Realizing families of Landweber exact homology theories

PAUL G GOERSS

I discuss the problem of realizing families of complex orientable homology theories as families of E_∞–ring spectra, including a recent result of Jacob Lurie emphasizing the role of p–divisible groups.

55N22; 55N34, 14H10

A few years ago, I wrote a paper [10] discussing a realization problem for families of Landweber exact spectra. Since Jacob Lurie [27] now has a major positive result in this direction, it seems worthwhile to revisit these ideas.

In brief, the realization problem can be stated as follows. Suppose we are given a flat morphism

$$g\colon \mathrm{Spec}(R) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

from an affine scheme to the moduli stack of smooth 1-dimensional formal groups. Then we get a 2–periodic homology theory $E(R,G)$ with $E(R,G)_0 \cong R$ and associated formal group

$$G = \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)$$

isomorphic to the formal group classified by g. The higher homotopy groups of $E(R,G)$ are zero in odd degrees and

$$E(R,G)_{2n} \cong \omega_G^{\otimes n}$$

where ω_G is the module of invariant differentials for G. The module ω_G is locally free of rank 1 over R, and free of rank 1 if G has a coordinate. In this case $E(R,G)_* = R[u^{\pm 1}]$ where $u \in E(R,G)_2$ is a generator. The fact that g was flat implies $E(R,G)$ is Landweber exact, even if G doesn't have a coordinate.

Now suppose we are given a flat morphism of stacks

$$g\colon X \longrightarrow \mathcal{M}_{\mathbf{fg}}.$$

Then for each affine open $\mathrm{Spec}(R) \to X$, we get a spectrum $E(R,G)$ and, because there are no phantom maps between these spectra (see Hovey and Strickland [18]), we get a presheaf $\mathcal{O}_X^\top$ on X in the homotopy category with

$$\mathcal{O}_X^\top(\mathrm{Spec}(R) \to X) = E(R,G).$$

Published: 16 June 2009 DOI: 10.2140/gtm.2009.16.49

A naïve version of the realization problem is this: can this presheaf $\mathcal{E}_X$ be lifted to a presheaf (or sheaf) $\mathcal{O}_X^{\mathbf{top}}$ of E_∞–ring spectra? If so, how unique is this lift?

I call this naïve because, at the very least, we want some hypotheses on X and the morphism $g\colon X \longrightarrow \mathcal{M}_{\mathbf{fg}}$. For example, we might want to specify that X be an algebraic stack, or a Deligne–Mumford stack, and we might want to specify that the morphism g be representable. Even having done this, I don't suppose anyone expects a positive answer in this generality; there are simply too many flat maps to $\mathcal{M}_{\mathbf{fg}}$. Indeed, Lurie's result in Theorem 4.9 below requires that g factor as

$$X \xrightarrow{f} \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

where $\mathcal{M}_p(n)$ is a moduli stack of p–divisible groups and f is appropriately étale. As a consequence, we don't just have a family of formal groups over X, but a very special family of p–divisible groups: a much more rigid requirement. See Remark 4.10 for more on this point.

Nonetheless, the original problem has its allure and its motivation in stable homotopy theory, and it's worth remembering this. One flat map to $\mathcal{M}_{\mathbf{fg}}$ is the identity map $\mathcal{M}_{\mathbf{fg}} \to \mathcal{M}_{\mathbf{fg}}$ itself, and we could ask whether the realization problem can be solved for all of $\mathcal{M}_{\mathbf{fg}}$. Put aside, for the moment, the fact that $\mathcal{M}_{\mathbf{fg}}$ is not an algebraic stack, let alone a Deligne–Mumford stack.

If the realization problem could be solved, we would have an equivalence

$$S^0 \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{M}_{\mathbf{fg}}} \mathcal{O}_{\mathbf{fg}}^{\mathbf{top}}$$

where S^0 is the stable sphere and the homotopy limit is over the category of flat morphisms $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$. We'd also get a descent spectral sequence

$$H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s} S^0.$$

By considering the Čech complex of the cover $\mathrm{Spec}(L) \to \mathcal{M}_{\mathbf{fg}}$, where L is the Lazard ring, we have

$$\mathrm{Ext}^s_{MU_* MU}(\Sigma^{2t} MU_*, MU_*) \cong H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t})$$

and we would have a derived algebraic geometry version of the Adams–Novikov spectral sequence.

Alas, $\mathcal{O}_{\mathbf{fg}}^{\mathbf{top}}$ probably doesn't exist: we would have to use the *fpqc*–topology on $\mathcal{M}_{\mathbf{fg}}$ and, as I mentioned above, there are too many flat maps. Nonetheless, stable homotopy theory often behaves as if $\mathcal{O}_{\mathbf{fg}}^{\mathbf{top}}$ exists. For example, the Hopkins–Ravenel chromatic convergence theorem [33] and various chromatic fracture squares are predicted exactly by the geometry of $\mathcal{M}_{\mathbf{fg}}$. A useful comparison chart is in Hopkins and Gross [16, Section 2] and much of the algebra is expanded in the later sections of Goerss [9]

There are four sections below. The first two sections are background on formal groups and $\mathcal{M}_{\mathbf{fg}}$. The third section makes a precise statement of the realization problem and discusses the Hopkins–Miller theorem. This theorem states that the realization problem has a positive answer for the moduli stack of generalized elliptic curves. The final section discusses p–divisible groups and Lurie's result.

This paper is a fleshed out version of a talk given at the conference *New Topological Contexts for Galois Theory and Algebraic Geometry* at the Banff International Research Station in March of 2008. The conference was organized by Andrew Baker and Birgit Richter. I would like to thank the referee and the editors for careful proofreading of this paper; any mistakes which remain are mine.

1 Cohomology theories and formal groups

Let's begin with a discussion of how formal groups and their invariant differentials arise in stable homotopy theory. The following is a slight generalization of the usual notion of a 2-periodic cohomology theory.

1.1 Definition Let $E^*(-)$ be a cohomology theory. Then E^* is *2–periodic* if

(1) the functor $X \mapsto E^*(X)$ is a functor to graded commutative rings;

(2) for all integers k, $E^{2k+1} = E^{2k+1}(*) = 0$;

(3) E^2 is a projective module of rank 1 over E^0; and

(4) for all integers k, the cup product map $(E^2)^{\otimes k} \to E^{2k}$ is an isomorphism.

Note that E^2 is an invertible module over E^0 and E^{-2} is the dual module. If E^2 is actually free, then so is $E_2 = E^{-2}$ and a choice of generator $u \in E_2$ defines an isomorphism $E_0[u^{\pm 1}] \cong E_*$. (The shift from E^2 to E_2 will be explained in a moment.) This happens in many important examples — complex K–theory is primordial. However, there are elliptic cohomology theories for which E_2 does not have a global generator, so we insist on this generality.

From such cohomology theories we automatically get a formal group. Recall that if R is a ring and $I \subseteq R$ is an ideal, then the *formal spectrum*

$$\mathrm{Spf}(R, I) = \mathrm{Spf}(R)$$

is the functor which assigns to each commutative ring A the set of homomorphisms

$$f\colon R \longrightarrow A$$

so that $f(I)$ is nilpotent. We have an isomorphism of functors

$$\operatorname{colim} \operatorname{Spec}(R/I^n) = \operatorname{Spf}(R).$$

In many cases, I is understood and dropped from the notation. Also, R and the I–adic completion of R have the same formal spectrum, so we usually assume R is complete. As a simple example, for $\operatorname{Spf}(\mathbb{Z}[\![x]\!])$ we have $I = (x)$ and this functor assigns to A the nilpotent elements of A.

If E^* is a 2–periodic homology theory, then $E^0\mathbb{C}\mathrm{P}^\infty$ is complete with respect to the augmentation ideal

$$I(e) \overset{\text{def}}{=} \widetilde{E}^0\mathbb{C}\mathrm{P}^\infty = \operatorname{Ker}\{\, E^0\mathbb{C}\mathrm{P}^\infty \to E^0(*)\,\}$$

and, using the H–space structure on $\mathbb{C}\mathrm{P}^\infty$, we get a commutative group object in formal schemes

$$G_E = \operatorname{Spf}(E^0\mathbb{C}\mathrm{P}^\infty).$$

This formal group is smooth and one-dimensional in the following sense. Define the E_0–module ω_G by

$$\omega_G = I(e)/I(e)^2 \cong \widetilde{E}^0 S^2 \cong E_2.$$

This module is locally free of rank 1, hence projective, and any choice of splitting of $I(e) \to \omega_G$ defines an homomorphism out of the symmetric algebra

$$S_{E_0}(\omega_G) \longrightarrow E^0\mathbb{C}\mathrm{P}^\infty.$$

which becomes an isomorphism after completion. For example, if E_2 is actually free we get a noncanonical isomorphism

$$E^0\mathbb{C}\mathrm{P}^\infty \cong E^0[\![x]\!].$$

Such an x is called a *coordinate*.

There is also a natural correspondence between morphisms of cohomology theories and morphisms of formal groups.

Let $\psi\colon D^*(X) \to E^*(X)$ be a natural ring operation between two 2-periodic cohomology theories. By evaluating at $X = *$ we obtain a ring homomorphism $f\colon D^0 \to E^0$ and by evaluating at $\mathbb{C}\mathrm{P}^\infty$ we obtain a homomorphism of formal groups

$$\phi\colon G_E \longrightarrow f^*G_D.$$

The following result can be found in Kashiwabara [20] and Butowiez and Turner [4]. The notion of Landweber exactness is taken up below.

1.2 Proposition *Let D^* and G^* be two Landweber exact 2-periodic cohomology theories. Then the assignment*

$$\psi \mapsto (f, \phi)$$

induces a one-to-one correspondence between ring operations

$$\psi\colon D^*(-) \to E^*(-)$$

and homomorphisms of pairs

$$(f, \phi)\colon (D^0, G_D) \to (E^0, G_E).$$

Furthermore, ψ is a stable operation if and only if ϕ is an isomorphism.

1.3 Remark **(Formal group laws)** The standard literature on chromatic homotopy theory, such as Adams [1] and Ravenel [32], emphasizes formal group laws. If $E^*(-)$ is a two-periodic theory with a coordinate, then the group multiplication

$$G_E \times G_E = \mathrm{Spf}(E^0(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)) \to \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty) = G_E$$

determines and is determined by a power series

$$x +_F y = F(x, y) \in E^0[\![x, y]\!] \cong \mathrm{Spf}(E^0(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty).$$

This power series is a 1–dimensional formal group law. With 2–periodic theories, we can insist that the formal group law be in degree zero. For complex oriented cohomology theories in general, the gradings become important.

Homomorphisms can also be described by power series. If G_1 and G_2 are two formal groups with coordinates over a base ring R, then a homomorphism of formal groups $\phi\colon G_1 \to G_2$ is determined by a power series $\phi(x) \in R[\![x]\!]$ so that

$$\phi(x +_{F_1} y) = \phi(x) +_{F_2} \phi(y)$$

where F_1 and F_2 are the associated formal groups. The homomorphism ϕ is an isomorphism if $\phi'(0)$ is a unit.

1.4 Remark (Invariant differentials) We have defined the module ω_{G_E} as the conormal module of the embedding

$$e\colon \mathrm{Spec}\,(E^0) \to \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty) = G_E$$

defined by the basepoint. This definition extends to any formal group over a base ring R. At first glance, this doesn't look very invariant or very differential. We address these points.

First, ω_G has the following invariance property. If $\phi\colon G_1 \to G_2$ is a homomorphism of formal groups over a ring R, then we get an induced map

$$d\phi\colon \omega_{G_2} \to \omega_{G_1}$$

described locally as follows. If a formal group has a coordinate x, then $I(e) \subseteq R[\![x]\!]$ is the ideal of power series with $f(0) = 0$ and any element of ω_G can be written

$$f(x) + I(e)^2 = f'(0)x + I(e)^2.$$

Then, writing $\phi\colon G_1 \to G_2$ as a power series we have

$$d\phi(f(x) + I(e)^2) = f(\phi(x)) + I(e)^2 = \phi'(0)f'(0) + I(e)^2. \tag{1-1}$$

Thus $d\phi$ is multiplication by $\phi'(0)$.

Second, while the last formula looks slightly differential, but we can do better: ω_G is naturally isomorphic to the module of invariant differentials on G. This can be defined as follows. Let Ω_G denote the module of continuous differentials on G; for example, if G has a coordinate x, then there is an isomorphism

$$\Omega_G \cong R[\![x]\!]dx.$$

There are then three maps

$$dp_1, dm, dp_2\colon \Omega_G \to \Omega_{G\times G}$$

induced by the two projections and multiplication. A differential η is invariant if

$$dm(\eta) = dp_1(\eta) + dp_2(\eta).$$

Invariant differentials form an E^0–module; call this module $\overline{\omega}_{G_E}$ for the moment.

If G has a coordinate x, then $\overline{\omega}_G$ is the free R–module generated by the *canonical* invariant differential

$$\eta_G = \frac{dx}{F_y(x,0)}$$

where $F_y(x,y)$ is the partial derivative of the associated formal group law. It is an exercise to calculate that if $\phi\colon G_1 \to G_2$ is a homomorphism of formal groups with coordinate, then $d\phi\colon \overline{\omega}_{G_2} \to \overline{\omega}_{G_1}$ is determined by

$$d\phi(\eta_{G_2}) = \phi'(0)\eta_{G_1}. \tag{1-2}$$

Finally, when G has a coordinate, then we get an isomorphism

$$\begin{aligned} \omega_G = I(e)/I(e)^2 &\to \bar{\omega}_G \\ f(x) + I(e)^2 &\mapsto f'(0)\eta_x. \end{aligned}$$

This isomorphisms is natural in homomorphisms (by Equations (1-1) and (1-2)). In particular, it doesn't depend on the choice of coordinate and thus extends to the more general case as well. Because of this we drop the notation $\bar{\omega}_G$

2 The moduli stack of formal groups

Let $\mathcal{M}_{\mathbf{fg}}$ be the moduli stack of formal groups: this is the algebraic geometric object which classifies all smooth 1-dimensional formal groups and their isomorphisms. Thus, if R is a commutative ring, the morphisms

$$G\colon \mathrm{Spec}\,(R) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

are in one-to-one correspondence with formal groups G over R. Furthermore, the 2-commutative diagrams

$$\begin{tikzcd} \mathrm{Spec}\,(S) \arrow[d, "f"'] \arrow[dr, "H"] & \\ \mathrm{Spec}\,(R) \arrow[r, "G"'] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

correspond to pairs $(f\colon R \to S,\ \phi\colon H \xrightarrow{\cong} f^*G\ )$.

2.1 Remark Recall that schemes can be defined in at least two equivalent ways. First, schemes are defined as locally ringed spaces $(X, \mathcal{O}_X)$ which have an open cover, as locally ringed spaces, by affine schemes. This is the point of view of Grothendieck [12]. Equivalently, schemes can be defined as functors from rings to sets which are sheaves in the Zariski topology and have an open cover, as functors, by functors of the form

$$A \mapsto \mathbf{Rings}(R, A).$$

If X is a scheme, in the first sense, we get a scheme in the second sense by defining $X(R)$ to the set of all morphisms of schemes

$$\mathrm{Spec}\,(R) \to X.$$

This is the point of view of Demazure and Gabriel [8]. It is the second definition that generalizes well. A stack is then a sheaf of groupoids on commutative rings satisfying an effective descent condition (see Laumon [24, Section 3]). For example, $\mathcal{M}_{\mathbf{fg}}$ assigns to each ring R the groupoid of smooth one-dimensional formal groups over $\mathrm{Spec}(R)$.[1]

2.2 Remark A scheme is more than a sheaf, of course, in that it must have an open cover by affine schemes. Similarly, we have *algebraic stacks*, which have a suitable cover by schemes. Here is a short explanation.

A morphism $\mathcal{M} \to \mathcal{N}$ of stacks is *representable* if for all morphisms $X \to \mathcal{N}$ with X a scheme, the 2–category pull-back (or homotopy pull-back)

$$X \times_{\mathcal{N}} \mathcal{M}$$

is equivalent to a scheme. A representable morphism then has algebraic property P (flat, smooth, surjective, étale, etc.) if all the resulting morphisms

$$X \times_{\mathcal{N}} \mathcal{M} \to X$$

have that property.

A stack $\mathcal{M}$ is then called algebraic[2] if

(1) every morphism $Y \to \mathcal{M}$ with Y a scheme is representable; and
(2) there is a smooth surjective map $q\colon X \to \mathcal{M}$ with X a scheme.

The morphism q is called a presentation. Note that an algebraic stack may have many presentations; indeed, flexibility in the choice of presentations leads to interesting theorems. See Neumann [31], for an example of this phenomenon. If the presentation can be chosen to be étale, we have a *Deligne–Mumford stack*.

2.3 Remark The stack $\mathcal{M}_{\mathbf{fg}}$ is not algebraic, in this sense, as it only has a flat presentation, not a smooth presentation. If we define **fgl** to be the functor which assigns to each ring R the set of formal group laws over R, then Lazard's theorem [25] says that

$$\mathbf{fgl} = \mathrm{Spec}(L)$$

where L is (noncanonically) isomorphic to $\mathbb{Z}[t_1, t_2, \ldots]$. The map

$$\mathbf{fgl} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

[1] As in [24, Section 2], we should really speak of categories fibered in groupoids, rather than sheaves of groupoids — for f^*g^*G is only isomorphic to $(gf)^*G$. However, there are standard ways to pass between the two notions, so I will ignore the difference.

[2] The notion defined here is stronger than what is usually called an algebraic (or Artin) stack, which requires a cover only by an algebraic space. Algebraic spaces are sheaves which themselves have an appropriate cover by a scheme. Details are in [24].

which assigns a formal group law to its underlying formal group is flat and surjective, but not smooth since it's not finitely presented. This difficulty can be surmounted in two ways: enlarge the notion of an algebraic stack to include flat presentations or note that $\mathcal{M}_{\mathbf{fg}}$ can be written as the 2-category inverse limit of a tower of the algebraic stacks of "buds" of formal groups and is, thus, proalgebraic; see Smith [37].

2.4 Remark A sheaf in the *fpqc*–topology on $\mathcal{M}_{\mathbf{fg}}$ is a functor $\mathcal{F}$ on the category of affine schemes over $\mathcal{M}_{\mathbf{fg}}$ which satisfies faithfully flat descent. Thus, for each formal group G over R we get a set (or ring, or module, etc.)

$$\mathcal{F}(R,G) = \mathcal{F}(G\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}})$$

and for each 2–commuting diagram

$$\begin{tikzcd} \mathrm{Spec}(S) \arrow[d, "f"'] \arrow[dr, "H"] & \\ \mathrm{Spec}(R) \arrow[r, "G"'] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

a restriction map $\mathcal{F}(S,H) \to \mathcal{F}(R,G)$. This must be a sheaf in the sense that if $q\colon S \to R$ is faithfully flat, then there is an equalizer diagram

$$\mathcal{F}(R,G) \longrightarrow \mathcal{F}(S,q^*G) \rightrightarrows \mathcal{F}(S \otimes_R S, p^*G)$$

where I have written p for the inclusion $R \to S \otimes_R S$.

For example, define the structure sheaf $\mathcal{O}_{\mathbf{fg}}$ to be the functor on affine schemes over $\mathcal{M}_{\mathbf{fg}}$ with

$$\mathcal{O}_{\mathbf{fg}}(R,G) = \mathcal{O}_{\mathbf{fg}}(G\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}) = R.$$

More generally, we consider module sheaves $\mathcal{F}$ over $\mathcal{O}_{\mathbf{fg}}$. Such a sheaf is called *quasicoherent* if, for each 2–commutative diagram, the restriction map $\mathcal{F}(R,G) \to \mathcal{F}(S,H)$ extends to an isomorphism

$$S \otimes_R \mathcal{F}(R,G) \cong \mathcal{F}(S,H).$$

This isomorphism can be very nontrivial, as it depends on the choice of isomorphism $\phi\colon H \to f^*G$ which makes the diagram 2–commute.

A fundamental example of a quasicoherent sheaf is the sheaf of invariant differentials ω with

$$\omega(R,G) = \omega_G$$

the invariant differentials on G. This is locally free of rank 1 and hence all powers $\omega^{\otimes n}$, $n \in \mathbb{Z}$, are also quasicoherent sheaves. The effect of the choice of isomorphism in the 2–commuting diagram on the transition maps for $\omega^{\otimes n}$ is displayed in Equations (1-1) and (1-2).

2.5 Remark Consider the 2–category pull-back

$$\mathbf{fgl} \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl}$$

where $\mathbf{fgl} = \mathrm{Spec}\,(L)$ is the functor of formal group laws. The pull-back is the functor which assigns to each commutative ring the set of triples (F_1, F_2, ϕ) where F_1 and F_2 are formal group laws and ϕ is an isomorphism of their underlying formal groups. Given that these formal groups have a chosen coordinate, the isomorphism ϕ can be expressed as an invertible power series $\phi(x) = a_0 x + a_1 x^2 + \cdots$. Thus the pull-back is the affine scheme on the ring

$$W = L[a_0^{\pm 1}, a_1, \ldots].$$

The pair (L, W) forms a *Hopf algebroid*; that is, a groupoid in affine schemes. Furthermore, the category of quasicoherent sheaves on $\mathcal{M}_{\mathbf{fg}}$ is equivalent to the category of (L, W)–comodules.

To get a functor in one direction, let $\mathcal{F}$ be a quasicoherent sheaf. Then

$$M = \mathcal{F}(\mathbf{fgl} \to \mathcal{M}_{\mathbf{fg}})$$

is an L–module. One of the two projections $\mathbf{fgl} \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \to \mathbf{fgl}$ shows

$$\mathcal{F}(\mathbf{fgl} \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \to \mathcal{M}_{\mathbf{fg}}) \cong W \otimes_L M$$

and the other projection supplies the comodule structure map. I will say something about how you pass from comodules to sheaves at the beginning of the next section.

It is here we see the flexibility of choosing the presentation. For example, if we work localized at some prime p and consider the stack

$$\mathrm{Spec}\,(\mathbb{Z}_{(p)}) \times \mathcal{M}_{\mathbf{fg}} \stackrel{\mathrm{def}}{=} \mathbb{Z}_{(p)} \otimes \mathcal{M}_{\mathbf{fg}}$$

then we could use the scheme of p–typical formal group laws as our cover and obtain a different category of comodules closely related (up to issues of grading) to (BP_*, BP_*BP)–comodules. This would then be equivalent to the category of $\mathbb{Z}_{(p)} \otimes (L, W)$–comodules as both would be equivalent to quasicoherent sheaves on $\mathbb{Z}_{(p)} \otimes \mathcal{M}_{\mathbf{fg}}$.

2.6 Remark (The height filtration) If G is a formal group over a field $\mathbb{F}$ we can always choose a coordinate. If $\mathbb{F}$ is of characteristic p for some prime p, then the homomorphism $p\colon G \to G$ can be written

$$p(x) = ux^{p^n} + \cdots$$

for some $u \neq 0$ and $1 \leq n \leq \infty$. (If $n = \infty$, $p = 0\colon G \to G$.) The number n is an isomorphism invariant and, if $\mathbb{F}$ is separably closed, a complete invariant, by the theorem of Lazard [25]. This notion of *height* can be extended to formal groups over an arbitrary $\mathbb{F}_p$–algebra or even to formal groups over schemes over $\mathbb{F}_p$, but some care is needed if G does not have a coordinate.

Consider a formal group G over an $\mathbb{F}_p$–algebra R. If we let $f\colon R \to R$ be the Frobenius, we get a new formal group $G^{(p)} = f^*G$. We then have a diagram

$$\begin{tikzcd} G \arrow[r, "F"'] \arrow[rr, bend left, "f"] \arrow[dr] & G^{(p)} \arrow[r] \arrow[d] & G \arrow[d] \\ & \mathrm{Spec}(R) \arrow[r, "f"'] & \mathrm{Spec}(R) \end{tikzcd}$$

where the square is a pull-back. The homomorphism F is the *relative Frobenius*. We know that if

$$\phi\colon G \to H$$

is a homomorphism of formal groups over R for which $d\phi = 0\colon \omega_H \to \omega_G$, there is then a factoring

$$\begin{tikzcd} G \arrow[r, "F"'] \arrow[rr, bend left, "\phi"] & G^{(p)} \arrow[r, "\psi"'] & H \end{tikzcd}$$

Then we can test $d\psi$ to see if we can factor further.[3]

For example, let $\phi = p\colon G \to G$ be pth power map. Then we obtain a factoring

$$\begin{tikzcd} G \arrow[r, "F"'] \arrow[rr, bend left, "p"] & G^{(p)} \arrow[r, "V_1"'] & G \end{tikzcd}$$

This yields an element

$$dV_1 \in \mathrm{Hom}(\omega_G, \omega_{G^{(p)}})$$

[3] See [9, Section 5] for a proof of these facts in this language.

and we can factor further if $dV_1 = 0$. Since G is of dimension 1, $\omega_{G^{(p)}} \cong \omega_G^{\otimes p}$; since ω_G is invertible,

$$\mathrm{Hom}(\omega_G, \omega_G^{\otimes p}) = \mathrm{Hom}(R, \omega_G^{\otimes p-1}).$$

Thus dV defines an element[4] $v_1(G)$ of $\omega_G^{\otimes p-1}$. If $v_1(G) = 0$, then we obtain a further factorization

$$\begin{tikzcd} G \arrow[r, "F"'] \arrow[rrr, bend left, "p"] & G^{(p)} \arrow[r, "F^{(p)}"'] & G^{(p^2)} \arrow[r, "V_2"'] & G \end{tikzcd}$$

and an element $v_2(G) \in \omega_G^{\otimes p^2-1}$. This can be continued to define elements $v_n(G) \in \omega_G^{\otimes p^n-1}$ and G has height at least n if $v_1(G) = \cdots = v_{n-1}(G) = 0$. We say G has height *exactly* n if $v_n(G) \in \omega_G^{\otimes p^n-1}$ is a generator.

The elements $v_n(G)$ defined in this way are isomorphism invariants. For example if G is a formal group over an $\mathbb{F}_p$ algebra R, then

$$p(x) = u_1 x^p + \cdots.$$

The element u_1 is not an isomorphism invariant, but if $\eta = dx/F_y(x,0)$ is the standard invariant differential, then

$$v_1(G) = u_1 \eta^{\otimes p-1} \in \omega_G^{\otimes p-1}$$

is an invariant.

Because of this invariance property, the assignment $G \mapsto v_1(G)$ defines a global section v_1 of the sheaf $\omega^{\otimes p-1}$ on the closed substack

$$\mathbb{F}_p \otimes \mathcal{M}_{\mathbf{fg}} \stackrel{\text{def}}{=} \mathcal{M}(1) \subseteq \mathcal{M}_{\mathbf{fg}}$$

In this way we obtain a sequence of closed substacks

$$\cdots \subseteq \mathcal{M}(n+1) \subseteq \mathcal{M}(n) \subseteq \cdots \subseteq \mathcal{M}(1) \subseteq \mathcal{M}_{\mathbf{fg}}$$

where $\mathcal{M}(n+1) \subseteq \mathcal{M}(n)$ is defined by the vanishing of the global section v_n of $\omega^{\otimes p^n-1}$. Thus $\mathcal{M}(n)$ classifies formal groups of height at least n. The relative open

$$\mathcal{H}(n) = \mathcal{M}(n) - \mathcal{M}(n+1)$$

classifies formal groups of height exactly n. Lazard's theorem, rephrased, says that $\mathcal{H}(n)$ has a single geometric point given by a formal group G of height n over any algebraically closed field $\mathbb{F}$. The pair $(\mathbb{F}, G)$ has plenty of isomorphisms, however, so $\mathcal{H}(n)$ is not a scheme; indeed, in the language of [24] it is a *neutral gerbe*; see [37].

[4] Over a base scheme which was not affine, $v_1(G)$ is a global section.

One way to think of neutral gerbes is as the *classifying stack* which assigns to any commutative ring R the groupoid of torsors for some group scheme. In this case, we can take the following group scheme. Let Γ_n be some fixed formal group defined over the finite field $\mathbb{F}_p$ and let $\mathrm{Aut}(\Gamma_n)$ be the group scheme which assigns to each $\mathbb{F}_p$ algebra R the group of automorphisms of $i^*\Gamma_n$ over R. Here I write $i\colon \mathbb{F}_p \to R$ for the inclusion. Then $\mathcal{H}(n)$ is the classifying stack for $\mathrm{Aut}(\Gamma_n)$.

This group scheme is quite familiar to homotopy theorists. To be specific, let Γ_n be the Honda formal group over $\mathbb{F}_p$; this is the p–typical formal group law with $p(x) = x^{p^n}$. Then $\mathrm{Aut}(\Gamma_n)$ is the affine group scheme obtained from the Morava stabilizer algebra (See [32, Section 6.2]) and the group of $\mathbb{F}_{p^n}$–points of $\mathrm{Aut}(\mathbb{F}_p, \Gamma_n)$ is the Morava stabilizer group S_n. By definition, S_n is the automorphisms of Γ_n over $\mathbb{F}_{p^n}$.

2.7 Remark (Landweber's criterion for flatness) We will be concerned with morphisms $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ of stacks which are representable and flat. We defined this notion above, but the Landweber exact functor theorem uses the height filtration to give an easily checked criterion for flatness. This we now state, first giving a global formulation, then giving a way to check this locally.

We begin by noting that the closed inclusion $j\colon \mathcal{M}(n+1) \to \mathcal{M}(n)$ is actually an effective Cartier divisor. This means the following. Let $\mathcal{O}(n)$ be the structure sheaf of $\mathcal{M}(n)$. Then the global section $v_n \in H^0(\mathcal{M}(n), \omega^{\otimes p^n - 1})$ defines an *injection* of sheaves

$$0 \to \mathcal{O}(n) \xrightarrow{v_n} \omega^{\otimes p^n - 1} .$$

This yields a short exact sequence

$$0 \to \omega^{\otimes -(p^n - 1)} \xrightarrow{v_n} \mathcal{O}(n) \longrightarrow j_*\mathcal{O}(n+1) \to 0.$$

This identifies $\omega^{\otimes -(p^n-1)}$ with the ideal defining the closed inclusion $\mathcal{M}(n+1)$ in $\mathcal{M}(n)$.

Now let $f\colon \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable morphism of stacks and let

$$\mathcal{N}(n) = \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N} \subseteq \mathcal{N}.$$

Then $\mathcal{N}(n+1) \subseteq \mathcal{N}(n)$ remains a closed inclusion and if f is flat, it remains an effective Cartier divisor; that is,

$$0 \to \mathcal{O}_{\mathcal{N}(n)} \xrightarrow{v_n} \omega^{\otimes p^n - 1}$$

remains an injection. Landweber's theorem [23] now says that this is sufficient.[5] That is, suppose that for all primes p and all integers n, the morphism

$$v_n\colon \mathcal{O}_{\mathcal{N}(n)} \to \omega^{\otimes p^n - 1}$$

is an injection. Then $f\colon \mathcal{N} \to \mathcal{M}$ is flat. For proofs in the language of stacks see Miller [30], Neumann [31], and Hollander [14]. The first of these (which has an extra hypothesis) was inspired directly by Mike Hopkins, the second had input from Mark Behrens.

Locally this can be unwound as follows. Let $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ be a formal group with a coordinate x. Define $u_0 = p$ and recursively define elements u_n by

$$p(x) = u_n x^{p^n} + \cdots$$

modulo $(p, u_1, \ldots, u_{n-1})$. Then we can rewrite the equations above as saying that, for all primes p and all n, the multiplication

$$u_n\colon R/(p, u_1, \ldots, u_{n-1}) \to R/(p, u_1, \ldots, u_{n-1})$$

is an injection.

Much of the proof of Landweber's result is formal, using only that the closed inclusions $\mathcal{M}(n+1) \subseteq \mathcal{M}(n)$ are effective Cartier divisors. But in the end, one must use something about formal groups, and Neil Strickland has pointed out crucial ingredient turns out to be Lazard's uniqueness theorem in the following strengthened form.

2.8 Proposition *Let G_1 and G_2 be two formal groups of the same height over an $\mathbb{F}_p$ algebra R. Then there is a sequence of étale extensions*

$$R \subseteq R_1 \subseteq R_2 \subseteq \cdots$$

so that G_1 and G_2 become isomorphic over $R_\infty = \mathrm{colim}\, R_n$.

This is what is actually proved by Lazard in [25]. See also [32, Appendix 2.2], and [9, Theorem 5.25] for this result over arbitrary base schemes. There are additional references in [9]. If $R = \mathbb{F}$ is field, the extension adjoins roots of certain separable polynomials; hence if $\mathbb{F}$ is separably closed, G_1 and G_2 were already isomorphic.

[5] For proofs in the language of stacks, see Miller [30], Goerss [9], Neumann [31], and Hollander [14]. The first two of these were inspired directly by Mike Hopkins, the third had input from Mark Behrens.

3 The realization problem

The Landweber Exact Functor Theorem was originally proved to provide homology theories. This begins with periodic complex cobordism MUP_*, which is obtained from ordinary complex cobordism by adjoining an invertible element of degree 2:

$$MUP_* X = \mathbb{Z}[u^{\pm 1}] \otimes_{\mathbb{Z}} MU_* X$$

The representing spectrum is $\vee_n \Sigma^{2n} MU$, the Thom spectrum of the universal bundle over $\mathbb{Z} \times BU$. Notice that the wedge summands keep track of the virtual dimension over the individual components. We have

$$MUP_0 = L \qquad \text{and} \qquad MUP_0 MUP = W.$$

Now suppose we are given a ring R and a formal group G with a coordinate over R. The choice of coordinate defines a map of rings $L \to R$ and we can examine the functor

$$X \mapsto R \otimes_L MUP_* X.$$

Landweber's criterion guarantees that this functor yields a homology theory $E(R,G)_*$. A theorem of Hovey and Strickland [18] says that there are no phantom maps between such theories and this implies that $E(R,G)$ is actually a homotopy commutative ring spectrum. To get the multiplication map, for example, note that

$$E(R,G)_0 E(R,G) \cong R \otimes_L W \otimes_L R$$

still satisfies Landweber's criterion; hence the morphism

$$R \otimes_L W \otimes_L R \to R$$

which classifies the identity from G to itself defines the multiplication. This is as natural as can be: we get a functor from formal groups with coordinate to the stable homotopy category.

I'd next like to eliminate the reliance on the coordinate. A formal group need not have a coordinate and, even if it does, I'd rather not choose one.

3.1 Remark (From comodules to sheaves) Suppose we are given an (L,W) comodule M; we'd like to produce a quasicoherent sheaf $\mathcal{F}_M$ on $\mathcal{M}_{\mathbf{fg}}$. Let

$$G\colon \operatorname{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$$

be flat. Consider the diagram with all squares pull-backs:

$$\begin{tikzcd} X_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_0 \arrow[r, "f"] \arrow[d] & \mathrm{Spec}(R) \arrow[d, "G"] \\ \mathbf{fgl} \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \arrow[r, shift left] \arrow[r, shift right] & \mathbf{fgl} \arrow[r, "q"'] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

Here I have written $\mathbf{fgl}$ for $\mathrm{Spec}(L)$ and $\mathbf{fgl} \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl}$ for $\mathrm{Spec}(W)$. The scheme X_0 represents the functor which assigns to each commutative R–algebra A the set of coordinates of G over A. If I could choose a coordinate for G, we'd get a noncanonical isomorphism

$$X_0 \cong \mathrm{Spec}(W \otimes_L R).$$

Since such a choice is always possible locally, we conclude f is an affine morphism of schemes. Since q is faithfully flat, so is f and to specify $\mathcal{F}_M(R,G)$ I need only specify a quasicoherent sheaf on X_0 together with descent data. This sheaf is the pull-back of the sheaf on $\mathbf{fgl}$ determined by M; the descent data is determined by the comodule structure and the commutative diagram.

This elaborate description is a choice-free way of naming $\mathcal{F}(R,G)$. If we can choose a coordinate for G, then we get an isomorphism

$$\mathcal{F}_M(R,G) \cong R \otimes_L M.$$

Now suppose $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat and classifies the formal group G. Define a homology theory by

$$E(R,G)_*X = \mathcal{F}_{MUP_*X}(R,G)$$

where $\mathcal{F}_M$ is the sheaf associated to the comodule M. By Hovey and Strickland's result, quoted above, this is a homotopy commutative ring theory. Moreover we have

$$E(R,G)_0 \cong R, \quad E(R,G)_{2k+1} = 0,$$

the associated formal group is G and $E(R,G)_{2k} \cong \omega_G^{\otimes k}$. In this way, we get a presheaf

$$E(-,-)\colon \mathrm{Flat}/\mathcal{M}_{\mathbf{fg}} \longrightarrow \mathbf{Ho}(\mathrm{Spectra})$$

from the category of flat maps with affine source over $\mathcal{M}_{\mathbf{fg}}$ to the stable homotopy category realizing the graded structure sheaf $\mathcal{O}_* = \{\omega^{\otimes *}\}$. Here and throughout we assume ω is in degree 2, for topological reasons.

The realization problem asks to what extent the presheaf $E(-,-)$ can be lifted to the category of E_∞ ring spectra.

Since the geometry of $\mathcal{M}_{\mathbf{fg}}$ is not so good — it is not an algebraic stack, for example (see 2.3) — I don't suppose anyone expects an affirmative answer to this question; there are simply too many flat maps. (For further comments on this point, see the introduction.) One way to cut down the class of morphisms is to restrict attention to stacks with more structure. Over an algebraic stack, for example, we can work with the smooth-étale topology (see Laumon [24, Section 12]), and over a Deligne–Mumford stack we can work with the étale topology. Thus, I will formulate the question as follows:

3.2 The realization problem Let $\mathcal{M} \to \mathcal{M}_{\mathbf{fg}}$ be a representable and flat morphism from an algebraic stack and let

$$\mathcal{O}_{\mathcal{M}*} = \{\omega^{\otimes *}\}$$

denote the graded structure sheaf on $\mathcal{M}$ in an appropriate topology. Is there a presheaf of E_∞–ring spectra $\mathcal{O}_S^{\mathbf{top}}$ with an isomorphism of associated sheaves

$$(\mathcal{O}_S^{\mathbf{top}})_* \cong \mathcal{O}_{\mathcal{M}*}?$$

If so, how unique is this? What is the homotopy type of the space of all realizations?

3.3 Example Even in this generality, the problem might not have a general solution. For example, we could take $\mathcal{M} = \mathrm{Spec}(R)$ and $G\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ to be any flat map and the Zariski topology on $\mathrm{Spec}(R)$. Then a positive solution to the realization problem would say that representing spectrum $E(R,G)$ of the resulting Landweber exact homology theory had the structure of an E_∞–ring spectrum. This is not very likely. More on this point below in Remark 4.10.

There is a very important example of a positive solution of the realization problem: the moduli stack of elliptic curves. Standard references on elliptic curves include [36] and [21]; the stack was introduced in [6] and thoroughly studied in [7].

3.4 Remark (The moduli stack of elliptic curves) Let S be a scheme. Then an elliptic curve over S

$$C \underset{e}{\overset{q}{\rightleftarrows}} S$$

is a proper, smooth curve over S, with geometrically connected fibers of genus 1 and with a given section e. Such curves have a natural structure as an abelian group scheme S with e as the identity section. By taking a formal neighborhood of e in C we obtain a formal group C_e. There is a stack $\mathcal{M}_{ell}$ (called $\mathcal{M}_{1,1}$ in the algebraic geometry literature or *genus 1 with 1 marked point*) classifying elliptic curves; that is, morphisms

$$C\colon S \to \mathcal{M}_{ell}$$

are in one-to-one correspondence with elliptic curves over S. This stack was produced in [6] and is one of the original examples of a stack. The assignment $C \to C_e$ produces a morphism of stacks

$$\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

which is representable and flat and we could ask about the realization problem for $\mathcal{M}_{ell}$. The sheaf ω on $\mathcal{M}_{\mathbf{fg}}$ restricts to the sheaf on $\mathcal{M}_{ell}$ which assigns to each elliptic curve C over S the sheaf ω_C on S of invariant differentials of C. The global sections

$$H^0(\mathcal{M}_{ell}, \omega^{\otimes t})$$

are the modular forms of weight t (and level one); they assemble into a graded ring. From [5] we have an isomorphism

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3\Delta) \cong H^0(\mathcal{M}_{ell}, \omega^{\otimes *}).$$

where c_4, c_6, and Δ are the standard modular forms of weight (degrees) 4, 6, and 12 respectively.

3.5 Remark (The compactification of $\mathcal{M}_{ell}$) There is a canonical compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack $\mathcal{M}_{ell}$. One way to construct this as follows.

Locally in S any elliptic curve is a nonsingular subscheme of $\mathbb{P}^2$ obtained from a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Any such curve is called a Weierstrass curve; more generally, we define a Weiertrass curve C over a scheme S to be a pointed morphism of schemes

$$\begin{tikzcd} C \arrow[r, shift left] & S \arrow[l, shift left, "e"] \end{tikzcd}$$

which can be given Zariski-locally by a Weierstrass equation. (The marked point of a Weierstrass curve is the point $e = [0, 1, 0]$). Although not every Weierstrass curve is an elliptic curve, we do get an embedding

$$\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{\mathrm{Weier}}$$

into a moduli stack of Weierstrass curves and the morphism $\mathcal{M}_{ell} \to \mathcal{M}_{\mathbf{fg}}$ factors through this embedding:

$$\mathcal{M}_{ell} \to \mathcal{M}_{\mathrm{Weier}} \to \mathcal{M}_{\mathbf{fg}}.$$

This is a consequence of the fact that, for any Weierstrass curve C, the marked point $e = [0,1,0]$ is always smooth and the smooth locus on C has a natural structure as an abelian group scheme with e as the identity.

The morphism $\mathcal{M}_{\mathrm{Weier}} \to \mathcal{M}_{\mathbf{fg}}$ is not good, however, for two reasons: first, it is not flat (see Rezk [34, Section 20]) and, second, the geometry of $\mathcal{M}_{\mathrm{Weier}}$ is not very good. For example, because the automorphism group of the cusp curve $y^2 = x^3$ is not an étale group scheme, this stack cannot be a Deligne–Mumford stack. (See [24], Théorème 8.1.) However, the sheaves $\omega^{\otimes t}$ yield sheaves on $\mathcal{M}_{\mathrm{Weier}}$ and a calculation from [34] and [2] (following Deligne [5], of course) implies that there is an isomorphism of graded rings

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = (12)^3\Delta) \cong H^0(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *}).$$

A Weierstrass curve C is an elliptic curve and, hence, smooth if $\Delta(C)$ is invertible. We define

$$\overline{\mathcal{M}}_{ell} \longrightarrow \mathcal{M}_{\mathrm{Weier}}$$

to be the substack of curves C so that the sections $c_4^3(C)$, $c_6^2(C)$, and $\Delta(C)$ generate $\omega_C^{\otimes 12}$; that is, in formulas, we have:

$$(c_4^3(C), c_6^2(C), \Delta(C)) = \omega_C^{\otimes 12}.$$

There is an inclusion $\mathcal{M}_{ell} \subseteq \overline{\mathcal{M}}_{ell}$; however, we also allow other curves — for example, we allow curves where $c_4(C)$ is invertible. In effect, we allow nodal, but not cusp, singularities. Thus

$$y^2 = x^2(x-1)$$

is allowed, but $y^2 = x^3$ is not.[6] The resulting map

$$\overline{\mathcal{M}}_{ell} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

is flat and $\overline{\mathcal{M}}_{ell}$ has good geometry.

3.6 Remark (Étale maps to $\overline{\mathcal{M}}_{ell}$) To get some feel for the geometry of $\overline{\mathcal{M}}_{ell}$, define the j–invariant

$$\begin{aligned} j\colon \overline{\mathcal{M}}_{ell} &\longrightarrow \mathbb{P}^1 \\ C &\longmapsto [c_4^3(C), \Delta(C)]. \end{aligned}$$

Then from [7, Section V1.1] we learn that j identifies $\mathbb{P}^1$ as the *coarse moduli stack* of $\overline{\mathcal{M}}_{ell}$ — the scheme which most closely approximates the sheaf of isomorphism classes

[6] One important generalized elliptic now allowed is the Tate curve over $\mathbb{Z}[\![q]\!]$ which is singular at $q = 0$; see [7, Section VII]. The morphism $\mathrm{Spf}(\mathbb{Z}[\![q]\!]) \to \overline{\mathcal{M}}_{ell}$ classifying the Tate curve identifies $\mathrm{Spf}(\mathbb{Z}[\![q]\!])$ as formal neighborhood of the singular generalized elliptic curves.

of generalized elliptic curves.(See [7, Section I.8] for precise definitions.) Furthermore, the fiber at any j–value is the classifying stack of a finite group scheme.

Note that while the geometry of the j–invariant is somewhat complicated, $\overline{\mathcal{M}}_{ell}$ is still smooth of dimension 1. (See [7].) In particular, if $\mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$ is étale, then R is smooth of dimension 1 over $\mathbb{Z}$. There are classical examples of such affine morphisms; see, for example, [19]. The one emphasized in usual sources ([36, Section III.1]) is the Legendre curve

$$y^2 = x(x-1)(x-\lambda)$$

over $\mathbb{Z}[1/2][\lambda, (\lambda^2-\lambda)^{-1}]$. There is also the Deuring curve ([36, Proposition A.1.3])

$$y^2 + 3\nu + xy = x^3$$

over $\mathbb{Z}[1/3][\nu, 1/(\nu^3+1)]$. Both the Legendre curve and the Deuring curve are smooth, because we've inverted the discriminant Δ. A wide example of nonsmooth curves can be obtained by base change from the curve

$$y^3 + xy = x^3 + \tau$$

over $\mathbb{Z}[\tau, 1/(1+2^4 3^3\tau)]$. (We invert $1+2^4 3^3\tau$ to make the singular locus of this curve exactly $\tau = 0$.) An observation, which I learned from Hopkins, is that these three curves form an affine étale cover of $\overline{\mathcal{M}}_{ell}$.

Here is the famous positive answer to the realization problem; see Hopkins [15].

3.7 Theorem (Hopkins–Miller) *The realization problem for*

$$\overline{\mathcal{M}}_{ell} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

has a solution in the étale topology: there is a presheaf $\mathcal{O}_{ell}^{\mathbf{top}}$ of E_∞–ring spectra realizing the graded structure sheaf $\mathcal{O}_{ell*}$. The space of all realizations is path connected.

If we define tmf to be the homotopy global sections

$$\mathrm{tmf} \overset{\mathrm{def}}{=} \operatorname*{holim}_{\overline{\mathcal{M}}_{ell}} \mathcal{O}_{ell}^{\mathbf{top}}$$

where the homotopy limit is over all étale morphisms $\mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$. There is a descent spectral sequence

(3-1) $$H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s}\mathrm{tmf}$$

and modular forms are, by definition,

$$H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes *}) \cong H^0(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *}) \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = (12)^3\Delta).$$

Hence *topological modular forms*. The question of which modular forms are homotopy classes is quite interesting. For example, c_6 and the discriminant Δ are not; however $2c_6$ and 24Δ are. See [15].

The calculation of $\pi_*\mathrm{tmf}$ has been made completely. While not yet explicitly in print, it can be easily deduced from [34] and [2], both of which follow [17]. There is a curious feature of the answer: while the E_2 term does not display any obvious duality, the homotopy groups of tmf have a very strong duality very similar to Serre duality for projective schemes. I know of no good explanation for this — the differentials and extensions in the spectral sequence conspire in an almost miraculous fashion to give the result — but there must be one in derived algebraic geometry. Compare also Mahowald–Rezk duality [28].

3.8 Warning Note that topological modular forms have often been defined to be the zero-connected cover of what I've called tmf. However, to even decide if this makes sense, you need to calculate $\pi_*\mathrm{tmf}$ and notice that the resulting answer takes a very special form.

3.9 Example (Topological automorphic forms) Work of Mark Behrens and Tyler Lawson [3] solve the realization problem for certain Shimura varieties, which are moduli stacks of highly structured abelian varieties. The extra structure is needed to get formal groups of higher heights. The problem is that the only abelian group schemes of dimension one are the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, and elliptic curves; from these we only get formal groups of height ∞, 1, and 2. To get formal groups of height greater than 2, one must use higher dimensional abelian group schemes A, but then one must add enough structure so that the formal completion A_e of A at the identity splits off a natural summand of dimension one. It takes a while to define such objects — so I won't do it here — but it turns out they've been heavily studied in number theory. See, for example, [22].

3.10 Remark (The role of E_∞–ring spectra) Why do I (following my betters, notably Mike Hopkins) insist on highly structured ring spectra in the realization problem? There are two reasons.

(1) (Practical) Asking for E_∞–ring spectra allows for algebraic geometry (ie, ring theoretic) input into the constructions.

(2) (Aesthetic) The stack $\mathcal{M}_{ell}$ with its E_∞ structure sheaf becomes a central exhibit in the world of derived algebraic geometry: we learn something inherently new about elliptic curves.

4 Lurie's theorem and p–divisible groups

The formal group of an elliptic curve is part of a richer and more rigid structure. At this point we pick a prime p and work over $\mathrm{Spf}(\mathbb{Z}_p)$; that is, p is implicitly nilpotent in all our rings. This has the implication that we will we working in the p–complete stable category.

4.1 Definition Let R be a ring and G a sheaf of abelian groups on R–algebras. Then G is a *p–divisible group* of height n if

(1) $p^k\colon G \to G$ is surjective for all k;

(2) $G(p^k) = \mathrm{Ker}(p^k\colon G \to G)$ is a finite and flat group scheme over R of rank p^{kn};

(3) $\operatorname{colim} G(p^k) \cong G$.

4.2 Remark (1) If G is a p–divisible group, then completion at $e \in G$ gives an abelian formal group $G_{\mathrm{for}} \subseteq G$, not necessarily of dimension 1. The quotient G/G_{for} is étale over R; thus we get a natural short exact sequence

$$0 \to G_{\mathrm{for}} \to G \to G_{\mathrm{et}} \to 0.$$

This is split over fields, but not in general.

(2) If C is a smooth elliptic curve, then $C(p^\infty) = \operatorname{colim} C(p^n)$ is p–divisible of height 2 with formal part of dimension 1.

(3) Formal groups need not be p–divisible groups as there is no reason to suppose

$$\operatorname{colim} G(p^k) \cong G$$

over a ring which is not local and complete. Nor can one assume that a formal group is a subgroup a p–divisible group.

(4) (Rigidity) If G is a p–divisible group over a scheme S, the function which assigns to each geometric point x of S the height of the fiber G_x of G at x is constant. This is not true of formal groups, as the example of elliptic curves shows. Indeed, if G is p–divisible of height n with G_{for} of dimension 1, then the height of G_{for} can be any integer between 1 and n.

For a simple example of this phenomenon, take $n > 1$ and let

$$S = \mathrm{Spec}\,(\mathbb{Z}/(p^n)[u_1])$$

and G the formal group obtained from the p–typical formal group law F with p–series

$$p_F(x) = px +_F u_1 x^p +_F x^{p^2}.$$

Then if x is the point given by the maximal ideal (p, u_1), G_x has height 2; however, if x is point given the ideal $(p, u_1 - 1)$, then G_x has height 1. This example is closely related to the Johnson–Wilson theory $E(2)_*$. It is not at all clear that this formal group has anything to do with a p–divisible group. See, however, [13], where the authors do have some success at the prime 3.

4.3 Example (p–divisible groups and localization) The following example, which I learned from Charles Rezk, shows that p–divisible groups arise naturally in homotopy theory.

Let $E = E_n$ be a Morava E–theory; this is a 2–periodic theory with a noncanonical isomorphism

$$E_0 = W(\mathbb{F}_{p^t})[\![u_1, \ldots, u_{n-1}]\!]$$

and whose formal group is a universal deformation of a height n–formal group. (See Examples 4.11 and 4.12 below for more on deformations.) Since E_0 is complete

$$G = G_E = \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)$$

is a p–divisible group of height n. Indeed,

$$\mathrm{map}(\mathbb{C}\mathrm{P}^\infty, E) \simeq \mathrm{map}(\mathrm{colim}\, BC_{p^k}, E) \simeq \lim \mathrm{map}(BC_{p^k}, E)$$

and, by applying π_0 we get

$$\mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{colim}\,\mathrm{Spec}\,(E^0 BC_{p^k}) = \mathrm{colim}\, G(p^k).$$

When we apply the localization functor $L_{n-1} = L_{E(n-1)}$, we have

$$\mathrm{map}(\mathbb{C}\mathrm{P}^\infty, L_{n-1}E) \longleftarrow L_{n-1}\,\mathrm{map}(\mathbb{C}\mathrm{P}^\infty, E) \longrightarrow L_{n-1}\,\mathrm{map}(BC_{p^k}, E)$$

yielding

$$G_{L_{n-1}E} \longrightarrow \mathrm{colim}\,\mathrm{Spec}\,(\pi_0 L_{n-1}\,\mathrm{map}(BC_{p^k}, E))$$

as the inclusion of the formal part of a p–divisible group. This map is not an isomorphism; indeed, the rank of

$$G_{L_{n-1}E}(p)$$

over $\pi_0 L_{n-1}E$ is p^{n-1} while the rank of

$$\mathrm{Spec}\,(\pi_0 L_{n-1}\,\mathrm{map}(BC_p, E))$$

is p^n. This last group scheme is the p–torsion in the p–divisible group.

Dealing with examples such as this is one of the deeper technical aspects of the original Hopkins–Miller proof of the existence of tmf.

4.4 Definition Let $\mathcal{M}_p(n)$ be the moduli stack of p–divisible groups of height n and with dim $G_{\mathrm{for}} = 1$.

4.5 Remark The stack $\mathcal{M}_p(n)$ is not an algebraic stack, but rather proalgebraic. This can be deduced from the material in the first chapter of Messing [29].

4.6 Remark There is a morphism of stacks:

$$\begin{aligned} \mathcal{M}_p(n) &\longrightarrow \mathcal{M}_{\mathbf{fg}} \\ G &\longmapsto G_{\mathrm{for}} \end{aligned}$$

By definition, there is a factoring of this map as

$$\mathcal{M}_p(n) \longrightarrow \mathcal{U}(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

through the open substack of formal groups of height at most n. It is worth noting right away that the map $\mathcal{M}_p(n) \to \mathcal{U}(n)$ doesn't have a section. See Remark 4.2.

4.7 Remark (The geometry of $\mathcal{M}_p(n)$) This stack is something of an mysterious object, despite years of work by many people. Basic references include [29]. As an example of what is known, it has one geometric point (ie, isomorphism class of an algebraically closed field) for each integer h, $1 \le h \le n$, given by the p-divisible group

$$G_h = \Gamma_h \times (\mathbb{Z}/p^\infty)^{n-h}.$$

Here Γ_h is a formal group of height h and $\mathbb{Z}/p^\infty$ is the colimit of the étale group schemes

$$\mathbb{Z}/p^n = \mathrm{Spec}\,(\mathbb{F}[x]/(x^{p^n} - x)).$$

The morphism $\mathcal{M}_p(n) \to \mathcal{U}(n)$ is then surjective on geometric points, but it is far from being an isomorphism. For example, the automorphism group of G_h is

$$\mathrm{Aut}(\Gamma_h) \times \mathrm{Gl}_{n-h}(\mathbb{Z}_p).$$

4.8 Remark The morphism $\mathcal{M}_p(n) \to \mathcal{M}_{\mathbf{fg}}$ is *not* representable. This follows from the statement about automorphisms in the previous remark, but let's go into some detail. Consider the two-category pull-back

$$\begin{tikzcd} P_H \arrow[r] \arrow[d] & \mathcal{M}_{\mathbf{fg}}(n) \arrow[d] \\ \mathrm{Spec}\,(R) \arrow[r, "H"'] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

where H is a formal group. By definition, P_H is the functor which assigns to each commutative ring A, the groupoid of triples (f, G, ϕ) where $f\colon R \to A$ is a ring map, G is a p–divisible group of height n over A and $\phi\colon G_{\mathrm{for}} \to f^*H$ is an isomorphism. Put another way, $P_H(A)$ is the groupoid sheaf of extensions

$$0 \to f^*H \to G \to G_{\mathrm{et}} \to 0 \tag{4-1}$$

over A. Isomorphisms fix f^*H, but not G_{et}. If P_H was actually equivalent to a scheme, then a short exact sequence of the form (4-1) would have no automorphisms, but this is evidently not the case. To be specific, let $R = \overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ and let Γ_h be a formal group of height h, $1 \le h < n$ over $\overline{\mathbb{F}}_p$. Then there is a split extension

$$0 \to \Gamma_h \to \Gamma_h \times (\mathbb{Z}/p^\infty)^{n-h} \to (\mathbb{Z}/p^\infty)^{n-h} \to 0.$$

There are no maps from $(\mathbb{Z}/p^\infty)^{n-h}$ to Γ_n; therefore, the automorphisms of this extension are $\mathrm{Gl}_{n-h}(\mathbb{Z}_p)$.

We now can state Lurie's realization result [27]. Since we are working over $\mathbb{Z}_p$, one must take care with the hypotheses of here: the notions of algebraic stack and étale must be the appropriate notions over $\mathrm{Spf}(\mathbb{Z}_p)$.

4.9 Theorem (Lurie) *Let $\mathcal{M}$ be an algebraic stack equipped with an étale morphism*

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n).$$

Then the realization problem for the composition

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

has a canonical solution; that is, the space of all solutions is connected and has a preferred basepoint.

4.10 Remark This theorem directly confronts the conundrum of Example 3.3. We can use this result to realize a Landweber exact theory as an E_∞ ring spectrum only if the associated formal group is the formal part of p–divisible group. This is a strong hypothesis. It works, for example, for p–complete K–theory, but not evidently for the p–completed analog of Johnson–Wilson theories $E(n)_*$. See Remark 4.2 (4).

There is a deeper point, which I have put off discussing until now. The homotopy groups E_* of an E_∞–ring spectum E support more structure than that of a graded commutative ring. In particular, the operad action maps

$$(E\Sigma_n)_+ \wedge_{\Sigma_n} E^{\wedge n} \to E$$

have induced maps in homotopy, which give rise to power operations in E_*. This has a significant impact on the realization problem for if $g\colon \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is a flat map classifying a formal group G, there is no particular reason to suppose the geometry of the formal group would specify the structure of the power operations. However, if g factors

$$\mathrm{Spec}(R) \longrightarrow \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

these power operations should be specified by the subgroup structure of the p–divisible group.

4.11 Example (Serre–Tate theory) As an addendum to this theorem, Lurie points out the morphism $\epsilon\colon \mathcal{M} \to \mathcal{M}_p(n)$ is étale if it satisfies the Serre–Tate theorem; thus, for example, we recover the Hopkins–Miller Theorem 3.7, at least for smooth elliptic curves.[7]

To state the Serre–Tate theorem we need the language of deformation theory. Let $\mathcal{M}$ be a stack over $\mathcal{M}_p(n)$ and $A_0/\mathbb{F}$ be an $\mathcal{M}$–object over a field $\mathbb{F}$, necessarily of characteristic p since we are working over $\mathrm{Spf}(\mathbb{Z}_p)$. Recall that an Artin local ring $(R,\mathfrak{m})$ is a local ring with nilpotent maximal ideal $\mathfrak{m}$. If $q\colon R \to \mathbb{F}$ be a surjective morphism of rings, then a *deformation* of A_0 to R is an $\mathcal{M}$–object A over R and a pull-back diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

Deformations form a groupoid functor $\mathrm{Def}_{\mathcal{M}}(\mathbb{F}, A_0)$ on an appropriate category of Artin local rings. The Serre–Tate theorem holds if the evident morphism

$$\mathrm{Def}_{\mathcal{M}}(\mathbb{F}, A_0) \longrightarrow \mathrm{Def}_{\mathcal{M}_p(n)}(\mathbb{F}, \epsilon A_0)$$

is an equivalence. This result holds for elliptic curves, but actually in much wider generality (see [29]).

4.12 Remark (Deformations of p–divisible groups) The deformation theory of p–divisible groups and formal groups is well understood and a simple application of Schlessinger's general theory [35]. For formal groups, this is Lubin–Tate theory [26]. If Γ is a formal group of height n over a perfect field $\mathbb{F}$, then Lubin–Tate theory says that the groupoid-valued functor $\mathrm{Def}_{\mathcal{M}_{\mathbf{fg}}}(\mathbb{F}, \Gamma)$ is discrete; that is, the natural map

$$\mathrm{Def}_{\mathcal{M}_{\mathbf{fg}}}(\mathbb{F}, \Gamma) \to \pi_0 \mathrm{Def}_{\mathcal{M}_{\mathbf{fg}}}(\mathbb{F}, \Gamma)$$

[7] Lurie [27] suggests an argument for completing the proof of the Hopkins–Miller Theorem, once we know the realization result for the open substack $\mathcal{M}_{ell}$ of smooth elliptic curves.

is an equivalence. Furthermore, $\pi_0\mathrm{Def}_{\mathcal{M}_{\mathrm{fg}}}(\mathbb{F},\Gamma)$ is prorepresented by a complete local ring $R(\mathbb{F},\Gamma)$; that is, there is a natural isomorphism

$$\pi_0\mathrm{Def}_{\mathcal{M}_{\mathrm{fg}}}(\mathbb{F},\Gamma) \cong \mathrm{Spf}(R(\mathbb{F},\Gamma)).$$

A choice of p–typical coordinate for the universal deformation of Γ over $R(\mathbb{F},\Gamma)$ defines an isomorphism

$$W(\mathbb{F})[\![u_1,\dots,u_{n-1}]\!] \cong R(\mathbb{F},\Gamma)$$

where $W(-)$ is the Witt vector functor.

A similar result holds for p–divisible groups. Let G be a p–divisible group over an algebraically closed field $\mathbb{F}$. Then we have split short exact sequence

$$0 \to G_{\mathrm{for}} \to G \to G_{\mathrm{et}} \to 0.$$

Since $\mathbb{F}$ is algebraically closed, there is an isomorphism

$$G_{\mathrm{et}} \cong (\mathbb{Z}/p^\infty)^{n-h} \tag{4-2}$$

where h is the height of G_{for}. Since G_{et} has a unique deformation up to isomorphism, by the definition of étale, a choice of isomorphism (4-2) now identifies deformations of G as extensions

$$0 \to H_{\mathrm{for}} \to H \to (\mathbb{Z}/p^\infty)^{n-h} \to 0$$

where H_{for} is a deformation of G_{for}. The exact sequence of sheaves of groups

$$0 \to \mathbb{Z}^{n-h} \to \mathbb{Q}^{n-h} \to (\mathbb{Z}/p^\infty)^{n-h} \to 0$$

now identifies the isomorphism class of extension as an element in

$$\mathrm{Hom}(\mathbb{Z}^{n-h}, H_{\mathrm{for}}).$$

Thus we conclude that the groupoid-valued functor $\mathrm{Def}_{\mathcal{M}_p(n)}(\mathbb{F},G)$ is discrete and $\pi_0\mathrm{Def}_{\mathcal{M}_p(n)}(\mathbb{F},G)$ is prorepresented by

$$R(\mathbb{F},\Gamma)[\![t_1,\dots,t_{n-h}]\!] \cong W(\mathbb{F})[\![u_1,\dots,u_{h-1},t_1,\dots,t_{n-h}]\!].$$

Note that this is always a power series in $n-1$ variables. Similar results hold for perfect fields by Galois descent.

Using this remark it is possible to give a local criterion for when a morphism of stacks $\mathcal{M} \to \mathcal{M}_p(n)$ is étale. It is in this guise that Lurie's theorem appears in [3].

4.13 Remark The proof of Theorem 4.9 has two large steps. The first is to define and prove the existence of an analog of the stack $\mathcal{M}$ appropriate for *derived algebraic geometry* — which can be thought of as algebraic geometry with E_∞–ring spectra as the basic object. This yields a stack $(\mathcal{M}, \mathcal{O}^{\mathbf{top}})$ where the structure sheaf is now a sheaf of E_∞–ring spectra. The second is to show that the resulting algebraic object $(\mathcal{M}, \mathcal{O}^{\mathbf{top}})$ is the realization required; that is, to construct an isomorphism $(\mathcal{M}, \mathcal{O}^{\mathbf{top}}_*) \cong (\mathcal{M}, \mathcal{O}_{\mathcal{M}*})$. For this, there must be some homotopy theoretic input; this is the local Hopkins–Miller theorem. This says that the Lubin–Tate theory $E(\mathbb{F}, \Gamma)$ obtained from the deformations of a height n–formal group over a perfect field $\mathbb{F}$ is an E_∞–ring spectrum and the space of all E_∞–structures is contractible (see Goerss and Hopkins [11]).

Acknowledgement

The author was partially supported by the National Science Foundation.

References

[1] **J F Adams**, *Stable homotopy and generalised homology*, University of Chicago Press (1974) MR0402720

[2] **T Bauer**, *Computation of the homotopy of the spectrum tmf*, from: "Groups, homotopy and configuration spaces (Tokyo 2005)", (N Iwase, T Kohno, R Levi, D Tamaki, J Wu, editors), Geom. Topol. Monogr. 13 (2008) 11–40

[3] **M Behrens**, **T Lawson**, *Topological automorphic forms* arXiv:math/072719

[4] **J-Y Butowiez**, **P Turner**, *Unstable multiplicative cohomology operations*, Q. J. Math. 51 (2000) 437–449 MR1806451

[5] **P Deligne**, *Courbes elliptiques: formulaire d'après J. Tate*, from: "Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)", Lecture Notes in Math. 476, Springer (1975) 53–73 MR0387292

[6] **P Deligne**, **D Mumford**, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969) 75–109 MR0262240

[7] **P Deligne**, **M Rapoport**, *Les schémas de modules de courbes elliptiques*, from: "Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)", Lecture Notes in Math. 349, Springer (1973) 143–316 MR0337993

[8] **M Demazure**, **P Gabriel**, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson (1970) MR0302656 Avec un appendice 'Corps de classes local' par Michiel Hazewinkel

[9] **P G Goerss**, *Quasi-coherent sheaves on the moduli stack of formal groups* arXiv:0802.0996

[10] **P G Goerss**, *(Pre-)sheaves of ring spectra over the moduli stack of formal group laws*, from: "Axiomatic, enriched and motivic homotopy theory", NATO Sci. Ser. II Math. Phys. Chem. 131, Kluwer (2004) 101–131 MR2061853

[11] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from: "Structured ring spectra", London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR2125040

[12] **R Hartshorne**, *Algebraic geometry*, Graduate Texts in Math. 52, Springer (1977) MR0463157

[13] **M Hill**, **T Lawson**, *Automorphic forms and cohomology theories on Shimura curves of small discriminant* arXiv:0902.2603

[14] **S Hollander**, *Geometric criteria for Landweber exactness*, preprint (2008) to appear in Proc. London Math. Soc. Available at http://www.math.ist.utl.pt/~sjh/le10.pdf

[15] **M J Hopkins**, *Algebraic topology and modular forms*, from: "Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)", Higher Ed. Press (2002) 291–317 MR1989190

[16] **M J Hopkins**, **B H Gross**, *The rigid analytic period mapping, Lubin–Tate space, and stable homotopy theory*, Bull. Amer. Math. Soc. 30 (1994) 76–86 MR1217353

[17] **M J Hopkins**, **M E Mahowald**, *From elliptic curves to homotopy theory*, preprint, Purdue University (1998) Available at http://tinyurl.com/co6twu

[18] **M Hovey**, **N P Strickland**, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. 139:666 (1999) MR1601906

[19] **J-i Igusa**, *Fibre systems of Jacobian varieties. III. Fibre systems of elliptic curves*, Amer. J. Math. 81 (1959) 453–476 MR0104669

[20] **T Kashiwabara**, *Hopf rings and unstable operations*, J. Pure Appl. Algebra 94 (1994) 183–193 MR1282839

[21] **N M Katz**, **B Mazur**, *Arithmetic moduli of elliptic curves*, Ann. of Math. Studies 108, Princeton University Press (1985) MR772569

[22] **R E Kottwitz**, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (1992) 373–444 MR1124982

[23] **P S Landweber**, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. 98 (1976) 591–610 MR0423332

[24] **G Laumon**, **L Moret-Bailly**, *Champs algébriques*, Ergebnisse Series 39, Springer, Berlin (2000) MR1771927

[25] **M Lazard**, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France 83 (1955) 251–274 MR0073925

[26] **J Lubin**, **J Tate**, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966) 49–59 MR0238854

[27] **J Lurie**, *Survey article on elliptic cohomology*, preprint, MIT (2007) Available at http://www-math.mit.edu/~lurie/papers/survey.pdf

[28] **M Mahowald**, **C Rezk**, *Brown–Comenetz duality and the Adams spectral sequence*, Amer. J. Math. 121 (1999) 1153–1177 MR1719751

[29] **W Messing**, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Math. 264, Springer, Berlin (1972) MR0347836

[30] **H R Miller**, *Sheaves, gradings, and the exact functor theorem* Available at http://www-math.mit.edu/~hrm/papers/papers.html

[31] **N Naumann**, *The stack of formal groups in stable homotopy theory*, Adv. Math. 215 (2007) 569–600 MR2355600

[32] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Math. 121, Academic Press, Orlando, FL (1986) MR860042

[33] **D C Ravenel**, *Nilpotence and periodicity in stable homotopy theory*, Ann. of Math. Studies 128, Princeton University Press (1992) MR1192553 Appendix C by Jeff Smith

[34] **C Rezk**, *Supplementary notes for Math 512*, notes of a course on topological modular forms given at Northwestern University (2001) Available at http://www.math.uiuc.edu/~rezk/papers.html

[35] **M Schlessinger**, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130 (1968) 208–222 MR0217093

[36] **J H Silverman**, *The arithmetic of elliptic curves*, Graduate Texts in Math. 106, Springer (1986) MR817210

[37] **B Smithling**, *On the moduli stack of commutative, 1-parameter formal Lie groups*, Thesis, University of Chicago (2007)

Department of Mathematics, Northwestern University
Evanston, IL 60208, USA

pgoerss@math.northwestern.edu

http://www.math.northwestern.edu/~pgoerss

Received: 30 October 2008