Odd-primary homotopy exponents of compact simple Lie groups

DONALD M DAVIS
STEPHEN D THERIAULT

We note that a recent result of the second author yields upper bounds for odd-primary homotopy exponents of compact simple Lie groups which are often quite close to the lower bounds obtained from $v_1$–periodic homotopy theory.

57T20, 55Q52

1 Statement of results

The homotopy $p$–exponent of a topological space $X$, denoted $\exp_p(X)$, is the largest $e$ such that some homotopy group $\pi_i(X)$ contains a $\mathbb{Z}/p^e$–summand.\footnote{Some authors (eg [12]) say that $p^e$ is the homotopy $p$–exponent.} In work dating back to 1989, the first author and collaborators have obtained lower bounds for $\exp_p(X)$ for all compact simple Lie groups $X$ and all primes $p$ by using $v_1$–periodic homotopy theory. Recently, the second author [12] proved a general result, stated here as Lemma 2.1, which can yield upper bounds for homotopy exponents of spaces which map to a sphere. In this paper, we show that these two bounds often lead to a quite narrow range of values for $\exp_p(X)$ when $p$ is odd and $X$ is a compact simple Lie group.

Our first new result, which will be proved in Section 2, combines Lemma 2.1 with a classical result of Borel and Hirzebruch.

1.1 Theorem  Let $p$ be odd.

(a) If $n < p^2 + p$, then $\exp_p(SU(n)) \leq n - 1 + v_p((n-1)!)$.

(b) If $n \geq p^2 + 1$, then $\exp_p(SU(n)) \leq n + p - 3 + \left\lfloor \frac{n+1}{p-1} \right\rfloor - p^2$.

Here and throughout, $v_p(-)$ denotes the exponent of $p$ in an integer, $p$ is an odd prime, and $\lfloor x \rfloor$ denotes the integer part of $x$. All spaces are localized at $p$. It is useful to note the elementary fact that

$$v_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \cdots,$$
and the well-known fact that $v_p(m!) \leq \left\lfloor \frac{m-1}{p-1} \right\rfloor$.

Theorem 1.1(a) compares nicely with the following known result.

1.2 Theorem

(a) (Davis and Sun [7, 1.1]) For any prime $p$, $\exp_p(SU(n)) \geq n - 1 + v_p(\left\lfloor \frac{n}{p} \right\rfloor!)$.

(b) (Davis and Yang [8, 1.8]) If $p$ is odd, $1 \leq t < p$, and $tp - t + 2 \leq n \leq tp + 1$, then $\exp_p(SU(n)) \geq n$.

Thus we have the following corollary, which gives the only values of $n > p$ in which the precise value of $\exp_p(SU(n))$ is known.

1.3 Corollary If $p$ is an odd prime, and $n = p + 1$ or $n = 2p$, then $\exp_p(SU(n)) = n$.

When $n = p + 1$, this was known (although perhaps never published) since, localized at $p$, we have $SU(p + 1) \simeq B(3, 2p + 1) \times S^5 \times \cdots \times S^{2p-1}$, the exponent of which follows from Proposition 1.4 together with the result of Cohen, Moore, and Neisendorfer [5] that if $p$ is odd, then $\exp_p(S^{2n+1}) = n$. Here and throughout, $B(2n + 1, 2n + 1 + q)$ denotes an $S^{2n+1}$-bundle over $S^{2n+1+q}$ with attaching map $\alpha_1$ a generator of $\pi_{2n+q}(S^{2n+1})$, and $q = 2p - 2$. Note also that the result of [5] implies that if $n \leq p$, then $\exp_p(SU(n)) = \exp_p(S^3 \times \cdots \times S^{2n-1}) = n - 1$.

1.4 Proposition If $p$ is odd, then $\exp_p(B(3, 2p + 1)) = p + 1$, while if $n > 1$, then $n + p - 1 \leq \exp_p(B(2n + 1, 2n + 1 + q)) \leq n + p$.

Proof This just combines Bendersky, Davis and Mimura [3, 1.3] for the lower bound and Theriault [12, 2.1] for the upper bound.

Upper and lower bounds for the $p$–exponents of $Sp(n)$ and $Spin(n)$ can be extracted from Theorems 1.1 and 1.2 using long-known relationships of their $p$–localizations to that of appropriate $SU(m)$. Indeed, Harris [9] showed that there are $p$–local equivalences

$$SU(2n) \simeq Sp(n) \times (SU(2n)/Sp(n))$$

$$Spin(2n + 1) \simeq Sp(n)$$

$$Spin(2n + 2) \simeq Spin(2n + 1) \times S^{2n+1}.$$

Combining this with Theorems 1.1 and 1.2 leads to the following corollary.

1.8 Corollary Let $p$ be odd.
Odd-primary exponents of Lie groups

197

(1) \( \exp_p(\text{Spin}(2n + 2)) = \exp_p(\text{Spin}(2n + 1)) = \exp_p(\text{Sp}(n)) \leq \exp_p(\text{SU}(2n)) \), which is bounded according to Theorem 1.1.

(2) \( \exp_p(\text{Sp}(n)) \geq 2n - 1 + v_p(\lfloor \frac{2n}{p} \rfloor) \).

(3) If \( 1 \leq t < p \) and \( tp - t + 2 \leq 2n \leq tp + 1 \), then \( \exp_p(\text{Sp}(n)) \geq 2n \).

Proof The second and third parts of (1) are immediate from (1.6) and (1.5), while the first equality of (1) follows from (1.7) and the fact that \( \exp_p(\text{Spin}(2n + 1)) \geq \exp_p(S^{2n+1}) \), which is a consequence of part (2) and (1.6). For parts (2) and (3), we need to know that the homotopy classes yielding the lower bounds for \( \exp_p(\text{SU}(2n)) \) given in Theorem 1.2 come from its \( \text{Sp}(n) \) factor in (1.5). To see this, we first note that in Bendersky and Davis [1, 1.2] it was proved that, if \( p \) is odd and \( k \) is odd, then

\[
(1.9) \quad v_1^{-1} \pi_{2k}(\text{Sp}(n); p) \approx v_1^{-1} \pi_{2k}(\text{SU}(2n); p).
\]

These denote the \( p \)-primary \( v_1 \)-periodic homotopy groups, which appear as summands of actual homotopy groups. The proofs of [7, 1.1] and [8, 1.8], which yielded Theorem 1.2, were obtained by computing \( v_1^{-1} \pi_{2k}(\text{SU}(n); p) \) for certain \( k \equiv n - 1 \mod 2 \). When applied to \( \text{SU}(2n) \), these groups are in \( v_1^{-1} \pi_{2k}(\text{SU}(2n); p) \) with \( k \) odd, and so by (1.9) they appear in the \( \text{Sp}(n) \) factor.

For all \( (X, p) \) with \( X \) an exceptional Lie group and \( p \) an odd prime, except \( (E_7, 3) \) and \( (E_8, 3) \), we can make an excellent comparison of bounds for \( \exp_p(X) \) using results in the literature. We use splittings of the torsion-free cases tabulated in [3, 1.1], but known much earlier (Mimura, Nishida and Toda [10]). In Table 1, we list the range of possible values of \( \exp_p(X) \) when the precise value is not known. We also list the factor in the product decomposition which accounts for the exponent. Finally, in cases in which the exponent bounds do not follow from results already discussed, we provide references. Here \( B(n_1, \ldots, n_r) \) denotes a space built from fibrations involving \( p \)-local spheres of the indicated dimensions and equivalent to a factor in a \( p \)-localization of a special unitary group or quotient of same. Also, \( B_2(3, 11) \) denotes a sphere-bundle with attaching map \( \alpha_2 \), and \( W \) denotes a space constructed by Wilkerson and shown in [11, 1.1] to fit into a fibration \( \Omega K_5 \to B(27, 35) \to W \). Finally, \( K_3 \) and \( K_5 \) denotes Harper’s space as described in Bendersky and Davis [2] and Theriault [12].

1.10 Theorem The homotopy \( p \)-exponents of exceptional Lie groups are as given in Table 1.
Donald M Davis and Stephen D Theriault

Table 1: Homotopy exponents of exceptional Lie groups

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( p )</th>
<th>( \exp_p(\chi) )</th>
<th>Factor</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_2 )</td>
<td>3</td>
<td>6</td>
<td>( B_2(3, 11) )</td>
<td>([3, 1.3], [12, 2.2])</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>5</td>
<td>6</td>
<td>( B(3, 11) )</td>
<td></td>
</tr>
<tr>
<td>( G_2 )</td>
<td>&gt; 5</td>
<td>5</td>
<td>( S^{11} )</td>
<td></td>
</tr>
<tr>
<td>( F_4, E_6 )</td>
<td>3</td>
<td>12</td>
<td>( K_3 )</td>
<td>([2, 1.6], [12, 1.2])</td>
</tr>
<tr>
<td>( F_4, E_6 )</td>
<td>5, 7</td>
<td>11, 12</td>
<td>( B(23 - q, 23) )</td>
<td></td>
</tr>
<tr>
<td>( F_4, E_6 )</td>
<td>11</td>
<td>12</td>
<td>( B(3, 23) )</td>
<td></td>
</tr>
<tr>
<td>( F_4, E_6 )</td>
<td>&gt; 11</td>
<td>11</td>
<td>( S^{23} )</td>
<td></td>
</tr>
<tr>
<td>( E_7 )</td>
<td>5</td>
<td>18, 19, 20</td>
<td>( B(3, 11, 19, 27, 35) )</td>
<td>factor of SU(18)</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>7</td>
<td>17, 18, 19</td>
<td>( B(11, 23, 35) )</td>
<td>factor of SU(18)</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>11, 13</td>
<td>17, 18</td>
<td>( B(35 - q, 35) )</td>
<td></td>
</tr>
<tr>
<td>( E_7 )</td>
<td>17</td>
<td>18</td>
<td>( B(3, 35) )</td>
<td></td>
</tr>
<tr>
<td>( E_7 )</td>
<td>&gt; 17</td>
<td>17</td>
<td>( S^{35} )</td>
<td></td>
</tr>
<tr>
<td>( E_8 )</td>
<td>5</td>
<td>30, 31</td>
<td>( W )</td>
<td>([6, 1.1], [11, 1.2])</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>7</td>
<td>29, 30, 31, 32</td>
<td>( B(23, 35, 47, 59) )</td>
<td>([3, 1.4], \text{Proposition 2.3})</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>11 – 23</td>
<td>29, 30</td>
<td>( B(59 - q, 59) )</td>
<td></td>
</tr>
<tr>
<td>( E_8 )</td>
<td>29</td>
<td>30</td>
<td>( B(3, 59) )</td>
<td></td>
</tr>
<tr>
<td>( E_8 )</td>
<td>&gt; 29</td>
<td>29</td>
<td>( S^{59} )</td>
<td></td>
</tr>
</tbody>
</table>

2 Proof of Theorem 1.1

In [12, Lemma 2.2], the second author proved the following result.

2.1 Lemma [12, 2.2,2.3] Suppose there is a homotopy fibration

\[
F \to E \xrightarrow{q} S^{2n+1}
\]

where \( E \) is simply-connected or an \( H \)-space and

\[
|\text{coker}(\pi_{2n+1}(E) \xrightarrow{q*} \pi_{2n+1}(S^{2n+1}))| \leq p^r.
\]

Then \( \exp_p(E) \leq r + \max(\exp_p(F), n) \).

In [12, 2.2], it was required that \( E \) be an \( H \)-space, but [12, 2.3] noted that if \( E \) is not an \( H \)-space, the desired conclusion can be obtained by applying the loop-space
functor to the fibration. We require \( E \) to be simply-connected so that we do not loop away a large fundamental group. We now use this lemma to prove Theorem 1.1.

Proof of Theorem 1.1 The proof is by induction on \( n \). Let the odd prime \( p \) be implicit, and let \( SU'(n) \) denote the factor in the \( p \)-local product decomposition \([10]\) of \( SU(n) \) which is built from spheres of dimension congruent to \( 2n-1 \mod 2p-2 \). By the induction hypothesis, the exponents of the other factors are \( \leq \) the asserted amount.

We will apply Lemma 2.1 to the fibration

\[
SU'(n - p + 1) \to SU'(n) \to S^{2n-1}.
\]

In order to determine \( |\ker(q_{2n-1})(SU'(n)) \to \pi_{2n-1}(S^{2n-1})| \), we use the classical result of Borel and Hirzebruch ([4, 26.7]) that

\[
\pi_{2n-2}(SU(n-1)) \approx \mathbb{Z}/(n-1)!.
\]

When localized at \( p \), it is clear that its \( p \)-component \( \mathbb{Z}/p^{v_p((n-1)!)} \) must come from the \( SU'(n - p + 1) \)-factor in the product decomposition of \( SU(n-1) \), since \( \pi_{2n-2}(SU(n-1)) \) is built from the classes \( \alpha_i \in \pi_{2n-2}(S^{2n-1-iq})(p) \). Thus

\[
\pi_{2n-2}(SU'(n - p + 1)) \approx \mathbb{Z}/p^{v_p((n-1)!)}
\]

implies

\[
(2.2) \quad v_p(|\ker(q_{2n-1})|) \leq v_p((n-1)!).
\]

(a) By the induction hypothesis, \( \exp_p(SU'(n - p + 1)) \leq n - p + v_p((n - p)!)) \). By hypothesis, \( n - p < p^2 \) and hence \( v_p((n - p)!)) \leq p - 1 \). Thus \( \exp_p(SU'(n - p + 1)) \leq n - 1 \), and so by Lemma 2.1 and (2.2)

\[
\exp_p(SU'(n)) \leq v_p(|\ker(q_{2n-1})|) + n - 1 \leq v_p((n - 1)!)) + n - 1,
\]

as claimed.

(b) By (a), part (b) is true if \( p^2 + 1 \leq n \leq p^2 + p - 1 \). Let \( n \geq p^2 + p \), and assume the theorem is true for \( SU'(n - p + 1) \). Then by Lemma 2.1 and the induction hypothesis

\[
\exp_p(SU'(n)) \leq v((n - 1)!)) + n - p + 1 + p - 3 + \left( \left\lfloor \frac{n-p-1}{p-1} \right\rfloor - p + 2 \right).
\]
Note that even if $\exp_p(SU'(n-p+1))$ happened to be less than $n-1$, our upper bound for it is $\geq n-1$, and so this bound for $\exp_p(SU'(n))$ is still a correct deduction from Lemma 2.1.

Since $v_p((n-1)!) \leq \left\lfloor \frac{n-2}{p-1} \right\rfloor$, we obtain

$$\exp_p(SU'(n)) \leq \left\lfloor \frac{n-2}{p-1} \right\rfloor + n - 2 + \left( \left\lfloor \frac{n-2}{p-1} \right\rfloor - p + 1 \right)$$

$$= \left\lfloor \frac{n-2}{p-1} \right\rfloor + n - 2 + \left( \left\lfloor \frac{n-2}{p-1} \right\rfloor - p + 2 \right) - \left( \left\lfloor \frac{n-2}{p-1} \right\rfloor - p + 1 \right)$$

$$= n + p - 3 + \left( \left\lfloor \frac{n-2}{p-1} \right\rfloor - p + 2 \right),$$

as desired. \qed

The result in part (b) could be improved somewhat by a more delicate numerical argument.

Part (b) of the following result was used in Table 1.

### 2.3 Proposition

Let $p = 7$.

(a) $\exp_7(B(23, 35, 47)) \leq 25$.

(b) $\exp_7(B(23, 35, 47, 59)) \leq 32$.

**Proof** The thing that makes this require special attention is that these spaces are not a factor of an $SU(n)$, because they do not contain an $S^{11}$. There are fibrations

$$B(23, 35) \to B(23, 35, 47) \to S^{47}$$

and

$$B(23, 35, 47) \to B(23, 35, 47, 59) \to S^{59}.$$
Odd-primary exponents of Lie groups

References


[8] D M Davis, H Yang, Tractable formulas for \( v_1 \)-periodic homotopy groups of \( SU(n) \) when \( n \leq p^2 - p + 1 \), Forum Math. 8 (1996) 585–619 MR1404804


Department of Mathematics, Lehigh University
Bethlehem, PA 18015, USA

Department of Mathematical Sciences, University of Aberdeen
Aberdeen AB24 3UE, UK

dmd1@lehigh.edu, s.theriault@maths.abdn.ac.uk

Received: 25 October 2005 Revised: 14 September 2006