ_1} and A_{σ_2} to intersect minimally. It follows that neither has inessential arcs of intersection with endpoints in ∂X. We claim that we can find essential disks D_1 and D_2 (possibly the same) that compress Σ towards ∂X and whose boundaries are disjoint from A_{σ_1} and A_{σ_2}, respectively. This would prove the lemma.

If $A_{\sigma_1} \cap A_{\sigma_2}$ contains an inessential arc, then it is inessential in both annuli, and its endpoints are on Σ. An outermost inessential arc on A_{σ_2} bounds a disk in A_{σ_2} and a disk in A_{σ_1}. The union of these disks is a disk D_1 that is essential (minimality) and disjoint from A_{σ_1}. A symmetric argument yields a disk D_2 disjoint from A_{σ_2}. We can therefore assume that the annuli meet only in essential arcs, so they are cut into rectangles.

If σ_1 and σ_2 do not have the same slope, then we can form a collection of disks that are shaped like a box without a top, each disjoint from both annuli. The four sides of the box are the sub-rectangles of the annuli and the bottom of a box is a rectangle in ∂X. Furthermore, it is not possible for all of our boxes to be inessential disks, this would imply that Σ is peripheral in X and, in turn, show that X is a solid torus, not a proper knot–manifold.

If σ_1 and σ_2 do have the same slope then the annuli are disjoint. In that case, we work in some $X(\alpha), \alpha \in L_{\sigma_1}$. Then the annulus $A = A_{\sigma_1} \cup A_{\sigma_2} \subset X(\alpha)$ is a properly embedded annulus in a handlebody (Σ_1 is a Heegaard surface). It can't be peripheral as this would imply that A_{σ_1} and A_{σ_2} are isotopic in X. So, it boundary compresses to a disk D that is essential and disjoint from both A_{σ_1} and A_{σ_2}. □

Suppose that Σ and Σ' are equivalent after a sequence of swaps of length n, but not for a sequence of shorter length: $\Sigma = \Sigma_0 \leftrightarrow \Sigma_1 \leftrightarrow \cdots \leftrightarrow \Sigma_n = \Sigma'$. If $n = 1$ Σ and Σ' are both γ–primitive, hence weakly γ–primitive, for some γ. If $n > 1$, then the previous lemma can be applied with $\Sigma = \Sigma_1$ (and $\Sigma = \Sigma_{n-1}$) to show that they are both weakly γ–primitive. It follows that only weakly γ–primitive Heegaard surfaces are swap equivalent to other Heegaard surfaces.

Corollary 6.7 *Suppose that Σ and Σ' are swap equivalent Heegaard surfaces. Then each is weakly γ–primitive for some γ.*

It is particularly interesting to see what happens when we perform a swap along a parallel stabilization. This case is covered by the following lemma:

Lemma 6.8 *Let Σ and Σ' be horizontal surfaces that are related by a swap along an annulus with slope σ. Then the parallel stabilizations of Σ and Σ' are swap-equivalent for every slope $\alpha \in L_\sigma$.*

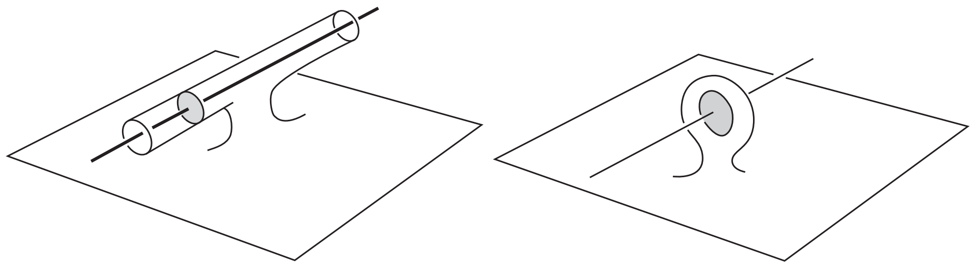

Figure 17: Parallel stabilizations that are swap equivalent along meridional annuli

Proof Choose a swap annulus A_α for the parallel stabilization of Σ where $\alpha \in L_\sigma$. Figure 17 indicates the situation in $X(\alpha)$, where A_α appears to be meridional. The parallel stabilization on the left and the surface on the right are swap equivalent along the meridional annuli pictured. It is easier to see the swap from right to left. In that case the swap just tubes the surface towards and then along the knot, yielding the surface and dual annulus on the left. But the surface on the right hand side is isotopic to an upside–down copy of the surface in Figure 15. Thus, the right hand surface is a parallel stabilization of Σ', the surface obtained if we had swapped Σ before parallel stabilizing. $\square$

Recall that if two surfaces are swap equivalent for some slope σ, then the surfaces are isotopic in $X(\alpha)$ for every $\alpha \in L_\sigma$.

Corollary 6.9 *Let Σ and Σ' be horizontal surfaces that are related by a swap along an annulus with slope σ. Then the parallel stabilizations of Σ and Σ' are isotopic in $X(\alpha)$ for every $\alpha \in LL_\sigma$.*

Remark Since the isotopy passes through the (filled) knot, it changes the sign of an oriented surface (see Section 2.3).

A parallel stabilization can also be swapped along the annulus with slope σ. The proof of the following lemma is just the observation that a parallel stabilization, see Figure 13, is just a regular stabilization followed by an annulus swap along the slope σ.

Lemma 6.10 *Suppose that Σ is a horizontal surface with annulus A_σ running from Σ to ∂X. Then the parallel stabilization of Σ is isotopic to a stabilization of Σ in $X(\alpha)$ for any $\alpha \in L_\sigma$.*

Remark If the horizonal surface is not a Heegaard surface, then its stabilization is also a horizontal surface that is not a Heegaard surface. (Stabilization and destabilization do not convert a Heegaard to a non-Heegaard surface.)

We now return our focus to the fillings on the (p,q)–torus knot exterior and show that annulus swaps explain when any pair of genus two Heegaard surfaces for the knot exterior become isotopic. (We will continue to restrict the class of torus knots as in Section 3.1).

The inner and outer splittings become isotopic, but remain distinct from the middle splitting, in fillings on the line of lines $LL_{pq/1}$. This is explained by the lemma and corollary above. First note that the Heegaard tori $T_i = \partial N(f_i)$ and $T_o = \partial N(f_o)$ are horizontal surfaces that are swap equivalent via an annulus with slope $\frac{pq}{1}$, hence they are isotopic in the lens space fillings on the line $L_{pq/1}$. The isotopy takes $T_i^{\pm}$ to $T_o^{\mp}$ and vice-versa. Since the Heegaard surfaces Σ_i and Σ_o are their respective parallel stabilizations, they are isotopic for all slopes $\beta \in LL_{pq/1}$ by Corollary 6.9. As oriented surfaces, this isotopy takes $\Sigma_i^{\pm}$ to $\Sigma_o^{\mp}$, and changes the partition in X from $\{f_i, \partial X \parallel f_o\}$ to $\{f_i \parallel \partial X, f_o\}$.

$$
\begin{tikzcd}
\Sigma_i^+ \arrow[r, dash] & S(T_i)^- \arrow[r, dash] \arrow[d, dash] & \Sigma_m^+ \arrow[r, dash] & S(T_o)^- \arrow[r, dash] \arrow[d, dash] & \Sigma_o^+ \\
\Sigma_o^- \arrow[r, dash] & S(T_o)^+ \arrow[r, dash] & \Sigma_m^- \arrow[r, dash] & S(T_i)^+ \arrow[r, dash] & \Sigma_i^-
\end{tikzcd}
$$

Figure 18: Genus two surfaces that are equivalent via swaps along annuli with slope $\frac{pq}{1}$. These surfaces are all isotopic in the lens spaces $X(\alpha)$ where $\alpha \in L_{pq/1}$.

All three genus two Heegaard surfaces become isotopic along the line $L_{pq/1}$ of lens space fillings. Lemma 6.10 shows that the inner and outer surfaces Σ_i^+ and Σ_o^+ become isotopic to stabilizations of the horizontal inner and outer Heegaard tori $S(T_i^-)$ and $S(T_o^-)$, respectively. Note that while $S(T_i^-)$ and $S(T_o^-)$ are not Heegaard surfaces, they are $\frac{pq}{1}$–primitive. The handlebody for the middle splitting Σ_m is a regular neighborhood of the union of the exceptional fibers f_i and f_o joined by a spanning arc for the annulus A_{io} running between them. So we can find two annuli from ∂X to Σ_m, each with slope $\frac{pq}{1}$. One runs between ∂X and T_i, the other between ∂X and T_o. Swapping along these annuli changes Σ_m^+ to the stabilized horizontal surfaces $S(T_i^-)$ and $S(T_o^-)$, respectively. These observations yield the swap diagram in Figure

18, all using annuli with slope $\frac{pq}{1}$. In particular, it explains how all three surfaces flip in the lens spaces. We suspect that this swap diagram is complete.

Annulus swaps explain every post–filling isotopy of Heegaard surfaces for the torus knot exteriors we consider. This leads to several questions:

Question 6.11 *Suppose that two Heegaard surfaces for a knot-manifold become isotopic after infinitely many fillings. Are they equivalent via annulus swaps?*

If so, then Lemma 6.6 would imply that the answers to the following is also "yes":

Question 6.12 *Suppose that two Heegaard surfaces for a knot-manifold become isotopic after infinitely many fillings. Are they weakly γ–primitive?*

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