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Preface

This book is devoted to the study of moduli spaces of Seiberg–Witten monopoles over spin^c Riemannian 4–manifolds with long necks and/or tubular ends. The original purpose of this work was to provide analytical foundations for a certain construction of Floer homology of rational homology 3–spheres; this is carried out in [23]. However, along the way the project grew, and except for some of the transversality results, most of the theory is developed more generally than is needed for that construction. Floer homology itself is hardly touched upon in this book, and to compensate for that I have included another application of the analytical machinery, namely a proof of a "generalized blow-up formula" which is an important tool for computing Seiberg–Witten invariants.

The book is divided into three parts. Part I is almost identical to my paper [22]. The only significant change is the addition of Proposition 3.4.2 and Lemma 8.1.2. The other two parts consist of previously unpublished material. Part II is an expository account of gluing theory including orientations. The main novelties here may be the formulation of the gluing theorem and the approach to orientations. In Part III the analytical results are brought together to prove the generalized blow-up formula. A detailed description of the contents of the book is provided by the introductions to each part.

At least on a formal level there are many analogies with the instanton theory, and at some places, most notably in Chapters 2 and 6, I have borrowed ideas from Donaldson's book [14]. Furthermore, the approach to orientations uses a concept of Benevieri–Furi [8] which I learnt about from Shuguang Wang [51].

About a year after the manuscript to this book was submitted, the book of Kronheimer–Mrowka [31] appeared, which takes the subject much further, using an entirely new approach involving certain blown-up configuration spaces. It is hoped that the present text may complement their work by giving a thorough discussion of ordinary moduli spaces (as opposed to the blown-up ones). Our setup is less general than that of [31] in that "balanced" perturbations of the Chern–Simons–Dirac functional (using their terminology) are ruled out when the underlying 3–manifold has Betti number $b_1 > 0$.

On the other hand, this book in some ways goes further in investigating compactness in the presence of "nonexact" perturbations, introducing a second technique in addition to the standard one based on the energy concept. As for the blow-up formula, this can be proved using Floer homology as is done in [31], but the proof given here is a lot more elementary.

Part of this work was carried out in 2001–2 during a stay at the Institut des Hautes Études Scientifiques, and the author is grateful for the hospitality and excellent research environment which he enjoyed there. This work was also partially supported by grants from the National Science Foundation and the DFG (German Research Foundation) as well as by the CRC 701 at the University of Bielefeld.

Part I

Compactness

Let Z be a closed, oriented 4–manifold equipped with a spin^c–structure. Suppose Z is separated by a closed hypersurface Y, say

$$Z = Z_1 \cup_Y Z_2.$$

Then one may attempt to express the Seiberg–Witten invariant of Z in terms of relative invariants of the two pieces Z_1, Z_2. The standard approach, familiar from instanton Floer theory (see Floer [19] and Donaldson [14]), is to construct a 1–parameter family $\{g_T\}$ of Riemannian metrics on Z by stretching along Y so as to obtain a neck $[-T, T] \times Y$, and study the monopole moduli space $M^{(T)}$ over (Z, g_T) for large T. There are different aspects of this problem: compactness, transversality and gluing. In this part of the book we will focus particularly on compactness, and also establish transversality results sufficient for the construction of Floer homology groups of rational homology 3–spheres.

Let the monopole equations over the neck $[-T, T] \times Y$ be perturbed by a closed 2–form η on Y, so that temporal gauge solutions to these equations correspond to downward gradient flow lines of the correspondingly perturbed Chern–Simons–Dirac functional ϑ_η over Y. Suppose all critical points of ϑ_η are nondegenerate. Because each moduli space $M^{(T)}$ is compact, one might expect, by analogy with Morse theory, that a sequence $\omega_n \in M^{(T_n)}$ where $T_n \to \infty$ has a subsequence which converges, in a suitable sense, to a pair of monopoles over the cylindrical-end manifolds associated to Z_1, Z_2 together with a broken gradient line of ϑ_η over $\mathbb{R} \times Y$. Unfortunately, this kind of compactness may fail when η is nonexact. (A simple class of counter-examples is described after Theorem 1.4.1 below.) It is then natural to seek topological conditions which ensure that compactness does hold. We will consider two approaches which provide different sufficient conditions. In the first approach, which is essentially well known, one first establishes global bounds on a certain energy functional and then derives local L^2 bounds on the curvature forms. In the second approach, which appears to be new, one begins by placing the connections in Coulomb gauge with respect to a given reference connection and then obtains global bounds on the corresponding connection forms in suitably weighted Sobolev norms, utilizing the apriori pointwise bounds on the spinors.

The energy concept is particularly well explained by Kronheimer–Mrowka [31], who discuss compactness (for blown-up moduli spaces) for exact and nonexact η. The important case when Y is a circle times a surface of genus g was studied by Morgan–Szabó–Taubes [40] (when $g > 1$) and Taubes [48] (when $g = 1$), in both cases with η nonexact. Other sources are Nicolaescu [41] (with $\eta = 0$) and Marcolli–Wang [36] (with η exact).

In the transversality theory of moduli spaces we mostly restrict ourselves, for the time being, to the case when all ends of the 4–manifold in question are modelled on rational homology spheres. The perturbations of the monopole equations on the ends are minor modifications of the ones introduced in [21]. (It is not clear to us that these perturbations immediately carry over to the case of more general ends, as has apparently been assumed by some authors, although we expect that a modified version may be shown to work with the aid of gluing theory.) In the language of finite dimensional Morse theory our approach is somewhat analogous to perturbing the gradient vector field away from the critical points. In contrast, [31] uses more general perturbations of the Chern–Simons–Dirac functional but retains the gradient flow property of the equations.

This part also contains expository chapters on configuration spaces and exponential decay.

CHAPTER 1

Compactness theorems

1.1 Vanishing results

Before describing our compactness results in more detail we will mention two applications to Seiberg–Witten invariants of closed 4–manifolds.

By a spin^c *manifold* we shall mean an oriented smooth manifold with a spin^c structure. If Z is a spin^c manifold then $-Z$ will refer to the same smooth manifold equipped with the opposite orientation and the corresponding spin^c structure.

If Z is a closed, oriented 4–manifold then by an *homology orientation* of Z we mean an orientation of the real vector space $H^0(Z)^* \oplus H^1(Z) \oplus H^+(Z)^*$, where $H^+(Z)$ is any maximal positive subspace for the intersection form on $H^2(Z)$. The dimension of H^+ is denoted b^+.

In [7] Bauer and Furuta introduced a refined Seiberg–Witten invariant for closed spin^c 4–manifolds Z. This invariant $\widetilde{\mathrm{SW}}(Z)$ lives in a certain equivariant stable cohomotopy group. If Z is connected and $b^+(Z) > 1$, and given an homology orientation of Z, then according to Bauer [5] there is a natural homomorphism from this stable cohomotopy group to $\mathbb{Z}$ which maps $\widetilde{\mathrm{SW}}(Z)$ to the ordinary Seiberg–Witten invariant $\mathrm{SW}(Z)$ defined by the homology orientation. In [6] Bauer showed that, unlike the ordinary Seiberg–Witten invariant, the refined invariant does not in general vanish for connected sums where both summands have $b^+ > 0$. However, $\widetilde{\mathrm{SW}}(Z) = 0$ provided there exists a metric and perturbation 2–form on Z for which the Seiberg–Witten moduli space M_Z is empty (see Bauer [5, Remark 2.2] and Ishida–LeBrun [27, Proposition 6]).

Theorem 1.1.1 *Let Z be a closed* spin^c *4–manifold and $Y \subset Z$ a 3–dimensional closed, orientable submanifold. Suppose*

(i) *Y admits a Riemannian metric with positive scalar curvature,*

(ii) *$H^2(Z;\mathbb{Q}) \to H^2(Y;\mathbb{Q})$ is nonzero.*

Then there exist a metric and perturbation 2–form on Z for which M_Z is empty, hence $\widetilde{\mathrm{SW}}(Z)=0$.

This generalizes a result of Fintushel–Stern [17] and Morgan–Szabó–Taubes [40] which concerns the special case when $Y \approx S^1 \times S^2$ is the link of an embedded 2–sphere of self-intersection 0. One can derive Theorem 1.1.1 from Nicolaescu's proof [41] of their result and the classification of closed orientable 3–manifolds admitting positive scalar curvature metrics (see Lawson–Michelson [33, p 325]). However, we shall give a direct (and much simpler) proof where the main idea is to perturb the monopole equations on Z by a suitable 2–form such that the corresponding perturbed Chern–Simons–Dirac functional on Y has no critical points. One then introduces a long neck $[-T,T]\times Y$. See Chapter 9 for details.

We now turn to another application, for which we need a little preparation. For any compact spin^c 4–manifold Z whose boundary is a disjoint union of rational homology spheres set

$$d(Z)=\frac{1}{4}(c_1(\mathcal{L}_Z)^2-\sigma(Z))+b_1(Z)-b^+(Z).$$

Here $\mathcal{L}_Z$ is the determinant line bundle of the spin^c structure, and $\sigma(Z)$ the signature of Z. If Z is closed then the moduli space M_Z has expected dimension $d(Z)-b_0(Z)$.

In [23] we will assign to every spin^c rational homology 3–sphere Y a rational number $h(Y)$. (A preliminary version of this invariant was introduced in [21].) In Chapter 9 this invariant will be defined in the case when Y admits a metric with positive scalar curvature. It satisfies $h(-Y)=-h(Y)$. In particular, $h(S^3)=0$.

Theorem 1.1.2 *Let Z be a closed, connected* spin^c *4–manifold, and let $W \subset Z$ be a compact, connected, codimension 0 submanifold whose boundary is a disjoint union of rational homology spheres $Y_1,\dots,Y_r$, $r \geq 1$, each of which admits a metric of positive scalar curvature. Suppose $b^+(W)>0$ and set $W^c = Z \setminus \operatorname{int} W$. Let each Y_j have the orientation and* spin^c *structure inherited from W. Then the following hold:*

(i) *If $2\sum_j h(Y_j) \leq -d(W)$ then there exist a metric and perturbation 2–form on Z for which M_Z is empty, hence $\widetilde{\mathrm{SW}}(Z)=0$.*

(ii) *If $b^+(Z)>1$ and $2\sum_j h(Y_j) < d(W^c)$ then $\mathrm{SW}(Z)=0$.*

Note that (ii) generalizes the classical theorem (see Salamon [44] and Nicolaescu [41]) which says that $\mathrm{SW}(Z)=0$ if Z is a connected sum where both sides have $b^+>0$.

1.2 The Chern–Simons–Dirac functional

Let Y be a closed, connected Riemannian spin^c 3–manifold. We consider the Seiberg–Witten monopole equations over $\mathbb{R}\times Y$, perturbed by adding a 2–form to the curvature part of these equations. This 2–form should be the pullback of a closed form η on Y. Recall from [30; 40] that in temporal gauge these perturbed monopole equations can be described as the downward gradient flow equation for a perturbed Chern–Simons–Dirac functional, which we will denote by ϑ_η, or just ϑ when no confusion can arise.

For transversality reasons we will add a further small perturbation to the monopole equations over $\mathbb{R}\times Y$, similar to that introduced in [21, Section 2]. This perturbation depends on a parameter $\mathfrak{p}$ (see Section 3.3). When $\mathfrak{p}\neq 0$ the perturbed monopole equations are no longer of gradient flow type. Therefore, $\mathfrak{p}$ has to be kept small in order for the perturbed equations to retain certain properties (see Section 4.2).

If S is a configuration over Y (ie a spin connection together with a section of the spin bundle) and $u\colon Y\to \mathrm{U}(1)$ then

$$\vartheta(u(S))-\vartheta(S)=2\pi\int_Y \tilde{\eta}\wedge[u], \tag{1.1}$$

where $[u]\in H^1(Y)$ is the pullback by u of the fundamental class of $\mathrm{U}(1)$, and

$$\tilde{\eta}=\pi c_1(\mathcal{L}_Y)-[\eta]\in H^2(Y). \tag{1.2}$$

Here $\mathcal{L}_Y$ is the determinant line bundle of the spin^c structure of Y.

Let $\mathcal{R}_Y$ be the space of (smooth) monopoles over Y (ie critical points of ϑ) modulo all gauge transformations $Y\to\mathrm{U}(1)$, and $\widetilde{\mathcal{R}}_Y$ the space of monopoles over Y modulo null-homotopic gauge transformations.

When no statement is made to the contrary, we will always make the following two assumptions:

(O1) $\tilde{\eta}$ is a real multiple of some rational cohomology class.

(O2) All critical points of ϑ are nondegenerate.

The second assumption implies that $\mathcal{R}_Y$ is a finite set. This rules out the case when $\tilde{\eta}=0$ and $b_1(Y)>0$, because if $\tilde{\eta}=0$ then the subspace of reducible points in $\mathcal{R}_Y$ is homeomorphic to a $b_1(Y)$–dimensional torus. If $\tilde{\eta}\neq 0$ or $b_1(Y)=0$ then the nondegeneracy condition can be achieved by perturbing η by an exact form (see Proposition 8.1.1).

For any $\alpha,\beta\in\widetilde{\mathcal{R}}_Y$ let $M(\alpha,\beta)$ denote the moduli space of monopoles over $\mathbb{R}\times Y$ that are asymptotic to α and β at $-\infty$ and ∞, respectively. Set $\check{M}=M/\mathbb{R}$, where

$\mathbb{R}$ acts by translation. By a *broken gradient line* from α to β we mean a sequence $(\omega_1,\dots,\omega_k)$ where $k\ge 0$ and $\omega_j\in\check{M}(\alpha_{j-1},\alpha_j)$ for some $\alpha_0,\dots,\alpha_k\in\widetilde{\mathcal{R}}_Y$ with $\alpha_0=\alpha$, $\alpha_k=\beta$, and $\alpha_{j-1}\ne\alpha_j$ for each j. If $\alpha=\beta$ then we allow the empty broken gradient line (with $k=0$).

1.3 Compactness

Let X be a spin^c Riemannian 4–manifold with tubular ends $\overline{\mathbb{R}}_+\times Y_j$, $j=1,\dots,r$, where $r\ge 0$ and each Y_j is a closed, connected Riemannian spin^c 3–manifold. Setting $Y=\bigcup_j Y_j$ this means that we are given

- an orientation preserving isometric embedding $\iota\colon\overline{\mathbb{R}}_+\times Y\to X$ such that
$$X_{:t}=X\setminus\iota((t,\infty)\times Y) \tag{1.3}$$
is compact for any $t\ge 0$,
- an isomorphism between the spin^c structure on $\overline{\mathbb{R}}_+\times Y$ induced from Y and the one inherited from X via the embedding ι.

Here $\mathbb{R}_+$ is the set of positive real numbers and $\overline{\mathbb{R}}_+=\mathbb{R}_+\cup\{0\}$. Usually we will just regard $\overline{\mathbb{R}}_+\times Y$ as a (closed) submanifold of X.

Let η_j be a closed 2–form on Y_j and define $\tilde\eta_j\in H^2(Y_j)$ in terms of η_j as in (1.2). We write ϑ instead of ϑ_{η_j} when no confusion is likely to arise. We perturb the curvature part of the monopole equations over X by adding a 2–form μ whose restriction to $\mathbb{R}_+\times Y_j$ agrees with the pullback of η_j. In addition we perturb the equations over $\mathbb{R}\times Y_j$ and the corresponding end of X using a perturbation parameter $\mathfrak{p}_j$. If $\vec\alpha=(\alpha_1,\dots,\alpha_r)$ with $\alpha_j\in\widetilde{\mathcal{R}}_{Y_j}$ let $M(X;\vec\alpha)$ denote the moduli space of monopoles over X that are asymptotic to α_j over $\mathbb{R}_+\times Y_j$.

Let $\lambda_1,\dots,\lambda_r$ be positive constants. We consider the following two equivalent conditions on the spin^c manifold X and $\tilde\eta_j,\lambda_j$:

(A) There exists a class $\tilde z\in H^2(X;\mathbb{R})$ such that $\tilde z|_{Y_j}=\lambda_j\tilde\eta_j$ for $j=1,\dots,r$.

(A′) For configurations S over $X_{:0}$ the sum $\sum_j\lambda_j\vartheta(S|_{\{0\}\times Y_j})$ depends only on the gauge equivalence class of S.

Note that if $\lambda_j=1$ for all j then (A) holds precisely when there exists a class $z\in H^2(X;\mathbb{R})$ such that $z|_{Y_j}=[\eta_j]$ for $j=1,\dots,r$.

Theorem 1.3.1 *If Condition (A) is satisfied and each $\mathfrak{p}_j$ has sufficiently small C^1 norm then the following holds. For $n = 1, 2, \dots$ let $\omega_n \in M(X; \vec{\alpha}_n)$, where $\vec{\alpha}_n = (\alpha_{n,1}, \dots, \alpha_{n,r})$. If*

$$\inf_n \sum_{j=1}^{r} \lambda_j \vartheta(\alpha_{n,j}) > -\infty \tag{1.4}$$

then there exists a subsequence of ω_n which chain-converges to an $(r+1)$–tuple $(\omega, \vec{v}_1, \dots, \vec{v}_r)$ where ω is an element of some moduli space $M(X; \vec{\beta})$ and $\vec{v}_j$ is a broken gradient line over $\mathbb{R} \times Y_j$ from β_j to some $\gamma_j \in \widetilde{\mathcal{R}}_{Y_j}$. Moreover, if ω_n chain-converges to $(\omega, \vec{v}_1, \dots, \vec{v}_r)$ then for sufficiently large n there is a gauge transformation $u_n \colon X \to \mathrm{U}(1)$ which is translationary invariant over the ends and maps $M(X; \vec{\alpha}_n)$ to $M(X; \vec{\gamma})$.

The Equation (1.4) imposes an "energy bound" over the ends of X, as we will show in Section 7.2. The notion of chain-convergence is defined in Section 7.1. The limit, if it exists, is unique up to gauge equivalence (see Proposition 7.1.2 below).

1.4 Compactness and neck-stretching

In this section cohomology groups will have real coefficients.

We consider again a spin^c Riemannian 4–manifold X as in the previous section, but we now assume that the ends of X are given by orientation preserving isometric embeddings

$$\begin{aligned} \iota'_j \colon \overline{\mathbb{R}}_+ \times Y'_j &\to X, \qquad j = 1, \dots, r', \\ \iota^{\pm}_j \colon \overline{\mathbb{R}}_+ \times (\pm Y_j) &\to X, \qquad j = 1, \dots, r, \end{aligned}$$

where $r, r' \geq 0$. Here each Y'_j, Y_j should be a closed, connected spin^c Riemannian 3–manifold, and as before there should be the appropriate identifications of spin^c structures. For every $T = (T_1, \dots, T_r)$ with $T_j > 0$ for each j, let $X^{(T)}$ denote the manifold obtained from X by gluing, for $j = 1, \dots, r$, the two ends $\iota^{\pm}_j(\overline{\mathbb{R}}_+ \times Y_j)$ to form a neck $[-T_j, T_j] \times Y_j$. To be precise, let $X^{\{T\}} \subset X$ be the result of deleting from X the sets $\iota^{\pm}_j([2T_j, \infty) \times Y_j)$, $j = 1, \dots, r$. Set

$$X^{(T)} = X^{\{T\}} / \sim,$$

where we identify

$$\iota^{+}_j(t, y) \sim \iota^{-}_j(2T_j - t, y)$$

for all $(t,y)\in(0,2T_j)\times Y_j$ and $j=1,\dots,r$. We regard $[-T_j,T_j]\times Y_j$ as a submanifold of $X^{(T)}$ by means of the isometric embedding $(t,y)\mapsto \pi_T\iota_j^+(t+T_j,y)$, where $\pi_T\colon X^{\{T\}}\to X^{(T)}$. Also, we write $\mathbb{R}_+\times(\pm Y_j)$ instead of $\iota_j^{\pm}(\mathbb{R}_+\times Y_j)$, and similarly for $\mathbb{R}_+\times Y'_j$, if this is not likely to cause any confusion.

Set $X^\#=X^{(T)}$ with $T_j=1$ for all j. The process of constructing $X^\#$ from X (as smooth manifolds) can be described by the unoriented graph γ which has one node for every connected component of X and, for each $j=1,\dots,r$, one edge representing the pair of embeddings $\iota_j^{\pm}$.

A node in an oriented graph is called a *source* if it has no incoming edges. If e is any node in γ let X_e denote the corresponding component of X. Let $Z_e=(X_e)_{:1}$ be the corresponding truncated manifold as in (1.3). Let $\gamma\setminus e$ be the graph obtained from γ by deleting the node e and all edges of which e is a boundary point. Given an orientation o of γ let $\partial^- Z_e$ denote the union of all boundary components of Z_e corresponding to incoming edges of (γ,o). Let F_e be the kernel of $H^1(Z_e)\to H^1(\partial^- Z_e)$, and set

$$\Sigma(X,\gamma,o)=\dim\,H^1(X^\#)-\sum_e \dim\,F_e.$$

It will follow from Lemma 5.3.1 below that $\Sigma(X,\gamma,o)\le 0$ if each connected component of γ is simply connected.

We will now state a condition on (X,γ) which is recursive with respect to the number of nodes of γ.

(C) If γ has more than one node then it should admit an orientation o such that the following two conditions hold:
- $\Sigma(X,\gamma,o)=0$.
- Condition (C) holds for $(X\setminus X_e,\gamma\setminus e)$ for all sources e of (γ,o).

We are only interested in this condition when each component of γ is simply connected. If γ is connected and has exactly two nodes e_1,e_2 then (C) holds if and only if $H^1(X^\#)\to H^1(Z_{e_j})$ is surjective for at least one value of j, as is easily seen from the Mayer–Vietoris sequence. See Section 5.3 and the proof of Proposition 5.4.2 for more information about Condition (C).

Let the Chern–Simons–Dirac functionals on Y_j,Y'_j be defined in terms of closed 2–forms η_j,η'_j respectively. Let $\widetilde{\eta}_j$ and $\widetilde{\eta}'_j$ be the corresponding classes as in (1.2). Let $\lambda_1,\dots,\lambda_r$ and $\lambda'_1,\dots,\lambda'_{r'}$ be positive constants. The following conditions on $X,\widetilde{\eta}_j,\widetilde{\eta}'_j,\lambda_j,\lambda'_j$ will appear in Theorem 1.4.1 below.

(B1) There exists a class in $H^2(X^\#)$ whose restrictions to Y_j and Y'_j are $[\eta_j]$ and $[\eta'_j]$, respectively, and all the constants λ_j,λ'_j are equal to 1.

(B2) There exists a class in $H^2(X^\#)$ whose restrictions to Y_j and Y'_j are $\lambda_j\widetilde{\eta}_j$ and $\lambda'_j\widetilde{\eta}'_j$, respectively. Moreover, the graph γ is simply connected, and Condition (C) holds for (X,γ).

Choose a 2–form μ on X whose restriction to each end $\mathbb{R}_+\times(\pm Y_j)$ is the pullback of η_j, and whose restriction to $\mathbb{R}_+\times Y'_j$ is the pullback of η'_j. Such a form μ gives rise, in a canonical way, to a form $\mu^{(T)}$ on $X^{(T)}$. We use the forms $\mu,\mu^{(T)}$ to perturb the curvature part of the monopole equations over X, $X^{(T)}$, respectively. We use the perturbation parameter $\mathfrak{p}'_j$ over $\mathbb{R}\times Y'_j$ and the corresponding ends, and $\mathfrak{p}_j$ over $\mathbb{R}\times Y_j$ and the corresponding ends and necks.

Moduli spaces over X will be denoted $M(X;\vec{\alpha}_+,\vec{\alpha}_-,\vec{\alpha}')$, where the j–th component of $\vec{\alpha}_\pm$ specifies the limit over the end $\mathbb{R}_+\times(\pm Y_j)$ and the j–th component of $\vec{\alpha}'$ specifies the limit over $\mathbb{R}_+\times Y'_j$. We set

$$T_{\min}:=\min(T_1,\dots,T_r).$$

If we are given a sequence $T(n)$ of r–tuples, we write

$$T_{\min}(n):=\min(T_1(n),\dots,T_r(n)).$$

Theorem 1.4.1 *Suppose at least one of the conditions (B1), (B2) holds, and for $n=1,2,\dots$ let $\omega_n\in M(X^{(T(n))};\vec{\alpha}'_n)$, where $\vec{\alpha}'_n=(\alpha'_{n,1},\dots,\alpha'_{n,r'})$ and $T_{\min}(n)\to\infty$ as $n\to\infty$. Suppose also that the perturbation parameters $\mathfrak{p}_j,\mathfrak{p}'_j$ are admissible for each $\vec{\alpha}'_n$, and that*

$$\inf_n\sum_{j=1}^{r'}\lambda'_j\vartheta(\alpha'_{n,j})>-\infty.$$

Then there exists a subsequence of ω_n which chain-converges to an $(r+r'+1)$–tuple $\mathbb{V}=(\omega,\vec{v}_1,\dots,\vec{v}_r,\vec{v}'_1,\dots,\vec{v}'_{r'})$, where

- *ω is an element of some moduli space $M(X;\vec{\alpha}_1,\vec{\alpha}_2,\vec{\beta}')$,*
- *$\vec{v}_j$ is a broken gradient line over $\mathbb{R}\times Y_j$ from α_{1j} to α_{2j},*
- *$\vec{v}'_j$ is a broken gradient line over $\mathbb{R}\times Y'_j$ from β'_j to some $\gamma'_j\in\widetilde{\mathcal{R}}_{Y'_j}$.*

Moreover, if ω_n chain-converges to $\mathbb{V}$ then for sufficiently large n there is a gauge transformation $u_n\colon X^{(T(n))}\to\mathrm{U}(1)$ which is translationary invariant over the ends and maps $M(X^{(T(n))};\vec{\alpha}'_n)$ to $M(X^{(T(n))};\vec{\gamma}')$.

The notion of chain-convergence is defined in Section 7.1. Note that the chain-limit is unique only up to gauge equivalence; see Proposition 7.1.2.

What it means for the perturbation parameters $\mathfrak{p}_j, \mathfrak{p}'_j$ to be "admissible" is defined in Definition 7.1.3. As in Theorem 1.3.1, if (B2) holds and the perturbation parameters have sufficiently small C^1 norm then they are admissible for any $\vec{\alpha}'$; see Proposition 5.4.2. If (B1) is satisfied but perhaps not (B2) then for any $C_1 < \infty$ there is a $C_2 > 0$ such that if the perturbation parameters have C^1 norm $< C_2$ then they are admissible for all $\vec{\alpha}'$ satisfying $\sum_{j=1}^{r'} \lambda'_j \vartheta(\alpha'_j) > -C_1$; see the remarks after Proposition 4.3.1.

The conditions (B1), (B2) in the theorem correspond to the two approaches to compactness referred to at the beginning of this introduction: If (B1) is satisfied then one can take the "energy approach", whereas if (B2) holds one can use the "Hodge theory approach"

The conclusion of the theorem does not hold in general when neither (B1) nor (B2) are satisfied. For in that case Theorem 1.1.1 would hold if instead of (ii) one merely assumed that $b_1(Y) > 0$. Since $\mathbb{R}^4$ contains an embedded $S^1 \times S^2$ this would contradict the fact that there are many spin^c 4–manifolds with $b^+ > 1$ and nonzero Seiberg–Witten invariant.

For the moment we will abuse language and say that (B2) holds if it holds for some choice of constants λ_j, λ'_j, and similarly for (B1). Then a simple example where (B1) is satisfied but not (B2) is $X = \mathbb{R} \times Y$, where one glues the two ends to obtain $X^{(T)} = (\mathbb{R}/2T\mathbb{Z}) \times Y$. There are also many examples where (B2) is satisfied but not (B1). For instance, consider the case when X consists of two copies of $\mathbb{R} \times Y$, say $X = \mathbb{R} \times Y \times \{1, 2\}$ with Y connected, and one glues $\mathbb{R}_+ \times Y \times \{1\}$ with $\mathbb{R}_- \times Y \times \{2\}$. In this case $r = 1$ and $r' = 2$, so we are given closed 2–forms η_1, η'_1, η'_2 on Y. Condition (B1) now requires that these three 2–forms represent the same cohomology class, while (B2) holds as long as there are $a_1, a_2 > 0$ such that $[\eta_1] = a_1[\eta'_1] = a_2[\eta'_2]$.

CHAPTER 2

Configuration spaces

2.1 Configurations and gauge transformations

Let X be a Riemannian n–manifold with tubular ends $\overline{\mathbb{R}}_+ \times Y_j$, $j = 1, \dots, r$, where $n \geq 1$, $r \geq 0$, and each Y_j is a closed, connected Riemannian $(n-1)$–manifold. This means that we are given for each j an isometric embedding

$$\iota_j \colon \overline{\mathbb{R}}_+ \times Y_j \to X;$$

moreover, the images of these embeddings are disjoint and their union have precompact complement. Usually we will just regard $\overline{\mathbb{R}}_+ \times Y_j$ as a submanifold of X. Set $Y = \bigcup_j Y_j$ and, for $t \geq 0$,

$$X_{:t} = X \setminus (t, \infty) \times Y.$$

Let $\mathbb{S} \to X$ and $\mathbb{S}_j \to Y_j$ be Hermitian complex vector bundles, and $L \to X$ and $L_j \to Y_j$ principal $\mathrm{U}(1)$–bundles. Suppose we are given, for each j, isomorphisms

$$\iota_j^* \mathbb{S} \xrightarrow{\approx} \overline{\mathbb{R}}_+ \times \mathbb{S}_j, \quad \iota_j^* L \xrightarrow{\approx} \overline{\mathbb{R}}_+ \times L_j.$$

By a *configuration* in $(L, \mathbb{S})$ we shall mean a pair (A, Φ) where A is a connection in L and Φ a section of $\mathbb{S}$. Maps $u\colon X \to \mathrm{U}(1)$ are referred to as *gauge transformations* and these act on configurations in the natural way:

$$u(A, \Phi) = (u(A), u\Phi).$$

The main goal of this chapter is to prove a "local slice" theorem for certain orbit spaces of configurations modulo gauge transformations.

We begin by setting up suitable function spaces. For $p \geq 1$ and any nonnegative integer m let $L^p_m(X)$ be the completion of the space of compactly supported smooth functions on X with respect to the norm

$$\|f\|_{m,p} = \|f\|_{L^p_m} = \left(\sum_{k=0}^{m} \int_X |\nabla^k f|^p \right)^{1/p}.$$

Here the covariant derivative is computed using some fixed connection in the tangent bundle TX which is translationary invariant over each end. Define the Sobolev space $L^p_m(X;\mathbb{S})$ of sections of $\mathbb{S}$ similarly.

We also need weighted Sobolev spaces. For any smooth function $w\colon X \to \mathbb{R}$ set $L^{p,w}_m(X) = e^{-w} L^p_m(X)$ and

$$\|f\|_{L^{p,w}_m} = \|e^w f\|_{L^p_m}.$$

In practice we require that w have a specific form over the ends, namely

$$w \circ \iota_j(t,y) = \sigma_j t,$$

where the σ_j's are real numbers.

The following Sobolev embeddings (which hold in $\mathbb{R}^n$, hence over X) will be used repeatedly:

$$\begin{aligned} L^p_{m+1} &\subset L^{2p}_m &&\text{if } p \geq n/2, m \geq 0,\\ L^p_2 &\subset C^0_B &&\text{if } p > n/2. \end{aligned}$$

Here C^0_B denotes the Banach space of bounded continuous functions, with the supremum norm. Moreover, if $pm > n$ then multiplication defines a continuous map $L^p_m \times L^p_k \to L^p_k$ for $0 \leq k \leq m$.

For the remainder of this chapter fix $p > n/2$.

Note that this implies $L^p_1 \subset L^2$ over compact n–manifolds.

We will now define an affine space $\mathcal{C}$ of $L^p_{1,\mathrm{loc}}$ configurations in $(L,\mathbb{S})$. Let A_o be a smooth connection in L. Choose a smooth section Φ_o of $\mathbb{S}$ whose restriction to $\mathbb{R}_+ \times Y_j$ is the pullback of a section ψ_j of $\mathbb{S}_j$. Suppose $\psi_j = 0$ for $j \leq r_0$, and $\psi_j \not\equiv 0$ for $j > r_0$, where r_0 is a nonnegative integer.. Fix a weight function w as above with $\sigma_j \geq 0$ small for all j, and $\sigma_j > 0$ for $j \leq r_0$. Set

$$\mathcal{C} = \{(A_o + a, \Phi_o + \phi) : a, \phi \in L^{p,w}_1\}.$$

We topologize $\mathcal{C}$ using the $L^{p,w}_1$ metric.

We wish to define a Banach Lie group $\mathcal{G}$ of $L^p_{2,\mathrm{loc}}$ gauge transformations over X such that $\mathcal{G}$ acts smoothly on $\mathcal{C}$ and such that if $S, S' \in \mathcal{C}$ and $u(S) = S'$ for some $L^p_{2,\mathrm{loc}}$ gauge transformations u then $u \in \mathcal{G}$. If $u \in \mathcal{G}$ then we must certainly have

$$(-u^{-1}du, (u-1)\Phi_o) = u(A_o, \Phi_o) - (A_o, \Phi_o) \in L^{p,w}_1.$$

Now

$$\|du\|_{L^{p,w}_1} \le \mathrm{const}\cdot(\|u^{-1}du\|_{L^{p,w}_1} + \|u^{-1}du\|^2_{L^{p,w}_1}), \tag{2.1}$$

and vice versa, $\|du\|_{L^{p,w}_1}$ controls $\|u^{-1}du\|_{L^{p,w}_1}$, so we try

$$\mathcal{G} = \{u \in L^p_{2,\mathrm{loc}}(X; \mathrm{U}(1)) \,:\, du, (u-1)\Phi_o \in L^{p,w}_1\}.$$

By $L^p_{2,\mathrm{loc}}(X; \mathrm{U}(1))$ we mean the set of elements of $L^p_{2,\mathrm{loc}}(X; \mathbb{C})$ that map into $\mathrm{U}(1)$. We will see that $\mathcal{G}$ has a natural smooth structure such that the above criteria are satisfied.

(This approach to the definition of $\mathcal{G}$ was inspired by Donaldson [14].)

2.2 The Banach algebra

Let $\widetilde{x}$ be a finite subset of X which contains at least one point from every connected component of X where Φ_o vanishes identically.

Definition 2.2.1 Set

$$\mathcal{E} = \{f \in L^p_{2,\mathrm{loc}}(X; \mathbb{C}) \,:\, df, f\Phi_o \in L^{p,w}_1\},$$

and let $\mathcal{E}$ have the norm

$$\|f\|_{\mathcal{E}} = \|df\|_{L^{p,w}_1} + \|f\Phi_o\|_{L^{p,w}_1} + \sum_{x\in\widetilde{x}} |f(x)|.$$

We will see in a moment that $\mathcal{E}$ is a Banach algebra (without unit if $r_0 < r$). The next lemma shows that the topology on $\mathcal{E}$ is independent of the choice of Φ_o and $\widetilde{x}$.

Lemma 2.2.1 *Let $Z \subset X$ be a compact, connected codimension 0 submanifold.*

(i) *If $\Phi_o|_Z \not\equiv 0$ then there is a constant C such that*

$$\int_Z |f|^p \le C \int_Z |df|^p + |f\Phi_o|^p \tag{2.2}$$

for all $f \in L^p_1(Z)$.

(ii) *There are constants* C_1, C_2 *such that*

$$|f(z_2) - f(z_1)| \le C_1 \|df\|_{L^{2p}(Z)} \le C_2 \|df\|_{L^p_1(Z)}$$

for all $f \in L^p_2(Z)$ *and* $z_1, z_2 \in Z$.

Proof Part (i) follows from the compactness of the embedding $L^p_1(Z) \to L^p(Z)$. The first inequality in (ii) can either be deduced from the compactness of $L^{2p}_1(Z) \to C^0(Z)$, or one can prove it directly, as a step towards proving the Rellich lemma, by considering the integrals of df along a suitable family of paths from z_1 to z_2. □

Lemma 2.2.2 *Let* Y *be a closed Riemannian manifold, and* $\sigma > 0$.

(i) *If* $q \ge 1$ *and* $f\colon \mathbb{R}_+ \times Y \to \mathbb{R}$ *is a* C^1 *function such that* $\lim_{t\to\infty} f(t, y) = 0$ *for all* $y \in Y$ *then*

$$\|f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)} \le \sigma^{-1} \|\partial_1 f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)}.$$

(ii) *If* $q > 1$, $T \ge 1$, *and* $f\colon [0, T] \times Y \to \mathbb{R}$ *is a* C^1 *function then*

$$\|f\|_{L^q([T-1,T]\times Y)} \le \|f_0\|_{L^q(Y)} + (\sigma r)^{-1/r} \|\partial_1 f\|_{L^{q,\sigma}([0,T]\times Y)},$$

where $f_0(y) = f(0, y)$ *and* $1/q + 1/r = 1$.

Here ∂_1 is the partial derivative in the first variable, ie in the $\mathbb{R}_+$ coordinate.

Proof Part (i) follows from:

$$\begin{aligned}
\|f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)} &= \left(\int_{\mathbb{R}_+\times Y} \left| \int_0^\infty e^{\sigma t} \partial_1 f(s+t, y)\, ds \right|^q dt\, dy \right)^{1/q} \\
&\le \int_0^\infty \left(\int_{\mathbb{R}_+\times Y} |e^{\sigma t} \partial_1 f(s+t, y)|^q\, dt\, dy \right)^{1/q} ds \\
&\le \left(\int_0^\infty e^{-\sigma s}\, ds \right) \left(\int_{\mathbb{R}_+\times Y} |e^{\sigma(s+t)} \partial_1 f(s+t, y)|^q\, dt\, dy \right)^{1/q} \\
&\le \sigma^{-1} \|\partial_1 f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)}.
\end{aligned}$$

Part (ii) follows by a similar computation. □

Parts (i)–(iv) of the following proposition are essentially due to Donaldson [14].

Proposition 2.2.1 (i) *There is a constant* C_1 *such that, for all* $f \in \mathcal{E}$,

$$\|f\|_\infty \le C_1 \|f\|_{\mathcal{E}}.$$

(ii) *For every* $f \in \mathcal{E}$ *and* $j = 1, \dots, r$ *the restriction* $f|_{\{t\}\times Y_j}$ *converges uniformly to a constant function* $f^{(j)}$ *as* $t \to \infty$, *and* $f^{(j)} = 0$ *for* $j > r_0$.

(iii) *There is a constant* C_2 *such that if* $f \in \mathcal{E}$ *and* $f^{(j)} = 0$ *for all* j *then*

$$\|f\|_{L^{p,w}_2} \le C_2 \|f\|_{\mathcal{E}}.$$

(iv) *There is an exact sequence*

$$0 \to L^{p,w}_2 \xrightarrow{\iota} \mathcal{E} \xrightarrow{e} \mathbb{C}^{r_0} \to 0,$$

where ι *is the inclusion and* $e(f) = (f^{(1)}, \dots, f^{(r_0)})$.

(v) $\mathcal{E}$ *is complete, and multiplication defines a continuous map* $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$.

Proof First observe that for any $f \in L^p_{2,\mathrm{loc}}(X)$ and $\epsilon > 0$ there exists a $g \in C^\infty(X)$ such that $\|g - f\|_{L^{p,w}_2} < \epsilon$. Therefore it suffices to prove (i)–(iii) when $f \in \mathcal{E}$ is smooth. Part (i) is then a consequence of Lemma 2.2.1 and Lemma 2.2.2 (ii), while Part (ii) for $r_0 < j \le r$ follows from Lemma 2.2.1.

We will now prove (ii) when $1 \le j \le r_0$. Let $f \in \mathcal{E}$ be smooth. Since $\int_{\mathbb{R}_+\times Y_j} |df| < \infty$ by the Hölder inequality, we have

$$\int_0^\infty |\partial_1 f(t,y)|\, dt < \infty \qquad \text{for a.e. } y \in Y_j.$$

For $n \in \mathbb{N}$ set $f_n = f|_{[n-1,n+1]\times Y_j}$, regarded as a function on $B = [n-1, n+1] \times Y_j$. Then $\{f_n\}$ converges a.e., so by Egoroff's theorem $\{f_n\}$ converges uniformly over some subset $T \subset B$ of positive measure. There is then a constant $C > 0$, depending on T, such that for every $g \in L^p_1(B)$ one has

$$\int_B |g|^p \le C\left(\int_B |dg|^p + \int_T |g|^p\right).$$

It follows that $\{f_n\}$ converges in L^p_2 over B, hence uniformly over B, to some constant function.

Part (iii) follows from Lemma 2.2.1 and Lemma 2.2.2 (i). Part (iv) is an immediate consequence of (ii) and (iii). It is clear from (i) that $\mathcal{E}$ is complete. The multiplication property follows easily from (i) and the fact that smooth functions are dense in $\mathcal{E}$. □

2.3 The infinitesimal action

If $f\colon X \to i\mathbb{R}$ and Φ is a section of $\mathbb{S}$ we define a section of $i\Lambda^1 \oplus \mathbb{S}$ by

$$\mathcal{I}_\Phi f := (-df, f\Phi)$$

whenever the expression on the right makes sense. Here Λ^k denotes the bundle of k–forms (on X, in this case). If $S = (A, \Phi)$ is a configuration then we will sometimes write $\mathcal{I}_S$ instead of $\mathcal{I}_\Phi$. Set

$$\mathcal{I} := \mathcal{I}_{\Phi_o}.$$

If Φ is smooth then the formal adjoint of the operator $\mathcal{I}_\Phi$ is

$$\mathcal{I}_\Phi^*(a, \phi) = -d^*a + i\langle i\Phi, \phi\rangle_{\mathbb{R}},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the real inner product on $\mathbb{S}$. Note that

$$\mathcal{I}_\Phi^*\mathcal{I}_\Phi = \Delta + |\Phi|^2$$

where Δ is the positive Laplacian on X.

Set
$$L\mathcal{G} := \{f \in \mathcal{E} : f \text{ maps into } i\mathbb{R}\}.$$

From Proposition 2.2.1 (i) we see that the operators

$$\mathcal{I}_\Phi \colon L\mathcal{G} \to L_1^{p,w}, \quad \mathcal{I}_\Phi^* \colon L_1^{p,w} \to L^{p,w}$$

are well defined and bounded for every $\Phi \in \Phi_o + L_1^{p,w}(X; \mathbb{S})$.

Lemma 2.3.1 *For every $\Phi \in \Phi_o + L_1^{p,w}(X; \mathbb{S})$, the operators $\mathcal{I}_\Phi^*\mathcal{I}_\Phi$ and $\mathcal{I}_\Phi$ have the same kernel in $L\mathcal{G}$.*

Proof Choose a smooth function $\beta \colon \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = 1$ for $t \le 1$, $\beta(t) = 0$ for $t \ge 2$. For $r > 0$ define a compactly supported function

$$\beta_r \colon X \to \mathbb{R}$$

by $\beta_r|_{X:0} = 1$, and $\beta_r(t, y) = \beta(t/r)$ for $(t, y) \in \mathbb{R}_+ \times Y_j$.

Now suppose $f \in L\mathcal{G}$ and $\mathcal{I}_\Phi^*\mathcal{I}_\Phi f = 0$. Proposition 2.2.1 (i) and elliptic regularity gives $f \in L^p_{2,\mathrm{loc}}$, so we certainly have

$$\mathcal{I}_\Phi f \in L^p_{1,\mathrm{loc}} \subset L^2_{\mathrm{loc}}.$$

Clearly,
$$\|\mathcal{I}_\Phi f\|_2 \le \liminf_{r\to\infty} \|\mathcal{I}_\Phi(\beta_r f)\|_2.$$

Over $\mathbb{R}_+ \times Y_j$ we have

$$\mathcal{I}_\Phi^*\mathcal{I}_\Phi = -\partial_1^2 + \Delta_{Y_j} + |\Phi|^2,$$

where $\partial_1 = \frac{\partial}{\partial t}$ and Δ_{Y_j} is the positive Laplacian on Y_j, so

$$\begin{aligned}
\|\mathcal{I}_\Phi(\beta_r f)\|_2^2 &= \int_X \mathcal{I}_\Phi^* \mathcal{I}_\Phi(\beta_r f)\cdot\beta_r \bar{f} \\
&= -\sum_j \int_{\mathbb{R}_+\times Y_j} ((\partial_1^2\beta_r)f + 2(\partial_1\beta_r)(\partial_1 f))\cdot\beta_r \bar{f} \\
&\le C_1\|f\|_\infty^2 \int_0^\infty |r^{-2}\beta''(t/r)|\,dt \\
&\qquad + C_1\|f\|_\infty\|df\|_p \times \left(\int_0^\infty |r^{-1}\beta'(t/r)|^q\,dt\right)^{1/q} \\
&\le C_2\|f\|_{\mathcal{E}}^2\left(r^{-1}\int_1^2|\beta''(u)|\,du + r^{1-q}\int_1^2|\beta'(u)|^q\,du\right) \\
&\to 0 \text{ as } r\to\infty,
\end{aligned}$$

where $C_1, C_2 > 0$ are constants and $1/p + 1/q = 1$. Hence $\mathcal{I}_\Phi f = 0$. □

Lemma 2.3.2 *$\mathcal{I}^*\mathcal{I}$: $L_2^{q,w}(X)\to L^{q,w}(X)$ is Fredholm of index $-r_0$, for $1<q<\infty$.*

Proof Because $\mathcal{I}^*\mathcal{I}$ is elliptic the operator in the Lemma is Fredholm if the operator

$$-\partial_1^2 + \Delta_{Y_j} + |\psi_j|^2\colon L_2^{q,\sigma_j} \to L^{q,\sigma_j}, \tag{2.3}$$

acting on functions on $\mathbb{R}\times Y_j$, is Fredholm for each j. The proof of [14, Proposition 3.21] (see also Lockhart–McOwen [34]) can be generalized to show that (2.3) is Fredholm if σ_j^2 is not an eigenvalue of $\Delta_{Y_j} + |\psi_j|^2$. Since we are taking $\sigma_j \ge 0$ small, and $\sigma_j > 0$ if $\psi_j = 0$, this establishes the Fredholm property in the Lemma.

We will now compute the index. Set

$$\mathrm{ind}^\pm = \mathrm{index}\{\mathcal{I}^*\mathcal{I}\colon L_2^{q,\pm w}(X)\to L^{q,\pm w}(X)\}.$$

Expressing functions on Y_j in terms of eigenvectors of $\Delta_{Y_j} + |\psi_j|^2$ as in Atiyah–Patodi–Singer [3] or Donaldson [14] one finds that the kernel of $\mathcal{I}^*\mathcal{I}$ in $L_{k'}^{q',\pm w}$ is the same for all $q' > 1$ and integers $k' \ge 0$. Combining this with the fact that $\mathcal{I}^*\mathcal{I}$ is formally self-adjoint we see that

$$\mathrm{ind}^+ = -\mathrm{ind}^-.$$

Now choose smooth functions $w_j\colon \mathbb{R}\to\mathbb{R}$ such that $w_j(t)=\sigma_j|t|$ for $|t|\ge 1$. The addition formula for the index (see Corollary C.0.1) gives

$$\begin{aligned}\mathrm{ind}^- &= \mathrm{ind}^+ + \sum_j \mathrm{index}\{\mathcal{I}^*_{\psi_j}\mathcal{I}_{\psi_j} : L^{q,-w_j}_2(\mathbb{R}\times Y_j)\to L^{q,-w_j}(\mathbb{R}\times Y_j)\}\\ &= \mathrm{ind}^+ + 2\sum_j \dim\ker(\Delta_{Y_j}+|\psi_j|^2)\\ &= \mathrm{ind}^+ + 2r_0.\end{aligned}$$

Therefore, $\mathrm{ind}^+=-r_0$ as claimed. □

Proposition 2.3.1 *For any $\Phi\in\Phi_o+L^{p,w}_1(X;\mathbb{S})$ the following hold:*

(i) *The operator*

$$\mathcal{I}^*_\Phi\mathcal{I}_\Phi\colon L\mathcal{G}\to L^{p,w} \tag{2.4}$$

is Fredholm of index 0, and it has the same kernel as $\mathcal{I}_\Phi\colon L\mathcal{G}\to L^{p,w}_1$ and the same image as $\mathcal{I}^*_\Phi\colon L^{p,w}_1\to L^{p,w}$.

(ii) *$\mathcal{I}_\Phi(L\mathcal{G})$ is closed in $L^{p,w}_1$ and*

$$L^{p,w}_1(i\Lambda^1\oplus\mathbb{S})=\mathcal{I}_\Phi(L\mathcal{G})\oplus\ker(\mathcal{I}^*_\Phi). \tag{2.5}$$

Proof It is easy to deduce Part (ii) from Part (i). We will now prove Part (i). Since

$$\mathcal{I}^*_\Phi\mathcal{I}_\Phi-\mathcal{I}^*\mathcal{I}=|\Phi|^2-|\Phi_o|^2\colon L^{p,w}_2\to L^{p,w}$$

is a compact operator, $\mathcal{I}^*_\Phi\mathcal{I}_\Phi$ and $\mathcal{I}^*\mathcal{I}$ have the same index as operators between these Banach spaces. It then follows from Lemma 2.3.2 and Proposition 2.2.1 (iv) that the operator (2.4) is Fredholm of index 0. The statement about the kernels is the same as Lemma 2.3.1. To prove the statement about the images, we may as well assume X is connected. If $\Phi\neq 0$ then the operator (2.4) is surjective and there is nothing left to prove. Now suppose $\Phi=0$. Then all the weights σ_j are positive, and the kernel of $\mathcal{I}_\Phi$ in $\mathcal{E}$ consists of the constant functions. Hence the image of (2.4) has codimension 1. But $\int_X d^*a=0$ for every 1–form $a\in L^{p,w}_1$, so $d^*\colon L^{p,w}_1\to L^{p,w}$ is not surjective. □

In the course of the proof of (i) we obtained:

Proposition 2.3.2 *If X is connected and $r_0=r$ then*

$$d^*d(\mathcal{E})=\left\{g\in L^{p,w}(X;\mathbb{C}) : \int_X g=0\right\}.$$ □

We conclude this section with a result that will be needed in the proofs of Proposition 5.2.1 and Lemma 5.4.2 below. Let $1 < q < \infty$ and for any $L^q_{1,\mathrm{loc}}$ function $f\colon X \to \mathbb{R}$ set

$$\delta_j f = \int_{\{0\}\times Y_j} \partial_1 f, \quad j = 1, \dots, r.$$

The integral is well defined because if n is any positive integer then there is a bounded restriction map $L^q_1(\mathbb{R}^n) \to L^q(\{0\}\times\mathbb{R}^{n-1})$.

Choose a point $x_0 \in X$.

Proposition 2.3.3 *If X is connected, $1 < q < \infty$, $r \geq 1$, and if $\sigma_j > 0$ is sufficiently small for each j then the operator*

$$\beta\colon L^{q,-w}_2(X;\mathbb{R}) \to L^{q,-w}(X;\mathbb{R}) \oplus \mathbb{R}^r,$$
$$f \mapsto (\Delta f, (\delta_1 f, \dots, \delta_{r-1} f, f(x_0)))$$

is an isomorphism.

Proof By the proof of Lemma 2.3.2, $\Delta\colon L^{q,-w}_2 \to L^{q,-w}$ has index r, hence $\mathrm{ind}(\beta) = 0$. We will show β is injective. First observe that $\sum_{j=1}^r \delta_j f = 0$ whenever $\Delta f = 0$, so if $\beta f = 0$ then $\delta_j f = 0$ for all j.

Suppose $\beta f = 0$. To simplify notation we will now assume Y is connected. Over $\mathbb{R}_+ \times Y$ we have $\Delta = -\partial_1^2 + \Delta_Y$. Let $\{h_\nu\}_{\nu=0,1,\dots}$ be a maximal orthonormal set of eigenvectors of Δ_Y, with corresponding eigenvalues λ_ν^2, where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Then

$$f(t, y) = a + bt + g(t, y),$$

where $a, b \in \mathbb{R}$, and g has the form

$$g(t, y) = \sum_{\nu\geq 1} c_\nu e^{-\lambda_\nu t} h_\nu(y)$$

for some real constants c_ν. Elliptic estimates show that g decays exponentially, or more precisely,

$$|(\nabla^j f)_{(t,y)}| \leq d_j e^{-\lambda_1 t}$$

for $(t, y) \in \mathbb{R}_+ \times Y$ and $j \geq 0$, where $d_j > 0$ is a constant. Now

$$\partial_1 f(t, y) = b - \sum_{\nu\geq 1} c_\nu \lambda_\nu e^{-\lambda_\nu t} h_\nu(y).$$

Since Δ_Y is formally self-adjoint we have $\int_Y h_\nu = 0$ if $\lambda_\nu \neq 0$, hence

$$b\,\mathrm{Vol}(Y) = \int_{\{\tau\}\times Y} \partial_1 f = 0.$$

It follows that f is bounded and df decays exponentially over $\mathbb{R}_+ \times Y$, so

$$0 = \int_X f \Delta f = \int_X |df|^2,$$

hence f is constant. Since $f(x_0) = 0$ we have $f = 0$. □

2.4 Local slices

Fix a finite subset $\mathfrak{b} \subset X$.

Definition 2.4.1 Set

$$\begin{aligned} \mathcal{G}_{\mathfrak{b}} &= \{u \in 1 + \mathcal{E} : u \text{ maps into } \mathrm{U}(1) \text{ and } u|_{\mathfrak{b}} \equiv 1\} \\ L\mathcal{G}_{\mathfrak{b}} &= \{f \in \mathcal{E} : f \text{ maps into } i\mathbb{R} \text{ and } f|_{\mathfrak{b}} \equiv 0\} \end{aligned}$$

and let $\mathcal{G}_{\mathfrak{b}}$ and $L\mathcal{G}_{\mathfrak{b}}$ have the subspace topologies inherited from $1 + \mathcal{E} \approx \mathcal{E}$ and $\mathcal{E}$, respectively.

By $1 + \mathcal{E}$ we mean the set of functions on X of the form $1 + f$ where $f \in \mathcal{E}$. If $\mathfrak{b}$ is empty then we write $\mathcal{G}$ instead of $\mathcal{G}_{\mathfrak{b}}$, and similarly for $L\mathcal{G}$.

Proposition 2.4.1 (i) *$\mathcal{G}_{\mathfrak{b}}$ is a smooth submanifold of $1+\mathcal{E}$ and a Banach Lie group with Lie algebra $L\mathcal{G}_{\mathfrak{b}}$.*

(ii) *The natural action $\mathcal{G}_{\mathfrak{b}} \times \mathcal{C} \to \mathcal{C}$ is smooth.*

(iii) *If $S \in \mathcal{C}$, $u \in L^p_{2,\mathrm{loc}}(X; \mathrm{U}(1))$ and $u(S) \in \mathcal{C}$ then $u \in \mathcal{G}$.*

Proof (i) If $r_0 < r$ then $1 \notin \mathcal{E}$, but in any case,

$$f \mapsto \sum_{k=1}^{\infty} \frac{1}{k!} f^k = \exp(f) - 1$$

defines a smooth map $\mathcal{E} \to \mathcal{E}$, by Proposition 2.2.1 (v). Therefore, the exponential map provides the local parametrization around 1 required for $\mathcal{G}_{\mathfrak{b}}$ to be a submanifold of $1 + \mathcal{E}$. The verification of (ii) and (iii) is left to the reader. □

Let $\mathcal{B}_{\mathfrak{b}} = \mathcal{C}/\mathcal{G}_{\mathfrak{b}}$ have the quotient topology. This topology is Hausdorff because it is stronger than the topology defined by the L^{2p} metric on $\mathcal{B}_{\mathfrak{b}}$ (see Donaldson–Kronheimer [15]). The image in $\mathcal{B}_{\mathfrak{b}}$ of a configuration $S \in \mathcal{C}$ will be denoted $[S]$, and we say S is a *representative* of $[S]$.

Let $\mathcal{C}^*_{\mathfrak{b}}$ be the set of all elements of $\mathcal{C}$ which have trivial stabilizer in $\mathcal{G}_{\mathfrak{b}}$. In other words, $\mathcal{C}^*_{\mathfrak{b}}$ consists of those $(A,\Phi)\in\mathcal{C}$ such that $\mathfrak{b}$ contains at least one point from every component of X where Φ vanishes almost everywhere. Let $\mathcal{B}^*_{\mathfrak{b}}$ be the image of $\mathcal{C}^*_{\mathfrak{b}}\to\mathcal{B}_{\mathfrak{b}}$. It is clear that $\mathcal{B}^*_{\mathfrak{b}}$ is an open subset of $\mathcal{B}_{\mathfrak{b}}$.

If $\mathfrak{b}$ is empty then $\mathcal{C}^*\subset\mathcal{C}$ and $\mathcal{B}^*\subset\mathcal{B}$ are the subspaces of *irreducible* configurations. As usual, a configuration that is not irreducible is called reducible.

We will now give $\mathcal{B}^*_{\mathfrak{b}}$ the structure of a smooth Banach manifold by specifying an atlas of local parametrizations. Let $S=(A,\Phi)\in\mathcal{C}^*_{\mathfrak{b}}$ and set

$$V=\mathcal{I}^*_{\Phi}(L^{p,w}_1),\quad W=\mathcal{I}^*_{\Phi}\mathcal{I}_{\Phi}(L\mathcal{G}_{\mathfrak{b}}).$$

By Proposition 2.3.1 we have

$$\dim(V/W)=|\mathfrak{b}|-\ell$$

where ℓ is the number of components of X where Φ vanishes a.e. Choose a bounded linear map $\rho\colon V\to W$ such that $\rho|_W=I$, and set

$$\mathcal{I}^{\#}_{\Phi}=\rho\mathcal{I}^*_{\Phi}.$$

Then
$$L^{p,w}_1(i\Lambda^1\oplus\mathbb{S})=\mathcal{I}_{\Phi}(L\mathcal{G}_{\mathfrak{b}})\oplus\ker(\mathcal{I}^{\#}_{\Phi})$$

by Proposition 2.3.1. Consider the smooth map

$$\Pi\colon L\mathcal{G}_{\mathfrak{b}}\times\ker(\mathcal{I}^{\#}_{\Phi})\to\mathcal{C},\quad (f,s)\mapsto\exp(f)(S+s).$$

The derivative of this map at $(0,0)$ is

$$D\Pi(0,0)(f,s)=\mathcal{I}_{\Phi}f+s,$$

which is an isomorphism by the above remarks. The inverse function theorem then says that Π is a local diffeomorphism at $(0,0)$.

Proposition 2.4.2 *In the situation above there is an open neighbourhood U of $0\in\ker(\mathcal{I}^{\#}_{\Phi})$ such that the projection $\mathcal{C}\to\mathcal{B}_{\mathfrak{b}}$ restricts to a homeomorphism of $S+U$ onto an open subset of $\mathcal{B}^*_{\mathfrak{b}}$.*

Remark It is clear that the collection of such local parametrizations $U\to\mathcal{B}^*_{\mathfrak{b}}$ is a smooth atlas for $\mathcal{B}^*_{\mathfrak{b}}$.

Proof It only remains to prove that $S+U\to\mathcal{B}_{\mathfrak{b}}$ is injective when U is sufficiently small. So suppose (a_k,ϕ_k), (b_k,ψ_k) are two sequences in $\ker(\mathcal{I}^{\#}_{\Phi})$ which both converge to 0 as $k\to\infty$, and such that

$$u_k(A+a_k,\Phi+\phi_k)=(A+b_k,\Phi+\psi_k)$$

for some $u_k\in\mathcal{G}_{\mathfrak{b}}$. We will show that $\|u_k-1\|_{\mathcal{E}}\to 0$. Since Π is a local diffeomorphism at $(0,0)$, this will imply that $u_k=1$ for $k\gg 0$.

Written out, the assumption on u_k is that

$$u_k^{-1}du_k=a_k-b_k,$$
$$(u_k-1)\Phi=\psi_k-u_k\phi_k.$$

By (2.1) we have $\|du_k\|_{L^{p,w}_1}\to 0$, which in turn gives $\|u_k\phi_k\|_{L^{p,w}_1}\to 0$, hence

$$\|(u_k-1)\Phi\|_{L^{p,w}_1}\to 0. \tag{2.6}$$

Because u_k is bounded and du_k converges to 0 in L^p_1 over compact subsets, we can find a subsequence $\{k_j\}$ such that u_{k_j} converges in L^p_2 over compact subsets to a locally constant function u. Then $u|_{\mathfrak{b}}=1$ and $u\Phi=\Phi$, hence $u=1$. Set $f_j=u_{k_j}-1$ and $\phi=\Phi-\Phi_o\in L^{p,w}_1$. Then $\|df_j\otimes\phi\|_{L^{p,w}}\to 0$. Furthermore, given $\epsilon>0$ we can find $t>0$ such that

$$\int_{[t,\infty)\times Y}|e^w\phi|^p<\frac{\epsilon}{4}$$

and N such that

$$\int_{X_{:t}}|e^w f_j\phi|^p<\frac{\epsilon}{2}$$

for $j>N$. Then $\int_X|e^w f_j\phi|^p<\epsilon$ for $j>N$. Thus $\|f_j\phi\|_{L^{p,w}}\to 0$, and similarly $\|f_j\nabla\phi\|_{L^{p,w}}\to 0$. Altogether this shows that $\|f_j\phi\|_{L^{p,w}_1}\to 0$. Combined with (2.6) this yields

$$\|(u_{k_j}-1)\Phi_0\|_{L^{p,w}_1}\to 0,$$

hence $\|u_{k_j}-1\|_{\mathcal{E}}\to 0$. But we can run the above argument starting with any subsequence of $\{u_k\}$, so $\|u_k-1\|_{\mathcal{E}}\to 0$. □

2.5 Manifolds with boundary

Let Z be a compact, connected, oriented Riemannian n–manifold, perhaps with boundary, and $\mathfrak{b}\subset Z$ a finite subset. Let $\mathbb{S}\to Z$ be a Hermitian vector bundle and $L\to Z$ a principal $\mathrm{U}(1)$–bundle. Fix $p>n/2$ and let $\mathcal{C}$ denote the space of L^p_1 configurations (A,Φ) in $(L,\mathbb{S})$. Let $\mathcal{G}_{\mathfrak{b}}$ be the group of those L^p_2 gauge transformations $Z \to \mathrm{U}(1)$ that restrict to 1 on $\mathfrak{b}$, and $\mathcal{C}^*_{\mathfrak{b}}$ the set of all elements of $\mathcal{C}$ that have trivial stabilizer in $\mathcal{G}_{\mathfrak{b}}$. Then

$$\mathcal{B}^*_{\mathfrak{b}} = \mathcal{C}^*_{\mathfrak{b}}/\mathcal{G}_{\mathfrak{b}}$$

is again a (Hausdorff) smooth Banach manifold. As for orbit spaces of connections (see [15, p 192]) the main ingredient here is the solution to the Neumann problem over Z, according to which the operator

$$\begin{aligned} T_\Phi\colon L^p_2(Z) &\to L^p(Z) \oplus \partial L^p_1(\partial Z),\\ f &\mapsto (\Delta f + |\Phi|^2 f, \partial_\nu f) \end{aligned}$$

is a Fredholm operator of index 0 (see Taylor [49, Section 5.7] and Hamilton [25, pp 85–6]). Here ν is the inward-pointing unit normal along ∂Z, and $\partial L^p_1(\partial Z)$ is the space of boundary values of L^p_1 functions on Z. Henceforth we work with imaginary-valued functions, and on ∂Z we identify 3–forms with functions by means of the Hodge $*$–operator. Then $T_\Phi = J_\Phi \mathcal{I}_\Phi$, where

$$J_\Phi(a,\phi) = (\mathcal{I}^*_\Phi(a,\phi), (*a)|_{\partial Z}).$$

Choose a bounded linear map

$$\rho\colon L^p(Z) \oplus \partial L^p_1(\partial Z) \to W := T_\Phi(L\mathcal{G}_{\mathfrak{b}})$$

which restricts to the identity on W, and set $J^\#_\Phi = \rho J_\Phi$. An application of Stokes' theorem shows that

$$\ker(T_\Phi) \subset \ker(\mathcal{I}_\Phi) \quad \text{in } L^p_2(Z),$$

hence
$$T_\Phi = J^\#_\Phi \mathcal{I}_\Phi\colon L\mathcal{G}_{\mathfrak{b}} \to W$$

is an isomorphism. In general, if $V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3$ are linear maps between vector spaces such that $T_2 T_1$ is an isomorphism, then $V_2 = \operatorname{im}(T_1) \oplus \ker(T_2)$. Therefore, for any $(A,\Phi) \in \mathcal{C}^*_{\mathfrak{b}}$ we have

$$L^p_1(Z; i\Lambda^1 \oplus \mathbb{S}) = \mathcal{I}_\Phi(L\mathcal{G}_{\mathfrak{b}}) \oplus \ker(J^\#_\Phi),$$

where both summands are closed subspaces. Thus we obtain the analogue of Proposition 2.4.2 with local slices of the form $(A,\Phi) + U$, where U is a small neighbourhood of $0 \in \ker(J^\#_\Phi)$.

CHAPTER 3

Moduli spaces

3.1 Spin^c structures

It will be convenient to have a definition of spin^c structure that does not refer to Riemannian metrics. So let X be an oriented n–dimensional manifold and P_{GL^+} its bundle of positive linear frames. Let $\widetilde{\mathrm{GL}}^+(n)$ denote the 2–fold universal covering group of the identity component $\mathrm{GL}^+(n)$ of $\mathrm{GL}(n,\mathbb{R})$, and denote by -1 the nontrivial element of the kernel of $\widetilde{\mathrm{GL}}^+(n) \to \mathrm{GL}^+(n)$. Set

$$\mathrm{GL}^c(n) = \widetilde{\mathrm{GL}}^+(n) \underset{\pm(1,1)}{\times} \mathrm{U}(1).$$

Then there is a short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{GL}^c(n) \to \mathrm{GL}^+(n) \times \mathrm{U}(1) \to 1,$$

and $\mathrm{Spin}^c(n)$ is canonically isomorphic to the preimage of $\mathrm{SO}(n)$ by the projection $\mathrm{GL}^c(n) \to \mathrm{GL}^+(n)$.

Definition 3.1.1 By a spin^c structure $\mathfrak{s}$ on X we mean a principal $\mathrm{GL}^c(n)$–bundle $P_{\mathrm{GL}^c} \to X$ together with a $\mathrm{GL}^c(n)$ equivariant map $P_{\mathrm{GL}^c} \to P_{\mathrm{GL}^+}$ which covers the identity on X. If $\mathfrak{s}'$ is another spin^c structure on X given by $P'_{\mathrm{GL}^c} \to P_{\mathrm{GL}^+}$ then $\mathfrak{s}$ and $\mathfrak{s}'$ are called *isomorphic* if there is a $\mathrm{U}(1)$ equivariant map $P'_{\mathrm{GL}^c} \to P_{\mathrm{GL}^c}$ which covers the identity on P_{GL^+}.

The natural $\mathrm{U}(1)$–bundle associated to P_{GL^c} is denoted $\mathcal{L}$, and the Chern class $c_1(\mathcal{L})$ is called the *canonical class* of the spin^c structure.

Now suppose X is equipped with a Riemannian metric, and let P_{SO} be its bundle of positive orthonormal frames, which is a principal $\mathrm{SO}(n)$–bundle. Then the preimage

P_{Spin^c} of P_{SO} by the projection $P_{\mathrm{GL}^c} \to P_{\mathrm{GL}^+}$ is a principal $\mathrm{Spin}^c(n)$–bundle over X, ie a spin^c structure of X in the sense of Lawson–Michelsohn [33]. Conversely, P_{GL^c} is isomorphic to $P_{\mathrm{Spin}^c} \times_{\mathrm{Spin}^c n} \mathrm{GL}^c(n)$. Thus there is a natural 1–1 correspondence between (isomorphism classes of) spin^c structures of the smooth oriented manifold X as defined above, and spin^c structures of the oriented Riemannian manifold X in the sense of [33].

By a *spin connection* in P_{Spin^c} we shall mean a connection in P_{Spin^c} that maps to the Levi-Civita connection in P_{SO}. If A is a spin connection in P_{Spin^c} then $\hat{F}_A$ will denote the $i\mathbb{R}$ component of the curvature of A with respect to the isomorphism of Lie algebras

$$\mathrm{spin}(n) \oplus i\mathbb{R} \xrightarrow{\approx} \mathrm{spin}^c(n)$$

defined by the double cover $\mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathrm{Spin}^c(n)$. In terms of the induced connection $\check{A}$ in $\mathcal{L}$ one has

$$\hat{F}_A = \frac{1}{2} F_{\check{A}}.$$

If A, A' are spin connections in P_{Spin^c} then we regard $A - A'$ as an element of $i\Omega^1_X$.

The results of Chapter 2 carry over to spaces of configurations (A, Φ) where A is a spin connection in P_{Spin^c} and Φ a section of some complex vector bundle $\mathbb{S} \to X$.

When the spin^c structure on X is understood then we will say "spin connection over X" instead of "spin connection in P_{Spin^c}".

If n is even then the complex Clifford algebra $\mathbb{C}\ell(n)$ has up to equivalence exactly one irreducible complex representation. Let $\mathbb{S}$ denote the associated spin bundle over X. Then the eigenspaces of the complex volume element $\omega_{\mathbb{C}}$ in $\mathbb{C}\ell(n)$ defines a splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ (see [33]).

If n is odd then $\mathbb{C}\ell(n)$ has up to equivalence two irreducible complex representations ρ_1, ρ_2. These restrict to equivalent representations of $\mathrm{Spin}^c(n)$, so one gets a well-defined spin bundle $\mathbb{S}$ for any spin^c structure on X [33]. If α is the unique automorphism of $\mathbb{C}\ell(n)$ whose restriction to $\mathbb{R}^n$ is multiplication by -1 then $\rho_1 \approx \rho_2 \circ \alpha$. Hence if A is any spin connection over X then the sign of the Dirac operator D_A depends on the choice of ρ_j. To remove this ambiguity we decree that Clifford multiplication of TX on $\mathbb{S}$ is to be defined using the representation ρ_j satisfying $\rho_j(\omega_{\mathbb{C}}) = 1$.

In the case of a Riemannian product $\mathbb{R} \times X$ there is a natural 1–1 correspondence between (isomorphism classes of) spin^c–structures on $\mathbb{R} \times X$ and spin^c–structures on X, and we can identify

$$\mathcal{L}_{\mathbb{R}\times X} = \pi_2^*(\mathcal{L}_X),$$

where $\pi_2\colon \mathbb{R}\times X\to X$ is the projection.

If A is a spin connection over $\mathbb{R}\times X$ then $A|_{\{t\}\times X}$ will denote the spin connection B over X satisfying $\check{A}|_{\{t\}\times X}=\check{B}$.

When n is odd then we can also identify

$$\mathbb{S}^+_{\mathbb{R}\times X}=\pi_2^*(\mathbb{S}_X). \tag{3.1}$$

If e is a tangent vector on X then Clifford multiplication with e on $\mathbb{S}_X$ corresponds to multiplication with e_0e on $\mathbb{S}^+_{\mathbb{R}\times X}$, where e_0 is the positively oriented unit tangent vector on $\mathbb{R}$. Therefore, reversing the orientation of X changes the sign of the Dirac operator on X.

From now on, to avoid confusion we will use ∂_B to denote the Dirac operator over a 3–manifold with spin connection B, while the notation D_A will be reserved for Dirac operators over 4–manifolds.

By a *configuration* over a spin^c 3–manifold Y we shall mean a pair (B,Ψ) where B is a spin connection over Y and Ψ a section of the spin bundle $\mathbb{S}_Y$. By a configuration over a spin^c 4–manifold X we mean a pair (A,Φ) where A is a spin connection over X and Φ a section of the positive spin bundle $\mathbb{S}^+_X$.

3.2 The Chern–Simons–Dirac functional

Let Y be a closed Riemannian spin^c 3–manifold and η a closed 2–form on Y of class C^1. Fix a smooth reference spin connection B_o over Y and for any configuration (B,Ψ) over Y define the Chern–Simons–Dirac functional $\vartheta=\vartheta_\eta$ by

$$\vartheta(B,\Psi)=-\frac{1}{2}\int_Y(\hat{F}_B+\hat{F}_{B_o}+2i\eta)\wedge(B-B_o)-\frac{1}{2}\int_Y\langle\partial_B\Psi,\Psi\rangle.$$

Here and elsewhere $\langle\cdot,\cdot\rangle$ denotes Euclidean inner products, while $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes Hermitian inner products. Note that reversing the orientation of Y changes the sign of ϑ. Let $\mathcal{C}=\mathcal{C}_Y$ denote the space of L^2_1 configurations (B,Ψ). Then ϑ defines a smooth map $\mathcal{C}_Y\to\mathbb{R}$ which has an L^2 gradient

$$\nabla\vartheta_{(B,\Psi)}=\Big(*(\hat{F}_B+i\eta)-\frac{1}{2}\sigma(\Psi,\Psi),-\partial_B\Psi\Big).$$

If $\{a_j\}$ is a local orthonormal basis of imaginary-valued 1–forms on Y then

$$\sigma(\phi,\psi)=\sum_{j=1}^{3}\langle a_j\phi,\psi\rangle a_j.$$

Here and elsewhere the inner products are Euclidean unless otherwise specified. Since $\nabla\vartheta$ is independent of B_o, ϑ is independent of B_o up to additive constants. If $u\colon Y \to \mathrm{U}(1)$ then

$$\vartheta(u(S)) - \vartheta(S) = \int_Y (\hat{F}_B + i\eta) \wedge u^{-1}\,du = 2\pi \int_Y \tilde{\eta} \wedge [u],$$

where $[u] \in H^1(Y)$ is the pullback by u of the fundamental class of $\mathrm{U}(1)$, and $\tilde{\eta}$ is as in (1.2).

The invariance of ϑ under null-homotopic gauge transformations imply

$$\mathcal{I}_\Psi^* \nabla\vartheta_{(B,\Psi)} = 0. \tag{3.2}$$

Let $H_{(B,\Psi)}\colon L^2_1 \to L^2$ be the derivative of $\nabla\vartheta\colon \mathcal{C} \to L^2$ at (B,Ψ), ie

$$H_{(B,\Psi)}(b,\psi) = (*db - \sigma(\Psi,\psi), -b\Psi - \partial_B\psi).$$

Note that $H_{(B,\Psi)}$ is formally self-adjoint, and $H_{(B,\Psi)}\mathcal{I}_\Psi = 0$ if $\partial_B\Psi = 0$. As in [21], a critical point (B,Ψ) of ϑ is called *nondegenerate* if the kernel of $\mathcal{I}_\Psi^* + H_{(B,\Psi)}$ in L^2_1 is zero, or equivalently, if $\mathcal{I}_\Psi + H_{(B,\Psi)}\colon L^2_1 \to L^2$ is surjective. Note that if η is smooth then any critical point of ϑ_η has a smooth representative.

Let $\mathcal{G}$ be the Hilbert Lie group of L^2_2 maps $Y \to \mathrm{U}(1)$, and $\mathcal{G}_0 \subset \mathcal{G}$ the subgroup of null-homotopic maps. Set

$$\mathcal{B} = \mathcal{C}/\mathcal{G}, \quad \tilde{\mathcal{B}} = \mathcal{C}/\mathcal{G}_0.$$

Then ϑ descends to a continuous map $\tilde{\mathcal{B}} \to \mathbb{R}$ which we also denote by ϑ. If Condition (O1) holds (which we always assume when no statement to the contrary is made) then there is a real number q such that

$$\vartheta(\mathcal{G}S) = \vartheta(S) + q\mathbb{Z}$$

for all configurations S. If (O1) does not hold then $\vartheta(\mathcal{G}S)$ is a dense subset of $\mathbb{R}$.

If S is any smooth configuration over a band $(a,b)\times Y$, with $a<b$, let $\nabla\vartheta_S$ be the section of the bundle $\pi_2^*(\mathbb{S}_Y \oplus i\Lambda^1_Y)$ over $(a,b)\times Y$ such that $\nabla\vartheta_S|_{\{t\}\times Y} = \nabla\vartheta_{S_t}$. Here $\pi_2\colon \mathbb{R}\times Y \to Y$ is the projection. Note that $S \mapsto \nabla\vartheta_S$ extends to a smooth map $L^2_1 \to L^2$.

Although we will normally work with L^2_1 configurations over Y, the following lemma is sometimes useful.

Lemma 3.2.1 *ϑ extends to a smooth function on the space of $L^2_{1/2}$ configurations over Y.*

Proof The solution to the Dirichlet problem provides bounded operators

$$E\colon L^2_{1/2}(Y) \to L^2_1(\mathbb{R}_+ \times Y)$$

such that, for any $f \in L^2_{1/2}(Y)$, the function Ef restricts to f on $\{0\} \times Y$ and vanishes on $(1,\infty)\times Y$, and Ef is smooth whenever f is smooth. (see [49, p 307]). Similar extension maps can clearly be defined for configurations over Y. The lemma now follows from the observation that if S is any smooth configuration over $[0,1]\times Y$ then

$$\vartheta(S_1) - \vartheta(S_0) = \int_{[0,1]\times Y} \left\langle \nabla\vartheta_S, \frac{\partial S}{\partial t} \right\rangle,$$

and the right hand side extends to a smooth function on the space of L^2_1 configurations S over $[0,1]\times Y$. □

We will now relate the Chern–Simons–Dirac functional to the 4–dimensional monopole equations, cf [30; 40]. Let X be a spin^c Riemannian 4–manifold. Given a parameter $\mu \in \Omega^2(X)$ there are the following Seiberg–Witten equations for a configuration (A,Φ) over X:

$$\begin{aligned} (\widehat{F}_A + i\mu)^+ &= Q(\Phi) \\ D_A\Phi &= 0, \end{aligned} \tag{3.3}$$

where
$$Q(\Phi) = \frac{1}{4}\sum_{j=1}^{3} \langle \alpha_j \Phi, \Phi\rangle \alpha_j$$

for any local orthonormal basis $\{\alpha_j\}$ of imaginary-valued self-dual 2–forms on X. If Ψ is another section of $\mathbb{S}^+_X$ then one easily shows that

$$Q(\Phi)\Psi = \langle \Psi, \Phi\rangle_{\mathbb{C}}\Phi - \frac{1}{2}|\Phi|^2\Psi.$$

Now let $X = \mathbb{R}\times Y$ and for present and later use recall the standard bundle isomorphisms

$$\begin{aligned} \rho^1\colon \pi_2^*(\Lambda^0(Y)\oplus\Lambda^1(Y)) &\to \Lambda^1(\mathbb{R}\times Y), \quad (f,a)\mapsto f\,dt + a, \\ \rho^+\colon \pi_2^*(\Lambda^1(Y)) &\to \Lambda^+(\mathbb{R}\times Y), \quad a \mapsto \frac{1}{2}(dt\wedge a + *_Y a). \end{aligned} \tag{3.4}$$

Here $*_Y$ is the Hodge $*$–operator on Y. Let μ be the pullback of a 2–form η on Y. Set $\vartheta = \vartheta_\eta$. Let $S = (A,\Phi)$ be any smooth configuration over $\mathbb{R}\times Y$ such that A is in temporal gauge. Under the identification $\mathbb{S}^+_{\mathbb{R}\times Y} = \pi_2^*(\mathbb{S}_Y)$ we have

$$\rho^+(\sigma(\Phi,\Phi)) = 2Q(\Phi).$$

Let $\nabla_1\vartheta$, $\nabla_2\vartheta$ denote the 1–form and spinor parts of $\nabla\vartheta$, respectively. Then

$$\begin{aligned} \rho^+\left(\frac{\partial A}{\partial t}+\nabla_1\vartheta_S\right) &= (\hat{F}_A+i\pi_2^*\eta)^+-Q(\Phi) \\ \frac{\partial \Phi}{\partial t}+\nabla_2\vartheta_S &= -dt\cdot D_A\Phi, \end{aligned} \tag{3.5}$$

Thus, the downward gradient flow equation

$$\frac{\partial S}{\partial t}+\nabla\vartheta_S=0$$

is equivalent to the Seiberg–Witten equations (3.3).

3.3 Perturbations

For transversality reasons we will, as in [21], add further small perturbations to the Seiberg–Witten equation over $\mathbb{R}\times Y$. The precise shape of these perturbations will depend on the situation considered. At this point we will merely describe a set of properties of these perturbations which will suffice for the Fredholm, compactness and gluing theory.

To any L^2_1 configuration S over the band $(-1/2,1/2)\times Y$ there will be associated an element $h(S)\in\mathbb{R}^N$, where $N\ge 1$ will depend on the situation considered. If S is an L^2_1 configuration over $\mathbf{B}^+=(a-1/2,b+1/2)\times Y$ where $-\infty<a<b<\infty$ then the corresponding function

$$h_S\colon [a,b]\to\mathbb{R}^N$$

given by $h_S(t)=h(S|_{(t-1/2,t+1/2)\times Y})$ will be smooth. These functions h_S will have the following properties. Let S_o be a smooth reference configuration over $\mathbf{B}^+$.

(P1) For $0\le k<\infty$ the assignment $s\mapsto h_{S_o+s}$ defines a smooth map $L^2_1\to C^k$ whose image is a bounded set.

(P2) If $S_n\to S$ weakly in L^2_1 then $\|h_{S_n}-h_S\|_{C^k}\to 0$ for every $k\ge 0$.

(P3) h_S is gauge invariant, ie $h_S=h_{u(S)}$ for any smooth gauge transformation u.

We will also choose a compact codimension 0 submanifold $\Xi\subset\mathbb{R}^N$ which does not contain $h(\underline{\alpha})$ for any critical point α, where $\underline{\alpha}$ is the translationary invariant configuration over $\mathbb{R}\times Y$ (in temporal gauge) determined by α. Let $\widetilde{\mathfrak{P}}=\widetilde{\mathfrak{P}}_Y$ denote the space of all (smooth) 2–forms on $\mathbb{R}^N\times Y$ supported in $\Xi\times Y$. For any S as above and any $\mathfrak{p}\in\widetilde{\mathfrak{P}}$ let $h_{S,\mathfrak{p}}\in\Omega^2([a,b]\times Y)$ denote the pullback of $\mathfrak{p}$ by the map $h_S\times\mathrm{Id}$. It is clear that $h_{S,\mathfrak{p}}(t,y)=0$ if $h_S(t)\notin\Xi$. Moreover,

$$\|h_{S,\mathfrak{p}}\|_{C^k}\le\gamma_k\|\mathfrak{p}\|_{C^k} \tag{3.6}$$

where the constant γ_k is independent of $S, \mathfrak{p}$.

Now let $-\infty \le a < b \le \infty$ and $\mathbf{B} = (a,b) \times Y$. If $\mathfrak{q}\colon \mathbf{B} \to \mathbb{R}$ is a smooth function then by a $(\mathfrak{p},\mathfrak{q})$*–monopole* over $\mathbf{B}$ we shall mean a configuration $S = (A, \Phi)$ over $\mathbf{B}^+ = (a-1/2, b+1/2) \times Y$ (smooth, unless otherwise stated) which satisfies the equations

$$\begin{aligned}(\hat{F}_A + i\pi_2^*\eta + i\mathfrak{q}h_{S,\mathfrak{p}})^+ &= Q(\Phi)\\ D_A\Phi &= 0,\end{aligned} \tag{3.7}$$

over $\mathbf{B}$, where η is as before. If A is in temporal gauge then these equations can also be expressed as

$$\frac{\partial S_t}{\partial t} = -\nabla\vartheta_{S_t} + E_S(t), \tag{3.8}$$

where the perturbation term $E_S(t)$ depends only on the restriction of S to $(t-1/2, t+1/2) \times Y$.

To reduce the number of constants later, we will always assume that $\mathfrak{q}$ and its differential $d\mathfrak{q}$ are pointwise bounded (in norm) by 1 everywhere. Note that if $\mathfrak{q}$ is constant then the equations (3.7) are translationary invariant. A $(\mathfrak{p},\mathfrak{q})$–monopole with $\mathfrak{q}h_{S,\mathfrak{p}} = 0$ is called a *genuine* monopole. In expressions like $\|F_A\|_2$ and $\|\Phi\|_\infty$ the norms will usually be taken over $\mathbf{B}$.

For the transversality theory in Chapter 8 we will need to choose a suitable Banach space $\mathfrak{P} = \mathfrak{P}_Y$ of forms $\mathfrak{p}$ as above (of some given regularity). It will be essential that

$$\mathcal{C}(\mathbf{B}^+) \times \mathfrak{P} \to L^p(\mathbf{B}, \Lambda^2), \quad (S, \mathfrak{p}) \mapsto h_{S,\mathfrak{p}} \tag{3.9}$$

be a smooth map when a, b are finite (here $p > 2$ is the exponent used in defining the configuration space $\mathcal{C}(\mathbf{B}^+)$). Now, one cannot expect $h_{S,\mathfrak{p}}$ to be smooth in S unless $\mathfrak{p}$ is smooth in the $\mathbb{R}^N$ direction (this point was overlooked in [21]). It seems natural then to look for a suitable space $\mathfrak{P}$ consisting of *smooth* forms $\mathfrak{p}$. Such a $\mathfrak{P}$ will be provided by Lemma 8.2.1. The topology on $\mathfrak{P}$ will be stronger than the C^∞ topology, ie stronger than the C^k topology for every k. The smoothness of the map (3.9) is then an easy consequence of property (P1) above and the next lemma.

Lemma 3.3.1 *Let A be a topological space, U a Banach space, and $K \subset \mathbb{R}^n$ a compact subset. Then the composition map*

$$C_B(A, \mathbb{R}^n) \times C^k(\mathbb{R}^n, U)_K \to C_B(A, U)$$

is of class C^{k-1} for any natural number k. Here $C_B(A, \cdot)$ denotes the supremum-normed space of bounded continuous maps from A into the indicated space, and $C^k(\mathbb{R}^n, U)_K$ is the space of C^k maps $\mathbb{R}^n \to U$ with support in K.

Proof This is a formal exercise in the differential calculus. □

3.4 Moduli spaces

Consider the situation of Section 1.3. (We do not assume here that (A) holds.) We will define the moduli space $M(X;\vec{\alpha})$. In addition to the parameter μ this will depend on a choice of perturbation forms $\mathfrak{p}_j \in \widetilde{\mathfrak{P}}_{Y_j}$ and a smooth function $\mathfrak{q}\colon X \to [0,1]$ such that $\|d\mathfrak{q}\|_\infty \le 1$, $\mathfrak{q}^{-1}(0) = X_{:(3/2)}$ and $\mathfrak{q} = 1$ on $[3,\infty)\times Y$.

Choose a smooth reference configuration $S_o = (A_o, \Phi_o)$ over X which is translationary invariant and in temporal gauge over the ends and such that $S_o|_{\{t\}\times Y_j}$ represents $\alpha_j \in \widetilde{\mathcal{R}}_{Y_j}$. Let $p > 4$ and choose w as in Section 2.1. Let $\mathfrak{b}$ be a finite subset of X and define $\mathcal{C}, \mathcal{G}_\mathfrak{b}, \mathcal{B}_\mathfrak{b}$ as in Sections 2.1 and 2.4. For clarity we will sometimes write $\mathcal{C}(X;\vec{\alpha})$ etc. Set $\vec{\mathfrak{p}} = (\mathfrak{p}_1, \dots, \mathfrak{p}_r)$ and let

$$M_\mathfrak{b}(X;\vec{\alpha}) = M_\mathfrak{b}(X;\vec{\alpha};\mu;\vec{\mathfrak{p}}) \subset \mathcal{B}_\mathfrak{b}$$

be the subset of gauge equivalence classes of solutions $S = (A, \Phi)$ (which we simply refer to as monopoles) to the equations

$$\begin{aligned} \Big(\widehat{F}_A + i\mu + i\mathfrak{q}\sum_{j=1}^r h_{S,\mathfrak{p}_j}\Big)^+ - Q(\Phi) &= 0 \\ D_A\Phi &= 0. \end{aligned} \tag{3.10}$$

It is clear that $\mathfrak{q}\sum_j h_{S,\mathfrak{p}_j}$ vanishes outside a compact set in X. If it vanishes everywhere then S is called a *genuine* monopole. If $\mathfrak{b}$ is empty then we write $M = M_\mathfrak{b}$.

Note that different choices of S_o give canonically homeomorphic moduli spaces $M_\mathfrak{b}(X;\vec{\alpha})$ (and similarly for $\mathcal{B}_\mathfrak{b}(X;\vec{\alpha})$).

Unless otherwise stated the forms μ and $\mathfrak{p}_j$ will be smooth. In that case every element of $M_\mathfrak{b}(X;\vec{\alpha};\mu;\vec{\mathfrak{p}})$ has a smooth representative, and in notation like $[S] \in M$ we will often implicitly assume that S is smooth.

We define the moduli spaces $M(\alpha,\beta) = M(\alpha,\beta;\mathfrak{p})$ of Section 1.2 similarly, except that we here use the equations (3.7) with $\mathfrak{q} \equiv 1$.

The following estimate will be crucial in compactness arguments later.

Proposition 3.4.1 *For any element $[A,\Phi] \in M(X;\vec{\alpha})$ one has that either*

$$\Phi = 0 \quad \text{or} \quad \|\Phi\|_\infty^2 \le -\frac{1}{2}\inf_{x\in X}\mathbf{s}(x) + 4\|\mu\|_\infty + 4\gamma_0 \max_j \|\mathfrak{p}_j\|_\infty,$$

where $\mathbf{s}$ is the scalar curvature of X and the constant γ_0 is as in (3.6).

Proof Let ψ_j denote the spinor field of $A_o|_{\{t\}\times Y_j}$. If $|\Phi|$ has a global maximum then the conclusion of the proposition holds by the proof of [30, Lemma 2]. Otherwise one must have $\|\Phi\|_\infty = \max_j \|\psi_j\|_\infty$ because of the Sobolev embedding $L^p_1 \subset C^0$ on compact 4–manifolds. But the argument in [30] applied to $\mathbb{R}\times Y$ yields

$$\psi_j = 0 \quad \text{or} \quad \|\psi_j\|^2_\infty \le -\frac{1}{2}\inf_{x\in X}\mathbf{s}(x) + 4\|\mu\|_\infty$$

for each j, and the proposition follows. $\square$

The left hand side of (3.10) can be regarded as a section $\Theta(S) = \widetilde{\Theta}(S,\mu,\vec{\mathfrak{p}})$ of the bundle $\Lambda^+ \oplus \mathbb{S}^-$ over X. It is clear that Θ defines a smooth map

$$\Theta\colon \mathcal{C} \to L^{p,w},$$

which we call the *monopole map*. Let $D\Theta$ denote the derivative of Θ. We claim that

$$\mathcal{I}^*_\Phi + D\Theta(S)\colon L^{p,w}_1 \to L^{p,w} \tag{3.11}$$

is a Fredholm operator for every $S=(A,\Phi)\in\mathcal{C}$. Note that the $\mathfrak{p}_j$–perturbations in (3.10) only contributes a compact operator, so we can take $\mathfrak{p}_j = 0$ for each j. We first consider the case $X=\mathbb{R}\times Y$, with $\mu=\pi_2^*\eta$ as before. By means of the isomorphisms (3.4), (3.1) and the isomorphism $\mathbb{S}^+\to\mathbb{S}^-$, $\phi\mapsto dt\cdot\phi$ we can think of the operator (3.11) as acting on sections of $\pi_2^*(\Lambda^0_Y\oplus\Lambda^1_Y\oplus\mathbb{S}_Y)$. If A is in temporal gauge then a simple computation yields

$$\mathcal{I}^*_\Phi + D\Theta(S) = \frac{d}{dt} + P_{S_t}, \tag{3.12}$$

where
$$P_{(B,\Psi)} = \begin{pmatrix} 0 & \mathcal{I}^*_\Psi \\ \mathcal{I}_\Psi & H_{(B,\Psi)} \end{pmatrix}$$

for any configuration (B,Ψ) over Y. Note that $P_{(B,\Psi)}$ is elliptic and formally self-adjoint, and if (B,Ψ) is a nondegenerate critical point of ϑ_η then $\ker P_{(B,\Psi)} = \ker\mathcal{I}_\Psi$. Thus, the structure of the linearized equations over a cylinder is analogous to that of the instanton equations studied in [14], and the results of [14] carry over to show that (3.11) is a Fredholm operator.

The index of (3.11) is independent of S and is called the *expected dimension* of $M(X;\vec{\alpha})$. If $S\in\mathcal{C}$ is a monopole and $D\Theta(S)\colon L^{p,w}_1\to L^{p,w}$ is surjective then $[S]$ is called a *regular* point of $M_{\mathfrak{b}}(X;\vec{\alpha})$. If in addition $S\in\mathcal{C}^*_{\mathfrak{b}}$ then $[S]$ has an open neighbourhood in $M_{\mathfrak{b}}(X;\vec{\alpha})$ which is a smooth submanifold of $\mathcal{B}^*_{\mathfrak{b}}$ of dimension

$$\dim M_{\mathfrak{b}}(X;\vec{\alpha}) = \operatorname{index}(\mathcal{I}^*_\Phi + D\Theta(S)) + |\mathfrak{b}|.$$

The following regularity result to some extent makes up for the fact that we only work with $L^p_{1,\mathrm{loc}}$ configurations.

Proposition 3.4.2 *Let ω_0 be a regular, irreducible point of $M(X;\vec{\alpha})$. Let $Z \subset X$ be a smooth compact codimension 0 submanifold and $\mathcal{C}(Z)_{C^\ell}$ the space of configurations over Z of class C^ℓ, where ℓ is a natural number. Then there is an open neighbourhood U of ω_0 and a smooth map $f\colon U \to \mathcal{C}(Z)_{C^\ell}$ such that $f(\omega)$ is a representative of $\omega|_Z$ for every $\omega \in U$.*

Proof Let S_0 be a smooth representative of ω_0. For any natural number k let V_k denote the space of all $L^p_{k,\mathrm{loc}}$ configurations S over X such that $S - S_0 \in L^{p,w}_k$ and the local slice condition $\mathcal{I}^*_{S_0}(S - S_0) = 0$ holds. Let V_k have the $L^{p,w}_k$ metric. Because S_0 is a regular monopole, V_k is smooth in a neighbourhood of S_0. By elliptic regularity V_k consists of smooth configurations. The inclusion $\iota_k\colon V_k \to V_1$ induces the identity map between the tangent spaces at S_0, so by the inverse function theorem ι_k is a local diffeomorphism at S_0 . By the local slice theorem the projection $V_1 \to M(X;\vec{\alpha})$ is a local diffeomorphism at S_0. Taking $k > \ell + 4/p$ the proposition now follows from the Sobolev embedding $L^p_k(Z) \to C^\ell(Z)$. □

CHAPTER 4

Local compactness I

This chapter provides the local compactness results needed for the proof of Theorem 1.4.1 assuming (B1).

4.1 Compactness under curvature bounds

For the moment let B be an arbitrary compact, oriented Riemannian manifold with boundary, and v the outward unit normal vector field along ∂B. Then

$$\Omega^*(B) \to \Omega^*(B) \oplus \Omega^*(\partial B), \quad \phi \mapsto ((d + d^*)\phi, \iota(v)\phi) \tag{4.1}$$

is an elliptic boundary system in the sense of Hörmander [26] and Atiyha [2]. Here $\iota(v)$ is contraction with v. By [26, Theorems 20.1.2, 20.1.8] we then have:

Proposition 4.1.1 *For $k \geq 1$ the map (4.1) extends to a Fredholm operator*

$$L^2_k(B, \Lambda^*_B) \to L^2_{k-1}(B, \Lambda^*_B) \oplus L^2_{k-1/2}(\partial B, \Lambda^*_{\partial B})$$

whose kernel consists of C^∞ forms.

Lemma 4.1.1 *Let X be a spin^c Riemannian 4–manifold and $V_1 \subset V_2 \subset \cdots$ precompact open subsets of X such that $X = \bigcup_j V_j$. For $n = 1, 2, \ldots$ let μ_n be a 2–form on V_n, and $S_n = (A_n, \Phi_n)$ a smooth solution to the Seiberg–Witten equations (3.3) over V_n with $\mu = \mu_n$. Let $q > 4$. Then there exist a subsequence $\{n_j\}$ and for each j a smooth $u_j \colon V_j \to \mathrm{U}(1)$ with the following significance. If k is any nonnegative integer such that*

$$\sup_{n \geq j} \left(\|\Phi_n\|_{L^q(V_j)} + \|\hat{F}(A_n)\|_{L^2(V_j)} + \|\mu_n\|_{C^k(V_j)} \right) < \infty \tag{4.2}$$

for every positive integer j then for every $p \geq 1$ one has that $u_j(S_{n_j})$ converges weakly in L^p_{k+1} and strongly in L^p_k over compact subsets of X as $j \to \infty$.

Before giving the proof, note that the curvature term in (4.2) cannot be omitted. For if ω is any nonzero, closed, anti-self-dual 2–form over the 4–ball B then there is a sequence A_n of U(1) connections over B such that $F(A_n) = in\omega$. If $S_n = (A_n, 0)$ then there are clearly no gauge transformations u_n such that $u_n(S_n)$ converges (in any reasonable sense) over compact subsets of B.

Proof of Lemma 4.1.1 Let $B \subset X$ be a compact 4–ball. After trivializing $\mathcal{L}$ over B we can write $A_n|_B = A_o + a_n$, where A_o is the spin connection over B corresponding to the product connection in $\mathcal{L}|_B$. By the solution to the Neumann problem (see [49]) there is a smooth $\xi_n\colon B \to i\mathbb{R}$ such that $b_n = a_n - d\xi$ satisfies

$$d^* b_n = 0; \quad *b_n|_{\partial B} = 0.$$

Using the fact that $H^1(B) = 0$ one easily proves that the map (4.1) is injective on $\Omega^1(B)$. Hence there is a constant C such that

$$\|b\|_{L^2_1(B)} \leq C(\|(d + d^*)b\|_{L^2(B)} + \|{*b}|_{\partial B}\|_{L^2_{1/2}(\partial B)})$$

for all $b \in \Omega^1(B)$. This gives

$$\|b_n\|_{L^2_1(B)} \leq C\|db_n\|_{L^2(B)} = C\|\widehat{F}(A_n)\|_{L^2(B)}.$$

Set $v_n = \exp(\xi_n)$. It is now an exercise in bootstrapping, using the Seiberg–Witten equations for S_n and interior elliptic estimates, to show that, for every $k \geq 0$ for which (4.2) holds and for every $p \geq 1$, the sequence $v_n(S_n) = (A_o + b_n, v_n\Phi_n)$ is bounded in L^p_{k+1} over compact subsets of $\mathrm{int}(B)$.

To complete the proof, choose a countable collection of such balls such that the corresponding balls of half the size cover X, and apply Lemma A.0.1. □

4.2 Small perturbations

If S is any smooth configuration over a band $(a,b) \times Y$ with $a < b$, the *energy* of S is by definition

$$\mathrm{energy}(S) = \int_{[a,b]\times Y} |\nabla\vartheta_S|^2.$$

If S is a genuine monopole then $\partial_t\vartheta(S_t) = -\int_Y |\nabla\vartheta_{S_t}|^2$, and so the energy equals $\vartheta(S_a) - \vartheta(S_b)$. If S is a $(\mathfrak{p},\mathfrak{q})$–monopole then one no longer expects these identities to hold, because the equation (3.8) is not of gradient flow type. The main object of this section is to show that if $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then, under suitable assumptions, the variation of $\vartheta(S_t)$ still controls the energy locally (Proposition 4.2.1), and there is a monotonicity result for $\vartheta(S_t)$ (Proposition 4.2.2).

It may be worth mentioning that the somewhat technical Lemma 4.2.3 and Proposition 4.2.1 are not needed in the second approach to compactness which is the subject of Chapter 5.

In this section $\mathfrak{q}\colon \mathbb{R}\times Y\to\mathbb{R}$ may be any smooth function satisfying $\|\mathfrak{q}\|_\infty, \|d\mathfrak{q}\|_\infty\le 1$. Constants will be independent of $\mathfrak{q}$. The perturbation forms $\mathfrak{p}$ may be arbitrary elements of $\widetilde{\mathfrak{P}}$.

Lemma 4.2.1 *There is a constant $C_0>0$ such that if $-\infty<a<b<\infty$ and $S=(A,\Phi)$ is any $(\mathfrak{p},\mathfrak{q})$–monopole over $(a,b)\times Y$ then there is a pointwise bound*

$$|\widehat{F}(A)|\le 2|\nabla\vartheta_S|+|\eta|+C_0|\Phi|^2+\gamma_0\|\mathfrak{p}\|_\infty.$$

Proof Note that both sides of the inequality are gauge invariant, and if A is in temporal gauge then

$$F(A)=dt\wedge\frac{\partial A_t}{\partial t}+F_Y(A_t),$$

where F_Y stands for the curvature of a connection over Y. Now use inequalities (3.8) and (3.6). ◻

Lemma 4.2.2 *There exists a constant $C_1>0$ such that for any $\tau>0$ and any $(\mathfrak{p},\mathfrak{q})$–monopole S over $(0,\tau)\times Y$ one has*

$$\int_{[0,\tau]\times Y}|\nabla\vartheta_S|^2\le 2(\vartheta(S_0)-\vartheta(S_\tau))+C_1^2\tau\|\mathfrak{p}\|_\infty^2.$$

Recall that by convention a $(\mathfrak{p},\mathfrak{q})$–monopole over $(0,\tau)\times Y$ is actually a configuration over $(-1/2,\tau+1/2)\times Y$, so the lemma makes sense.

Proof We may assume S is in temporal gauge. Then

$$\begin{aligned}\vartheta(S_\tau)-\vartheta(S_0)&=\int_0^\tau \partial_t\vartheta(S_t)\,dt\\&=\int_{[0,\tau]\times Y}\langle\nabla\vartheta_S,-\nabla\vartheta_S+E_S\rangle\,dt\\&\le\|\nabla\vartheta_S\|_2(\|E_S\|_2-\|\nabla\vartheta_S\|_2),\end{aligned}$$

where the norms on the last line are taken over $[0,\tau]\times Y$. If a,b,x are real numbers satisfying $x^2-bx-a\le 0$ then

$$x^2\le 2x^2-2bx+b^2\le 2a+b^2.$$

Putting this together we obtain

$$\|\nabla\vartheta_S\|_2^2\le 2(\vartheta(S_0)-\vartheta(S_\tau))+\|E_S\|_2^2,$$

and the lemma follows from the estimate (3.6). □

Lemma 4.2.3 *For all $C>0$ there exists an $\epsilon>0$ with the following significance. Let $\tau\ge 4$, $\mathfrak{p}\in\widetilde{\mathfrak{P}}$ with $\|\mathfrak{p}\|_\infty\tau^{1/2}\le\epsilon$, and let $S=(A,\Phi)$ be a $(\mathfrak{p},\mathfrak{q})$–monopole over $(0,\tau)\times Y$ satisfying $\|\Phi\|_\infty\le C$. Then at least one of the following two statements must hold:*

(i) $\partial_t\vartheta(S_t)\le 0$ *for* $2\le t\le\tau-2$.

(ii) $\vartheta(S_{t_2})<\vartheta(S_{t_1})$ *for* $0\le t_1\le 1$, $\tau-1\le t_2\le\tau$.

Proof Given $C>0$, suppose that for $n=1,2,\dots$ there exist $\tau_n\ge 4$, $\mathfrak{p}_n\in\widetilde{\mathfrak{P}}$ with $\|\mathfrak{p}_n\|_\infty\tau_n^{1/2}\le 1/n$, and a $(\mathfrak{p}_n,\mathfrak{q}_n)$–monopole $S_n=(A_n,\Phi_n)$ over $(0,\tau_n)\times Y$ satisfying $\|\Phi_n\|_\infty\le C$ such that (i) is violated at some point $t=t_n$ and (ii) also does not hold. By Lemma 4.2.2 the last assumption implies

$$\|\nabla\vartheta_{S_n}\|_{L^2([1,\tau_n-1]\times Y)}\le C_1\|\mathfrak{p}_n\|_\infty\tau_n^{1/2}\le C_1/n.$$

For $s\in\mathbb{R}$ let $\mathcal{T}_s\colon\mathbb{R}\times Y\to\mathbb{R}\times Y$ be translation by s:

$$\mathcal{T}_s(t,y)=(t+s,y).$$

Given $p>2$ then by Lemma 4.2.1 and Lemma 4.1.1 we can find $u_n\colon(-1,1)\times Y\to\mathrm{U}(1)$ in $L^p_{2,\mathrm{loc}}$ such that a subsequence of $u_n(\mathcal{T}^*_{t_n}(S_n))$ converges weakly in L^p_1 over $(-1/2,1/2)\times Y$ to an L^p_1 solution S' to the equations (3.3) with $\mu=\pi_2^*\eta$. Then $\nabla\vartheta_{S'}=0$. After modifying the gauge transformations we can even arrange that S' is smooth and in temporal gauge, in which case there is a critical point α of ϑ such that $S'(t)\equiv\alpha$. After relabelling the subsequence above consecutively we then have

$$h_{S_n}(t_n)\to h_{S'}(0)\notin\Xi.$$

Since Ξ is closed, $h_{S_n}(t_n)\notin\Xi$ for n sufficiently large. Therefore, $\partial_t|_{t_n}\vartheta(S_n(t))=-\|\nabla\vartheta_{S_n(t_n)}\|^2\le 0$, which is a contradiction. □

Proposition 4.2.1 *For any constant $C>0$ there exist $C',\delta>0$ such that if $S=(A,\Phi)$ is any $(\mathfrak{p},\mathfrak{q})$–monopole over $(-2,T+4)\times Y$ where $T\ge 2$, $\|\mathfrak{p}\|_\infty\le\delta$, and $\|\Phi\|_\infty\le C$, then for $1\le t\le T-1$ one has*

$$\int_{[t-1,t+1]\times Y}|\nabla\vartheta_S|^2\le 2\Bigl(\sup_{0\le r\le 1}\vartheta(S_{-r})-\inf_{0\le r\le 4}\vartheta(S_{T+r})\Bigr)+C'\|\mathfrak{p}\|_\infty^2.$$

Proof Choose $\epsilon>0$ such that the conclusion of Lemma 4.2.3 holds (with this constant C), and set $\delta=\epsilon/\sqrt{6}$. We construct a sequence $t_0,\dots,t_m$ of real numbers, for some $m\ge 1$, with the following properties:

(i) $-1\le t_0\le 0$ and $T\le t_m\le T+4$.

(ii) For $i=1,\dots,m$ one has $1\le t_i-t_{i-1}\le 5$ and $\vartheta(S_{t_i})\le\vartheta(S_{t_{i-1}})$.

The lemma will then follow from Lemma 4.2.2. The t_i 's will be constructed inductively, and this will involve an auxiliary sequence $t'_0,\dots,t'_{m+1}$. Set $t_{-1}=t'_0=0$.

Now suppose t_{i-1},t'_i have been constructed for $0\le i\le j$. If $t'_j\ge T$ then we set $t_j=t'_j$ and $m=j$, and the construction is finished. If $t'_j<T$ then we define t_j,t'_{j+1} as follows:

If $\partial_t\vartheta(S_t)\le 0$ for all $t\in[t'_j,t'_j+2]$ set $t_j=t'_j$ and $t'_{j+1}=t'_j+2$; otherwise set $t_j=t'_j-1$ and $t'_{j+1}=t'_j+4$.

Then (i) and (ii) are satisfied, by Lemma 4.2.3. □

Proposition 4.2.2 *For all $C>0$ there exists a $\delta>0$ such that if $S=(A,\Phi)$ is any $(\mathfrak{p},\mathfrak{q})$–monopole in temporal gauge over $(-1,1)\times Y$ such that $\|\mathfrak{p}\|_{C^1}\le\delta$, $\|\Phi\|_\infty\le C$ and $\|\nabla\vartheta_S\|_2\le C$ then the following holds: Either $\partial_t|_0\vartheta(S_t)<0$, or there is a critical point α such that $S_t=\alpha$ for $|t|\le 1/2$.*

Proof First observe that if S is any C^1 configuration over $\mathbb{R}\times Y$ then

$$\vartheta(S_{t_2})-\vartheta(S_{t_1})=\int_{t_1}^{t_2}\int_Y\langle\nabla\vartheta_{S_t},\partial_tS_t\rangle\,dy\,dt,$$

hence $\vartheta(S_t)$ is a C^1 function of t whose derivative can be expressed in terms of the L^2 gradient of ϑ as usual.

Now suppose there is a $C>0$ and for $n=1,2,\dots$ a $\mathfrak{p}_n\in\widetilde{\mathfrak{P}}$ and a $(\mathfrak{p}_n,\mathfrak{q}_n)$–monopole $S_n=(A_n,\Phi_n)$ over $(-1,1)\times Y$ such that $\|\mathfrak{p}_n\|_{C^1}\le 1/n$, $\|\Phi_n\|_\infty\le C$, $\|\nabla\vartheta_{S_n}\|_2\le C$ and $\partial_t|_0\vartheta(S_n(t))\ge 0$. Let $p>4$ and $0<\epsilon<1/2$. After passing to a subsequence and relabelling consecutively we can find $u_n\colon(-1,1)\times Y\to\mathrm{U}(1)$ in $L^p_{3,\mathrm{loc}}$ such that

$\widetilde{S}_n = u_n(S_n)$ converges weakly in L^p_2, and strongly in C^1, over $(-1/2-\epsilon, 1/2+\epsilon)\times Y$ to a smooth solution S' of (3.3) with $\mu = \pi_2^*\eta$. We may arrange that S' is in temporal gauge. Then

$$0 \le \partial_t|_0 \vartheta(\widetilde{S}_n(t)) = \int_Y \langle \nabla\vartheta_{\widetilde{S}_n(0)}, \partial_t|_0 \widetilde{S}_n(t)\rangle \to \partial_t|_0\vartheta(S'_t).$$

But S' is a genuine monopole, so $\partial_t \vartheta(S'_t) = -\|\nabla\vartheta_{S'_t}\|_2^2$. It follows that $\nabla\vartheta_{S'(0)} = 0$, hence $\nabla\vartheta_{S'} = 0$ in $(-1/2-\epsilon, 1/2+\epsilon)\times Y$ by unique continuation as in [21, Appendix]. Since $h_{S_n} \to h_{S'}$ uniformly in $[-\epsilon,\epsilon]$, and $h_{S'} \equiv \text{const} \notin \Xi$, the function h_{S_n} maps $[-\epsilon,\epsilon]$ into the complement of Ξ when n is sufficiently large. In that case, S_n restricts to a genuine monopole on $[-\epsilon,\epsilon]\times Y$, and the assumption $\partial_t|_0\vartheta(S_n(t)) \ge 0$ implies that $\nabla\vartheta_{S_n} = 0$ on $[-\epsilon,\epsilon]\times Y$. Since this holds for any $\epsilon \in (0,1/2)$, the proposition follows. □

We say a $(\mathfrak{p},\mathfrak{q})$–monopole S over $\mathbb{R}_+\times Y$ has *finite energy* if $\inf_{t>0}\vartheta(S_t) > -\infty$. A monopole over a 4–manifold with tubular ends is said to have finite energy if it has finite energy over each end.

Proposition 4.2.3 *Let C,δ be given such that the conclusion of Proposition 4.2.2 holds. If $S = (A,\Phi)$ is any finite energy $(\mathfrak{p},\mathfrak{q})$–monopole over $\mathbb{R}_+\times Y$ with $\|\mathfrak{p}\|_{C^1} \le \delta$, $\mathfrak{q} \equiv 1$, $\|\Phi\|_\infty \le C$ and*

$$\sup_{t\ge 1} \|\nabla\vartheta_S\|_{L^2((t-1,t+1)\times Y)} \le C$$

then the following hold:

(i) *There is a $t > 0$ such that S restricts to a genuine monopole on $(t,\infty)\times Y$.*

(ii) *$[S_t]$ converges in $\mathcal{B}_Y$ to some critical point as $t \to \infty$.*

Proof Let $p > 4$. If $\{t_n\}$ is any sequence with $t_n \to \infty$ as $n \to \infty$ then by Lemma 4.1.1 and Lemma 4.2.1 there exist $u_n \in L^p_{3,\mathrm{loc}}(\mathbb{R}\times Y; \mathrm{U}(1))$ such that a subsequence of $u_n(\mathcal{T}^*_{t_n} S)$ converges weakly in L^p_2 (hence strongly in C^1) over compact subsets of $\mathbb{R}\times Y$ to a smooth $(\mathfrak{p},\mathfrak{q})$–monopole S' in temporal gauge. Proposition 4.2.2 guarantees that $\partial_t\vartheta(S_t) \le 0$ for $t \ge 1$, so the finite energy assumption implies that $\vartheta(S'_t)$ is constant. By Proposition 4.2.2 there is a critical point α such that $S'_t = \alpha$ for all t. This implies (i) by choice of the set Ξ (see Section 3.3). Part (ii) follows by a continuity argument from the facts that $\mathcal{B}_Y$ contains only finitely many critical points, and the topology on $\mathcal{B}_Y$ defined by the L^2–metric is weaker than the usual topology. □

The following corollary of Lemma 3.2.1 shows that elements of the moduli spaces defined in Section 3.4 have finite energy.

Lemma 4.2.4 *Let S be a configuration over $\overline{\mathbb{R}}_+ \times Y$ and α a critical point of ϑ such that $S - \underline{\alpha} \in L^p_1$ for some $p \geq 2$. Then*

$$\vartheta(S_t) \to \vartheta(\alpha) \quad \text{as } t \to \infty. \qquad \square$$

4.3 Neck-stretching I

This section contains the crucial step in the proof of Theorem 1.4.1 assuming (B1), namely what should be thought of as a global energy bound.

Lemma 4.3.1 *Let X be as in Section 1.3 and set $Z = X_{:1}$. We identify $Y = \partial Z$. Let $\mu_1, \mu_2 \in \Omega^2(Z)$, where $d\mu_1 = 0$. Set $\eta = \mu_1|_Y$ and $\mu = \mu_1 + \mu_2$. Let A_o be a spin connection over Z, and let the Chern–Simons–Dirac functional ϑ_η over Y be defined in terms of the reference connection $B_o = A_o|_Y$. Then for all configurations $S = (A, \Phi)$ over Z which satisfy the monopole equations (3.3) one has*

$$\left| 2\vartheta_\eta(S|_Y) + \int_Z \left(|\nabla_A \Phi|^2 + |\widehat{F}_A + i\mu_1|^2 \right) \right|$$
$$\leq C\,\mathrm{Vol}(Z) \left(1 + \|\Phi\|^2_\infty + \|F_{A_o}\|_\infty + \|\mu_1\|_\infty + \|\mu_2\|_\infty + \|\mathbf{s}\|_\infty \right)^2,$$

for some universal constant C, where $\mathbf{s}$ is the scalar curvature of Z.

The upper bound given here is not optimal but suffices for our purposes.

Proof Set $F'_A = \widehat{F}_A + i\mu_1$ and define F'_{A_o} similarly. Set $B = A|_Y$. Without the assumption $d\mu_1 = 0$ we have

$$\begin{aligned}
\int_Z |F'_A|^2 &= \int_Z (2|(F'_A)^+|^2 + F'_A \wedge F'_A) \\
&= \int_Z \left(2|Q(\Phi) - i\mu_2^+|^2 + F'_{A_o} \wedge F'_{A_o} - 2i\, d\mu_1 \wedge (A - A_o) \right) \\
&\quad + \int_Y (\widehat{F}_B + \widehat{F}_{B_o} + 2i\eta) \wedge (B - B_o).
\end{aligned}$$

Without loss of generality we may assume A is in temporal gauge over the collar $\iota([0,1] \times Y)$. By the Weitzenböck formula we have

$$0 = D_A^2 \Phi = \nabla_A^* \nabla_A \Phi + \widehat{F}_A^+ + \frac{\mathbf{s}}{4}.$$

This gives

$$\begin{aligned}\int_Z |\nabla_A\Phi|^2 &= \int_Z \langle \nabla_A^*\nabla_A\Phi, \Phi\rangle + \int_Y \langle \partial_t\Phi, \Phi\rangle \\ &= \int_Z \left(-\frac{1}{2}|\Phi|^4 - \frac{\mathbf{s}}{4}|\Phi|^2 + \langle i\mu^+\Phi, \Phi\rangle\right) + \int_Y \langle \partial_B\Phi, \Phi\rangle. \qquad \square\end{aligned}$$

Consider now the situation of Section 1.4. If (B1) holds then we can find a closed 2–form μ_1 on X whose restriction to $\mathbb{R}_+\times(\pm Y_j)$ is the pullback of η_j, and whose restriction to $\mathbb{R}_+\times Y'_j$ is the pullback of η'_j. From Lemma 4.3.1 we deduce:

Proposition 4.3.1 *For every constant $C_1<\infty$ there exists a constant $C_2<\infty$ with the following significance. Suppose we are given*

- *$\tau, C_0 < \infty$ and an r–tuple T such that $\tau \le T_j$ for each j,*
- *real numbers $\tau_j^\pm$, $1\le j\le r$ and τ'_j, $1\le j\le r'$ satisfying $0\le T_j-\tau_j^\pm\le\tau$ and $0\le\tau'_j\le\tau$.*

Let Z be the result of deleting from $X^{(T)}$ all the necks $(-\tau_j^-,\tau_j^+)\times Y_j$, $1\le j\le r$ and all the ends $(\tau'_j,\infty)\times Y'_j$, $1\le j\le r'$. Then for any configuration $S=(A,\Phi)$ representing an element of a moduli space $M(X^{(T)};\vec{\alpha}';\mu;\vec{\mathfrak{p}},\vec{\mathfrak{p}}')$ where $\sum_{j=1}^{r'}\vartheta(\alpha'_j)>-C_0$ and $\mathfrak{p}_j,\mathfrak{p}'_j,\mu$ all have L^∞ norm $<C_1$ one has that

$$\begin{aligned}\int_Z (|\nabla_A\Phi|^2 + |\widehat{F}_A + i\mu_1|^2) + 2\sum_{j=1}^{r}\left(\vartheta(S|_{\{-\tau_j^-\}\times Y_j}) - \vartheta(S|_{\{\tau_j^+\}\times Y_j})\right)& \\ + 2\sum_{j=1}^{r'}\left(\vartheta(S|_{\{\tau'_j\}\times Y'_j}) - \vartheta(\alpha'_j)\right) &< C_2(1+\tau)+2C_0.\end{aligned}$$

Thus, if each $\mathfrak{p}_j$, $\mathfrak{p}'_j$ has sufficiently small L^∞ norm then Lemma 4.2.4 and Proposition 4.2.1 provides local energy bounds over necks and ends for such monopoles. (To apply Proposition 4.2.1 one can take $\tau_j^\pm$ to be the point t in a suitable interval where $\pm\vartheta(S|_{\{t\}\times Y_j})$ attains its maximum, and similarly for τ'_j.) Moreover, if these perturbation forms have sufficiently small C^1 norms then we can apply Proposition 4.2.2 over necks and ends. (How small the C^1 norms have to be depends on C_0.)

CHAPTER 5

Local compactness II

This chapter, which is logically independent from Chapter 4, provides the local compactness results needed for the proof of Theorem 1.4.1 assuming (B2).

While the main result of this chapter, Proposition 5.4.1, is essentially concerned with local convergence of monopoles, the arguments will, in contrast to those of Chapter 4, be of a global nature. In particular, function spaces over manifolds with tubular ends will play a central role.

5.1 Hodge theory for the operator $-d^* + d^+$

In this section we will study the kernel (in certain function spaces) of the elliptic operator

$$\mathcal{D} = -d^* + d^+ \colon \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X), \tag{5.1}$$

where X is an oriented Riemannian 4–manifold with tubular ends. The notation $\ker(\mathcal{D})$ will refer to the kernel of $\mathcal{D}$ in the space $\Omega^1(X)$ of all smooth 1–forms, where X will be understood from the context. The results of this section complement those of [14].

We begin with the case of a half-infinite cylinder $X = \mathbb{R}_+ \times Y$, where Y is any closed, oriented, connected Riemannian 3–manifold. Under the isomorphisms (3.4) there is the identification $\mathcal{D} = \frac{\partial}{\partial t} + P$ over $\mathbb{R}_+ \times Y$, where P is the self-adjoint elliptic operator

$$P = \begin{pmatrix} 0 & -d^* \\ -d & *d \end{pmatrix}$$

acting on sections of $\Lambda^0(Y)\oplus\Lambda^1(Y)$ (cf (3.12)). Since P^2 is the Hodge Laplacian,

$$\ker(P)=H^0(Y)\oplus H^1(Y).$$

Let $\{h_\nu\}$ be a maximal orthonormal set of eigenvectors of P, say $Ph_\nu=\lambda_\nu h_\nu$.

Given a smooth 1–form a over $\mathbb{R}_+\times Y$ we can express it as $a=\sum_\nu f_\nu h_\nu$, where $f_\nu\colon \mathbb{R}_+\to\mathbb{R}$. If $\mathcal{D}a=0$ then $f_\nu(t)=c_\nu e^{-\lambda_\nu t}$ for some constant c_ν. If in addition $a\in L^p$ for some $p\geq 1$ then $f_\nu\in L^p$ for all ν, hence $f_\nu\equiv 0$ when $\lambda_\nu\leq 0$. Elliptic inequalities for $\mathcal{D}$ then show that a decays exponentially, or more precisely,

$$|(\nabla^j a)_{(t,y)}|\leq \beta_j e^{-\delta t}$$

for $(t,y)\in\mathbb{R}_+\times Y$ and $j\geq 0$, where β_j is a constant and δ the smallest positive eigenvalue of P.

Now let $\sigma>0$ be a small constant and $a\in\ker(\mathcal{D})\cap L^{p,-\sigma}$. Arguing as above we find that

$$a=b+c\,dt+\pi^*\psi, \tag{5.2}$$

where b is an exponentially decaying form, c a constant, $\pi\colon\mathbb{R}_+\times Y\to Y$, and $\psi\in\Omega^1(Y)$ harmonic.

We now turn to the case when X is an oriented, connected Riemannian 4–manifold with tubular end $\overline{\mathbb{R}}_+\times Y$ (so $X\setminus\mathbb{R}_+\times Y$ is compact). Let $Y_1,\dots,Y_r$ be the connected components of Y and set

$$Y'=\bigcup_{j=1}^{s}Y_j,\qquad Y''=Y\setminus Y',$$

where $0\leq s\leq r$. Let $\sigma>0$ be a small constant and $\kappa\colon X\to\mathbb{R}$ a smooth function such that

$$\kappa=\begin{cases}-\sigma t & \text{on } \mathbb{R}_+\times Y'\\ \sigma t & \text{on } \mathbb{R}_+\times Y'',\end{cases}$$

where t is the $\mathbb{R}_+$ coordinate. Our main goal in this section is to describe $\ker(\mathcal{D})\cap L^{p,\kappa}$.

We claim that all elements $a\in\ker(\mathcal{D})\cap L^{p,\kappa}$ are closed. To see this, note first that the decomposition (5.2) shows that a is pointwise bounded, and da decays exponentially over the ends. Applying the proof of [15, Proposition 1.1.19] to $X_{:T}=X\setminus(T,\infty)\times Y$

we get

$$\int_{X:T} (|d^+a|^2 - |d^-a|^2) = \int_{X:T} da \wedge da = \int_{X:T} d(a \wedge da)$$
$$= \int_{\partial X:T} a \wedge da \to 0 \quad \text{as } T \to \infty.$$

Since $d^+a = 0$, we conclude that $da = 0$.

Fix $\tau \geq 0$ and for any $a \in \Omega^1(X)$ and $j = 1, \ldots, r$ set

$$R_j a = \int_{\{\tau\}\times Y_j} *a. \tag{5.3}$$

Recall that $d^* = -*d*$ on 1–forms, so if $d^*a = 0$ then $R_j a$ is independent of τ. Therefore, if $a \in \ker(\mathcal{D}) \cap L^{p,\kappa}$ then $R_j a = 0$ for $j > s$, hence

$$\sum_{j=1}^{s} R_j a = \int_{\partial X:\tau} *a = \int_{X:\tau} d*a = 0.$$

Set
$$\Xi = \{(z_1, \ldots, z_s) \in \mathbb{R}^s : \sum_j z_j = 0\}.$$

Proposition 5.1.1 *In the situation above the map*

$$\alpha\colon \ker(\mathcal{D}) \cap L^{p,\kappa} \to \ker(H^1(X) \to H^1(Y'')) \oplus \Xi,$$
$$a \mapsto ([a], (R_1 a, \ldots, R_s a))$$

is an isomorphism.

Proof We first prove α is injective. Suppose $\alpha(a) = 0$. Then $a = df$ for some function f on X. From the decomposition (5.2) we see that a decays exponentially over the ends. Hence f is bounded, in which case

$$0 = \int_X f\, d^*a = \int_X |a|^2.$$

This shows α is injective.

Next we prove α is surjective. Suppose $b \in \Omega^1(X)$, $db = 0$, $[b|_{Y''}] = 0$, and $(z_1, \ldots, z_s) \in \Xi$. Let $\psi \in \Omega^1(Y)$ be the harmonic form representing $[b|_Y] \in H^1(Y)$. Then

$$b|_{\mathbb{R}_+\times Y} = \pi^*\psi + df,$$

for some $f\colon \mathbb{R}_+ \times Y \to \mathbb{R}$. Choose a smooth function $\rho\colon X \to \mathbb{R}$ which vanishes in a neighbourhood of $X_{:0}$ and satisfies $\rho \equiv 1$ on $[\tau,\infty)\times Y$. Set $z_j = 0$ for $j > s$ and let z be the function on Y with $z|_{Y_j} \equiv \mathrm{Vol}(Y_j)^{-1} z_j$. Define

$$\widetilde{b} = b + d(\rho(tz - f)).$$

Then over $[\tau,\infty)\times Y$ we have $\widetilde{b} = \pi^*\psi + z\,dt$, so $d^*\widetilde{b} = 0$ in this region, and

$$\int_X d^*\widetilde{b} = -\int_{\partial X_{:\tau}} *\widetilde{b} = -\int_Y z = 0.$$

Let $\bar{\kappa}\colon X \to \mathbb{R}$ be a smooth function which agrees with $|\kappa|$ outside a compact set. By Proposition 2.3.2 we can find a smooth $\xi\colon X \to \mathbb{R}$ such that $d\xi \in L^{p,\bar{\kappa}}_1$ and

$$d^*(\widetilde{b} + d\xi) = 0.$$

Set $a = \widetilde{b} + d\xi$. Then $(d + d^*)a = 0$ and $\alpha(a) = ([b], (z_1, \dots, z_s))$. □

The following proposition is essentially [14, Proposition 3.14] and is included here only for completeness.

Proposition 5.1.2 *If $b_1(Y) = 0$ and $s = 0$ then the operator*

$$\mathcal{D}\colon L^{p,\kappa}_1 \to L^{p,\kappa} \tag{5.4}$$

has index $-b_0(X) + b_1(X) - b^+(X)$.

Proof By Proposition 5.1.1 the dimension of the kernel of (5.4) is $b_1(X)$. From Proposition 2.3.1 (ii) with $\mathbb{S} = 0$ we see that the image of (5.4) is the sum of $d^*L^{p,\kappa}_1$ and $d^+L^{p,\kappa}_1$. The codimensions of these spaces in $L^{p,\kappa}$ are $b_0(X)$ and $b^+(X)$, respectively. □

5.2 The case of a single moduli space

Consider the situation of Section 1.3. Initially we do not assume Condition (A).

Proposition 5.2.1 *Fix $1 < q < \infty$. Let $\sigma > 0$ be a small constant and $\kappa\colon X \to \mathbb{R}$ a smooth function such that $\kappa(t,y) = -\sigma t$ for all $(t,y) \in \mathbb{R}_+ \times Y$. Let A_o be a spin connection over X which is translationary invariant over the ends of X. For $n = 1, 2, \dots$ let $S_n = (A_o + a_n, \Phi_n)$ be a smooth configuration over X which satisfies the monopole equations (3.10) with $\mu = \mu_n$, $\vec{\mathfrak{p}} = \vec{\mathfrak{p}}_n$. Suppose $a_n \in L^{q,\kappa}_1$ for every n,* *and* $\sup_n \|\Phi_n\|_\infty < \infty$. *Then there exist smooth* $u_n\colon X \to \mathrm{U}(1)$ *such that if* k *is any nonnegative integer with*

$$\sup_{j,n} \left(\|\mu_n\|_{C^k} + \|\mathfrak{p}_{n,j}\|_{C^k}\right) < \infty \tag{5.5}$$

then the sequence $u_n(S_n)$ *is bounded in* $L^{p'}_{k+1}$ *over compact subsets of* X *for every* $p' \geq 1$.

Before giving the proof we record the following two elementary lemmas:

Lemma 5.2.1 *Let* E, F, G *be Banach spaces, and* $E \xrightarrow{S} F$ *and* $E \xrightarrow{T} G$ *bounded linear maps. Set*

$$S + T\colon E \to F \oplus G, \quad x \mapsto (Sx, Tx).$$

Suppose S *has finite-dimensional kernel and closed range and that* $S + T$ *is injective. Then* $S + T$ *has closed range, hence there is a constant* $C > 0$ *such that*

$$\|x\| \leq C(\|Sx\| + \|Tx\|)$$

for all $x \in E$.

Proof Exercise. $\square$

Lemma 5.2.2 *Let* X *be a smooth, connected manifold and* $x_0 \in X$. *Denote by* $\mathit{Map}_0(X, \mathrm{U}(1))$ *the set of smooth maps* $u\colon X \to \mathrm{U}(1)$ *such that* $u(x_0) = 1$, *and let* V *denote the set of all closed* 1*–forms* ϕ *on* X *such that* $[\phi] \in H^1(X;\mathbb{Z})$. *Then*

$$\mathit{Map}_0(X, \mathrm{U}(1)) \to V, \quad u \mapsto \frac{1}{2\pi i} u^{-1} du$$

is an isomorphism of Abelian groups.

Proof If $\phi \in V$ define

$$u(x) = \exp\left(2\pi i \int_{x_0}^{x} \phi\right),$$

where $\int_{x_0}^{x} \phi$ denotes the integral of ϕ along any path from x_0 to x. Then we have $(1/(2\pi i))u^{-1} du = \phi$. The details are left to the reader. $\square$

Proof of Proposition 5.2.1 We may assume X is connected and that (5.5) holds at least for $k = 0$. Choose closed 3–forms $\omega_1, \ldots, \omega_{b_1(X)}$ which are supported in the interior of $X_{:0}$ and represent a basis for $H^3_c(X)$. For any $a \in \Omega^1(X)$ define the coordinates of $Ja \in \mathbb{R}^{b_1(X)}$ by

$$(Ja)_k = \int_X a \wedge \omega_k.$$

Then J induces an isomorphism $H^1(X) \to \mathbb{R}^{b_1(X)}$, by Poincaré duality. By Lemma 5.2.2 we can find smooth $v_n\colon X \to \mathrm{U}(1)$ such that $J(a_n - v_n^{-1} dv_n)$ is bounded as $n \to \infty$. We can arrange that $v_n(t, y)$ is independent of $t \geq 0$ for every $y \in Y$. Then there are $\xi_n \in L^{q,\kappa}_2(X; i\mathbb{R})$ such that $b_n = a_n - v_n^{-1} dv_n - d\xi_n$ satisfies

$$d^* b_n = 0; \qquad R_j b_n = 0, \quad j = 1, \dots, r-1$$

where R_j is as in (5.3). If $r \geq 1$ this follows from Proposition 2.3.3, while if $r = 0$ (ie if X is closed) it follows from Proposition 2.3.2. (By Stokes' theorem we have $R_r b_n = 0$ as well, but we don't need this.) Note that ξ_n must be smooth, by elliptic regularity for the Laplacian d^*d. Set $u_n = \exp(\xi_n) v_n$. Then $u_n(A_o + a_n) = A_o + b_n$. By Proposition 5.1.1 and Lemma 5.2.1 there is a $C_1 > 0$ such that

$$\|b\|_{L^{q,\kappa}_1} \leq C_1 \left(\|(d^* + d^+) b\|_{L^{q,\kappa}} + \sum_{j=1}^{r-1} |R_j b| + \|Jb\| \right)$$

for all $b \in L^{q,\kappa}_1$. From inequality (3.6) and the curvature part of the Seiberg–Witten equations we find that $\sup_n \|d^+ b_n\|_\infty < \infty$, hence

$$\|b_n\|_{L^{q,\kappa}_1} \leq C_1 (\|d^+ b_n\|_{L^{q,\kappa}} + \|Jb_n\|) \leq C_2$$

for some constant C_2. We can now complete the proof by bootstrapping over compact subsets of X, using alternately the Dirac and curvature parts of the Seiberg–Witten equation. □

Combining Proposition 5.2.1 (with $k \geq 1$) and Proposition 3.4.1 we obtain, for fixed closed 2–forms η_j on Y_j:

Corollary 5.2.1 *If (A) holds then for every constant $C_0 < \infty$ there exists a constant $C_1 < \infty$ with the following significance. Suppose $\|\mu\|_{C^1}, \|\mathfrak{p}_j\|_{C^1} \leq C_0$ for each j. Then for any $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ with $\alpha_j \in \widetilde{\mathcal{R}}_{Y_j}$, and any $[S] \in M(X; \vec{\alpha}; \mu; \vec{\mathfrak{p}})$ and $t_1, \dots, t_r \in [0, C_0]$ one has*

$$\left| \sum_{j=1}^{r} \lambda_j \vartheta(S|_{\{t_j\} \times Y_j}) \right| \leq C_1.$$

Note that if $\sum_j \lambda_j \vartheta(\alpha_j) \geq -C_0$ then this gives

$$\sum_{j=1}^{r} \lambda_j \left(\vartheta(S|_{\{t_j\}\times Y_j}) - \vartheta(\alpha_j)\right) \leq C_0 + C_1. \tag{5.6}$$

5.3 Condition (C)

Consider the situation in Section 1.4 and suppose γ is simply connected and equipped with an orientation o. Throughout this section (and the next) (co)homology groups will have real coefficients, unless otherwise indicated.

We associate to (γ, o) a "height function", namely the unique integer valued function h on the set of nodes of γ whose minimum value is 0 and which satisfies $h(e') = h(e)+1$ whenever there is an oriented edge from e to e'.

Let Z^k and $Z^{[k}$ denote the union of all subspaces $Z_e \subset X^\#$ where e has height k and $\geq k$, respectively. Set

$$\partial^- Z^k = \bigcup_{h(e)=k} \partial^- Z_e.$$

For each node e of γ choose a subspace $G_e \subset H_1(Z_e)$ such that

$$H_1(Z_e) = G_e \oplus \operatorname{im}\left(H_1(\partial^- Z_e) \to H_1(Z_e)\right).$$

Then the natural map $F_e \to G_e^*$ is an isomorphism, where G_e^* is the dual of the vector space G_e.

Lemma 5.3.1 *The natural map $H^1(X^\#) \to \bigoplus_e G_e^*$ is injective. Therefore, this map is an isomorphism if and only if $\Sigma(X, \gamma, o) = 0$.*

Proof Let N be the maximum value of h and suppose $z \in H^1(X^\#)$ lies in the kernel of the map in the lemma. It is easy to show, by induction on $k = 0, \dots, N$, that $H^1(X^\#) \to H^1(Z^k)$ maps z to zero for each k. We now invoke the Mayer–Vietoris sequence for the pair of subspaces $(Z^{k-1}, Z^{[k})$ of $X^\#$:

$$H^0(Z^{k-1}) \oplus H^0(Z^{[k}) \xrightarrow{a} H^0(\partial^- Z^k) \to H^1(Z^{[k-1}) \xrightarrow{b} H^1(Z^{k-1}) \oplus H^1(Z^{[k}).$$

Using the fact that γ is simply connected it is not hard to see that a is surjective, hence b is injective. Arguing by induction on $k = N, N-1, \dots, 0$ we then find that $H^1(X^\#) \to H^1(Z^{[k})$ maps z to zero for each k. □

We will now formulate a condition on (X,γ) which is stronger than (C) and perhaps simpler to verify. A connected, oriented graph is called a *tree* if it has a unique node (the *root node*) with no incoming edge, and any other node has a unique incoming edge.

Proposition 5.3.1 *Suppose there is an orientation o of γ such that (γ,o) is a tree and*

$$H^1(Z^{[k})\to H^1(Z^k)$$

is surjective for all k. Then Condition (C) holds.

Proof It suffices to verify that $\Sigma(X,\gamma,o)=0$. Set

$$F^{[k}=\ker(H^1(Z^{[k})\to H^1(\partial^-Z^k)),$$
$$F^k=\ker(H^1(Z^k)\to H^1(\partial^-Z^k)).$$

The Mayer–Vietoris sequence for $(Z^k,Z^{[k+1})$ yields an exact sequence

$$0\to F^{[k}\to F^k\oplus H^1(Z^{[k+1})\to H^1(\partial^-Z^{[k+1}).$$

If $H^1(Z^{[k})\to H^1(Z^k)$ is surjective then so is $F^{[k}\to F^k$, hence $\ker(F^{[k}\to F^k)\to F^{[k+1}$ is an isomorphism, in which case

$$\dim\ F^{[k}=\dim\ F^{[k+1}+\dim\ F^k.$$

Therefore
$$\begin{aligned}\Sigma(X,\gamma,o)&=\dim\ H^1(X^\#)-\textstyle\sum_k\dim\ F^k\\&=\dim\ H^1(X^\#)-\dim\ F^{[0}=0.\end{aligned}$$
□

5.4 Neck-stretching II

Consider again the situation in Section 1.4. The following set-up will be used in the next two lemmas. We assume that γ is simply connected and that o is an orientation of γ with $\Sigma(X,\gamma,o)=0$. Let $1<p<\infty$.

An end of X that corresponds to an edge of γ is either *incoming* or *outgoing* depending on the orientation o. (These are the ends $\mathbb{R}_+\times(\pm Y_j)$, but the sign here is unrelated to o.) All other ends (ie $\mathbb{R}_+\times Y'_j$, $1\le j\le r'$) are called *neutral*.

Choose subspaces G_e of $H_1(Z_e)\approx H_1(X_e)$ as in the previous section, and set $g_e=\dim\ G_e$. For each component X_e of X let $\{q_{ek}\}$ be a collection of closed 3–forms on X_e supported in the interior of $Z'_e=(X_e)_{:0}$ which represents a basis for the image of G_e in $H^3_c(X_e)$ under the Poincaré duality isomorphism. For any $a\in\Omega^1(Z'_e)$ define $J_ea\in\mathbb{R}^{g_e}$ by

$$(J_ea)_k=\int_{X_e}a\wedge q_{ek}.$$

For each e let $\mathbb{R}_+ \times Y_{em}$, $m = 1, \dots, h_e$ be the outgoing ends of X_e. For any $a \in \Omega^1(Z'_e)$ define $R_e a \in \mathbb{R}^{h_e}$ by

$$(R_e a)_m = \int_{\{0\}\times Y_{em}} *a.$$

Set $n_e = g_e + h_e$ and

$$L_e a = (J_e a, R_e a) \in \mathbb{R}^{n_e}.$$

For any $a \in \Omega^1(X^{(T)})$ let $La \in V = \bigoplus_e \mathbb{R}^{n_e}$ be the element with components $L_e a$.

For any tubular end $\mathbb{R}_+ \times P$ of X let $t\colon \mathbb{R}_+ \times P \to \mathbb{R}_+$ be the projection. Choose a small $\sigma > 0$ and for each e a smooth function $\kappa_e\colon X_e \to \mathbb{R}$ such that

$$\kappa_e = \begin{cases} \sigma t & \text{on incoming ends,} \\ -\sigma t & \text{on outgoing and neutral ends.} \end{cases}$$

Let $X^{\{T\}} \subset X$ be as in Section 1.4 and let $\kappa = \kappa_T\colon X^{(T)} \to \mathbb{R}$ be a smooth function such that $\kappa_T - \kappa_e$ is constant on $X_e \cap X^{\{T\}}$ for each e. (Such a function exists because γ is simply connected.) This determines κ_T up to an additive constant.

Fix a point $x_e \in X_e$ and define a norm $\|\cdot\|_T$ on V by

$$\|v\|_T = \sum_e \exp(\kappa_T(x_e))\|v_e\|,$$

where $\|\cdot\|$ is the Euclidean norm on R^{n_e} and $\{v_e\}$ the components of v.

Let $\mathcal{D}$ denote the operator $-d^* + d^+$ on $X^{(T)}$.

Lemma 5.4.1 *There is a constant C such that for every r–tuple T with $T_{\min}$ sufficiently large and every $L^{p,\kappa}_1$ 1–form a on $X^{(T)}$ we have*

$$\|a\|_{L^{p,\kappa}_1} \le C(\|\mathcal{D}a\|_{L^{p,\kappa}} + \|La\|_T).$$

Note that adding a constant to κ rescales all norms in the above inequality by the same factor.

Proof Let τ be a function on X which is equal to $2T_j$ on the ends $\mathbb{R}_+ \times (\pm Y_j)$ for each j. Choose smooth functions $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ such that $(f_1(t))^2 + (f_2(1-t))^2 = 1$ for all t, and $f_k(t) = 1$ for $t \le 1/3$, $k = 1, 2$. For each e define $\beta_e\colon X_e \to \mathbb{R}$ by

$$\beta_e = \begin{cases} f_1(t/\tau) & \text{on outgoing ends,} \\ f_2(t/\tau) & \text{on incoming ends,} \\ 1 & \text{elsewhere.} \end{cases}$$

Let $\overline{\beta}_e$ denote the smooth function on $X^{(T)}$ which agrees with β_e on $X_e \cap X^{\{T\}}$ and is zero elsewhere.

In the following, $C, C_1, C_2, \dots$ will be constants that are independent of T. Assume $T_{\min} \ge 1$.

Note that $|\nabla\beta_e| \le C_1 T_{\min}^{-1}$ everywhere, and similarly for $\overline{\beta}_e$. Therefore

$$\|\beta_e a\|_{L^{p,\kappa_e}_1} \le C_2 \|a\|_{L^{p,\kappa_e}_1}$$

for 1–forms a on X_e.

Let $\mathcal{D}_e$ denote the operator $-d^* + d^+$ on X_e. By Proposition 5.1.1 the Fredholm operator

$$\mathcal{D}_e \oplus L_e\colon L^{p,\kappa_e}_1 \to L^{p,\kappa_j} \oplus \mathbb{R}^{n_e}$$

is injective, hence it has a bounded left inverse P_e,

$$P_e(\mathcal{D}_e \oplus L_e) = \mathrm{Id}.$$

If a is a 1–form on $X^{(T)}$ and $v \in V$ set

$$\overline{\beta}_e(a, v) = (\overline{\beta}_e a, v_e).$$

Here we regard $\overline{\beta}_e a$ as a 1–form on X_e. Define

$$P = \sum_e \beta_e P_e \overline{\beta}_e\colon L^{p,\kappa} \oplus V \to L^{p,\kappa}_1.$$

If we use the norm $\|\cdot\|_T$ on V then $\|P\| \le C_3$. Now

$$\begin{aligned}
P(\mathcal{D} \oplus L)a &= \sum_e \beta_e P_e(\overline{\beta}_e \mathcal{D} a, L_e a) \\
&= \sum_e \beta_e P_e(\mathcal{D}_e \overline{\beta}_e a + [\overline{\beta}_e, \mathcal{D}]a, L_e \overline{\beta}_e a) \\
&= \sum_e (\beta_e \overline{\beta}_e a + \beta_e P_e([\overline{\beta}_e, \mathcal{D}]a, 0)) \\
&= a + Ea,
\end{aligned}$$

where
$$\|Ea\|_{L^{p,\kappa}_1} \le C_4 T_{\min}^{-1} \|a\|_{L^{p,\kappa}}.$$

Therefore,
$$\|P(\mathcal{D} \oplus L) - I\| \le C_4 T_{\min}^{-1},$$

so if $T_{\min} > C_4$ then $z = P(\mathcal{D} \oplus L)$ will be invertible, with

$$\|z^{-1}\| \le (1 - \|z - I\|)^{-1}.$$

In that case we can define a left inverse of $\mathcal{D}\oplus L$ by

$$Q=(P(\mathcal{D}\oplus L))^{-1}P.$$

If $T_{\min}\ge 2C_4$ say, then $\|Q\|\le 2C_3$, whence for any $a\in L^{p,\kappa}_1$ we have

$$\|a\|_{L^{p,\kappa}_1}=\|Q(\mathcal{D}a,La)\|_{L^{p,\kappa}_1}\le C(\|\mathcal{D}a\|_{L^{p,\kappa}}+\|La\|_T). \qquad \square$$

Lemma 5.4.2 *Let e be a node of γ and for $n=1,2,\dots$ let $T(n)$ be an r–tuple and a_n an L^{p,κ_n}_1 1–form on $X^{(T(n))}$, where $\kappa_n=\kappa_{T(n)}$. Suppose*

(i) $\Sigma(X,\gamma,o)=0$,

(ii) $T_{\min}(n)\to\infty$ as $n\to\infty$,

(iii) *there is a constant $C'<\infty$ such that*

$$\kappa_n(x_{e'})\le\kappa_n(x_e)+C'$$

for all nodes e' of γ and all n,

(iv) $\sup_n\|d^+a_n\|_\infty<\infty$.

Then there are smooth $u_n\colon X^{(T(n))}\to \mathrm{U}(1)$ such that the sequence $b_n=a_n-u_n^{-1}du_n$ is bounded in L^p_1 over compact subsets of X_e, and $b_n\in L^{p,\kappa_n}_1$ and $d^*b_n=0$ for every n.

Note that (iii) implies that e must be a source of (γ,o).

Proof Without loss of generality we may assume that $\kappa_n(x_e)=1$ for all n, in which case

$$\sup_n\|1\|_{L^{p,\kappa_n}}<\infty.$$

By Lemma 5.2.2 and Lemma 5.3.1 we can find smooth $v_n\colon X^{(T(n))}\to\mathrm{U}(1)$ such that

$$\sup_n\|J_{e'}(a_n-v_n^{-1}dv_n)\|<\infty$$

for every node e', where $\|\cdot\|$ is the Euclidean norm. (Compare the proof of Proposition 5.2.1.) Moreover, we can take v_n translationary invariant over each end of $X^{(T(n))}$. Proposition 2.3.3 then provides smooth $\xi_n\in L^{p,\kappa_n}_2(X;i\mathbb{R})$ such that

$$b_n=a_n-v_n^{-1}dv_n-d\xi_n\in L^{p,\kappa_n}_1$$

satisfies
$$d^*b_n=0,\qquad \int_{\{0\}\times Y'_j}*b_n=0$$

for $j=1,\dots,r'-1$. Stokes' theorem shows that the integral vanishes for $j=r'$ as well, and since γ is simply connected we obtain, for $j=1,\dots,r$,

$$\int_{\{t\}\times Y_j} *b_n = 0 \qquad \text{for } |t|\le T_j(n).$$

In particular, $R_{e'}b_n=0$ for all nodes e' of γ.

Set $u_n=v_n\exp(\xi_n)$, so that $b_n=a_n-u_n^{-1}du_n$. By Lemma 5.4.1 we have

$$\begin{aligned}\|b_n\|_{L_1^{p,\kappa_n}} &\le C\Big(\|d^+b_n\|_{L^{p,\kappa_n}}+\sum_{e'}\exp(\kappa_n(x_{e'}))\|J_{e'}(b_n)\|\Big)\\ &\le C\Big(\|d^+a_n\|_{L^{p,\kappa_n}}+\exp(1+C')\sum_{e'}\|J_{e'}(a_n-v_n^{-1}dv_n)\|\Big),\end{aligned}$$

which is bounded as $n\to\infty$. □

Proposition 5.4.1 *Suppose γ is simply connected and that Condition (C) holds for (X,γ). For $n=1,2,\dots$ let $[S_n]\in M(X^{(T(n))};\vec{\alpha}'_n;\mu_n;\vec{\mathfrak{p}}_n;\vec{\mathfrak{p}}'_n)$, where $T_{\min}(n)\to\infty$. Then there exist smooth maps $w_n\colon X\to\mathrm{U}(1)$ such that if k is any positive integer with*

$$\sup_{j,j',n}\,(\|\mu_n\|_{C^k}+\|\mathfrak{p}_{j,n}\|_{C^k}+\|\mathfrak{p}'_{j',n}\|_{C^k})<\infty$$

then the sequence $w_n(S_n)$ is bounded in $L^{p'}_{k+1}$ over compact subsets of X for every $p'\ge 1$.

Proof Consider the set-up in the beginning of this section where now $p>4$ is the exponent used in defining configuration spaces, and o is an orientation of γ for which (C) is fulfilled. By passing to a subsequence we can arrange that $\kappa_n(x_e)-\kappa_n(x_{e'})$ converges to a point $\ell(e,e')\in[-\infty,\infty]$ for each pair of nodes e,e' of γ. Define an equivalence relation $\sim$ on the set $\mathcal{N}$ of nodes of γ by declaring that $e\sim e'$ if and only if $\ell(e,e')$ is finite. Then we have a linear ordering on $\mathcal{N}/\sim$ such that $[e]\ge[e']$ if and only if $\ell(e,e')>-\infty$. Here $[e]$ denotes the equivalence class of e.

Choose e such that $[e]$ is the maximum with respect to this linear ordering. Let $S_n=(A_o+a_n,\Phi_n)$. Then all the hypotheses of Lemma 5.4.2 are satisfied. If u_n is as in that lemma then, as in the proof of Proposition 5.2.1, $u_n(S_n)=(A_o+b_n,u_n\Phi_n)$ will be bounded in $L^{p'}_{k+1}$ over compact subsets of X_e for every $p'\ge 1$.

For any r–tuple T let $W^{(T)}$ be the result of gluing ends of $X\setminus X_e$ according to the graph $\gamma\setminus e$ and (the relevant part of) the vector T. To simplify notation let us assume that the outgoing ends of X_e are $\mathbb{R}_+\times(-Y_j)$, $j=1,\dots,r_1$. Then $\mathbb{R}_+\times Y_j$ is an end of $W^{T(n)}$ for $j=1,\dots,r_1$. Let b'_n be the 1–form on $W^{T(n)}$ which away from the ends $\mathbb{R}_+\times Y_j$, $1\le j\le r_1$ agrees with b_n, and on each of these ends is defined by cutting off b_n:

$$b'_n(t,y)=\begin{cases} f_1(t-2T_j(n)+1)\cdot b_n(t,y), & 0\le t\le 2T_j(n),\\ 0, & t\ge 2T_j(n).\end{cases}$$

Here f_1 is as in the proof of Lemma 5.4.1. Then $\sup_n \|d^+b'_n\|_\infty<\infty$. After choosing an orientation of $\gamma\setminus e$ for which (C) holds we can apply Lemma 5.4.2 to each component of $\gamma\setminus e$, with b'_n in place of a_n. Repeating this process proves the proposition. □

Corollary 5.4.1 *If (B2) holds then for every constant $C_0<\infty$ there exists a constant $C_1<\infty$ such that for any element $[S]$ of a moduli space $M(X^{(T)};\vec\alpha';\mu;\vec{\mathfrak{p}};\vec{\mathfrak{p}}')$ where $T_{\min}>C_1$ and $\mu,\mathfrak{p}_j,\mathfrak{p}'_j$ all have C^1–norm $<C_0$ one has*

$$\left|\sum_{j=1}^{r}\lambda_j\left(\vartheta(S|_{\{-T_j\}\times Y_j})-\vartheta(S|_{\{T_j\}\times Y_j})\right)+\sum_{j=1}^{r'}\lambda'_j\vartheta(S|_{\{0\}\times Y'_j})\right|<C_1. \qquad \square$$

The next proposition, which is essentially a corollary of Proposition 5.4.1, exploits the fact that Condition (C) is preserved under certain natural extensions of (X,γ).

Proposition 5.4.2 *Suppose γ is simply connected and (C) holds for (X,γ). Then for every constant $C_0<\infty$ there is a constant $C_1<\infty$ such that if $S=(A,\Phi)$ represents an element of a moduli space $M(X^{(T)};\vec\alpha';\mu;\vec{\mathfrak{p}};\vec{\mathfrak{p}}')$ where $T_{\min}>C_1$ and $\mu,\mathfrak{p}_j,\mathfrak{p}'_j$ all have L^∞ norm $<C_0$ then*

$$\begin{aligned}\|\nabla\vartheta_S\|_{L^2((t-1,t+1)\times Y_j)}&<C_1 \quad \text{for } |t|\le T_j-1,\\ \|\nabla\vartheta_S\|_{L^2((t-1,t+1)\times Y'_j)}&<C_1 \quad \text{for } t\ge 1.\end{aligned}$$

Proof Given an edge v of γ corresponding to a pair of ends $\mathbb{R}_+\times(\pm Y_j)$ of X, we can form a new pair $(X_{(j)},\gamma_{(j)})$ where $X_{(j)}=X\coprod(\mathbb{R}\times Y_j)$, and $\gamma_{(j)}$ is obtained from γ by splitting v into two edges with a common endpoint representing the component $\mathbb{R}\times Y_j$ of $X_{(j)}$.

Similarly, if e is a node of γ and $\mathbb{R}_+\times Y'_j$ an end of X_e then we can form a new pair $(X^{(j)},\gamma^{(j)})$ where $X^{(j)}=X\coprod(\mathbb{R}\times Y'_j)$, and $\gamma^{(j)}$ is obtained from γ by adding one node e'_j representing the component $\mathbb{R}_+\times Y'_j$ of $X^{(j)}$ and one edge joining e and e'_j.

One easily shows, by induction on the number of nodes of γ, that if (C) holds for (X,γ) then (C) also holds for each of the new pairs $(X_{(j)},\gamma_{(j)})$ and $(X^{(j)},\gamma^{(j)})$. Given this observation, the proposition is a simple consequence of Proposition 5.4.1. □

CHAPTER 6

Exponential decay

In this chapter we will prove exponential decay results for genuine monopoles over half-cylinders $\mathbb{R}_+ \times Y$ and long bands $[-T, T] \times Y$. The overall scheme of proof will be the same as that for instantons in [14], and Section 6.1 and Section 6.3 follow [14] quite closely. On the other hand, the proof of the main result of Section 6.2, Proposition 6.2.1, is special to monopoles.

Throughout this chapter Y will be a closed, connected Riemannian spin^c 3–manifold, and $\eta \in \Omega^2(Y)$ closed. We will study exponential decay towards a nondegenerate critical point α of $\vartheta = \vartheta_\eta$. We make no nondegeneracy assumptions on any other (gauge equivalence classes of) critical points, and we do not assume that (O1) holds, except implicitly in Proposition 6.4.1. All monopoles will be genuine (ie $\mathfrak{p} = 0$).

Earlier treatments of exponential decay can be found in Nicolaescu [41] (in the case $\eta = 0$) and Kronheimer–Mrowka [31] (in the context of "blown-up" configurations).

6.1 A differential inequality

We begin by presenting an argument from [14] in a more abstract setting, so that it applies equally well to the Chern–Simons and the Chern–Simons–Dirac functionals.

Let E be a real Banach space with norm $\|\cdot\|$ and E' a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $E \to E'$ be an injective, bounded operator with dense image. We will identify E as a vector space with its image in E'. Set $|x| = \langle x, x \rangle^{1/2}$ for $x \in E'$.

Let $U \subset E$ be an open set containing 0 and

$$f \colon U \to \mathbb{R}, \qquad g \colon U \to E'$$

smooth maps satisfying $f(0)=0$, $g(0)=0$ and

$$Df(x)y = \langle g(x), y\rangle$$

for all $x \in U$, $y \in E$. Here $Df(x)\colon E \to \mathbb{R}$ is the derivative of f at x. Suppose $H = Dg(0)\colon E \to E'$ is an isomorphism (of topological vector spaces). Note that H can be thought of as a symmetric operator in E'. Suppose E contains a countable set $\{e_j\}$ of eigenvectors for H which forms an orthonormal basis for E'. Suppose σ, λ are real numbers satisfying $0 \le \lambda < \sigma$ and such that H has no positive eigenvalue less than σ.

Lemma 6.1.1 *In the above situation there is a constant $C > 0$ such that for every $x \in U$ with $\|x\| \le C^{-1}$ one has*

$$\begin{aligned} 2\sigma f(x) &\le |g(x)|^2 + C|g(x)|^3, \\ 2\lambda f(x) &\le |g(x)|^2. \end{aligned}$$

Proof It clearly suffices to establish the first inequality for some C. By Taylor's formula (Dieudonné [10, 8.14.3]) there is a $C_1 > 0$ such that for all $x \in U$ with $\|x\| \le C_1^{-1}$ one has

$$\begin{aligned} |f(x) - \frac{1}{2}\langle Hx, x\rangle| &\le C_1\|x\|^3, \\ |g(x) - Hx| &\le C_1\|x\|^2. \end{aligned}$$

Let $He_j = \lambda_j e_j$, and set $x_j = \langle x, e_j\rangle e_j$. Then

$$\sigma\langle Hx, x\rangle = \sigma\sum_j \lambda_j|x_j|^2 \le \sum_j \lambda_j^2|x_j|^2 = |Hx|^2.$$

By assumption, there is a $C_2 > 0$ such that

$$\|x\| \le C_2|Hx|$$

for all $x \in E$. Putting the above inequalities together we get, for $r = \|x\| \le C_1^{-1}$,

$$\begin{aligned} |Hx| &\le |g(x)| + C_1 r^2 \\ &\le |g(x)| + rC_1C_2|Hx|. \end{aligned}$$

If $r < (C_1C_2)^{-1}$ this gives

$$|Hx| \le (1 - rC_1C_2)^{-1}|g(x)|,$$

hence

$$\begin{aligned}2\sigma f(x) &\le \sigma\langle Hx,x\rangle + 2\sigma C_1 r^3\\ &\le |Hx|^2 + 2\sigma r C_1 C_2^2 |Hx|^2\\ &\le \frac{1+2\sigma r C_1 C_2^2}{(1-rC_1C_2)^2}|g(x)|^2\\ &\le (1+C_3 r)|g(x)|^2\\ &\le |g(x)|^2 + C_4|g(x)|^3\end{aligned}$$

for some constants C_3, C_4. □

We will now apply this to the Chern–Simons–Dirac functional. Let $\alpha = (B_0, \Psi_0)$ be a nondegenerate critical point of ϑ. Set $K_\alpha = \ker(\mathcal{I}_\alpha^*) \subset \Gamma(i\Lambda^1 \oplus \mathbb{S})$ and $\widetilde{H}_\alpha = H_\alpha|_{K_\alpha}\colon K_\alpha \to K_\alpha$. Note that any eigenvalue of $\widetilde{H}_\alpha$ is also an eigenvalue of the self-adjoint elliptic operator

$$\begin{pmatrix} 0 & \mathcal{I}_\alpha^* \\ \mathcal{I}_\alpha & H_\alpha \end{pmatrix}$$

over Y acting on sections of $i\Lambda^0 \oplus i\Lambda^1 \oplus \mathbb{S}$. Let $\lambda^\pm$ be positive real numbers such that $\widetilde{H}_\alpha$ has no eigenvalue in the interval $[-\lambda^-, \lambda^+]$.

In the following lemma, Sobolev norms of sections of the spinor bundle $\mathbb{S}_Y$ over Y will be taken with respect to B_0 and some fixed connection in the tangent bundle TY. This means that the same constant ϵ will work if α is replaced with some monopole gauge equivalent to α.

Lemma 6.1.2 *In the above situation there exists an $\epsilon > 0$ such that if S is any smooth monopole over the band $(-1,1)\times Y$ satisfying $\|S_0 - \alpha\|_{L^2_1(Y)} \le \epsilon$ then*

$$\pm 2\lambda^\pm(\vartheta(S_0) - \vartheta(\alpha)) \le -\partial_t|_0 \vartheta(S_t).$$

Proof Choose a smooth $u\colon (-1,1)\times Y \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge. Then

$$\partial_t\vartheta(S_t) = \partial_t\vartheta(u_t(S_t)) = -\|\nabla\vartheta(u_t(S_t))\|_2^2 = -\|\nabla\vartheta(S_t)\|_2^2.$$

If $\epsilon > 0$ is sufficiently small then by the local slice theorem we can find a smooth $v\colon Y \to \mathrm{U}(1)$ which is L^2_2 close to 1 and such that $\mathcal{I}_\alpha^*(v(S_0) - \alpha) = 0$. We now apply Lemma 6.1.1 with E the kernel of $\mathcal{I}_\alpha^*$ in L^2_1, E' the kernel of $\mathcal{I}_\alpha^*$ in L^2 and $f(x) = \pm(\vartheta(\alpha + x) - \vartheta(\alpha))$. The assumption that α be nondegenerate means that $H = \widetilde{H}_\alpha\colon E \to E'$ is an isomorphism, so the lemma follows. □

6.2 Estimates over $[0, T] \times Y$

Let α be a nondegenerate critical point of ϑ and $\underline{\alpha} = (B, \Psi)$ the monopole over $\mathbb{R} \times Y$ that α defines. Throughout this section, the same convention for Sobolev norms of sections of $\mathbb{S}_Y$ will apply as in Lemma 6.1.2. For Sobolev norms of sections of the spinor bundles over (open subsets of) $\mathbb{R} \times Y$ we will use the connection B.

Throughout this section $S = (A, \Phi)$ will be a monopole over a band $\mathbf{B} = [0, T] \times Y$ where $T \geq 1$. Set $s = (a, \phi) = S - \underline{\alpha}$ and

$$\begin{aligned} \delta &= \|s\|_{L^2_2(\mathbf{B})}, \\ \nu^2 &= \|\nabla\vartheta_S\|^2_{L^2(\mathbf{B})} = \vartheta(S_0) - \vartheta(S_T). \end{aligned} \tag{6.1}$$

The main result of this section is Proposition 6.2.1, which asserts in particular that if δ is sufficiently small then S is gauge equivalent to a configuration $\widetilde{S}$ which is in Coulomb gauge with respect to $\underline{\alpha}$ and satisfies $\|\widetilde{S} - \underline{\alpha}\|_{L^2_1(\mathbf{B})} \leq \text{const} \cdot \nu$.

We will assume $\delta \leq 1$. Let a' denote the contraction of a with the vector field $\partial_1 = \frac{\partial}{\partial t}$. Quantities referred to as constants or denoted "const" may depend on $Y, \eta, [\alpha], T$ but not on S. Note that

$$\nu \leq \text{const} \cdot (\|s\|_{1,2} + \|s\|^2_{1,2}) \leq \text{const}, \tag{6.2}$$

the last inequality because $\delta \leq 1$.

For real numbers t set

$$i_t \colon Y \to \mathbb{R} \times Y, \quad y \mapsto (t, y).$$

If ω is any differential form over $\mathbf{B}$ set $\omega_t = i_t^*\omega$, $0 \leq t \leq T$. Similar notation will be used for connections and spinors over $\mathbf{B}$.

Lemma 6.2.1 *There is a constant $C_0 > 0$ such that*

$$\|\partial_1\phi\|_2 \leq C_0(\nu + \|a'\|_3).$$

Proof We have

$$\partial_1\phi = \partial_1\Phi = \nabla^A_1\Phi - a'\Phi,$$

where ∇^A_1 is the covariant derivative with respect to A in the direction of the vector field $\partial_1 = \frac{\partial}{\partial t}$. Now $|\nabla^A_1\Phi|$ depends only on the gauge equivalence class of $S = (A, \Phi)$, and if A is in temporal gauge (ie if $a' = 0$) then $(\nabla^A_1\Phi)_t = \partial_{A_t}\Phi_t$. The lemma now follows because $\delta \leq 1$. □

Lemma 6.2.2 *There is a constant* $C_1>0$ *such that if* δ *is sufficiently small then the following hold:*

(i) $\|\phi\|_{1,2}\le C_1(\|\mathcal{I}^*_{\underline{\alpha}}s\|_2+\|a'\|_{1,2}+\nu)$.

(ii) *There is a smooth* $\check f\colon \mathbf{B}\to i\mathbb{R}$ *such that* $\check s=(\check a,\check\phi)=\exp(\check f)(S)-\underline{\alpha}$ *satisfies*

$$\|\check s_t\|_{1,2}\le C_1\|\nabla\vartheta_{S_t}\|_2,\quad 0\le t\le T.$$

(iii) $\|da\|_2\le C_1\nu$.

In (i) and (iii) all norms are taken over $\mathbf{B}$.

Proof The proof will use an elliptic inequality over Y, the local slice theorem for Y, and the gradient flow description of the Seiberg–Witten equations over $\mathbb{R}\times Y$.

(i) Since α is nondegenerate we have

$$\|z\|_{1,2}\le\text{const}\cdot\|(\mathcal{I}^*_\alpha+H_\alpha)z\|_2$$

for L^2_1 sections z of $(i\Lambda\oplus\mathbb{S})_Y$. Recall that

$$\nabla\vartheta_{\alpha+z}=H_\alpha z+z\otimes z$$

where $z\otimes z$ represents a pointwise quadratic function of z. Furthermore, $\|z\otimes z\|_2\le\text{const}\cdot\|z\|^2_{1,2}$. If $\|z\|_{1,2}$ is sufficiently small then we can rearrange to get

$$\|z\|_{1,2}\le\text{const}\cdot(\|\mathcal{I}^*_\alpha z\|_2+\|\nabla\vartheta_{\alpha+z}\|_2). \tag{6.3}$$

By the Sobolev embedding theorem we have

$$\|s_t\|_{L^2_1(Y)}\le\text{const}\cdot\|s\|_{L^2_2(\mathbf{B})},\quad t\in[0,T],$$

for some constant independent of t, so we can apply inequality (6.3) with $z=s_t$ when δ is sufficiently small. Because

$$(\mathcal{I}^*_{\underline{\alpha}}s-\partial_1a')_t=\mathcal{I}^*_\alpha s_t$$

we then obtain

$$\int_0^T\|s_t\|^2_{L^2_1(Y)}\,dt\le\text{const}\cdot(\|\mathcal{I}^*_{\underline{\alpha}}s\|^2_{L^2(\mathbf{B})}+\|\partial_1a'\|^2_{L^2(\mathbf{B})}+\nu^2).$$

This together with Lemma 6.2.1 establishes (i).

(ii) Choose a base-point $y_0\in Y$. By the local slice theorem there is a constant C such that if δ is sufficiently small then for each $t\in[0,T]$ there is a unique smooth $\check f_t\colon Y\to i\mathbb{R}$ such that

- $\|\check{f}_t\|_{2,2} \le C\delta$,
- $\check{f}_t(y_0) = 0$ if α is reducible,
- $\check{s}_t = \exp(\check{f}_t)(S_t) - \alpha$ satisfies $\mathcal{I}_\alpha^* \check{s}_t = 0$.

It is not hard to see that the function $\check{f}\colon \mathbf{B} \to i\mathbb{R}$ given by $\check{f}(t,y) = \check{f}_t(y)$ is smooth. Moreover, $\|\check{s}_t\|_{1,2} \le \mathrm{const} \cdot \|s_t\|_{1,2}$. Part (ii) then follows by taking $z = \check{s}_t$ in (6.3).

(iii) Choose a smooth $u\colon \mathbf{B} \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge, and set $(\underline{a}, \underline{\phi}) = u(S) - \underline{\alpha}$. Then

$$da = d\underline{a} = dt \wedge \partial_1 \underline{a} + d_Y \underline{a} = -dt \wedge \nabla_1 \vartheta_{S_t} + d_Y \check{a}_t,$$

where $\nabla_1 \vartheta$ is the first component of $\nabla\vartheta$. This yields the desired estimate on da. □

Lemma 6.2.3 *Let $\{v_1, \dots, v_{b_1(Y)}\}$ be a family of closed 2–forms on Y which represents a basis for $H^2(Y;\mathbb{R})$. Then there is a constant C such that*

$$\|b\|_{L^2_1(\mathbf{B})} \le C\left(\|(d^* + d)b\|_{L^2(\mathbf{B})} + \|(*b)|_{\partial\mathbf{B}}\|_{L^2_{1/2}(\partial\mathbf{B})} + \sum_j \left| \int_{\mathbf{B}} dt \wedge \pi^* v_j \wedge b \right| \right) \tag{6.4}$$

for all L^2_1 1–forms b on $\mathbf{B}$, where $\pi\colon \mathbf{B} \to Y$ is the projection.

Proof Let K denote the kernel of the operator

$$\Omega^1_{\mathbf{B}} \to \Omega^0_{\mathbf{B}} \oplus \Omega^2_{\mathbf{B}} \oplus \Omega^0_{\partial\mathbf{B}}, \quad b \mapsto (d^*b, db, *b|_{\partial\mathbf{B}}).$$

Then we have an isomorphism

$$\rho\colon K \xrightarrow{\approx} H^1(Y;\mathbb{R}), \quad b \mapsto [b_0].$$

For on the one hand, an application of Stokes' theorem shows that ρ is injective. On the other hand, any $c \in H^1(Y;\mathbb{R})$ can be represented by an harmonic 1–form ω, and $\pi^*\omega$ lies in K, hence ρ is surjective.

It follows that every element of K is of the form $\pi^*(\omega)$. Now apply Proposition 4.1.1 and Lemma 5.2.1. □

Lemma 6.2.4 *There is a smooth map $\hat{f}\colon \mathbf{B} \to i\mathbb{R}$, unique up to an additive constant, such that $\hat{a} = a - d\hat{f}$ satisfies*

$$d^*\hat{a} = 0, \quad (*\hat{a})|_{\partial\mathbf{B}} = 0.$$

Given any such $\widehat{f}$*, if we set* $\widehat{s}=(\widehat{a},\widehat{\phi})=\exp(\widehat{f})(S)-\underline{\alpha}$ *then*

$$\|\widehat{a}\|_{L^2_1(\mathbf{B})}\le C_2\nu,\quad \|\widehat{s}\|_{L^2_2(\mathbf{B})}\le C_2\delta$$

for some constant $C_2>0$.

Proof The first sentence of the lemma is just the solution to the Neumann problem. If we fix $x_0\in\mathbf{B}$ then there is a unique $\widehat{f}$ as in the lemma such that $\widehat{f}(x_0)=0$, and we have $\|\widehat{f}\|_{3,2}\le\text{const}\cdot\|a\|_{2,2}$. Writing

$$\widehat{\phi}=\exp(\widehat{f})\Phi-\Psi=(\exp(\widehat{f})-1)\Phi+\phi$$

and recalling that, for functions on $\mathbf{B}$, multiplication is a continuous map $L^2_3\times L^2_k\to L^2_k$ for $0\le k\le 3$, we get

$$\begin{aligned}\|\widehat{\phi}\|_{2,2}&\le C\|\exp(\widehat{f})-1\|_{3,2}\|\Phi\|_{2,2}+\|\phi\|_{2,2}\\&\le C'\|\widehat{f}\|_{3,2}\exp(C''\|\widehat{f}\|_{3,2})+\|\phi\|_{2,2}\\&\le C'''\|s\|_{2,2}\end{aligned}$$

for some constants $C,\dots,C'''$, since we assume $\delta\le 1$. There is clearly a similar L^2_2 bound on $\widehat{a}$, so this establishes the L^2_2 bound on $\widehat{s}$.

We now turn to the L^2_1 bound on $\widehat{a}$. Let $\check{a}$ be as in Lemma 6.2.2. Since $\widehat{a}-\check{a}$ is exact we have

$$\left|\int_Y v\wedge\widehat{a}_t\right|=\left|\int_Y v\wedge\check{a}_t\right|\le\text{const}\cdot\|v\|_2\|\check{a}_t\|_2$$

for any closed $v\in\Omega^2_Y$. Now take $b=\widehat{a}$ in Lemma 6.2.3 and use Lemma 6.2.2, remembering that $d\widehat{a}=da$. □

Definition 6.2.1 For any smooth $h\colon Y\to i\mathbb{R}$ define $\underline{h}$, $P(h)\colon\mathbf{B}\to i\mathbb{R}$ by $\underline{h}(t,y)=h(y)$ and

$$P(h)=\Delta\underline{h}+i\langle i\Psi,\exp(\underline{h})\Phi\rangle,$$

where $\Delta=d^*d$ is the Laplacian over $\mathbb{R}\times Y$. Let $P_t(h)$ be the restriction of $P(h)$ to $\{t\}\times Y$.

Note that
$$\mathcal{I}^*_{\underline{\alpha}}(\exp(\underline{h})(S)-\underline{\alpha})=-d^*a+P(h).$$

Lemma 6.2.5 *If* α *is irreducible then the following hold:*

(i) *There is a* $C_3>0$ *such that if* δ *is sufficiently small then there exists a unique smooth* $h\colon Y\to i\mathbb{R}$ *satisfying* $\|h\|_{3,2}\le C_3\delta$ *and* $P_0(h)=0$.

(ii) *If $h\colon Y \to i\mathbb{R}$ is any smooth function satisfying $P_0(h) = 0$ then*

$$\|P(h)\|_{L^2(\mathbf{B})} \le \mathrm{const}\cdot(\nu + \|a'\|_{L^3(\mathbf{B})}).$$

Proof (i) We will apply Proposition B.0.2 (ie the inverse function theorem) to the smooth map

$$P_0\colon L^2_3 \to L^2_1, \quad h \mapsto \Delta_Y h + i\langle i\Psi_0, \exp(h)\Phi_0\rangle.$$

The first two derivatives of this map are

$$DP_0(h)k = \Delta_Y k + k\langle \Psi_0, \exp(h)\Phi_0\rangle,$$

$$D^2P_0(h)(k,\ell) = ik\ell\langle i\Psi_0, \exp(h)\Phi_0\rangle.$$

The assumption $\delta \le 1$ gives

$$\|D^2P_0(h)\| \le \mathrm{const}\cdot(1 + \|\nabla h\|_3).$$

Set $L = DP_0(0)$. Then

$$(L - \Delta_Y - |\Psi_0|^2)k = k\langle \Psi_0, \phi_0\rangle,$$

hence
$$\|L - \Delta_Y - |\Psi_0|^2\| \le \mathrm{const}\cdot\delta.$$

Thus if δ is sufficiently small then L is invertible and

$$\|L^{-1}\| \le \|(\Delta_Y + |\Psi_0|^2)^{-1}\| + 1.$$

Furthermore, we have $P_0(0) = i\langle i\Psi_0, \phi_0\rangle$, so

$$\|P_0(0)\|_{1,2} \le \mathrm{const}\cdot\delta.$$

By Proposition B.0.2 (i) there exists a constant $C > 0$ such that if δ is sufficiently small then there is a unique $h \in L^2_3$ such that $\|h\|_{3,2} \le C$ and $P_0(h) = 0$ (which implies that h is smooth). Proposition B.0.2 (ii) then yields

$$\|h\|_{3,2} \le \mathrm{const}\cdot\|P_0(0)\|_{1,2} \le \mathrm{const}\cdot\delta.$$

(ii) Setting $Q = P(h)$ we have, for $0 \le t \le T$,

$$\int_Y |Q(t,y)|^2\,dy = \int_Y \left|\int_0^t \partial_1 Q(s,y)\,ds\right|^2 dy \le \mathrm{const}\cdot\int_{\mathbf{B}} |\partial_1 Q|^2.$$

Now, $\partial_1 Q = i\langle i\Psi, \exp(\underline{h})\partial_1\Phi\rangle$, hence

$$\|\partial_1 Q\|_2 \le \mathrm{const}\cdot\|\partial_1\Phi\|_2 \le \mathrm{const}\cdot(\nu + \|a'\|_3)$$

by Lemma 6.2.1. □

Proposition 6.2.1 *There is a constant* C_4 *such that if* δ *is sufficiently small then there exists a smooth* $\widetilde{f}\colon \mathbf{B}\to i\mathbb{R}$ *such that* $\widetilde{s}=(\widetilde{a},\widetilde{\phi})=\exp(\widetilde{f})(S)-\underline{\alpha}$ *satisfies*

$$\mathcal{I}^*_{\underline{\alpha}}\widetilde{s}=0,\quad (*\widetilde{a})|_{\partial\mathbf{B}}=0,\quad \|\widetilde{s}\|_{L^2_1(\mathbf{B})}\le C_4\nu,\quad \|\widetilde{s}\|_{L^2_2(\mathbf{B})}\le C_4\delta,$$

where δ,ν *are as in* (6.1).

This is analogous to Uhlenbeck's theorem [50, Theorem 1.3] (with $p=2$), except that we assume a bound on δ rather than on ν.

Proof To simplify notation we will write $\mathcal{I}=\mathcal{I}_{\underline{\alpha}}$ in this proof.

Case 1: α reducible In that case the operator $\mathcal{I}^*$ is given by $\mathcal{I}^*(b,\psi)=-d^*b$. Let $\widetilde{f}$ be the $\widehat{f}$ provided by Lemma 6.2.4. Then apply Lemma 6.2.2 (ii), taking the S of that lemma to be the present $\exp(\widetilde{f})(S)$.

Case 2: α irreducible Let $\widehat{f},\widehat{S}$ etc be as in Lemma 6.2.4. Choose $h\colon Y\to i\mathbb{R}$ such that the conclusions of Lemma 6.2.5 (i) holds with the S of that lemma taken to be the present $\widehat{S}$. Set $\acute{S}=(\acute{A},\acute{\Phi})=\exp(\underline{h})(\widehat{S})$ and $\acute{s}=(\acute{a},\acute{\phi})=\acute{S}-\underline{\alpha}$. By Lemma 6.2.5 and Lemma 6.2.2 (ii) we have

$$\|\mathcal{I}^*\acute{s}\|_2\le\text{const}\cdot\nu,\quad \|\acute{s}\|_{2,2}\le\text{const}\cdot\delta,\quad \|\acute{\phi}\|_{1,2}\le\text{const}\cdot\nu.$$

Since $-d^*\acute{a}=\mathcal{I}^*\acute{s}-i\langle i\Psi,\acute{\phi}\rangle$ we also get

$$\|d^*\acute{a}\|_2\le\text{const}\cdot\nu.$$

Applying Lemma 6.2.3 as in the proof of Lemma 6.2.4 we see that

$$\|\acute{a}\|_{1,2}\le\text{const}\cdot\nu.$$

It now only remains to make a small perturbation to $\acute{S}$ so as to fulfil the Coulomb gauge condition. To this end we invoke the local slice theorem for $\mathbf{B}$. This says that there is a $C>0$ such that if δ is sufficiently small then there exists a unique smooth $f\colon\mathbf{B}\to i\mathbb{R}$ such that setting $\widetilde{s}=(\widetilde{a},\widetilde{\phi})=\exp(f)(\acute{S})-\underline{\alpha}$ one has

$$\|f\|_{3,2}\le C\delta,\quad \mathcal{I}^*\widetilde{s}=0,\quad *\widetilde{a}|_{\partial\mathbf{B}}=0.$$

We will now estimate first $\|f\|_{2,2}$, then $\|\widetilde{s}\|_{1,2}$ in terms of ν. First note that $*\acute{a}|_{\partial\mathbf{B}}=*\widehat{a}|_{\partial\mathbf{B}}=0$, and

$$\widetilde{a}=\acute{a}-df,\qquad \widetilde{\phi}=\exp(f)\acute{\Phi}-\Psi$$

by definition. Write the imaginary part of $\exp(f)$ as $f + f^3 u$. Then f satisfies the equations $(\partial_t f)|_{\partial \mathbf{B}} = 0$ and

$$\begin{aligned} 0 &= -d^*\tilde{a} + i\langle i\Psi, \tilde{\phi}\rangle_{\mathbb{R}} \\ &= \Delta f - d^*\acute{a} + i\langle i\Psi, \exp(f)(\acute{\phi} + \Psi)\rangle_{\mathbb{R}} \\ &= \Delta f + |\Psi|^2 f - d^*\acute{a} + i\langle i\Psi, \exp(f)\acute{\phi} + f^3 u\Psi\rangle_{\mathbb{R}}. \end{aligned}$$

By the Sobolev embedding theorem we have

$$\|f\|_\infty \le \mathrm{const}\cdot\|f\|_{3,2} \le \mathrm{const}\cdot\delta,$$

and we assume $\delta \le 1$, so $\|u\|_\infty \le \mathrm{const}$. Therefore,

$$\begin{aligned} \|f\|_{2,2} &\le \mathrm{const}\cdot\|\Delta f + |\Psi|^2 f\|_2 \\ &\le \nu + \mathrm{const}\cdot\|f^3\|_2 \\ &\le \nu + \mathrm{const}\cdot\|f\|_{2,2}^3, \end{aligned}$$

cf Section 2.5 for the first inequality. If δ is sufficiently small then we can rearrange to get $\|f\|_{2,2} \le \mathrm{const}\cdot\nu$. Consequently, $\|\tilde{a}\|_{1,2} \le \mathrm{const}\cdot\nu$. To estimate $\|\tilde{\phi}\|_{1,2}$ we write

$$\tilde{\phi} = g\Psi + \exp(f)\acute{\phi},$$

where $g = \exp(f) - 1$. Then $|dg| = |df|$ and $|g| \le \mathrm{const}\cdot|f|$. Now

$$\begin{aligned} \|\tilde{\phi}\|_2 &\le \mathrm{const}\cdot\|f\|_2 + \|\acute{\phi}\|_2 \le \mathrm{const}\cdot\nu, \\ \|\nabla\tilde{\phi}\|_2 &\le \mathrm{const}\cdot(\|g\|_{1,2} + \|df\otimes\acute{\phi}\|_2 + \|\nabla\acute{\phi}\|_2) \\ &\le \mathrm{const}\cdot(\nu + \|df\|_4\|\acute{\phi}\|_4) \\ &\le \mathrm{const}\cdot(\nu + \|f\|_{2,2}\|\acute{\phi}\|_{1,2}) \\ &\le \mathrm{const}\cdot(\nu + \nu^2) \\ &\le \mathrm{const}\cdot\nu \end{aligned}$$

by (6.2). Therefore, $\|\tilde{\phi}\|_{1,2} \le \mathrm{const}\cdot\nu$. Thus, the proposition holds with

$$\tilde{f} = \hat{f} + \underline{h} + f. \qquad \square$$

Proposition 6.2.2 *Let k be a positive integer and $V \Subset \mathrm{int}(\mathbf{B})$ an open subset. Then there are constants $\epsilon_k, C_{k,V}$, where ϵ_k is independent of V, such that if*

$$\mathcal{I}^*_{\underline{\alpha}} s = 0, \quad \|s\|_{L^2_1(\mathbf{B})} \le \epsilon_k$$

then

$$\|s\|_{L^2_k(V)} \le C_{k,V}\|s\|_{L^2_1(\mathbf{B})}.$$

Proof The argument in [15, pp 62-3] carries over, if one replaces the operator $d^* + d^+$ with $\mathcal{I}^*_{\underline{\alpha}} + D\Theta_{\underline{\alpha}}$, where $D\Theta_{\underline{\alpha}}$ is the linearization of the monopole map at $\underline{\alpha}$. Note that $\mathcal{I}^*_{\underline{\alpha}} + D\Theta_{\underline{\alpha}}$ is injective over $S^1 \times Y$ because α is nondegenerate, so if $\gamma\colon \mathbf{B} \to \mathbb{R}$ is a smooth function supported in $\mathrm{int}(\mathbf{B})$ then

$$\|\gamma s\|_{k,2} \le C'_k \|(\mathcal{I}^*_{\underline{\alpha}} + D\Theta_{\underline{\alpha}})(\gamma s)\|_{k-1,2}$$

for some constant C'_k. □

6.3 Decay of monopoles

The two theorems in this section are analogues of Propositions 4.3 and 4.4 in [14], respectively.

Let β be a nondegenerate monopole over Y, and $U \subset \mathcal{B}_Y$ an L^2–closed subset which contains no monopoles except perhaps $[\beta]$. Choose $\lambda^{\pm} > 0$ such that $\widetilde{H}_\beta$ has no eigenvalue in the interval $[-\lambda^-, \lambda^+]$, and set $\lambda = \min(\lambda^-, \lambda^+)$. Define

$$\mathbf{B}_t = [t-1, t+1] \times Y.$$

Theorem 6.3.1 *For any $C > 0$ there are constants $\epsilon, C_0, C_1, \dots$ such that the following holds. Let $S = (A, \Phi)$ be any monopole in temporal gauge over $(-2, \infty) \times Y$ such that $[S_t] \in U$ for some $t \ge 0$. Set*

$$\overline{\nu} = \|\nabla \vartheta_S\|_{L^2((-2,\infty)\times Y)}, \qquad \nu(t) = \|\nabla \vartheta_S\|_{L^2(\mathbf{B}_t)}.$$

If $\|\Phi\|_\infty \le C$ and $\overline{\nu} \le \epsilon$ then there is a smooth monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\underline{\alpha}$ then for every $t \ge 1$ and nonnegative integer k one has

$$\sup_{y \in Y} |\nabla^k_B (S - \underline{\alpha})|_{(t,y)} \le C_k \sqrt{\nu(0)}\, e^{-\lambda^+ t}. \tag{6.5}$$

Proof It follows from the local slice theorem that $\widetilde{\mathcal{B}}_Y \to \mathcal{B}_Y$ is a (topological) principal $H^1(Y;\mathbb{Z})$–bundle. Choose a small open neighbourhood V of $[\beta] \in \mathcal{B}_Y$ which is the image of a convex set in $\mathcal{C}_Y$. We define a continuous function $\overline{f}\colon V \to \mathbb{R}$ by

$$\overline{f}(x) = \vartheta(\sigma(x)) - \vartheta(\sigma([\beta]))$$

where $\sigma\colon V \to \widetilde{\mathcal{B}}_Y$ is any continuous cross-section. It is clear $\overline{f}$ is independent of σ.

Given $C > 0$, let $S = (A, \Phi)$ be any monopole over $(-2, \infty) \times Y$ such that $\|\Phi\|_\infty \le C$ and $[S_t] \in U$ for some $t \ge 0$. If $\delta > 0$, and k is any nonnegative integer, then provided

$\overline{\nu}$ is sufficiently small, our local compactness results (Lemma 4.2.1 and Lemma 4.1.1) imply that for every $t \geq 0$ we can find a smooth $u\colon \mathbf{B}_0 \to \mathrm{U}(1)$ such that

$$\|u(S|_{\mathbf{B}_t}) - \underline{\beta}\|_{C^k(\mathbf{B}_0)} < \delta.$$

In particular, if $\overline{\nu}$ is sufficiently small then

$$f\colon \overline{\mathbb{R}}_+ \to \mathbb{R}, \quad t \mapsto \overline{f}([S_t])$$

is a well-defined smooth function. Since $f(t) - \vartheta(S_t)$ is locally constant, and $f(t) \to 0$ as $t \to \infty$, we have

$$f(t) = \vartheta(S_t) - L,$$

where $L = \lim_{t\to\infty} \vartheta(S_t)$. If $\overline{\nu}$ is sufficiently small then Lemma 6.1.2 gives $2\lambda^+ f \leq -f'$, hence

$$0 \leq f(t) \leq e^{-2\lambda^+ t} f(0), \quad t \geq 0.$$

This yields

$$\nu(t)^2 = f(t-1) - f(t+1) \leq \mathrm{const} \cdot e^{-2\lambda^+ t} f(0), \quad t \geq 1.$$

If $\overline{\nu}$ is sufficiently small then by Propositions 6.2.1 and 6.2.2 we have

$$f(t) \leq \mathrm{const} \cdot \nu(t), \quad \sup_{y \in Y} |\nabla_A^k (\nabla \vartheta_S)|_{(t,y)} \leq C_k' \nu(t) \tag{6.6}$$

for every $t \geq 0$ and nonnegative integer k, where C_k' is some constant. Here we are using the simple fact that if E, E' are Banach spaces, $W \subset E$ an open neighbourhood of 0, and $h\colon W \to E'$ a differentiable map with $h(0) = 0$ then $\|h(x)\| \leq (\|Dh(0)\| + 1)\|x\|$ in some neighbourhood of 0. For instance, to deduce the second inequality in (6.6) we can apply this to the map

$$h\colon L^2_{k+j+1} \to L^2_j, \quad s = (a, \phi) \mapsto \nabla^k_{B'+a}(\nabla \vartheta_{\underline{\beta}+s})$$

where $j \geq 3$, say, and B' is the connection part of $\underline{\beta}$.

Putting the inequalities above together we get

$$\sup_{y \in Y} |\nabla_A^k (\nabla \vartheta_S)|_{(t,y)} \leq C_k'' \sqrt{\nu(0)} e^{-\lambda^+ t}, \quad t \geq 1$$

for some constants C_k''. If S is in temporal gauge we deduce, by taking $k = 0$, that S_t converges uniformly to some continuous configuration α. One can now prove by induction on k that α is of class C^k and that (6.5) holds. □

Theorem 6.3.2 *For any $C>0$ there are constants $\epsilon, C_0, C_1, \dots$ such that the following holds for every $T>1$. Let $S=(A,\Phi)$ be any smooth monopole in temporal gauge over the band $[-T-2, T+2]\times Y$, and suppose $[S_t]\in U$ for some $t\in[-T,T]$. Set*

$$\overline{\nu}=\|\nabla\vartheta_S\|_{L^2((-T-2,T+2)\times Y)},\quad \nu(t)=\|\nabla\vartheta_S\|_{L^2(\mathbf{B}_t)}.$$

If $\|\Phi\|_\infty\le C$ and $\overline{\nu}\le\epsilon$ then there is a smooth monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\underline{\alpha}$ then for $|t|\le T-1$ and every nonnegative integer k one has

$$\sup_{y\in Y}|\nabla_B^k(S-\underline{\alpha})|_{(t,y)}\le C_k(\nu(-T)+\nu(T))^{1/2}e^{-\lambda(T-|t|)}.$$

Proof Given $C>0$, let $S=(A,\Phi)$ be any monopole over $[-T-2,T+2]\times Y$ such that $\|\Phi\|_\infty\le C$ and $[S_t]\in U$ for some $t\in[-T,T]$. If $\overline{\nu}$ is sufficiently small then we can define the function $f(t)$ for $|t|\le T$ as in the proof of Theorem 6.3.1, and (6.6) will hold with $f(t)$ replaced by $|f(t)|$, for $|t|\le T$. Again, $f(t)=\vartheta(S_t)-L$ for some constant L. Lemma 6.1.2 now gives

$$e^{-2\lambda_-(T-t)}f(T)\le f(t)\le e^{-2\lambda_+(T+t)}f(-T),\quad |t|\le T,$$

which implies

$$|f(t)|\le(|f(-T)|+|f(T)|)e^{-2\lambda(T-|t|)},\quad |t|\le T,$$
$$\nu(t)^2\le\text{const}\cdot(\nu(-T)+\nu(T))e^{-2\lambda(T-|t|)},\quad |t|\le T-1.$$

By Proposition 6.2.1 and Proposition 6.2.2 there is a critical point α gauge equivalent to β such that

$$\|\nabla_B^k(S_0-\alpha)\|_{L^\infty(Y)}\le C_k'''\nu(0)$$

for some constants C_k'''. It is now easy to complete the proof by induction on k. □

6.4 Global convergence

The main result of this section is Proposition 6.4.1, which relates local and global convergence of monopoles over a half-cylinder. First some lemmas.

Lemma 6.4.1 *Let Z be a compact Riemannian n–manifold (perhaps with boundary), m a nonnegative integer, and $q\ge n/2$. Then there is a real polynomial $P_{m,q}(x)$ of degree $m+1$ satisfying $P_{m,q}(0)=0$, such that for any smooth $u\colon Z\to\mathrm{U}(1)$ one has*

$$\|du\|_{m,q}\le P_{m,q}(\|u^{-1}du\|_{m,q}).$$

Proof Argue by induction on m and use the Sobolev embedding $L^r_k(Z) \subset L^{2r}_{k-1}(Z)$ for $k \geq 1$, $r \geq n/2$. $\square$

Lemma 6.4.2 *Let Z be a compact, connected Riemannian n–manifold (perhaps with boundary), $z \in Z$, m a positive integer, and $q \geq 1$. Then there is a $C > 0$ such that for any smooth $f\colon Z \to \mathbb{C}$ one has*

(i) $\|f - f_{\mathrm{av}}\|_{m,q} \leq C\|df\|_{m-1,q}$,

(ii) $\|f\|_{m,q} \leq C(\|df\|_{m-1,q} + |f(z)|)$,

where $f_{\mathrm{av}} = \mathrm{Vol}(Z)^{-1}\int_Z f$ is the average of f.

Proof Exercise. $\square$

Lemma 6.4.3 *Let Z be a compact Riemannian n–manifold (perhaps with boundary), m a positive integer, and q a real number such that $mq > n$. Let Φ be a smooth section of some Hermitian vector bundle $E \to Z$, $\Phi \not\equiv 0$. Then there exists a $C > 0$ with the following significance. Let ϕ_1 be a smooth section of E satisfying $\|\phi_1\|_q \leq C^{-1}$ and $w\colon Z \to \mathbb{C}$ a smooth map. Define another section ϕ_2 by*

$$w(\Phi + \phi_1) = \Phi + \phi_2.$$

Then $$\|w - 1\|_{m,q} \leq C(\|dw\|_{m-1,q} + \|\phi_2 - \phi_1\|_q).$$

Proof The equation

$$(w-1)\Phi = \phi_2 - \phi_1 - (w-1)\phi_1$$

gives

$$\begin{aligned}\|w-1\|_{m,q} &\leq \mathrm{const}\cdot(\|dw\|_{m-1,q} + \|(w-1)\Phi\|_q)\\ &\leq \mathrm{const}\cdot(\|dw\|_{m-1,q} + \|\phi_2 - \phi_1\|_q + \|w-1\|_{m,q}\|\phi_1\|_q).\end{aligned}$$

Here the first inequality is analogous to Lemma 6.4.2 (ii). If $\|\phi_1\|_q$ is sufficiently small then we can rearrange to get the desired estimate on $\|w-1\|_{m,q}$. $\square$

Now let α be a nondegenerate critical point of ϑ. Note that if $S = (A, \Phi)$ is any finite energy monopole in temporal gauge over $\mathbb{R}_+ \times Y$ such that $\|\Phi\|_\infty < \infty$ and

$$\liminf_{t\to\infty} \int_{[t,t+1]\times Y} |S - \underline{\alpha}|^r = 0$$

for some $r > 1$ then by the results of Chapter 4 we have $[S_t] \to [\alpha]$ in $\mathcal{B}_Y$, hence $S - \underline{\alpha}$ decays exponentially by Theorem 6.3.1. In this situation we will simply say that S is *asymptotic to* α.

(Here we used the fact that for any $p > 2$ and $1 < r \le 2p$, say, the L^r metric on the L^p_1 configuration space $\mathcal{B}([0,1]\times Y)$ is well defined.)

Lemma 6.4.4 *If $S = (A, \Phi)$ is any smooth monopole over $\mathbb{R}_+ \times Y$ such that $\|\Phi\|_\infty < \infty$ and $S - \underline{\alpha} \in L^p_1$ for some $p > 2$ then there exists a null-homotopic smooth $u\colon \mathbb{R}_+ \times Y \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge and asymptotic to α.*

Proof By Theorem 6.3.1 there exists a smooth $u\colon \mathbb{R}_+ \times Y \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge and asymptotic to α. Lemma 6.4.1, Lemma 6.4.2 (i), and the assumption $S - \underline{\alpha} \in L^p_1$ then gives

$$\|u - u_{\mathrm{av}}\|_{L^\infty([t,t+1]\times Y)} \to 0 \quad \text{as } t \to \infty,$$

hence u is null-homotopic. □

It follows that all elements of the moduli spaces defined in Section 3.4 have smooth representatives that are in temporal gauge over the ends.

Proposition 6.4.1 *Let $\delta > 0$ and suppose $\vartheta\colon \widetilde{\mathcal{B}}_Y \to \mathbb{R}$ has no critical value in the half-open interval $(\vartheta(\alpha), \vartheta(\alpha) + \delta]$ (this implies Condition (O1)). For $n = 1, 2, \ldots$ let $S_n = (A_n, \Phi_n)$ be a smooth monopole over $\overline{\mathbb{R}}_+ \times Y$ such that*

$$S_n - \underline{\alpha} \in L^p_1, \quad \sup_n \|\Phi_n\|_\infty < \infty, \quad \vartheta(S_n(0)) \le \vartheta(\alpha) + \delta,$$

for some $p > 2$. Let $v_n\colon \overline{\mathbb{R}}_+ \times Y \to \mathrm{U}(1)$ be a smooth map such that the sequence $v_n(S_n)$ converges in C^∞ over compact subsets of $\overline{\mathbb{R}}_+ \times Y$ to a configuration S in temporal gauge. Then the following hold:

(i) *S is asymptotic to a critical point α' gauge equivalent to α.*

(ii) *If $\alpha = \alpha'$ then v_n is null-homotopic for all sufficiently large n, and there exist smooth $u_n\colon \overline{\mathbb{R}}_+ \times Y \to \mathrm{U}(1)$ with the following significance: For every $t \ge 0$ one has $u_n = 1$ on $[0, t] \times Y$ for all sufficiently large n. Moreover, for any $\sigma < \lambda^+$, $q \ge 1$ and nonnegative integer m one has*

$$\|u_n v_n(S_n) - S\|_{L^{q,\sigma}_m} \to 0 \quad \text{as } n \to \infty.$$

Here λ^+ is as in Section 6.1.

Proof It clearly suffices to prove the proposition when $q \ge 2$ and $mq > 4$, which we assume from now on.

By Lemma 4.2.4 we have

$$\int_{\mathbb{R}_+\times Y} |\nabla\vartheta_{S_n}|^2 = \vartheta(S_n(0)) - \vartheta(\alpha) \le \delta \tag{6.7}$$

for each n, hence $\int_{\mathbb{R}_+\times Y} |\nabla\vartheta_S|^2 \le \delta$. Part (i) of the proposition is now a consequence of Theorem 6.3.1 and the following:

Claim 6.4.1 *$[S(t)]$ converges in $\mathcal{B}_Y$ to $[\alpha]$ as $t\to\infty$.*

Proof of claim For $r>0$ let $B_r\subset\mathcal{B}_Y$ denote the open r–ball around $[\alpha]$ in the L^2 metric, and let $\bar{B}_r$ be the corresponding closed ball. Choose $r>0$ such that $\bar{B}_{2r}$ contains no monopole other than $[\alpha]$. Assuming the claim does not hold then by Lemma 4.1.1 one can find a sequence t'_j such that $t'_j\to\infty$ as $j\to\infty$ and $[S(t'_j)]\notin\bar{B}_{2r}$ for each j. Because of the convergence of $v_n(S_n)$ it follows by a continuity argument that there are sequences n_j, t_j with $t_j, n_j\to\infty$ as $j\to\infty$, such that

$$[S_{n_j}(t_j)]\in\bar{B}_{2r}\setminus B_r$$

for all j. For $s\in\mathbb{R}$ let $\mathcal{T}_s\colon \mathbb{R}\times Y\to\mathbb{R}\times Y$ be translation by s:

$$\mathcal{T}_s(t,y)=(t+s,y).$$

Again by Lemma 4.1.1 there are smooth $\omega_j\colon\bar{\mathbb{R}}_+\times Y\to \mathrm{U}(1)$ such that a subsequence of $(\mathcal{T}_{t_j})^*(\omega_j(S_{n_j}))$ converges in C^∞ over compact subsets of $\mathbb{R}\times Y$ to some finite energy monopole S' whose spinor field is pointwise bounded. Moreover, it is clear that $\vartheta\circ\omega_j(0)-\vartheta\in\mathbb{R}$ must be bounded as $j\to\infty$, so by passing to a subsequence and replacing ω_j by $\omega_j\omega_{j_0}^{-1}$ for some fixed j_0 we may arrange that $\vartheta\circ\omega_j(0)=\vartheta$ for all n. Then $\ell=\lim_{t\to-\infty}\vartheta(S'(t))$ must be a critical value of ϑ. Since

$$[S'(0)]\in\bar{B}_{2r}\setminus B_r,$$

$S'(0)$ is not a critical point, whence $\partial_t|_0\vartheta(S'(t))<0$. Therefore,

$$\vartheta(\alpha)+\delta\ge\ell>\vartheta(S'(0))>\vartheta(\alpha),$$

contradicting our assumptions. This proves the claim. □

We will now prove Part (ii). For $\tau\ge 0$ let

$$B^-_\tau=[0,\tau]\times Y,\quad B^+_\tau=[\tau,\infty)\times Y,\quad \mathcal{O}_\tau=[\tau,\tau+1]\times Y.$$

By Lemma 6.4.4 there is, for every n, a null-homotopic, smooth $\tilde{v}_n\colon\bar{\mathbb{R}}_+\times Y\to\mathrm{U}(1)$ such that $S''_n=\tilde{v}_n(S_n)$ is in temporal gauge and asymptotic to α.

Note that
$$\lim_{t\to\infty}\limsup_{n\to\infty}\vartheta(S_n(t))=\vartheta(\alpha).$$

For otherwise we could find an $\epsilon > 0$ and for every natural number j a pair $t_j, n_j \geq j$ such that

$$\vartheta(S_{n_j}(t_j)) \geq \vartheta(\alpha) + \epsilon,$$

and we could then argue as in the proof of Claim 6.4.1 to produce a critical value of ϑ in the interval $(\alpha, \alpha + \delta]$. Since $|\nabla\vartheta_{S_n}| = |\nabla\vartheta_{S''_n}|$ it follows from (6.7) and Theorem 6.3.1 that there exists a $t_1 \geq 0$ such that if $\tau \geq t_1$ then

$$\limsup_{n\to\infty} \|S''_n - \underline{\alpha}\|_{L^{q,\sigma}_m(B^+_\tau)} \leq \text{const} \cdot e^{(\sigma-\lambda^+)\tau}$$

where the constant is independent of τ. Then we also have

$$\limsup_{n\to\infty} \|S''_n - S\|_{L^{q,\sigma}_m(B^+_\tau)} \leq \text{const} \cdot e^{(\sigma-\lambda^+)\tau}.$$

Set $S'_n = v_n(S_n)$ and $w_n = \widetilde{v}_n v_n^{-1}$. Then we get

$$\limsup_{n\to\infty} \|S''_n - S'_n\|_{L^q_m(\mathcal{O}_\tau)} \leq \text{const} \cdot e^{-\lambda^+\tau},$$

which gives
$$\limsup_{n\to\infty} \|dw_n\|_{L^q_m(\mathcal{O}_\tau)} \leq \text{const} \cdot e^{-\lambda^+\tau}$$

by Lemma 6.4.1. In particular, w_n is null-homotopic for all sufficiently large n.

Fix $y_0 \in Y$ and set $x_\tau = (\tau, y_0)$. Choose a sequence τ_n such that $\tau_n \to \infty$ as $n \to \infty$ and

$$\begin{aligned} &\|S'_n - S\|_{L^{q,\sigma}_m(B^-_{\tau_n+1})} \to 0, \\ &\|S''_n - S\|_{L^{q,\sigma}_m(B^+_{\tau_n})} \to 0 \end{aligned} \tag{6.8}$$

as $n \to \infty$. If α is reducible then by multiplying each $\widetilde{v}_n$ by a constant and redefining w_n, S''_n accordingly we may arrange that $w_n(x_{\tau_n}) = 1$ for all n. (If α is irreducible we keep $\widetilde{v}_n$ as before.) Then (6.8) still holds. Applying Lemma 6.4.1 together with Lemma 6.4.2 (ii) (if α is reducible) or Lemma 6.4.3 (if α is irreducible) we see that

$$e^{\sigma\tau_n} \|w_n - 1\|_{L^q_{m+1}(\mathcal{O}_{\tau_n})} \to 0$$

as $n \to \infty$.

Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\beta(t) = 0$ for $t \leq 1/3$ and $\beta(t) = 1$ for $t \geq 2/3$. Set $\beta_\tau(t) = \beta(t - \tau)$. Given any function $w\colon \mathcal{O}_\tau \to \mathbb{C} \setminus (-\infty, 0]$ define

$$\mathcal{U}_{w,\tau} = \exp(\beta_\tau \log w)$$

where $\log(\exp(z)) = z$ for complex numbers z with $|\operatorname{Im}(z)| < \pi$. Let m' be any integer such that $m'q > 4$. If $\|w-1\|_{m',q}$ is sufficiently small then

$$\begin{aligned} \|\mathcal{U}_{w,\tau} - 1\|_{m',q} &\le \mathrm{const} \cdot \|w-1\|_{m',q}, \\ \|w^{-1}dw\|_{m'-1,q} &\le \mathrm{const} \cdot \|w-1\|_{m',q}. \end{aligned} \tag{6.9}$$

To see this recall that for functions on $\mathbb{R}^4$, multiplication defines a continuous map $L^q_{m'} \times L^q_k \to L^q_k$ for $0 \le k \le m'$. Therefore, if V is the set of all functions in $L^q_{m'}(\mathcal{O}_\tau, \mathbb{C})$ that map into some fixed small open ball about $1 \in \mathbb{C}$ then $w \mapsto \mathcal{U}_{w,\tau}$ defines a C^∞ map $V \to L^q_{m'}$. This yields the first inequality in (6.9), and the proof of the second inequality is similar.

Combining (6.8) and (6.9) we conclude that Part (ii) of the proposition holds with

$$u_n = \begin{cases} 1 & \text{in } B^-_{\tau_n}, \\ \mathcal{U}_{w_n,\tau_n} & \text{in } \mathcal{O}_{\tau_n}, \\ w_n & \text{in } B^+_{\tau_n+1}. \end{cases}$$

This completes the proof of Proposition 6.4.1. $\square$

CHAPTER 7

Global compactness

In this chapter we will prove Theorems 1.3.1 and 1.4.1. Given the results of Chapters 4 and 5, what remains to be understood is convergence over ends and necks. We will use the following terminology:

c-convergence = C^∞ convergence over compact subsets.

7.1 Chain-convergence

We first define the notion of chain-convergence. For simplicity we only consider two model cases: first the case of one end and no necks, then the case of one neck and no ends. It should be clear how to extend the notion to the case of multiple ends and/or necks.

Definition 7.1.1 Let X be a spin^c Riemannian 4–manifold with one tubular end $\mathbb{R}_+ \times Y$, where Y is connected. Let $\alpha_1, \alpha_2, \dots$ and $\beta_0, \dots, \beta_k$ be elements of $\widetilde{\mathcal{R}}_Y$, where $k \geq 0$ and $\vartheta(\beta_{j-1}) > \vartheta(\beta_j)$ for $j = 1, \dots, k$. Let $\omega \in M(X; \beta_0)$ and $\vec{v} = (v_1, \dots, v_k)$, where $v_j \in \check{M}(\beta_{j-1}, \beta_j)$. We say a sequence $[S_n] \in M(X; \alpha_n)$ *chain-converges* to $(\omega, \vec{v})$ if there exist, for each n,

- a smooth map $u_n\colon X \to \mathrm{U}(1)$,
- for $j = 1, \dots, k$ a smooth map $u_{n,j}\colon \mathbb{R} \times Y \to \mathrm{U}(1)$,
- a sequence $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k}$,

such that

(i) $u_n(S_n)$ c-converges over X to a representative of ω (in the sense of Section 2.4),

(ii) $t_{n,j} - t_{n,j-1} \to \infty$ as $n \to \infty$,

(iii) $u_{n,j}(\mathcal{T}^*_{t_{n,j}} S_n)$ c-converges over $\mathbb{R} \times Y$ to a representative of v_j,

(iv) $\limsup_{n\to\infty} \big[\vartheta(S_n(t_{n,j-1}+\tau)) - \vartheta(S_n(t_{n,j}-\tau))\big] \to 0$ as $\tau \to \infty$,

(v) $\limsup_{n\to\infty} \big[\vartheta(S_n(t_{n,k}+\tau)) - \vartheta(\alpha_n)\big] \to 0$ as $\tau \to \infty$,

where (ii), (iii) and (iv) should hold for $j = 1, \dots, k$.

Conditions (iv), (v) mean, in familiar language, that "no energy is lost in the limit". As before, $\mathcal{T}_s$ denotes translation by s, ie $\mathcal{T}_s(t, y) = (t+s, y)$.

We now turn to the case of one neck and no ends.

Definition 7.1.2 In the situation of Section 1.4, suppose $r = 1$ and $r' = 0$. Let $\beta_0, \dots, \beta_k \in \widetilde{\mathcal{R}}_Y$, where $k \geq 0$ and $\vartheta(\beta_{j-1}) > \vartheta(\beta_j)$, $j = 1, \dots, k$. Let $\omega \in M(X; \beta_0, \beta_k)$ and $\vec{v} = (v_1, \dots, v_k)$, where $v_j \in \check{M}(\beta_{j-1}, \beta_j)$. Let $T(n) \to \infty$ as $n \to \infty$. We say a sequence $[S_n] \in M(X^{(T(n))})$ *chain-converges* to $(\omega, \vec{v})$ if there exist, for every n,

- a smooth map $u_n \colon X^{(T(n))} \to \mathrm{U}(1)$,
- for $j = 1, \dots, k$ a smooth map $u_{n,j} \colon \mathbb{R} \times Y \to \mathrm{U}(1)$,
- a sequence $-T(n) = t_{n,0} < t_{n,1} < \dots < t_{n,k+1} = T(n)$,

such that (i)–(iv) of Definition 7.1.1 hold for the values of j for which they are defined (in other words, (ii) and (iv) should hold for $1 \leq j \leq k+1$ and (iii) for $1 \leq j \leq k$).

In the notation of Section 1.2, if $J \subset \mathbb{R}$ is an interval with nonempty interior then a smooth configuration S over $J \times Y$ is called *normal* (with respect to ϑ) if either $\partial_t \vartheta(S_t) < 0$ for every $t \in J$, or S is gauge equivalent to the translationary invariant configuration $\underline{\alpha}$ determined by some critical point α of ϑ. Proposition 4.2.2 guarantees the normality of certain $(\mathfrak{p}, \mathfrak{q})$–monopoles when $\mathfrak{p}$ is sufficiently small. In particular, genuine monopoles are always normal.

Consider now the situation of Section 1.4 (without assuming (B1) or (B2)), and let the 2–form μ on X be fixed.

Definition 7.1.3 A set of perturbation parameters $\vec{\mathfrak{p}}, \vec{\mathfrak{p}}'$ is *admissible* for a vector $\vec{\alpha}'$ of critical points if for some $t_0 \geq 1$ the following holds. Let $\mathcal{M}$ be the disjoint union of all moduli spaces $M(X^{(T)}; \vec{\alpha}'; \vec{\mathfrak{p}}; \vec{\mathfrak{p}}')$ with $T_{\min} \geq t_0$. Then we require, for all j, k, that the following hold:

(i) If $\widetilde{S}$ is any configuration over $[-1,1]\times Y_j$ which is a C^∞ limit of configurations of the form $S|_{[t-1,t+1]\times Y_j}$ with $S\in\mathcal{M}$ and $|t|\le T_j-1$, then $\widetilde{S}$ is normal.

(ii) If $\widetilde{S}$ is any configuration over $[-1,1]\times Y'_k$ which is a C^∞ limit of configurations of the form $S|_{[t-1,t+1]\times Y'_k}$ with $S\in\mathcal{M}$ and $t\ge 1$, then $\widetilde{S}$ is normal.

In particular, the zero perturbation parameters are always admissible.

The next two propositions describe some properties of chain-convergence.

Proposition 7.1.1 *In the notation of Theorem 1.3.1, suppose ω_n chain-converges to $(\omega,\vec{v}_1,\dots,\vec{v}_r)$, where $\vec{\alpha}_n=\vec{\beta}$ for all n, each $\vec{v}_j$ is empty and $\vec{\mathfrak{p}}$ is admissible for $\vec{\beta}$. Then $\omega_n\to\omega$ in $M(X;\vec{\beta})$ with its usual topology.*

Proof This follows from Proposition 6.4.1. □

In other words, if a sequence ω_n in a moduli space M chain-converges to an element $\omega\in M$, then $\omega_n\to\omega$ in M provided the perturbations are admissible.

Proposition 7.1.2 *In the notation of Section 1.4, suppose $\omega_n\in M(X^{(T(n))};\vec{\alpha}'_n)$ chain-converges to $\mathbb{V}=(\omega,\vec{v}_1,\dots,\vec{v}_r,\vec{v}'_1,\dots,\vec{v}'_{r'})$, where $T_{\min}(n)\to\infty$. Suppose also that the perturbation parameters $\vec{\mathfrak{p}},\vec{\mathfrak{p}}'$ are admissible for each $\vec{\alpha}'_n$. Then the following hold:*

(i) *For sufficiently large n there is a smooth map $u_n\colon X^{(T(n))}\to \mathrm{U}(1)$ such that $v_{n,j}=u_n|_{\{0\}\times Y'_j}$ satisfies $v_{n,j}(\alpha'_{n,j})=\gamma'_j$, $j=1,\dots,r'$.*

(ii) *The chain limit is unique up to gauge equivalence, ie if $\mathbb{V}$, $\mathbb{V}'$ are two chain limits of ω_n then there exists a smooth $u\colon X^{\#}\to\mathrm{U}(1)$ which is translationary invariant over the ends of $X^{\#}$, and such that $u(\mathbb{V})=\mathbb{V}'$.*

In (i), recall that moduli spaces are labelled by critical points modulo null-homotopic gauge transformations. Note that we can arrange that the maps u_n are translationary invariant over the ends. This allows us to identify the moduli spaces $M(X;\vec{\alpha}'_n)$ and $M(X;\vec{\gamma}')$, so that we obtain a sequence $u_n(\omega_n)$, $n\gg 0$ in a *fixed* moduli space.

In (ii) we define $u(\mathbb{V})$ as follows. Let $w_j\colon \mathbb{R}\times Y_j\to\mathrm{U}(1)$ and $w'_j\colon\mathbb{R}\times Y'_j\to\mathrm{U}(1)$ be the translationary invariant maps which agree with u on $\{0\}\times Y_j$ and $\mathbb{R}_+\times Y'_j$, respectively. Let $w\colon X\to\mathrm{U}(1)$ be the map which is translationary invariant over each end and agrees with u on $X_{:1}$. Then $u(\mathbb{V})$ is the result of applying the appropriate maps w,w_j,w'_j to the various components of $u(\mathbb{V})$.

Proof of proposition (i) For simplicity we only discuss the case of one end and no necks, ie the situation of Definition 7.1.1. The proof in the general case is similar.

Using Condition (v) of Definition 7.1.1 and a simple compactness argument it is easy to see that α_n is gauge equivalent to β_k for all sufficiently large n. Moreover, Conditions (iv) and (v) of Definition 7.1.1 ensure that there exist $\tau, n' > 0$ such that if $n > n'$ then ω_n restricts to a genuine monopole on $(t_{n,k} + \tau, \infty) \times Y$ and on $(t_{n,j-1} + \tau, t_{n,j} - \tau) \times Y$ for $j = 1, \dots, k$. It then follows from Proposition 6.4.1 that $v_n = u_{n,k}|_{\{0\}\times Y}$ satisfies $v_n(\alpha_n) = \beta_k$ for $n \gg 0$. (Recall again that $\alpha_n, \beta_k \in \widetilde{\mathcal{R}}_Y$ are critical points modulo null-homotopic gauge transformations, so $v_n(\alpha_n)$ depends only on the homotopy class of v_n.) Similarly, it follows from Theorems 6.3.1 and 6.3.2 that $u_{n,j-1}|_{\{0\}\times Y}$ is homotopic to $u_{n,j}|_{\{0\}\times Y}$ for $j = 1, \dots, r$ and $n \gg 0$, where $u_{n,0} = u_n$. Therefore, v_n extends over $X_{:0}$.

(ii) This is a simple exercise. □

7.2 Proof of Theorem 1.3.1

By Propositions 3.4.1,5.4.2 and 4.2.2, if each $\mathfrak{p}_j$ has sufficiently small C^1 norm then $\vec{\mathfrak{p}}$ will be admissible for all $\vec{\alpha}$. Choose $\vec{\mathfrak{p}}$ so that this is the case. Set

$$C_0 = -\inf_n \sum_j \lambda_j \vartheta(\alpha_{n,j}) < \infty.$$

Let S_n be a smooth representative for ω_n. The energy assumption on the asymptotic limits of S_n is unaffected if we replace S_n by $u_n(S_n)$ for some smooth $u_n\colon X \to \mathrm{U}(1)$ which is translationary invariant on $(t_n, \infty) \times Y$ for some $t_n > 0$. After passing to a subsequence we can therefore, by Proposition 5.2.1, assume that S_n c-converges over X to some monopole S' which is in temporal gauge over the ends. Because $\vec{\mathfrak{p}}$ is admissible we have that

$$\partial_t \vartheta(S_n|_{\{t\}\times Y_j}) \le 0$$

for all j, n and $t \ge 0$. From the energy bound (5.6) we then see that S' must have finite energy. Let γ_j denote the asymptotic limit of S' over the end $\mathbb{R}_+ \times Y_j$ as guaranteed by Proposition 4.2.3. Then

$$\limsup_n \vartheta(\alpha_{n,j}) \le \vartheta(\gamma_j)$$

for each j. Hence there is a constant $C_2 < \infty$ such that for $h = 1, \dots, r$ and all n one has

$$C_2 + \lambda_h \vartheta(\alpha_{n,h}) \ge \sum_j \lambda_j \vartheta(\alpha_{n,j}) \ge -C_0.$$

Consequently, $$\sup_{n,j} |\vartheta(\alpha_{n,j})| < \infty.$$

For the remainder of this proof we fix j and focus on one end $\mathbb{R}_+ \times Y_j$. For simplicity we drop j from notation and write Y, α_n instead of $Y_j, \alpha_{n,j}$ etc.

After passing to a subsequence we may arrange that $\vartheta(\alpha_n)$ has the same value L for all n (here we use Condition (O1)). If $\vartheta(\gamma) = L$ then we set $k = 0$ and the proof is complete. Now suppose $\vartheta(\gamma) > L$. Then there is an n' such that $\partial_t \vartheta(S_n(t)) < 0$ for all $n \geq n'$, $t \geq 0$. Set

$$\delta = \frac{1}{2} \min\{|x - y| \,:\, x, y \text{ are distinct critical values of } \vartheta\colon \widetilde{\mathcal{B}}_Y \to \mathbb{R}\}.$$

The minimum exists by (O1). For sufficiently large n we define $t_{n,1} \gg 0$ implicitly by

$$\vartheta(S_n(t_{n,1})) = \vartheta(\gamma) - \delta.$$

It is clear that $t_{n,1} \to \infty$ as $n \to \infty$. Moreover, Definition 7.1.1 (iv) must hold for $j = 1$. For otherwise we can find $\epsilon > 0$ and sequences τ_ℓ, n_ℓ with $\tau_\ell, n_\ell \to \infty$ as $\ell \to \infty$, such that

$$\vartheta(S_{n_\ell}(\tau_\ell)) - \vartheta(S_{n_\ell}(t_{n_\ell,1} - \tau_\ell)) > \epsilon \tag{7.1}$$

for every ℓ. As in the proof of Claim 6.4.1 there are smooth $\widetilde{u}_\ell\colon \mathbb{R} \times Y \to \mathrm{U}(1)$ satisfying $\vartheta \circ \widetilde{u}_\ell(0) = \vartheta$ such that a subsequence of

$$\widetilde{u}_\ell(\mathcal{T}^*_{t_{n_\ell,1}} S_{n_\ell})$$

c-converges over $\mathbb{R} \times Y$ to a finite energy monopole $\widetilde{S}$ in temporal gauge. The asymptotic limit $\widetilde{\gamma}$ of $\widetilde{S}$ at $-\infty$ must satisfy

$$\epsilon \leq \vartheta(\gamma) - \vartheta(\widetilde{\gamma}) < \delta,$$

where the first inequality follows from (7.1). This contradicts the choice of δ. Therefore, Definition 7.1.1 (iv) holds for $j = 1$ as claimed.

After passing to a subsequence we can find $u_{n,1}\colon \mathbb{R} \times Y \to \mathrm{U}(1)$ such that $u_{n,1}(\mathcal{T}^*_{t_{n,1}} S_n)$ c-converges over $\mathbb{R} \times Y$ to some finite energy monopole S'_1 in temporal gauge. Let $\beta_1^\pm$ denote the limit of S'_1 at $\pm\infty$. A simple compactness argument shows that γ and β_1^- are gauge equivalent, so we can arrange that $\gamma = \beta_1^-$ by modifying the $u_{n,1}$ by a fixed gauge transformation $\mathbb{R} \times Y \to \mathrm{U}(1)$. As in the proof of Proposition 7.1.2 (i) we see that $u_{n,1}$ must be null-homotopic for all sufficiently large n. Hence $\vartheta(\beta_1^+) \geq L$. If $\vartheta(\beta_1^+) = L$ then we set $k = 1$ and the proof is finished. If on the other hand $\vartheta(\beta_1^+) > L$ then we continue the above process. The process ends when, after passing successively to subsequences and choosing $u_{n,j}, t_{n,j}, \beta_j^\pm$ for $j = 1, \ldots, k$ (where

$\beta_{j-1}^+ = \beta_j^-$, and $u_{n,j}$ is null-homotopic for $n \gg 0$) we have $\vartheta(\beta_k^+) = L$. This must occur after finitely many steps; in fact $k \le (2\delta)^{-1}(\vartheta(\gamma) - L)$.

7.3 Proof of Theorem 1.4.1

For simplicity we first consider the case when there is exactly one neck (ie $r = 1$), and we write $Y = Y_1$ etc. We will make repeated use of the local compactness results proved earlier.

Let S_n be a smooth representative of ω_n. After passing to a subsequence we can find smooth maps

$$u_n\colon X^{(T(n))} \setminus (\{0\} \times Y) \to \mathrm{U}(1)$$

such that $\widetilde{S}_n = u_n(S_n)$ c-converges over X to some finite energy monopole S' which is in temporal gauge over the ends. Introduce the temporary notation $S_n(t) = S_n|_{\{t\}\times Y}$, and similarly for $\widetilde{S}_n$ and u_n. For $0 \le \tau < T(n)$ set

$$\Theta_{\tau,n} = \vartheta(S_n(-T(n)+\tau)) - \vartheta(S_n(T(n)-\tau)).$$

Let $u_n^\pm = u_n(\pm T(n))$ and

$$I_n^\pm = 2\pi \int_Y \widetilde{\eta}_j \wedge [u_n^\pm],$$

cf Equation (1.1). Since $\Theta_{0,n}$ is bounded as $n \to \infty$, it follows that $I_n^+ - I_n^-$ is bounded as $n \to \infty$. By Condition (O1) there is a $q > 0$ such that $qI_n^\pm$ is integral for all n. Hence we can arrange, by passing to a subsequence, that $I_n^+ - I_n^-$ is constant. In particular,

$$I_n^+ - I_1^+ = I_n^- - I_1^-.$$

Choose a smooth map $w\colon X \to \mathrm{U}(1)$ which is translationary invariant over the ends, and homotopic to u_1^{-1} over $X_{:0}$. After replacing u_n by wu_n for every n we then obtain $I_n^+ = I_n^-$. Set $I_n = I_n^\pm$. We now have

$$\Theta_{\tau,n} = \vartheta(\widetilde{S}_n(-T(n))) - \vartheta(\widetilde{S}_n(T(n))).$$

Let β_0 and β' denote the asymptotic limits of S' over the ends $\iota^+(\mathbb{R}_+ \times Y)$ and $\iota^-(\mathbb{R}_+ \times Y)$, respectively. Set

$$L = \lim_{\tau\to\infty} \lim_{n\to\infty} \Theta_{\tau,n} = \vartheta(\beta_0) - \vartheta(\beta').$$

Since $\Theta_{\tau,n} \ge 0$ for $\tau \ge 0$ we have $L \ge 0$.

Suppose $L = 0$. Then a simple compactness argument shows that there is a smooth $v\colon Y \to \mathrm{U}(1)$ such that $v(\beta_0) = \beta'$. Moreover, there is an n_0 such that $v\, u_n^- \sim u_n^+$ for

$n \geq n_0$, where $\sim$ means "homotopic". Therefore, we can find a smooth $z\colon X \to \mathrm{U}(1)$ which is translationary invariant over the ends and homotopic to $u_{n_0}^{-1}$ over $X_{:0}$, such that after replacing u_n by zu_n for every n we have that $\beta_0 = \beta'$ and $u_n^+ \sim u_n^-$. In that case we can in fact assume that u_n is a smooth map $X^{(T(n))} \to \mathrm{U}(1)$. The remainder of the proof when $L = 0$ (dealing with convergence over the ends) is now a repetition of the proof of Theorem 1.3.1.

We now turn to the case $L > 0$. For large n we must then have $\partial_t S_n(t) < 0$ for $|t| \leq T(n)$. Let δ be as in the proof of Theorem 1.3.1. We define $t_{n,1} \in (-T(n), T(n))$ implicitly for large n by

$$\vartheta(\beta_0) = \vartheta(S_n(t_{n,1})) + I_n + \delta.$$

Then $|t_{n,1} \pm T(n)| \to \infty$ as $n \to \infty$. As in the proof of Theorem 1.3.1 one sees that

$$\limsup_{n\to\infty} \left[\vartheta(S_n(-T(n)+\tau)) - \vartheta(S_n(t_{n,1}-\tau))\right] \to 0$$

as $\tau \to \infty$, and after passing to a subsequence we can find smooth $u_{n,1}\colon \mathbb{R}\times Y \to \mathrm{U}(1)$ such that $u_{n,1}(\mathcal{T}^*_{t_{n,1}} S_n)$ c-converges over $\mathbb{R}\times Y$ to a finite energy monopole S_1' in temporal gauge whose asymptotic limit at $-\infty$ is β_0. Let β_1 denote the asymptotic limit of S_1' at ∞. We now repeat the above process. The process ends when, after passing successively to subsequences and choosing $u_{n,j}, t_{n,j}, \beta_j$ for $j = 1, \dots, k$ one has that

$$\limsup_{n\to\infty} \left[\vartheta(S_n(t_{n,k}+\tau)) - \vartheta(S_n(T(n)-\tau))\right] \to 0$$

as $\tau \to \infty$. As in the case $L = 0$ one sees that β_k, β' must be gauge equivalent, and after modifying $u_n, u_{n,j}$ one can arrange that $\beta_k = \beta'$. This establishes chain-convergence over the neck. As in the case $L = 0$ we can in fact assume that u_n is a smooth map $X^{(T(n))} \to \mathrm{U}(1)$, and the rest of the proof when $L > 0$ is again a repetition of the proof of Theorem 1.3.1.

In the case of multiple necks one applies the above argument successively to each neck. In this case, too, after passing to a subsequence one ends up with smooth maps $u_n\colon X^{(T(n))} \to \mathrm{U}(1)$ such that $u_n(S_n)$ c-converges over X. One can then deal with convergence over the ends as before. $\square$

CHAPTER 8

Transversality

We will address two kinds of transversality problems: nondegeneracy of critical points of the Chern–Simons–Dirac functional and regularity of moduli spaces over 4–manifolds.

In this chapter we do not assume Condition (O1).

Recall that a subset of a topological space Z is called *residual* if it contains a countable intersection of dense open subsets of Z.

8.1 Nondegeneracy of critical points

Lemma 8.1.1 *Let Y be a closed, connected, Riemannian* spin^c *3–manifold and η any closed (smooth) 2–form on Y. Let G^* be the set of all $\nu \in \Omega^1(Y)$ such that all irreducible critical points of $\vartheta_{\eta+d\nu}$ are nondegenerate. Then $G^* \subset \Omega^1(Y)$ is residual, hence dense (with respect to the C^∞ topology).*

Proof The proof is a slight modification of the argument in [21]. For $2 \le k \le \infty$ and $\delta > 0$ let $W_{k,\delta}$ be the space of all 1–forms ν on Y of class C^k which satisfy $\|d\nu\|_{C^1} < \delta$. Let $W_{k,\delta}$ have the C^k topology. For $1 \le k < \infty$ we define a $\mathcal{G}$–equivariant smooth map

$$\Upsilon_k \colon \mathcal{C}^* \times L^2_1(Y; i\mathbb{R}) \times W_{k,\delta} \to L^2(Y; i\Lambda^1 \oplus \mathbb{S}),$$
$$(B, \Psi, \xi, \nu) \mapsto \mathcal{I}_\Psi \xi + \nabla\vartheta_{\eta+d\nu}(B, \Psi),$$

where $\mathcal{G}$ acts trivially on forms and by multiplication on spinors. If $\Upsilon_k(B, \Psi, \xi, \nu) = 0$ then

$$\|\mathcal{I}_\Psi \xi\|_2^2 = -\int_Y \langle \nabla\vartheta_{\eta+d\nu}(B, \Psi), \mathcal{I}_\Psi \xi \rangle = 0$$

by (3.2), which implies $\xi = 0$ since $\Psi \neq 0$. The derivative of Υ_k at a point $x = (B, \Psi, 0, \nu)$ is

$$D\Upsilon_k(x)(b, \psi, f, v) = H_{(B,\Psi)}(b, \psi) + \mathcal{I}_\Psi f + (i*dv, 0). \tag{8.1}$$

Let (B, Ψ) be any irreducible critical point of ϑ_η. We show that $P = D\Upsilon_k(B, \Psi, 0, 0)$ is surjective. Note that altering (B, Ψ) by an L^2_2 gauge transformation u has the effect of replacing P by uPu^{-1}. We may therefore assume that (B, Ψ) is smooth. Since $P_1 = \mathcal{I}_\Psi + H_{(B,\Psi)}$ has surjective symbol, the image of the induced operator $L^2_1 \to L^2$ is closed and has finite codimension. The same must then hold for $\mathrm{im}(P)$. Suppose $(b, \psi) \in L^2$ is orthogonal to $\mathrm{im}(P)$, ie $db = 0$ and $P_1^*(b, \psi) = 0$. The second equation implies that b and ψ are smooth, by elliptic regularity. Writing out the equations we find as in [21] that on the complement of $\Psi^{-1}(0)$ we have $-b = i\,dr$ for some smooth function $r\colon Y \setminus \Psi^{-1}(0) \to \mathbb{R}$. We now invoke a result of Bär [4] which says that, because B is smooth and $\Psi \not\equiv 0$, the equation $\partial_B \Psi = 0$ implies that the zero-set of Ψ is contained in a countable union of smooth 1–dimensional submanifolds of Y. In particular, any smooth loop in Y can be deformed slightly so that it misses $\Psi^{-1}(0)$. Hence b is exact. From Bär's theorem (or unique continuation for ∂_B, which holds when B is of class C^1; see Kazdan [28]) we also deduce that the complement of $\Psi^{-1}(0)$ is dense and connected. Therefore, f has a smooth extension to all of Y, and as in [21] this gives $(b, \psi) = 0$. Hence P is surjective.

Consider now the vector bundle

$$E = (\mathcal{C}^* \underset{\mathcal{G}}{\times} L^2(Y; i\Lambda^1 \oplus \mathbb{S})) \times L^2_1(Y; i\mathbb{R}) \to \mathcal{B}^* \times L^2_1(Y; i\mathbb{R}).$$

For $1 \le k < \infty$ the map Υ_k defines a smooth section $\sigma_{k,\delta}$ of the bundle

$$E \times W_{k,\delta} \to \mathcal{B}^* \times L^2_1(Y; i\mathbb{R}) \times W_{k,\delta}.$$

By the local slice theorem, a zero of Υ_k is a regular point of Υ_k if and only if the corresponding zero of $\sigma_{k,\delta}$ is regular. Since surjectivity is an open property for bounded operators between Banach spaces, a simple compactness argument shows that the zero-set of $\sigma_{2,\delta}$ is regular when $\delta > 0$ is sufficiently small. Fix such a δ. Observe that the question of whether the operator (8.1) is surjective for a given x is independent of k. Therefore, the zero-set $M_{k,\delta}$ of $\sigma_{k,\delta}$ is regular for $2 \le k < \infty$. In the remainder of the proof assume $k \ge 2$.

For any $\rho > 0$ let $\mathcal{B}_\rho$ be the set of elements $[B, \Psi] \in \mathcal{B}$ satisfying

$$\int_Y |\Psi| \ge \rho.$$

Define $M_{k,\delta,\rho} \subset M_{k,\delta}$ similarly. For any given ν, the formula for Υ_k defines a Fredholm section of E which we denote by σ_ν. Let $G_{k,\delta,\rho}$ be the set of those $\nu \in W_{k,\delta}$ such that σ_ν has only regular zeros in $\mathcal{B}_\rho \times \{0\}$. For $k < \infty$ let

$$\pi\colon M_{k,\delta} \to W_{k,\delta}$$

be the projection, and $\Sigma \subset M_{k,\delta}$ the closed subset consisting of all singular points of π. A compactness argument shows that π restricts to a closed map on $M_{k,\delta,\rho}$, hence

$$G_{k,\delta,\rho} = W_{k,\delta} \setminus \pi(M_{k,\delta,\rho} \cap \Sigma)$$

is open in $W_{k,\delta}$. On the other hand, applying the Sard–Smale theorem as in [15, Section 4.3] we see that $G_{k,\delta,\rho}$ is residual (hence dense) in $W_{k,\delta}$. Because $W_{\infty,\delta}$ is dense in $W_{k,\delta}$, we deduce that $G_{\infty,\delta,\rho}$ is open and dense in $W_{\infty,\delta}$. But then

$$\bigcap_{n\in\mathbb{N}} G_{\infty,\delta,1/n}$$

is residual in $W_{\infty,\delta}$, and this is the set of all $\nu \in W_{\infty,\delta}$ such that σ_ν has only regular zeros.

An irreducible critical point of $\vartheta_{\eta+d\nu}$ is nondegenerate if and only if the corresponding zero of σ_ν is regular. Thus we have proved that among all smooth 1–forms ν with $\|d\nu\|_{C^1} < \delta$, those ν for which all irreducible critical points of $\vartheta_{\eta+d\nu}$ are nondegenerate make up a residual subset in the C^∞ topology. The same must hold if η is replaced with $\eta + d\nu$ for any $\nu \in \Omega^1(Y)$, so we conclude that G^* is locally residual in $\Omega^1(Y)$, ie any point in G^* has a neighbourhood V such that $G^* \cap V$ is residual in V. Hence G^* is residual in $\Omega^1(Y)$. (This last implication holds if $\Omega^1(Y)$ is replaced with any second countable, regular space.) □

Lemma 8.1.2 *Let Y be a closed, connected, Riemannian* spin^c *3–manifold with $b_1(Y) = 0$, and let B be a spin connection over Y. Let $K \subset Y$ be a compact subset with nonempty interior and W the set of all smooth 1–forms on Y which are supported in K. Let G' be the set of all $\nu \in W$ such that* $\ker \partial_{B-i\nu} = 0$. *Then G' is open and dense in W with respect to the C^∞ topology.*

Proof For $k \geq 2$ let W_k denote the closure of W in the space of all 1–forms of class C^k on Y (with the C^k–topology). Consider the smooth map

$$\begin{aligned}\Upsilon_k\colon \big(L^2_1(Y;\mathbb{S})\backslash\{0\}\big) \times W_k \times \mathbb{R} &\to L^2(Y;\mathbb{S}),\\ (\phi,\nu,t) &\mapsto \partial_{B-i\nu}(\phi) + ti\phi.\end{aligned}$$

We first show that 0 is a regular value of Υ_k. Let $\Upsilon_k(\phi,\nu,t) = 0$. Because $\partial_{B-i\nu}$ is self-adjoint, we must have $t = 0$ and $\partial_{B-i\nu}(\phi) = 0$. Let P denote the derivative of Υ_k at $(\phi, \nu, 0)$ and P_j the partial derivative with the respect to the j–th variable, $j = 1, 2, 3$. Suppose $\psi \in L^2$ is orthogonal to the image of P. Since $\psi \perp \mathrm{im}(P_1)$, we have $\partial_{B-i\nu}(\psi) = 0$. By elliptic regularity, both ϕ and ψ are of class C^2. By unique continuation (see Kazdan [28]) there is a point y in the interior of K where ϕ does not vanish. Because $\psi \perp \mathrm{im}(P_2)$ we can express $\psi = ri\phi$ in a neighbourhood of y for some real function r. Then

$$0 = \partial_{B-i\nu}(\psi) = dr \cdot i\phi,$$

hence r is equal to a constant C in some neighbourhood U of y. But then $\psi - Ci\phi$ lies in the kernel of $\partial_{B-i\nu}$ and vanishes in U, so by unique continuation, $\psi = Ci\phi$ in Y. But $\psi \perp \mathrm{im}(P_3)$, so $C = 0$. This shows that 0 is a regular value of Υ_k as claimed.

For every $\nu \in W$ the map $\Upsilon_{k,\nu} := \Upsilon_k(\cdot, \nu, \cdot)$ is Fredholm of index 1. Let N_ν denote the kernel of $\partial_{B-i\nu}$ in L^2_1, and let G'_k be the set of those $\nu \in W_k$ for which $N_\nu = 0$. By the Sard–Smale theorem there is a residual set of ν's in W_k for which $\Upsilon^{-1}_{k,\nu}(0)$ (which we can identify with $N_\nu \backslash \{0\}$) is a smooth submanifold of real dimension 1. Since $\partial_{B-i\nu}$ is complex linear, this is only possible when $N_\nu = 0$. Thus, G'_k is residual in W_k. Since G'_k is obviously open in W_k, the lemma follows. $\square$

Proposition 8.1.1 *Let Y be a closed, connected, Riemannian* spin^c *3–manifold and η any closed 2–form on Y such that either $b_1(Y) = 0$ or $\tilde{\eta} \neq 0$. Let G be the set of all $\nu \in \Omega^1(Y)$ such that all critical points of $\vartheta_{\eta+d\nu}$ are nondegenerate. Then G is open and dense in $\Omega^1(Y)$ with respect to the C^∞ topology.*

Proof A compactness argument shows that G is open. If $b_1(Y) > 0$ then $\vartheta_{\eta+d\nu}$ has no reducible critical points and the proposition follows from Lemma 8.1.1.

Now suppose $b_1(Y) = 0$. Then we may assume $\eta = 0$. For any $\nu \in \Omega^1(Y)$ the functional $\vartheta_{d\nu}$ has up to gauge equivalence a unique reducible critical point, represented by $(B - i\nu, 0)$ for any spin connection B over Y with $\check{B}$ flat. This critical point is nondegenerate precisely when

$$\ker \partial_{B-i\nu} = 0,$$

which by Lemma 8.1.2 holds for an open, dense subset of ν's in $\Omega^1(Y)$. Now apply Lemma 8.1.1. $\square$

Marcolli [35] proved a weaker result in the case $b_1(Y) > 0$, allowing η to vary freely among the closed 2–forms.

8.2 Regularity of moduli spaces

The following lemma will provide us with suitable Banach spaces of perturbation forms.

Lemma 8.2.1 *Let X be a smooth n–manifold, $K \subset X$ a compact, codimension 0 submanifold, and $E \to X$ a vector bundle. Then there exists a separable Banach space W consisting of smooth sections of E supported in K, such that the following hold:*

(i) *The natural map $W \to \Gamma(E|_K)$ is continuous with respect to the C^∞ topology on $\Gamma(E|_K)$.*

(ii) *For every point $x \in \mathrm{int}(K)$ and every $v \in E_x$ there exists a section $s \in \Gamma(E)$ with $s(x) = v$ and a smooth embedding $g\colon \mathbb{R}^n \to X$ with $g(0) = x$ such that for arbitrarily small $\epsilon > 0$ there are elements of W of the form fs where $f\colon X \to [0,1]$ is a smooth function which vanishes outside $g(\mathbb{R}^n)$ and satisfies*

$$f(g(z)) = \begin{cases} 0, & |z| \geq 2\epsilon, \\ 1, & |z| \leq \epsilon. \end{cases}$$

Proof Fix connections in E and TX, and a Euclidean metric on E. For any sequence $a = (a_0, a_1, \dots)$ of positive real numbers and any $s \in \Gamma(E)$ set

$$\|s\|_a = \sum_{k=0}^{\infty} a_k \|\nabla^k s\|_\infty$$

and
$$W_a = \{s \in \Gamma(E) : \mathrm{supp}(s) \subset K,\ \|s\|_a < \infty\}.$$

Then $W = W_a$, equipped with the norm $\|\cdot\|_a$, clearly satisfies (i) for any a. We claim that one can choose a such that (ii) also holds. To see this, first observe that there is a finite dimensional subspace $V \subset \Gamma(E)$ such that

$$V \to E_x, \quad s \mapsto s(x)$$

is surjective for every $x \in K$. Fix a smooth function $b\colon \mathbb{R} \to [0,1]$ satisfying

$$b(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 4. \end{cases}$$

We use functions f that in local coordinates have the form

$$f_r(z) = b(r|z|^2),$$

where $r \gg 0$. Note that for each k there is a bound $\|f_r\|_{C^k} \leq \mathrm{const} \cdot r^k$ where the constant is independent of $r \geq 1$. It is now easy to see that a suitable sequence a can be found. □

In the next two propositions, $X, \vec{\alpha}, \mu$ will be as in Section 1.3. Let $K \subset X$ be any nonempty compact codimension 0 submanifold. Let W be a Banach space of smooth self-dual 2–forms on X supported in K, as provided by Lemma 8.2.1. The following proposition will be used in the proof of Theorem 1.1.2.

Proposition 8.2.1 *In the above situation, let G be set of all $\nu \in W$ such that all irreducible points of the moduli space $M(X; \vec{\alpha}; \mu + \nu; 0)$ are regular (here $\mathfrak{p}_j = 0$ for each j). Then $G \subset W$ is residual, hence dense.*

There is another version of this proposition where W is replaced with the Fréchet space of all smooth self-dual 2–forms on X supported in K, at least if one assumes that (O1) holds for each pair Y_j, η_j and that (A) holds for $X, \widetilde{\eta}_j, \lambda_j$. The reason for the extra assumptions is that the proof then seems to require global compactness results (cf the proof of Lemma 8.1.1).

Proof We may assume X is connected. Let $\widetilde{\Theta}$ be as in Section 3.4. Then

$$(S, \nu) \mapsto \widetilde{\Theta}(S, \mu + \nu, 0)$$

defines a smooth map

$$f \colon \mathcal{C}^* \times W \to L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-),$$

where $\mathcal{C}^* = \mathcal{C}^*(X; \vec{\alpha})$. We will show that 0 is a regular value of f. Suppose $f(S, \nu) = 0$ and write $S = (A, \Phi)$. We must show that the derivative $P = Df(S, \nu)$ is surjective. Because of the gauge equivariance of f we may assume that S is smooth. Let P_1 denote the derivative of $f(\cdot, \nu)$ at S. Since the image of P_1 in $L^{p,w}$ is closed and has finite codimension, the same holds for the image of P. Let p' be the exponent conjugate to p and suppose $(z, \psi) \in L^{p',-w}(X; i\Lambda^+ \oplus \mathbb{S}^-)$ is L^2 orthogonal to the image of P, ie

$$\int_X \langle P(a, \phi, \nu'), (z, \psi) \rangle = 0$$

for all $(a, \phi) \in L^{p,w}_1$ and $\nu' \in W$. Taking $\nu' = 0$ we see that $P_1^*(z, \psi) = 0$. Since P_1^* has injective symbol, z, ψ must be smooth. On the other hand, taking $a, \phi = 0$ and varying ν' we find that $z|_K = 0$ by choice of W. By assumption, Φ is not identically zero. Since $D_A \Phi = 0$, the unique continuation theorem in [28] applied to D_A^2 says that Φ cannot vanish in any nonempty open set. Hence Φ must be nonzero at some point x in the interior of K. Varying a alone near x one sees that ψ vanishes in some neighbourhood of x. But $P_1 P_1^*$ has the same symbol as $D_A^2 \oplus d^+(d^+)^*$, so another application of the same unique continuation theorem shows that $(z, \psi) = 0$. Hence P is surjective.

Consider now the vector bundle

$$E = \mathcal{C}^* \times_{\mathcal{G}} L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-)$$

over $\mathcal{B}^*$. The map f defines a smooth section σ of the bundle

$$E \times W \to \mathcal{B}^* \times W.$$

Because of the local slice theorem and the gauge equivariance of f, the fact that 0 is a regular value of f means precisely that σ is transverse to the zero-section. Since $\sigma(\cdot,\nu)$ is a Fredholm section of E for any ν, the proposition follows by another application of the Sard-Smale theorem. □

We will now establish transversality results for moduli spaces of the form $M(X,\vec{\alpha})$ or $M(\alpha,\beta)$ involving perturbations of the kind discussed in Section 3.3. For the time being we limit ourselves to the case where the 3–manifolds Y, Y_j are all rational homology spheres. We will use functions h_S that are a small modification of those in [21]. To define these, let Y be a closed Riemannian spin^c 3–manifold satisfying $b_1(Y)=0$, and ϑ the Chern–Simons–Dirac functional on Y defined by some closed 2–form η. Choose a smooth, nonnegative function $\chi\colon \mathbb{R}\to\mathbb{R}$ which is supported in the interval $(-1/4,1/4)$ and satisfies $\int\chi=1$. If S is any L^2_1 configuration over a band $(a-1/4,b+1/4)$ where $a\le b$ define the smooth function $\widetilde{\vartheta}_S\colon [a,b]\to\mathbb{R}$ by

$$\widetilde{\vartheta}_S(T) = \int_{\mathbb{R}} \chi(T-t)\vartheta(S_t)\,dt,$$

where we interpret the right hand side as an integral over $\mathbb{R}\times Y$. A simple exercise, using the Sobolev embedding theorem, shows that if $S_n\to S$ weakly in L^2_1 over $(a-1/4,b+1/4)\times Y$ then $\widetilde{\vartheta}_{S_n}\to\widetilde{\vartheta}_S$ in C^∞ over $[a,b]$.

Choose a smooth function $c\colon \mathbb{R}\to\mathbb{R}$ with the following properties:

- $c'>0$,
- c and all its derivatives are bounded,
- $c(t)=t$ for all critical values t of ϑ,

where c' is the derivative of c. The last condition is added only for convenience.

For any L^2_1 configuration S over $(a-1/2,b+1/2)\times Y$ with $a\le b$ define

$$h_S(t) = \int_{\mathbb{R}} \chi(t_1)c(\widetilde{\vartheta}_S(t-t_1))\,dt_1.$$

It is easy to verify that h_S satisfies the properties (P1)–(P3).

It remains to choose Ξ and $\mathfrak{P}$. Choose one compact subinterval (with nonempty interior) of each bounded connected component of $\mathbb{R} \setminus \mathrm{crit}(\vartheta)$, where $\mathrm{crit}(\vartheta)$ is the set of critical values of ϑ. Let Ξ be the union of these compact subintervals. Let $\mathfrak{P} = \mathfrak{P}_Y$ be a Banach space of 2–forms on $\mathbb{R} \times Y$ supported in $\Xi \times Y$ as provided by Lemma 8.2.1.

We now return to the situation described in the paragraph preceding Proposition 8.2.1. Let $W' \subset W$ be the open subset consisting of those elements ν that satisfy $\|\nu\|_{C^1} < 1$. Let Π_δ denote the set of all $\vec{\mathfrak{p}} = (\mathfrak{p}_1, \dots, \mathfrak{p}_r)$ where $\mathfrak{p}_j \in \mathfrak{P}_{Y_j}$ and $\|\mathfrak{p}_j\|_{C^1} < \delta$ for each j.

Proposition 8.2.2 *Suppose each Y_j is a rational homology sphere and $K \subset X_{:0}$. Then there exists a $\delta > 0$ such that the following holds. Let G be the set of all $(\nu, \vec{\mathfrak{p}}) \in W' \times \Pi_\delta$ such that every irreducible point of the moduli space $M(X; \vec{\alpha}; \mu + \nu; \vec{\mathfrak{p}})$ is regular. Then $G \subset W' \times \Pi_\delta$ is residual, hence dense.*

It seems necessary here to let $\vec{\mathfrak{p}}$ vary as well, since if any of the $\mathfrak{p}_j$ is nonzero then the linearization of the monopole map is no longer a differential operator, and it is not clear whether one can appeal to unique continuation as in the proof of Proposition 8.2.1.

Proof To simplify notation assume $r = 1$ and set $Y = Y_1$, $\alpha = \alpha_1$ etc. (The proof in the general case is similar.) Note that (A) is trivially satisfied, since each Y_j is a rational homology sphere. Therefore, by Propositions 4.2.2 and 5.4.2, if $\delta > 0$ is sufficiently small then for any $(\nu, \mathfrak{p}) \in W' \times \Pi_\delta$ and $[S] \in M(X; \alpha; \mu + \nu; \mathfrak{p})$ one has that either

(i) $[S_t] = \alpha$ for $t \geq 0$, or

(ii) $\partial_t \vartheta(S_t) < 0$ for $t \geq 0$.

As in the proof of Proposition 8.2.1 it suffices to prove that 0 is a regular value of the smooth map

$$\begin{aligned} \tilde{f}\colon \mathcal{C}^* \times W' \times \Pi_\delta &\to L^{p,w}, \\ (S, \nu, \mathfrak{p}) &\mapsto \tilde{\Theta}(S, \mu + \nu, \mathfrak{p}). \end{aligned}$$

The smoothness of the perturbation term $g(S, \mathfrak{p}) = \mathfrak{q} h_{S,\mathfrak{p}}$ follows from the smoothness of the map (3.9), for by (P1) there exist a t_0 and a neighbourhood $U \subset \mathcal{C}$ of S such that $h_{S'}(t) \notin \Xi$ for all $t > t_0$ and $S' \in U$.

Now suppose $\tilde{f}(S, \nu, \mathfrak{p}) = 0$ and $(z, \psi) \in L^{p', -w}$ is orthogonal to the image of $D\tilde{f}(S, \nu, \mathfrak{p})$. We will show that z is orthogonal to the image of $T = Dg(S, \mathfrak{p})$, or equivalently, that (z,ψ) is orthogonal to the image of $Df(S,\nu)$, where $f=\tilde{f}-g$ as before. The latter implies $(z,\psi)=0$ by the proof of Proposition 8.2.1.

Let $h_S\colon [1/2,\infty)\to\mathbb{R}$ be defined in terms of the restriction of S to $\mathbb{R}_+\times Y$. If $h_S(J)\subset\mathbb{R}\setminus\Xi$ for some compact interval J then by (P1) one has that $h_{S'}(J)\subset\mathbb{R}\setminus\Xi$ for all S' in some neighbourhood of S in $\mathcal{C}$. Therefore, all elements of $\mathrm{im}(T)$ vanish on $h_S^{-1}(\mathbb{R}\setminus\Xi)\times Y$.

We now digress to recall that if u is any locally integrable function on $\mathbb{R}^n$ then the complement of the Lebesgue set of u has measure zero, and if v is any continuous function on $\mathbb{R}^n$ then any Lebesgue point of u is also a Lebesgue point of uv. The notion of Lebesgue set also makes sense for sections τ of a vector bundle of finite rank over a finite dimensional smooth manifold M. In that case a point $x\in M$ is called a Lebesgue point of τ if it is a Lebesgue point in the usual sense for some (hence any) choice of local coordinates and local trivialization of the bundle around x.

Returning to our main discussion, there are now two cases: If (i) above holds then $h_S(t)=\vartheta(\alpha)\notin\Xi$ for $t\ge 1/2$, whence $T=0$ and we are done (recall the overall assumption $\mathfrak{q}^{-1}(0)=X_{:(3/2)}$ made in Section 3.4). Otherwise (ii) must hold. In that case we have $\partial_t c(\tilde{\vartheta}(t))<0$ for $t\ge 1/4$ and $\partial_t h_S(t)<0$ for $t\ge 1/2$. Since z is orthogonal to $\mathfrak{q}h_{S,\mathfrak{p}'}$ for all $\mathfrak{p}'\in\mathfrak{P}_Y$ we conclude that $z(t,y)=0$ for every Lebesgue point (t,y) of z with $t>3/2$ and $h_S(t)\in\mathrm{int}(\Xi)$. Since $h_S^{-1}(\partial\Xi)\cap(3/2,\infty)$ is a finite set, z must vanish almost everywhere in $[h_S^{-1}(\Xi)\cap(3/2,\infty)]\times Y$. Combining this with our earlier result we deduce that z is orthogonal to $\mathrm{im}(T)$. $\square$

In the next proposition (which is similar to [21, Proposition 5]) let Π_δ be as above with $r=1$, and set $Y=Y_1$.

Proposition 8.2.3 *In the situation of Section 1.2, suppose Y is a rational homology sphere and $\alpha,\beta\in\mathcal{R}_Y=\tilde{\mathcal{R}}_Y$. Then there exists a $\delta>0$ such that the following holds. Let G be the set of all $\mathfrak{p}\in\Pi_\delta$ such that every point in $M(\alpha,\beta;\mathfrak{p})$ is regular. Then $G\subset\Pi_\delta$ is residual, hence dense.*

Proof If $\alpha=\beta$ then an application of Proposition 4.2.2 shows that if $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then $M(\alpha,\beta;\mathfrak{p})$ consists of a single point represented by $\underline{\alpha}$, which is regular because α is nondegenerate.

If $\alpha\ne\beta$ and $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then for any $[S]\in M(\alpha,\beta;\mathfrak{p})$ one has $\partial_t\vartheta(S_t)<0$ for all t. Moreover, the moduli space contains no reducibles, since α,β cannot both be reducible. The proof now runs along the same lines as that of Proposition 8.2.2. Note that the choice of Ξ is now essential: it ensures that $\mathrm{im}(h_S)=(\vartheta(\alpha),\vartheta(\beta))$ contains interior points of Ξ. $\square$

CHAPTER 9

Proofs of Theorems 1.1.1 and 1.1.2

In these proofs we will only use genuine monopoles.

Proof of Theorem 1.1.1 We may assume Y is connected. Let η be a closed nonexact 2–form on Y which is the restriction of a closed form on Z. Let Y have a metric of positive scalar curvature. If $s \neq 0$ is a small real number then $\vartheta_{s\eta}$ will have no irreducible critical points, by the apriori estimate on the spinor fields and the positive scalar curvature assumption. If in addition $s[\eta] - \pi c_1(\mathcal{L}_Y) \neq 0$ then $\vartheta_{s\eta}$ will have no reducible critical points either.

Choose a spin^c Riemannian 4–manifold X as in Section 1.4, with $r = 1$, $r' = 0$, such that there exists a diffeomorphism $X^{\#} \to Z$ which maps $\{0\} \times Y_1$ isometrically onto Y. Let η_1 be the pullback of $s\eta$. Then (B1) is satisfied (but perhaps not (B2)), so it follows from Theorem 1.4.1 that $M(X^{(T)})$ is empty for $T \gg 0$. $\square$

We will now define an invariant h for closed spin^c 3–manifolds Y that satisfy $b_1(Y) = 0$ and admit metrics with positive scalar curvature. Let g be such a metric on Y. Recall that for the unperturbed Chern–Simons–Dirac functional ϑ the space $\mathcal{R}_Y$ of critical points modulo gauge equivalence consists of a single point θ, which is reducible. Let $(B, 0)$ be a representative for θ. Let $Y_1, \dots, Y_r$ be the connected components of Y and choose a spin^c Riemannian 4–manifold X with tubular ends $\mathbb{R}_+ \times Y_j$, $j = 1, \dots, r$ (in the sense of Section 1.3) and a smooth spin connection A over X such that the restriction of $\check{A}$ to $\mathbb{R}_+ \times Y$ is equal to the pullback of $\check{B}$. (The notation here is explained in Section 3.1.) Define

$$\begin{aligned} h(Y, g) &= \mathrm{ind}_{\mathbb{C}}(D_A) - \frac{1}{8}(c_1(\mathcal{L}_X)^2 - \sigma(X)) \\ &= \frac{1}{2}(\dim M(X;\theta) - d(X) + b_0(X)), \end{aligned}$$

where $D_A\colon L^2_1 \to L^2$, "dim" is the expected dimension, and $d(X)$ is the quantity defined in Section 1.1. Since $\mathrm{ind}_{\mathbb{C}}(D_A) = (1/8)(c_1(\mathcal{L}_X)^2 - \sigma(X))$ when X is closed, it follows easily from the addition formula for the index (see Corollary C.0.1) that $h(Y,g)$ is independent of X and that

$$h(-Y,g) = -h(Y,g).$$

Clearly,

$$h(Y,g) = \sum_j h(Y_j, g_j),$$

where g_j is the restriction of g to Y_j. To show that $h(Y,g)$ is independent of g we may therefore assume Y is connected. Suppose g' is another positive scalar curvature metric on Y and consider the spin^c Riemannian manifold $X = \mathbb{R}\times Y$ where the metric agrees with $1\times g$ on $(-\infty,-1]\times Y$ and with $1\times g'$ on $[1,\infty)\times Y$. As we will prove later (see Lemma 14.2.2 below) the moduli space $M(X;\theta,\theta)$ must have negative odd dimension. Thus,

$$h(Y,g') + h(-Y,g) = \frac{1}{2}(\dim\ M(X;\theta,\theta)+1) \le 0.$$

This shows $h(Y) = h(Y,g)$ is independent of g.

Proof of Theorem 1.1.2 Let each Y_j have a positive scalar curvature metric. Choose a spin^c Riemannian 4–manifold X as in Section 1.4, with $r' = 0$ and with the same r, such that there exists a diffeomorphism $f\colon X^{\#} \to Z$ which maps $\{0\}\times Y_j$ isometrically onto Y_j. Then (B1) is satisfied (but perhaps not (B2)). Let X_0 be the component of X such that $W = f((X_0)_{:1})$. For each j set $\eta_j = 0$ and let $\alpha_j \in \mathcal{R}_{Y_j}$ be the unique (reducible) critical point. Choose a reference connection A_o as in Section 3.4 and set $A_0 = A_o|_{X_0}$. Since each α_j has representatives of the form $(B,0)$ where $\check{B}$ is flat it follows that $\hat{F}(A_o)$ is compactly supported. In the following, μ will denote the (compactly supported) perturbation 2–form on X and μ_0 its restriction to X_0.

Let $\mathcal{H}^+$ be the space of self-dual closed L^2 2–forms on X_0. Then $\dim \mathcal{H}^+ = b^+(X_0) > 0$, so $\mathcal{H}^+$ contains a nonzero element z. By unique continuation for harmonic forms we can find a smooth 2–form μ_0 on X_0, supported in any given small ball, such that $\hat{F}^+(A_0) + i\mu_0^+$ is not L^2 orthogonal to z. (Here $\hat{F}^+$ is the self-dual part of $\hat{F}$.) Then

$$\hat{F}^+(A_0) + i\mu_0^+ \notin \mathrm{im}(d^+\colon L^{p,w}_1 \to L^{p,w}),$$

where w is the weight function used in the definition of the configuration space. Hence $M(X_0;\vec{\alpha})$ contains no reducible monopoles. After perturbing μ_0 in a small ball we can arrange that $M(X_0;\vec{\alpha})$ is transversally cut out as well, by Proposition 8.2.1.

To prove (i), recall that

$$\dim\ M(X_0;\vec{\alpha}) = d(W) - 1 + 2\sum_j h(Y_j), \tag{9.1}$$

so the inequality in (i) simply says that

$$\dim\ M(X_0;\vec{\alpha}) < 0,$$

hence $M(X_0;\vec{\alpha})$ is empty. Since there are no other moduli spaces over X_0, it follows from Theorem 1.4.1 that $M(X^{(T)})$ is empty when $T_{\min} \gg 0$.

We will now prove (ii). If $M(X^{(T)})$ has odd or negative dimension then there is nothing to prove, so suppose this dimension is $2m \geq 0$. Since $M(X_0;\vec{\alpha})$ contains no reducibles we deduce from Theorem 1.4.1 that $M(X^{(T)})$ is also free of reducibles when $T_{\min}$ is sufficiently large. Let $\mathbf{B} \subset X_0$ be a compact 4–ball and $\mathcal{B}^*(\mathbf{B})$ the Banach manifold of irreducible L^p_1 configurations over $\mathbf{B}$ modulo L^p_2 gauge transformations. Here $p > 4$ should be an even integer to ensure the existence of smooth partitions of unity. Let $\mathbb{L} \to \mathcal{B}^*(\mathbf{B})$ be the natural complex line bundle associated to some base-point in $\mathbf{B}$, and s a generic section of the m–fold direct sum $m\mathbb{L}$. For $T_{\min} \gg 0$ let

$$S^{(T)} \subset M(X^{(T)}), \qquad S_0 \subset M(X_0;\vec{\alpha})$$

be the subsets consisting of those elements ω that satisfy $s(\omega|_{\mathbf{B}}) = 0$. By assumption, S_0 is a submanifold of codimension $2m$. For any T for which $S^{(T)}$ is transversely cut out the Seiberg–Witten invariant of Z is equal to the number of points in $S^{(T)}$ counted with sign. Now, the inequality in (ii) is equivalent to

$$d(W) + 2\sum_j h(Y_j) < d(Z) = 2m + 1,$$

which by (9.1) gives

$$\dim\ S_0 = \dim\ M(X_0;\vec{\alpha}) - 2m < 0.$$

Therefore, S_0 is empty. By Theorem 1.4.1, $S^{(T)}$ is empty too when $T_{\min} \gg 0$, hence $\mathrm{SW}(Z) = 0$. □

Part II

Gluing theory

There are many different hypotheses under which one can consider the gluing problem. Here we will not aim at the utmost generality, but rather give an expository account of gluing in what might be called the favourable cases. More precisely, we will glue precompact families of regular monopoles over 4–manifolds with tubular ends, under similar general assumptions as in Part I. Although obstructed gluing is not discussed explicitly, we will show in Part III how the parametrized version of our gluing theorem can be used to handle one kind of gluing obstructions.

One source of difficulty when formulating a gluing theorem is that gluing maps are in general not canonical, but rather depend on various choices hidden in their construction. We have therefore chosen to express our gluing theorem as a statement about an *ungluing map*, which is explicitly defined in terms of data that appear naturally in applications.

If X is a 4–manifold with tubular ends and $X^{(T)}$ the glued manifold as in Section 1.4, then the first component of the ungluing map involves restricting monopoles over $X^{(T)}$ to some fixed compact subset $K \subset X$ (which may also be regarded as a subset of $X^{(T)}$ when each T_j is large). In the case of gluing along a reducible critical point, the ungluing map has an additional component which reads off the $\mathrm{U}(1)$ gluing parameter by measuring the holonomy along a path running once through the corresponding neck in $X^{(T)}$.

Ungluing maps of a different kind were studied already in Donaldson [11] and Freed–Uhlenbeck [20] but later authors have mostly formulated gluing theorems in terms of gluing maps, usually without characterizing these maps uniquely.

The proof of the gluing theorem is divided into two parts: surjectivity and injectivity of the ungluing map. In the first part the (quantitative) inverse function theorem is used to construct a smooth local right inverse $\widehat{\zeta}$ of an "extended monopole map" $\widehat{\Xi}$. In the second part the inverse function theorem is applied a second time to show, essentially, that the image of $\widehat{\zeta}$ is not too small. There are many similarities with the proof of the gluing theorem in [15], but also some differences. For instance, we do not use the method of continuity, and we handle gluing parameters differently.

It may be worth mentioning that the proof does not depend on unique continuation for monopoles (only for harmonic spinors), as we do not know whether solutions to our perturbed monopole equations satisfy any such property. (Unique continuation for genuine monopoles was used in Proposition 4.2.2 in the discussion of perturbations, but this has little to do with gluing theory.) Therefore, in the injectivity part of the proof, we argue by contradiction, restricting monopoles to ever larger subsets $\widetilde{K} \subset X$. This is also reflected in the statement of the theorem, which would have been somewhat simpler if unique continuation were available.

In Chapter 12 we give a detailed account of orientations of moduli spaces, using Benevieri–Furi's concept of orientations of Fredholm operators of index 0 [8]. This seems simpler to us than the standard approach using determinant line bundles (see [11; 15; 45; 14]). Our main result here, Theorem 12.4.1, says that ungluing maps are orientation preserving. The length of this chapter is much due to the fact that we allow gluing along reducible critical points and that we work with (multi)framed moduli spaces (as a means of handling reducibles over the 4–manifolds).

There is now a large literature on gluing theory for instantons and monopoles. The theory was introduced by Taubes [46; 47], who used it to obtain existence results for self-dual connections over closed 4–manifolds. It was further developed in seminal work of Donaldson [11]; see also Freed–Uhlenbeck [20]. General gluing theorems for instantons over connected sums were proved by Donaldson [12] and Donaldson–Kronheimer [15]. In the setting of instanton Floer theory there is a highly readable account in [14]; see also Floer [19] and Fukaya [24]. Gluing with degenerate asymptotic limits was studied by Morgan–Mrowka [39]; part of their work was adapted to the context of monopoles by Safari [43]. Nicolaescu [41] established gluing theorems for monopoles in certain situations, including one involving gluing obstructions. Marcolli–Wang [36] discuss gluing theory in connection with monopole Floer homology. For monopoles over closed 3–manifolds split along certain tori, see Chen [9]. Product formulae for Seiberg–Witten invariants of 4–manifolds split along a circle times a surface of genus g were established by Morgan–Szabó–Taubes [40] (for $g > 1$) and by Taubes [48] (for $g = 1$). Gluing theory is a key ingredient in a large programme of Feehan–Leness [16] for proving Witten's conjecture relating Donaldson and Seiberg–Witten invariants. Gluing theory in the context of blown-up moduli spaces was developed by Kronheimer–Mrowka [31].

CHAPTER 10

The gluing theorem

10.1 Statement of theorem

Consider the situation of Section 1.4, but without assuming any of the conditions (B1), (B2), (C). We now assume that every component of X contains an end $\mathbb{R}_+ \times Y_j$ or $\mathbb{R}_+ \times (-Y_j)$ (ie an end that is being glued). Fix nondegenerate monopoles α_j over Y_j and α'_j over Y'_j. (These should be smooth configurations rather than gauge equivalence classes of such.) Suppose α_j is reducible for $1 \le j \le r_0$ and irreducible for $r_0 < j \le r$, where $0 \le r_0 \le r$. We consider monopoles over X and $X^{(T)}$ that are asymptotic to α'_j over $\mathbb{R}_+ \times Y'_j$ and (in the case of X) asymptotic to α_j over $\mathbb{R}_+ \times (\pm Y_j)$. These monopoles build moduli spaces

$$M_{\mathfrak{b}} = M_{\mathfrak{b}}(X; \vec{\alpha}, \vec{\alpha}, \vec{\alpha}'), \qquad M_{\mathfrak{b}}^{(T)} = M_{\mathfrak{b}}(X^{(T)}; \vec{\alpha}').$$

Here $\mathfrak{b} \subset X$ is a finite subset to be specified in a moment, and the subscript indicates that we only divide out by those gauge transformations that restrict to the identity on $\mathfrak{b}$; see Section 3.4. The ungluing map $\mathbf{f}$ will be a diffeomorphism between certain open subsets of $M_{\mathfrak{b}}^{(T)}$ and $M_{\mathfrak{b}}$ when

$$T_{\min} := \min(T_1, \dots, T_r)$$

is large.

When gluing along the critical point α_j, the stabilizer of α_j in $\mathcal{G}_{Y_j}$ appears as a "gluing parameter". This stabilizer is a copy of $\mathrm{U}(1)$ if α_j is reducible and trivial otherwise. When α_j is reducible we will read off the gluing parameter by means of the holonomy of the connection part of the glued monopole along a path γ_j in $X^{(T)}$ which runs once through the neck $[-T_j, T_j]\times Y_j$. To make this precise, for $1\le j\le r_0$ fix $y_j\in Y_j$ and smooth paths

$$\gamma_j^\pm\colon [-1,\infty)\to X$$

such that $\gamma_j^\pm(t)=\iota_j^\pm(t,y_j)$ for $t\ge 0$ and $\gamma_j^\pm([-1,0])\subset X_{:0}$. Let $\mathfrak{b}$ denote the collection of all the start-points $o_j^\pm:=\gamma_j^\pm(-1)$. (We do not assume that these are distinct.) Note in passing that we then have

$$M^*_{\mathfrak{b}}=M_{\mathfrak{b}}.$$

Define the smooth path

$$\gamma_j\colon I_j=[-T_j-1,T_j+1]\to X^{(T)}$$

by
$$\gamma_j(t)=\begin{cases}\pi_T\gamma_j^+(T_j+t), & -T_j-1\le t<T_j,\\ \pi_T\gamma_j^-(T_j-t), & -T_j<t\le T_j+1,\end{cases}$$

where $\pi_T\colon X^{\{T\}}\to X^{(T)}$ is as in Section 1.4.

Choose a reference configuration $S_o=(A_o,\Phi_o)$ over X with limits α_j, α'_j over $\mathbb{R}_+\times(\pm Y_j)$, $\mathbb{R}\times Y'_j$, resp. Let $S'_o=(A'_o,\Phi'_o)$ denote the reference configuration over $X^{(T)}$ obtained from S_o in the obvious way when gluing the ends. Precisely speaking, S'_o is the unique smooth configuration over $X^{(T)}$ which agrees with S_o over $\mathrm{int}(X_{:T})$ (which can also be regarded as a subset of X).

If $P\to X^{(T)}$ temporarily denotes the principal $\mathrm{Spin}^c(4)$–bundle defining the spin^c structure, then the holonomy of a spin^c connection A in P along γ_j is a $\mathrm{Spin}^c(4)$–equivariant map

$$\mathrm{hol}_{\gamma_j}(A)\colon P_{o_j^+}\to P_{o_j^-}.$$

Because A and A'_o map to the same connection in the tangent bundle of $X^{(T)}$, there is a unique element $\mathrm{Hol}_j(A)$ in $\mathrm{U}(1)$ (identified with the kernel of $\mathrm{Spin}^c(4)\to\mathrm{SO}(4)$) such that

$$\mathrm{hol}_{\gamma_j}(A)=\mathrm{Hol}_j(A)\cdot\mathrm{hol}_{\gamma_j}(A'_o). \tag{10.1}$$

Explicitly
$$\mathrm{Hol}_j(A)=\exp\left(-\int_{I_j}\gamma_j^*(A-A'_o)\right),$$

where as usual $A-A'_o$ is regarded as an imaginary valued 1–form on $X^{(T)}$. For gauge transformations $u\colon X^{(T)}\to\mathrm{U}(1)$ we have

$$\mathrm{Hol}_j(u(A))=u(o_j^-)\cdot\mathrm{Hol}_j(A)\cdot u(o_j^+)^{-1}. \tag{10.2}$$

In particular, there is a natural smooth map

$$\mathrm{Hol}\colon M_{\mathfrak{b}}^{(T)} \to \mathrm{U}(1)^{r_0}, \quad [A,\Phi] \mapsto (\mathrm{Hol}_1(A),\dots,\mathrm{Hol}_{r_0}(A))$$

which is equivariant with respect to the appropriate action of

$$\mathbb{T} := \mathrm{Map}(\mathfrak{b}, \mathrm{U}(1)) \approx \mathrm{U}(1)^b,$$

where $b = |\mathfrak{b}|$.

Consider for the moment an arbitrary compact codimension 0 submanifold $K \subset X$ containing $\mathfrak{b}$. Let $D^{(T)}$ be the subgroup of $H^1(X^{(T)};\mathbb{Z})$ consisting of those classes whose restriction to each Y_j' is zero. Let D_K be the cokernel of the restriction map $D^{(T)} \to H^1(K;\mathbb{Z})$. Here $T_{\min}$ should be so large that K may be regarded as a subset of $X^{(T)}$, and D_K is then obviously independent of T. In the following we use the L_1^p configuration spaces etc introduced in Section 2.5. Let $\breve{\mathcal{G}}_{\mathfrak{b}}(K)$ be the kernel of the (surjective) group homomorphism

$$\mathcal{G}(K) \to \mathbb{T} \times D_K, \quad u \mapsto (u|_{\mathfrak{b}}, [u]),$$

where $[u]$ denotes the image in D_K of the homotopy class of u regarded as an element of $H^1(K;\mathbb{Z})$. Set

$$\breve{\mathcal{B}}_{\mathfrak{b}}(K) = \mathcal{C}(K)/\breve{\mathcal{G}}_{\mathfrak{b}}(K), \qquad \breve{\mathcal{B}}^*_{\mathfrak{b}}(K) = \mathcal{C}^*_{\mathfrak{b}}(K)/\breve{\mathcal{G}}_{\mathfrak{b}}(K).$$

On both these spaces there is a natural action of $\mathbb{T} \times D_K$. Note that D_K acts freely and properly discontinuously on the (Hausdorff) Banach manifold $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$ with quotient $\mathcal{B}^*_{\mathfrak{b}}(K)$.

It is convenient here to agree once and for all that the Sobolev exponent $p > 4$ is to be an even integer. This ensures that our configuration spaces admit smooth partitions of unity, which are needed in Sections 11.1 and 10.4 (but not in the proof of Theorem 10.1.1).

Fix a $\mathbb{T}$–invariant open subset $G \subset M_{\mathfrak{b}}$ whose closure $\overline{G}$ is compact and contains only regular points. (Of course, G is the preimage of an open set G' in M, but G' may not be a smooth manifold due to reducibles and we therefore prefer to work with G.)

Definition 10.1.1 By a *kv-pair* we mean a pair (K,V) where

- $K \subset X$ is a compact codimension 0 submanifold which contains $\mathfrak{b}$ and intersects every component of X,
- $V \subset \breve{\mathcal{B}}_{\mathfrak{b}}(K)$ is a $\mathbb{T}$–invariant open subset containing $R_K(\overline{G})$, where R_K denotes restriction to K.

We define a partial ordering $\le$ on the set of all kv-pairs, by decreeing that

$$(K', V') \le (K, V)$$

if and only if $K \subset K'$ and $R_K(V') \subset V$.

Now fix a kv-pair (K, V) which satisfies the following two additional assumptions: firstly, that $V \subset \breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$; secondly, that if X_e is any component of X which contains a point from $\mathfrak{b}$ then $X_e \cap K$ is connected. The second condition ensures that the image of $R_K\colon M_{\mathfrak{b}} = M^*_{\mathfrak{b}} \to \breve{\mathcal{B}}_{\mathfrak{b}}(K)$ lies in $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$.

Suppose we are given a $\mathbb{T}$–equivariant smooth map

$$q\colon V \to M_{\mathfrak{b}} \tag{10.3}$$

such that $q(\omega|_K) = \omega$ for all $\omega \in \bar{G}$. (If $\mathbb{T}$ acts freely on $\bar{G}$ then such a map always exists when K is sufficiently large; see Section 10.4. In concrete applications there is often a natural choice of q; see Sections 11.1–11.2.)

Let $X^\#$ and the forms $\tilde{\eta}_j, \tilde{\eta}'_j$ be as in Section 1.4, and choose $\lambda_j, \lambda'_j > 0$.

Theorem 10.1.1 *Suppose there is class in $H^2(X^\#)$ whose restrictions to Y_j and Y'_j are $\lambda_j \tilde{\eta}_j$ and $\lambda'_j \tilde{\eta}'_j$, respectively, and suppose the perturbation parameters $\vec{\mathfrak{p}}, \vec{\mathfrak{p}}'$ are admissible for $\vec{\alpha}'$. Then there exists a kv-pair $(\tilde{K}, \tilde{V}) \le (K, V)$ such that if (K', V') is any kv-pair $\le (\tilde{K}, \tilde{V})$ then the following holds when $T_{\min}$ is sufficiently large. Set*

$$H^{(T)} := \left\{\omega \in M^{(T)}_{\mathfrak{b}} \; : \; \omega|_{K'} \in V'\right\},$$
$$\mathbf{q}\colon H^{(T)} \to M_{\mathfrak{b}}, \quad \omega \mapsto q(\omega|_K).$$

Then $\mathbf{q}^{-1}G$ consists only of regular monopoles (hence is a smooth manifold), and the $\mathbb{T}$–equivariant map $\mathbf{f} := \mathbf{q} \times Hol$ restricts to a diffeomorphism

$$\mathbf{q}^{-1}G \to G \times \mathrm{U}(1)^{r_0}.$$

Remarks (1) When $T_{\min}$ is large then $K' \subset X$ can also be regarded as a subset of $X^{(T)}$, in which case the expression $\omega|_{K'}$ in the definition of $H^{(T)}$ makes sense.

(2) Except for the equivariance of $\mathbf{f}$, the theorem remains true if one leaves out all assumptions on $\mathbb{T}$–invariance resp. –equivariance on G and q, and on V in Definition 10.1.1, above. However, it is hard to imagine any application that would not require equivariance of $\mathbf{f}$.

(3) The theorem remains true if one replaces $\breve{\mathcal{B}}_{\mathfrak{b}}(K)$ and $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$ by $\mathcal{B}_{\mathfrak{b}}(K)$ and $\mathcal{B}^*_{\mathfrak{b}}(K)$ above. However, working with $\breve{\mathcal{B}}$ gives more flexibility in the construction of maps q; see Section 11.2.

(4) Concerning admissibility of perturbation parameters, see the remarks after Theorem 1.4.1. Note that the assumption on $\lambda_j\tilde{\eta}_j$ and $\lambda'_j\tilde{\eta}'_j$ in the theorem above is weaker than either of the conditions (B1) and (B2) in Section 1.4. However, in practice the gluing theorem is only useful in conjunction with a compactness theorem, so one may still have to assume (B1) or (B2).

The proof of Theorem 13.3.1 has two parts. The first part consists in showing that $\mathbf{f}$ has a smooth local right inverse around every point in $\overline{G}\times\mathrm{U}(1)^{r_0}$ (Proposition 10.2.1 below). In the second part we will prove that $\mathbf{f}$ is injective on $\mathbf{q}^{-1}\overline{G}$. (Proposition 10.3.1 below).

10.2 Surjectivity

The next two sections are devoted to the proof of Theorem 10.1.1. Both parts of the proof make use of the same set-up, which we now introduce.

We first choose weight functions for our Sobolev spaces over X and $X^{(T)}$. Let $\sigma_j,\sigma'_j\ge 0$ be small constants and $w\colon X\to\mathbb{R}$ a smooth function which is equal to $\sigma_j t$ on $\mathbb{R}_+\times(\pm Y_j)$ and equal to $\sigma'_j t$ on $\mathbb{R}_+\times Y'_j$. As usual, we require $\sigma_j>0$ if α_j is reducible (ie for $j=1,\dots,r_0$), and similarly for σ'_j. For $j=1,\dots,r$ choose a smooth function $w_j\colon\mathbb{R}\to\mathbb{R}$ such that $w_j(t)=-\sigma_j|t|$ for $|t|\ge 1$. We will always assume $T_{\min}\ge 4$, in which case we can define a weight function $\kappa\colon X^{(T)}\to\mathbb{R}$ by

$$\begin{aligned}&\kappa=w\quad\text{on } X^{(T)}\setminus\textstyle\bigcup_j[-T_j,T_j]\times Y_j,\\&\kappa(t,y)=\sigma_jT_j+w_j(t)\quad\text{for }(t,y)\in[-T_j,T_j]\times Y_j.\end{aligned}$$

Let $\mathcal{C}$ denote the $L^{p,w}_1$ configuration space over X defined by the reference configuration S_o, and let $\mathcal{C}'$ denote the $L^{p,\kappa}_1$ configuration space over $X^{(T)}$ defined by S'_o. Let $\mathcal{G}_{\mathfrak{b}},\mathcal{G}'_{\mathfrak{b}}$ be the corresponding groups of gauge transformations and $\mathcal{B}_{\mathfrak{b}},\mathcal{B}'_{\mathfrak{b}}$ the corresponding orbit spaces.

Now fix $(\omega_0,z)\in\overline{G}\times\mathrm{U}(1)^{r_0}$. Our immediate goal is to construct a smooth local right inverse of $\mathbf{f}$ around this point, but the following set-up will also be used in the injectivity part of the proof.

Choose a smooth representative $S_0\in\mathcal{C}$ for ω_0 which is in temporal gauge over the ends of X. (This assumption is made in order to ensure exponential decay of S_0.) Set $d=\dim M_{\mathfrak{b}}$ and let $\pi\colon\mathcal{C}\to\mathcal{B}_{\mathfrak{b}}$ be the projection. By the local slice theorem we can find a smooth map

$$\mathbf{S}\colon\mathbb{R}^d\to\mathcal{C}$$

such that $\mathbf{S}(0) = S_0$ and such that $\varpi := \pi \circ \mathbf{S}$ is a diffeomorphism onto an open subset of $M_{\mathfrak{b}}$.

We will require one more property of $\mathbf{S}$, involving holonomy. If $a \in L^{p,w}_1(X; i\mathbb{R})$ then we define $\mathrm{Hol}^{\pm}_j(A_o + a) \in \mathrm{U}(1)$ by

$$\mathrm{Hol}^{\pm}_j(A_o + a) = \exp\left(-\int_{[-1,\infty)} (\gamma^{\pm}_j)^* a\right).$$

The integral exists because, by the Sobolev embedding $L^p_1 \subset C^0_B$ in $\mathbb{R}^4$ for $p > 4$, we have

$$\|e^w a\|_\infty \le C\|e^w a\|_{L^p_1} = C\|a\|_{L^{p,w}_1} \tag{10.4}$$

for some constant C. It is clear that $\mathrm{Hol}^{\pm}_j$ is a smooth function on $\mathcal{C}$. Because any smooth map $\mathbb{R}^d \to \mathrm{U}(1)$ factors through exp: $\mathbb{R}i \to \mathrm{U}(1)$, we can arrange, after perhaps modifying $\mathbf{S}$ by a smooth family of gauge transformations that are all equal to 1 outside the ends $\mathbb{R}_+ \times Y_j$ and constant on $[1,\infty) \times Y_j$, that

$$\mathrm{Hol}^{+}_j(\mathbf{S}(v)) \cdot (\mathrm{Hol}^{-}_j(\mathbf{S}(v)))^{-1} = z_j \tag{10.5}$$

for $j = 1, \dots, r_0$ and every $v \in \mathbb{R}^d$. Here $\mathrm{Hol}^{\pm}_j(\mathbf{S}(v))$ denotes the holonomy, as defined above, of the connection part of the configuration $\mathbf{S}(v)$, and the z_j are the coordinates of z.

Lemma 10.2.1 *Let E, F, G be Banach spaces, S: $E \to F$ a bounded operator and T: $E \to G$ a surjective bounded operator such that*

$$S + T\colon E \to F \oplus G, \quad x \mapsto (Sx, Tx)$$

is Fredholm. Then T has a bounded right inverse.

Proof Because $S + T$ is Fredholm there is a bounded operator A: $F \oplus G \to E$ such that $(S+T)A - I$ is compact. Set $A(x,y) = A_1 x + A_2 y$ for $(x,y) \in F \oplus G$. Then

$$TA_2 - I\colon G \to G$$

is compact, hence TA_2 is Fredholm of index 0. Using the surjectivity of T and the fact that any closed subspace of finite dimension or codimension in a Banach space is complemented, it is easy to see that there is a bounded operator K: $G \to E$ (with finite-dimensional image) such that $T(A_2 + K)$ is an isomorphism. □

Let $$\Theta\colon \mathcal{C} \to L^{p,w}$$

be the Seiberg–Witten map over X. By assumption, every point in $\overline{G}$ is regular, so in particular ω_0 is regular, which means that $D\Theta(S_0)\colon L_1^{p,w} \to L^{p,w}$ is surjective. Let Φ be the spinor part of S_0 and define $\mathcal{I}_\Phi$ as in Section 2.3. Then

$$\mathcal{I}_\Phi^* + D\Theta(S_0)\colon L_1^{p,w} \to L^{p,w}$$

is Fredholm, so by Lemma 10.2.1 $D\Theta(S_0)$ has a bounded right inverse Q. (This can also be deduced from Proposition 2.3.1 (ii).)

Let $\mathsf{r}\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\mathsf{r}(t) = 1$ for $t \le 0$ and $\mathsf{r}(t) = 0$ for $t \ge 1$. For $\tau \ge 1$ set $\mathsf{r}_\tau(t) = \mathsf{r}(t - \tau)$ and let $S_{v,\tau}$ be the configuration over X which agrees with $\mathbf{S}(v)$ away from the ends $\mathbb{R}_+ \times (\pm Y_j)$ and satisfies

$$S_{v,\tau} = (1 - \mathsf{r}_\tau)\underline{\alpha}_j + \mathsf{r}_\tau \mathbf{S}(v)$$

over $\mathbb{R}_+ \times (\pm Y_j)$. Here $\underline{\alpha}_j$ denotes, as before, the translationary invariant monopole over $\mathbb{R} \times Y_j$ determined by α_j. For each v we have

$$\|S_{v,\tau} - \mathbf{S}(v)\|_{L_1^{p,w}} \to 0 \qquad \text{as } \tau \to \infty.$$

Therefore, when τ is sufficiently large, the operator

$$D\Theta(S_{0,\tau}) \circ Q\colon L^{p,w} \to L^{p,w}$$

will be invertible, and we set

$$Q_\tau = Q(D\Theta(S_{0,\tau}) \circ Q)^{-1}\colon L^{p,w} \to L_1^{p,w},$$

which is then a right inverse of $D\Theta(S_{0,\tau})$. It is clear that the operator norm $\|Q_\tau - Q\| \to 0$ as $\tau \to \infty$.

For the remainder of the proof of Theorem 10.1.1, the term "constant" will always refer to a quantity that is independent of τ, T, unless otherwise indicated. The symbols $C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ will each denote at most one constant, while other symbols may denote different constants in different contexts.

Consider the configuration space

$$\mathcal{C}_j = \underline{\alpha}_j + L_1^{p,-w_j}$$

over $\mathbb{R} \times Y_j$ and the Seiberg–Witten map

$$\Theta_j\colon \mathcal{C}_j \to L^{p,-w_j}.$$

As explained in Section 3.4 there is an identification

$$\mathcal{I}_{\underline{\alpha}}^* + D\Theta_j(\underline{\alpha}_j) = \frac{d}{dt} + P_\alpha.$$

By the results of [14] the operator on the right hand side defines a Fredholm operator $L^{p,w_j}_1 \to L^{p,w_j}$, and this must be surjective because of the choice of weight function w_j. In particular,

$$D\Theta_j(\underline{\alpha}_j)\colon L^{p,w_j}_1 \to L^{p,w_j} \tag{10.6}$$

is surjective, hence has a bounded right inverse P_j by Lemma 10.2.1. (Here one cannot appeal to Proposition 2.3.1 (ii).)

Let $\Theta'\colon \mathcal{C}' \to L^{p,\kappa}$ be the Seiberg–Witten map over $X^{(T)}$. When $T_{\min} > \tau + 1$ then by splicing $S_{v,\tau}$ in the natural way one obtains a smooth configuration $S_{v,\tau,T}$ over $X^{(T)}$. There is a constant $C_0 \gg 0$ such that if

$$T_{\min} > \tau + C_0 \tag{10.7}$$

then we can splice the right inverses Q_τ and $P_1, \dots, P_r$ to obtain a right inverse $Q_{\tau,T}$ of

$$D\Theta'(S_{0,\tau,T})\colon L^{p,\kappa}_1 \to L^{p,\kappa}$$

which satisfies

$$\|Q_{\tau,T}\| \le C\left(\|Q_\tau\| + \sum_j \|P_j\|\right)$$

for some constant C; see Appendix C. Since $\|Q_\tau\|$ is bounded in τ (ie as a function of τ), we see that $\|Q_{\tau,T}\|$ is bounded in τ, T.

The inequality (10.7) will be assumed from now on.

We now introduce certain 1–forms that will be added to the configurations $S_{v,\tau,T}$ in order to make small changes to the holonomies Hol_j. For any $c = (c_1, \dots, c_{r_0}) \in \mathbb{R}^{r_0}$ define the 1–form $\theta_{c,\tau}$ over $X^{(T)}$ by

$$\theta_{c,\tau} = \begin{cases} 0 & \text{outside } \bigcup_{j=1}^{r_0} [-T_j, T_j] \times Y_j, \\ i c_j \mathrm{r}'_{\tau+1-T_j}\, dt & \text{on } [-T_j, T_j] \times Y_j, \quad j = 1, \dots, r_0, \end{cases}$$

where $\mathrm{r}'_s(t) = \frac{d}{dt}\mathrm{r}_s(t)$. Set

$$E = \mathbb{R}^d \times \mathbb{R}^{r_0} \times L^{p,\kappa}(X^{(T)}; i\Lambda^+ \oplus \mathbb{S}^-).$$

For $0 < \epsilon < 1$ let $B_\epsilon \subset E$ be the open ϵ–ball about 0. Define a smooth map $\zeta\colon E \to \mathcal{C}'$ by

$$\zeta(v, c, \xi) = S_{v,\tau,T} + \theta_{c,\tau} + Q_{\tau,T}\xi, \tag{10.8}$$

where $\theta_{c,\tau}$ is added to the connection part of $S_{v,\tau,T}$.

When deciding where to add the perturbation 1–form $\theta_{c,\tau}$ one has to balance two concerns. One the one hand, because the weight function κ increases exponentially as one approaches the middle of the necks $[-T_j, T_j]\times Y_j$, $j = 1, \dots, r_0$, it is desirable to add $\theta_{c,\tau}$ as close to the boundaries of these necks as possible. On the other hand, in order for Lemma 10.2.4 below to work, the spinor field of $S_{v,\tau,T}$ needs to be "small" in the perturbation region. We have chosen to add $\theta_{c,\tau}$ at the negative end of the cutoff region, where the spinor field is zero.

Although we will sometimes use the notation $\zeta(x)$, we shall think of ζ as a function of three variables v, c, ξ, and $D_j\zeta$ will denote the derivative of ζ with respect to the j–th variable. Similarly for other functions on (subsets of) E that we will define later. Set

$$\sigma = \max(\sigma_1, \dots, \sigma_r).$$

Notice that if $r_0 = 0$, ie if we are not gluing along any reducible critical point, then we may take $\sigma = 0$.

Lemma 10.2.2 *There exists a constant $C_1 > 0$ such that for $x \in E$ the following hold:*

(i) $\|D_1\zeta(x)\|, \|D^2\zeta(x)\| < C_1$ *if* $\|x\| < 1$.

(ii) $\|D_2\zeta(x)\| < C_1 e^{\sigma\tau}$.

(iii) $\|D_3\zeta(x)\| < C_1$.

Proof To prove (ii), note that if $r_0 > 0$ and $c = (c_1, \dots, c_{r_0})$ then

$$\left\| \frac{\partial\zeta(v,c,\xi)}{\partial c_j} \right\|_{L^{p,\kappa}_1} = \mathrm{const}\cdot e^{\sigma_j\tau}.$$

The other two statements are left to the reader. □

Let $\mathcal{C}'_1$ be the set of all $S \in \mathcal{C}'$ such that $[S|_K] \in V$, $q(S|_K) \in \varpi(\mathbb{R}^d)$ and $\mathrm{Hol}_j(S) \neq -z_j$ for $j = 1, \dots, r_0$. Then $\mathcal{C}'_1$ is an open subset of $\mathcal{C}'$, and there are unique smooth functions

$$\eta_j \colon \mathcal{C}'_1 \to (-\pi, \pi)$$

such that $\mathrm{Hol}_j(S) = z_j \exp(i\eta_j(S))$. Set $\eta = (\eta_1, \dots, \eta_{r_0})$ and define

$$\widehat{\Xi} = (\varpi^{-1} \circ q \circ R_K, \eta, \Theta') \colon \mathcal{C}'_1 \to E.$$

A crucial point in the proof of Theorem 10.1.1 will be the construction of a smooth local right inverse of $\widehat{\Xi}$, defined in a neighbourhood of 0. The map ζ is a first approximation to such a local right inverse. The construction of a genuine local right inverse will involve an application of the quantitative inverse function theorem (see Lemma 10.2.7 below).

From now on we will take τ so large that $K \subset X_{:\tau}$ and

$$\mathrm{Hol}_j^+(S_{0,\tau}) \cdot (\mathrm{Hol}_j^-(S_{0,\tau}))^{-1} \neq -z_j$$

for $j = 1, \dots, r_0$. Note that the left hand side of this equation is equal to $\mathrm{Hol}_j(S_{0,\tau,T})$ whenever $T_j > \tau + 1$. There is then a constant $\epsilon > 0$ such that $\zeta(B_\epsilon) \subset \mathcal{C}'_1$, in which case we have a composite map

$$\Xi = \widehat{\Xi} \circ \zeta \colon B_\epsilon \to E.$$

Choose $\lambda > 0$ so that none of the operators $\widetilde{H}_{\alpha_j}$ ($j = 1, \dots, r$) and $\widetilde{H}_{\alpha'_j}$ ($j = 1, \dots, r'$) has any eigenvalue of absolute value $\leq \lambda$. (The notation $\widetilde{H}_\alpha$ was introduced in Section 6.1.) Recall that we assume the σ_j are small and nonnegative, so in particular we may assume $6\sigma < \lambda$.

Lemma 10.2.3 *There is a constant $C_2 < \infty$ such that*

$$\|\Xi(0)\| \leq C_2 e^{(\sigma-\lambda)\tau}.$$

Proof The first two components of $\Xi(0) = \widehat{\Xi}(S_{0,\tau,T})$ are in fact zero: the first one because $S_{0,\tau,T} = S_0$ over K, the second one because the dt–component of

$$S_{0,\tau} - S_0 = (1 - \mathrm{r}_\tau)(\underline{\alpha}_j - S_0)$$

vanishes on $[1,\infty) \times (\pm Y_j)$ since S_0 and $\underline{\alpha}_j$ are both in temporal gauge there.

The third component of $\Xi(0)$ is $\Theta'(S_{0,\tau,T})$. It suffices to consider τ so large that the $\mathfrak{p}$–perturbations do not contribute to $\Theta'(S_{0,\tau,T})$, which then vanishes outside the two bands of length 1 in $[-T_j, T_j] \times Y_j$ centred at $t = \pm(T_j - \tau - 1/2)$, $j = 1, \dots, r$. Our exponential decay results say that for every $k \geq 0$ there is a constant C'_k such that for every $(t, y) \in \mathbb{R}_+ \times (\pm Y_j)$ we have

$$|\nabla^k(S_0 - \underline{\alpha}_j)|_{(t,y)} \leq C'_k e^{-\lambda t}.$$

Consequently,

$$\|\Theta'(S_{0,\tau,T})\|_\infty \leq \mathrm{const} \cdot (e^{-\lambda\tau} + e^{-2\lambda\tau}) \leq \mathrm{const} \cdot e^{-\lambda\tau}.$$

This yields

$$\|\Xi(0)\| = \|\Theta'(S_{0,\tau,T})\|_{L^{p,\kappa}} \leq \mathrm{const} \cdot e^{(\sigma-\lambda)\tau}. \qquad \square$$

Lemma 10.2.4 *There is a constant $C_3 < \infty$ such that for sufficiently large τ the following hold:*

(i) $\|D\Xi(0)\| \le C_3$.

(ii) $D\Xi(0)$ *is invertible and* $\|D\Xi(0)^{-1}\| \le C_3$.

Proof By construction, the derivative of Ξ at 0 has the form

$$D\Xi(0) = \begin{pmatrix} I & 0 & \beta_1 \\ \delta_2 & I & \beta_2 \\ \delta_3 & 0 & I \end{pmatrix},$$

where the k–th column is the k–th partial derivative and I the identity map.

The middle top entry in the above matrix is zero because $\theta_{c,\tau}$ vanishes on K. The middle bottom entry is zero because $S_{0,\tau,T} = \underline{\alpha}_j$ on the support of $\theta_{c,\tau}$ and the spinor field of $\underline{\alpha}_j$ is zero (for $j = 1,\dots,r_0$). Adding $\theta_{c,\tau}$ to $S_{0,\tau,T}$ therefore has the effect of altering the latter by a gauge transformation over $[-T_j+\tau+1, -T_j+\tau+2]\times Y_j$, $j = 1,\dots,r_0$.

We claim that β_k is bounded in τ, T for $k = 1,2$. For $k = 1$ this is obvious from the boundedness of $Q_{\tau,T}$. For $k = 2$ note that the derivative of $\eta_j\colon \mathcal{C}'_1 \to (-\pi,\pi)$ at any $S \in \mathcal{C}'_1$ is

$$D\eta_j(S)(a,\phi) = i\int_{I_j} \gamma_j^* a \tag{10.9}$$

where a is an imaginary valued 1–form and ϕ a positive spinor. Because of the weights used in the Sobolev norms, $D\eta(S)$ is (independent of S and) bounded in τ, T (see (10.4)). This together with the bound on $Q_{\tau,T}$ gives the desired bound on β_2.

Note that, for $k = 2,3$, $\|\delta_k\|$ is independent of T when $\tau \gg 0$, and routine calculations show that $\|\delta_k\| \to 0$ as $\tau \to \infty$. (In the case of δ_2 this depends on the normalization (10.5) of the holonomy of $\mathbf{S}(v)$.)

Write $D\Xi(0) = x - y$, where

$$x = \begin{pmatrix} I & 0 & \beta_1 \\ 0 & I & \beta_2 \\ 0 & 0 & I \end{pmatrix}, \qquad x^{-1} = \begin{pmatrix} I & 0 & -\beta_1 \\ 0 & I & -\beta_2 \\ 0 & 0 & I \end{pmatrix}.$$

When τ is so large that $\|y\|\,\|x^{-1}\| < 1$ then of course $\|yx^{-1}\| \le \|y\|\,\|x^{-1}\| < 1$, hence $x - y = (I - yx^{-1})x$ is invertible. Moreover,

$$(x-y)^{-1} - x^{-1} = x^{-1}[(I - yx^{-1})^{-1} - I] = x^{-1}\sum_{k=1}^{\infty}(yx^{-1})^k,$$

which gives

$$\|(x-y)^{-1}-x^{-1}\| \le \frac{\|x^{-1}\|^2\,\|y\|}{1-\|x^{-1}\|\,\|y\|} \to 0 \quad \text{as } \tau\to\infty. \qquad \square$$

We now record some basic facts that will be used in the proof of Lemma 10.2.6 below.

Lemma 10.2.5 *If E_1, E_2, E_3 are Banach spaces, $U_j \subset E_j$ an open set for $j=1,2$, and $f\colon U_1 \to U_2$, $g\colon U_2 \to E_3$ smooth maps then the second derivate of the composite map $g\circ f\colon U_1 \to E_3$ is given by*

$$\begin{aligned} D^2(g\circ f)(x)(y,z) &= D^2 g(f(x))(Df(x)y, Df(x)z) \\ &\quad + Dg(f(x))(D^2 f(x)(y,z)) \end{aligned}$$

for $x\in U_1$ and $y,z\in E_1$.

Proof Elementary. $\square$

It is also worth noting that embedding and multiplication theorems for L^q_k Sobolev spaces on $\mathbb{R}^4$ ($k\ge 0$, $1\le q<\infty$) carry over to $X^{(T)}$, and that the embedding and multiplication constants are bounded functions of T.

Furthermore, a differential operator of degree d over $X^{(T)}$ which is translationary invariant over necks and ends induces a bounded operator $L^q_{k+d} \to L^q_k$ whose operator norm is a bounded function of T.

Lemma 10.2.6 *There is a constant $C_4 > 0$ such that $\|D^2\Xi(x)\| \le C_4$ whenever $\|x\| \le C_4^{-1}$ and $\tau \ge C_4$.*

Proof We will say a quantity depending on x,τ is *s-bounded* if the lemma holds with this quantity in place of $D^2\Xi$.

Let Ξ_1, Ξ_2, Ξ_3 be the components of Ξ.

The assumption $K\subset X_{:\tau}$ ensures that $\Xi_1(v,c,\xi)$ is independent of c. It then follows from Lemma 10.2.5 and the bound on $Q_{\tau,T}$ that $D^2\Xi_1$ is s-bounded.

When v,c,ξ are small we have

$$\Xi_2(v,c,\xi) = \eta(S_{v,\tau,T} + \theta_{c,\tau} + Q_{\tau,T}\xi) = c + \eta(S_{v,\tau,T} + Q_{\tau,T}\xi).$$

Since $D\eta$ is constant, as noted above, we have $D^2\eta = 0$. From the bounds on $D\eta$ and $Q_{\tau,T}$ we then deduce that $D^2\Xi_2$ is s-bounded.

To estimate Ξ_3, we fix $h \gg 0$ and consider only $\tau \geq h$. It is easy to see that

$$\Xi_3(x)|_{X_{:h}}$$

is s-bounded. By restricting to small x and choosing h large we may arrange that the $\mathfrak{p}$–perturbations do not contribute to

$$\Xi_{3,j} := \Xi_3|_{[-T_j+h,T_j-h]\times Y_j}$$

for $j = 1,\dots,r$. We need to show that each $D^2\Xi_{3,j}$ is s-bounded, but to simplify notation we will instead prove the same for $D^2\Xi_3$ under the assumption that the $\mathfrak{p}$ perturbations are zero.

First observe that for any configuration (A,Φ) over $X^{(T)}$ and any closed, imaginary valued 1–form a we have

$$\Theta'(A+a,\Phi) = \Theta'(A,\Phi) + (0, a\cdot\Phi).$$

Moreover,

$$\|a\cdot\Phi\|_{L^{p,\kappa}} = C\|a\cdot e^{\kappa}\Phi\|_p \leq C\|a\|_{2p}\,\|e^{\kappa}\Phi\|_{2p} \leq C'\|a\|_{2p}\,\|\Phi\|_{L^{p,\kappa}_1}$$

for some constants $C, C' < \infty$. Taking $(A,\Phi) = S_{v,\tau,T} + Q_{\tau,T}\xi$ and $a = \theta_{c,\tau}$ we see that $D_k D_2 \Xi_3(x)$ is s-bounded for $k = 1,2,3$.

Next note that the derivative of the Seiberg–Witten map $\Theta'\colon \mathcal{C}' \to L^{p,\kappa}$ at a point $S'_o + s_1$ has the form

$$D\Theta'(S'_o + s_1)s_2 = Ls_2 + B(s_1,s_2)$$

where B is a pointwise bilinear operator, and L a first order operator which is independent of s_1 and translationary invariant over necks and ends. This yields

$$\|D\Theta'(S'_o + s)\| \leq \mathrm{const}\cdot(1+\|s\|_{2p}). \tag{10.10}$$

Moreover, $D^2\Theta'(S) = B$ for all S, hence there is a constant $C'' < \infty$ such that

$$\|D^2\Theta'(S)\| \leq C''$$

for all T.

Combining the above results on Θ' with Lemma 10.2.5 we see that $D_j D_k \Xi_3$ is s-bounded also when $j,k \neq 2$. $\square$

Lemma 10.2.7 *There exist $c_5 > 0$ and $C_6 < \infty$ such that if $0 < \epsilon' < c_5\epsilon < c_5^2$ then for sufficiently large τ the following hold:*

(i) *$\Xi\colon B_\epsilon \to E$ is injective.*

(ii) *There is a (unique) smooth map* $\Xi^{-1}\colon B_{\epsilon'} \to B_\epsilon$ *such that* $\Xi \circ \Xi^{-1} = I$.

(iii) $\|D(\Xi^{-1})(x)\| \le C_6$ *for all* $x \in B_{\epsilon'}$.

(iv) $\|D^2(\Xi^{-1})(x)\| \le C_6$ *for all* $x \in B_{\epsilon'}$.

(v) $\|\Xi^{-1}(0)\| \le C_6 e^{(\sigma-\lambda)\tau}$.

Proof For sufficiently large τ we have

$$\epsilon' + \|\Xi(0)\| < c_5\epsilon$$

by Lemma 10.2.3. Statements (i)–(iv) now follow from the inverse function theorem, Proposition B.0.2, applied to the function $x \mapsto \Xi(x) - \Xi(0)$, together with Lemmas 10.2.4 and 10.2.6. To prove (v), set $h = \Xi^{-1}$, $x = \Xi(0)$ and take τ so large that $x \in B_{\epsilon'}$. Since Ξ is injective on B_ϵ we must have $h(x) = 0$, so

$$\|h(0)\| = \|h(x) - h(0)\| \le \|x\| \sup_{\|y\|\le\epsilon'} \|Dh(y)\|.$$

Now (v) follows from (iii) and Lemma 10.2.3. □

From now on we assume that $\epsilon, \epsilon', \tau$ are chosen so that the conclusions of the lemma are satisfied. Define

$$\hat\zeta = \zeta \circ \Xi^{-1}\colon B_{\epsilon'} \to \mathcal{C}_1'.$$

Then clearly

$$\widehat{\Xi} \circ \hat\zeta = I.$$

Thus, $(v,c) \mapsto \hat\zeta(v,c,0)$ is a "gluing map", ie for small v, c it solves the problem of gluing the monopole $\mathbf{S}(v)$ over X to get a monopole over $X^{(T)}$ with prescribed holonomy $z_j e^{ic_j}$ along the path γ_j for $j = 1, \dots, r_0$.

Lemma 10.2.8 *There is a constant* $C_7 < \infty$ *such that for* $x \in B_{\epsilon'}$ *one has*

$$\|D\hat\zeta(x)\|, \|D^2\hat\zeta(x)\| \le C_7 e^{\sigma\tau}.$$

Proof This follows from Lemma 10.2.2 and Lemma 10.2.7 and the chain rule. □

The following proposition refers to the situation of Section 10.1 and uses the notation of Theorem 10.1.1.

Proposition 10.2.1 *If* (K', V') *is any kv-pair* $\le (K, V)$ *then* $\bar G \times \mathrm{U}(1)^{r_0}$ *can be covered by finitely many connected open sets* W *in* $M_{\mathfrak{b}} \times \mathrm{U}(1)^{r_0}$ *such that if* $T_{\min}$ *is sufficiently large then for each* W *there exists a smooth map* $\mathbf{h}\colon W \to H^{(T)}$ *whose image consists only of regular points and which satisfies* $\mathbf{f} \circ \mathbf{h} = I$.

Here we do not need any assumptions on $\tilde{\eta}_j, \tilde{\eta}'_j$ or on $\mathfrak{p}_j, \mathfrak{p}'_j$.

Proof Let $(\omega_0, z) \in \bar{G} \times \mathrm{U}(1)^{r_0}$ and consider the set-up above, with τ so large that $K' \subset X_{:\tau}$ and ϵ so small that

$$\zeta(x)|_{K'} \in V' \qquad \text{for every } x \in B_\epsilon. \tag{10.11}$$

Note that taking ϵ small may require taking τ (and hence $T_{\min}$) large; see Lemma 10.2.7. For any sufficiently small open neighbourhood $W \subset M_{\mathfrak{b}} \times \mathrm{U}(1)^{r_0}$ of (ω_0, z) we can define a smooth map $\nu\colon W \to \mathcal{C}'_1$ by the formula

$$\nu(\omega, a) = \hat{\zeta}(\varpi^{-1}(\omega), -i\log(a/z), 0).$$

Here $\log e^u = u$ for any complex number u with $|\mathrm{Im}\, u| < \pi$, and $i\log(a/z) \in \mathbb{R}^{r_0}$ denotes the vector whose j–th component is $i\log(a_j/z_j)$. Because $\hat{\Xi} \circ \hat{\zeta} = I$ and the Seiberg–Witten map is the third component of $\hat{\Xi}$, the image of ν consists of regular monopoles. Let $\mathbf{h}\colon W \to \mathcal{B}'_{\mathfrak{b}}$ be the composition of ν with the projection $\mathcal{C}' \to \mathcal{B}'_{\mathfrak{b}}$. Unravelling the definitions involved and using (10.11) one finds that $\mathbf{h}$ has the required properties.

How large $T_{\min}$ must be for this to work might depend on (ω_0, z). But $\bar{G} \times \mathrm{U}(1)^{r_0}$ is compact, hence it can be covered by finitely many such open sets W. If $T_{\min}$ is sufficiently large then the above construction will work for each of these W. □

10.3 Injectivity

We now continue the discussion that was interrupted by Proposition 10.2.1. Set

$$\tilde{S} = S_{0,\tau,T}, \quad \hat{S} = \hat{\zeta}(0).$$

Lemma 10.3.1 *There is a constant $C_8 < \infty$ such that for sufficiently large τ one has*

$$\|\hat{S} - \tilde{S}\|_{L^{p,\kappa}_1} \le C_8 e^{(2\sigma-\lambda)\tau}, \quad \|\hat{S} - S'_o\|_{L^{p,\kappa}_1} \le C_8.$$

Proof Set $\Xi^{-1}(0) = (v, c, \xi) \in B_\epsilon$. For sufficiently large τ we have

$$\begin{aligned}
\|\hat{S} - \tilde{S}\|_{L^{p,\kappa}_1} &\le \|S_{v,\tau,T} - S_{0,\tau,T}\|_{L^{p,\kappa}_1} + \|\theta_{c,\tau} + Q_{\tau,T}\xi\|_{L^{p,\kappa}_1} \\
&\le \mathrm{const}\cdot(\|v\| + e^{\sigma\tau}\|c\| + \|\xi\|) \\
&\le \mathrm{const}\cdot e^{(2\sigma-\lambda)\tau},
\end{aligned}$$

where we used Lemma 10.2.7 (v) to obtain the last inequality. Because $\|\tilde{S} - S'_o\|_{L^{p,\kappa}_1}$ is bounded in τ, T, and we assume $6\sigma < \lambda$, the second inequality of the lemma follows as well. □

For positive spinors Φ on $X^{(T)}$ it is convenient to extend the definition of $\mathcal{I}_\Phi$ to complex valued functions on $X^{(T)}$:

$$\mathcal{I}_\Phi f=(-df,f\Phi).$$

(However, $\mathcal{I}_\Phi^*$ will always refer to the formal adjoint of $\mathcal{I}_\Phi$ acting on imaginary valued functions.) When Φ is the spinor part of $S'_o,\widetilde{S},\widehat{S}$ then the corresponding operators $\mathcal{I}_\Phi$ will be denoted $\mathcal{I}'_o,\widetilde{\mathcal{I}},\mathcal{I}$, respectively. (We omit the ˆ on $\mathcal{I}$ to simplify notation.) As in Section 2.2 we define

$$\mathcal{E}'=\{f\in L^p_{2,\mathrm{loc}}(X^{(T)};\mathbb{C}):\mathcal{I}'_of\in L^{p,\kappa}_1\}.$$

We can take the norm to be

$$\|f\|_{\mathcal{E}'}=\|\mathcal{I}'_of\|_{L^{p,\kappa}_1}+\sum_{x\in\mathfrak{b}}|f(x)|.$$

Lemma 10.3.2 *There is a constant $C_9<\infty$ such that if $\mathbf{I}$ is any of the operators $\mathcal{I}'_o,\widetilde{\mathcal{I}},\mathcal{I}$ then for all $f\in\mathcal{E}'$ one has*

$$\|f\|_\infty\le C_9\Bigl(\|\mathbf{I}f\|_{L^{p,\kappa}}+\sum_{x\in\mathfrak{b}}|f(x)|\Bigr).$$

Proof We first prove the inequality for $\mathbf{I}=\widetilde{\mathcal{I}}$ (the case of $\mathcal{I}'_o$ is similar, or easier). If X_e is any component of X and $0\le\overline{\tau}\le\tau$ then for some constant $C_{\overline{\tau}}<\infty$ one has

$$\|f\|_\infty\le\mathrm{const}\cdot\|f\|_{L^p_1}\le C_{\overline{\tau}}\Bigl(\|\widetilde{\mathcal{I}}f\|_p+\sum_{x\in\mathfrak{b}\cap X_e}|f(x)|\Bigr)$$

for all L^p_1 functions $f\colon(X_e)_{:\overline{\tau}}\to\mathbb{C}$. Here the Sobolev inequality holds because $p>4$, whereas the second inequality follows from Lemma 2.2.1. We use part (i) of that lemma if the spinor field of S_0 is not identically zero on X_e, and part (ii) otherwise. (In the latter case $\mathfrak{b}\cap X_e$ is nonempty.)

When $\overline{\tau},\tau$ are sufficiently large we can apply part (i) of the same lemma in a similar fashion to the band $[t,t+1]\times Y'_j$ provided $t\ge\overline{\tau}$ and α'_j is irreducible, and to the band $[t-1,t+1]\times Y_j$ provided $|t|\le T_j-\overline{\tau}-1$ and α_j is irreducible. To estimate $|f|$ over these bands when α'_j resp. α_j is reducible one can use Lemma 2.2.2 (ii). This proves the lemma for $\mathbf{I}=\widetilde{\mathcal{I}}$ (and for $\mathbf{I}=\mathcal{I}'_o$).

We now turn to the case $\mathbf{I}=\mathcal{I}$. Let ϕ denote the spinor part of $\widehat{S}-\widetilde{S}$. Then

$$\begin{aligned}\|\widetilde{\mathcal{I}}f\|_{L^{p,\kappa}} &\le \|\mathcal{I}f\|_{L^{p,\kappa}}+\|f\phi\|_{L^{p,\kappa}}\\ &\le \|\mathcal{I}f\|_{L^{p,\kappa}}+\text{const}\cdot\left(\|\widetilde{\mathcal{I}}f\|_{L^{p,\kappa}}+\sum_{x\in\mathfrak{b}}|f(x)|\right)\cdot\|\phi\|_{L^{p,\kappa}}.\end{aligned}$$

By Lemma 10.3.1 we have $\|\phi\|_{L^{p,\kappa}_1}\to 0$ as $\tau\to 0$, so for sufficiently large τ we get

$$\|\widetilde{\mathcal{I}}f\|_{L^{p,\kappa}}\le\text{const}\cdot\left(\|\mathcal{I}f\|_{L^{p,\kappa}}+\sum_{x\in\mathfrak{b}}|f(x)|\right).$$

Therefore, the lemma holds with $\mathbf{I}=\mathcal{I}$ as well. □

Lemma 10.3.3 *There is a constant $C_{10}<\infty$ such that for all $f,g\in\mathcal{E}'$ and $\phi\in L^{p,\kappa}_1(X^{(T)};\mathbb{S}^+)$ one has*

(i) $\|fg\|\le C_{10}\|f\|\,\|g\|$,

(ii) $\|f\phi\|\le C_{10}\|f\|\,\|\phi\|$,

where we use the $L^{p,\kappa}_1$ norm on spinors and the $\mathcal{E}'$ norm on elements of $\mathcal{E}'$.

Proof By routine calculation using Lemma 10.3.2 with $\mathbf{I}=\mathcal{I}'_o$ one easily proves (ii) and the inequality

$$\|d(fg)\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\|g\|_{\mathcal{E}'}.$$

Now observe that by definition $g\Phi'_o\in L^{p,\kappa}_1$, where as before Φ'_o denotes the spinor field of the reference configuration S'_o. Applying (ii) we then obtain

$$\|fg\Phi'_o\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\,\|g\Phi'_o\|_{L^{p,\kappa}_1}\le\text{const}\cdot\|f\|_{\mathcal{E}'}\,\|g\|_{\mathcal{E}'},$$

completing the proof of (i). □

Recall from Section 2.4 that the Lie algebra $L\mathcal{G}'_{\mathfrak{b}}$ is the space of imaginary valued functions in $\mathcal{E}'$ that vanish on $\mathfrak{b}$.

Lemma 10.3.4 *There is a constant $C_{11}>0$ such that for $\tau>C_{11}$ and all $f\in L\mathcal{G}'_{\mathfrak{b}}$ one has*

$$C_{11}^{-1}\|\mathcal{I}'_of\|_{L^{p,\kappa}_1}\le\|\mathcal{I}f\|_{L^{p,\kappa}_1}\le C_{11}\|\mathcal{I}'_of\|_{L^{p,\kappa}_1}.$$

Proof Let ψ denote the spinor part of $\widehat{S}-S'_o$. Then

$$\begin{aligned}\|f\psi\|_{L^{p,\kappa}_1}&\le\text{const}\cdot(\|f\|_\infty\|\psi\|_{L^{p,\kappa}_1}+\|df\|_{L^{2p,\kappa}}\|\psi\|_{2p})\\ &\le\text{const}\cdot\|\mathcal{I}f\|_{L^{p,\kappa}_1}\|\psi\|_{L^{p,\kappa}_1},\end{aligned}$$

and similarly with $\mathcal{I}'_o$ instead of $\mathcal{I}$. The lemma now follows from Lemma 10.3.1. □

We are going to use the inverse function theorem a second time, to show that the image of the smooth map

$$\begin{aligned} \Pi\colon\ L\mathcal{G}'_{\mathfrak{b}} \times B_{\epsilon'} &\to \mathcal{C}'_1, \\ (f,x) &\mapsto \exp(f)(\hat{\zeta}(x)) \end{aligned}$$

contains a "not too small" neighbourhood of $\hat{S}$. The derivative of Π at $(0,0)$ is

$$\begin{aligned} D\Pi(0,0)\colon\ L\mathcal{G}'_{\mathfrak{b}} \oplus E &\to L^{p,\kappa}_1, \\ (f,x) &\mapsto \mathcal{I}f + D\hat{\zeta}(0)x. \end{aligned}$$

To be concrete, let $L\mathcal{G}'_{\mathfrak{b}} \oplus E$ have the norm $\|(f,x)\| = \|f\|_{\mathcal{E}'} + \|x\|_E$.

Lemma 10.3.5 $D\Pi(0,0)$ *is a linear homeomorphism.*

Proof By Proposition 2.3.1, $\mathcal{I}^*\mathcal{I}\colon L\mathcal{G}'_{\mathfrak{b}} \to L^{p,\kappa}$ is a Fredholm operator with the same kernel as $\mathcal{I}$. Now, $\mathcal{I}$ is injective on $L\mathcal{G}'_{\mathfrak{b}}$, because $\hat{\zeta}$ maps into $\mathcal{C}'_1$ and therefore $[\hat{S}|_K] \in V \subset \mathcal{B}^*_{\mathfrak{b}}$. Since

$$W = \mathcal{I}^*\mathcal{I}(L\mathcal{G}'_{\mathfrak{b}})$$

is a closed subspace of $L^{p,\kappa}$ of finite codimension, we can choose a bounded operator

$$\pi\colon\ L^{p,\kappa} \to W$$

such that $\pi|_W = I$. Set

$$\mathcal{I}^\# = \pi\mathcal{I}^*\colon\ L^{p,\kappa}_1 \to W.$$

Then

$$\mathcal{I}^\#\mathcal{I}\colon\ L\mathcal{G}'_{\mathfrak{b}} \to W$$

is an isomorphism. Furthermore,

$$\operatorname{index}(\mathcal{I}^\# + D\Theta'(\hat{S})) = \dim\ M^{(T)}_{\mathfrak{b}} = \dim M_{\mathfrak{b}} + r_0,$$

where "dim" refers to expected dimension (which in the case of $M_{\mathfrak{b}}$ is equal to the actual dimension of G), and the second equality follows from the addition formula for the index (see Corollary C.0.1). Consequently,

$$\operatorname{index}(\mathcal{I}^\# + D\hat{\Xi}(\hat{S})) = 0.$$

We now compute

$$(\mathcal{I}^\# + D\hat{\Xi}(\hat{S})) \circ D\Pi(0,0) = \begin{pmatrix} \mathcal{I}^\#\mathcal{I} & B \\ 0 & I \end{pmatrix}\colon\ L\mathcal{G}'_{\mathfrak{b}} \oplus E \to W \oplus E, \tag{10.12}$$

where $B\colon E \to W$. The zero in the matrix above is due to the fact that

$$D\widehat{\Xi}(\widehat{S})\mathcal{I}f = \frac{d}{dt}\bigg|_0 \widehat{\Xi}(e^{tf}(\widehat{S})) = 0,$$

which holds because $\widehat{\Xi}_1, \widehat{\Xi}_2$ are $\mathcal{G}'_{\mathfrak{b}}$–invariant, $\widehat{\Xi}_3$ is $\mathcal{G}'_{\mathfrak{b}}$–equivariant, and $\widehat{\Xi}(\widehat{S}) = 0$.

Since the right hand side of (10.12) is invertible, it follows that $\mathcal{I}^\# + D\widehat{\Xi}(\widehat{S})$ is a surjective Fredholm operator of index 0, hence invertible. Of course, this implies that $D\Pi(0,0)$ is also invertible. □

Lemma 10.3.6 *There is a constant $C_{12} < \infty$ such that for sufficiently large τ,*

$$\|D\Pi(0,0)^{-1}\| \le C_{12}e^{\sigma\tau}.$$

Proof In this proof all unqualified norms are $L^{p,\kappa}_1$ norms. It follows from (10.9), (10.10) and Lemma 10.3.1 that $D\widehat{\Xi}(\widehat{S})$ is bounded in τ, T. Therefore there exists a constant $C < \infty$ such that

$$\|x\|_E = \|D\widehat{\Xi}(\widehat{S})(\mathcal{I}f + D\hat{\zeta}(0)x)\| \le C\|D\Pi(0,0)(f,x)\|$$

for all $f \in L\mathcal{G}'_{\mathfrak{b}}$ and $x \in E$. From Lemma 10.3.4 and Lemma 10.2.8 we get

$$\begin{aligned} C_{11}^{-1}\|\mathcal{I}'_o f\| &\le \|\mathcal{I}f\| \\ &\le \|D\Pi(0,0)(f,x)\| + \|D\hat{\zeta}(0)x\| \\ &\le \|D\Pi(0,0)(f,x)\| + C_7e^{\sigma\tau}\|x\|_E \\ &\le (1 + CC_7e^{\sigma\tau})\|D\Pi(0,0)(f,x)\|. \end{aligned}$$

This yields

$$\|f\|_{\mathcal{E}'} + \|x\|_E \le \mathrm{const}\cdot e^{\sigma\tau}\|D\Pi(0,0)(f,x)\|. \qquad \square$$

Lemma 10.3.7 *There is a constant $C_{13} < \infty$ such that for sufficiently large τ one has*

$$\|D^2\Pi(f,x)\| \le C_{13}e^{\sigma\tau}$$

for all $f \in L\mathcal{G}'_{\mathfrak{b}}$, $x \in E$ such that $\|f\| < 1$ and $\|x\| < \epsilon'$.

Proof For the purposes of this proof it is convenient to rescale the norm on $\mathcal{E}'$ so that we can take $C_{10} = 1$ in Lemma 10.3.3.

If $f, g \in \mathcal{E}'$ then $e^f g \in \mathcal{E}'$, and from Lemma 10.3.3 we obtain

$$\|e^f g\| \le \sum_{n=0}^{\infty} \frac{1}{n!}\|f^n g\| \le e^{\|f\|}\|g\|,$$

and similarly with a spinor $\phi \in L^{p,\kappa}_1$ instead of g.

The first two derivatives of $\exp\colon \mathcal{E}' \to 1+\mathcal{E}'$ are

$$D\exp(f)g = g\exp(f), \qquad D^2\exp(f)(g,h) = gh\exp(f),$$

so
$$\|D\exp(f)\|, \|D^2\exp(f)\| \le \exp(\|f\|).$$

Let $\hat{\zeta}_1, \hat{\zeta}_2$ be the connection and spinor parts of ζ, respectively, and define Π_1, Π_2 similarly. Then

$$\Pi(f,x) = (\hat{\zeta}_1(x) - df, e^f \cdot \hat{\zeta}_2(x)).$$

We regard $\Pi(f,x)$ as a function of the two variables f, x. Let $D_j\Pi$ denote the derivative of Π with respect to the j–th variable. Similarly for the second derivatives $D_jD_k\Pi$.

Applying Lemma 10.2.8 and Lemma 10.3.3 we now find that

$$\|D_2^2\Pi_1(f,x)\| = \|D^2\hat{\zeta}_1(x)\| \le \mathrm{const}\cdot e^{\sigma\tau},$$
$$\|D_jD_k\Pi_2(f,x)\| \le \mathrm{const}\cdot e^{\sigma\tau}, \quad j,k = 1,2$$

for $\|f\| < 1$ and $\|x\| < \epsilon'$. Since $D_jD_1\Pi_1 = 0$ for $j = 1,2$, the lemma is proved. $\square$

In the following, $B(x;r)$ will denote the open r–ball about x (both in various Banach spaces and in $\mathcal{C}'$).

Lemma 10.3.8 *There exist constants $r_1, r_2 > 0$ such that for sufficiently large τ the image of Π contains the ball of radius $r_2e^{-3\sigma\tau}$ about $\hat{S}$ in $\mathcal{C}'$; more precisely one has*

$$B(\hat{S}; r_2e^{-3\sigma\tau}) \subset \Pi(B(0; r_1e^{-2\sigma\tau})).$$

Proof We wish to apply the inverse function theorem Proposition B.0.2 to the map Π restricted to a ball $B(0;R_1)$, where $R_1 \in (0,\epsilon']$ is to be chosen. For the time being let M, L, κ have the same meaning as in that proposition. By Lemma 10.3.7 we can take $M = C_{13}e^{\sigma\tau}$, and by Lemma 10.3.6 we have $\|L^{-1}\| \le C_{12}e^{\sigma\tau}$. We need

$$0 \le \kappa = \|L^{-1}\|^{-1} - R_1M.$$

This will hold if

$$R_1 \le (C_{12}C_{13})^{-1}e^{-2\sigma\tau}.$$

When τ is large we can take R_1 to be the right hand side of this inequality. By Proposition B.0.2, $\Pi(B(0;R_1))$ contains the ball $B(\hat{S};R_2)$ where

$$R_2 = \frac{1}{2}R_1C_{12}^{-1}e^{-\sigma\tau} = \frac{1}{2}C_{12}^{-2}C_{13}^{-1}e^{-3\sigma\tau}. \qquad \square$$

Theorem 10.1.1 is a consequence of Proposition 10.2.1 and the following proposition:

Proposition 10.3.1 *Under the assumptions of Theorem 10.1.1, and using the same notation, there is a kv-pair $(K',V')\le(K,V)$ such that $\mathbf{f}$ is injective on $\mathbf{q}^{-1}(\overline{G})$ for all sufficiently large $T_{\min}$.*

The proof of Proposition 10.3.1 occupies the remainder of this section.

For any natural number m which is so large that $K\subset X_{:m}$, let V'_m be the set of all $\omega\in\check{\mathcal{B}}_{\mathfrak{b}}(X_{:m})$ such that there exist a representative S of ω, and a configuration $\overline{S}=(\overline{A},\overline{\Phi})$ over X representing an element of $\overline{G}$, such that

$$d_m(S,\overline{S}):=\int_{X_{:m}}|\overline{S}-S|^p+|\nabla_{\overline{A}}(\overline{S}-S)|^p<\frac{1}{m}. \tag{10.13}$$

Note that
$$d_m(u(S),u(\overline{S}))=d_m(S,\overline{S})$$
for any gauge transformation u over $X_{:m}$. In particular, V'_m is $\mathbb{T}$–invariant.

Lemma 10.3.9 *Let $\omega_n\in V'_{m_n}$ for $n=1,2,\dots$, where $m_n\to\infty$. Then there exists for each n a representative S_n of ω_n such that a subsequence of S_n converges locally in L^p_1 over X to a smooth configuration representing an element of $\overline{G}$.*

Proof By assumption there exist for each n a representative S_n of ω_n and a configuration $\overline{\mathsf{S}}_n$ over X representing an element of $\overline{G}$ such that

$$d_{m_n}(\mathsf{S}_n,\overline{\mathsf{S}}_n)<\frac{1}{m_n}. \tag{10.14}$$

After passing to a subsequence we may assume (since $\overline{G}$ is compact) that $[\overline{\mathsf{S}}_n]$ converges in $\overline{G}$ to some element $[\overline{S}]$, and we can choose $\overline{S}$ smooth. Since $M_{\mathfrak{b}}=M^*_{\mathfrak{b}}$, the local slice theorem guarantees that for large n we can find $u_n\in\mathcal{G}_{\mathfrak{b}}$ such that $\overline{S}_n=u_n(\overline{\mathsf{S}}_n)$ satisfies
$$\|\overline{S}_n-\overline{S}\|_{L^{p,w}_1}\to 0.$$
Set $S_n=u_n(\mathsf{S}_n)$, which is again a representative of ω_n. Let $\overline{A},\overline{A}_n$ be the connection parts of $\overline{S},\overline{S}_n$, respectively. Then (10.14) implies that $\overline{S}_n-S_n\to 0$ and $\nabla_{\overline{A}_n}(\overline{S}_n-S_n)\to 0$ locally in L^p over X, hence also $S_n\to\overline{S}$ locally in L^p over X. Now

$$\nabla_{\overline{A}}(S_n-\overline{S})=\nabla_{\overline{A}_n}(S_n-\overline{S}_n)+\nabla_{\overline{A}}(\overline{S}_n-\overline{S})+(\overline{A}-\overline{A}_n)(S_n-\overline{S}_n),$$

and each of the three terms on the right hand side converges to 0 locally in L^p over X (the third term because of the continuous multiplication $L^p_1\times L^p\to L^p$ in $\mathbb{R}^4$ for $p>4$). Hence $S_n\to\overline{S}$ locally in L^p_1 over X. □

Corollary 10.3.1 *For sufficiently large n one has that $R_K(V'_n) \subset V$.*

Lemma 10.3.10 *Let $\omega_n \in V'_{m_n}$ for $n = 1, 2, \dots,$ where $m_n \to \infty$. Suppose $q(\omega_n|_K)$ converges in $M_{\mathfrak{b}}$ to an element g as $n \to \infty$. Then $g \in \overline{G}$, and there exists for each n a representative S_n of ω_n such that the sequence S_n converges locally in L^p_1 over X to a smooth configuration representing g.*

Proof Let $\mathsf{S}_n, \overline{\mathsf{S}}_n$ be as in the proof of Lemma 10.3.9. First suppose that $[\overline{\mathsf{S}}_n]$ converges in $\overline{G}$ to some element $[\overline{S}]$, where $\overline{S}$ is smooth. Choosing $S_n, \overline{S}_n$ as in that proof we find again that $S_n \to \overline{S}$ locally in L^p_1 over X, hence

$$g = \lim_n q(S_n|_K) = q(\overline{S}|_K) = [\overline{S}].$$

We now turn to the general case when $[\overline{\mathsf{S}}_n]$ is not assumed to converge. Because $\overline{G}$ is compact, every subsequence of $[\overline{\mathsf{S}}_n]$ has a convergent subsequence whose limit must be g by the above argument. Hence $[\overline{\mathsf{S}}_n] \to g$. □

Suppose we are given a sequence $\{m_n\}_{n=1,2,\dots}$ of natural numbers tending to infinity, and for each n an r–tuple $T(n)$ of real numbers such that

$$T_{\min}(n) := \min_j T_j(n) > m_n.$$

Define $\mathbf{q}_n$ and $\mathbf{f}_n$ as in Theorem 10.1.1, with $K' = X_{:m_n}$ and $V' = V'_{m_n}$.

Lemma 10.3.11 *For $n = 1, 2, \dots$ suppose S_n is a smooth configuration over $X^{(T(n))}$ representing an element $\omega_n \in \mathbf{q}_n^{-1}(\overline{G})$, and such that*

$$\mathbf{f}_n(\omega_n) \to (\omega_0, z) \in \overline{G} \times \mathrm{U}(1)^{r_0}$$

as $n \to \infty$. There exists a constant $C_{14} < \infty$ such that for sufficiently large τ the following holds for sufficiently large n. Let the map $\widehat{\zeta} = \widehat{\zeta}_n$ be defined as above and set $\widehat{S}_n = \widehat{\zeta}_n(0)$. Then there exists a smooth gauge transformation $u_n \in \mathcal{G}'_{\mathfrak{b}}$ such that

$$\|u_n(S_n) - \widehat{S}_n\|_{L^{p,\kappa}_1} \le C_{14} e^{(3\sigma - \lambda)\tau}.$$

Note: This constant C_{14} depends on (ω_0, z) but not on the sequence S_n.

Before proving the lemma, we will use it to show that $\mathbf{f}_n$ is injective on $\mathbf{q}_n^{-1}(\overline{G})$ for some n. This will prove Proposition 10.3.1. Suppose $\omega_n, \omega'_n \in \mathbf{q}_n^{-1}(\overline{G})$ and $\mathbf{f}_n(\omega_n) = \mathbf{f}_n(\omega'_n)$, $n = 1, 2, \dots$. After passing to a subsequence we may assume that $\mathbf{f}_n(\omega_n)$ converges to some point $(\omega_0, z) \in \overline{G} \times \mathrm{U}(1)^{r_0}$. Combining Lemma 10.3.8, Lemma 10.3.11 and the assumption $6\sigma < \lambda$ we conclude that if τ is sufficiently large then for sufficiently large n we can represent ω_n and ω'_n by configurations $\widehat{\zeta}(x_n)$ and $\widehat{\zeta}(x'_n)$, respectively, where $x_n, x'_n \in B_{\epsilon'}$. Now recall that $\widehat{\Xi}\circ\widehat{\zeta} = I$, and that the components $\widehat{\Xi}_1, \widehat{\Xi}_2$ are $\mathcal{G}'_{\mathfrak{b}}$–invariant whereas $\widehat{\Xi}_3$ is the Seiberg–Witten map. Comparing the definitions of $\mathbf{f}_n$ and $\widehat{\Xi}$ we conclude that

$$x_n = \widehat{\Xi}(\widehat{\zeta}(x_n) = \widehat{\Xi}(\widehat{\zeta}(x'_n) = x'_n,$$

hence $\omega_n = \omega'_n$ for large n. To complete the proof of Proposition 10.3.1 it therefore only remains to prove Lemma 10.3.11.

Proof of Lemma 10.3.11 In this proof, constants will be independent of the sequence S_n (as well as of τ as before).

By Lemma 10.3.10 we can find for each n an $L^p_{2,\mathrm{loc}}$ gauge transformation v_n over X with $v_n|_{\mathfrak{b}} = 1$ such that $S'_n = v_n(S_n)$ converges locally in L^p_1 over X to a smooth configuration S' representing ω_0. A moment's thought shows that we can choose the v_n smooth, and we can clearly arrange that $S' = S_0$. Then for any $t \ge 0$ we have

$$\limsup_n \|S'_n - \widehat{S}_n\|_{L^{p,\kappa}_1(X_{:t})} = \limsup_n \|S_0 - \widehat{S}_n\|_{L^{p,\kappa}_1(X_{:t})} \le \mathrm{const}\cdot e^{(2\sigma-\lambda)\tau} \quad (10.15)$$

when τ is so large that Lemma 10.3.1 applies.

For $t \ge 0$ and any smooth configurations S over $X_{:t}$ consider the functional

$$\begin{aligned} E(S,t) = &\sum_{j=1}^{r} \lambda_j \big(\vartheta(S|_{\{t\}\times Y_j}) + \vartheta(S|_{\{t\}\times(-Y_j)})\big) \\ &+ \sum_{j=1}^{r'} \lambda'_j \big(\vartheta(S|_{\{t\}\times Y'_j}) - \vartheta(\alpha'_j)\big), \end{aligned}$$

where in this formula $\{t\}\times(\pm Y_j)$ has the boundary orientation inherited from $X_{:t}$. (Recall that the Chern–Simons–Dirac functional ϑ changes sign when the orientation of the 3–manifold in question is reversed.) The assumption on λ_j, λ'_j and $\widetilde{\eta}_j, \widetilde{\eta}'_j$ in Theorem 10.1.1 implies that $E(S,t)$ depends only on the gauge equivalence class of S. Since ϑ is a smooth function on the $L^2_{1/2}$ configuration space by Lemma 3.2.1, we obtain

$$E(S_n,t) = E(S'_n,t) \to E(S_0,t)$$

as $n \to \infty$. By our exponential decay results (see the proof of Theorem 6.3.1),

$$E(S_0,t) < \mathrm{const}\cdot e^{-2\lambda t} \quad \text{for } t \ge 0.$$

It follows that

$$E(S_n,t) < \mathrm{const}\cdot e^{-2\lambda t} \quad \text{for } n > N(t)$$

for some positive function N. By assumption the perturbation parameters $\vec{\mathfrak{p}}, \vec{\mathfrak{p}}'$ are admissible, hence there is a constant $C<\infty$ such that when $T_{\min}(m_n)>C$, each of the $(r+r')$ summands appearing in the definition of $E(S_n,t)$ is nonnegative. Explicitly, this yields

$$0 \le \vartheta(S_n|_{\{-T_j(n)+t\}\times Y_j}) - \vartheta(S_n|_{\{T_j(n)-t\}\times Y_j}) < \mathrm{const}\cdot e^{-2\lambda t},$$
$$0 \le \vartheta(S_n|_{\{t\}\times Y'_j}) - \vartheta(\alpha'_j) < \mathrm{const}\cdot e^{-2\lambda t},$$

where the first line holds for $0\le t\le T_j(n)$ and $j=1,\dots,r$, the second line for $t\ge 0$ and $j=1,\dots,r'$, and in both cases we assume $T_{\min}(n)>C$ and $n>N(t)$.

In the following we will ignore the ends $\mathbb{R}_+\times Y'_{j'}$ of X, ie we will pretend that $X^\#$ is compact. If $\alpha'_{j'}$ is irreducible then the argument for dealing with the end $\mathbb{R}_+\times Y'_{j'}$ is completely analogous to the one given below for a neck $[-T_j,T_j]\times Y_j$, while if $\alpha'_{j'}$ is reducible it is simpler. (Compare the proof of Proposition 6.4.1 (ii).)

For the remainder of the proof of this lemma we will focus on one particular neck $[-T_j(n),T_j(n)]\times Y_j$ where $1\le j\le r$. To simplify notation we will therefore mostly omit j from notation and write $T(n), Y, \alpha$ etc instead of $T_j(n), Y_j, \alpha_j$.

For $0\le t\le T(n)$ set

$$B_t=[-T(n)+t,T(n)-t]\times Y,$$

regarded as a subset of $X^{(T(n))}$. By the above discussion there is a constant $t_1>0$ such that when n is sufficiently large, S_n will restrict to a genuine monopole over the band B_{t_1+3} by Lemmas 4.1.1, 4.2.1 and 4.2.2 and will have small enough energy over this band for Theorem 6.3.2 to apply. That theorem then provides a smooth

$$\widetilde{v}_n\colon B_{t_1}\to \mathrm{U}(1)$$

such that $S''_n=\widetilde{v}_n(S_n|_{B_{t_1}})$ is in temporal gauge and

$$\|S''_n-\underline{\alpha}\|_{L^{p,\kappa}_1(B_t)}\le \mathrm{const}\cdot e^{(\sigma-\lambda)t},\quad t\ge t_1.$$

Writing
$$S''_n-\widehat{S}_n=(S''_n-\underline{\alpha})+(\underline{\alpha}-\widetilde{S})+(\widetilde{S}-\widehat{S}_n)$$

we get

$$\limsup_{n\to\infty}\|S''_n-\widehat{S}_n\|_{L^{p,\kappa}_1(B_t)}\le \mathrm{const}\cdot\left(e^{(2\sigma-\lambda)\tau}+e^{(\sigma-\lambda)t}\right) \tag{10.16}$$

when $t\ge t_1$ and τ is so large that Lemma 10.3.1 applies.

To complete the proof of the lemma we interpolate between v_n and $\widetilde{v}_n$ in the overlap region $\mathcal{O}_\tau=X_{:\tau}\cap B_{\tau-1}$. (This requires $\tau\ge t_1+1$.) The choice of this overlap region is somewhat arbitrary but simplifies the exposition. Define

$$w_n = \widetilde{v}_n v_n^{-1} \colon \mathcal{O}_\tau \to \mathrm{U}(1).$$

Then
$$w_n(S'_n) = S''_n \quad \text{on } \mathcal{O}_\tau.$$

Set $x^\pm = \gamma(\pm(T-\tau))$, where $\gamma = \gamma_j$ is the path introduced in the beginning of this chapter. If α is reducible then by multiplying each $\widetilde{v}_n$ by a constant and redefining w_n, S''_n accordingly we can arrange that $w_n(x^+) = 1$ for all n. These changes have no effect on the estimates above.

Lemma 10.3.11 is a consequence of the estimates (10.15)–(10.16) together with the following sublemma (see the proof of Proposition 6.4.1 (ii).)

Sublemma 10.3.1 *There is a constant $C_{15} < \infty$ such that if $\tau \geq C_{15}$ then*

$$\limsup_{n\to\infty} \|w_n - 1\|_{L^p_2(\mathcal{O}_\tau)} \leq C_{15} e^{(2\sigma-\lambda)\tau}.$$

Proof of sublemma If α is irreducible then the sublemma follows from inequalities (10.15), (10.16) and Lemmas 6.4.2, 6.4.4. (In this case the sublemma holds with $C_{15} e^{(\sigma-\lambda)\tau}$ as upper bound.)

Now suppose α is reducible. We will show that

$$\limsup_{n\to\infty} |w_n(x^-) - 1| \leq \mathrm{const} \cdot e^{(2\sigma-\lambda)\tau} \tag{10.17}$$

for large τ. Granted this, we can prove the sublemma by applying Lemma 6.4.2 and Lemma 6.4.3 (ii) to each component of $\mathcal{O}_\tau$.

In the remainder of the proof of the sublemma we will omit n from subscripts. To prove (10.17), define intervals

$$J_0 = [-T-1, -T+\tau], \quad J_1 = [-T+\tau, T-\tau], \quad J_2 = [T-\tau, T+1]$$

and for $k = 0, 1, 2$ set $\gamma^{(k)} = \gamma|_{J_k}$. Let $\mathrm{Hol}^{(k)}$ denote holonomy along $\gamma^{(k)}$ in the same sense as (10.1), ie $\mathrm{Hol}^{(k)}$ is the result of replacing the domain of integration I_j in that formula with J_k. Define $\delta^{(k)} \in \mathbb{C}$ by

$$\begin{aligned} \mathrm{Hol}^{(k)}(\widehat{S}) &= \mathrm{Hol}^{(k)}(S')(1+\delta^{(k)}), \quad k = 0, 2, \\ \mathrm{Hol}^{(1)}(\widehat{S}) &= \mathrm{Hol}^{(1)}(S'')(1+\delta^{(1)}) \end{aligned}$$

where as usual we mean holonomy with respect to the connection parts of the configurations. For large τ the estimates (10.15) and (10.16) give

$$|\delta^{(k)}| \leq \mathrm{const} \cdot e^{(2\sigma-\lambda)\tau}$$

when n is sufficiently large.

Writing $h=\prod_{k=0}^{2}(1+\delta^{(k)})$ we have

$$z=\mathrm{Hol}(\widehat{S})=\prod_{k=0}^{2}\mathrm{Hol}^{(k)}(\widehat{S})=h\,\mathrm{Hol}^{(0)}(S')\,\mathrm{Hol}^{(1)}(S'')\,\mathrm{Hol}^{(2)}(S').$$

Now, by the definition of holonomy,

$$\mathrm{Hol}^{(1)}(S'')=\frac{\widetilde{v}(x^+)}{\widetilde{v}(x^-)}\,\mathrm{Hol}^{(1)}(S),$$

and there are similar formulas for $\mathrm{Hol}^{(k)}(S')$. Because $w(x^+)=1$ we obtain

$$z=h\ \mathrm{Hol}(S)\,w(x^-)^{-1}.$$

Setting $a=\mathrm{Hol}(S)z^{-1}$ we get

$$w(x^-)-1=ah-1=(a-1)h+h-1.$$

Since by assumption $a\to 1$ as $n\to\infty$, we have

$$|w(x^-)-1|\le\mathrm{const}\cdot\left(|a-1|+\sum_k|\delta^{(k)}|\right)\le\mathrm{const}\cdot e^{(2\sigma-\lambda)\tau}$$

for large n, proving the sublemma and hence also Lemma 10.3.11. □

This completes the proof of Proposition 10.3.1 and thus also the proof of Theorem 10.1.1.

10.4 Existence of maps q

Let $G\subset M_{\mathfrak{b}}$ be as in Section 10.1. In this section we will show that there is always a map q as in (10.3) provided $\mathbb{T}$ acts freely on $\overline{G}$ and K is sufficiently large. It clearly suffices to prove the same with $\mathcal{B}^*_{\mathfrak{b}}(K)$ in place of $\breve{\mathcal{B}}^*_{\mathfrak{b}}(K)$.

Let $\mathcal{B}$, M denote the configuration and moduli spaces over X with the same asymptotic limits as $\mathcal{B}_{\mathfrak{b}}$, $M_{\mathfrak{b}}$, but using the full group of gauge transformations $\mathcal{G}$ rather than $\mathcal{G}_{\mathfrak{b}}$.

Because $\mathcal{G}_{\mathfrak{b}}$ acts freely on $\mathcal{C}$, an element in $\mathcal{B}_{\mathfrak{b}}$ has trivial stabilizer in $\mathbb{T}$ if and only if its image in $\mathcal{B}$ is irreducible, ie when its spinor field does not vanish identically on any component of X.

Throughout this section, K will be a compact codimension 0 submanifold of X which contains $\mathfrak{b}$ and intersects every component of X.

Proposition 10.4.1 *If $\mathbb{T}$ acts freely on $\bar{G}$ then for sufficiently large K there exist a $\mathbb{T}$–invariant open neighbourhood $V \subset \mathcal{B}^*_{\mathfrak{b}}(K)$ of $R_K(\bar{G})$ and a $\mathbb{T}$–equivariant smooth map $q\colon V \to M_{\mathfrak{b}}$ such that $q(\omega|_K) = \omega$ for all $\omega \in G$.*

We first prove three lemmas. Let $H \subset M^*$ be the image of G. Because $\mathbb{T}$ is compact, the projection $\mathcal{B}_{\mathfrak{b}} \to \mathcal{B}$ is a closed map and therefore maps $\bar{G}$ to $\bar{H}$. Let $H_0 \subset M^*$ be any precompact open subset which contains $\bar{H}$ and whose closure consists only of regular points.

Lemma 10.4.1 *If K is sufficiently large then $R_K\colon M^* \to \mathcal{B}^*(K)$ restricts to an immersion on an open neighbourhood of $\bar{H}_0$.*

By "immersion" we mean the same as in Lang [32]. Since a finite-dimensional subspace of a Banach space is always complemented, the condition in our case is simply that the derivative of the map be injective at every point.

Proof Fix $\omega = [S] \in \bar{H}_0$. We will show that R_K is an immersion at ω (hence in a neighbourhood of ω) when K is large enough. Since $\bar{H}_0$ is compact, this will prove the lemma.

Let $W \subset L^{p,w}_1$ be a linear subspace such that the derivative at S of the projection $S + W \to \mathcal{B}^*$ is a linear isomorphism onto the tangent space of M at ω. Let δ denote that derivative. For $t \geq 0$ so large that $\mathfrak{b} \subset X_{:t}$ let δ_t be the derivative at S of the natural map $S + W \to \mathcal{B}^*(X_{:t})$. We claim that δ_t is injective for $t \gg 0$. For suppose $\{w_n\}$ is a sequence in W such that $\|w_n\|_{L^{p,w}_1} = 1$ and $\delta_{t_n}(w_n) = 0$ for each n, where $t_n \to \infty$. Set $K_n = X_{:t_n}$. Then

$$w_n|_{K_n} = \mathcal{I}_\Phi f_n$$

for some $f_n \in L\mathcal{G}(K_n)$, where Φ is the spinor field of S. After passing to a subsequence we may assume that w_n converges in $L^{p,w}_1$ to some $w \in W$ (since W is finite-dimensional). By Lemma 2.2.1 there exists for each n a constant $C_n < \infty$ such that for all $h \in L\mathcal{G}(K_n)$ one has

$$\|h\|_{L^p_2} \leq C_n \|\mathcal{I}_\Phi h\|_{L^p_1}.$$

It follows that f_n converges in L^p_2 over compact subsets of X to some function f. We obviously have $\mathcal{I}_\Phi f = w$, hence $f \in L\mathcal{G}$ and $\delta(w) = 0$. But this is impossible, since w has norm 1. This proves the lemma. □

Lemma 10.4.2 *If K is sufficiently large then the restriction map $H_0 \to \mathcal{B}^*(K)$ is a smooth embedding.*

Proof Because of Lemma 10.4.1 it suffices to show that R_K is injective on $\bar{H}_0$ when K is large.

Suppose $\omega_n, \omega'_n \in \bar{H}_0$ restrict to the same element in $\mathcal{B}(X_{:t_n})$ for $n = 1, 2, \dots$, where $t_n \to \infty$. Since $\bar{H}_0$ is compact we may assume, after passing to a subsequence, that ω_n, ω'_n converge in $\bar{H}_0$ to ω, ω' respectively. By the local slice theorem we can find representatives $S_n = (A_n, \Phi_n)$, $S'_n = (A'_n, \Phi'_n)$ for ω_n, ω'_n respectively, such that the sequence $\{S_n\}$ converges in $\mathcal{C}$ to some configuration S, and similarly $S'_n \to S'$. By assumption, $S'_n = u_n(S_n)$ over $X_{:t_n}$ for some $u_n \in L^p_2$. In particular, $du_n = u_n \cdot (A_n - A'_n)$, so the sequence $\{u_n\}$ is locally bounded in L^p_2. After passing to a subsequence we may assume that $\{u_n\}$ converges weakly in L^p_2 over compact subsets of X to some gauge transformation u. Then $u(S) = S'$. By Proposition 2.4.1 (iii) we have $u \in \mathcal{G}$, hence $\omega = \omega'$. When n is large then $\bar{H}_0 \to \mathcal{B}(X_{:t_n})$ will be injective in a neighbourhood of ω by Lemma 10.4.1, hence $\omega_n = \omega'_n$ for n sufficiently large. □

For the present purposes, we will call a Banach space E *admissible*, if $x \mapsto \|x\|^r$ is a smooth function on E for some $r > 0$. (The examples we have in mind are L^p_k Sobolev spaces where p is an even integer.)

Lemma 10.4.3 *Let B be any second countable (smooth) Banach manifold modelled on an admissible Banach space. Then any submanifold Z of B possesses a tubular neighbourhood (in the sense of [32]).*

Proof According to [32, p 96], if a Banach manifold admits partitions of unity then any *closed* submanifold possesses a tubular neighborhood. Now observe that Z is by definition locally closed, hence $C = \bar{Z} \setminus Z$ is closed in B. But then Z is a closed submanifold of $B \setminus C$. In general, any second countable, regular T_1–space is metrizable, hence paracompact (see Kelley [29]). Because $B \setminus C$ is modelled on an admissible Banach space, the argument in [32] carries over to show that $B \setminus C$ admits partitions of unity. Therefore, Z possesses a tubular neighbourhood in $B \setminus C$, which also serves as a tubular neighbourhood of Z in B. □

Proof of Proposition 10.4.1 Choose K so large that $H_0 \to \mathcal{B}^*(K)$ is an embedding, with image Z, say. Let G_0 denote the preimage of H_0 in $M_{\mathfrak{b}}$.

Let $\mathcal{B}^{**}_{\mathfrak{b}}(K)$ be the open subset of $\mathcal{B}_{\mathfrak{b}}(K)$ consisting of those elements whose spinor does not vanish identically on any component of K. Then the projection $\pi\colon \mathcal{B}^{**}_{\mathfrak{b}}(K) \to \mathcal{B}^*(K)$ is a principal $\mathbb{T}$–bundle, and restriction to K defines a diffeomorphism

$$\iota\colon G_0 \to \pi^{-1} Z.$$

By Lemma 10.4.3 there is an open neighbourhood U of H_0 in $\mathcal{B}^*(K)$ and a smooth map

$$\rho\colon U \times [0,1] \to \mathcal{B}^*(K)$$

such that $\rho(x,1) \in Z$ for all x, and $\rho(x,t) = x$ if $x \in Z$ or $t = 0$. (In other words, ρ is a strong deformation retraction of U to Z.) After choosing a connection in the $\mathbb{T}$–bundle $\mathcal{B}^{**}_{\mathfrak{b}}(K)$ we can then construct a $\mathbb{T}$–invariant smooth retraction

$$\tilde{\rho}\colon \pi^{-1}(U) \to \pi^{-1}(Z)$$

by means of holonomy along the paths $t \mapsto \rho(t,x)$. Now set

$$q = \iota^{-1} \circ \tilde{\rho}\colon \pi^{-1} U \to G_0. \qquad \square$$

CHAPTER 11

Applications

11.1 A model application

In this section we will show in a model case how the gluing theorem may be applied in combination with the compactness results of Part I. Here we only consider gluing along irreducible critical points. Examples of gluing along reducible critical points will be given in Part III and in [23]. The main result of this section, Theorem 11.1.1, encompasses both the simplest gluing formulae for Seiberg–Witten invariants (in situations where reducibles are not encountered) and, as we will see in the next section, the formula $d \circ d = 0$ for the standard Floer differential.

Recall that the Seiberg–Witten invariant of a closed spin^c 4–manifold (with $b^+ > 1$) can be defined as the number of points (counted with sign) in the zero-set of a generic section of a certain vector bundle over the moduli space. To obtain a gluing formula, this vector bundle and its section should be expressed as the pullback of a vector bundle $E \to \breve{\mathcal{B}}^*(K)$ with section s, where $K \subset X$. In the proof of Theorem 11.1.1 below we will see how the section s gives rise in a natural way to a map q as in Theorem 10.1.1. Thus, the section s is being incorporated into the equations that the gluing map is required to solve. (We owe this idea to [14, p 99].)

We will now describe the set-up for our model application. Let X be as in Section 1.4 with $r = 1$ and $r' \geq 0$, and set $Y = Y_1$. In other words, we will be gluing one single pair of ends $\mathbb{R}_+ \times (\pm Y)$ of X, but X may have other ends $\mathbb{R}_+ \times Y'_j$ not involved in the gluing. We assume $X^\#$ is connected, which means that X has one or two connected components. For $j = 1, \ldots, r'$ fix a critical point $\alpha'_j \in \widetilde{\mathcal{R}}_{Y'_j}$. Let μ be a 2–form and $\mathfrak{p}$ a perturbation parameter for Y, and let $\mu'_j, \mathfrak{p}'_j$ be similar data for Y'_j. Let each $\mathfrak{p}, \mathfrak{p}'_j$

have small C^1 norm. To simplify notation we write, for $\alpha,\beta\in\widetilde{\mathcal{R}}_Y$,

$$M_{\alpha,\beta}=M(X;\alpha,\beta,\vec{\alpha}'),\qquad M^{(T)}=M(X^{(T)};\vec{\alpha}').$$

We make the following assumptions:

- (Compactness) At least one of the conditions (B1), (B2) of Section 1.4 holds for some $\lambda_j,\lambda'_j>0$,
- (Regularity) All moduli spaces over $\mathbb{R}\times Y$, $\mathbb{R}\times Y'_j$ and X contain only regular points, and
- (No reducibles) Given $\alpha_1,\alpha_2\in\widetilde{\mathcal{R}}_Y$ and $\alpha'_j\in\widetilde{\mathcal{R}}_{Y'_j}$, if there exist a broken gradient line over $\mathbb{R}\times Y$ from α_1 to α_2 and for each j a broken gradient line over $\mathbb{R}\times Y'_j$ from α'_j to β'_j then $M(X;\alpha_1,\alpha_2,\vec{\alpha}')$ contains no reducible. (It then follows by compactness that $M^{(T)}$ contains no reducible when T is large.)

The regularity condition is stronger than necessary, because there are energy constraints on the moduli spaces that one may encounter in the situation to be considered, but we will not elaborate on this here.

Note that we have so far only developed a full transversality theory in the case when Y and each Y'_j are rational homology spheres; in the remaining cases the discussion here is therefore somewhat theoretical at this time.

Let $K\subset X$ be a compact codimension 0 submanifold which intersects every component of X. When $T\gg 0$ then K may also be regarded as a submanifold of $X^{(T)}$, and we have restriction maps

$$R_{\alpha,\beta}\colon M^*_{\alpha,\beta}\to\breve{\mathcal{B}}^*(K),\qquad R'\colon M^{(T)}\to\breve{\mathcal{B}}^*(K).$$

These take values in $\breve{\mathcal{B}}^*(K)$ rather than just in $\breve{\mathcal{B}}(K)$ because of the unique continuation property of harmonic spinors.

Suppose $E\to\breve{\mathcal{B}}^*(K)$ is an oriented smooth real vector bundle whose rank d is equal to the (expected) dimension of $M^{(T)}$. Choose a smooth section s of E such that the pullback section $s_{\alpha,\beta}=R^*_{\alpha,\beta}s$ is transverse to the zero-section of the pullback bundle $E_{\alpha,\beta}=R^*_{\alpha,\beta}E$ over $M^*_{\alpha,\beta}$ for each pair α,β. (Here the Sobolev exponent $p>4$ should be an even integer to ensure the existence of smooth partitions of unity.) Set $s'=(R')^*s$, which is a section of $E'=(R')^*E$. We write $M_\alpha=M_{\alpha,\alpha}=M^*_{\alpha,\alpha}$ and $s_\alpha=s_{\alpha,\alpha}$ etc. Let $\widehat{M}_\alpha$, $\widehat{M}^{(T)}$ denote the zero-sets of s_α, s' respectively. By index theory we have

$$0=\dim\,\widehat{M}^{(T)}=\dim\,\widehat{M}_\alpha+n_\alpha,$$

where $n_\alpha=0$ if α is irreducible and $n_\alpha=1$ otherwise. Thus, $\widehat{M}_\alpha$ is empty if α is reducible.

Lemma 11.1.1 *If $\omega_n \in \widehat{M}^{(T(n))}$ for $n = 1, 2, \dots$, where $T(n) \to \infty$, then a subsequence of ω_n chain-converges to an element of $\widehat{M}_\alpha$ for some $\alpha \in \widetilde{\mathcal{R}}^*_Y$. Moreover, if $\omega_n = [S_n]$ chain-converges to $[S] \in \widehat{M}_\alpha$ then there exists for each n a smooth $u_n \colon X^{(T(n))} \to \mathrm{U}(1)$ whose restriction to each end $\mathbb{R}_+ \times Y'_j$ is null-homotopic and such that the sequence $u_n(S_n)$ c-converges over X to S.*

Proof The statement of the first sentence follows from Theorem 1.4.1 by dimension counting. Such maps u_n exist in general for chain-convergent sequences when the ω_n all have the same asymptotic limits over the ends $\mathbb{R}_+ \times Y'_j$. □

Let $J \subset H^1(Y;\mathbb{Z})$ be the subgroup consisting of elements of the form $z|_Y$ where z is an element of $H^1(X^\#;\mathbb{Z})$ satisfying $z|_{Y'_j} = 0$ for $j = 1, \dots, r'$. This group J acts on the disjoint union

$$\widehat{M}^u = \bigcup_{\alpha \in \widetilde{\mathcal{R}}^*_Y} \widehat{M}_\alpha,$$

permuting the sets in the union.

Lemma 11.1.2 *The quotient $\widehat{M} = \widehat{M}^u / J$ is a finite set.*

Proof By Theorem 1.3.1 any sequence $\omega_n \in \widehat{M}_{\alpha_n}$, $n = 1, 2, \dots$ has a chain-convergent subsequence, and for dimensional reasons the limit (well-defined up to gauge equivalence) must lie in some moduli space $\widehat{M}_\beta$. Furthermore, if ω_n chain-converges to an element in $\widehat{M}_\beta$ then $\widehat{M}_{\alpha_n}$ is contained in the orbit $J \cdot \widehat{M}_\beta$ for $n \gg 0$. Therefore, each $\widehat{M}_\alpha$ is a finite set, and only finitely many orbits $J \cdot \widehat{M}_\alpha$ are nonempty. This is equivalent to the statement of the lemma. □

Note that J is the largest subgroup of $H^1(Y;\mathbb{Z})$ which acts on $\widehat{M}^u$ in a natural way. On the other hand, if $\widehat{M}^u$ is nonempty then, since $H^1(Y;\mathbb{Z})$ acts freely on $\widetilde{\mathcal{R}}_Y$, only subgroups $J' \subset J$ of finite index have the property that $\widehat{M}^u / J'$ is finite.

Lemma 11.1.3 *There is a compact codimension 0 submanifold $K_0 \subset X$ such that the restriction map $\widehat{M} \to \mathcal{B}(K_0)$ is injective.*

Proof Let $[S_j] \in \widehat{M}_{\beta_j}$, $j = 1, 2$, where each S_j is in temporal gauge over the ends of X (and therefore decays exponentially). Suppose there exists a sequence of smooth gauge transformations $u_n \colon X_{:t_n} \to \mathrm{U}(1)$ where $t_n \to \infty$, such that $u_n(S_1) = S_2$ over $X_{:t_n}$. By passing to a subsequence we can arrange that u_n c-converges over X to some gauge transformation u with $u(S_1) = S_2$. If $t \gg 0$ then $u|_{\{t\}\times(\pm Y)}$ will both be homotopic to a smooth $v \colon Y \to \mathrm{U}(1)$ with $v(\alpha_1) = \alpha_2$. Hence $\widehat{M}_{\alpha_1}, \widehat{M}_{\alpha_2}$ lie in the same J–orbit, and S_1, S_2 represent the same element of $\widehat{M}$ by Proposition 2.4.1 (iii).

Thus we can take $K_0 = X_{:t}$ for $t \gg 0$. □

Now fix K_0 as in Lemma 11.1.3 and with $K \subset K_0$. Let $\{b_1, \dots, b_m\}$ be the image of the restriction map $R_{K_0}\colon \widehat{M} \to \breve{\mathcal{B}}(K_0)$. Choose disjoint open neighbourhoods $W_j \subset \breve{\mathcal{B}}(K_0)$ of the points b_j. If $T \gg 0$ then

$$R_{K_0}(\widehat{M}^{(T)}) \subset \textstyle\bigcup_j W_j$$

by Lemma 11.1.1. For such T we get a natural map

$$g\colon \widehat{M}^{(T)} \to \widehat{M}.$$

It is clear that if g' is the map corresponding to a different choice of K_0 and neighbourhoods W_j then $g = g'$ for T sufficiently large.

Theorem 11.1.1 *For sufficiently large T the following hold:*

(i) *Every element of $\widehat{M}^{(T)}$ is a regular point in $M^{(T)}$ and a regular zero of s'.*

(ii) *g is a bijection.*

Proof If $\widehat{M}$ is empty then, by Lemma 11.1.1, $\widehat{M}^{(T)}$ is empty as well for $T \gg 0$, and there is nothing left to prove.

We now fix b_j and for the remainder of the proof omit j from notation. (Thus $b = b_j$, $W = W_j$ etc.) We will show that for $T \gg 0$ the set

$$\widehat{B}^{(T)} = \{\omega \in \widehat{M}^{(T)} : \omega|_{K_0} \in W\}$$

consists of precisely one element, and that this element is regular in the sense of (i). This will prove the theorem.

By definition, b is the restriction of some $\omega_0 \in \widehat{M}_\alpha$. Choose an open neighbourhood $V \subset \breve{\mathcal{B}}^*(K)$ of $b|_K$ and a smooth map

$$\pi\colon E|_V \to \mathbb{R}^d$$

which restricts to a linear isomorphism on every fibre. Choose an open neighbourhood $V_0 \subset W$ of b such that $R_K(V_0) \subset V$. Let $G_+ \subset M_\alpha$ be a precompact open neighbourhood of ω_0 such that $R_{K_0}(\overline{G}_+) \subset V_0$. The assumption that ω_0 be a regular zero of s_α means that the composite map

$$G_+ \xrightarrow{R_K} V \xrightarrow{\pi \circ s} \mathbb{R}^d$$

is a local diffeomorphism at ω_0. We can then find an injective smooth map

$$p\colon \mathbb{R}^d \to M_\alpha$$

such that $p\circ\pi\circ s\circ R_K=\mathrm{Id}$ in some open neighbourhood $G\subset G_+$ of ω_0. In particular, $p^{-1}(\omega_0)=\{0\}$ and p is a local diffeomorphism at 0. Set

$$q=p\circ\pi\circ s\colon V\to M_\alpha.$$

By Theorem 10.1.1 there is a kv-pair $(K',V')\le(K_0,V_0)$ such that if $T\gg 0$ then $\mathbf{q}^{-1}G$ consists only of regular monopoles and

$$\mathbf{f}=q\circ R_K\colon \mathbf{q}^{-1}G\to G$$

is a diffeomorphism. By Lemma 11.1.1 one has

$$\widehat{B}^{(T)}=\mathbf{q}^{-1}G\cap(s')^{-1}(0)=\mathbf{f}^{-1}(\omega_0)$$

for $T\gg 0$. For such T the set $\widehat{B}^{(T)}$ consists of precisely one point, and this point is regular in the sense of (i). □

11.2 The Floer differential

Consider the situation of Section 1.2. Suppose a perturbation parameter $\mathfrak{p}$ of small C^1 norm has been chosen for which all moduli spaces $M(\alpha_1,\alpha_2)$ over $\mathbb{R}\times Y$ are regular. (This is possible at least when Y is a rational homology sphere, by Proposition 8.2.3.) Fix $\alpha_1,\alpha_2\in\widetilde{\mathcal{R}}^*_Y$ with

$$\dim\, M(\alpha_1,\alpha_2)=2.$$

We will show that the disjoint union

$$\check{M}:=\bigcup_{\beta\in\widetilde{\mathcal{R}}^*_Y\setminus\{\alpha_1,\alpha_2\}}\check{M}(\alpha_1,\beta)\times\check{M}(\beta,\alpha_2)$$

is the boundary of a compact 1–manifold. (In other words, the standard Floer differential d satisfies $d\circ d=0$ at least with $\mathbb{Z}/2$ coefficients.) To this end we will apply Theorem 11.1.1 to the case when X consists of two copies of $\mathbb{R}\times Y$, say

$$X=\mathbb{R}\times Y\times\{1,2\},$$

and we glue $\mathbb{R}_+\times Y\times\{1\}$ with $\mathbb{R}_-\times Y\times\{2\}$. Thus $r=1$, $r'=2$. We take $K=K_1\cup K_2$, where $K_j=[0,1]\times Y\times\{j\}$. In this case, $\breve{\mathcal{B}}^*(K)$ is the quotient of $\mathcal{C}^*(K)$ by the null-homotopic gauge transformations. The bundle E over $\breve{\mathcal{B}}^*(K)$ will be the product bundle with fibre $\mathbb{R}^2$. To define the section s of E, choose $\delta_1,\delta_2>0$ such that ϑ has no critical value in the set

$$(\vartheta(\alpha_2),\vartheta(\alpha_2)+\delta_2]\cup[\vartheta(\alpha_1)-\delta_1,\vartheta(\alpha_1)).$$

This is possible because we assume Condition (O1) of Section 1.2. For any configuration S over $[0,1]\times Y$ set

$$s_j(S)=\int_0^1\vartheta(S_t)\,dt-\vartheta(\alpha_j)-(-1)^j\delta_j.$$

Note $s_j(S)$ does not change if we apply a null-homotopic gauge transformation to S.

A configuration over K consists of a pair (S_1,S_2) of configurations over $[0,1]\times Y$. Define a smooth function $s\colon \breve{\mathcal{B}}^*(K)\to\mathbb{R}^2$ (ie a section of E) by

$$s([S_1],[S_2])=(s_1(S_1),s_2(S_2)).$$

If $[S]$ belongs to some moduli space $M(\beta_1,\beta_2)$ over $\mathbb{R}\times Y$ with $\beta_1\neq\beta_2$ in $\widetilde{\mathcal{R}}_Y$ then $\frac{d}{dt}\vartheta(S_t)<0$ for all t by choice of $\mathfrak{p}$. Since $J=0$, the natural map $\widehat{M}\to\check{M}$ is therefore a bijection.

Let s_j' be the pullback of s_j to $M^{(T)}$. Here $M^{(T)}$ is defined using Equation (3.7) with $\mathfrak{q}=0$, and so can be identified with $M(\alpha_1,\alpha_2)$. By Theorem 1.3.1 the set

$$Z^{(T)}=\{\omega\in M^{(T)}\,:\,s_1'(\omega)=0,\quad s_2'(\omega)\le 0\}$$

is compact for all $T>0$. If $T\gg 0$ then, by Theorem 11.1.1, $Z^{(T)}$ is a smooth submanifold of $M^{(T)}$, and the composition of the two bijections

$$\partial Z^{(T)}=\widehat{M}^{(T)}\xrightarrow{g}\widehat{M}\to\check{M}$$

yields the desired identification of $\check{M}$ with the boundary of a compact 1–manifold.

CHAPTER 12

Orientations

In this chapter we discuss orientations of moduli spaces and explain the sense in which the ungluing map of Theorem 10.1.1 is orientation preserving (after reordering the factors in the target space).

We will adopt the approach to orientations of Fredholm operators (and families of such) introduced by Benevieri–Furi [8], which was brought to our attention by Shuguang Wang [51]. This approach is more economical than the traditional one using determinant line bundles in the sense that it produces the orientation double cover directly. It also fits well in with gluing theory.

After reviewing Benevieri–Furi orientations in Section 12.1 we study orientations of unframed and (multi)framed moduli spaces and the relationship between these in Section 12.2. The framings require some extra care because of reducibles. The orientation cover $\lambda \to \mathcal{B} = \mathcal{B}(X;\vec{\alpha})$ is defined by the family of Fredholm operators $\mathcal{I}_S^* + D\Theta_S$ parametrized by $S \in \mathcal{C}$ (cf [14, p 130]). Any section of λ (which is always trivial; see Proposition 12.4.1 below) defines an orientation of the regular part of the moduli space $M_{\mathfrak{b}}^*$ for any finite, oriented subset $\mathfrak{b} \subset X$. If all limits α_j are reducible then any homology orientation of X determines a section of λ; see Proposition 12.2.1 below. To relate ungluing maps to orientations we show that, in the notation of Section 10.2, any section of $\lambda \to \mathcal{B}$ determines a section of the orientation cover $\lambda' \to \mathcal{B}'$. (Here $\mathcal{B}, \mathcal{B}'$ are configuration spaces over $X, X^{(T)}$, respectively.) This is explained in Section 12.4 after some preparation in Section 12.3 concerning framings. With this background material in place, the result on ungluing maps, Theorem 12.4.1, is an easy consequence of earlier estimates. Section 12.5 addresses the question of whether gluing of orientations in the above sense is compatible with gluing of homology orientations in the case when all limits α_j, α_j' are reducible.

12.1 Benevieri–Furi orientations

We first review Benevieri–Furi's concept of orientability of a Fredholm operator $L\colon E \to F$ of index 0 between real Banach spaces. A *corrector* of L is a bounded operator $A\colon E \to F$ with finite dimensional image such that $L + A$ is an isomorphism. We introduce the following equivalence relation in the set $\mathcal{C}(L)$ of correctors of L. Given $A, B \in \mathcal{C}(L)$ set

$$P = L + A, \quad Q = L + B.$$

Let F_0 be any finite dimensional subspace of F containing the image of $A - B$. Then QP^{-1} is an automorphism of F which maps F_0 into itself. We call A and B *equivalent* if the map $F_0 \to F_0$ induced by QP^{-1} is orientation preserving (which holds by convention if $F_0 = 0$). This condition is independent of F_0. The set $\mathcal{C}(L)$ is now partitioned into two equivalence classes (unless $E = F = 0$), and we define an *orientation* of L to be a choice of an equivalence class, the elements of which are then called *positive* correctors. A corrector which is not positive is called *negative*. Given $\epsilon = \pm 1$, a corrector is called an *ϵ–corrector* if it is positive or negative according to the sign of ϵ.

Benevieri–Furi consider $Q^{-1}P$ instead of QP^{-1}, but it is easy to see that this yields the same equivalence relation.

Note that the equivalence classes are open and closed subsets of $\mathcal{C}(L)$ with respect to the operator norm. To see this, observe that $\mathcal{C}(L)$ is open among the bounded operators $E \to F$ of finite rank. Therefore, if B is a corrector sufficiently close to a given corrector A, then $A_t = (1-t)A + tB$ is a corrector for $0 \le t \le 1$. Since $\mathrm{im}(A_t - A) \subset \mathrm{im}(A) + \mathrm{im}(B)$, it follows by continuity that the A_t are all equivalent. In particular, A and B are equivalent.

If $L\colon E \to F$ is a Fredholm operator of arbitrary index then for any nonnegative integers m, n we can form the operator

$$L_{m,n}\colon E \oplus \mathbb{R}^m \to F \oplus \mathbb{R}^n, \quad (x, 0) \mapsto (Lx, 0). \tag{12.1}$$

If L has index 0 then for any m there is a canonical correspondence between orientations of L and orientations of $L_{m,m}$ such that if A is a positive corrector of L then $A \oplus I_{\mathbb{R}^m}$ is a positive corrector of $L_{m,m}$. If L has index $k \neq 0$ then we define an orientation of L to be an orientation (in the above sense) of $L_{0,k}$ (if $k > 0$) or $L_{-k,0}$ (if $k < 0$).

Note that if A is a corrector of $L_{m,n}$ where $n - m = \mathrm{index}(L)$, and C an automorphism of $\mathbb{R}^m$ then A is equivalent to $A \circ (I_E \oplus C)$ if and only if $\det(C) > 0$, and similarly for automorphisms of $\mathbb{R}^n$.

A complex linear Fredholm operator carries a canonical orientation (in this case we replace $\mathbb{R}$ by $\mathbb{C}$ in (12.1) and the orientation is then given by any complex linear corrector).

We will now associate to any pair of oriented Fredholm operators $L_j\colon E_j \to F_j$, $j = 1, 2$ an orientation of their direct sum

$$L_1 \oplus L_2\colon E_1 \oplus E_2 \to F_1 \oplus F_2.$$

Let the orientation of L_j be given by a corrector A_j of $(L_j)_{m_j,n_j}$, where $n_j - m_j = \operatorname{index}(L_j)$. Then we decree that

$$E_1 \oplus E_2 \oplus \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2} \to F_1 \oplus F_2 \oplus \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2},$$
$$(x_1, x_2, y_1, y_2) \mapsto A_1(x_1, y_1) + A_2(x_2, y_2)$$

is a $(-1)^{m_1 \cdot \operatorname{ind}(L_2)}$ corrector of $(L_1 \oplus L_2)_{m_1+m_2, n_1+n_2}$. The sign is chosen so that the induced orientation of $L_1 \oplus L_2$ is independent of the choice of m_j, n_j. It is easily verified that under the natural identification of the operators $L_1 \oplus L_2$ and $L_2 \oplus L_1$ their orientations differ by the sign $(-1)^{\operatorname{ind}(L_1)\operatorname{ind}(L_2)}$. If L_3 is a third oriented Fredholm operator then the natural identification of the operators $(L_1 \oplus L_2) \oplus L_3$ and $L_1 \oplus (L_2 \oplus L_3)$ respects orientations.

We now consider families of Fredholm operators. Let $\mathbf{E}, \mathbf{F}$ be Banach vector bundles over a topological space T, with fibres $\mathbf{E}_t, \mathbf{F}_t$ over $t \in T$. (We require that these satisfy the analogues of the vector bundle axioms VB 1–3 in Lang [32, pp 41-2] in the topological category.) Let $L(\mathbf{E}, \mathbf{F})$ denote the Banach vector bundle over T whose fibre over t is the Banach space of bounded operators $\mathbf{E}_t \to \mathbf{F}_t$. Suppose h is a (continuous) section of $L(\mathbf{E}, \mathbf{F})$ such that $h(t)\colon \mathbf{E}_t \to \mathbf{F}_t$ is a Fredholm operator of index 0 for every $t \in T$. If $\mathbf{E}_t \neq 0$ for every t then there is a natural double cover $\widetilde{h} \to T$, the *orientation cover* of h, whose fibre over t consists of the two orientations of $h(t)$. If $U \subset T$ is an open subset and a a section of $L(\mathbf{E}, \mathbf{F})$ such that $a(t)$ has finite rank for all $t \in U$ then a defines a trivialization of $\widetilde{h}$ over the open set of those $t \in U$ for which $h(t) + a(t)$ is an isomorphism. An *orientation* of h is by definition a section of $\widetilde{h}$. If instead each $h(t)$ has index $k \neq 0$ then we define the orientation cover $\widetilde{h}$ and orientations of h by first turning h into a family of index 0 operators as above and then applying the definitions just given for such families.

If $h_1(t), h_2(t)$ are two families of Fredholm operators parametrized by $t \in T$, and $h(t) = h_1(t) \oplus h_2(t)$, then the above direct sum construction of orientations yields an isomorphism of $\mathbb{Z}/2$–bundles over T,

$$\widetilde{h}_1 \otimes \widetilde{h}_2 \xrightarrow{\approx} \widetilde{h}, \tag{12.2}$$

where $\otimes$ refers to the operation on $\mathbb{Z}/2$–bundles which corresponds to tensor product of the associated real line bundles.

Wang [51] established a 1–1 correspondence between orientations of any family of index 0 Fredholm operators (between fixed Banach spaces) and orientations of its determinant line bundle. While we will make no use of determinant line bundles in this book, we need to fix our convention for passing between orientations of a Fredholm operator $L\colon E \to F$ of arbitrary index and orientations of its determinant line,

$$\det(L) = \Lambda^{\max} \ker(L) \otimes \Lambda^{\max} \operatorname{coker}(L)^*.$$

(This will only be used to decide how a homology orientation of a 4–manifold induces orientations of its moduli spaces.) Set $n = \dim \ker(L)$ and $m = \dim \operatorname{coker}(L)$. Choose bounded operators $A_1\colon E \to \mathbb{R}^n$ and $A_2\colon \mathbb{R}^m \to F$ which induce isomorphisms

$$\widetilde{A}_1\colon \ker(L) \to \mathbb{R}^n, \quad \widetilde{A}_2\colon \mathbb{R}^m \to \operatorname{coker}(L).$$

Then

$$A = \begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}$$

is a corrector of $L_{m,n}$ which also defines an isomorphism $J_A\colon \det(L) \to \mathbb{R}$. Moreover, two such correctors A, B are equivalent if and only if $J_A J_B^{-1}$ preserves orientation. (To see this, note that after altering A_j, B_j by automorphisms of $\mathbb{R}^m$ or $\mathbb{R}^n$ as appropriate one can assume that $\widetilde{A}_j = \widetilde{B}_j$, in which case $(1-t)A + tB$ is a corrector of $L_{m,n}$ for every $t \in \mathbb{R}$.) This provides a 1–1 correspondence between orientations of L and orientations of $\det(L)$.

12.2 Orientations of moduli spaces

In the situation of Section 3.4 set

$$\mathcal{S} = L^{p,w}_1(X; i\Lambda^1 \oplus \mathbb{S}^+), \quad \mathcal{F}_1 = L^{p,w}(X; i\mathbb{R}), \quad \mathcal{F}_2 = L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-) \tag{12.3}$$

and consider the family of Fredholm operators

$$\delta_S = \mathcal{I}^*_\Phi + D\Theta_S\colon \mathcal{S} \to \mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2$$

parametrized by $S = (A, \Phi) \in \mathcal{C}(X; \vec{\alpha})$. This family is gauge equivariant in the sense that

$$\delta_{u(S)}(us) = u\delta_S(s)$$

for any $s \in L^{p,w}_1$, $u \in \mathcal{G}$, where as usual u acts trivially on differential forms and by complex multiplication on spinors. Thus, if C is a corrector of $(\delta_S)_{\ell,m}$, where

$m-\ell=\mathrm{ind}(\delta_S)$, then uCu^{-1} is a corrector of $(\delta_{u(S)})_{\ell,m}$. This defines a continuous action of $\mathcal{G}$ on the orientation cover $\widetilde{\delta}$ such that the projection $\widetilde{\delta}\to\mathcal{C}(X;\vec{\alpha})$ is $\mathcal{G}$–equivariant. The local slice theorem and Lemma 12.2.1 below then show that $\widetilde{\delta}$ descends to a double cover $\lambda\to\mathcal{B}(X;\vec{\alpha})$.

(Note that in the situation of Section 2.4, the local slice theorem for the group $\mathcal{G}$ at a reducible point $(A,0)\in\mathcal{C}$ is easily deduced from the version of Proposition 2.4.2 with $\mathfrak{b}$ consisting of one point from each component of X where Φ vanishes a.e.)

Lemma 12.2.1 *Let the topological group G act continuously on the spaces $Z,\widetilde{Z}$, and let $\pi\colon\widetilde{Z}\to Z$ be a G–equivariant covering map. Suppose any point in Z has arbitrarily small open neighbourhoods U such that for any $z\in U$ the set*

$$\{g\in G\;:\;gz\in U\}$$

is connected. Then the natural map $\widetilde{\pi}\colon\widetilde{Z}/G\to Z/G$ is a covering whose pullback to Z is canonically isomorphic to π. Pull-back defines a 1–1 correspondence between (continuous) sections of $\widetilde{\pi}$ and G–equivariant sections of π. If in addition G is connected then any section of π is G–equivariant.

Proof Let $p\colon Z\to Z/G$ and $q\colon\widetilde{Z}\to\widetilde{Z}/G$. If U is as in the lemma and s is a section of π over U then for all $z\in U$, $g\in G$ with $gz\in U$ one has

$$s(gz)=gs(z).$$

Hence s descends to a section of $\widetilde{\pi}$ over $p(U)$. If in addition $\pi^{-1}U$ is the disjoint union of open sets V_j each of which is mapped homeomorphically onto U by π then $q(V_j)\cap q(V_k)=\varnothing$ when $j\neq k$. Moreover, $\bigcup_j q(V_j)=\widetilde{\pi}^{-1}p(U)$. □

The following proposition extends a well-known result in the case when X is closed (see Morgan [38] and Salamon [45]). A related result was proved in Nicolaescu [41, Proposition 4.4.18].

Proposition 12.2.1 *If each Y_j is a rational homology sphere and each α_j is reducible then any homology orientation of X canonically determines a section of $\lambda\to\mathcal{B}(X;\vec{\alpha})$.*

Proof We may assume $\vec{\mathfrak{p}}=0$, since rescaling $\vec{\mathfrak{p}}$ yields a homotopy of families δ. For any $(A,0)\in\mathcal{C}(X;\vec{\alpha})$ the operator $\delta_{(A,0)}$ is the connected sum of the operators $-d^*+d^+$ and D_A. While the homology orientation of X determines an orientation of $-d^*+d^+$ (whose cokernel we identify with $H^0\oplus H^+$ rather than $H^+\oplus H^0$, where H^+ now denotes the space of self-dual closed L^2 2–forms on X), the family of complex linear operators D_A carries a natural orientation which is preserved by the action of $\mathcal{G}$. This yields a section of λ over the reducible part $\mathcal{B}^{\mathrm{red}} \subset \mathcal{B} = \mathcal{B}(X;\vec{\alpha})$. Since the map $([A,\Phi],t) \mapsto [A,t\Phi]$, $0 \le t \le 1$ is a deformation retraction of $\mathcal{B}$ to $\mathcal{B}^{\mathrm{red}}$, we also obtain a section of λ over $\mathcal{B}$. □

Returning to the situation discussed before Lemma 12.2.1, a section of λ determines an orientation of the regular part of the moduli space $M^*(X;\vec{\alpha})$. As we will now explain, it also determines an orientation of the regular part of $M^*_{\mathfrak{b}}(X;\vec{\alpha})$ for any finite oriented subset $\mathfrak{b} \subset X$. (By an orientation of $\mathfrak{b}$ we mean an equivalence class of orderings, two orderings being equivalent if they differ by an even permutation.)

Let $\mathcal{W}$ be the space of spinors that may occur in elements of $\mathcal{B}^*_{\mathfrak{b}}(X;\vec{\alpha})$; more precisely, $\mathcal{W}$ is the open subset of $\Phi_o + L^{p,w}_1$ consisting of those elements Φ such that $\mathfrak{b} \cup \operatorname{supp}(\Phi)$ intersects every component of X. For any such Φ the operator

$$\Delta_\Phi := \mathcal{I}^*_\Phi \mathcal{I}_\Phi = \Delta + |\Phi|^2 \colon L\mathcal{G} \to \mathcal{F}_1 \tag{12.4}$$

is injective on $L\mathcal{G}_{\mathfrak{b}}$, hence

$$\mathcal{V}_\Phi := \mathcal{F}_1 / \Delta_\Phi(L\mathcal{G}_{\mathfrak{b}})$$

has dimension $b := |\mathfrak{b}|$ by Proposition 2.3.1 (i). Since $\Phi \mapsto \Delta_\Phi$ is a smooth map from $\Phi_o + L^{p,w}_1$ into the space of bounded operators $L\mathcal{G} \to \mathcal{F}_1$, the spaces $\mathcal{V}_\Phi$ form a smooth vector bundle $\mathcal{V}$ over $\mathcal{W}$. Because $\mathcal{W}$ is simply connected, $\mathcal{V}$ is orientable. To specify an orientation it suffices to consider those Φ that do not vanish identically on any component of X. Given such a Φ, the operator (12.4) is an isomorphism, and we decree that a b–tuple $g_1,\dots,g_b \in \mathcal{F}_1$ spanning a linear complement of $\Delta_\Phi(L\mathcal{G}_{\mathfrak{b}})$ is *positive* if the determinant of the matrix

$$\big(-i\Delta_\Phi^{-1}(g_j)(x_k)\big)_{j,k=1,\dots,b}$$

is positive, where $(x_1,\dots,x_b)$ is any positive ordering of $\mathfrak{b}$.

It is natural to ask what it means for $\{g_j\}$ to be a positive basis for $\mathcal{V}_\Phi$ when $\Phi = 0$. Section 12.7 answers this question in the case $b = 1$.

For the purpose of understanding ungluing maps it is convenient to introduce local slices for the action of $\mathcal{G}_{\mathfrak{b}}$ that are defined by compactly supported functions on X. Given $S = (A,\Phi) \in \mathcal{C}^*_{\mathfrak{b}}(X;\vec{\alpha})$, choose compactly supported smooth functions $g_j, h_j \colon X \to i\mathbb{R}$, $j = 1,\dots,b$, such that

$$\int_X g_j h_k = \delta_{jk}, \tag{12.5}$$

where δ_{jk} is the Kronecker symbol, and such that $(g_1,\dots,g_b)$ represents a positive basis for $\mathcal{V}_\Phi$. (Note that there is a preferred choice of h_k, which lies in the linear span of the g_j's.) We define the operator $\mu\colon \mathcal{F}_1 \to \mathcal{F}_1$ by

$$\mu f = f - \sum_{j=1}^{b} g_j \int_X f h_j. \tag{12.6}$$

Clearly, this is a projection operator whose kernel is spanned by $g_1, \dots, g_b$. Furthermore, μ restricts to an isomorphism

$$\Delta_\Phi(L\mathcal{G}_\mathfrak{b}) \to \mathrm{im}(\mu). \tag{12.7}$$

Set $\mathcal{I}^\#_\Phi = \mu \circ \mathcal{I}^*_\Phi$ and

$$\delta_{\mu,S} := \mathcal{I}^\#_\Phi + D\Theta_S\colon \mathcal{S} \to \mathrm{im}(\mu) \oplus \mathcal{F}_2. \tag{12.8}$$

After composing with the inverse of (12.7), $\mathcal{I}^\#_\Phi$ becomes an operator of the same kind as considered in Section 3.4. Therefore, the local slice theorem Proposition 2.4.2 applies, and if S represents a regular point of $M^*_\mathfrak{b}(X;\vec{\alpha})$ then an orientation of $\delta_{\mu,S}$ defines an orientation of the tangent space $T_{[S]}M^*_\mathfrak{b}(X;\vec{\alpha})$.

We will now relate orientations of δ_S to orientations of $\delta_{\mu,S}$. For any imaginary valued function f on X let $\mu' f \in \mathbb{R}^b$ have coordinates $\int_X f h_j$, $j = 1, \dots, b$. Choose nonnegative integers ℓ, m with $m - \ell = \mathrm{index}(\delta_S)$ and set

$$\begin{gathered} \nu : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathbb{R}^m \xrightarrow{\approx} \mathrm{im}(\mu) \oplus \mathcal{F}_2 \oplus \mathbb{R}^b \oplus \mathbb{R}^m, \\ (x_1, x_2, y) \mapsto (\mu x_1, x_2, \mu' x_1, y). \end{gathered}$$

To any corrector C of $(\delta_S)_{\ell,m}$ we associate a corrector $\mathsf{C}_\mathfrak{b}$ of $(\delta_{\mu,S})_{\ell,b+m}$ given by

$$(\delta_{\mu,S})_{\ell,b+m} + \mathsf{C}_\mathfrak{b} = \nu \circ ((\delta_S)_{\ell,m} + \mathsf{C}).$$

For gauge transformations u one has

$$(u\mathsf{C}u^{-1})_\mathfrak{b} = u\mathsf{C}_\mathfrak{b}u^{-1},$$

where u acts by multiplication on spinors and trivially on the other components. Moreover, the map $\mathsf{C} \mapsto \mathsf{C}_\mathfrak{b}$ clearly respects the equivalence relation for correctors. We define a 1–1 correspondence between orientations of δ_S and orientations of $\delta_{\mu,S}$ as follows: If C is a positive corrector of $(\delta_S)_{\ell,m}$ then $\mathsf{C}_\mathfrak{b}$ is a $(-1)^{b\cdot\mathrm{ind}(\delta_S)}$–corrector of $(\delta_{\mu,S})_{\ell,b+m}$. (The sign is chosen so as to make Diagram (12.13) below commutative.)

If $[S]$ is a regular point of $M^*_\mathfrak{b}(X;\vec{\alpha})$ then the above defined correspondence between orientations of δ_S and orientations of $M^*_\mathfrak{b}(X;\vec{\alpha})$ at $[S]$ does not depend on the choice of the $2b$–tuple $g_1, \dots, g_b, h_1, \dots, h_b$, because the space of such $2b$–tuples supported in a given compact subset of X is path-connected in the C^∞–topology.

The relationship between the orientations of $M^*=M^*(X;\vec{\alpha})$ and $M^*_{\mathfrak{b}}=M^*_{\mathfrak{b}}(X;\vec{\alpha})$ can be described explicitly as follows (assuming M^* is regular). Let $M^{**}_{\mathfrak{b}}$ be the part of $M^*_{\mathfrak{b}}$ that lies above M^*. Then $\pi\colon M^{**}_{\mathfrak{b}}\to M^*$ is a principal $\mathrm{U}(1)^b$–bundle whose fibres inherit orientations from $\mathrm{U}(1)^b$. If $(v_1,\dots,v_{d-b})$ is a $(d-b)$–tuple of elements of the tangent space $T_\omega M^{**}_{\mathfrak{b}}$ which maps to a positive basis for $T_{\pi(\omega)}M^*$, and $(v_{d-b+1},\dots,v_d)$ a positive basis for the vertical tangent space of $M^{**}_{\mathfrak{b}}$ at ω, then $(v_1,\dots,v_d)$ is a positive basis for $T_\omega M^{**}_{\mathfrak{b}}$.

12.3 Gluing and the Laplacian

We continue the discussion of the previous section, but we now consider the situation of Section 10.1, so that the ends of X are labelled as in Section 1.4 and $\mathfrak{b}\subset X$ is the set of start-points of the paths $\gamma_j^{\pm}$, $j=1,\dots,r_0$.

We define the function spaces $\mathcal{S}'$ and $\mathcal{F}'=\mathcal{F}'_1\oplus\mathcal{F}'_2$ over $X^{(T)}$ just as the corresponding spaces $\mathcal{S},\mathcal{F}$ etc over X, replacing the weight function w by κ. We also define the space $\mathcal{W}'$ of spinors over $X^{(T)}$ and the oriented vector bundle $\mathcal{V}'\to\mathcal{W}'$ in the same way as $\mathcal{V}\to\mathcal{W}$, using the same set $\mathfrak{b}$.

Let $S=(A,\Phi)\in\mathcal{C}$ be a configuration over X such that $S-S_o$ is compactly supported, where S_o is the reference configuration over X. For large $T_{\min}$ consider the glued configuration $S'=(A',\Phi')$ over $X^{(T)}$; this is the smooth configuration over $X^{(T)}$ which agrees with S over $\mathrm{int}(X_{:T})$. (This notation will also be used in later sections. For the time being we are only interested in the spinors.) Let $g=(g_1,\dots,g_b)$ be as in the previous section.

Lemma 12.3.1 *If g represents a positive basis for $\mathcal{V}_\Phi$ then g also represents a positive basis for $\mathcal{V}'_{\Phi'}$ when $T_{\min}$ is sufficiently large.*

(In this lemma it is not essential that X be a 4–manifold or that $\mathbb{S}^+$ be a spinor bundle, one could just as well use the more general set-up in Section 2.1, at least if $p>\dim X$.)

Proof There is one case where the lemma is obvious, namely when Φ does not vanish on any component of X and $g_j=\Delta_\Phi f_j$, where f_j is compactly supported. We will prove the general case by deforming a given set of data Φ,g to one of this special form. We begin by establishing a version of the lemma where "positive basis" is replaced by "basis" and one considers compact families of such data Φ,g. To make this precise, choose $\rho>0$ such that $\mathrm{supp}(g_j)\subset X_{:\rho}$ for each j, and let $T_{\min}>\rho$. Let $\Gamma'\subset L^p_1(X;\mathbb{S}^+)$ and $\Gamma''\subset C^\infty(X;(i\mathbb{R})^b)$ be the subspaces consisting of those elements that vanish outside $X_{:\rho}$. Let Γ (resp. Γ_T) be the set of pairs $(\phi,g)\in\Gamma'\times\Gamma''$

such that $\Phi_o+\phi\in\mathcal{W}$ (whence $\Phi'_o+\phi\in\mathcal{W}'$) and such that g represents a basis for $\mathcal{V}_{\Phi_o+\phi}$ (resp. $\mathcal{V}'_{\Phi'_o+\phi}$).

Sublemma 12.3.1 *If K is any compact subset of Γ then $K\subset\Gamma_T$ for $T_{\min}\gg 0$.*

Assuming the sublemma for the moment, choose $\phi\in\Gamma'$ such that on each component of X exactly one of Φ,ϕ is zero. Choose smooth functions $f_j\colon X\to i\mathbb{R}$, $j=1,\dots,b$, which are supported in $X_{:\rho}$ and satisfy $f_j(x_k)=i\delta_{jk}$, where $(x_1,\dots,x_b)$ is a positive ordering of $\mathfrak{b}$. Choose a small $\epsilon>0$ and set $\widetilde{g}_j=\Delta_{\Phi+\epsilon\phi}f_j$. Choose a path $((t),\mathbf{g}(t))$, $0\le t\le 1$ in Γ from (Φ,g) to $(\Phi+\epsilon\phi,\widetilde{g})$ such that $\mathbf{g}(t)=g$ for $0\le t\le\epsilon$ and

$$\mathbf{\Phi}(t)=\begin{cases}\Phi+t\phi, & 0\le t\le\epsilon,\\ \Phi+\epsilon\phi, & \epsilon\le t\le 1.\end{cases}$$

Let $\mathbf{\Phi}'(t)$ be the glued spinor over $X^{(T)}$ obtained from $\mathbf{\Phi}(t)$. By the sublemma, if $T_{\min}\gg 0$ then for $0\le t\le 1$ one has $(\mathbf{\Phi}'(t),\mathbf{g}(t))\in\Gamma_T$. Since $\widetilde{g}$ represents a positive basis for $\mathcal{V}'_{\Phi'+\epsilon\phi}$, it follows by continuity that g must represent a positive basis for $\mathcal{V}'_{\Phi'}$. This proves the lemma assuming the sublemma.

Proof of Sublemma 12.3.1 Suppose to the contrary that for $n=1,2,\dots$ there are $(\phi(n),g(n))\in K\setminus\Gamma_{T(n)}$, where $T_{\min}(n)\to\infty$. We may assume $(\phi(n),g(n))\to(\phi,g)$ in K. Let V be the linear span of $g_1,\dots,g_b$ and V_n the linear span of $g_1(n),\dots,g_b(n)$. Set

$$\Phi'_n=\Phi'_o+\phi(n).$$

By assumption there exists a nonzero $f_n\in L\mathcal{G}'_{\mathfrak{b}}$ with $\Delta_{\Phi'_n}f_n\in V_n$. Choose real numbers σ,τ with $\rho<\sigma<\tau$. Since $\Delta_{\Phi'_o}f_n=0$ outside $X_{:\rho}$, unique continuation implies that f_n cannot vanish identically on $X_{:\tau}$, so we may assume that

$$\|f_n\|_{L^p_2(X_{:\tau})}=1.$$

We digress briefly to consider an injective bounded operator $J\colon E\to F$ between normed vector spaces and for fixed m a sequence of linear maps $P_n\colon\mathbb{R}^m\to E$ which converges in the operator norm to an injective linear map P. Then there is a constant $C<\infty$ such that $\|e\|<C\|Je\|$ for all e in a neighbourhood U of $P(S^{m-1})$. For large n one must have $P_n(S^{m-1})\subset U$, hence

$$\|P_nx\|\le C\|JP_nx\|$$

for all $x\in\mathbb{R}^m$.

We apply this result with $m = b$, $E = L^p_1(X)$, $F = L^p(X)$, J the inclusion map, $P_n x = \sum_j x_j g_j(n)$, and $Px = \sum_j x_j g_j$. We conclude that there is a constant $C < \infty$ such that for sufficiently large n one has

$$\|v\|_{L^p_1} \le C\|v\|_{L^p}$$

for all $v \in V_n$. For such n,

$$\begin{aligned}
\|f_n\|_{L^p_3(X_{:\sigma})} &\le \text{const}\cdot\big(\|\Delta f_n\|_{L^p_1} + \|f_n\|_{L^p_2}\big)\\
&\le \text{const}\cdot\big(\|\Delta_{\Phi'_n} f_n\|_{L^p_1} + \||\Phi'_n|^2 f_n\|_{L^p_1} + 1\big)\\
&\le \text{const}\cdot\big(\|\Delta_{\Phi'_n} f_n\|_{L^p} + \|\Phi'_n\|^2_{L^p_1}\|f_n\|_{L^p_1} + 1\big)\\
&\le \text{const}\cdot\big(\|\Delta f_n\|_{L^p} + 1\big)\\
&\le \text{const},
\end{aligned}$$

where except in the first term all norms are taken over $X_{:\tau}$.

Let ψ_j, ψ'_j be the spinor parts of α_j, α'_j, respectively. Fix n for the moment and write $\overline{\rho} = T_j(n) - \rho$. Define $\overline{\sigma}$ similarly. Over $[-\overline{\rho}, \overline{\rho}] \times Y_j$ we then have

$$(-\partial_1^2 + \Delta_{\psi_j}) f_n = 0,$$

where $\partial_1 = \frac{\partial}{\partial t}$ and $\Delta_{\psi_j} = \Delta_{Y_j} + |\psi_j|^2$. If h is any continuous real function on $[-\overline{\rho}, \overline{\rho}] \times Y_j$ satisfying $(-\partial_1^2 + \Delta_{\psi_j})h = 0$ on $(-\overline{\rho}, \overline{\rho}) \times Y_j$ then for any nonnegative integer k and $t \in [-\overline{\sigma}+1, \overline{\sigma}-1]$ one has

$$\|h\|_{C^k([t-1,t+1]\times Y_j)} \le \text{const}\cdot\big(\|h\|_{L^2(\{-\overline{\rho}\}\times Y_j)} + \|h\|_{L^2(\{\overline{\rho}\}\times Y_j)}\big),$$

where the constant is independent of n, t. (To see this, expand h in terms of eigenvectors of Δ_{ψ_j} and note that each coefficient function c satisfies an equation $c'' = \lambda^2 c$, $\lambda \in \mathbb{R}$, which yields $(c^2)'' = 2(c')^2 + 2(\lambda c)^2 \ge 0$. Combine this with the usual elliptic estimates.) Similarly, if h is any bounded continuous function on $[\rho, \infty) \times Y'_j$ satisfying $(-\partial_1^2 + \Delta_{\psi'_j})h = 0$ on $(\rho, \infty) \times Y'_j$ then for any nonnegative integer k and $t \ge \sigma$ one has

$$\|h\|_{C^k([t,t+1]\times Y'_j)} \le \text{const}\cdot\|h\|_{L^2(\{\rho\}\times Y'_j)},$$

for some constant independent of t.

After passing to a subsequence, we may therefore assume that f_n converges in L^p_2 over compact subsets of X to some function f, whose restriction to each end of X must be the sum of a constant function and an exponentially decaying one, the constant function being zero if the limiting spinor over that end (ψ_j or ψ'_j) is nonzero. In particular, $f \in L\mathcal{G}_{\mathfrak{b}}$. Furthermore,

$$\Delta_{\Phi_o+\phi} f \in V, \qquad \|f\|_{L^p_2(X_{:\tau})} = 1.$$

Since $\Delta_{\Phi_o+\phi}$ is injective on $L\mathcal{G}_{\mathfrak{b}}$ this contradicts the assumption that V is a linear complement of $\Delta_{\Phi_o+\phi}(L\mathcal{G}_{\mathfrak{b}})$ in $\mathcal{F}_1$.

This completes the proof of Sublemma 12.3.1 and thus the proof of Lemma 12.3.1. □

12.4 Orientations and gluing

Let S, S' be as in the beginning of Section 12.3. For the time being we will consider a map μ defined by fixed but arbitrary b–tuples $\{g_j\}$, $\{h_j\}$ of imaginary valued, compactly supported, smooth functions on X satisfying the duality relation (12.5), where b is any nonnegative integer. We will show that an orientation of $\delta_{\mu,S}$ canonically determines an orientation of $\delta_{\mu,S'}$ for large $T_{\min}$. Set

$$\mathcal{F}_\mu = \mathrm{im}(\mu) \oplus \mathcal{F}_2, \quad \mathcal{F}'_\mu = \mathrm{im}(\mu) \oplus \mathcal{F}'_2.$$

Choose $\tau > 1$ so large that the functions g_j, h_j are all supported in $X_{:\tau}$, and define

$$\mathcal{S}_{:\tau} = L^p_1(X_{:\tau}; i\Lambda^1 \oplus \mathbb{S}^+).$$

Let $\mathcal{S}^{:\tau}$ be the subspace of $\mathcal{S}$ consisting of those elements that are supported in $X_{:\tau}$, and define $\mathcal{F}^{:\tau}_\mu \subset \mathcal{F}_\mu$ similarly. Set

$$\mathcal{C}^{:\tau} = S_o + \mathcal{S}^{:(\tau-1)}.$$

In other words, $\mathcal{C}^{:\tau}$ is the set of all $L^p_{1,\mathrm{loc}}$ configurations S over X such that $S - S_o$ is supported in $X_{:(\tau-1)}$. (The $\tau - 1$ is chosen here because of the nonlocal nature of our perturbations.) Suppose we are given a bounded operator

$$\mathsf{C}\colon \mathcal{S}_{:\tau} \oplus \mathbb{R}^\ell \to \mathcal{F}^{:\tau}_\mu \oplus \mathbb{R}^m \tag{12.9}$$

with finite dimensional image, where $m - \ell = \mathrm{index}(\delta_{\mu,S})$. Clearly, C induces linear maps

$$\mathcal{S} \oplus \mathbb{R}^\ell \to \mathcal{F}_\mu \oplus \mathbb{R}^m, \quad \mathcal{S}' \oplus \mathbb{R}^\ell \to \mathcal{F}'_\mu \oplus \mathbb{R}^m$$

(the latter when $T_{\min} \geq \tau$); these will also be denoted by C. Fix an r_0–tuple of paths $\gamma = (\gamma_1, \dots, \gamma_{r_0})$ as in Section 10.1, and for any imaginary valued 1–form a on $X^{(T)}$ let $H_\gamma(a) \in \mathbb{R}^{r_0}$ have coordinates

$$H_{\gamma_j}(a) := \int_{\gamma_j} ia, \quad j = 1, \dots, r_0.$$

Lemma 12.4.1 *There exists a constant $C<\infty$ with the property that if C is any map as above and S any element of $\mathcal{C}^{:\tau}$ such that*

$$D:=\delta_{\mu,S}+\mathsf{C}\colon \mathcal{S}\oplus\mathbb{R}^{\ell}\to\mathcal{F}_{\mu}\oplus\mathbb{R}^{m}$$

is invertible, then

$$E:=\delta_{\mu,S'}+H_{\gamma}+\mathsf{C}\colon \mathcal{S}'\oplus\mathbb{R}^{\ell}\to\mathcal{F}'_{\mu}\oplus\mathbb{R}^{r_0}\oplus\mathbb{R}^{m} \tag{12.10}$$

is invertible when $T_{\min}>C(\|D^{-1}\|+1)$.

Proof Let P_j be a bounded right inverse of the operator (10.6). As in Appendix C, if $T_{\min}>\mathrm{const}\cdot(\|D^{-1}\|+\sum_j\|P_j\|)$ then we can splice $D^{-1},P_1,\dots,P_r$ to obtain a right inverse R of

$$\delta_{\mu,S'}+\mathsf{C}\colon \mathcal{S}'\oplus\mathbb{R}^{\ell}\to\mathcal{F}'_{\mu}\oplus\mathbb{R}^{m}.$$

(The present situation is slightly different from that in the appendix, but the construction there carries over.) Furthermore,

$$\|R\|\le\mathrm{const}\cdot(\|D^{-1}\|+\sum_j\|P_j\|).$$

Let $\mathsf{f}\colon\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\mathsf{f}(t)=0$ for $t\le 1/2$ and $\mathsf{f}(t)=1$ for $t\ge 1$. Set

$$q_j(t)=\mathsf{f}(T_j-\tau+t)\mathsf{f}(T_j-\tau-t).$$

Thus, q_j approximates the characteristic function of the interval $[-T_j+\tau,T_j-\tau]$. For $c=(c_1,\dots,c_{r_0})\in\mathbb{R}^{r_0}$ let $\eta(c)$ be the imaginary valued 1–form on $X^{(T)}$ given by

$$\eta(c)=\begin{cases}0 & \text{outside } \bigcup_{j=1}^{r_0}[-T_j,T_j]\times Y_j,\\ -(2T_j)^{-1}c_jq_j\,i\,dt & \text{on } [-T_j,T_j]\times Y_j,\quad j=1,\dots,r_0.\end{cases}$$

For the present purposes it is convenient to rearrange summands and regard E as mapping into $(\mathcal{F}'_{\mu}\oplus\mathbb{R}^{m})\oplus\mathbb{R}^{r_0}$. Set

$$L=R+\eta\colon(\mathcal{F}'_{\mu}\oplus\mathbb{R}^{m})\oplus\mathbb{R}^{r_0}\to\mathcal{S}'\oplus\mathbb{R}^{\ell}.$$

Then EL takes the matrix form

$$\begin{pmatrix} I & 0\\ \beta & I\end{pmatrix}+o, \tag{12.11}$$

where for large $T_{\min}$ one has $\|\beta\|\le\mathrm{const}\cdot\|R\|$ and $\|o\|\le\mathrm{const}\cdot T_{\min}^{-1}$, the constants being independent of S,T. As in the proof of Lemma 10.2.4 we conclude that EL is invertible when $T_{\min}>\mathrm{const}\cdot(\|\beta\|+1)$, which holds if $T_{\min}>\mathrm{const}\cdot(\|D^{-1}\|+1)$. Since E has index 0, it is invertible whenever EL is surjective. □

Lemma 12.4.2 *Suppose* $\mathsf{C}, \widetilde{\mathsf{C}}$ *are two maps as in* (12.9) *which define correctors of* $(\delta_{\mu,S})_{\ell,m}$, *and let* $\gamma, \widetilde{\gamma}$ *be two* r_0*–tuples of paths as in* Section 10.1. *Then for sufficiently large* $T_{\min}$ *the following holds:* C *and* $\widetilde{\mathsf{C}}$ *define equivalent correctors of* $(\delta_{\mu,S})_{\ell,m}$ *if and only if* $H_\gamma + \mathsf{C}$ *and* $H_{\widetilde{\gamma}} + \widetilde{\mathsf{C}}$ *define equivalent correctors of* $(\delta_{\mu,S'})_{\ell,r_0+m}$.

Proof We will use the same notation as in Lemma 12.4.1 and its proof. Let $\widetilde{D}, \widetilde{E}$ be defined as D, E, replacing C, γ by $\widetilde{\mathsf{C}}, \widetilde{\gamma}$. Observe that the image of $D - \widetilde{D}$ is contained in $N \oplus \mathbb{R}^m$ for some finite dimensional subspace $N \subset \mathcal{F}_\mu^{:\tau}$, and the image of $E - \widetilde{E}$ is then contained in $(N \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0}$ (again rearranging summands). Moreover,

$$\widetilde{E} E^{-1} = \widetilde{E} L (EL)^{-1},$$

and $\widetilde{E} L$ has the form

$$\begin{pmatrix} \widetilde{D}' R & 0 \\ \beta_1 & I \end{pmatrix} + o_1, \tag{12.12}$$

where $\|\beta_1\|$ is bounded and $\|o_1\| \to 0$ as $T_{\min} \to \infty$, and $\widetilde{D}' = \delta_{\mu,S'} + \widetilde{\mathsf{C}}$. From the description (12.11) of EL we see that $\widetilde{E} E^{-1}$ also has the shape (12.12).

If $s \in \mathcal{F}_\mu^{:\tau}$ and ρ denotes restriction to $X_{:\tau}$, then

$$\begin{aligned} \|\rho \widetilde{D}' R s - \rho \widetilde{D} D^{-1} s\|_{L^{p,w}} &\le \mathrm{const} \cdot \|\widetilde{D}\| \cdot \|\rho R s - \rho D^{-1} s\|_{L^{p,w}} \\ &\le \mathrm{const} \cdot \|T_{\min}^{-1}\| \cdot \|\widetilde{D}\| \cdot \Big(\|D^{-1}\| + \sum \|P_j\| \Big) \cdot \|s\|. \end{aligned}$$

It follows that as $T_{\min} \to \infty$, the determinant of the endomorphism of $(N \oplus \mathbb{R}^m) \oplus \mathbb{R}^{r_0}$ induced by $\widetilde{E} E^{-1}$ approaches the determinant of the endomorphism of $N \oplus \mathbb{R}^m$ induced by $\widetilde{D} D^{-1}$. □

Consider again the situation before Lemma 12.4.1. Given an orientation of $\delta_{\mu,S}$ we define a *glued* orientation of $\delta_{\mu,S'}$ for $T_{\min} \gg 0$ as follows. Let C be a positive corrector $(\delta_{\mu,S})_{\ell,m}$ of the kind (12.9). Then we decree that $H_\gamma + \mathsf{C}$ is a positive corrector of $(\delta_{\mu,S'})_{\ell,r_0+m}$, where the summands are ordered as in (12.10). By continuity we can extend this to an orientation of $\delta_{\mu,S'}$ for $T_{\min} > 2\tau$. Now fix T with $T_{\min} > 2\tau$, and let $\lambda_{\mu,S}$ (resp. $\lambda_{\mu,S'}$) denote the set consisting of the two orientations of $\delta_{\mu,S}$ (resp. $\delta_{\mu,S'}$). From Lemma 12.4.1 and Lemma 12.4.2 we obtain a natural map

$$\lambda_{\mu,S} \to \lambda_{\mu,S'}.$$

There are two cases that we are interested in: One is when $b = 0$ (so that $\lambda_{\mu,S} = \lambda_S$). The other is when $b = |\mathfrak{b}|$ and $(g_1, \ldots, g_b)$ defines a positive basis for $\mathcal{V}_\Phi$. Now letting

μ refer to the second case, the preceeding discussion yields the following commutative diagram of bijections:

$$\begin{array}{ccc} \lambda_S & \longrightarrow & \lambda_{S'} \\ \downarrow & & \downarrow \\ \lambda_{\mu,S} & \longrightarrow & \lambda_{\mu,S'}. \end{array} \tag{12.13}$$

Turning to the global picture, and taking $b=0$, let $\mathcal{C}'^{[\tau]}$ denote the set of all $L^p_{1,\mathrm{loc}}$ configurations $\widetilde{S}$ over $X^{(T)}$ such that $\widetilde{S}-S'_o$ is supported in $X_{:\tau}$. Then the gluing operation $S\mapsto S'$ defines a homeomorphism $u\colon \mathcal{C}^{:\tau}\to\mathcal{C}'^{[\tau]}$, and Lemma 12.4.1 and Lemma 12.4.2 establish an isomorphism between the orientation cover of $\mathcal{C}^{:\tau}$ and the pullback by u of the orientation cover of $\mathcal{C}'^{[\tau]}$. Combining this with Proposition 12.4.1 below we see that any section of $\lambda\to\mathcal{B}$ determines a section of the orientation cover $\lambda'\to\mathcal{B}'$. (Here $\mathcal{B},\mathcal{B}'$ mean the same as in the beginning of Section 10.2 with $\mathfrak{b}=\varnothing$).

Proposition 12.4.1 *If $X,\vec{\alpha}$ are as in Section 3.4 then the orientation cover $\lambda\to\mathcal{B}(X;\vec{\alpha})$ is trivial.*

Proof We may assume X is connected. Let $\mathfrak{b}\subset X$ consist of a single point. Let $\pi\colon\mathcal{B}_\mathfrak{b}\to\mathcal{B}$ be the projection, where $\mathcal{B}=\mathcal{B}(X;\vec{\alpha})$ etc. Since $\mathcal{B}$ is the quotient of $\mathcal{B}_\mathfrak{b}$ by the natural $\mathrm{U}(1)$ action, the local slice theorem and Lemma 12.2.1 imply that any section of $\pi^*\lambda$ descends to a section of λ. It therefore suffices to show that $\pi^*\lambda$ is trivial, or equivalently, that for any loop ℓ in $\mathcal{B}$ that lifts to $\mathcal{B}_\mathfrak{b}$ the pullback $\ell^*\lambda$ is trivial. Since $\mathcal{C}\to\mathcal{B}_\mathfrak{b}$ is a (principal) fibre bundle, such a loop is the image of a path $z\colon[0,1]\to\mathcal{C}$ such that $z(1)=u(z(0))$ for some $u\in\mathcal{G}_\mathfrak{b}$. After altering the loop ℓ by a homotopy one can arrange that $u=1$ and $z(t)=S_o$ (for all t) outside a compact subset of X.

(Here is one way to construct such a homotopy. For $0\le s\le 1$ let $\xi_s=\zeta_s*\widetilde{\zeta}_s$ be the composite of the two paths (both defined for $0\le t\le 1$)

$$\begin{aligned} \zeta_s(t) &= (1-s)z(t)+sS_o, \\ \widetilde{\zeta}_s(t) &= (1-t)\zeta_s(1)+t v_s(\zeta_s(0)), \end{aligned}$$

where v_s is a path in $\mathcal{G}$ such that $v_0=u$, and $v_1=1$ outside a compact subset of X. Clearly, $\xi_0=\zeta_0$ is homotopic to z relative to $\{0,1\}$. Moreover, $v_s(\xi_s(0))=\xi_s(1)$, and $\xi_1(t)=S_o$ where $v_1=1$.)

Now let $-X$ be the Riemannian manifold X with the opposite orientation and corresponding spin^c structure. Starting with $X\cup(-X)$ we form, for any $T>0$, a compact manifold $W^{(T)}$ by gluing the j–th end of X with the j–th end of $-X$ to obtain a neck $[-T_j,T_j]\times Y_j$. Let S be any configuration over $-X$ which agrees with $\underline{\alpha}_j$ over the j–th end, and $z_1(t)$ the configuration over $W^{(T)}$ obtained by gluing S and $z(t)$. Then z_1 maps to a loop ℓ_1 in $\mathcal{B}(W^{(T)})$. By Proposition 12.4.1 the orientation cover $\lambda_1 \to \mathcal{B}(W^{(T)})$ is trivial. Now Lemma 12.4.1 and Lemma 12.4.2 yield an isomorphism of $\mathbb{Z}/2$–bundles $\ell^*\lambda \to \ell_1^*\lambda_1$ when $T_{\min}$ is large, hence $\ell^*\lambda$ is trivial. □

We now consider the situation of Theorem 10.1.1. Let $b = |\mathfrak{b}|$. Choose an orientation of $\lambda \to \mathcal{B}$, and let $\lambda' \to \mathcal{B}'$ have the glued orientation. Given an orientation of $\mathfrak{b}$, this orients the regular parts of $M_{\mathfrak{b}}$ and $M_{\mathfrak{b}}^{(T)}$.

Theorem 12.4.1 *In the situation of Theorem 10.1.1, if $T_{\min}$ is sufficiently large then the diffeomorphism*

$$\mathbf{F}\colon \mathbf{q}^{-1}G \to \mathrm{U}(1)^{r_0} \times G, \quad \omega \mapsto (\mathrm{Hol}(\omega), \mathbf{q}(\omega))$$

is orientation preserving.

Proof In view of Proposition 10.2.1 it suffices to show that, for any given point $(z,\omega) \in \mathrm{U}(1)^{r_0} \times G$, the inverse $\mathbf{F}^{-1}$ is orientation preserving at (z,ω) when $T_{\min}$ is sufficiently large.

Consider the set-up in Section 10.2, with $\varpi\colon \mathbb{R}^d \to M_{\mathfrak{b}}$ orientation preserving. Let $\pi\colon \mathcal{C}(K) \to \breve{\mathcal{B}}(K)$ be the projection. Then $f := \varpi^{-1} \circ q \circ \pi$ maps a small neighbourhood of $S_0|_K$ in $\mathcal{C}(K)$ to $\mathbb{R}^d$. Let

$$\mathsf{C}\colon L^p_1(K; i\Lambda^1 \oplus \mathbb{S}^+) \to \mathbb{R}^d$$

be the derivative of f at S_0. Let μ be as in (12.6), with Φ the spinor part of S_0. For $0 \le t \le 1$ set

$$\begin{aligned} S(t) &= (1-t)S_0 + tS_{0,\tau-2}, \quad \delta(t) = \delta_{\mu,S(t)}, \\ \widehat{S}(t) &= (1-t)\widehat{S} + tS_{0,\tau-2,T}, \quad \delta'(t) = \delta_{\mu,\widehat{S}(t)}. \end{aligned}$$

(Thus, the τ in the proof of Theorem 10.1.1 corresponds to the present $\tau - 2$.) In the following, constants will be independent of τ, T. Because $q(\omega'|_K) = \omega'$ for all $\omega' \in G$, we see that C defines a positive corrector of

$$\delta(t)_{0,d}\colon \mathcal{S} \to \mathcal{F}_\mu \oplus \mathbb{R}^d$$

for $t = 0$. Hence, if $\tau >$ const (for a suitable constant) then C will define a positive corrector of $\delta(t)_{0,d}$ for $0 \le t \le 1$. We want to show that if $\tau >$ const then

$$E_t := \delta'(t) + H_\gamma + \mathsf{C}\colon \mathcal{S}' \to \mathcal{F}'_\mu \oplus \mathbb{R}^{r_0} \oplus \mathbb{R}^d$$

is an isomorphism for $0 \le t \le 1$ when $T_{\min} \gg 0$. This is a Fredholm operator of index 0, so it suffices to show that it is surjective. As in the proof of Lemma 12.4.1 we can, for $\tau > \text{const}$ and $T_{\min} > \tau + \text{const}$, construct a right inverse R of

$$\delta'(1) + \mathsf{C}\colon \mathcal{S}' \to \mathcal{F}'_\mu \oplus \mathbb{R}^d$$

such that $\|R\|$ is bounded independently of τ, T. Set $L = R + \eta$ as in the said proof. For notational convenience we will here regard $E_t L$ as acting on

$$(\mathcal{F}'_\mu \oplus \mathbb{R}^d) \oplus \mathbb{R}^{r_0}.$$

Then there is the matrix representation

$$E_t L = \begin{pmatrix} (\delta'(t)) + \mathsf{C})R & \delta'(t)\eta \\ H_\gamma R & H_\gamma \eta \end{pmatrix}.$$

By Lemma 10.3.1 one has, for $\tau > \text{const}$,

$$\begin{aligned} \|(\delta'(t) + \mathsf{C})R - I\| &= \|(\delta'(t) - \delta'(1))R\| \\ &\le \text{const} \cdot \|\widehat{S}(t) - \widehat{S}(1)\|_{L^{p,\kappa}_1} \le \text{const} \cdot e^{(2\sigma-\lambda)\tau}. \end{aligned}$$

Furthermore, for $\tau > \text{const}$,

$$\begin{aligned} \|\delta'(t)\eta\| &\le \text{const} \cdot e^{\sigma\tau} T_{\min}^{-1}, \\ \|H_\gamma R\| &\le \text{const}, \\ \|H_\gamma \eta - I\| &\le (\tau - \text{const}) \cdot T_{\min}^{-1}. \end{aligned}$$

Recalling the assumption $0 \le 6\sigma < \lambda$, we see that if $\tau > \text{const}$ then $E_t L$ (and hence E_t) will be invertible for $0 \le t \le 1$ when $T_{\min} \gg 0$. Since $H_\gamma + \mathsf{C}$ is a positive corrector of $\delta'(1)_{0,r_0+d}$, it must also be a positive corrector of $\delta'(0)_{0,r_0+d}$, which in turn is equivalent to $\mathbf{F}$ being orientation preserving at $\mathbf{F}^{-1}(z,\omega)$. □

12.5 Homology orientations and gluing

In this section we will describe the "gluing of orientations" of Section 12.4 in terms of homology orientations in the simplest cases. This result will be needed in Part III.

Let X be as in Section 1.4 with $r = 1$, ie only one pair of ends $\mathbb{R}_+ \times (\pm Y)$ is being glued. Suppose Y and each Y'_j are rational homology spheres. We assume the glued manifold $X^\#$ is connected, so that X has at most two components. As in Section 10.1 let γ be a path in $X^{(T)}$ running once through the neck $[-T, T] \times Y$, with starting-point x_0 and endpoint x_1. If X is connected then we assume $x_0 = x_1$.

As before in this chapter, we will denote by $H^+(X)$ the space of self-dual closed L^2 2–forms on X. It is useful to observe here that orientations of $H^+(X)$ can be specified solely in terms of the intersection form on X. (We made implicit use of this already in the definition of homology orientation in Section 1.1.) To see this, let V be any real vector space with a nondegenerate symmetric bilinear form $B\colon V\times V\to\mathbb{R}$ of signature (m,n), where $m>0$ (the case $m=0$ being trivial). Let $\mathcal{V}^+$ denote the space of all linearly independent m–tuples $(v_1,\dots,v_m)$ of elements of V such that B is positive definite on the linear span of $v_1,\dots,v_m$. Then $\mathcal{V}^+$ has exactly two path-components, and two such m–tuples $(v_1,\dots,v_m)$ and $(w_1,\dots,w_m)$ lie in the same component if and only if the matrix $(B(v_j,w_k))_{j,k=1,\dots,m}$ has positive determinant. In the case when B is the intersection form of X, a choice of a component of $\mathcal{V}^+$ determines orientations of both $H^+(X)$ and $H^+(X^{(T)})$ (since the intersection forms of X and $X^{(T)}$ are canonically isomorphic).

Given the ordering of the ends $\mathbb{R}_+\times(\pm Y)$ of X there is a natural 1–1 correspondence between homology orientations of X and of $X^\#$. In general one can specify a homology orientation by choosing ordered bases for H^0, H^1 and H^+, or equivalently, for the dual groups. (If the 4–manifold in question is connected then we will usually take (1) as basis for H^0.) If X has two components then the correspondence is given by replacing the basis (x_0) for $H_0(X^\#)$ with the ordered basis (x_0,x_1) for $H_0(X)$. If X is connected then we replace a given ordered basis $(e_1,\dots,e_\ell)$ for $H_1(X)$ (where $\ell=b_1(X)$) with the ordered basis $(-[\gamma],e_1,\dots,e_\ell)$ for $H_1(X^\#)$, and we call this the *glued* homology orientation of $X^\#$. (The sign in front of $[\gamma]$ is related to a sign appearing in the formula for Hol_j in (10.1).)

Now fix homology orientations of $X, X^\#$ which are compatible in the above sense. Let $\mathcal{B},\mathcal{B}'$ be the configuration spaces over $X, X^{(T)}$ with reducible limits. According to Proposition 12.2.1 the chosen homology orientations determine an orientation o of $\lambda\to\mathcal{B}$ and an orientation o' of $\lambda'\to\mathcal{B}'$. On the other hand, λ' inherits a glued orientation $\tilde{o}$ from (λ,o) as specified in Section 12.4.

Proposition 12.5.1 (i) *If X is connected then $o'=\tilde{o}$.*

(ii) *If X has two components, then $o'=\tilde{o}$ if and only if $b_1(X)+b^+(X)$ is odd.*

The sign in (ii) will be dealt with in Section 12.6 by introducing appropriate sign conventions.

Proof Let $S_o=(A_o,0)$ be a reference configuration over X as in Section 10.1 with reducible limit over each end. To simplify notation we will now write S, A instead of S_o, A_o. Let $S' = (A', 0)$ be the glued reference configuration over $X^{(T)}$. Set $L_A = d^+ \oplus D_A$, so that

$$\delta_S = -d^* + L_A \colon \mathcal{S} \to \mathcal{F}$$

Set $b_1 = b_1(X)$, $b^+ = b^+(X)$, $m = \dim \ker(\delta_S)$ and $\ell = \dim \operatorname{coker}(L_A)$.

Choose smooth loops $\ell_1, \dots, \ell_{b^1}$ in $X_{:0}$ representing a positive basis for $H_1(X;\mathbb{R})$ and define

$$\mathsf{B}_1 \colon L^p_1(X_{:0}, i\Lambda^1) \to \mathbb{R}^{b_1}, \qquad a \mapsto \left(-\int_{\ell_j} ia \right)_{j=1,\dots,b_1}.$$

Choose a bounded complex-linear map

$$\mathsf{B}_2 \colon L^p_1(X_{:0}, \mathbb{S}^+) \to \mathbb{C}^{m_2}$$

whose composition with the restriction to $X_{:0}$ defines an isomorphism $\ker(D_A) \to \mathbb{C}^{m_2}$. Set

$$\mathsf{B} = \mathsf{B}_1 + \mathsf{B}_2 \colon \mathcal{S}_{:0} \to \mathbb{R}^{b_1} \oplus \mathbb{C}^{m_2} = \mathbb{R}^m.$$

Choose smooth imaginary-valued closed 2–forms $\omega_1, \dots, \omega_{b^+}$ on X which are supported in $X_{:0}$ and such that the cohomology classes $[-i\omega_1], \dots, [-i\omega_{b^+}]$ form a positive basis of a positive subspace for the intersection form of X. Then the self-dual parts $\omega_1^+, \dots, \omega_{b^+}^+$ map to a basis for $\operatorname{coker}(d^+)$ on both X and $X^{(T)}$, which in both cases is compatible with the chosen orientation of $H^+ = \operatorname{coker}(d^+)^*$. Choose smooth sections $\omega_{b^++1}, \dots, \omega_\ell$ of $\mathbb{S}^-_X$ which are supported in $X_{:0}$ and map to a positive basis for the real vector space $\operatorname{coker}(D_A)$ (with its complex orientation).

The remainder of the proof deals separately with the two cases.

Case (i) X is connected.

Let $g \colon X \to i\mathbb{R}$ be a smooth function supported in $X_{:0}$ and with $\int g = i$. Then for large T the orientations $o', \widetilde{o}$ of $\delta_{S'}$ are both represented by the following corrector of $(\delta_{S'})_{\ell+1, m+1}$:

$$\mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R}^\ell \to \mathcal{F}' \oplus \mathbb{R} \oplus \mathbb{R}^m,$$

$$(\xi, t, z) \mapsto (tg + \sum_{j=1}^{\ell} z_j \omega_j, H_\gamma \xi, \mathsf{B}\xi),$$

where $H_\gamma \xi$ means H_γ applied to the 1–form part of ξ.

Case (ii) X has two components X_0, X_1, where $x_j \in X_j$.

Thus, $\mathbb{R}_+ \times Y \subset X_0$ and $\mathbb{R}_+ \times (-Y) \subset X_1$. For $j = 0, 1$ choose a smooth function $g_j \colon X_j \to i\mathbb{R}$ supported in $(X_j)_{:0}$ and with $\int g_j = i$. Set

$$\mathsf{C}' \colon \mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R}^\ell \oplus \mathbb{R} \to \mathcal{F}' \oplus \mathbb{R}^m \oplus \mathbb{R},$$
$$(\xi, t, z, t') \mapsto (tg_0 + \sum_{j=1}^{\ell} z_j \omega_j, \mathsf{B}\xi, t'),$$

$$\mathsf{C}_\gamma \colon \mathcal{S}' \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^\ell \to \mathcal{F}' \oplus \mathbb{R} \oplus \mathbb{R}^m,$$
$$(\xi, t, t', z) \mapsto (tg_0 + t'g_1 + \sum_{j=1}^{\ell} z_j \omega_j, H_\gamma \xi, \mathsf{B}\xi).$$

When T is large, C' and C_γ are both correctors of $(\delta_{S'})_{\ell+2,m+1}$ which represent the orientations $o', \widetilde{o}$ of $\delta_{S'}$, respectively. Let C be the corrector of $(\delta_{S'})_{\ell+2,m+1}$ which has the same domain and target spaces as C_γ, and which is obtained from C' by interchanging summands as follows. If

$$(x, y, z) \in (\mathcal{S}' \oplus \mathbb{R}) \oplus \mathbb{R}^\ell \oplus \mathbb{R},$$
$$\mathsf{C}'(x, y, z) = (u, v, w) \in \mathcal{F}' \oplus \mathbb{R}^m \oplus \mathbb{R}$$

then $\mathsf{C}(x, z, y) = (u, w, v)$. As explained in Section 12.1, the correctors C, C' are equivalent if and only if $\ell + m$ is even. Set

$$E = \delta_{S'} + \mathsf{C}, \quad E_\gamma = \delta_{S'} + \mathsf{C}_\gamma.$$

We have
$$(\mathsf{C}_\gamma - \mathsf{C})(\xi, t, t', z) = (t'g_1, H_\gamma \xi - t', 0),$$

so the image N of $\mathsf{C}_\gamma - \mathsf{C}$ has dimension 2. We need to compute the determinant of the automorphism of N induced by $E_\gamma E^{-1}$. Let $s, s' \in \mathbb{R}$ and set

$$(\xi, t, t', z) = E^{-1}(sg_1, s', 0).$$

Write $\xi = (a, \phi) \in \Gamma(i\Lambda^1 \oplus \mathbb{S}^+)$. Then

$$-d^* a + tg_0 = sg_1, \quad t' = s'. \tag{12.14}$$

Integrating the first equation gives $t = s$. The equation

$$E_\gamma E^{-1} = (\mathsf{C}_\gamma - \mathsf{C}) E^{-1} + I$$

now yields
$$E_\gamma E^{-1}(sg_1, s', 0) = ((s + s')g_1, H_\gamma \xi, 0).$$

Note that ξ depends on s alone, so we can write $a = a(s)$. Set $\eta = H_\gamma(a(1))$. Then $E_\gamma E^{-1}|_N$ is represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ \eta & 0 \end{pmatrix},$$

so $\mathsf{C}, \mathsf{C}_\gamma$ are equivalent correctors if and only if $\eta < 0$. We will show that $\eta > 0$ when T is large. This implies that $\mathsf{C}', \mathsf{C}_\gamma$ are equivalent correctors if and only if $\ell + m$ is odd. Since D_A is complex linear, this will prove part (ii) of the proposition.

Let $\{T_n\}$ be a sequence tending to ∞ and, working over $X^{(T_n)}$, set

$$(\xi_n, s_n, 0, z_n) = E^{-1}(s_n g_1, 0, 0),$$

or more explicitly,

$$\delta_{S'}(\xi_n) + \sum_j z_{n,j}\omega_j = s_n(g_1 - g_0), \quad \mathsf{B}\xi_n = 0,$$

where $s_n > 0$ is chosen such that

$$\|\xi_n\|_{L^2(X_{:2})} = 1.$$

(Because the supports of g_0 and g_1 are disjoint, Equation (12.14) shows that $a \neq 0$ over $X_{:0}$ when $s \neq 0$.) Write $\xi_n = (a_n, \phi_n)$. Equation (12.14) yields

$$s_n \|g_1\|_2^2 = -\int \langle a_n, dg_1 \rangle \leq \|dg_1\|_2,$$

hence the sequence s_n is bounded. An analogous argument applied to the equation

$$L_A(\xi_n) + \sum_j z_{n,j}\omega_j = 0$$

shows that the sequence z_n is bounded as well. Thus, $\delta_{S'}(\xi_n)$ is supported in $X_{:0}$, and for each $k \geq 0$ the C^k–norm of $\delta_{S'}(\xi_n)$ is bounded independently of n. Now recall from Section 3.4 that over the neck $[-T_n, T_n] \times Y$ the operator $\delta_{S'}$ can be expressed in the form $\frac{\partial}{\partial t} + P$, where

$$P = \begin{pmatrix} 0 & -d^* & 0 \\ -d & *d & 0 \\ 0 & 0 & -\partial_B \end{pmatrix}$$

for some spin^c connection B over Y. Because of our nondegeneracy assumption on the critical points, the kernel of P consists of the constant functions in $i\Omega^0(Y)$. There is also a similar description of $\delta_{S'}$ over the ends $\mathbb{R}_+ \times Y'_j$. In general, if $(\frac{\partial}{\partial t} + P)\zeta = 0$ over a band $[0, \tau] \times Y$ and ζ involves only eigenvectors of P corresponding to positive eigenvalues then for any nonnegative integer k and $1 \le t \le \tau - 2$, say, there is an estimate

$$\|\zeta\|_{C^k([t,t+1]\times Y)} \le \text{const}\cdot e^{-\rho t}\|\zeta\|_{L^2(\{0\}\times Y)},$$

where ρ is the smallest positive eigenvalue of P. This result immediately applies to ξ_n over the ends $\mathbb{R}_+ \times Y'_j$. Over the neck $[-T_n, T_n]\times Y$ one can write $\xi_n = \text{const}\cdot i\,dt + \xi_n^+ + \xi_n^-$ where $\xi_n^\pm$ involves only eigenvectors corresponding to positive/negative eigenvalues of P. One then obtains C^k–estimates on $\xi_n^\pm$ in terms of its L^2–norm over $\{\mp T_n\}\times Y$. It follows that after passing to a subsequence we may assume that ξ_n c-converges over X to some pair $\xi = (a,\phi)$ satisfying $\|\xi\|_{L^2(X_{:2})} = 1$. Of course, we may also assume that the sequences s_n, z_n converge, with limits s, z, say. Then

$$\delta_S\xi + \sum_j z_j\omega_j = s(g_1 - g_0), \quad \mathsf{B}\xi = 0.$$

Moreover,

$$\xi = \pm ci\,dt + \zeta_\pm \quad \text{on } \mathbb{R}_+ \times (\pm Y),$$

where $\zeta_\pm$ decays exponentially and

$$c = -\lim_{n\to\infty}\frac{H_\gamma(a_n)}{2T_n}.$$

On the other hand, Stokes' theorem yields

$$\int_{\{-T_n\}\times Y} *a_n = -\int_{(X_0)_{:0}} d^*a_n = -\int_{(X_0)_{:0}} s_n g_0 = -s_n i,$$

Hence $$ci\cdot\text{Vol}(Y) = \int_{\{0\}\times Y} *a = \lim_n \int_{\{-T_n\}\times Y} *a_n = -si.$$

Thus, $c\cdot\text{Vol}(Y) = -s \le 0$. If $c = 0$ then $\xi \ne 0$ would decay exponentially on all ends of X and satisfy $\delta_S\xi = 0$, contradicting $\mathsf{B}\xi = 0$. Therefore, $c < 0$, and $H_\gamma(a_n) > 0$ for large n.

This shows that $\eta > 0$ when T is large. □

12.6 Components

Let again $X, \vec{\alpha}$ be as in Section 3.4 and suppose X is the disjoint union of open subsets $X_1, \dots, X_q$. If S is a configuration over X and S_j its restriction to X_j then δ_S is the direct sum of the operators δ_{S_j}. Moreover,

$$\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_q,$$

where $\mathcal{B}_j = \mathcal{B}(X_j;\vec{\alpha}(j))$ for a suitable vector $\vec{\alpha}(j)$ of critical points. If we denote by $\overline{\lambda}_j$ the pullback to $\mathcal{B} = \mathcal{B}(X;\vec{\alpha})$ of the orientation cover $\lambda_j \to \mathcal{B}_j$ then by (12.2) we have an isomorphism of $\mathbb{Z}/2$–bundles

$$\overline{\lambda}_1 \otimes \cdots \otimes \overline{\lambda}_q \xrightarrow{\approx} \lambda, \tag{12.15}$$

where $\lambda \to \mathcal{B}$ is the orientation cover. If now X and its asymptotic limits are as in Section 10.1 and o_j an orientation of λ_j then (12.15) defines a *direct sum* orientation of λ, which in turn yields a glued orientation $\widetilde{o}$ of $\lambda' \to \mathcal{B}(X^{(T)};\vec{\alpha}')$. Suppose now that $q = 2$ and that for each pair of ends $\mathbb{R}_+ \times (\pm Y_j)$ being glued ($j = 1,\dots,r$), one of these ends lies in X_1 and the other one in X_2. In this case we define

$$o_1 \# o_2 := (-1)^{r_0(i_1+1)}\widetilde{o}, \tag{12.16}$$

where $i_1 = \mathrm{index}(\delta_{S_1})$. The sign is chosen so that (among other things) the operation # is *associative* in the following sense: Suppose $q = 3$ and that for each pair of ends being glued, one is contained in X_2 and the other one is contained in either X_1 or X_3. Then it makes sense to glue first X_1 and X_2 and then add X_3, or one can start with X_2 and X_3. It is now easy to check that

$$(o_1 \# o_2) \# o_3 = o_1 \# (o_2 \# o_3) \tag{12.17}$$

as orientations of λ.

Returning to the situation of (12.16), suppose X_1, X_2 are the connected components of X and that all ends of X are modelled on rational homology spheres. Let a homology orientation of X_j be given by an ordered basis U_j for $H^1(X_j)$ and a maximal linearly independent subset V_j of $H^2(X_j)$ on which the intersection form is positive. Then we define the *glued* homology orientation of $X^\#$ to be $(-1)^{b^+(X_1)(b_1(X_2)+b^+(X_2))}$ times the one given by the concatenated ordered basis U_1U_2 of $H^1(X^\#)$ and the ordered subset V_1V_2 of $H^2(X^\#)$. Let o_j and o' be the orientations of λ_j and λ' given by the homology orientations of X_j and $X^\#$, respectively. If all asymptotic limits over X are reducible then one easily checks, using Proposition 12.5.1, that

$$o_1 \# o_2 = o'.$$

If one thinks of homology orientations as orientations of the operator $\mathcal{D} = -d^* + d^+$ (acting on Sobolev spaces with small positive weights) then the above defined gluing of homology orientations corresponds to the #–operation on orientations of $\mathcal{D}$, hence it is associative in the sense of (12.17).

12.7 Orientation of $\mathcal{V}_0$

In Section 12.2 the question arose what it means for $\{g_j\}$ to be a positive basis for $\mathcal{V}_\Phi$ when $\Phi = 0$. The following proposition answers this question when $b = 1$. This result is not needed elsewhere in Part II, but will be used in Part III.

Proposition 12.7.1 *Let X be as in Section 1.3. Suppose X is connected, $b = 1$, and consider the bundle $\mathcal{V} \to \mathcal{W} = L^{p,w}_1(X; \mathbb{S}^+)$ with all weights σ_j positive. Then $g \in C^\infty_c(X; i\mathbb{R})$, represents a positive basis for $\mathcal{V}_0$ if and only if $\int_X g/i > 0$.*

Proof It suffices to prove that $g = ih$ represents a positive basis when $\int_X h > 0$. Let $\mathfrak{b} = \{x\}$. By Proposition 2.3.2 we may assume $h \geq 0$, $h(x) > 0$ and $\operatorname{supp}(h) \subset X_{:0}$. The proposition is then a consequence of the following lemma.

Lemma 12.7.1 *Let X, h be as above and v a smooth positive function on X whose restriction to each end $\mathbb{R}_+ \times Y_j$ is the pullback of a function v_j on $\mathbb{R}_+$. Suppose f is a real function on X satisfying*

$$(\Delta + v) f = h, \quad df \in L^{p,w}_1.$$

Then $f \geq 0$, and $f > 0$ where $h > 0$.

The proof will make use of the following elementary result, whose proof is left to the reader.

Sublemma 12.7.1 *Suppose a, u are smooth real functions on $[0, \infty)$ such that $u'' = au$, $a > 0$, $u(0) > 0$ and u is bounded. Then $u > 0$ and $u' < 0$.* □

Proof of Lemma 12.7.1 We first study the behaviour of f on an end $\mathbb{R}_+ \times Y_j$. We omit j from notation and write $Y = Y_j$ etc. Set $\mathsf{f} = f|_{\mathbb{R}_+ \times Y}$. By Proposition 2.2.1 the assumption $df \in L^{p,w}_1$ implies that $\mathsf{f}(t, \cdot)$ converges uniformly towards a constant function c as $t \to \infty$. Let $\{e_\nu\}$ be a maximal orthonormal set of eigenvectors of Δ_Y with corresponding eigenvalues λ^2_ν. Write

$$\mathsf{f}(t, y) = \sum_\nu u_\nu(t) e_\nu(y).$$

Then
$$u''_\nu = (\lambda^2_\nu + \mathsf{v}) u_\nu.$$

By the sublemma, either $u_\nu = 0$ or $u_\nu u'_\nu < 0$. Consequently,

$$\int_Y \mathsf{f}^2(t, y)\, dy = \sum_\nu u^2_\nu(t)$$

is a decreasing function of t, and

$$\max_{y \in Y} |\mathsf{f}(t,y)| \geq c$$

for all $t \geq 0$. In particular, if $c < 0$ then there exists a $(t,y) \in \mathbb{R}_+ \times Y$ with $\mathsf{f}(t,y) \leq c$. Hence, if $\inf f < 0$ then the infimum is attained.

Now, at any local minimum of f one has

$$vf = h - \Delta f \geq h,$$

so $f \geq 0$ everywhere. But then every zero of f is an absolute minimum, so $f > 0$ where $h > 0$. This proves the lemma and thereby also Proposition 12.7.1. $\square$

CHAPTER 13

Parametrized moduli spaces

Parametrized moduli spaces appear in many different situations in gauge theory, eg in the construction of 4–manifold invariants [15; 45] and Floer homology [14], and in connection with gluing obstructions (see Part III). A natural setting here would involve certain fibre bundles whose fibres are 4–manifolds. We feel, however, that gauge theory for such bundles in general deserves a separate treatment, and will therefore limit ourselves, at this time, to the case of a product bundle over a vector space. However, we take care to set up the theory in such a way that it would easily carry over to more general situations.

The main goal of this chapter is to extend the gluing theorem and the discussion of orientations to the parametrized case.

13.1 Moduli spaces

As in Section 1.3 let X be a spin^c 4–manifold with Riemannian metric $\bar{g}$ and tubular ends $\overline{\mathbb{R}}_+ \times Y_j$, $j = 1, \dots, r$. Let W be a finite-dimensional Euclidean vector space and $\mathbf{g} = \{g_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}$ a smooth family of Riemannian metrics on X all of which agree with $\bar{g}$ outside $X_{:0}$. We then have a principal SO(4)–bundle $P_{\mathrm{SO}}(\mathbf{g}) \to X \times \mathsf{W}$ whose fibre over (x, w) consists of all positive g_{w}–orthonormal frames in $T_x X$.

In the notation of Section 3.1 let $P_{\mathrm{GL}^c} \to P_{\mathrm{GL}^+}$ be the spin^c structure on X. Denote by $P_{\mathrm{Spin}^c}(\mathbf{g})$ the pullback of $P_{\mathrm{SO}}(\mathbf{g})$ under the projection $P_{\mathrm{GL}^c} \times \mathsf{W} \to P_{\mathrm{GL}^+} \times \mathsf{W}$. Then $P_{\mathrm{Spin}^c}(\mathbf{g})$ is a principal Spin$^c(4)$–bundle over $X \times \mathsf{W}$.

For $j = 1, \dots, r$ let $\alpha_j \in \mathcal{C}(Y_j)$ be a nondegenerate smooth monopole. Let $\mathcal{C}(g_{\mathsf{w}})$ denote the $L^{p,w}_1$ configuration space over X for the metric g_{w} and limits α_j, where

p, w are as in Section 3.4. We will provide the disjoint union

$$\mathcal{C}(\mathbf{g}) = \bigcup_{\mathsf{w}\in\mathsf{W}} \mathcal{C}(g_{\mathsf{w}}) \times \{\mathsf{w}\}$$

with a natural structure of a (trivial) smooth fibre bundle over W. Let

$$\mathsf{v}\colon P_{\mathrm{Spin}^c}(\mathbf{g}) \to P_{\mathrm{Spin}^c}(g_0) \times \mathsf{W} \tag{13.1}$$

be any isomorphism of $\mathrm{Spin}^c(4)$–bundles which covers the identity on $X \times \mathsf{W}$ and which outside $X_{:1} \times \mathsf{W}$ is given by the identification $P_{\mathrm{Spin}^c}(g_{\mathsf{w}}) = P_{\mathrm{Spin}^c}(g_0)$. There is then an induced isomorphism of $\mathrm{SO}(4)$–bundles

$$P_{\mathrm{SO}}(\mathbf{g}) \to P_{\mathrm{SO}}(g_0) \times \mathsf{W},$$

since these are quotients of the corresponding $\mathrm{Spin}^c(4)$–bundles by the $\mathrm{U}(1)$–action. Such an isomorphism v can be constructed by means of the holonomy along rays of the form $\{x\} \times \overline{\mathbb{R}}_+\mathsf{w}$ where $(x, \mathsf{w}) \in X \times \mathsf{W}$, with respect to any connection in $P_{\mathrm{Spin}^c}(\mathbf{g})$ which outside $X_{:1} \times \mathsf{W}$ is the pullback of a connection in $P_{\mathrm{Spin}^c}(g_0)$. Then v induces a $\mathcal{G} = \mathcal{G}(X;\vec{\alpha})$–equivariant diffeomorphism

$$\mathcal{C}(g_{\mathsf{w}}) \to \mathcal{C}(g_0) \tag{13.2}$$

for each w, where the map on the spin connections is obtained by identifying these with connections in the respective determinant line bundles and applying the isomorphism between these bundles induced by v. Putting together the maps (13.2) for all w yields a bijection

$$\mathsf{v}_*\colon \mathcal{C}(\mathbf{g}) \to \mathcal{C}(g_0) \times \mathsf{W}.$$

If $\tilde{\mathsf{v}}$ is another isomorphism as in (13.1) then $\mathsf{v}_*(\tilde{\mathsf{v}}_*)^{-1}$ is smooth, hence we have obtained the desired structure on $\mathcal{C}(\mathbf{g})$. Furthermore, because of the gauge equivariance of v_* we also get a similar smooth fibre bundle structure on

$$\mathcal{B}^*_{\mathfrak{b}}(\mathbf{g}) = \bigcup_{\mathsf{w}\in\mathsf{W}} \mathcal{B}^*_{\mathfrak{b}}(g_{\mathsf{w}}) \times \{\mathsf{w}\} \tag{13.3}$$

for any finite subset $\mathfrak{b} \subset X$. The image of $(S, \mathsf{w}) \in \mathcal{C}(\mathbf{g})$ in $\mathcal{B}(\mathbf{g})$ will be denoted $[S, \mathsf{w}]$.

We consider the natural smooth action of $\mathbb{T}$ on $\mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})$ where an element of $\mathbb{T}$ maps each fibre $\mathcal{B}_{\mathfrak{b}}(g_{\mathsf{w}})$ into itself in the standard way. (There is another version of the gluing theorem where $\mathbb{T}$ acts nontrivially on W; see below.)

The principal bundle $P_{\mathrm{Spin}^c}(\mathbf{g})$ also gives rise to Banach vector bundles $\mathcal{S}(\mathbf{g})$, $\mathcal{F}(\mathbf{g})$, $\mathcal{F}_2(\mathbf{g})$ over W whose fibres over $\mathsf{w} \in \mathsf{W}$ are the spaces $\mathcal{S}(g_{\mathsf{w}}), \mathcal{F}(g_{\mathsf{w}}), \mathcal{F}_2(g_{\mathsf{w}})$ resp. defined as in (12.3) using the metric g_{w} on X.

Let $\dot{\Theta}\colon \mathcal{C}(\mathbf{g}) \to \mathcal{F}_2(\mathbf{g})$ be the fibre-preserving monopole map whose effect on the fibre over $\mathsf{w} \in \mathsf{W}$ is the left hand side of Equation (3.10), interpreted in terms of the metric g_{w}. If we conjugate $\dot{\Theta}$ by the appropriate diffeomorphisms induced by v then we obtain the smooth $\mathcal{G}$–equivariant map

$$\dot{\Theta}_{\mathsf{v}}\colon \mathcal{C}(g_0) \times \mathsf{W} \to \mathcal{F}_2(g_0),$$
$$(A, \Phi, \mathsf{w}) \mapsto \left((\mathsf{v}_{\mathsf{w}}(\hat{F}_A + \mathfrak{m}(A, \Phi)))^+ - Q(\Phi)\,,\, \sum_j \mathsf{v}_{\mathsf{w}}(e_j) \cdot \nabla_{e_j}^{A+\mathsf{a}_{\mathsf{w}}}(\Phi) \right)$$

where the perturbation $\mathfrak{m}$ is smooth, hence $\dot{\Theta}$ is smooth. Here v_{w} denotes the isomorphism that v induces from the Clifford bundle of (X, g_{w}) to the Clifford bundle of (X, g_0), and $\{e_j\}$ is a local g_{w}–orthonormal frame on X. Finally, if we temporarily let $\nabla^{(\mathsf{w})}$ denote the g_{w}–Riemannian connection in the tangent bundle of X then

$$\mathsf{a}_{\mathsf{w}} = \mathsf{v}_{\mathsf{w}}(\nabla^{(\mathsf{w})}) - \nabla^{(0)}.$$

Note that a_{w} is supported in $X_{:1}$.

In situations involving parametrized moduli spaces there will often be an additional perturbation which affects the equations only over some compact part of X. For the gluing theory one can consider quite generally perturbations given by an isomorphism v and a smooth $\mathcal{G}$–equivariant map

$$\mathfrak{o}\colon \mathcal{C}(X_{:\mathfrak{t}}, g_0) \times \mathsf{W} \to (\mathcal{F}_2)^{:\mathfrak{t}}(g_0)$$

for some $\mathfrak{t} \geq 0$, using notation introduced in Section 12.4. We require that the derivative of $\mathfrak{o}$ at any point be a compact operator. Let

$$\Theta\colon \mathcal{C}(\mathbf{g}) \to \mathcal{F}_2(\mathbf{g}) \tag{13.4}$$

be the map corresponding to $\Theta_{\mathsf{v}} := \dot{\Theta}_{\mathsf{v}} + \mathfrak{o}$. We define the parametrized moduli space $M_{\mathfrak{b}}(\mathbf{g})$ to be the image of $\Theta^{-1}(0)$ in $\mathcal{B}_{\mathfrak{b}}(\mathbf{g})$. By construction, v_* induces a homeomorphism

$$M_{\mathfrak{b}}(\mathbf{g}) \xrightarrow{\approx} \Theta_{\mathsf{v}}^{-1}(0)/\mathcal{G}_{\mathfrak{b}}\,.$$

A point in $M_{\mathfrak{b}}(\mathbf{g})$ is called *regular* if the corresponding zeros of Θ_{v} are regular (a regular zero being one where the derivative of Θ_{v} is surjective). This notion is independent of v. By the local slice theorem, the set of regular points in $M_{\mathfrak{b}}^*(\mathbf{g})$ is a smooth submanifold of $\mathcal{B}_{\mathfrak{b}}^*(\mathbf{g})$.

13.2 Orientations

Fix orientations of the vector space W and of the set $\mathfrak{b}$. For any $S \in \mathcal{C}(g_{\mathsf{w}})$ let

$$\delta_{S,\mathsf{w}}\colon \mathcal{S}(g_{\mathsf{w}}) \to \mathcal{F}(g_{\mathsf{w}})$$

be the Fredholm operator δ_S defined in terms of the metric g_{w}, now using the perturbed monopole map (13.4). The orientation cover of this family descends to a double cover $\lambda(\mathbf{g}) \to \mathcal{B}(\mathbf{g})$. (Note that the perturbation $\mathfrak{o}$ can be scaled down, so that an orientation of $\lambda(\mathbf{g})$ for $\mathfrak{o} = 0$ determines an orientation for any other $\mathfrak{o}$.) Clearly, any section of $\lambda(\mathbf{g})$ over $\mathcal{B}(g_0)$ extends uniquely to all of $\mathcal{B}(\mathbf{g})$. On the other hand, a section of $\lambda(\mathbf{g})$ determines an orientation of the regular part of $M^*_{\mathfrak{b}}(\mathbf{g})$, as we will now explain.

Let $T^v\mathcal{C}(\mathbf{g}) \subset T\mathcal{C}(\mathbf{g})$ be the subbundle of vertical tangent vectors. We can identify $T^v_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) = \mathcal{S}(g_{\mathsf{w}})$. A choice of an isomorphism v_1 as in (13.1) determines a bundle homomorphism

$$P_1\colon T\mathcal{C}(\mathbf{g}) \to T^v\mathcal{C}(\mathbf{g})$$

which is the identity on vertical tangent vectors. This yields a splitting

$$T_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) = \mathcal{S}(g_{\mathsf{w}}) \oplus \mathsf{W}$$

into vertical and horizontal vectors (the latter making up the kernel of P_1 and being identified with W through the projection).

In general, a connection in a vector bundle $E \to \mathsf{W}$ determines for every element u of a fibre E_{w} a linear map $T_uE \to E_{\mathsf{w}}$, namely the projection onto the vertical part of the tangent space. Moreover, if $u = 0$ then this projection is independent of the connection. Together these projections form a smooth map $TE \to E$. Let

$$P_2\colon T\mathcal{F}_2(\mathbf{g}) \to \mathcal{F}_2(\mathbf{g})$$

be such a map for $E = \mathcal{F}_2(\mathbf{g})$ determined by some isomorphism v_2.

Now let
$$\mathcal{I}^*\colon T^v\mathcal{C}(\mathbf{g}) \to \mathcal{F}_1(\mathbf{g})$$

be the map which sends $s \in T^v_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g})$ to $(\mathcal{I}^*_S(s), \mathsf{w})$, where the * refers to the metric g_{w}. Set

$$\underline{\delta} := \mathcal{I}^* \circ P_1 + P_2 \circ D\Theta\colon T\mathcal{C}(\mathbf{g}) \to \mathcal{F}(\mathbf{g}),$$

where $D\Theta$ is the derivative of the map (13.4). By restriction of $\underline{\delta}$ we obtain bounded operators

$$\underline{\delta}_{S,\mathsf{w}}\colon T_{(S,\mathsf{w})}\mathcal{C}(\mathbf{g}) \to \mathcal{F}(g_{\mathsf{w}}).$$

Since the restriction of $\underline{\delta}_{S,\mathsf{w}}$ to the vertical tangent space $\mathcal{S}(g_{\mathsf{w}})$ is equal to the Fredholm operator $\delta_{S,\mathsf{w}}$, we conclude that $\underline{\delta}_{S,\mathsf{w}}$ is also Fredholm, and

$$\mathrm{ind}(\underline{\delta}_{S,\mathsf{w}}) = \mathrm{ind}(\delta_{S,\mathsf{w}}) + \mathsf{d},$$

where $\mathsf{d} = \dim \mathsf{W}$.

Choose nonnegative integers ℓ, m with $\mathrm{ind}(\delta_{S,\mathsf{w}}) = m - \ell$ and an orientation preserving linear isomorphism $h\colon \mathsf{W} \to \mathbb{R}^{\mathsf{d}}$. If C is any corrector of $(\delta_{S,\mathsf{w}})_{\ell,m}$ then

$$\underline{\delta}_{S,\mathsf{w}} + h + \mathsf{C}\colon \mathcal{S}(g_{\mathsf{w}}) \oplus \mathsf{W} \oplus \mathbb{R}^{\ell} \to \mathcal{F}(g_{\mathsf{w}}) \oplus \mathbb{R}^{\mathsf{d}} \oplus \mathbb{R}^{m}$$

is an injective Fredholm operator of index 0, hence an isomorphism. The map $\mathsf{C} \mapsto h + \mathsf{C}$ respects the equivalence relation for correctors. We can therefore define a 1–1 correspondence between orientations of $\delta_{S,\mathsf{w}}$ and orientations of $\underline{\delta}_{S,\mathsf{w}}$ by saying that if C is a positive corrector of $(\delta_{S,\mathsf{w}})_{\ell,m}$ then $h + \mathsf{C}$ is a $(-1)^{\mathsf{d}\cdot\mathrm{ind}(\delta_{S,\mathsf{w}})}$–corrector of $(\underline{\delta}_{S,\mathsf{w}})_{\ell,\mathsf{d}+m}$.

Fix $S \in \mathcal{C}^*_{\mathfrak{b}}(g_{\mathsf{w}})$, choose a map μ as in (12.6), and let

$$\underline{\delta}_{\mu,S,\mathsf{w}} := \mathcal{S}(g_{\mathsf{w}}) \oplus \mathsf{W} \to \mathcal{F}_{\mu}(g_{\mathsf{w}})$$

be the operator obtained from $\underline{\delta}_{S,\mathsf{w}}$ by replacing $\mathcal{I}^*_S$ by $\mu \circ \mathcal{I}^*_S$ (cf (12.8)). Just as in the unparametrized case we define a 1–1 correspondence between orientations of $\underline{\delta}_{S,\mathsf{w}}$ and orientations of $\underline{\delta}_{\mu,S,\mathsf{w}}$ by decreeing that if C is a positive corrector of $(\underline{\delta}_{S,\mathsf{w}})_{\ell-\mathsf{d},m}$ (where we now assume $\ell \geq \mathsf{d}$) then $\mathsf{C}_{\mathfrak{b}}$ is a $(-1)^{b\cdot\mathrm{ind}(\underline{\delta}_{S,\mathsf{w}})}$–corrector of $(\underline{\delta}_{\mu,S,\mathsf{w}})_{\ell-b,\mathsf{d}+m}$.

Now suppose $[S,\mathsf{w}] \in M^*_{\mathfrak{b}}(\mathbf{g})$. Working in the trivialization v_1 and using the local slice theorem for the metric g_{w} one finds that $[S,\mathsf{w}]$ is a regular point of $M^*_{\mathfrak{b}}(\mathbf{g})$ if and only if $\underline{\delta}_{\mu,S,\mathsf{w}}$ is surjective, and in that case the projection $\mathcal{C}^*_{\mathfrak{b}}(\mathbf{g}) \to \mathcal{B}^*_{\mathfrak{b}}(\mathbf{g})$ induces an isomorphism

$$\ker(\underline{\delta}_{\mu,S,\mathsf{w}}) \xrightarrow{\approx} T_{[S,\mathsf{w}]} M^*_{\mathfrak{b}}(\mathbf{g}).$$

This establishes a 1–1 correspondence between orientations of $\delta_{S,\mathsf{w}}$ and orientations of $T_{[S,\mathsf{w}]} M^*_{\mathfrak{b}}(\mathbf{g})$. This correspondence is obviously independent of P_2, and it is independent of P_1 because the set of such operators form an affine space. It is also independent of μ for reasons explained earlier.

This associates to any orientation of $\lambda(\mathbf{g})$ an orientation of the regular part of $M^*_{\mathfrak{b}}(\mathbf{g})$.

13.3 The gluing theorem

We continue the discussion of the previous section, but we now specialize to the case when the ends of X are $\mathbb{R}_+ \times (\pm Y_j)$, $j = 1, \dots, r$ and $\mathbb{R}_+ \times Y'_j$, $j = 1, \dots, r'$, with nondegenerate limits α_j over $\mathbb{R}_+ \times (\pm Y_j)$ and α'_j over $\mathbb{R}_+ \times Y'_j$, as in Section 10.1. Let the paths $\gamma_j^\pm, \gamma_j$ and $\mathfrak{b} \subset X$ be as in that section. The family of metrics $\mathbf{g}$ on X defines, in a natural way, a smooth family of metrics $\{g(T, \mathsf{w})\}_{\mathsf{w} \in \mathsf{W}}$ on $X^{(T)}$ for any T. We retain our previous notation for configuration and moduli spaces over X, whereas those over $X^{(T)}$ will be denoted $\mathcal{C}'(\mathbf{g}), \mathcal{B}'_{\mathfrak{b}}(\mathbf{g}), M_{\mathfrak{b}}^{(T)}(\mathbf{g})$ etc. Fix an isomorphism v as in (13.1).

We first discuss gluing of orientations. The isomorphism v defines a corresponding isomorphism over $X^{(T)}$ and operators P_1, P_2 over both X and $X^{(T)}$. We then get families of Fredholm operators $\underline{\delta}, \underline{\delta}'$ parametrized by $\mathcal{C}(\mathbf{g}), \mathcal{C}'(\mathbf{g})$ resp. The procedure in Section 12.4 for gluing orientations carries over to this situation and yields a 1–1 correspondence between orientations of $\underline{\delta}$ and orientations of $\underline{\delta}'$. Given $S \in \mathcal{C}^{:0}(g_{\mathsf{w}})$, if $\lambda_{S,\mathsf{w}}$ and $\underline{\lambda}_{S,\mathsf{w}}$ denote the set of orientations of $\delta_{S,\mathsf{w}}$ and $\underline{\delta}_{S,\mathsf{w}}$ resp., and similarly for the glued configuration $S' \in \mathcal{C}'(g_{\mathsf{w}})$, then we have a commutative diagram of bijections:

$$\begin{array}{ccc} \lambda_{S,\mathsf{w}} & \longrightarrow & \lambda_{S',\mathsf{w}} \\ \downarrow & & \downarrow \\ \underline{\lambda}_{S,\mathsf{w}} & \longrightarrow & \underline{\lambda}_{S',\mathsf{w}} \end{array}$$

The analogue of Diagram (12.13) in the parametrized situation also commutes.

Now fix an orientation of $\lambda(\mathbf{g}) \to \mathcal{B}_{\mathfrak{b}}(\mathbf{g})$ and let $\lambda'(\mathbf{g}) \to \mathcal{B}'_{\mathfrak{b}}(\mathbf{g})$ have the glued orientation. These orientations determine orientations of the regular parts of the moduli spaces $M_{\mathfrak{b}}(\mathbf{g})$ and $M_{\mathfrak{b}}^{(T)}(\mathbf{g})$, respectively, as specified in the previous section.

As before, a choice of reference configuration in $\mathcal{C}(g_0)$ gives rise to a glued reference configuration in $\mathcal{C}'(g_0)$ and a holonomy map

$$\mathcal{B}'_{\mathfrak{b}}(g_0) \to \mathrm{U}(1)^{r_0}.$$

Composing this with the map $M_{\mathfrak{b}}^{(T)}(\mathbf{g}) \to \mathcal{B}'_{\mathfrak{b}}(g_0)$ defined by the chosen isomorphism v yields a holonomy map

$$\mathrm{Hol}\colon M_{\mathfrak{b}}^{(T)}(\mathbf{g}) \to \mathrm{U}(1)^{r_0}.$$

Fix an open $\mathbb{T}$–invariant subset $G \subset M_{\mathfrak{b}}(\mathbf{g})$ whose closure is compact and contains only regular points.

By a kv-pair we mean as before a pair (K,V), where $K\subset X$ is a compact codimension 0 submanifold which contains $\mathfrak{b}$ and intersects every component of X, and V is an open $\mathbb{T}$–invariant neighbourhood of $R_K(\bar{G})$ in

$$\breve{\mathcal{B}}_{\mathfrak{b}}(K,\mathbf{g})=\bigcup_{\mathsf{w}\in\mathsf{W}}\breve{\mathcal{B}}_{\mathfrak{b}}(K,g_{\mathsf{w}})\times\{\mathsf{w}\}.$$

Now fix a kv-pair (K,V) satisfying similar additional assumptions as before: firstly, that $V\subset\breve{\mathcal{B}}^*_{\mathfrak{b}}(K,\mathbf{g})$; secondly, that if X_e is any component of X which contains a point from $\mathfrak{b}$ then $X_e\cap K$ is connected.

Suppose
$$q\colon V\to M_{\mathfrak{b}}(\mathbf{g})$$
is a smooth $\mathbb{T}$–equivariant map such that $q(\omega|_K)=\omega$ for all $\omega\in G$. (We do not require that q commute with the projections to W.) Choose $\lambda_j,\lambda'_j>0$. Let “admissibility of $\vec{\alpha}'$” be defined in terms of the parametrized moduli spaces $M_{\mathfrak{b}}^{(T)}(\mathbf{g})$ (see Definition 7.1.3).

Theorem 13.3.1 *Theorem 10.1.1 holds in the present situation if one replaces $M_{\mathfrak{b}}$ and $M_{\mathfrak{b}}^{(T)}$ by $M_{\mathfrak{b}}(\mathbf{g})$ and $M_{\mathfrak{b}}^{(T)}(\mathbf{g})$, respectively. Moreover, the diffeomorphism $\mathbf{F}$ defined as in Theorem 12.4.1 is orientation preserving.*

Proof The proofs carry over without any substantial changes. □

There is another version of the theorem (which will be used in Part III) where the family of metrics $\mathbf{g}$ is constant (ie $g_{\mathsf{w}}=g_0$ for every w) and $\mathbb{T}$ acts smoothly on the manifold W. One then has a product action of $\mathbb{T}$ on

$$M_{\mathfrak{b}}(\mathbf{g})=M_{\mathfrak{b}}\times\mathsf{W},$$

and the theorem holds in this setting as well. In fact, the action of $\mathbb{T}$ affects the proof in only one way, namely the requirement that $\tilde{K}$ be $\mathbb{T}$–invariant. To obtain this, let dist be a $\mathbb{T}$–invariant metric on the set W compatible with the given topology (arising for instance from a $\mathbb{T}$–invariant Riemannian metric) and replace the definition of d_m in (10.13) by

$$d_m((S,\mathsf{w}),(\bar{S},\bar{\mathsf{w}}))=\int_{X_{:m}}|\bar{S}-S|^p+|\nabla_{\bar{A}}(\bar{S}-S)|^p+\operatorname{dist}(\mathsf{w},\bar{\mathsf{w}}).$$

Then V'_m will be $\mathbb{T}$–invariant.

13.4 Compactness

In contrast to gluing theory, compactness requires more specific knowledge of the perturbation $\mathfrak{o}$, so we will here take $\mathfrak{o} = 0$. We observe that the notion of chain-convergence has a natural generalization to the parametrized situation, and that the compactness theorem Theorem 1.4.1 carries over to sequences

$$[A_n, \Phi_n, \mathsf{w}_n] \in M_{\mathfrak{b}}(X^{(T(n))}, g(T(n), \mathsf{w}_n); \vec{\alpha}'_n)$$

provided the sequence w_n is bounded (and similarly for Theorem 1.3.1). The only new ingredient in the proof is the following simple fact: Suppose B is a Banach space, E, F vector bundles over a compact manifold, $L, L'\colon \Gamma(E) \to \Gamma(F)$ differential operators of order d and $K\colon \Gamma(E) \to B$ a linear operator. If L satisfies an inequality

$$\|f\|_{L^p_k} \le C\big(\|Lf\|_{L^p_{k-d}} + \|Kf\|_B\big)$$

and L, L' are sufficiently close in the sense that

$$\|(L-L')f\|_{L^p_{k-d}} \le \epsilon \|f\|_{L^p_k}$$

for some constant $\epsilon > 0$ with $\epsilon C < 1$, then L' obeys the inequality

$$\|f\|_{L^p_k} \le (1-\epsilon C)^{-1} C\big(\|L'f\|_{L^p_{k-d}} + \|Kf\|_B\big).$$

Part III

An application

We will now use the analytical results that we have obtained to prove a gluing formula for Seiberg–Witten invariants of certain 4–manifolds containing a negative definite piece. The formula describes in particular the behaviour of the Seiberg–Witten invariant under blow-up and under the rational blow-down procedure introduced by Fintushel–Stern [18]. The formula was first proved by Fintushel–Stern for blow-up in [17] and for rational blow-down in [18]. Their results were extended to generalized rational blow-down by Park [42]. Detailed proofs of various versions of the formula have been given by Nicolaescu [41], Bauer [6] (using refined Seiberg–Witten invariants) and Kronheimer–Mrowka [31] (using Floer homology).

Apart from providing a detailed and elementary proof of a version sufficient for many applications, this part will show how the parametrized version of our gluing theorem can be used to handle at least the simplest cases of obstructed gluing, thereby providing a unified approach to a wide range of gluing problems.

CHAPTER 14

A generalized blow-up formula

14.1 Statement of result

We first explain how the Seiberg–Witten invariant, usually defined for closed 4–manifolds, can easily be generalized to compact, connected spin^c 4–manifolds Z whose boundary $Y' = \partial Z$ satisfies $b_1(Y') = 0$ and admits a metric g of positive scalar curvature. As usual we assume that $b^+(Z) > 1$. Let $\{Y'_j\}$ be the components of Y', which are rational homology spheres. Let $\hat{Z}$ be the manifold with tubular ends obtained from Z by adding a half-infinite tube $\mathbb{R}_+ \times Y'$. Choose a Riemannian metric on $\hat{Z}$ which agrees with $1 \times g$ on the ends. We consider the monopole equations on $\hat{Z}$ perturbed solely by means of a smooth 2–form μ on $\hat{Z}$ supported in Z as in Equation (3.3). Let $M = M(\hat{Z})$ denote the moduli space of monopoles over $\hat{Z}$ that are asymptotic over $\mathbb{R}_+ \times Y'_j$ to the unique (reducible) monopole over Y'_j. For generic μ the moduli space M will be free of reducibles and a smooth compact manifold of dimension

$$\dim M = 2h(Y') + \frac{1}{4}(c_1(\mathcal{L}_Z)^2 - \sigma(Z)) - 1 + b_1(Z) - b^+(Z),$$

(see Chapter 9). Choose a base-point $x \in \hat{Z}$ and let M_x be the framed moduli space defined just as M except that we now only divide out by those gauge transformations u for which $u(x) = 1$. Let $\mathbb{L} \to M$ be the complex line bundle whose sections are given by maps $s\colon M_x \to \mathbb{C}$ satisfying

$$s(u(\omega)) = u(x) \cdot s(\omega) \tag{14.1}$$

for all $\omega \in M_x$ and gauge transformations u. A choice of homology orientation of Z determines an orientation of M, and we can then define the Seiberg–Witten invariant of Z just as for closed 4–manifolds:

$$\mathrm{SW}(Z) = \begin{cases} \langle c_1(\mathbb{L})^k, [M] \rangle & \text{if} \quad \dim M = 2k \geq 0, \\ 0 & \text{if} \quad \dim M \text{ is negative or odd.} \end{cases}$$

The use of $\mathbb{L}$ rather than $\mathbb{L}^{-1}$ prevents a sign in Theorem 14.1.1 below. (Another justification is that, although $M_x \to M$ is a principal bundle with respect to the canonical $\mathrm{U}(1)$–action, it seems more natural to regard that action as a *left* action.) This invariant $\mathrm{SW}(Z)$ depends only on the homology oriented spin^c–manifold Z, not on the choice of positive scalar curvature metric g on Y'; the proof of this is a special case of the proof of the generalized blow-up formula, which we are now ready to state.

Theorem 14.1.1 *Let Z be a compact, connected, homology oriented* spin^c *4–manifold whose boundary Y' satisfies $b_1(Y') = 0$ and admits a metric of positive scalar curvature. Let $b^+(Z) > 1$, and suppose Z is separated by an embedded rational homology sphere Y admitting a metric of positive scalar curvature,*

$$Z = Z_0 \cup_Y Z_1,$$

where $b_1(Z_0) = b^+(Z_0) = 0$. Let Z_1 have the orientation, homology orientation and spin^c *structure inherited from Z. Then*

$$\mathrm{SW}(Z) = \mathrm{SW}(Z_1) \quad \textit{if} \quad \dim M(\hat{Z}) \geq 0.$$

We will show in Section 14.2 that $\dim M(\hat{Z}_0) \leq -1$. (A particular case of this was proved by different methods in [18, Lemma 8.3].) The addition formula for the index then yields

$$\dim M(\hat{Z}) = \dim M(\hat{Z}_0) + 1 + \dim M(\hat{Z}_1) \leq \dim M(\hat{Z}_1).$$

The following corollary describes the effect on the Seiberg–Witten invariant of both ordinary blow-up and rational blow-down:

Corollary 14.1.1 *Let Z_0, Z_0', Z_1 be compact, connected, homology oriented* spin^c *4–manifolds with $-\partial Z_1 = \partial Z_0 = \partial Z_0' = Y$ as* spin^c *manifolds, where Y is a* spin^c *rational homology sphere admitting a metric of positive scalar curvature. Suppose $b^+(Z_1) > 1$, $b_1(Z_0) = b_1(Z_0') = 0$ and $b_2(Z_0) = b^+(Z_0') = 0$. Let*

$$Z = Z_0 \cup_Y Z_1, \quad Z' = Z_0' \cup_Y Z_1$$

have the orientation, homology orientation and spin^c *structure induced from Z_0, Z_0', Z_1. Then*

$$\mathrm{SW}(Z) = \mathrm{SW}(Z') \quad \textit{if} \quad \dim M(Z') \geq 0.$$

Proof Set $n_\pm = \dim M(\pm\hat{Z}_0)$ and $W = Z_0 \cup_Y (-Z_0)$. Then

$$-1 = \dim M(W) = n_+ + 1 + n_-,$$

hence $n_\pm = -1$. Thus

$$\dim M(Z) = \dim M(\hat{Z}_1) \geq \dim M(Z') \geq 0.$$

The theorem now yields

$$\mathrm{SW}(Z) = \mathrm{SW}(Z_1) = \mathrm{SW}(Z'). \qquad \square$$

14.2 Preliminaries

Let X be a connected spin^c Riemannian 4–manifold with tubular ends $\mathbb{R}_+ \times Y_j$, $j = 1, \dots, r$, as in Section 1.3. Suppose each Y_j is a rational homology sphere and $b_1(X) = 0 = b^+(X)$. We consider the monopole equations on X perturbed only by means of a 2–form μ as in Equation (3.3), where now μ is supported in a given nonempty, compact, codimension 0 submanifold $K \subset X$. Let $\alpha_j \in \mathcal{R}_{Y_j}$ be the reducible monopole over Y_j and $M_\mu = M(X; \vec{\alpha}; \mu; 0)$ the moduli space of monopoles over X with asymptotic limits $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$. This moduli space contains a unique reducible point $\omega(\mu) = [A(\mu), 0]$. Let $\Omega^+_{X,K}$ denote the space of (smooth) self-dual 2–forms on X supported in K, with the C^∞ topology. Let p and w be the exponent and weight function used in the definition of the moduli space M_μ, as in Section 3.4.

Lemma 14.2.1 *Let R be the set of all $\mu \in \Omega^+_{X,K}$ such that the operator*

$$D_{A(\mu)}\colon L^{p,w}_1(\mathbb{S}^+_X) \to L^{p,w}(\mathbb{S}^-_X) \tag{14.2}$$

is either injective or surjective. Then R is open and dense in $\Omega^+_{X,K}$.

Of course, whether the operator is injective or surjective for a given $\mu \in R$ is determined by its index, which is independent of μ.

Proof By Proposition 2.3.1 (ii) and the proof of Proposition 5.1.2, the operator

$$d^+\colon \ker(d^*) \cap L^{p,w}_1 \to L^{p,w}$$

is an isomorphism. Therefore, if A_o is a reference connection over X with limits α_j as in Section 3.4 then there is a unique (smooth) $a = a(\mu) \in L^{p,w}_1$ with

$$d^*a = 0, \qquad d^+a = -\hat{F}^+(A_o) - i\mu.$$

Hence we can take $A(\mu) = A_o + a(\mu)$. Since the operator (14.2) has closed image, it follows by continuity of the map $\mu \mapsto A(\mu)$ that R is open in $\Omega^+_{X,K}$.

To see that R is dense, fix $\mu \in \Omega^+_{X,K}$ and write $A = A(\mu)$. Let W be a Banach space of smooth 1–forms on X supported in K as provided by Lemma 8.2.1. Using the unique continuation property of the Dirac operator it is easy to see that 0 is a regular value of the smooth map

$$\begin{aligned} h\colon W \times (L^{p,w}_1(\mathbb{S}^+_X) \setminus \{0\}) &\to L^{p,w}(\mathbb{S}^-_X), \\ (\eta, \Phi) &\mapsto D_{A+i\eta}\Phi. \end{aligned}$$

In general, if $f_1\colon E \to F_1$ and $f_2\colon E \to F_2$ are surjective homomorphisms between vector spaces then $f_1|_{\ker f_2}$ and $f_2|_{\ker f_1}$ have identical kernels and isomorphic cokernels. In particular, the projection $\pi\colon h^{-1}(0) \to W$ is a Fredholm map whose index at every point agrees with the index m of D_A. By the Sard-Smale theorem the regular values of π form a residual (hence dense) subset of W. If $\eta \in W$ is a regular value then we see that $D_{A+i\eta}$ is injective when $m \le 0$ and surjective when $m > 0$. Since the topology on W is stronger than the C^∞ topology it follows that R contains points of the form $\mu + d^+\eta$ arbitrarily close to μ. □

Lemma 14.2.2 *Suppose the metric on each Y_j has positive scalar curvature. Let R' be the set of all $\mu \in \Omega^+_{X,K}$ such that the irreducible part M^*_μ is empty and the operator $D_{A(\mu)}$ in (14.2) is injective. Then R' is open and dense in $\Omega^+_{X,K}$.*

Proof Recall that M^*_μ has expected dimension $2m-1$, where $m = \mathrm{ind}_{\mathbb{C}} D_{A(\mu)}$.

Suppose $m > 0$. We will show that this leads to a contradiction. Let R'' be the set of all $\mu \in \Omega^+_{X,K}$ for which M_μ is regular. (Note that the reducible point is regular precisely when $D_{A(\mu)}$ is surjective.) From Lemma 14.2.1 and Proposition 8.2.1 one finds that R'' is dense in $\Omega^+_{X,K}$. (Starting with a given μ, first perturb it a little to make the reducible point regular, then a little more to make also the irreducible part regular.) But for any $\mu \in R''$ the moduli space M_μ would be compact with one reducible point, which yields a contradiction as in [21]. Therefore, $m \le 0$.

We now see, exactly as for R'', that R' is dense in $\Omega^+_{X,K}$. To prove that R' is open we use a compactness argument together with the following fact: For any given $\mu_0 \in R'$ there is a neighbourhood U of $\omega(\mu_0)$ in $\mathcal{B}(X;\vec{\alpha})$ such that

$$M^*_\mu \cap U = \varnothing$$

for any $\mu \in \Omega^+_{X,K}$ with $\|\mu - \mu_0\|_p$ sufficiently small. To prove this we work in a slice at $(A(\mu_0), 0)$, ie we represent $\omega(\mu)$ (uniquely) by $(A, 0)$ where $d^*(A - A(\mu_0)) = 0$, and we consider a point in M^*_μ represented by $(A+a,\phi)$ where $d^*a=0$. Note that since $b_1(X)=0$, the latter representative is unique up to multiplication of ϕ by unimodular constants.

Observe that there is a constant $C_1<\infty$ such that if $\|\mu-\mu_0\|_p$ is sufficiently small then

$$\|\psi\|_{L^{p,w}_1}\le C_1\|D_A\psi\|_{L^{p,w}}$$

for all $\psi\in L^{p,w}_1$. Hence if $L=(d^*+d^+,D_A)$ then for such μ one has

$$\|s\|_{L^{p,w}_1}\le C_2\|Ls\|_{L^{p,w}}$$

for all $s\in L^{p,w}_1$. Denoting by SW_μ the Seiberg–Witten map over X for the perturbation form μ we have

$$0=\mathrm{SW}_\mu(A+a,\phi)-\mathrm{SW}_\mu(A,0)=(d^+a-Q(\phi),D_A\phi+a\phi),$$

where Q is as in (3.3). Taking $s=(a,\phi)$ we obtain

$$\|s\|_{L^{p,w}_1}\le C_2\|Ls\|_{L^{p,w}}\le C_3\|s\|^2_{L^{2p,w}}\le C_4\|s\|^2_{L^{p,w}_1}.$$

Since $s\neq 0$ we conclude that

$$\|s\|_{L^{p,w}_1}\ge C_4^{-1}.$$

Choose $\delta\in(0,C_4^{-1})$ and define

$$U=\{[A(\mu_0)+b,\psi]\,:\,\|(b,\psi)\|_{L^{p,w}_1}<\delta,\ d^*b=0\}.$$

If $\|\mu-\mu_0\|_p$ is so small that $\|A-A(\mu_0)\|_{L^{p,w}_1}\le C_4^{-1}-\delta$ then

$$\|(A+a-A(\mu_0),\phi)\|_{L^{p,w}_1}\ge\|s\|_{L^{p,w}_1}-\|A-A(\mu_0)\|_{L^{p,w}_1}\ge\delta,$$

hence $[A+a,\phi]\notin U$. □

14.3 The extended monopole equations

We now return to the situation in Theorem 14.1.1. Set $X_j=\widehat{Z}_j$ for $j=0,1$. Choose metrics of positive scalar curvature on Y and Y' and a metric on the disjoint union $X=X_0\cup X_1$ which agrees with the corresponding product metrics on the ends. Let Y be oriented as the boundary of Z_0, so that X_0 has an end $\mathbb{R}_+\times Y$ and X_1 an end $\mathbb{R}_+\times(-Y)$. Gluing these two ends of X we obtain as in Section 1.4 a manifold $X^{(T)}$ for each $T>0$.

Choose smooth monopoles α over Y and α'_j over Y'_j (these are reducible and unique up to gauge equivalence). Let $S_o=(A_o,\Phi_o)$ be a reference configuration over X

with these limits over the ends, and S'_o the associated reference configuration over $X^{(T)}$. Adopting the notation introduced in the beginning of Section 10.2, let $\mathcal{C}$ be the corresponding $L^{p,w}_1$ configuration space over X and $\mathcal{C}'$ the corresponding $L^{p,\kappa}_1$ configuration space over $X^{(T)}$. For any finite subset $\mathfrak{b} \subset X_{:0} = Z_0 \cup Z_1$ let $\mathcal{G}_{\mathfrak{b}}, \mathcal{G}'_{\mathfrak{b}}$ be the corresponding groups of gauge transformations that restrict to 1 on $\mathfrak{b}$.

As in Section 14.2 we first consider the monopole equations over X and $X^{(T)}$ perturbed only by means of a self-dual 2–form $\mu = \mu_0 + \mu_1$, where μ_j is supported in Z_j. The corresponding moduli spaces will be denoted $M(X)$ and $M^{(T)} = M(X^{(T)})$. Of course, $M(X)$ is a product of moduli spaces over X_0 and X_1:

$$M(X) = M(X_0) \times M(X_1).$$

By Lemma 14.2.2 we can choose μ_0 such that $M(X_0)$ consists only of the reducible point (which we denote by $\omega_{\mathrm{red}} = [A_{\mathrm{red}}, 0]$), and such that the operator

$$D_{A_{\mathrm{red}}}\colon L^{p,w}_1(\mathbb{S}^+_{X_0}) \to L^{p,w}(\mathbb{S}^-_{X_0}) \tag{14.3}$$

is injective. By Proposition 8.2.1 and unique continuation for self-dual closed 2–forms we can then choose μ_1 such that

- $M(X_1)$ is regular and contains no reducibles,
- the irreducible part of $M^{(T)}$ is regular for all natural numbers T.

Set
$$k = -\mathrm{ind}_{\mathbb{C}}(D_{A_{\mathrm{red}}}) \ge 0.$$

If $k > 0$ then ω_{red} is not a regular point of $M(X_0)$ and we cannot appeal to the gluing theorem, Theorem 10.1.1, for describing $M^{(T)}$ when T is large. We will therefore introduce an extra parameter $z \in \mathbb{C}^k$ into the Dirac equation on Z_0, to obtain what we will call the "extended monopole equations", such that ω_{red} becomes a regular point of the resulting parametrized moduli space over X_0. This will allow us to apply the gluing theorem for parametrized moduli spaces, Theorem 13.3.1.

We are going to add to the Dirac equation an extra term $\beta(A, \Phi, z)$ which will be a product of three factors:

(i) a holonomy term h_A (to achieve gauge equivariance),

(ii) a cutoff function $g(A, \Phi)$ (to retain an apriori pointwise bound on Φ),

(iii) a linear combination $\sum z_j \psi_j$ of certain negative spinors (to make ω_{red} regular).

We will now describe these terms more precisely.

(i) Choose an embedding $f\colon \mathbb{R}^4 \to \mathrm{int}(Z_0)$, and set $x_0 = f(0)$ and $U_0 = f(\mathbb{R}^4)$. For each $x \in U_0$ let $\gamma_x\colon [0,1] \to U_0$ be the path from x_0 to x given by

$$\gamma_x(t) = f(tf^{-1}(x)).$$

For any spin^c connection A over U_0 define the function $h_A\colon U_0 \to \mathrm{U}(1)$ by

$$h_A(x) = \exp\left(-\int_{[0,1]} \gamma_x^*(A - A_{\mathrm{red}})\right),$$

cf Equation (10.1). Note that h_A depends on the choice of A_{red}, which is only determined up to modification by elements of $\mathcal{G}$.

(ii) Set $K_0 = f(D^4)$, where $D^4 \subset \mathbb{R}^4$ is the closed unit disk. Choose a smooth function $g\colon \mathcal{B}^*(K_0) \to [0,1]$ such that $g(A,\Phi) = 0$ when $\|\Phi\|_{L^\infty(K_0)} \geq 2$ and $g(A,\Phi) = 1$ when $\|\Phi\|_{L^\infty(K_0)} \leq 1$. Extend g to $\mathcal{B}(K_0)$ by setting $g(A,0) = 1$ for all A.

(iii) By unique continuation for the formal adjoint $D^*_{A_{\mathrm{red}}}$ there are smooth sections $\psi_1, \dots, \psi_k$ of $\mathbb{S}^-_{X_0}$ supported in K_0 and spanning a linear complement of the image of the operator $D_{A_{\mathrm{red}}}$ in (14.3).

For any configuration (A,Φ) over X and $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ define

$$\beta(A,\Phi,z) = g(A,\Phi)\, h_A \sum_{j=1}^k z_j \psi_j .$$

Note that for gauge transformations u over X one has

$$u(x_0)\, h_{u(A)} = u\, h_A .$$

Since g is gauge invariant, this yields

$$\beta(u(A), u\,\Phi, u(x_0)z) = u \cdot \beta(A,\Phi,z).$$

The following lemma is useful for estimating the holonomy term h_A:

Lemma 14.3.1 *Let $a = \sum a_j\,dx_j$ be a 1–form on the closed unit disk D^n in $\mathbb{R}^n$, $n > 1$. For each $x \in D^n$ let $J(x)$ denote the integral of a along the line segment from 0 to x, ie*

$$J(x) = \sum_{j=1}^n x_j \int_0^1 a_j(tx)\,dt.$$

Then for any $q \ge 1$ and $r > qn$ and nonnegative integer k there is a constant $C < \infty$ independent of a such that

$$\|J\|_{L^q_k(D^n)} \le C\|a\|_{L^r_k(D^n)}.$$

Proof If b is a function on D^n and χ the characteristic function of the interval $[0,1]$ then

$$\begin{aligned}\int_{D^n}\int_0^1 b(tx)\,dt\,dx &= \int_{D^n} b(x)\int_0^1 t^{-n}\chi(t^{-1}|x|)\,dt\,dx\\ &= \frac{1}{n-1}\int_{D^n}(|x|^{1-n}-1)\,b(x)\,dx.\end{aligned}$$

From this basic calculation the lemma is easily deduced. □

It follows from the lemma that $a \mapsto h_{A_{\text{red}}+a}$ defines a smooth map $L^p_1(K_0; i\Lambda^1) \to L^q_1(K_0)$ provided $p > 4q > 16$. Hence, if $p > 16$ (which we henceforth assume) then

$$\mathcal{C}(K_0)\times\mathbb{C}^k \to L^p(K_0;\mathbb{S}^-), \quad ((A,\Phi),z)\mapsto \beta(A,\Phi,z)$$

is a smooth map whose derivative at every point is a compact operator. Here $\mathcal{C}(K_0)$ is the L^p_1 configuration space over K_0.

The extended monopole equations for $((A,\Phi),z)\in\mathcal{C}\times\mathbb{C}^k$ are

$$\begin{aligned}\hat{F}^+_A + i\mu - Q(\Phi) &= 0,\\ D_A\Phi + \beta(A,\Phi,z) &= 0.\end{aligned} \tag{14.4}$$

(Compare the holonomy perturbations of the instanton equations constructed by Donaldson [13, 2 (b)].) Define actions of $\mathcal{G}$ and $\mathcal{G}'$ on $\mathcal{C}\times\mathbb{C}^k$ and $\mathcal{C}'\times\mathbb{C}^k$ respectively by

$$u(S,z) = (u(S), u(x_0)z).$$

Then the left hand side of (14.4) describes a $\mathcal{G}$–equivariant smooth map $\mathcal{C}\times\mathbb{C}^k \to L^{p,w}$.

For $\epsilon > 0$ let $B^{2k}_\epsilon \subset \mathbb{C}^k$ denote the open ball of radius ϵ about the origin, and D^{2k}_ϵ the corresponding closed ball. For $0 < \epsilon \le 1$ set

$${}_\epsilon M_{\mathfrak{b}}(X) = \{\text{solutions } ((A,\Phi),z)\in\mathcal{C}\times B^{2k}_\epsilon \text{ to (14.4)}\}\Big/\mathcal{G}_{\mathfrak{b}},$$

This moduli space is clearly a product of moduli spaces over X_0 and X_1:

$${}_\epsilon M_{\mathfrak{b}}(X) = {}_\epsilon M_{\mathfrak{b}_0}(X_0)\times M_{\mathfrak{b}_1}(X_1),$$

where $\mathfrak{b}_j = \mathfrak{b}\cap X_j$.

Noting that the equations (14.4) also make sense over $X^{(T)}$ we define

$$ {}_\epsilon M_{\mathfrak{b}}^{(T)} = \{\text{solutions } ((A,\Phi),z) \in \mathcal{C}' \times B_\epsilon^{2k} \text{ to (14.4)}\} \Big/ \mathcal{G}'_{\mathfrak{b}}. $$

We define ${}^\epsilon M_{\mathfrak{b}}(X)$ and ${}^\epsilon M_{\mathfrak{b}}^{(T)}$ in a similar way as ${}_\epsilon M_{\mathfrak{b}}(X)$ and ${}_\epsilon M_{\mathfrak{b}}^{(T)}$, but with D_ϵ^{2k} in place of B_ϵ^{2k}.

Choose a base-point $x_1 \in Z_1$. We will only consider the cases when $\mathfrak{b}$ is a subset of $\{x_0, x_1\}$, and we indicate $\mathfrak{b}$ by listing its elements (writing ${}_\epsilon M_{x_0,x_1}$ and ${}_\epsilon M$ etc).

Lemma 14.3.2 *Any element of ${}^1M(X_0)$ or ${}^1M^{(T)}$ has a smooth representative.*

Proof Given Lemma 14.3.1 this is proved in the usual way. □

Lemma 14.3.3 *There is a $C < \infty$ independent of T such that $\|\Phi\|_\infty < C$ for all elements $[A,\Phi,z]$ of ${}^1M(X)$ or ${}^1M^{(T)}$.*

Proof Suppose $|\Phi|$ achieves a local maximum ≥ 2 at some point x. If $x \notin K_0$ then one obtains a bound on $|\Phi(x)|$ using the maximum principle as in [30, Lemma 2]. If $x \in K_0$ then the same works because then $g(A,\Phi) = 0$. □

Lemma 14.3.4 *${}^1M(X)$ and ${}^1M^{(T)}$ are compact for all $T > 0$.*

Proof Given Lemma 14.3.1 and Lemma 14.3.3, the second approach to compactness carries over. □

We identify $M_{\mathfrak{b}_0}(X_0)$ with the set of elements of ${}_1M_{\mathfrak{b}_0}(X_0)$ with $z = 0$, and similarly for moduli spaces over $X, X^{(T)}$. It is clear from the definition of $\beta(A,\Phi,z)$ that ω_{red} is a regular point of ${}_1M(X_0)$. Since ${}_1M_{x_0}(X_0)$ has expected dimension 0, it follows that ω_{red} is an isolated point of ${}_1M_{x_0}(X_0)$. Because ${}^1M_{x_0}(X_0)$ is compact, there is an ϵ such that ${}_\epsilon M_{x_0}(X_0)$ consists only of the point ω_{red}. Fix such an ϵ for the remainder of the chapter.

Lemma 14.3.5 *If $\omega_n \in {}_\epsilon M^{(T_n)}$ with $T_n \to \infty$ then a subsequence of $\{\omega_n\}$ chain-converges to $(\omega_{\mathrm{red}}, \omega)$ for some $\omega \in M(X_1)$.*

Proof Again, this is proved using the second approach to compactness. □

Corollary 14.3.1 *If $T \gg 0$ then ${}_\epsilon M^{(T)}$ contains no element which is reducible over Z_1.*

14.4 Applying the gluing theorem

Let $\mathrm{Hol}=\mathrm{Hol}_1$ be defined as in Equation (10.1) in terms of a path in $X^{(T)}$ from x_0 to x_1 running once through the neck.

By Proposition 10.4.1, if $K_1=(X_1)_{:\mathfrak{t}}$ with $\mathfrak{t}\gg 0$ then there is a $\mathrm{U}(1)$–invariant open subset $V_1\subset\mathcal{B}^*_{x_1}(K_1)=\mathcal{B}_{x_1}(K_1)$ containing $R_{K_1}(M_{x_1}(X_1))$, and a $\mathrm{U}(1)$–equivariant smooth map

$$q_1\colon V_1\to M_{x_1}(X_1)$$

such that $q_1(\omega|_{K_1})=\omega$ for all $\omega\in M_{x_1}(X_1)$. Here R_{K_1} denotes restriction to K_1. It follows from Lemma 14.3.5 that if T is sufficiently large then $\omega|_{K_1}\in V_1$ for all $\omega\in{}_\epsilon M^{(T)}_{x_1}$.

Proposition 14.4.1 *For all sufficiently large T the moduli space ${}_\epsilon M^{(T)}_{x_1}$ is regular and the map*

$${}_\epsilon M^{(T)}_{x_1}\to M_{x_1}(X_1),\quad \omega\mapsto q_1(\omega|_{K_1}) \tag{14.5}$$

is an orientation preserving $\mathrm{U}(1)$–equivariant diffeomorphism.

Proof We will apply the version of Theorem 13.3.1 with (in the notation of that chapter) $\mathbb{T}$ acting nontrivially on W. Set

$$G={}_\epsilon M_{x_0,x_1}(X)=\{\omega_{\mathrm{red}}\}\times M_{x_1}(X_1),$$
$$K=K_0\cup K_1,$$
$$V=\mathcal{B}_{x_0}(K_0)\times V_1\times B^{2k}_\epsilon.$$

Note that G is compact and $\breve{\mathcal{G}}_{\mathfrak{b}}(K)=\mathcal{G}_{\mathfrak{b}}(K)$. Define

$$q\colon V\to G,\quad (\omega_0,\omega_1,z)\mapsto(\omega_{\mathrm{red}},q_1(\omega_1)).$$

In general, an element $(u_0,u_1)\in\mathrm{U}(1)^2$ acts on appropriate configuration and moduli spaces like any gauge transformation u with $u(x_j)=u_j$, $j=0,1$, and it acts on B^{2k}_ϵ by multiplication with u_0. Then clearly, q is $\mathrm{U}(1)^2$–equivariant, so by the gluing theorem there is a compact, codimension 0 submanifold $K'\subset X$ containing K and a $\mathrm{U}(1)^2$–equivariant open subset $V'\subset\mathcal{B}^*_{x_0,x_1}(K')\times B^{2k}_\epsilon$ containing $R_{K'}(G)$ and satisfying $R_K(V')\subset V$ and such that for all sufficiently large T the space

$$H^{(T)}=\{(\omega,z)\in{}_\epsilon M^{(T)}_{x_0,x_1}\;:\;(\omega|_{K'},z)\in V'\}$$

consists only of regular points, and the map

$$H^{(T)}\to\mathrm{U}(1)\times{}_\epsilon M_{x_0,x_1}(X),\quad (\omega,z)\mapsto(\mathrm{Hol}(\omega),(\omega_{\mathrm{red}},q_1(\omega|_{K_1}))) \tag{14.6}$$

is a $\mathrm{U}(1)^2$–equivariant diffeomorphism. But it follows from Lemma 14.3.5 that $H^{(T)} = {}_\epsilon M^{(T)}_{x_0,x_1}$ for $T \gg 0$, and dividing out by the action of $\mathrm{U}(1)\times\{1\}$ in (14.6) we see that (14.5) is a $\mathrm{U}(1)$–equivariant diffeomorphism.

We now discuss orientations. Let X have the *direct sum* homology orientation inherited from X_0 and X_1 (corresponding to the direct sum orientation of the operators $-d^* + d^+$; see Section 12.6). Then the map (14.6) is orientation preserving by Theorem 13.3.1. Using Proposition 12.7.1 it is a simple exercise to show that ${}_\epsilon M_{x_0,x_1}(X) \to M_{x_1}(X_1)$ preserves orientations if and only if $b_1(X_1) + b^+(X_1)$ is even. On the other hand, $(u_0, 1) \in \mathrm{U}(1)\times\{1\}$ acts on $\mathrm{U}(1)$ in (14.6) by multiplication with u_0^{-1}. Recalling our convention for orienting framed moduli spaces (see the last paragraph of Section 12.2) we find that the signs cancel and the map (14.5) does preserve orientations. □

Proof of Theorem 14.1.1 For large T let $\mathbb{L} \to {}_\epsilon M^{(T)}$ be the complex line bundle associated to the base-point x_1 as in Section 14.1. For $j = 1, \dots, k$ the map

$$s_j\colon {}_\epsilon M^{(T)}_{x_1} \to \mathbb{C}, \quad [A, \Phi, z] \mapsto \mathrm{Hol}(A)\cdot z_j$$

is $\mathrm{U}(1)$–equivariant in the sense of (14.1) and therefore defines a smooth section of $\mathbb{L}$. The sections s_j together form a section s of the bundle $\mathbb{E} = \bigoplus^k \mathbb{L}$ whose zero set is the unparametrized moduli space $M^{(T)}$. It is easy to see that s is a regular section precisely when $M^{(T)}$ is a regular moduli space, which by Corollary 14.3.1 and the choice of μ_1 holds at least when T is a sufficiently large natural number. In that case $s^{-1}(0) = M^{(T)}$ as oriented manifolds. Set

$$\ell = \frac{1}{2}\dim\, M^{(T)} \geq 0,$$

so that $\dim M(X_1) = 2(k+\ell)$. If ℓ is not integral then $\mathrm{SW}(Z_1) = 0 = \mathrm{SW}(Z)$ and we are done. Now suppose ℓ is integral and let T be a large natural number. Choose a smooth section s' of $\mathbb{E}' = \bigoplus^\ell \mathbb{L}$ such that $\sigma = s'|_{M^{(T)}}$ is a regular section of $\mathbb{E}'|_{M^{(T)}}$, or equivalently, such that $s \oplus s'$ is a regular section of $\mathbb{E} \oplus \mathbb{E}' = \bigoplus^{k+\ell}\mathbb{L}$. Then

$$\mathrm{SW}(Z_1) = \#(s\oplus s')^{-1}(0) = \#\sigma^{-1}(0) = \mathrm{SW}(Z),$$

where the first equality follows from Proposition 14.4.1, and # as usual means a signed count. □

APPENDIX A

Patching together gauge transformations

In the proof of Lemma 4.1.1 we encounter sequences S_n of configurations such that for any point x in the base-manifold there is a sequence v_n of gauge transformations defined in a neighbourhood of x such that $v_n(S_n)$ converges (in some Sobolev norm) in a (perhaps smaller) neighbourhood of x. The problem then is to find a sequence u_n of global gauge transformations such that $u_n(S_n)$ converges globally. If v_n, w_n are two such sequences of local gauge transformations then $v_n w_n^{-1}$ will be bounded in the appropriate Sobolev norm, so the problem reduces to the lemma below.

This issue was discussed by Uhlenbeck in [50, Section 3]. Our approach has the advantage that it does not involve any "limiting bundles".

Lemma A.0.1 *Let X be a Riemannian manifold and $P \to X$ a principal G–bundle, where G is a compact subgroup of some matrix algebra $M_r(\mathbb{R})$. Let $M_r(\mathbb{R})$ be equipped with an Ad_G–invariant inner product, and fix a connection in the Euclidean vector bundle $E = P \times_{\mathrm{Ad}_G} M_r(\mathbb{R})$ (which we use to define Sobolev norms of automorphisms of E). Let $\{U_i\}_{i=1}^{\infty}$, $\{V_i\}_{i=1}^{\infty}$ be open covers of X such that $U_i \Subset V_i$ for each i. We also assume that each V_i is the interior of a compact codimension 0 submanifold of X, and that ∂V_i and ∂V_j intersect transversally for all $i \neq j$. For each i and $n = 1, 2, \dots$ let $v_{i,n}$ be a continuous automorphism of $P|_{V_i}$. Suppose $v_{i,n} v_{j,n}^{-1}$ converges uniformly over $V_i \cap V_j$ for each i, j (as maps into E). Then there exist*

- *a sequence of positive integers $n_1 \le n_2 \le \cdots$,*
- *for each positive integer k an open subset $W_k \subset X$ with*

$$\bigcup_{i=1}^{k} U_i \Subset W_k \subset \bigcup_{i=1}^{k} V_i,$$

- *for each k and $n \geq n_k$ a continuous automorphism $w_{k,n}$ of $P|_{W_k}$,*

such that the following hold:

(i) *If $1 \leq j \leq k$ and $n \geq n_k$ then $w_{j,n} = w_{k,n}$ on $\bigcup_{i=1}^{j} U_i$.*

(ii) *For each i, k the sequence $w_{k,n} v_{i,n}^{-1}$ converges uniformly over $W_k \cap V_i$.*

(iii) *If $1 \leq p < \infty$, and $m > n/p$ is an integer such that $v_{i,n} \in L^p_{m,\mathrm{loc}}$ for all i, n then $w_{k,n} \in L^p_{m,\mathrm{loc}}$ for all k and $n \geq n_k$. If in addition*

$$\sup_n \|v_{i,n} v_{j,n}^{-1}\|_{L^p_m(V_i \cap V_j)} < \infty \qquad \text{for all } i, j$$

then
$$\sup_{n \geq n_k} \|w_{k,n} v_{i,n}^{-1}\|_{L^p_m(W_k \cap V_i)} < \infty \qquad \text{for all } k, i.$$

The transversality condition ensures that the Sobolev embedding theorem holds for $V_i \cap V_j$ (see Adams [1]). Note that this condition can always be achieved by shrinking the V_i's a little.

Proof Let $N' \subset LG$ be a small Ad_G invariant open neighbourhood of 0. Then $\exp\colon LG \to G$ maps N' diffeomorphically onto an open neighbourhood N of 1. Let $f\colon N \to N'$ denote the inverse map. Let $\mathrm{Aut}(P)$ the bundle of fibre automorphisms of P and $\mathfrak{g}_P$ the corresponding bundle of Lie algebras. Set $\mathbf{N} = P \times_{\mathrm{Ad}_G} N \subset \mathfrak{g}_P$ and let $\exp^{-1}\colon \mathbf{N} \to \mathrm{Aut}(P)$ be the map defined by f.

Set $w_{1,n} = v_{1,n}$ and $W_1 = V_1$. Now suppose $w_{k,n}$, W_k have been chosen for $1 \leq k < \ell$, where $\ell \geq 2$, such that (i)–(iii) hold for these values of k. Set $z_n = w_{\ell-1,n}(v_{\ell,n})^{-1}$ on $W_{\ell-1} \cap V_\ell$. According to the induction hypothesis the sequence z_n converges uniformly over $W_{\ell-1} \cap V_\ell$, hence there exists an integer $n_\ell \geq n_{\ell-1}$ such that $y_n = (z_{n_\ell})^{-1} z_n$ takes values in $\mathbf{N}$ for $n \geq n_\ell$.

Choose an open subset $\mathcal{W} \subset X$ which is the interior of a compact codimension 0 submanifold of X, and which satisfies

$$\bigcup_{i=1}^{\ell-1} U_i \Subset \mathcal{W} \Subset W_{\ell-1}.$$

We also require that $\partial\mathcal{W}$ intersect $\partial V_i \cap \partial V_j$ transversally for all i, j. (For instance, one can take $\mathcal{W} = \alpha^{-1}([0,\epsilon])$ for suitable ϵ, where $\alpha\colon X \to [0,1]$ is any smooth function with $\alpha = 0$ on $\bigcup_{i=1}^{\ell-1} U_i$ and $\alpha = 1$ on $W_{\ell-1}$.) Choose also a smooth, compactly supported function $\phi\colon W_{\ell-1} \to \mathbb{R}$ with $\phi|_{\mathcal{W}} = 1$. Set $W_\ell = \mathcal{W} \cup V_\ell$ and for $n \geq n_\ell$ define an automorphism $w_{\ell,n}$ of $P|_{W_\ell}$ by

$$w_{\ell,n} = \begin{cases} w_{\ell-1,n} & \text{on } \mathcal{W}, \\ z_{n_\ell} \exp(\phi \exp^{-1} y_n) v_{\ell,n} & \text{on } W_{\ell-1} \cap V_\ell, \\ z_{n_\ell} v_{\ell,n} & \text{on } V_\ell \setminus \operatorname{supp}(\phi). \end{cases}$$

Then (i)–(iii) hold for $k = \ell$ as well. To see that (iii) holds, note that our transversality assumptions guarantee that the Sobolev embedding theorem holds for $W_{\ell-1} \cap V_\ell$ and for all $V_i \cap V_j$. Since $mp > n$, L^p_m is therefore a Banach algebra for these spaces (see [1]). Recalling the proof of this fact, and the behaviour of L^p_m under composition with smooth maps on the left (see McDuff–Salamon [37, p 184]), one obtains (iii). □

APPENDIX B

A quantitative inverse function theorem

In this appendix E, E' will be Banach spaces. We denote by $\mathcal{B}(E, E')$ the Banach space of bounded operators from E to E'. If $T \in \mathcal{B}(E, E')$ then $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$. If $U \subset E$ is open and $f\colon U \to E'$ smooth then $Df(x) \in \mathcal{B}(E, E')$ is the derivative of f at $x \in U$. The second derivative $D(Df)(x) \in \mathcal{B}(E, \mathcal{B}(E, E'))$ is usually written $D^2 f(x)$ and can be identified with the symmetric bilinear map $E \times E \to E'$ given by

$$D^2 f(x)(y, z) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} f(x + sy + tz).$$

The norm of the second derivative is

$$\|D^2 f(x)\| = \sup_{\|y\|, \|z\| \le 1} \|D^2 f(x)(y, z)\|.$$

For $r > 0$ let $B_r = \{x \in E : \|x\| < r\}$.

Lemma B.0.2 *Let $\epsilon, M > 0$ be positive real numbers such that $\epsilon M < 1$, and suppose $f\colon B_\epsilon \to E$ is a smooth map satisfying*

$$f(0) = 0; \quad Df(0) = I; \quad \|D^2 f(x)\| \le M \quad \text{for } x \in B_\epsilon.$$

Then f restricts to a diffeomorphism $f^{-1} B_{\epsilon/2} \xrightarrow{\approx} B_{\epsilon/2}$.

The conclusion of the lemma holds even when $\epsilon M = 1$; see Proposition B.0.2 below.

Proof The estimate on $D^2 f$ gives

$$\|Df(x) - I\| = \|Df(x) - Df(0)\| \le M\|x\|. \tag{B.1}$$

Therefore the map

$$h(x) = f(x) - x = \int_0^1 (Df(tx) - I)x\,dt$$

satisfies

$$\|h(x)\| \le \frac{M}{2}\|x\|^2,$$
$$\|h(x_2) - h(x_1)\| \le \epsilon M \|x_2 - x_1\|$$

for all $x, x_1, x_2 \in B_\epsilon$. Hence for every $y \in B_{\epsilon/2}$ the assignment $x \mapsto y - h(x)$ defines a map $B_\epsilon \to B_\epsilon$ which has a unique fix-point. In other words, f maps $f^{-1}B_{\epsilon/2}$ bijectively onto $B_{\epsilon/2}$. Moreover, $Df(x)$ is an isomorphism for every $x \in B_\epsilon$, by (B.1). Applying the contraction mapping argument above to f around an arbitrary point in B_ϵ then shows that f is an open map. It is then a simple exercise to prove that the inverse $g\colon B_{\epsilon/2} \to f^{-1}B_{\epsilon/2}$ is differentiable and $Dg(y) = (Df(g(y)))^{-1}$ (see Dieudonné [10, 8.2.3]). Repeated application of the chain rule then shows that g is smooth. □

For $r > 0$ let $B_r \subset E$ be as above, and define $B'_r \subset E'$ similarly.

Proposition B.0.2 *Let ϵ, M be positive real numbers and $f\colon B_\epsilon \to E'$ a smooth map such that $f(0) = 0$, $L = Df(0)$ is invertible and*

$$\|D^2 f(x)\| \le M \quad \text{for all } x \in B_\epsilon.$$

Set $\kappa = \|L^{-1}\|^{-1} - \epsilon M$ and $\epsilon' = \epsilon\|L^{-1}\|^{-1}$. Then the following hold:

(i) *If $\kappa \ge 0$ then f is a diffeomorphism onto an open subset of E' containing $B'_{\epsilon'/2}$.*

(ii) *If $\kappa > 0$ and $g\colon B'_{\epsilon'/2} \to B_\epsilon$ is the smooth map satisfying $f \circ g = I$ then for all $x \in B_\epsilon$ and $y \in B'_{\epsilon'/2}$ one has*

$$\|Df(x)^{-1}\|, \|Dg(y)\| < \kappa^{-1}, \quad \|D^2 g(y)\| < M\kappa^{-3}.$$

The reader may wish to look at some simple example (such as a quadratic polynomial) to understand the various ways in which these results are optimal.

Proof (i) For every $x \in B_\epsilon$ we have

$$\|Df(x)L^{-1} - I\| \le \|Df(x) - L\| \cdot \|L^{-1}\| < \epsilon M\|L^{-1}\| \le 1,$$

hence $Df(x)$ is invertible. Thus f is a local diffeomorphism by Lemma B.0.2. Set $h(x) = f(x) - Lx$. If $x_1, x_2 \in B_\epsilon$ and $x_1 \neq x_2$ then

$$\|f(x_2) - f(x_1)\| \geq \|L(x_2 - x_1)\| - \|h(x_2) - h(x_1)\| > \kappa \|x_2 - x_1\|,$$

hence f is injective. By choice of ϵ' the map

$$\tilde{f} = f \circ L^{-1} \colon B'_{\epsilon'/2} \to E'$$

is well defined, and for every $y \in B'_{\epsilon'/2}$ one has

$$\|D^2 \tilde{f}(y)\| \leq M \|L^{-1}\|^2.$$

Because
$$\epsilon' M \|L^{-1}\|^2 = \epsilon M \|L^{-1}\| \leq 1,$$

Lemma B.0.2 says that the image of $\tilde{f}$ contains every ball $B'_{\delta/2}$ with $0 < \delta < \epsilon'$, hence also $B'_{\epsilon'/2}$.

(ii) Set $c = I - Df(x)L^{-1}$. Then

$$Df(x)^{-1} = L^{-1} \sum_{n=0}^{\infty} c^n,$$

hence
$$\|Df(x)^{-1}\| \leq \frac{\|L^{-1}\|}{1 - \|c\|} < \frac{\|L^{-1}\|}{1 - \epsilon M \|L^{-1}\|} = \kappa^{-1}.$$

This also gives the desired bound on $Dg(y) = Df(g(y))^{-1}$.

To estimate $D^2 g$, let $\mathrm{Iso}(E, E') \subset \mathcal{B}(E, E')$ be the open subset of invertible operators, and let $\iota \colon \mathrm{Iso}(E, E') \to \mathcal{B}(E', E)$ be the inversion map: $\iota(a) = a^{-1}$. Then ι is smooth, and its derivative is given by

$$D\iota(a)b = -a^{-1} b a^{-1}$$

(see Dieudonné [10]). The chain rule says that

$$\begin{aligned} Dg &= \iota \circ Df \circ g, \\ D(Dg)(y) &= D\iota(Df(g(y))) \circ D(Df)(g(y)) \circ Dg(y). \end{aligned}$$

This gives

$$\begin{aligned} \|D(Dg)(y)\| &\leq \|Df(g(y))^{-1}\|^2 \cdot \|D(Df)(g(y))\| \cdot \|Dg(y)\| \\ &< \kappa^{-2} \cdot M \cdot \kappa^{-1}. \end{aligned}$$
□

APPENDIX C

Splicing left or right inverses

Let X be a Riemannian manifold with tubular ends as in Section 1.4 but of arbitrary dimension. Let $E \to X$ be a vector bundle which over each end $\mathbb{R}_+ \times (\pm Y_j)$ (resp. $\mathbb{R}_+ \times Y'_j$) is isomorphic (by a fixed isomorphism) to the pullback of a bundle $E_j \to Y_j$ (resp. $E'_j \to Y'_j$). Let $F \to X$ be another bundle of the same kind. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a differential operator of order $d \geq 1$ which is translationary invariant over each end and such that for each j the restrictions of D to $\mathbb{R}_+ \times Y_j$ and $\mathbb{R}_+ \times (-Y_j)$ agree in the obvious sense. The operator D gives rise to a glued differential operator $D'\colon \Gamma(E') \to \Gamma(F')$ over $X^{(T)}$, where E', F' are the bundles over $X^{(T)}$ formed from E, F resp. Let k, ℓ, m be nonnegative integers and $1 \leq p < \infty$. Let $L^p_k(X; F)_{:0}$ denote the subspace of $L^p_k(X; F)$ consisting of those elements that vanish a.e. outside $X_{:0}$. We can clearly also identify $L^p_k(X; F)_{:0}$ with a subspace of $L^p_k(X^{(T)}; F')$. Let

$$V\colon L^p_{k+d}(X_{:0}; E) \oplus \mathbb{R}^\ell \to L^p_k(X; F)_{:0} \oplus \mathbb{R}^m$$

be a bounded operator and set

$$\begin{aligned} P = D + V\colon L^p_{k+d}(X; E) \oplus \mathbb{R}^\ell &\to L^p_k(X; F) \oplus \mathbb{R}^m, \\ (s, x) &\mapsto (Ds, 0) + V(s|_{X_{:0}}, x). \end{aligned}$$

Define the operator $P' = D' + V$ over $X^{(T)}$ similarly.

Proposition C.0.3 *If P has a bounded left (resp. right) inverse Q then for $T_{\min} > C_1\|Q\|$ the operator P' has a bounded left (resp. right) inverse Q' with $\|Q'\| < C_2\|Q\|$. Here the constants $C_1, C_2 < \infty$ depend on the restriction of D to the ends $\mathbb{R}_+ \times (\pm Y_j)$ but are otherwise independent of P.*

For left inverses this was proved in a special case in Lemma 5.4.1, and the general case is not very different. However, we would like to have the explicit expression for the right inverse on record, since this is used both in Section 10.2 and in Section 12.4.

Proof Choose smooth functions $f_1, f_2\colon \mathbb{R}\to\mathbb{R}$ such that $(f_1(t))^2+(f_2(1-t))^2=1$ for all t, and $f_k(t)=1$ for $t\le 1/3$, $k=1,2$. Define $\beta\colon X\to\mathbb{R}$ by

$$\beta=\begin{cases} f_1(t/(2T_j)) & \text{on each end } \mathbb{R}_+\times Y_j,\\ f_2(t/(2T_j)) & \text{on each end } \mathbb{R}_+\times(-Y_j),\\ 1 & \text{elsewhere,}\end{cases}$$

where t is the first coordinate on $\mathbb{R}_+\times(\pm Y_j)$. If s' is a section of F' let the section $\overline{\beta}(s')$ of F be the result of pulling s' back to $X^{\{T\}}$ by means of $\pi^{\{T\}}$, multiplying by β, and then extending trivially to all of X. (The notation $\pi^{\{T\}}$ was introduced in Section 1.4.) If $x\in\mathbb{R}^m$ set $\overline{\beta}(s',x)=(\overline{\beta}(s'),x)$. For any section s of E we define a section $\underline{\beta}(s)$ of E' as follows when $T_{\min}\ge 3/2$. Outside $[-T_j+1,T_j-1]\times Y_j$ we set $\underline{\beta}(s)=s$. Over $[-T_j,T_j]\times Y_j$ let $\underline{\beta}(s)$ be the sum of the restrictions of the product βs to $[0,2T_j]\times Y_j$ and $[0,2T_j]\times(-Y_j)$, identifying both these bands with $[-T_j,T_j]\times Y_j$ by means of the projection $\pi^{\{T\}}\colon X^{\{T\}}\to X^{(T)}$. If $x\in\mathbb{R}^\ell$ set $\underline{\beta}(s,x)=(\underline{\beta}(s),x)$. Note that

$$\underline{\beta}\,\overline{\beta}=I.$$

Now suppose Q is a left or right inverse of P. Define

$$R'=\underline{\beta}\,Q\,\overline{\beta}:L^p_k(X^{(T)};F')\oplus\mathbb{R}^m\to L^p_{k+d}(X^{(T)};E')\oplus\mathbb{R}^\ell.$$

If $QP=I$ then a simple calculation yields

$$\|R'P'-I\|\le C\,T_{\min}^{-1}\|Q\|.$$

Therefore, if $T_{\min}>C\|Q\|$ then $R'P'$ is invertible and $Q'=(R'P')^{-1}R'$ is a left inverse of P'. Similarly, if $PQ=I$ then

$$\|P'R'-I\|\le C\,T_{\min}^{-1}\|Q\|,$$

hence $Q'=R'(P'R')^{-1}$ is a right inverse of P' when $T_{\min}>C\|Q\|$. In both cases the constant C depends on the restriction of D to the ends $\mathbb{R}_+\times(\pm Y_j)$ but is otherwise independent of P. As for the bound on $\|Q'\|$; see the proof of Lemma 10.2.4. □

From the proposition one easily deduces the following version of the addition formula for the index, which was proved for first order operators in [14].

Corollary C.0.1 *If*

$$D\colon L^p_{k+d}(X;E) \to L^p_k(X;F)$$

is Fredholm, then for sufficiently large $T_{\min}$,

$$D'\colon L^p_{k+d}(X^{(T)};E') \to L^p_k(X^{(T)};F')$$

is Fredholm with $\operatorname{ind}(D') = \operatorname{ind}(D)$. □

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