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On Wigner's theorem

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Wigner's theorem asserts that any symmetry of a quantum system is unitary or antiunitary. In this short note we give two proofs based on the geometry of the Fubini–Study metric.

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For Mike Freedman, on the occasion of his 60th birthday

The space of pure states of a quantum mechanical system is the projective space $\mathbb{P}\mathscr{H}$ of lines in a separable complex Hilbert space $\big(\mathscr{H}, \langle -,-\rangle\big)$, which may be finite or infinite dimensional. It carries a symmetric function $p\colon \mathbb{P}\mathscr{H}\times\mathbb{P}\mathscr{H}\to[0,1]$ whose value $p(L_1,L_2)$ on states $L_1,L_2\in\mathbb{P}\mathscr{H}$ is the *transition probability*: if $\psi_i\in L_i$ is a unit norm vector in the line L_i, then

$$p(L_1,L_2)=|\langle\psi_1,\psi_2\rangle|^2.$$

Let $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathscr{H})$ denote the group of symmetries of $(\mathbb{P}\mathscr{H},p)$, the group of quantum symmetries. A fundamental theorem of Wigner[1] [12, Sections 20A and 26] (see also Bargmann [2] and Weinberg [11, Section 2A]) expresses $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathscr{H})$ as a quotient of linear and antilinear symmetries of $\mathscr{H}$. This note began with the rediscovery of a formula which relates the quantum geometry of $(\mathbb{P}\mathscr{H},p)$ to a more familiar structure in differential geometry: the Fubini–Study Kähler metric on $\mathbb{P}\mathscr{H}$. It leads to two proofs of Wigner's theorem, Theorem 8 of this note, based on the differential geometry of projective space.

The proofs here use more geometry than the elementary proofs [2], [11, Section 2A]. We take this opportunity to draw attention to Wigner's theorem and to the connection between quantum mechanics and projective geometry. It is a fitting link for a small tribute to Mike Freedman, whose dual careers in topology and condensed matter physics continue to inspire.

Let $d\colon \mathbb{P}\mathscr{H}\times\mathbb{P}\mathscr{H}\to\mathbb{R}^{\ge 0}$ be the distance function associated to the Fubini–Study metric.

[1] As I learned in Bonolis [3, page 74], this theorem was first asserted in a 1928 joint paper [10, page 207] of von Neumann and Wigner, though with only a brief justification. A more complete account appeared in Wigner's book (in the original German) in 1931.

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1 Theorem *The functions p and d are related by*

$$\cos(d) = 2p - 1. \tag{2}$$

As a gateway into the literature on 'geometric quantum mechanics', where (2) can be found,[2] see Brody and Hughston [4] and the references therein.

3 Corollary $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ *is the group of isometries of* $\mathbb{P}\mathcal{H}$ *with the Fubini–Study distance function.*

4 Remark If $\mathcal{H}$ is infinite dimensional, then $\mathbb{P}\mathcal{H}$ is an infinite dimensional smooth manifold modeled on a Hilbert space. Basic notions of calculus and differential geometry carry over to Hilbert manifolds (Lang [8]). The Myers–Steenrod theorem asserts that a distance-preserving map between two Riemannian manifolds is smooth and preserves the Riemannian metric. That theorem is also true on Riemannian manifolds modeled on Hilbert manifolds (Garrido, Jaramillo and Rangel [6]).[3] So in the sequel we use that a distance-preserving map $\phi\colon \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ is smooth and is an isometry in the sense of Riemannian geometry.

The tangent space to $\mathbb{P}\mathcal{H}$ at a line $L \subset \mathcal{H}$ is canonically $T_L\mathbb{P}\mathcal{H} \cong \mathrm{Hom}_{\mathbb{C}}(L, L^{\perp})$, where $L^{\perp} \subset \mathcal{H}$ is the orthogonal complement to L, a closed subspace and therefore itself a Hilbert space. If $f_1, f_2\colon L \to L^{\perp}$, then the Fubini–Study hermitian metric is defined by

$$\langle f_1, f_2 \rangle = \mathrm{Tr}(f_1^* f_2). \tag{5}$$

The adjoint f_1^* is computed using the inner products on L and $L^{\perp}$. The composition $f_1^* f_2$ is an endomorphism of L, hence multiplication by a complex number which we identify as the trace of the endomorphism. If $\ell \in L$ has unit norm, then the map

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(L, L^{\perp}) &\longrightarrow L^{\perp} \\ f &\longmapsto f(\ell) \end{aligned} \tag{6}$$

is a linear isometry for the induced metric on $L^{\perp} \subset \mathcal{H}$. The underlying Riemannian metric is the real part of the hermitian metric (5); it only depends on the real part of the inner product on $\mathcal{H}$.

[2] Notice that (2) is equivalent to $p = \cos^2(d/2)$.

[3] The proof depends on the existence of geodesic convex neighborhoods, proved in [8, Section VIII.5]. For the Fubini–Study metric on $\mathbb{P}\mathcal{H}$ such neighborhoods may easily be constructed explicitly. I thank Karl-Hermann Neeb for his inquiry about the Myers–Steenrod theorem in infinite dimensions.

Proof of Theorem 1 Equation (2) is obvious on the diagonal in $\mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H}$, as well as if $\dim \mathcal{H} = 1$. Henceforth we rule out both possibilities. Fix $L_1 \neq L_2 \in \mathbb{P}\mathcal{H}$ and let V be the 2–dimensional space $L_1 + L_2 \subset \mathcal{H}$. The unitary automorphism of $\mathcal{H} = V \oplus V^\perp$ which is $+1$ on V and -1 on $V^\perp$ induces an isometry of $\mathbb{P}\mathcal{H}$ which has $\mathbb{P}V$ as a component of its fixed point set. It follows that $\mathbb{P}V$ is totally geodesic. Therefore, to compute $d(L_1, L_2)$ we are reduced to the case of the complex projective line with its Fubini–Study metric: the round 2–sphere.

Let $e_1 \in L_1$ have unit norm and choose $e_2 \in V$ to fill out a unitary basis $\{e_1, e_2\}$. Then $\lambda e_1 + e_2 \in L_2$ for a unique $\lambda \in \mathbb{C}$. If $\lambda = 0$ then it is easy to check that $d = \pi$ and $p = 0$, consistent with (2), so we now assume $\lambda \neq 0$. Identify $\mathbb{P}V \setminus \{\mathbb{C} \cdot e_2\} \approx \mathbb{C}$ by $\mathbb{C} \cdot (e_1 + \mu e_2) \leftrightarrow \mu$. Use stereographic projection from the north pole $(1, 0)$ in Euclidean 3–space $\mathbb{R} \times \mathbb{C}$ to identify $\{0\} \times \mathbb{C} \approx S^2 \setminus \{(1, 0)\}$, where $S^2 \subset \mathbb{R} \times \mathbb{C}$ is the unit sphere. Under these identifications we have

$$
\begin{aligned}
L_1 &\longleftrightarrow (-1\,,\,0) \\
L_2 &\longleftrightarrow \left(-\frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}\,,\,\frac{2|\lambda|^2}{|\lambda|^2 + 1}\,\frac{1}{\lambda}\right)
\end{aligned}
$$

from which $\cos(d) = (|\lambda|^2 - 1)/(|\lambda|^2 + 1)$ can be computed as the inner product of vectors in the 3–dimensional vector space $\mathbb{R} \oplus \mathbb{C}$. Since $p = |\lambda|^2/(|\lambda|^2 + 1)$, equation (2) is satisfied. □

A *real* linear map $S\colon \mathcal{H} \to \mathcal{H}$ is *antiunitary* if it is conjugate linear and

$$\langle S\psi_1, S\psi_2\rangle = \overline{\langle \psi_1, \psi_2\rangle} \qquad \text{for all } \psi_1, \psi_2 \in \mathcal{H}.$$

Let $G(\mathcal{H})$ denote the group consisting of all unitary and antiunitary operators on $\mathcal{H}$. In the norm topology it is a Banach Lie group (Milnor [9]) with two contractible components; the same is true in the compact–open topology (Freed and Moore [5, Appendix D]). The identity component is the group $U(\mathcal{H})$ of unitary transformations. Any $S \in G(\mathcal{H})$ maps complex lines to complex lines, so induces a diffeomorphism of $\mathbb{P}\mathcal{H}$, and since S preserves the real part of $\langle -, - \rangle$ the induced diffeomorphism is an isometry. The unit norm scalars $\mathbb{T} \subset G(\mathcal{H})$ act trivially on $\mathbb{P}\mathcal{H}$, so there is an exact[4] sequence of Lie groups

(7)
$$1 \longrightarrow \mathbb{T} \longrightarrow G(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}).$$

Note that $\mathbb{T}$ is not central since antiunitary maps conjugate scalars.

[4]We assume $\dim \mathcal{H} > 1$.

8 Theorem (Wigner [12]) *The homomorphism* $G(\mathscr{H}) \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathscr{H})$ *is surjective: every quantum symmetry of* $\mathbb{P}\mathscr{H}$ *lifts to a unitary or antiunitary operator on* $\mathscr{H}$.

By Corollary 3 the same is true for isometries of the Fubini–Study metric, and indeed we prove Wigner's Theorem by computing the group of isometries.

9 Remark If $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathscr{H})$ is any group of quantum symmetries, then the surjectivity of $G(\mathscr{H}) \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathscr{H})$ implies the extension (7) pulls back to a twisted central extension of G. The twist is the homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$ which tells whether a symmetry lifts to be unitary or antiunitary. The isomorphism class of this twisted central extension is then an invariant of ρ. This is the starting point for joint work with Greg Moore [5] about quantum symmetry classes and topological phases in condensed matter physics.

10 Example $\mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1$ with the Fubini–Study metric is the round 2–sphere of unit radius. Its isometry group is the group $O(3)$ of orthogonal transformations of $\mathrm{SO}(3)$. The identity component $\mathrm{SO}(3)$ is the image of the group $U(2)$ of unitary transformations of $\mathbb{C}^2$. The other component of $O(3)$ consists of orientation-reversing orthogonal transformations, such as reflections, and they lift to antiunitary symmetries of $\mathbb{C}^2$. In this case the group $G(\mathscr{H})$ is also known as $\mathrm{Pin}^c(3)$; see Atiyah, Bott and Shapiro [1].

We present two proofs of Theorem 8. The first is based on the following standard fact in Riemannian geometry.

11 Lemma *Let* M *be a Riemannian manifold,* $p \in M$, *and* $\phi\colon M \to M$ *an isometry with* $\phi(p) = p$. *Suppose* $B_r \subset T_pM$ *is the open ball of radius* r *centered at the origin and assume the Riemannian exponential map* $\exp_p$ *maps* B_r *diffeomorphically into* M. *Then in exponential coordinates* $\phi\big|_{B_r}$ *equals the restriction of the linear isometry* $d\phi_p$ *to* B_r.

Proof If $\xi \in B_r$, then $\exp_p(\xi) = \gamma_\xi(1)$, where $\gamma_\xi\colon [0,1] \to M$ is the unique geodesic which satisfies $\gamma_\xi(0) = p$, $\dot\gamma_\xi(0) = \xi$. Since ϕ maps geodesics to geodesics, $\phi \circ \exp_p = \exp_p \circ d\phi_p$ on B_r, as desired. □

If $\rho\colon [0,r') \to [0,r)$ is a diffeomorphism for some $r' > 0$, then

$$\xi \longmapsto \exp_p\bigl(\rho(|\xi|)\xi\bigr) \tag{12}$$

maps $B_{r'}$ diffeomorphically into M, and ϕ in this coordinate system is also linear.

First Proof of Theorem 8 Let $\phi\colon \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ be an isometry. Composing with an isometry in $G(\mathcal{H})$ we may assume $\phi(L) = L$ for some $L \in \mathbb{P}\mathcal{H}$. The tangent space $T_L\mathbb{P}\mathcal{H}$ is canonically $\mathrm{Hom}_{\mathbb{C}}(L, L^{\perp})$, and also $f \in \mathrm{Hom}_{\mathbb{C}}(L, L^{\perp})$ determines $\Gamma_f \in \mathbb{P}\mathcal{H}$ by $\Gamma_f \subset \mathcal{H} = L \oplus L^{\perp}$ is the graph of f. We claim $f \mapsto \Gamma_f$ has the form (12) for some $\rho\colon [0,\infty) \to [0,\pi)$. It suffices to show that for any $f \in \mathrm{Hom}_{\mathbb{C}}(L, L^{\perp})$ of unit norm, the map $t \mapsto \Gamma_{tf}$ traces out a (reparametrized) geodesic in a parametrization independent of f. As in the proof of Theorem 1 this reduces to $\dim \mathcal{H} = 2$ and so to an obvious statement about the round 2–sphere. It follows from Lemma 11 that ϕ is a *real* isometry $S \in \mathrm{End}_{\mathbb{R}}\big(\mathrm{Hom}_{\mathbb{C}}(L, L^{\perp})\big)$. It remains to prove that S is complex linear or antilinear; then we extend S by the identity on L to obtain a unitary or antiunitary operator on $\mathcal{H} = L \oplus L^{\perp}$.

If $\dim \mathcal{H} = 2$ then Theorem 8 can be verified (see Example 10), so assume $\dim \mathcal{H} > 2$. Identify $\mathrm{Hom}_{\mathbb{C}}(L, L^{\perp}) \approx L^{\perp}$ as in (6). Since $S \in \mathrm{End}_{\mathbb{R}}(L^{\perp})$ maps complex lines in $L^{\perp}$ to complex lines, there is a function $\alpha\colon L^{\perp}\backslash\{0\} \to \mathbb{C}$ such that $S(i\xi) = \alpha(\xi)S(\xi)$ for all nonzero $\xi \in L^{\perp}$. Fix $\xi \neq 0$ and choose $\eta \in L^{\perp}$ which is linearly independent. Then

$$\begin{aligned} S\big(i(\xi+\eta)\big) &= \alpha(\xi+\eta)\big[S(\xi) + S(\eta)\big] \\ &= \alpha(\xi)S(\xi) + \alpha(\eta)S(\eta) \end{aligned}$$

from which $\alpha(\xi) = \alpha(\eta)$. Applied to $i\xi, \eta$ we learn $\alpha(\xi) = \alpha(i\xi)$. On the other hand,

$$-S(\xi) = S(-\xi) = \alpha(i\xi)S(i\xi) = \alpha(i\xi)\alpha(\xi)S(\xi),$$

whence $\alpha(\xi)^2 = -1$. By continuity either $\alpha \equiv i$ or $\alpha \equiv -i$, which proves that S is linear or S is antilinear. □

The second proof leans on complex geometry.

13 Lemma *An isometry $\phi\colon \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ is either holomorphic or antiholomorphic.*

Proof Let $I\colon T\mathbb{P}\mathcal{H} \to T\mathbb{P}\mathcal{H}$ be the (almost) complex structure. Then I is parallel with respect to the Levi–Civita covariant derivative, since $\mathbb{P}\mathcal{H}$ is Kähler, and so therefore is ϕ^*I. We claim *any* parallel almost complex structure J equals $\pm I$; the lemma follows immediately.

If J is parallel, then it commutes with the Riemann curvature tensor R. Compute at $L \in \mathbb{P}\mathcal{H}$ and identify $T_L\mathbb{P}\mathcal{H} \approx L^{\perp}$, as in (6). Then if $\xi, \eta \in L^{\perp}$ and $\langle \xi, \eta\rangle = 0$, since $\mathbb{P}(L \oplus \mathbb{C}\cdot\xi \oplus \mathbb{C}\cdot\eta) \subset \mathbb{P}\mathcal{H}$ is totally geodesic and has constant holomorphic sectional curvature one (Kobayashi and Nomizu [7, Section IX.7]), we compute

$$R(\xi, I\xi)\xi = -|\xi|^2 I\xi,$$
$$R(\xi, I\xi)\eta = -\frac{1}{2}|\xi|^2 I\eta.$$

It follows that J preserves every complex line $K = \mathbb{C}\cdot\xi \subset L^\perp$ and commutes with I on K. Therefore, $J = \pm I$ on K. By continuity, the sign is independent of K and L. □

Second Proof of Theorem 8 First, recall that if U is finite dimensional, then every holomorphic symmetry of $\mathbb{P}U$ is linear. The proof is as follows. Let $\mathcal{L} \to \mathbb{P}U$ be the canonical holomorphic line bundle whose fiber at $L \in \mathbb{P}U$ is L. A holomorphic line bundle on $\mathbb{P}U$ is determined by its Chern class, so $\phi^*\mathcal{L} \cong \mathcal{L}$. Fix an isomorphism; it is unique up to scale. There is an induced linear map on the space $H^0(\mathbb{P}U;\mathcal{L}^*) \cong U^*$ of global holomorphic sections:

$$\phi^*\colon H^0(\mathbb{P}U;\mathcal{L}^*) \longrightarrow H^0(\mathbb{P}U;\phi^*\mathcal{L}^*) \cong H^0(\mathbb{P}U;\mathcal{L}^*). \tag{14}$$

The transpose $\hat\phi$ of (14) is the desired linear lift of ϕ.

Let $\phi\colon \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ be an isometry. After composition with an element of $G(\mathcal{H})$ we may, by Lemma 13, assume ϕ is holomorphic and fixes some $L \in \mathbb{P}\mathcal{H}$. Let $U \subset \mathcal{H}$ be a finite dimensional subspace containing L. Then the pullback of $\mathcal{L}_{\mathcal{H}} \to \mathbb{P}\mathcal{H}$ to $\phi^*\mathcal{L}_{\mathcal{H}}|_{\mathbb{P}U} \to \mathbb{P}U$ has degree one, so is isomorphic to $\mathcal{L}_U \to \mathbb{P}U$, and there is a unique isomorphism which is the identity on the fiber over L. A functional $\alpha \in \mathcal{H}^*$ restricts to a holomorphic section of $\phi^*\mathcal{L}^*_{\mathcal{H}}|_{\mathbb{P}U} \to \mathbb{P}U$, so by composition with the isomorphism $\phi^*\mathcal{L}^*_{\mathcal{H}}|_{\mathbb{P}U} \cong \mathcal{L}^*_U$ to an element of U^*. The resulting map $\mathcal{H}^* \to U^*$ is linear, and its transpose $\hat\phi\colon U \to \mathcal{H}$ is the identity on L. Let U run over all finite dimensional subspaces of $\mathcal{H}$ to define $\hat\phi\colon \mathcal{H} \to \mathcal{H}$. The uniqueness of the isomorphism $\phi^*\mathcal{L}_{\mathcal{H}}|_{\mathbb{P}U} \cong \mathcal{L}_U$ implies that $\hat\phi$ is well-defined and a linear lift of ϕ. It is unitary since ϕ is an isometry. □

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