

Approximating $C^{1,0}$ -foliations

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We extend the Eliashberg–Thurston theorem on approximations of taut oriented C^2 -foliations of 3-manifolds by both positive and negative contact structures to a large class of taut oriented $C^{1,0}$ -foliations, where by $C^{1,0}$ foliation we mean a foliation with continuous tangent plane field. These $C^{1,0}$ -foliations can therefore be approximated by weakly symplectically fillable, universally tight, contact structures. This allows applications of C^2 -foliation theory to contact topology and Floer theory to be generalized and extended to constructions of $C^{1,0}$ -foliations.

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1 Introduction

In [12], Eliashberg and Thurston introduce the notion of *confoliation* and prove that when $k \geq 2$, a transversely oriented C^k -foliation \mathcal{F} of a closed, oriented 3-manifold not equal to $S^1 \times S^2$ can be C^0 -approximated by a pair of C^∞ contact structures, one positive and one negative. They also prove that when \mathcal{F} is also taut, any contact structure sufficiently close to the plane field of \mathcal{F} is weakly symplectically fillable and universally tight. For the most part, they restrict attention to confoliations which are at least C^1 . For their main theorem, they restrict attention to confoliations which are at least C^2 . Their assumption that 2-plane fields and 1-forms are C^1 is necessary for it to be possible to take derivatives. Their assumption that 2-plane fields and 1-forms are C^2 is necessary for Sacksteder's theorem ([50], see also [12, Theorem 1.2.5]) to apply. A weakening of this C^2 assumption in the neighborhood of compact leaves is used by Kronheimer, Mrowka, Ozsváth, and Szabó [38] to show that the methods of [12] apply to those foliations constructed by Gabai [17; 21; 22] which are C^∞ except along torus leaves. In this paper we show that many of the techniques of Eliashberg and Thurston extend to transversely oriented, taut, $C^{1,0}$ -foliations satisfying a natural transitivity condition.

Eliashberg and Thurston's proof that sufficiently smooth transversely oriented taut foliations of 3-manifolds can be perturbed to weakly symplectically fillable contact structures gives a direct connection between foliation theory and symplectic topology via

contact topology. This connection has been most spectacularly exploited by Kronheimer and Mrowka [36] in their proof of the Property P conjecture.

Since the contact structures produced by Eliashberg and Thurston are weakly symplectically fillable and universally tight, their theorem is important in contact topology. To apply their theorem, of course one must start with a taut oriented foliation. The issue that we seek to address is that most new constructions of foliations produce foliations that are not smooth enough to apply the Eliashberg–Thurston theorem.

Constructions of foliations which fail even to be C^1 can be found in [7; 8; 17; 18; 19; 20; 21; 22; 23; 33; 40; 41; 42; 47; 48; 49]. In each of these papers, foliations are constructed using branched surfaces. Foliations constructed using branched surfaces will have smooth leaves but will often vary only continuously in a transverse direction because of the role and nonsmooth nature of leaf blow-up [23, Operation 2.1.1]. (To obtain the smooth foliations found in [17; 18; 19; 20; 21; 22], Gabai avoids appealing to Operation 2.1.1 and instead takes advantage of the fact that the branched surfaces involved in his construction are finite depth.)

The main results of this paper are, roughly, that $C^{1,0}$ -foliations satisfying a simple transitivity condition can be approximated by weakly symplectically fillable contact structures. Such foliations include most of those found in [7; 8; 18; 19; 20; 22; 23; 33; 42; 47; 48; 49] and many of those found in [17; 21; 40; 41]. In proving this, we give new methods for both initiating and propagating contact structures when beginning with a foliation. Before giving a more precise indication of our results, we give a brief description of some of the main ideas in Eliashberg and Thurston’s proof.

Eliashberg and Thurston interpolate between the notions of foliation and contact structure by introducing confoliations. This structure restricts to a contact structure on an open set, but is equal to the original foliation elsewhere. Their first step is to create a contact zone in the manifold, that is, to approximate the foliation so that it is a contact structure on a nonempty set. One place they accomplish this is in a neighborhood of a curve with attracting holonomy. We obtain a related result, [Theorem 7.2](#), for the larger class of curves for which the holonomy has a contracting interval. We also introduce the notion of L -bracketed foliation in [Definition 8.4](#) and show how a contact zone can be naturally introduced about a regular neighborhood of the corresponding link L . This has applications to spines of open book decompositions, and more generally to manifolds obtained by surgery or filling.

Their next step is to propagate the contact structure throughout the manifold using smooth foliation charts. See [Lemma 5.1](#), and [Corollary 5.2](#). We use a similar strategy, but we rely on the local existence of smooth approximations to a given $C^{1,0}$ -foliation.

To prove that the contact structure they produce is weakly symplectically fillable, they use the crucial structure of a smooth volume-preserving flow Φ transverse to the taut foliation. Such a smooth flow is the starting point for our work, for they exist even for $C^{1,0}$ -foliations.

There are several notions that describe the relationship between a foliation \mathcal{F} and “nearby” contact structures ξ . If there is a continuous family of contact structures ξ_t such that $\xi_t \rightarrow \mathcal{F}$, we say \mathcal{F} can be *perturbed* or *deformed* to a contact structure. A weaker notion is the existence of a sequence $\xi_n \rightarrow \mathcal{F}$, in which case \mathcal{F} can be *approximated* by contact structures. Both notions of convergence can be refined by defining convergence using a C^k norm on tangent planes for values of k ranging from 0 to ∞ . We use the flow Φ transverse to \mathcal{F} to define a weaker, topological notion of approximation.

Definition 1.1 Given an oriented foliation \mathcal{F} and a positively transverse flow Φ , we say an oriented 2-plane field χ , typically a contact structure or a confoliation, is a Φ -*approximation* of \mathcal{F} if χ is positively transverse to Φ . We say that two oriented 2-plane fields are Φ -*close* if both are positively transverse to Φ .

One application of our work is to complete the proof of a theorem of [32]. Honda, Kazez, and Matic show that sufficiently large fractional Dehn twisting for an open book decomposition with connected binding implies that the canonically associated contact structure is weakly symplectically fillable. Their proof requires the existence of contact structures approximating $C^{1,0}$ -foliations constructed by Roberts in [49], and thus needs a stronger version of the Eliashberg–Thurston theorem.

Another application of our work is to prove that 3-manifolds containing taut, oriented $C^{1,0}$ -foliations satisfying our transitivity condition cannot be L-spaces, thus extending the result of Ozsváth and Szabó [45, Theorem 1.4]. (See also Kronheimer, Mrowka, Ozsváth and Szabó [38, Theorem 2.1] and Kronheimer and Mrowka [37, Theorem 41.4.1].)

Our result also strengthens a result of Baldwin and Etnyre [1]. They give a set of examples showing that when an open book decomposition has multiple binding components, no fixed lower bounds on fractional Dehn twisting can guarantee weak symplectic fillability. This can now be viewed as a nonexistence theorem for taut, oriented $C^{1,0}$ -foliations satisfying our transitivity condition.

The techniques introduced in this paper are used by the authors in [34] to prove general C^0 approximation theorems; in particular, in Corollary 2.9, the condition that \mathcal{F} be Φ -bracketed can be replaced by the condition that \mathcal{F} be taut, with continuous tangent plane

field. Closely related C^0 approximation results have also been obtained independently by J Bowden [2] using different techniques.

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2 An overview

The transition from taut foliations to tight contact structures involves two auxiliary structures, volume-preserving flows and symplectic topology. We summarize the results we need from each field as follows.

Theorem 2.1 ([51, Theorem II.20]; see also [29, Theorem A1]) *Suppose \mathcal{F} is a taut codimension-1 $C^{1,0}$ -foliation of a smooth closed Riemannian 3-manifold M . Then there is a volume-preserving smooth flow Φ transverse to \mathcal{F} .*

For clarity, we break the theorem found in [12] into two statements:

Theorem 2.2 ([12, Corollaries 3.2.2, 3.2.4 and 3.2.8]; see also [37, Theorem 41.3.2]) *Let M be a smooth closed Riemannian 3-manifold with a volume-preserving flow Φ . Suppose there exist a smooth positive contact structure ξ_+ and a smooth negative contact structure ξ_- , both of which are transverse to Φ . Then each of ξ_{\pm} is weakly symplectically fillable and universally tight. Moreover, if ξ is any smooth (positive or negative) contact structure transverse to Φ , then ξ is weakly symplectically fillable and universally tight.*

Remark The statement of [Theorem 2.2](#) is meant to emphasize two things. First, given a smooth positive (respectively, negative) contact structure transverse to a volume-preserving flow, it is sufficient to produce a negative (respectively, positive) contact structure also transverse to the flow to conclude both are weakly symplectically fillable and universally tight. Next, once such ξ_+ and ξ_- are shown to exist, any contact structure ξ transverse to Φ is necessarily weakly symplectically fillable and universally tight.

Theorem 2.3 ([12, Corollary 3.2.5]; see also [37, Theorem 41.3.2]) *Let M be a smooth closed oriented 3-manifold which contains a taut, oriented C^2 -foliation \mathcal{F} . There exist a smooth positive contact structure ξ_+ and a smooth negative contact structure ξ_- , both C^0 -close to \mathcal{F} .*

After giving background definitions and facts about foliations in [Section 3](#), we describe in [Section 4](#) how flow boxes can be organized with an eye towards spreading an initial contact structure throughout the ambient manifold. Contact structures are propagated from one flow box to the next via a collection of local extension theorems described in [Section 5](#). This leads to an inductive construction of the desired contact structure in [Section 6](#). Throughout, it is helpful to keep in mind the following

Guiding principle Constructions must be kept transverse to the flow. Moreover, when constructing a positive contact structure, the slope of the characteristic foliation of a partially constructed confoliation must be greater than or equal the slope of the intersection of the given foliation of \mathcal{F} and vertical boundary of our flow boxes, with equality allowed only where \mathcal{F} is smooth. When constructing a negative contact structure, the slope inequality is reversed.

To explain this principle more formally, suppose that a closed oriented 3-manifold M is expressed as a union of smooth submanifolds V and W , possibly with corners, with $\partial V = \partial W$. Suppose, moreover, that W admits a codimension-1 foliation \mathcal{F}_W . These submanifolds will be chosen so that their common boundary decomposes into *horizontal* and *vertical* portions, that is, portions tangent and transverse, respectively, to \mathcal{F}_W . If a confoliation ξ_V has been constructed on V so that it is tangent to the horizontal portion of ∂V , transverse to the vertical portion, and contact on certain prescribed portions of V , then we call V a *contact zone*.

The guiding principle is a statement that the confoliation ξ_V on V must *dominate* the foliation \mathcal{F}_W along the vertical boundary (see [Definition 6.4](#)) both for an initial choice of V , and also for subsequent choices as V is expanded to all of M , and W is shrunk correspondingly.

To expand a contact zone V to the entire manifold M , we use the following structure.

Definition 2.4 A foliation \mathcal{F}_W is *V-transitive* if every point in W can be connected by a path in a leaf of \mathcal{F} to a point of V .

We will see in [Theorem 6.10](#) that the following structure is very useful.

Definition 2.5 A closed 3-manifold M admits a *positive (respectively, negative)* $(\xi_V, \mathcal{F}_W, \Phi)$ *decomposition* if M can be decomposed as a union

$$M = V \cup W,$$

where the horizontal portion of ∂W is tangent to \mathcal{F}_W and the vertical portion of ∂V is tangent to Φ , and:

- (1) \mathcal{F}_W is a V -transitive oriented foliation of W .
- (2) ξ_V is a smooth contact structure defined on V which positively (respectively, negatively) dominates \mathcal{F}_W .
- (3) For some choice of Riemannian metric, M admits a volume-preserving flow Φ transverse to both ξ_V and \mathcal{F}_W .

Note that the existence of a $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition does not require the existence of a codimension-1 foliation defined on all of M .

Theorem 6.10 *If M admits a positive $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition, then M admits a smooth positive contact structure which agrees with ξ_V on V and is Φ -close to \mathcal{F}_W on W . The analogous result holds if M admits a negative $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ decomposition. If M admits both a positive $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition and a negative $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ decomposition, then these contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

Question 2.6 *If a closed oriented 3-manifold admits both a positive $(\xi_V, \mathcal{F}_W, \Phi)$ and a negative $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ decomposition, does it contain a taut oriented foliation transverse to Φ ?*

Definition 2.7 *Given a splitting $M = V \cup W$ and a flow Φ , an oriented, codimension-1 foliation \mathcal{F} is compatible with (V, W, Φ) if \mathcal{F} is transverse to Φ , and the common boundary $\partial V = \partial W$ decomposes into subsurfaces which are either horizontal or vertical with respect to \mathcal{F} .*

Definition 2.8 *An oriented codimension-1 foliation \mathcal{F} of a 3-manifold M is bracketed if, for some volume-preserving flow Φ :*

- (1) \mathcal{F} is compatible with some (V, W, Φ) decomposition of M for which there exist \mathcal{F}_W and ξ_V such that $(\xi_V, \mathcal{F}_W, \Phi)$ is a positive decomposition.
- (2) \mathcal{F} is compatible with some (V', W', Φ) decomposition of M for which there exist $\mathcal{F}_{W'}$ and $\xi_{V'}$ such that $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ is a negative decomposition.

When we wish to specify the flow Φ , \mathcal{F} is called Φ -bracketed.

Corollary 2.9 *Let \mathcal{F} be an oriented codimension-1 foliation of a 3-manifold M which is Φ -bracketed. Then there exist a smooth positive contact structure ξ_+ and a smooth negative contact structure ξ_- , both Φ -close to \mathcal{F} . These contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

Sometimes \mathcal{F}_W and $\mathcal{F}_{W'}$ are obtained by restricting \mathcal{F} to W and W' respectively, and sometimes they are not. Very roughly speaking, when $W = W'$, we think of the restriction of \mathcal{F} to W as being *bracketed by* $\mathcal{F}_{W'}$ and \mathcal{F}_W as a generalization of the situation in which the slope of \mathcal{F} along boundary components of W lies between the corresponding boundary slopes of $\mathcal{F}_{W'}$ and \mathcal{F}_W .

As noted in [Corollary 7.3](#), all taut, oriented C^2 -foliations apart from the product foliation $S^1 \times S^2$ are bracketed. In [Example 8.8](#), we show that the product foliation of $S^1 \times S^2$ is not bracketed. In this paper, we show that many taut, oriented $C^{1,0}$ -foliations are bracketed. For each bracketed foliation considered in this paper, it is possible to choose $V = V'$ and $W = W'$.

Conjecture 2.10 *Let \mathcal{F} be a taut oriented $C^{1,0}$ -foliation of a closed oriented 3-manifold $M \neq S^1 \times S^2$. Then \mathcal{F} is bracketed.*

We have the following two closely related questions.

Question 2.11 Suppose \mathcal{F} is a taut, oriented foliation with no torus leaf. Is \mathcal{F} C^0 -close to a taut, oriented smooth foliation?

Question 2.12 Suppose \mathcal{F} is a taut, oriented foliation with no torus leaf. Is \mathcal{F} Φ -close to a taut, oriented smooth foliation for some volume-preserving flow Φ ?

Establishing the initial contact zone is of fundamental importance. In the context of $C^{1,0}$ -foliation theory we introduce, in [Section 7](#), the notion of *holonomy with a contracting interval* and define what we mean by *attracting neighborhood*. This is significantly weaker than the more familiar notion of linear attracting holonomy, yet it suffices to build an initial contact zone. The precise definition appears as [Definition 7.1](#). As a corollary to [Theorem 6.10](#), we obtain:

Theorem 7.2 *Let \mathcal{F} be a taut $C^{1,0}$ -foliation transverse to a flow Φ . If V is a disjoint union of attracting neighborhoods, and \mathcal{F} is V -transitive, then \mathcal{F} is bracketed and hence can be Φ -approximated by a pair of contact structures ξ_{\pm} , one positive and one negative. These contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

When working with $C^{1,0}$ -foliations, it can be difficult, or even impossible, to establish the existence of sufficient attracting holonomy. Therefore, we introduce a different way of creating an initial contact zone. Roughly speaking, instead of looking for loops tangent to the foliation and satisfying a nice property, we look for loops *transverse* to the foliation and satisfying a nice property. We make this precise in [Definition 8.4](#), where we define *L -bracketed foliation*. As a corollary to [Theorem 6.10](#), we obtain:

Theorem 8.5 *Suppose \mathcal{F} is a taut oriented codimension-1 foliation in M , and that \mathcal{F} is L -bracketed for some link L . Then \mathcal{F} is bracketed and hence can be Φ -approximated by a pair of smooth contact structures ξ_{\pm} , one positive and one negative. These contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

In Section 9.6, we consider the important special case that \mathcal{F} is transverse to a flow Φ that has been obtained by removing a fibered link L and doing a Dehn filling of a volume-preserving suspension flow. In this case, L forms the binding of an open book decomposition (S, h) of M and the contact structure $\xi_{(S, h)}$ compatible with (S, h) is Φ -close to \mathcal{F} .

In [32], Honda, Kazez and Matić introduced the use of foliations Φ -close to $\xi_{(S, h)}$ as a way of establishing universal tightness of $\xi_{(S, h)}$. In particular, they appealed to $C^{1,0}$ -foliations constructed in [48; 49] to claim that $\xi_{(S, h)}$ is universally tight whenever the binding of (S, h) is connected and the fractional Dehn twist coefficient at least one. Although the foliations constructed in [48; 49] are not smooth, and therefore the proof in [32] contained a gap, they are L -bracketed, and hence Theorem 8.5 reveals that the conclusions of [32] are correct.

In Section 9 we also include some background material relating language arising in the theory of open books with language arising in the theory of foliations. In particular, we give a translation between coordinates used in each subject together with a summary of our results related to open book decompositions.

To make the paper more self-contained there is an appendix containing an overview of the relationship between volume-preserving flows and closed dominating 2-forms, and giving some standard definitions from symplectic topology. Most of this material is present either implicitly or explicitly in [12].

We close this section with an application of Theorem 6.10 to the study of L -spaces.

Definition 2.13 [46, Definition 1.1] *A closed three-manifold Y is called an L -space if $H_1(Y; \mathbb{Q}) = 0$ and $\widehat{HF}(Y)$ is a free abelian group of rank $|H_1(Y; \mathbb{Z})|$.*

Theorem 2.14 [45, Theorem 1.4] *An L -space has no symplectic semifilling with disconnected boundary; and all its symplectic fillings have $b_2^+(W) = 0$. In particular, Y admits no taut smooth foliation.*

In other words, Ozsváth and Szabó show that if Y is an L -space then there is no symplectic manifold (X, ω) with weakly convex boundary such that $|\partial X| > 1$ and Y

is one of the boundary components. So an L-space cannot contain a pair of Φ -close contact structures, ξ_+ positive and ξ_- negative, where Φ is a volume-preserving flow. (For details, see the Appendix.) [Theorem 6.10](#) thus implies the following.

Corollary 2.15 *An L-space Y admits no bracketed foliation.*

In particular, the foliations constructed in [\[33; 42; 48; 49\]](#) never exist in an L-space.

3 Foliation basics

Definition 3.1 Let M be a smooth closed 3-manifold, and let k be a nonnegative integer or infinity. A C^k codimension-1 foliation \mathcal{F} of (or in) M is a union of disjoint connected surfaces L_i , called the leaves of \mathcal{F} , such that:

- (1) $\bigcup_i L_i = M$.
- (2) There exists a C^k atlas \mathcal{A} on M which contains all C^∞ charts and with respect to which \mathcal{F} satisfies the following local product structure:

For every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in \mathcal{A} about p such that $U \approx \mathbb{R}^3$ and the restriction of \mathcal{F} to U is the union of planes given by $z = \text{constant}$.

If the tangent plane field $T\mathcal{F}$ exists and is continuous, \mathcal{F} is called $C^{1,0}$. If \mathcal{F} is $C^{1,0}$ and all leaves are smoothly immersed, then \mathcal{F} is called $C^{\infty,0}$. (Compare with [\[5, Definition 1.2.22\]](#) and see [\[34, Definition 2.2\]](#).)

When $T\mathcal{F}$ is continuous, there is a continuous, and hence a smooth, 1-dimensional foliation transverse to \mathcal{F} . Moreover, it follows by [Theorem 3.3](#) that if \mathcal{F} is $C^{1,0}$, then it can be C^0 approximated by a $C^{\infty,0}$ foliation. Thus in seeking to approximate a $C^{1,0}$ foliation, no generality is lost in assuming the leaves are smoothly immersed.

A frequently used technique for constructing foliations is to start with a branched surface embedded in M that has product complementary regions. Since the embedding may be smoothed, a foliation resulting from thickening the branched surface and extending across the complementary regions can be constructed to be $C^{1,0}$.

[Definition 3.1](#) extends in an obvious way to define a codimension-1 foliation on a compact oriented smooth 3-manifold with nonempty boundary, where we insist that for each torus boundary component T , either T is a leaf of \mathcal{F} , or \mathcal{F} is everywhere transverse to T , and that any nontorus boundary component is a leaf of \mathcal{F} .

Recall that a smooth structure with corners on a topological 3–manifold M with nonempty boundary is a maximal collection of smoothly compatible charts with corners whose domains cover M , where a chart with corners is an open set diffeomorphic to one of \mathbb{R}^3 , $\{(x, y, z) \mid z \geq 0\}$, $\{(x, y, z) \mid y, z \geq 0\}$, or $\{(x, y, z) \mid x, y, z \geq 0\}$. Notice that the boundary of a manifold with corners naturally admits a stratification as a disjoint union of 0–, 1–, and 2–dimensional manifolds. The 0– and 1–manifolds of this stratification are referred to as the corners of M .

Definition 3.1 extends further in an obvious way to define a codimension-1 foliation on a compact smooth 3–manifold M with corners, where we insist that ∂M can be written as a union of two compact piecewise linear surfaces $\partial_v M$ and $\partial_h M$, where the intersection $\partial_v M \cap \partial_h M$ is a union of corners of M , the components of $\partial_h M$ are contained in leaves of \mathcal{F} , and $\partial_v M$ is everywhere transverse to \mathcal{F} .

Definition 3.2 A *flow* is an oriented 1–dimensional foliation of M ; namely, a decomposition Φ of a smooth compact 3–manifold M into a disjoint union of connected 1–manifolds, called the *flow curves* of Φ , such that there exists a C^k atlas \mathcal{A} on M which contains all C^∞ charts and with respect to which Φ satisfies the following local product structure:

For every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in \mathcal{A} about p such that $U \approx \mathbb{R}^3$, and the restriction of Φ to U is the union of lines given by $(x, y) = \text{constant}$.

When M has boundary a disjoint union of tori, we insist that for each torus boundary component T , either Φ is everywhere tangent to T or Φ is everywhere transverse to T . When M is smooth with corners, we insist that ∂M can be written as a union of two compact piecewise linear surfaces $\partial_v M$ and $\partial_h M$, where the intersection $\partial_v M \cap \partial_h M$ is a union of corners of M , $\partial_h M$ is everywhere transverse to Φ , and $\partial_v M$ is everywhere tangent to Φ .

Flows and oriented codimension-1 foliations coexist in interesting ways. A good overview can be found in [5]. In particular, given an oriented $C^{1,0}$ –foliation \mathcal{F} of an oriented 3–manifold M , possibly with nonempty boundary and possibly with corners, there is a C^∞ flow everywhere transverse to \mathcal{F} . From this we have the following:

Theorem 3.3 [4; 35] *Suppose \mathcal{F} is a $C^{1,0}$ foliation in M . Then there is an isotopy of M taking \mathcal{F} to a $C^{\infty,0}$ foliation \mathcal{G} with smoothly immersed leaves which is C^0 close to \mathcal{F} . If Φ is a smooth flow transverse to \mathcal{F} , the isotopy may be taken to map each flow line of Φ to itself.*

When the oriented foliation \mathcal{F} is taut, and M is Riemannian, the smooth transverse flow can be chosen to be volume-preserving.

Theorem 3.4 ([51, Theorem II.20]; see also [29, Theorem A1]) *Let \mathcal{F} be a codimension-1, taut $C^{1,0}$ -foliation of a closed smooth Riemannian 3-manifold M . Then there is a volume-preserving smooth flow everywhere transverse to \mathcal{F} . Equivalently, there is a smooth closed 2-form dominating \mathcal{F} .*

Given a 3-manifold M containing a taut $C^{1,0}$ -foliation, one can ask whether there is a closely related C^∞ -foliation. Interpreting “closely related” to mean any of C^0 - ϵ -close for some fixed $\epsilon > 0$, Φ -close, or topologically conjugate results in questions for which the answers are very little understood. There are certainly 3-manifolds which contain Reebless $C^{1,0}$ -foliations but not Reebless C^∞ -foliations [3, Theorem D]. The existence of a taut sutured manifold hierarchy guarantees the existence of two types of foliation, one $C^{1,0}$ and finite depth and the other C^∞ [17; 21; 22]. Fixing a Riemannian metric and some $\epsilon > 0$, these two types of foliation are not necessarily C^0 - ϵ -close. However, since they are carried by a common transversely oriented branched surface, they are Φ -close.

We will take advantage of the fact that it is always possible *locally* to C^0 -approximate $(\mathcal{F}, T\mathcal{F})$ by $(\tilde{\mathcal{F}}, T\tilde{\mathcal{F}})$, for some locally defined smooth foliation $\tilde{\mathcal{F}}$.

Proposition 3.5 *Let D be a smooth disk with corners and let \mathcal{F} be a $C^{1,0}$ -foliation of $D^2 \times [0, 1]$ which is positively transverse to the smooth 1-dimensional foliation by intervals $\{(x, y)\} \times [0, 1]$, $(x, y) \in D^2$. Given any $\epsilon > 0$, there is a smooth foliation $\tilde{\mathcal{F}}$ which is positively transverse to the smooth 1-dimensional foliation by intervals $\{(x, y)\} \times [0, 1]$, $(x, y) \in D^2$, and satisfies $(\tilde{\mathcal{F}}, T\tilde{\mathcal{F}})$ is C^0 ϵ -close to $(\mathcal{F}, T\mathcal{F})$. Moreover, if \mathcal{F} is smooth on some compact \mathcal{F} -saturated subset, then we may choose $\tilde{\mathcal{F}}$ to equal \mathcal{F} on this subset.*

Proof By identifying D with a subset of the plane, a point p in a leaf of \mathcal{F} determines both a point in \mathbb{R}^3 , and by choosing a unit vector perpendicular to the tangent plane of the leaf, a point \mathbf{u}_p in $T\mathbb{R}^3$. The standard metric on $T\mathbb{R}^3 = \mathbb{R}^6$ is used to measure the distance between two leaves of \mathcal{F} as follows.

By Theorem 3.3, we may assume that the leaves of \mathcal{F} are given by the graphs of $z = f_\theta(x, y)$, for some continuous family of smooth functions $f_\theta: D \rightarrow [0, 1]$, $0 \leq \theta \leq 1$. For any two such leaves, L_1 and L_2 say, given by $z = f_{\theta_1}(x, y)$ and $z = f_{\theta_2}(x, y)$ respectively, define the distance between them to be the maximum distance, computed in $T\mathbb{R}^3$, between $(x, y, f_{\theta_1}(x, y), \mathbf{u}_{(x,y,f_{\theta_1}(x,y))})$ and $(x, y, f_{\theta_2}(x, y), \mathbf{u}_{(x,y,f_{\theta_2}(x,y))})$

for $(x, y) \in D$. Since D is compact, uniform continuity guarantees that d is continuous and hence a metric on the leaf space of \mathcal{F} .

For any $\theta \in [0, 1]$, let U_θ denote the subset of $D \times [0, 1]$ which is the union of all graphs $z = f_\theta(x, y)$ which are of d -distance strictly less than $\epsilon/2$ from the leaf $z = f_\theta(x, y)$. Since U_θ is the pullback of an $\epsilon/2$ d -neighborhood in the leaf space, U_θ is open in $D \times [0, 1]$.

Pick a finite cover of $D \times [0, 1]$ by $U_{\theta_0}, U_{\theta_1}, \dots, U_{\theta_r}$ for some $r \geq 0$ and $0 = \theta_0 < \theta_1 < \dots < \theta_r = 1$. Now let $\tilde{\mathcal{F}}$ be the foliation of $D \times [0, 1]$ which includes the leaves given by the graphs of f_{θ_i} and, for each $i, 0 \leq i \leq r - 1$, the leaves given by the graphs of a damped straight line homotopy between f_{θ_i} and $f_{\theta_{i+1}}$. Thus if g is a smooth homeomorphism of $[0, 1]$ with derivatives at 0 and 1 vanishing to infinite order, the leaves of $\tilde{\mathcal{F}}$ are $z = (1 - g(t))f_{\theta_i}(x, y) + g(t)f_{\theta_{i+1}}(x, y), 0 \leq t \leq 1$, on the subset of $D \times [0, 1]$ bounded by the graphs of f_{θ_i} and $f_{\theta_{i+1}}$. By construction, $\tilde{\mathcal{F}}$ is smooth.

Moreover, $(\tilde{\mathcal{F}}, T\tilde{\mathcal{F}})$ and $(\mathcal{F}, T\mathcal{F})$ are ϵ -close. To see this, recall that a normal vector to a graph $z = f(x, y)$ is given by $\mathbf{n}_f = \langle -f_x, -f_y, 1 \rangle$, and a straight-line homotopy between f_{θ_1} and f_{θ_2} induces a straight-line homotopy between $\mathbf{n}_{f_{\theta_1}}$ and $\mathbf{n}_{f_{\theta_2}}$. Normalizing this straight-line homotopy of normal vectors gives a geodesic on the unit sphere joining $\mathbf{u}_{f_{\theta_1}}$ and $\mathbf{u}_{f_{\theta_2}}$. Since the leaves given by $z = f_{\theta_i}(x, y)$ and $z = f_{\theta_{i+1}}(x, y)$ are of d -distance at most $\epsilon/2$, it follows immediately from the triangle inequality that the leaves given by $z = f_{(1-g(t))\theta_i+g(t)\theta_{i+1}}(x, y)$ and $z = (1 - g(t))f_{\theta_i}(x, y) + g(t)f_{\theta_{i+1}}(x, y)$ are of d -distance strictly less than ϵ . So $(\tilde{\mathcal{F}}, T\tilde{\mathcal{F}})$ and $(\mathcal{F}, T\mathcal{F})$ are ϵ -close.

If \mathcal{F} is smooth on some compact \mathcal{F} -saturated subset A of $D \times [0, 1]$, each component of A is bounded by graphs of the form f_θ . By compactness of A , ∂A contains only finitely many such f_θ . For each $z = f_\theta$ in ∂A , include θ in the list $\theta_0, \theta_1, \dots, \theta_r$ and modify \mathcal{F} only on the complement of A . □

Next we recall Operation 2.1.1 of [23]. Let L_1, \dots, L_m be distinct leaves of a $C^{1,0}$ -foliation \mathcal{F} . Modify \mathcal{F} by thickening each of the leaves L_j . Thus, each L_j is blown up to an I -bundle $L_j \times [-1, 1]$. Let \mathcal{F}' denote the resulting foliation. We highlight the following observation.

Lemma 3.6 *The leaves L_1, \dots, L_m may be thickened so that the foliation \mathcal{F}' is $C^{1,0}$ and the restriction of \mathcal{F}' to*

$$L_j \times (-1, 1) \subset M$$

is a smooth foliation for each j .

4 Transitive flow box decompositions

Definition 4.1 A flow box, F , for a $C^{1,0}$ -foliation \mathcal{F} and smooth transverse flow Φ , is a smooth chart with corners that is of the form $D \times I$, where D is a polygon (a disk with at least three corners), Φ intersects F in the arcs $\{(x, y)\} \times I$, and \mathcal{F} intersects F in disks which are everywhere transverse to Φ and hence can be thought of as graphs over D . In particular, $D \times \partial I$ lies in leaves of \mathcal{F} , each component of $\mathcal{F} \cap F$ is a smoothly embedded disk, and these disks vary continuously in the I direction.

The vertical boundary of F , denoted $\partial_v F$, is $\partial D \times I$. The horizontal boundary of F is $D \times \partial I$ and is denoted $\partial_h F$. An arc in M is vertical if it is a subarc of a flow line and horizontal if it contained in a leaf of \mathcal{F} .

It is often useful to view the disk D as a 2-cell with ∂D the cell complex obtained by letting the vertices correspond exactly to the corners of D . Similarly, it is useful to view the flow box F as a 3-cell possessing the product cell complex structure of $D \times I$. Then the horizontal boundary $\partial_h F$ is a union of two (horizontal) 2-cells and the vertical boundary $\partial_v F$ is a union of c (vertical) 2-cells, where c is the number of corners of D .

A subset R of F is called a vertical rectangle if it has the form $\alpha \times [a, b]$, where $0 \leq a < b \leq 1$ and α is either a 1-cell of ∂D or else a properly embedded arc in D connecting distinct vertices of D . A subset e of F is called an edge if it is a compact interval contained in a 1-cell of F .

Given a vector \vec{w} tangent to $\partial_v F$, we choose a slope convention such that the leaves of $\mathcal{F} \cap \partial_v F$ have slope 0, the flow lines have slope ∞ , and the sign of the slope of \vec{w} is computed as viewed from outside of F .

Definition 4.2 Given a codimension-1 leafwise smooth foliation \mathcal{F} and transverse smooth flow Φ , let V be a compact codimension-0 submanifold of M , with $\partial V = \partial_v V \cup \partial_h V$, where $\partial_v V$ is a union of flow arcs or circles, and $\partial_h V$ is a union of subsurfaces of leaves of \mathcal{F} . In the case that $\partial V = \partial_v V$, \mathcal{F} and Φ need only be defined on the complement of V . A flow box decomposition of M rel V is a decomposition of $M \setminus \text{int } V$ as a finite union $M = V \cup (\bigcup_i F_i)$, where:

- (1) Each F_i is a standard flow box for \mathcal{F} .
- (2) If $i \neq j$, the interiors of F_i and F_j are disjoint.
- (3) If F_i and F_j are different flow boxes, then their intersection is connected and either empty, a 0-cell, an edge, a vertical rectangle, or a subdisk of $\partial_h F_i \cap \partial_h F_j$.

Definition 4.3 We call a flow box decomposition $M = V \cup F_1 \cup \cdots \cup F_n$ *transitive* if $V_0 = V$, $V_i = V_{i-1} \cup F_i$, and for $i = 1, \dots, n$:

- (1) Each 2–cell of $\partial_v F_i$ has interior disjoint from $\partial_h F_j$ for all $j < i$.
- (2) $V_{i-1} \cap F_i$ is a union of horizontal subsurfaces and vertical 2–cells of F_i , together possibly with some 0– and 1–cells.
- (3) $V_{i-1} \cap F_i$ contains a vertical 2–cell of F_i .

Proposition 4.4 *If M is V –transitive, then there is a transitive flow box decomposition of M rel V .*

Proof Since M is V –taut, for each point $x \in M \setminus V$, there exists γ an embedded arc in the leaf containing x that connects x to V . By taking a regular neighborhood of γ in its leaf and flowing along it, create a flow box F with the property that it has one vertical 2–cell contained in $\partial_v V$. The point x may or may not be in the interior of F . Using compactness of M , pick a finite collection of flow boxes, of the sort just described, F_1, F_2, \dots, F_r that cover $M \setminus V$. Assume no proper subset of the F_i , $1 \leq i \leq r$, cover.

Next, let L_1, L_2, \dots, L_m be the collection of leaves of \mathcal{F} that contain the horizontal boundaries of all F_i .

We proceed by induction to show that $V \cup F_1 \cup \cdots \cup F_i$ admits a flow box decomposition with respect to V for every i , $1 \leq i \leq r$. Certainly, $V \cup F_1$ does. So suppose that $V \cup F_1 \cup \cdots \cup F_{i-1}$ admits a flow box decomposition with respect to V . After renaming and reindexing as necessary, we assume that this flow box decomposition is given by $V \cup F_1 \cup \cdots \cup F_{i-1}$. We show that $V_i = V \cup F_1 \cup \cdots \cup F_{i-1} \cup F_i$ also admits a flow box decomposition with respect to V .

Begin by slightly increasing the size of F_i in $M \setminus \text{int } V$, as necessary, so that F_i is still a flow box and, for all $j < i$, $\partial_v F_j$ and $\partial_v F_i$ are transverse away from V , where they may overlap tangentially. Notice that this ensures that $V \cup F_1 \cup \cdots \cup F_{i-1} \cup F_i$ is a codimension-0 submanifold with corners and piecewise vertical and horizontal boundary. Also, cut F_i open along those (horizontal disk) components of $(\bigcup_i L_i) \cap F_i$ which have nonempty intersection with $\partial_h F_j$ for some $j < i$. Denote the resulting flow boxes by F_i^1, \dots, F_i^s ; so $F_i = F_i^1 \cup \cdots \cup F_i^s$.

Consider $F_j \cap F_i^1$ for some $j < i$. Since $\partial_v F_j$ and $\partial_v F_i^1$ are transverse away from V , each component of $F_j \cap F_i^1$ is a flow box. Consider any such component, X say, from the point of view of the flow box $F_i^1 = D_i \times [c, d]$. Notice that $X = D \times [c, d]$, where D is a subdisk (with corners) of D_i , and $X \cap \partial_v F_i^1$ is a nonempty union of vertical 2–cells.

Now, for all $j < i$, remove $F_j \cap F_i^1$ from F_i^1 . Taking the closure of the result we get a union $G_1 \cup \dots \cup G_b$ of flow boxes, where each $G_k \cap V_{i-1}$ is a union of horizontal subsurfaces and vertical 2-cells of G_k , together possibly with some 0- and 1-cells, and contains a vertical 2-cell of G_k . Notice that $G_k \cap G_l = \emptyset$ if $k \neq l$ and, by subdividing each G_k along finitely many vertical rectangles as necessary, we may assume $G_k \cap F_j$ is connected for all $j < i$. The resulting union

$$V \cup F_1 \cup \dots \cup F_{i-1} \cup G_1 \cup \dots \cup G_b$$

is then a transitive flow box decomposition of $V \cup F_1 \cup \dots \cup F_{i-1} \cup F_i^1$ with respect to V .

Repeat this process for each a , $2 \leq a \leq s$, to obtain a transitive flow box decomposition of $V \cup F_1 \cup \dots \cup F_i$ with respect to V . □

5 Basic extension results

In this section, we collect together an assortment of confoliation extension results important for the inductive construction to be described in Section 6. For the most part, it will be possible to restrict attention to flow boxes diffeomorphic to one of the flow boxes F , G or H , where F , G and H are defined as follows.

Let F denote the flow box given by

$$F = \{|x| \leq 1, 0 \leq y \leq 1, |z| \leq 1\}.$$

Let G denote the flow box given by $\Delta \times [0, 1]$, where Δ is the region in the xy -plane bounded by the triangle with vertices

$$\left(-\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right) \text{ and } \left(0, -\frac{1}{2}\right).$$

Let $\Delta^{(0)}$ denote the 0-skeleton

$$\Delta^{(0)} = \left\{\left(-\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right), \left(0, -\frac{1}{2}\right)\right\}.$$

Let $H = F \cap G$, a flow box with hexagonal horizontal cross-section. Notice that H is diffeomorphic to the complement in G of an open neighborhood of the 1-skeleton of $\partial_v G$.

We begin with the following elementary, and very useful, observations of Eliashberg and Thurston [12].

Lemma 5.1 [12, Proposition 1.1.5] *Let η be a C^k -confoliation with $k \geq 1$ and domain F given by a 1-form $dz - a(x, y, z)dx$. Then*

$$\frac{\partial a}{\partial y}(x, y, z) \geq 0$$

at all points of F , and

$$\frac{\partial a}{\partial y}(x, y, z) > 0,$$

where η is contact.

Eliashberg and Thurston use this lemma to approximate the confoliation by a contact structure with the following corollary.

Corollary 5.2 [12, Lemma 2.8.2] *Let η be a C^k -confoliation, with $k \geq 1$ and domain F , given by a 1-form $dz - a(x, y, z)dx$. If η is contact in a neighborhood of $y = 1$ in F , then η can be approximated by a confoliation $\hat{\eta}$ which coincides with η together with all of its derivatives along the boundary ∂F and which is contact inside F .*

Proof It is enough to approximate $a(x, y, z)$ along each interval $\{x\} \times [0, 1] \times \{z\}$ by a function $\hat{a}(x, y, z)$ that is strictly monotonic for $(x, z) \in (-1, 1) \times (-1, 1)$ but is damped to agree smoothly with $a(x, y, z)$ on ∂F . \square

Definition 5.3 If α and β are families of curves transverse to $\partial/\partial z$ and contained in a vertical 2-cell R of the vertical boundary of a flow box, we say α *strictly dominates* β along $A \subset R$ if at every $p \in A$, the slope of the tangent to α at p is greater than the slope of the tangent to β at p . It must be specified if the comparison of slopes is made from inside or outside of the flow box. If α strictly dominates A , and α and β are the characteristic foliations of 2-plane fields ξ_1 and ξ_2 respectively, we also say that ξ_1 strictly dominates ξ_2 along R . If $\xi_2 = T\mathcal{F}$ for some codimension-1 foliation \mathcal{F} , we also say that ξ_1 strictly dominates \mathcal{F} along R .

The statement of [Lemma 5.1](#) raises the question of whether flow box coordinates can always be chosen so that the contact form can be written as $dz - a(x, y, z)dx$. The next lemma points out that this is the case and gives a simple condition for a contact structure to dominate in such coordinates.

Lemma 5.4 *Let U be a regular neighborhood in F of the union of $x = \pm 1$ and $z = \pm 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of $y = 1$ in F which is everywhere transverse to the vertical segments $(x, y) = \text{constant}$,*

horizontal in \bar{U} , and contact on $V \setminus \bar{U}$. Then, after smoothly reparametrizing F as necessary, we may assume that η is given by a 1-form

$$dz - a(x, y, z)dx$$

with

- (1) $a(x, y, z) = 0$ on $V \cap \bar{U}$,
- (2) $\frac{\partial a}{\partial y}(x, y, z) > 0$ on $V \setminus \bar{U}$.

Moreover, the characteristic foliation of η along the complement of \bar{U} in $y = 1$ strictly dominates the horizontal foliation, when viewed from inside F , if and only if $a(x, y, z) > 0$ in $V' \setminus \bar{U}$, for some neighborhood $V' \subset V$ of $y = 0$.

Proof Since $dz - a(x, y, z)dx$ vanishes on $\partial/\partial y$, it is enough to choose coordinates x, y for leaves so that curves with constant x coordinate are Legendrian. At points where η is transverse to the horizontal foliation, there is a unique Legendrian direction. At all other points, any direction is Legendrian. The coordinate y can be constructed by choosing a section of the Legendrian directions. □

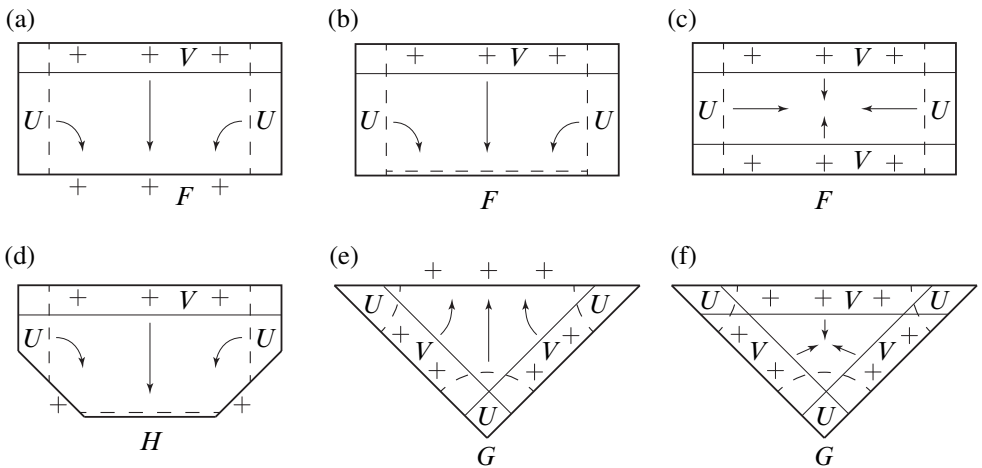


Figure 1: Each figure shows a $z = 0$ slice capturing the flow box setup of one of Corollaries 5.5–5.10. Plus signs are positioned on the side from which the contact structure dominates the horizontal foliation. Dashes, for instance along U , show where the confoliation slope is 0. The smooth foliation acts as a transport mechanism for contact structures in the direction shown by the arrows.

Corollary 5.5 (Figure 1a) Let U be a regular neighborhood in F of the union of $x = \pm 1$ and $z = \pm 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of $y = 1$ in F by a 1-form

$$dz - a(x, y, z)dx$$

with

- (1) $a(x, y, z) = 0$ on $V \cap \bar{U}$,
- (2) $a(x, y, z) > 0$ and $\frac{\partial a}{\partial y}(x, y, z) > 0$ on $V \setminus \bar{U}$.

Then η can be extended to a C^k -confoliation $\hat{\eta}$ on F that agrees with the horizontal foliation of F in \bar{U} , is contact on the complement of \bar{U} , and strictly dominates, when viewed from outside F , the horizontal foliation on the complement in $y = 0$ of \bar{U} .

Proof It is enough to extend $a(x, y, z)$ along each interval $\{x_0\} \times [0, 1] \times \{z_0\}$ to a C^k function $\hat{a}(x, y, z)$ such that

- (1) $\hat{a}(x, y, z) = 0$ in \bar{U} ,
- (2) $\hat{a}(x, y, z) > 0$ and $\frac{\partial \hat{a}}{\partial y}(x, y, z) > 0$ outside \bar{U} .

This completes the proof. □

Corollary 5.6 (Figure 1b) Let U be a regular neighborhood in F of the union of $x = \pm 1$ and $z = \pm 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of $y = 1$ in F by a 1-form

$$dz - a(x, y, z)dx$$

with

- (1) $a(x, y, z) = 0$ on $V \cap \bar{U}$,
- (2) $a(x, y, z) > 0$ and $\frac{\partial a}{\partial y}(x, y, z) > 0$ on $V \setminus \bar{U}$.

Then η can be extended to a C^k -confoliation $\hat{\eta}$ on F that agrees with the horizontal foliation of F in \bar{U} , is contact on the complement of \bar{U} in the interior of F , and smoothly agrees with the horizontal foliation at $y = 0$.

Proof Proceed as in the proof of Corollary 5.5 except insist that

$$\hat{a}(x, 0, z) \equiv 0. \quad \square$$

Corollary 5.7 (Figure 1c) *Let U be a regular neighborhood in F of the union of $x = \pm 1$ and $z = \pm 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of the union of $y = 0$ and $y = 1$ in F . Suppose that when viewed from inside F , η dominates the horizontal foliation along the vertical faces given by $y = 0$ and $y = 1$, with strict domination in the complement of \bar{U} . Then η can be extended to a C^k -confoliation $\hat{\eta}$ on F that agrees with the horizontal foliation of F in \bar{U} , and is contact on the complement of \bar{U} .*

Proof Decompose F as a union of two flow boxes diffeomorphic to F by cutting open along the plane $y = \frac{1}{2}$. Apply Corollary 5.6 to each of the resulting flow boxes. \square

The point of “smoothly agrees with” in the next corollary is that flow boxes are brick-like objects that, when sensibly glued together, should define a smooth confoliation. Thus we require smooth convergence of the confoliation to horizontal at $y = 0$.

Corollary 5.8 (Figure 1d) *Let U be a regular neighborhood in F of the union of $x = \pm 1$ and $z = \pm 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of $y = 1$ in F by a 1-form*

$$dz - a(x, y, z)dx$$

with

- (1) $a(x, y, z) = 0$ on $V \cap \bar{U}$,
- (2) $a(x, y, z) > 0$ and $\frac{\partial a}{\partial y}(x, y, z) > 0$ on $V \setminus \bar{U}$.

Then η can be extended to a C^k -confoliation $\hat{\eta}$ on H that agrees with the horizontal foliation of H in \bar{U} , is contact on the complement of \bar{U} in the interior of H , smoothly agrees with the horizontal foliation at $y = 0$, and dominates the horizontal foliation in the complement of \bar{U} along the lines

$$y = x - \frac{1}{2} \quad \text{and} \quad y = -\frac{1}{2} - x.$$

Proof This follows from Corollary 5.6. \square

Corollary 5.9 (Figure 1e) *Let U be a regular neighborhood in G of the union of $z = \pm 1$ and $\Delta^{(0)} \times [-1, 1]$. Let S denote the complement in $\partial_v G$ of the 2-cell given by $y = 1$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of S in G such that:*

- (1) η is contact on $V \setminus \bar{U}$.
- (2) η agrees with the horizontal foliation on $V \cap \bar{U}$.
- (3) η strictly dominates the horizontal foliation along $S \setminus \bar{U}$, when viewed from inside G .

Then η can be extended to a C^k -confoliation $\hat{\eta}$ on G that agrees with the horizontal foliation of G in \bar{U} , is contact on the complement of \bar{U} in the interior of G , and, when viewed from outside G , dominates the horizontal foliation in the complement of \bar{U} along the line $y = 1$.

Proof Let α be a smooth arc properly embedded in Δ which connects the vertices $(\frac{3}{2}, 1)$ and $(0, -\frac{1}{2})$ and is not tangent to a side of the triangle Δ at its endpoints. Let $R = \alpha \times [-1, 1]$, a vertical rectangle in G . Decompose G along R into two flow boxes as $G = G' \cup F'$, where G' is diffeomorphic to G and F' is diffeomorphic to F . First apply Corollary 5.8 to G' and then apply Corollary 5.5 to F' . \square

Corollary 5.10 (Figure 1f) *Let U be a regular neighborhood in G of the union of $z = \pm 1$ and $\Delta^{(0)} \times [-1, 1]$. Let η be a C^k -confoliation with $k \geq 1$ defined in a neighborhood V of $\partial_v G$ in G such that:*

- (1) η is contact on $V \setminus \bar{U}$.
- (2) η agrees with the horizontal foliation on $V \cap \bar{U}$.
- (3) η strictly dominates the horizontal foliation along $\partial_v G \setminus \bar{U}$, when viewed from inside G .

Then η can be extended to a C^k -confoliation $\hat{\eta}$ on G that agrees with the horizontal foliation of G in \bar{U} and is contact on the complement of \bar{U} in the interior of G .

Proof This time we cut G open along two vertical rectangles and consider instead the resulting union of flow boxes. This time we cut G open along two disjoint vertical rectangles to obtain

$$G = G' \cup F' \cup F'',$$

where G' is diffeomorphic to G and each of F' and F'' is diffeomorphic to F . Then apply Corollary 5.8 to G' . Finally, apply Corollary 5.5 to each of F' and F'' . \square

We now consider a case where the initial confoliation is defined on the entire vertical boundary of a solid cylinder. Let \mathcal{L} be a smooth 1-dimensional foliation on the cylinder $S^1 \times I$ such that the boundary components of $S^1 \times I$ are leaves of \mathcal{L} . Let $h: I \rightarrow I$ be the holonomy map of \mathcal{L} .

Proposition 5.11 *If $h'(z) < 0$ for $z \in (0, 1)$, there is a confoliation on $D^2 \times I$ that is contact on $D^2 \times (0, 1)$, tangent on $D^2 \times \partial I$, has characteristic foliation \mathcal{L} , and is everywhere transverse to $\partial/\partial z$.*

Proof This will follow from Lemmas 5.12–5.14. □

Lemma 5.12 *If $h'(z) < 0$ for $z \in (0, 1)$, then there exists a smooth 1–dimensional foliation \mathcal{K} of $S^1 \times I$ with the same holonomy map $h: I \rightarrow I$ such that for every $(\theta, z) \in S^1 \times (0, 1)$, the slope $s(\theta, z)$ of \mathcal{K} at (θ, z) is negative.*

Proof As a first approximation, let \mathcal{K}_1 be the foliation of $[0, 2\pi] \times I$ given by connecting each point $(0, z)$ to $(2\pi, h(z))$ by a straight line. Next create $S^1 \times I$ by identifying $(0, z)$ and $(2\pi, z)$ for $z \in I$. Let \mathcal{K}_2 denote the image of \mathcal{K}_1 in $S^1 \times I$. This has the desired properties, except that \mathcal{K}_2 is not smooth along $\{0\} \times I$. Carefully rounding these corners (see for example [43, Lemma 4.7]), yields the desired smooth foliation. □

Lemma 5.13 *There is a diffeomorphism, F , of $D^2 \times I$ that is the identity map on $D^2 \times \partial I$ and takes \mathcal{L} to \mathcal{K} .*

Proof Let $\theta = 0$ be a base point for S^1 so that the holonomy map for each foliation is $h: \{0\} \times I \rightarrow \{0\} \times I$. Let f be the diffeomorphism of $S^1 \times I$ such that f restricts to the identity map on $\{0\} \times I$, preserves the S^1 coordinate, and maps \mathcal{L} to \mathcal{K} .

Define $F: D^2 \times I \rightarrow D^2 \times I$ by $F(r, \theta, z) = (r, t(r)f(\theta, z) + (1 - t(r))(\theta, z))$, where t is a diffeomorphism of the interval smoothly damped at the endpoints. □

Lemma 5.14 *There is a confoliation ξ on $D^2 \times I$ that is contact on $D^2 \times (0, 1)$, tangent on $D^2 \times \partial I$, has characteristic foliation \mathcal{K} , and is everywhere transverse to $\partial/\partial z$.*

Proof Using cylindrical coordinates, define $\alpha = dz - r^2s(\theta, z)d\theta$. Then $d\alpha = -2rs(\theta, z)drd\theta - r^2s_z(\theta, z)dzd\theta$, from which it follows that

$$\alpha \wedge d\alpha = -2s(\theta, z)rdrd\theta dz.$$

Setting $\xi = \ker \alpha$ has the desired properties. □

6 The inductive construction of ξ

In this section, we show how a $C^{1,0}$ -foliation can be used to propagate a contact structure across M . Before describing this procedure, we highlight the role of smoothness in the approach used by Eliashberg and Thurston in performing this propagation.

First, recall [Lemma 5.1](#). Roughly speaking, a clever choice of foliation coordinates permits a confoliation along a Legendrian curve to be described by a monotone function. By taking advantage of a beginning contact zone, such a function can be approximated by a strictly monotone function, thereby creating a larger contact zone. This argument can be repeated on overlapping regions covering the manifold. The issue is whether strict monotonicity attained on a given region can be preserved under subsequent approximations. This is precisely where smoothness of the foliation becomes important, guaranteeing that derivatives are globally defined and continuous, and thus allowing one to preserve strict monotonicity under subsequent approximations.

We circumvent the issue of monotonicity with carefully chosen, minimally overlapping, flow boxes and a more discrete propagation technique, which we now describe.

Recall [Definition 5.3](#) and the slope convention chosen in [Definition 4.1](#). Let F be a flow box and let R be a rectangle in $\partial_v F$. Let χ_{ξ_1} and χ_{ξ_2} denote the characteristic foliations induced on R by two 2-plane fields ξ_1 and ξ_2 defined in a neighborhood of R and positively transverse to Φ . We write $\chi_{\xi_1} <_p \chi_{\xi_2}$ if the unit vector tangent to χ_{ξ_1} at p has slope less than the unit vector tangent to χ_{ξ_2} at p . Similarly, we write $\chi_{\xi_1} =_p \chi_{\xi_2}$ if the unit vector tangent to χ_{ξ_1} at p has slope equal to the unit vector tangent to χ_{ξ_2} at p . We write $\chi_{\xi_1} < \chi_{\xi_2}$ if $\chi_{\xi_1} <_p \chi_{\xi_2}$ for all $p \in \text{int}(R)$. Similarly, we write $\chi_{\xi_1} \leq \chi_{\xi_2}$ if $\chi_{\xi_1} \leq_p \chi_{\xi_2}$ for all $p \in \text{int}(R)$.

Definition 6.1 Let M be a closed oriented 3-manifold with smooth flow Φ . Suppose that M decomposes as a union

$$M = V \cup W,$$

where V and W are smooth 3-manifolds, possibly with corners, and $\partial V = \partial W$. We say that this decomposition is *compatible with the flow* Φ if ∂V (and hence ∂W) decomposes as a union of compact subsurfaces $\partial_v V \cup \partial_h V$, where $\partial_v V$ is a union of flow segments of Φ and, $\partial_h V$ is transverse to Φ . In the presence of a foliation transverse to the flow, the notation $\partial_h V$ will be used for the portion of ∂V tangent to the foliation.

Definition 6.2 Let M be a closed oriented 3-manifold with smooth flow Φ . Suppose that M can be expressed as a union

$$M = V \cup W,$$

where V and W are smooth 3-manifolds, possibly with corners, such that $\partial V = \partial W$. Suppose also that this decomposition is compatible with Φ , that V admits a smooth contact structure ξ_V , and that W admits a $C^{1,0}$ -foliation \mathcal{F}_W . We say that (V, ξ_V) is Φ -compatible with (W, \mathcal{F}_W) , and that M admits a positive $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition, if the following are satisfied:

- (1) ξ_V and \mathcal{F}_W are (positively) transverse to Φ on their domains of definition.
- (2) ξ_V is tangent to $\partial_h V$.
- (3) $\chi_{\xi_V} < \chi_{T\mathcal{F}_W}$ on the interior of $\partial_v V$, when viewed from outside V .

The main result of the section is [Theorem 6.10](#). The starting point is a transitive flow box decomposition $M = V \cup F_1 \cup \dots \cup F_n$ for \mathcal{F} . Set $V_0 = V$ and $V_i = V \cup F_1 \cup \dots \cup F_i$ for each $i, 1 \leq i \leq n$. For each $i, 0 \leq i \leq n$, set $W_i = M \setminus \text{int}(V_i)$. Thus, for $0 \leq i \leq n$, \mathcal{F} is compatible with (V_i, W_i, Φ) . When $i = 0$, $\partial_h V_i = \emptyset$, and when $i = n$, $V_n = M$ and hence $\partial V_n = \emptyset$.

Lemma 6.3 *After possibly blowing up finitely many leaves of \mathcal{F} , we may assume that:*

- (1) \mathcal{F} is smooth in a neighborhood of the horizontal faces $\partial_h F_i$.
- (2) \mathcal{F} is smooth in a neighborhood of the vertical edges of each F_i .

Proof We will define ξ inductively and in a piecewise fashion over each F_i . To guarantee that the resulting confoliation is everywhere smooth, we add a *smoothly foliated collar* about the horizontal boundary and the vertical 1-simplices of each F_i as follows. Let L_1, L_2, \dots, L_m be the collection of leaves of \mathcal{F} that contain the horizontal boundaries of all F_i . Modify the original foliation \mathcal{F} by thickening each of the leaves L_j . Thus each L_j is replaced with a product of leaves $L_j \times [-1, 1]$. The thickening should be performed so that for each i , if a component of $\partial_h F_i$ was originally contained in L_j , then it is now contained in $L_j \times \{0\}$. As noted in [Lemma 3.6](#), we may assume the interior of each thickening, $L_j \times (-1, 1)$, is smoothly immersed. Note that $M = V \cup F_1 \cup \dots \cup F_n$ remains a transitive flow box decomposition of M with respect to this new foliation. Moreover, $\mathcal{F} \cap F_i$ is smooth in a neighborhood of $\partial_h F_i$ for each i .

The vertical edges of the F_i are smooth transverse arcs, thus \mathcal{F} can be smoothed on $D^2 \times I$ product neighborhoods of these edges. □

Now we fix a preferred regular neighborhood of the union of the horizontal 2-cells and the vertical 1-cells of the flow box decomposition. We do this as follows. For each $i, 1 \leq i \leq n$, choose a regular neighborhood of $\partial_h F_i$ which is contained in the

thickening $L_i \times (-1, 1)$, and choose a regular neighborhood of the vertical 1–cells of F_i on which \mathcal{F} is smooth. Choose these neighborhoods so that the union U of these neighborhoods is a regular neighborhood of the union of all horizontal 2–cells and all vertical 1–cells of the flow box decomposition. We refer to both this preferred neighborhood U and the restriction of \mathcal{F} to U as the *smoothly foliated collar*.

Next, we modify [Definition 6.2](#) slightly to account for the smoothly foliated collar.

Definition 6.4 Let $0 \leq i < n$ and let ξ_i be a smooth confoliation on V_i such that ξ_i is tangent to $\partial_h V_i$ and everywhere transverse to $\partial_v V_i$. (Note that by smoothness, we may consider ξ_i to be defined on an open neighborhood of V_i .) We say the smooth confoliation ξ_i *dominates* \mathcal{F} if $\chi_{\xi_i} \geq \chi_{\mathcal{F}}$ on $\partial_v V_i$ when viewed from outside V_i , with equality permitted only on the closure \bar{U} of the smoothly foliated collar. Use *strictly dominates* if the inequality is strict.

Let ξ_i be a smooth confoliation defined on V_i . We say that (V_i, ξ_i) is *smoothly Φ –compatible* with (W_i, \mathcal{F}) if the following are satisfied:

- (1) ξ_i and \mathcal{F}_W are (positively) transverse to Φ on their domains of definition.
- (2) ξ_i is tangent to $\partial_h V_i$.
- (3) $\xi_i = T\mathcal{F}$ on $\bar{U} \cap V_i$.
- (4) ξ_i is a contact structure on $V_i \setminus U$.
- (5) ξ_i dominates \mathcal{F} (on $\partial_v V_i$).

Proposition 6.5 Let $M = V \cup F_1 \cup \dots \cup F_n$ be a transitive flow box decomposition of M , and let ξ_i , $i \geq 0$, be a smooth confoliation defined on V_i that is smoothly Φ –compatible with (W_i, \mathcal{F}) . Then there is a smooth confoliation ξ_{i+1} defined on V_{i+1} that is smoothly Φ –compatible with (W_{i+1}, \mathcal{F}) and restricts to ξ_i on V_i .

Proof Since any polygon admits a triangulation, any transitive flow box decomposition can be chosen to consist only of flow boxes diffeomorphic to the flow box $G = \Delta \times [0, 1]$ defined in [Section 5](#). In particular, we may assume that F_{i+1} is diffeomorphic to G . Such a diffeomorphism preserves the foliation and flow directions, that is, slopes 0 and ∞ ; thus we will make slope comparisons and approximations without reference to the change of coordinates.

By hypothesis, ξ strictly dominates \mathcal{F} on $X = \partial_v V_i \setminus U$. By compactness, there exists $\epsilon > 0$ such that

$$\text{slope}(\xi_i) - \text{slope}(\mathcal{F}) > 3\epsilon$$

on $X \cap \partial_v F_{i+1}$.

By Proposition 3.5, we may approximate the restriction of \mathcal{F} to F_{i+1} by a smooth foliation $\tilde{\mathcal{F}}$ such that $(\tilde{\mathcal{F}}, T\tilde{\mathcal{F}})$ is C^0 -close to $(\mathcal{F}, T\mathcal{F})$. Choose this approximation so that ξ_i dominates $\tilde{\mathcal{F}}$ on $\partial_v(F_{i+1}) \cap V_i$, and so that

$$|\text{slope}(\tilde{\mathcal{F}}) - \text{slope}(\mathcal{F})| < \epsilon$$

on all of $\partial_v F_{i+1}$. It follows that

$$\text{slope}(\xi_i) - \text{slope}(\tilde{\mathcal{F}}) > 2\epsilon$$

on $X \cap \partial_v F_{i+1}$. Choose smooth coordinates (x, y, z) on F_{i+1} so that the leaves of $\tilde{\mathcal{F}}$ are horizontal, given by $z = \text{constant}$. (Note that although this change of coordinates might change slope values, it doesn't affect the relative values of slopes.)

Next consider the number of 2-cells contained in $\partial_v(F_{i+1}) \cap V_i$. From the definition of a transitive flow box decomposition there is at least one. Depending on whether there are exactly one, two, or three such 2-cells, apply the corresponding Corollary 5.8, 5.9, or 5.10 to smoothly extend ξ_i across F_{i+1} and call the resultant confoliation ξ_{i+1} . Smoothness of the glued confoliation is assured by the construction of the confoliations near the boundaries of their flow boxes.

In constructing the extensions of Corollaries 5.7–5.10 the starting point is a 1-form on $\partial_v F_{i+1}$ given by $dz - a(x, y, z)dx$ with $a(x, y, z) > 0$ on $X \cap \partial_v F_{i+1}$. This is then extended across F_{i+1} while keeping $a(x, y, z) > 0$ and also $\frac{\partial a}{\partial y}(x, y, z) > 0$. Given any $\delta > 0$, these constructions can be performed while keeping the change in $a(x, y, z)$ along Legendrian curves less than δ . In other words, the extension ξ_{i+1} can be chosen so that the change in $\text{slope}(\xi_{i+1})$ along Legendrian curves is less than ϵ . Thus

$$\text{slope}(\xi_{i+1}) - \text{slope}(\tilde{\mathcal{F}}) > \epsilon$$

on $\partial_v V_{i+1} - (N_h \cup N_v)$, and consequently, on this set we also have

$$\text{slope}(\xi_{i+1}) - \text{slope}(\mathcal{F}) > 0. \quad \square$$

Corollary 6.6 *Let $M = V \cup F_1 \cup \dots \cup F_n$ be a transitive flow box decomposition of M , and let ξ_i , $i \geq 0$, be a smooth confoliation defined on V_i that is smoothly Φ -compatible with (W_i, \mathcal{F}) and lies within ϵ of \mathcal{F} on the intersection of the domain of \mathcal{F} with V_i . Then there is a smooth confoliation ξ_{i+1} defined on V_{i+1} that is smoothly Φ -compatible with (W_{i+1}, \mathcal{F}) , restricts to ξ_i on V_i , and lies within 3ϵ of $T\mathcal{F}$ on the intersection of the domain of \mathcal{F} with V_{i+1} .*

Proof It follows from Proposition 3.5 that a smooth foliation $\tilde{\mathcal{F}}$ may be chosen on F_{i+1} so that $T\tilde{\mathcal{F}}$ is within ϵ of $T\mathcal{F}$. Restricting attention to $\partial_v V \cap F_{i+1}$, $T\tilde{\mathcal{F}}$ lies within 2ϵ of ξ_i . From the proof of Proposition 6.5, ξ_{i+1} is constructed to be as close

or closer to $T\tilde{\mathcal{F}}$ on F_{i+1} than ξ_i is on $\partial_v V \cap F_{i+1}$. Thus ξ_{i+1} is within 2ϵ of $T\tilde{\mathcal{F}}$, and hence within 3ϵ of $T\mathcal{F}$. \square

Corollary 6.7 *Let $M = V \cup F_1 \cup \dots \cup F_n$ be a transitive flow box decomposition of M , and let ξ_0 be a smooth contact structure defined on V that is compatible with (W, \mathcal{F}) . Then there is a smooth contact structure defined on M that agrees with ξ_0 in V and is Φ -close to \mathcal{F} on W .*

Proof Inductively applying [Proposition 6.5](#) produces a transitive smooth confoliation that is Φ -close to \mathcal{F} on W and agrees with ξ_0 in V . By [[12](#), Proposition 2.8.1] (see also [[14](#)]), this transitive smooth confoliation can be smoothly deformed into a smooth contact structure. Thus there is a smooth contact structure defined on M that agrees with ξ_0 in V and is Φ -close to \mathcal{F} on W . \square

Corollary 6.8 *Let $M = V \cup F_1 \cup \dots \cup F_n$ be a transitive flow box decomposition of M , and let ξ_0 be a smooth confoliation defined on V that is Φ -compatible with (W, \mathcal{F}) . Then there is a smooth contact structure defined on M that agrees with ξ_0 in V and is Φ -close to \mathcal{F} on W . If ξ_0 is within ϵ of $T\mathcal{F}$ along $\partial V = \partial W$, then ξ can be chosen to lie within $f(n)\epsilon$ of $T\mathcal{F}$ in W , for some positive function f .*

Proof This follows immediately from [Corollary 6.6](#). \square

Lemma 6.9 *Let M be a closed oriented 3-manifold. Suppose that M can be expressed as a union*

$$M = V \cup W,$$

such that $\partial V = \partial W$, V admits a smooth contact structure ξ_0 , W admits a $C^{1,0}$ -foliation \mathcal{F} , and (V, ξ_0) and (W, \mathcal{F}) are Φ -compatible. Then ξ_0 can be modified in an arbitrarily small collar neighborhood of ∂V so that the restriction of ξ_0 to $\partial_v V$ lies arbitrarily C^0 -close to $T\mathcal{F}$.

Proof Fix any $\epsilon > 0$. Let $X = (\partial_v V) - (N_h \cup N_v)$. Let $X \times (-\delta, 0]$ be a collar neighborhood of X in V , with $X = X \times \{0\}$. Pick any smooth line field l on X which is tangent to \mathcal{F} along the smoothly foliated collar, is dominated by the projection of $\xi_0|_{X \times \{-\delta\}}$ to X , dominates the projection of $\mathcal{F}|_{X \times \{0\}}$ to X , and lies within ϵ of $\mathcal{F}|_{X \times \{0\}}$. Replace the restriction of ξ_0 to $X \times [-\delta, 0]$ by a straight line homotopy between ξ_0 restricted to $X \times \{-\delta\}$ and l , damped to fit smoothly with the original ξ_0 as defined on the complement of the collar. \square

Theorem 6.10 *If M admits a positive $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition, then M admits a smooth positive contact structure ξ_+ which agrees with ξ_V on V and is Φ -close to \mathcal{F}_W on W . Moreover, M also admits a smooth contact structure which agrees with ξ_0 on the complement in V of a collar neighborhood of ∂V and is arbitrarily C^0 -close to \mathcal{F} on W . The analogous result holds if M admits a negative $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ decomposition. If M admits both a positive $(\xi_V, \mathcal{F}_W, \Phi)$ decomposition and a negative $(\xi_{V'}, \mathcal{F}_{W'}, \Phi)$ decomposition, then these contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

Proof This follows from [Proposition 4.4](#), [Corollary 6.7](#), and [Lemma 6.9](#). □

7 Attracting holonomy

This section gives a generalization of the Eliashberg–Thurston result on perturbing foliations in a neighborhood of a curve in a leaf with sometimes attracting holonomy [[12](#), [Proposition 2.5.1](#)] to the larger class of $C^{1,0}$ -foliations with holonomy containing a *contracting interval*. Holonomy with a contracting interval will be defined below in terms of the existence of a particular smooth submanifold with corners. Holonomy with a contracting interval is a weaker condition than holonomy with an attracting leaf.

Let P be the prism in \mathbb{R}^3 given by $|x| \leq 1$, $|y| \leq 1$ and $|z| \leq y/2 + \frac{3}{2}$. The slanted top and bottom of P is denoted by $\partial_h P$, and the rest of ∂P is denoted by $\partial_v P$. Let V be the solid torus given by identifying each pair of points $(x, -1, z)$ and $(x, 1, z)$ where both are in P . Then $\partial_h V$ is defined to be the image of $\partial_h P$, and $\partial_v V$ is defined to be the portion of ∂V that is in the image of $\partial_v P$.

Definition 7.1 The holonomy of a $C^{1,0}$ -foliation \mathcal{F} transverse to a flow Φ in a 3-manifold M has a *contracting interval* if there is a subset of M diffeomorphic to V , called an *attracting neighborhood*, such that:

- (1) $\partial_h V$ is mapped into leaves of \mathcal{F} .
- (2) vertical intervals in V are mapped to flow lines of Φ .

Theorem 7.2 *Let \mathcal{F} be a taut oriented $C^{1,0}$ -foliation transverse to a flow Φ . If V is a disjoint union of attracting neighborhoods, and \mathcal{F} is V -transitive, then \mathcal{F} can be Φ -approximated both by a smooth positive contact structure ξ_+ and by a smooth negative contact structure ξ_- . These contact structures (M, ξ_+) and $(-M, \xi_-)$ are weakly symplectically fillable and universally tight.*

Proof In light of [Theorem 6.10](#), it is enough to construct a smooth confoliation ξ on V such that (V, ξ) is compatible with (W, \mathcal{F}) , in the sense of [Definition 6.4](#). It is sufficient to consider the case that ξ is positive. The construction of the desired ξ is the same on each component of V , so we will treat V as if it is connected. Let W denote the closure of the complement of V in M .

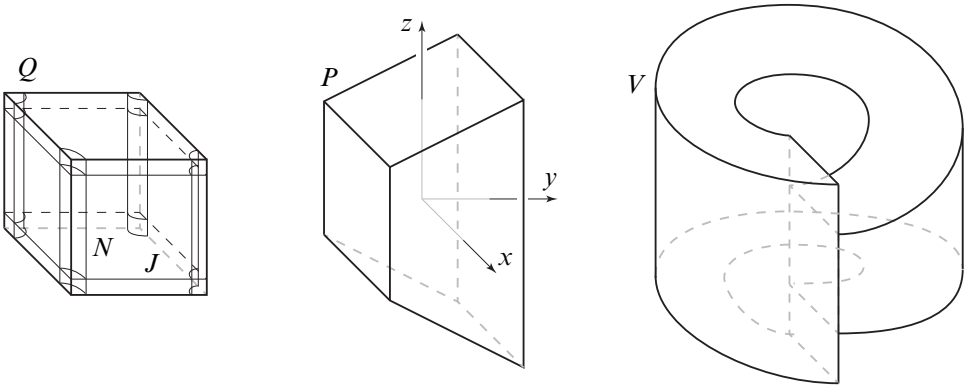


Figure 2: Foliated neighborhoods of horizontal faces and vertical edges are shown only in Q though they exist in P and $V = P/\sim$ as well

Consider the transformation of the prism P to the cube

$$Q = [-1, 1] \times [-1, 1] \times [-1, 1]$$

that fixes (x, y) and linearly scales the z -coordinate. View V as the quotient

$$V = Q/\sim$$

of Q obtained by identifying

$$(x, -1, z) \sim (x, 1, z/2).$$

Notice that $\partial_h Q$ is mapped into leaves of \mathcal{F} , and vertical intervals in Q are mapped to flow lines of Φ . Moreover, we may assume that the original parametrization of P was chosen so that \mathcal{F} meets the $y = \pm 1$ sides of $\partial_v Q$ in horizontal lines. Since \mathcal{F} is $C^{1,0}$, the leaves of the foliation meet each of the $x = \pm 1$ sides in a continuous family of smooth graphs.

To facilitate smooth gluings, thicken the leaf or leaves of \mathcal{F} which meet $\partial_h Q$. Fix $0 < \epsilon < \frac{1}{4}$. Choose the thickening of the leaves of \mathcal{F} intersecting $\partial_h V$ to replace the leaves $z = \pm 1$ in Q with a disjoint union, J , of I -bundles in Q , with J containing $[-1, 1] \times [-1, 1] \times [-1, -1 + \epsilon]$ and $[-1, 1] \times [-1, 1] \times [1 - \epsilon, 1]$ as components. Assume also that \mathcal{F} meets an ϵ -neighborhood, N , of the quotient of the vertical edges of Q

in a smooth horizontal foliation. Let $U_Q = \text{int}(J \cup N)$; U_Q is an ϵ -neighborhood of the union of $\partial_h Q$ with the 1-cells of $\partial_v Q$. We assume also that the pull-back of \mathcal{F} to $\overline{U_Q}$ is horizontal. We will abuse notation and let U_Q also refer to the projection of U_Q to M .

Since Q is a flow box it is amenable to the constructions of Section 5. We will Φ -approximate \mathcal{F} in Q , and thus in V , by a C^∞ -confoliation ξ_0 which smoothly respects both the identification \sim and the gluing of (V, ξ_0) and (W, \mathcal{F}) along $\partial_h V = \partial_h W$.

The confoliation ξ_0 will be chosen to agree with \mathcal{F} on U_Q and to be a contact structure on $Q \setminus \overline{U_Q}$ which, when viewed from outside Q , is strictly dominated by \mathcal{F} on the $y = -1$ side of $\partial_v Q$ and strictly dominates \mathcal{F} on the remaining three sides of $\partial_v Q$. See Figure 3. Since the interior of the $y = -1$ side lies in the interior of V , the resulting (V, ξ_0) will be compatible with (W, \mathcal{F}) in the sense of Definition 6.4.

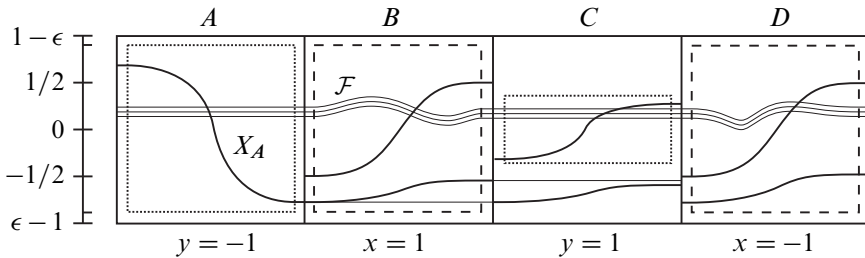


Figure 3: Some of the choices of flow lines of the vector fields X_A, \dots and their relationship to \mathcal{F} are shown. Not enough detail is drawn to show that the holonomy given by flowing from left to right, that is counterclockwise about $\partial_v Q$, is decreasing.

As a first step in constructing ξ_0 , we define ξ_0 along $\partial_v Q$. The vertical boundary $\partial_v Q$ consists of four vertical faces. Let A denote the face $y = -1$, let B denote the face $x = 1$, let C denote the face $y = 1$, and let D denote the face $x = -1$.

We construct ξ_0 by first specifying smooth unit vector fields X_A, X_B, X_C , and X_D along the faces A, B, C , and D respectively, and then declaring ξ_0 along $\partial_v Q$ to be the 2-plane field which is normal to $\partial_v Q$ and contains the corresponding tangent vector X_A, X_B, X_C , or X_D . We will choose X_A, X_B, X_C , and X_D to be horizontal on $\overline{U_Q}$ and hence in a neighborhood of the vertical 1-simplices of $\partial_v Q$; in particular, the 2-plane field ξ_0 will therefore be well-defined on the vertical edges.

Begin by defining the vector field X_B . Choose $X_B = X_B(1, y, z)$ to be a smooth unit vector field which satisfies the following:

- (1) X_B dominates \mathcal{F} .
- (2) X_B has positive slope when both y and z lie in $(-1 + \epsilon, 1 - \epsilon)$.
- (3) $X_B = \partial/\partial y$ when y or z lies in $[-1, -1 + \epsilon] \cup [1 - \epsilon, 1]$.

Let Ψ_B denote the flow generated by X_B . Abusing notation a bit, we denote by $\Psi_B(y, z)$ the intersection of the flow line of Ψ_B that starts at $(1, -1, z)$ with $\{(1, y)\} \times [-1, 1]$. (We use this notation when referring to all flows in this section.) Let $f_B: [-1, 1] \rightarrow [-1, 1]$ denote the diffeomorphism given by $f_B(z) = \Psi_B(1, z)$.

Note that since X_B has positive slope whenever both t and z lie in $(-1 + \epsilon, 1 - \epsilon)$, $f_B(z) > z$ whenever z lies in $(-1 + \epsilon, 1 - \epsilon)$. Indeed, by choosing the slope of X_B to be great enough, we may guarantee that $f_B(-\frac{1}{2}) \geq \frac{1}{2}$. Rechoose X_B as necessary so that X_B satisfies (1)–(3) and also:

- (4) $f_B(-\frac{1}{2}) \geq \frac{1}{2}$.

Now choose X_D along the side $x = -1$ by setting $X_D(-1, y, z) = X_B(1, -y, z)$.

Notice that if $f: [-1, 1] \rightarrow [-1, 1]$ is any orientation-preserving diffeomorphism, then there is a smooth flow $\Psi = \Psi(x, z)$ on $[-1, 1] \times [-1, 1]$ such that $\Psi(1, z) = f(z)$; simply set

$$\Psi(x, z) = \frac{1 - t(x)}{2}z + \frac{1 + t(x)}{2}f(z),$$

where $t(x)$ is a smooth function of x that is -1 for $x \in [-1, -1 + \epsilon]$, 1 for $x \in [1 - \epsilon, 1]$, and has positive derivative for all other x . Thus, by specifying a diffeomorphism $f: [-1, 1] \rightarrow [-1, 1]$, we specify a family of smooth flows $\Psi(x, z)$ and corresponding smooth unit tangent vector fields X .

We take advantage of this to define the vector field X_A along the side $y = -1$. Let $f_A: [-1, 1] \rightarrow [-1, 1]$ be the diffeomorphism given by

$$f_A^{-1}(z) = u \circ f_B \circ f_B \circ f_B(z),$$

where $u: [-1, 1] \rightarrow [-1, 1]$ is a diffeomorphism which is the identity on $z \in [-1, -1 + \epsilon] \cup [1 - \epsilon, 1]$ and strictly increasing elsewhere. Let Ψ_A be a smooth flow on $[-1, 1] \times [-1, 1]$ which satisfies the following:

- (1) Ψ_A has negative slope whenever $z \in (-1 + \epsilon, 1 - \epsilon)$.
- (2) Ψ_A has unit tangent vector field given by $\partial/\partial x$ when $z \in [-1, -1 + \epsilon] \cup [1 - \epsilon, 1]$.
- (3) $\Psi_A(1, z) = f_A(z)$.

Let X_A be the smooth unit tangent vector field to Ψ_A . Recall that \mathcal{F} is horizontal, and hence is dominated by X_A , along A .

Similarly, we use a diffeomorphism $f_C: [-1, 1] \rightarrow [-1, 1]$ to define a smooth vector field X_C along the side $y = 1$. Let $f_C: [-1, 1] \rightarrow [-1, 1]$ be a diffeomorphism which satisfies:

- (1) $f_C(z) = \frac{1}{2} f_A(2z)$ when $|z| \leq \frac{1}{2}$.
- (2) $f_C(z) = z$ when $z \in [-1, -1 + \epsilon] \cup [1 - \epsilon, 1]$.
- (3) $f_C(z) < f_B(z)$ when $\frac{1}{2} < |z| < 1 - \epsilon$.

Since $f_A(z) = z$ whenever $|z| \geq 1 - \epsilon$, $f_C(z) = z$ for $(1 - \epsilon)/2 \leq |z| \leq \frac{1}{2}$. Therefore, $f_C[-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, \frac{1}{2}]$. Since $f_B(-\frac{1}{2}) > \frac{1}{2}$, it follows that $f_C(z) \leq f_B(z)$ for all z , with equality only when $|z| \geq 1 - \epsilon$. So $f_B \circ f_C \circ f_B(z) < f_B \circ f_B \circ f_B(z)$, and hence

$$f_A \circ f_B \circ f_C \circ f_B(z) < z,$$

for all $z \in (-1 + \epsilon, 1 - \epsilon)$.

So the diffeomorphism $f_A \circ f_B \circ f_C \circ f_B: [-1, 1] \rightarrow [-1, 1]$ is strictly decreasing on $(-1 + \epsilon, 1 - \epsilon)$ and hence **Proposition 5.11** applies. □

Corollary 7.3 *Let \mathcal{F} be a taut, oriented, C^2 -foliation of a closed oriented 3-manifold. Suppose \mathcal{F} is not the product foliation $S^1 \times S^2$. Then \mathcal{F} is bracketed.*

Proof Let Φ be a volume-preserving flow transverse to \mathcal{F} . As noted in [12], either \mathcal{F} has a lot of nontrivial linear holonomy or it is C^0 close, and hence Φ -close, to a foliation \mathcal{F}' which has a lot of nontrivial linear holonomy. In other words, there exists a disjoint union V of attracting neighborhoods such that one of \mathcal{F} or \mathcal{F}' is V -transitive. In either case it follows that \mathcal{F} is bracketed. □

8 L -bracketed foliations

In this section, we introduce a new method for Φ -approximating a foliation \mathcal{F} by a pair of transitive confoliations, one positive and one negative. This method applies whenever there exists a link transverse to \mathcal{F} which satisfies the condition given in **Definition 8.4**. We also remark on some consequences yielding 3-manifolds containing weakly symplectically fillable contact structures.

Before stating **Definition 8.4**, we make some preliminary observations.

Proposition 8.1 *Suppose \mathcal{F} is a taut oriented codimension-1 foliation in M . Let L be any link transverse to \mathcal{F} . Then there is a choice of metric on M and volume-preserving flow Φ everywhere transverse to \mathcal{F} such that L is contained as a union of closed orbits of Φ . Moreover, given a choice of regular neighborhood $N(L) = \bigcup_i D_i \times S^1$ of L , the metric on M and Φ can be chosen so that Φ is a trivial product $\{p\} \times S^1$, $p \in D_i$, on this regular neighborhood; in particular, Φ restricts to a flow on $\partial N(L)$.*

Proof This follows immediately from the proof of Theorem A1 found in [29]. \square

So any link transverse to \mathcal{F} can be extended to a volume-preserving flow Φ transverse to \mathcal{F} . Alternatively, we may begin with a volume-preserving flow Φ and let L be a collection of closed orbits of Φ . Without loss of generality, we will restrict attention to the case that this flow Φ restricts to a flow on $\partial N(L)$ for some choice of regular neighborhood $N(L)$ of L . In either case, Φ determines a preferred, possibly noncompact, curve on each component of $\partial N(L)$:

Lemma 8.2 *Let T be a framed torus and let Φ be a flow on T . Then either Φ contains a simple closed curve of some rational slope m_Φ or Φ is topologically conjugate either to a foliation by lines of some irrational slope m_Φ or to a Denjoy blowup of a foliation by lines of some irrational slope m_Φ . In each case, the slope m_Φ is uniquely determined (by Φ).*

Proof As long as no leaf of Φ has slope $1/0$, the framing determines a unique realization of Φ as a suspension of some homeomorphism f of S^1 and the Poincaré rotation number of f determines the slope m_Φ . See, for example, [30, 4.3.1, 5.1.1 and 5.1.3]. \square

Denote this preferred isotopy class of curves, represented by either a simple closed leaf or an immersed \mathbb{R} , by m_Φ^T . We are interested in the case that this torus T is a component of $\partial N(L)$. In this case, there is also the isotopy class of the meridian, v^T say, and $v^T \neq m_\Phi^T$. We shall call an isotopy class of a nontrivial curve C in T *positive* if it has positive slope with respect to $\langle v, m_\Phi \rangle$ when viewed from inside N . Similarly, we shall call an isotopy class of curves *negative* if it has negative slope with respect to $\langle v, m_\Phi \rangle$ when viewed from inside N . This convention is illustrated in Figure 4.

Note that if \mathcal{F} is an oriented codimension-1 foliation which intersects a torus T transversely, then $\mathcal{F} \cap T$ is a flow on T , and $m_{\mathcal{F}}^T$ denotes the preferred isotopy class of this flow.

Definition 8.3 Define a triple (M, \mathcal{F}, Φ) to be *coherent* if the foliation \mathcal{F} is taut and oriented, the flow Φ is volume-preserving and positively transverse to \mathcal{F} , and the boundary of M , if nonempty, is a union of flow lines.

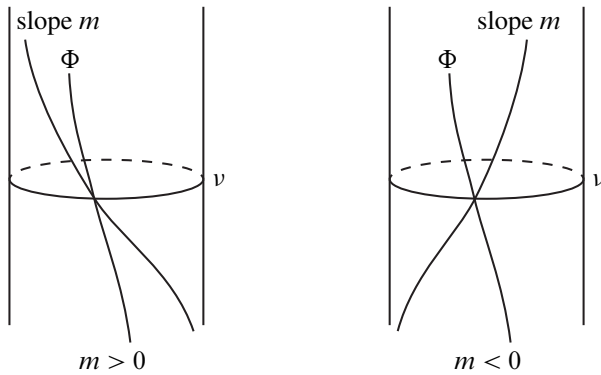


Figure 4: Slope convention on a component of $\partial N(L)$ in which slopes are designated as viewed from inside $N(L)$

Let L be a link in M . A foliation \mathcal{F} is L -taut if $\mathcal{F} \pitchfork L$, and \mathcal{F} is L -transitive, that is, each leaf of \mathcal{F} has nonempty intersection with L . Similarly, if $\partial M \neq \emptyset$, then \mathcal{F} is ∂M -taut if \mathcal{F} intersects ∂M transversely, with no Reeb annuli, and \mathcal{F} is ∂M -transitive, that is, each leaf of \mathcal{F} has nonempty intersection with ∂M .

Recall that a foliation \mathcal{F}_0 is said to realize slope m along a framed torus boundary component T if $\partial\mathcal{F}_0 \cap T$ consists of parallel curves, not necessarily compact, of slope x . When \mathcal{F}_0 is oriented, these curves $\partial\mathcal{F}_0 \cap T$ are necessarily consistently oriented. Notice that the condition that a foliation \mathcal{F}_0 be ∂M_0 -taut is weaker than the condition that $\partial\mathcal{F}_0$ realize slope $m_{\mathcal{F}_0}^T$ for each component T of ∂M_0 ; in other words, nontrivial holonomy is possible for $\mathcal{F}_0 \cap \partial M_0$.

Definition 8.4 Suppose \mathcal{F} is a taut oriented codimension-1 foliation in M . Let L be a link in M which is transverse to \mathcal{F} and let M_0 equal $M \setminus \text{int } N(L)$. Let \mathcal{F}_0 denote the restriction of \mathcal{F} to M_0 . The foliation \mathcal{F} is L -bracketed if, for some choice of metric on M_0 , there is a volume-preserving flow Φ_0 on M_0 such that $(M_0, \mathcal{F}_0, \Phi_0)$ is coherent and the following property is satisfied:

M_0 contains a pair of foliations \mathcal{F}_{\pm} such that:

- (1) $(M_0, \mathcal{F}_+, \Phi_0)$ and $(M_0, \mathcal{F}_-, \Phi_0)$ are coherent.
- (2) \mathcal{F}_{\pm} are ∂M_0 -taut.
- (3) For each component T of ∂M_0 , $m_{\mathcal{F}_-}^T$ is negative and $m_{\mathcal{F}_+}^T$ is positive with respect to $\langle v^T, \Phi|_T \rangle$, where v^T is the meridian slope of component T (and hence is the slope of $\mathcal{F}_0 \cap T$).

To make the flow explicit, we also say that M contains an (\mathcal{F}, Φ) -transitive link L , where Φ is the flow Φ_0 blown down to a (volume-preserving) flow on M .

The notion of L -bracketed is a special case of a bracketed foliation \mathcal{F} . The decomposition is given by setting $V = V' = N(L)$, $W = W' = M_0$. There is a canonical choice of positive or negative contact structure on $N(L)$ which is given by perturbing the meridional disks, and then the requirement is to find foliations $\mathcal{F}_W = \mathcal{F}_+$ and $\mathcal{F}_{W'} = \mathcal{F}_-$ on M_0 .

Theorem 8.5 *Suppose \mathcal{F} is a taut oriented codimension-1 foliation in M and that \mathcal{F} is L -bracketed for some link L . Then \mathcal{F} can be Φ -approximated by a pair of smooth contact structures ξ_{\pm} , one positive and one negative. These contact structures (M, ξ_+) and $(-M, \xi_-)$ are necessarily weakly symplectically fillable and universally tight.*

Proof Set $V = N(L)$, and let W denote the closure of the complement of V . Define a contact structure ξ_0 on $N(L)$ so that each component of L is a transverse knot and each component of $N(L)$ is a standard positive contact neighborhood of its core. Choosing the rate of rotation of the contact planes along each meridional disk to be small guarantees that the characteristic foliation of ξ_0 along $\partial N(L)$ is close to the meridian. It follows that we may choose ξ_0 so that it is strictly dominated by \mathcal{F}_+ . Apply [Theorem 6.10](#) to obtain ξ_+ . Similarly, each component of $N(L)$ can be modeled using the standard negative radial model, and [Theorem 6.10](#) can be applied to obtain ξ_- .

Since ξ_{\pm} are both positively transverse to the volume-preserving flow Φ , they are weakly symplectically fillable. \square

Since transitive links are somewhat mysterious, it is natural to ask:

Question 8.6 Given a foliation \mathcal{F} , does there exist a link L for which \mathcal{F} is L -bracketed?

It is not clear how the answer to this might change if the link is required to be connected.

Question 8.7 Given a foliation \mathcal{F} , does there exist a knot K for which \mathcal{F} is K -bracketed?

Example 8.8 The product foliation \mathcal{F} on $S^1 \times S^2$ is an example of a foliation which is not L -bracketed for any link L . The existence of such a link would imply, by [Theorem 8.5](#), that \mathcal{F} can be approximated by a tight contact structure ξ . This would imply that the underlying 2-plane bundles of \mathcal{F} and ξ are equivalent. The Euler class of the foliation, $e(\mathcal{F})$, evaluated on a spherical leaf S^2 of \mathcal{F} equals 2. On the other hand, this S^2 is homotopic to a convex surface S' [[24](#)] in a tight contact structure. It follows that S' has a connected dividing set [[25](#)] and that $e(\xi)$ vanishes on it.

A taut foliation \mathcal{F} is certainly L -taut for some link L . In fact, it is L -taut for some knot L . Moreover, as noted above, for some choice of metric there is a volume-preserving flow Φ transverse to \mathcal{F} and containing L as an orbit or union of closed orbits. And often, although not necessarily, foliations on M_0 that are Φ_0 -close to \mathcal{F} will also be L -taut. So a key question is the existence of a pair of Φ_0 -close foliations \mathcal{F}_\pm in M_0 such that $m_{\mathcal{F}_-}^T < \text{slope } \partial\mathcal{F}_0|_T < m_{\mathcal{F}_+}^T$ for each component T .

Example 8.9 Consider the case that M_0 is any compact orientable manifold with boundary a nonempty union of b tori. Suppose that Φ_0 is a volume-preserving flow which is tangent to ∂M_0 and that B is a transversely oriented branched surface transverse to Φ_0 .

If B fully carries a set of foliations which are ∂M_0 -taut and realize a nonempty open set J of boundary slopes (if $b = 1$) or multislopes (if $b \geq 2$), then Dehn-filling M_0 along any rational slope or multislope in J results in a foliation which is L -bracketed, where L is the link which is the core of the Dehn filling. Examples of such foliations can be found in the papers [7; 8; 18; 19; 20; 22; 33; 41; 42; 47; 48; 49].

One can ask whether the foliations constructed by Dehn filling more than one torus can be K -bracketed for some knot K .

In [17; 21; 22], Gabai constructs foliations in closed manifolds M with $H_2(M) \neq 0$. These foliations are fully carried by finite depth branched surfaces. One can ask whether such foliations are L -bracketed for some link L .

Finally, we note that the proof of [Theorem 8.5](#) doesn't actually require the existence of the foliation \mathcal{F}_0 . More precisely, we have the following.

Definition 8.10 Suppose (M, ξ) is a contact 3-manifold. Let L be a transverse link in (M, ξ) . Let M_0 equal $M \setminus \text{int } N(L)$. The contact structure ξ is L -bracketed if, for some choice of metric on M_0 , there is a volume-preserving flow Φ_0 on M_0 , tangent to ∂M_0 , such that the following property is satisfied:

M_0 contains a pair of foliations \mathcal{F}_\pm such that:

- (1) $(M_0, \mathcal{F}_+, \Phi_0)$ and $(M_0, \mathcal{F}_-, \Phi_0)$ are coherent.
- (2) \mathcal{F}_\pm are ∂M_0 -taut.
- (3) For each component T of ∂M_0 , $m_{\mathcal{F}_-}^T$ is negative and $m_{\mathcal{F}_+}^T$ is positive with respect to $\langle \nu^T, \Phi|_T \rangle$, where ν^T is the meridian slope of component T .

Theorem 8.11 Suppose (M, ξ) is a contact 3-manifold and that ξ is L -bracketed for some transverse link L . Then ξ can be Φ -approximated by a pair of smooth contact structures ξ_\pm , one positive and one negative. These contact structures are necessarily weakly symplectically fillable and universally tight.

9 Open book decompositions

An interesting class of L -bracketed foliations is obtained by considering the special case that L is a fibered link in M and \mathcal{F} is transverse to a flow Φ obtained by surgery from a volume-preserving suspension flow of the corresponding fiber bundle complement of L . In this case L forms the binding of an open book decomposition (S, h) of M and the contact structure $\xi_{(S, h)}$ compatible with (S, h) is Φ -close to \mathcal{F} .

For completeness, we begin with some standard definitions. Since we are relating ideas from the world of codimension-1 foliations and the world of contact structures, we will also provide some translations between the terminologies of these two worlds. The main results of this section appear in [Section 9.6](#).

9.1 Open book decompositions

Let S be a compact surface with nonempty boundary. A pair (S, h) , where h is a homeomorphism that restricts to the identity map on ∂S , determines a closed 3-manifold $M = S \times [0, 1] / \approx$, where the equivalence relation \approx identifies $(x, 1) \approx (h(x), 0)$ for all $x \in S$ and $(x, s) \approx (x, t)$ for all $x \in \partial S$ and $s, t \in [0, 1]$. The singular fibration with pages $S \times \{t\}$ is called the *open book determined by the data* (S, h) , and we write $M = (S, h)$.

9.2 Surface bundles over S^1

Corresponding to an open book decomposition of M is a description of M as a Dehn surgery along the binding $L = \bigcup_i l_i$ by meridional multislope (ν_1, \dots, ν_b) . Conversely, corresponding to such a Dehn filling description of M , we have a corresponding open book description of M . Since many existing constructions of foliations are described from the Dehn surgery perspective, it is useful to consider this correspondence more carefully.

Let M_0 denote the compact complement of L ; so $M_0 = M \setminus \text{int } N(L)$, where $N(L)$ is a regular neighborhood of L , and M_0 is homeomorphic to $S \times [0, 1] / h$.

Notice that if S is a disk, then necessarily h is isotopic rel boundary to the identity map. If S is an annulus, then h is isotopic rel boundary to some power of the Dehn twist about the core of S . Otherwise, S is hyperbolic. We therefore lose little by restricting attention to the case that S is hyperbolic and will now do so.

Recall Thurston's classification of surface automorphisms.

Theorem 9.1 [52; 6; 15] *Let S be an oriented hyperbolic surface with geodesic boundary, and let $h \in \text{Homeo}(S, \partial S)$. Then h is freely isotopic to one of the following:*

- (1) *A pseudo-Anosov homeomorphism θ .*
- (2) *A periodic homeomorphism θ , in which case there is a hyperbolic metric for which S has geodesic boundary and such that θ is an isometry of S .*
- (3) *A reducible homeomorphism h' that fixes, setwise, a maximal collection of disjoint simple closed geodesic curves $\{C_j\}$ in S .*

Recall that a pseudo-Anosov homeomorphism has finitely many prong singularities and is smooth and hyperbolic elsewhere [15]. To avoid overlap in the cases, we refer to a map as reducible only if it is not periodic. Since we will be considering homeomorphisms h in the context of open books (S, h) , we will be considering only homeomorphisms h which fix ∂S pointwise. Therefore, given a reducible map, splitting S along $\bigcup_j C_j$ gives a collection of surfaces $S_1, \dots, S_n \subset S$ with geodesic boundary that are fixed by h' . Maximality of $\{C_j\}$ implies that applying Thurston's classification theorem to each $h'|_{S_i} \in \text{Homeo}(S_i, \partial S_i)$ produces either a pseudo-Anosov or periodic representative. So we may assume that h' is either periodic or pseudo-Anosov away from some small neighborhood of the C_i .

Definition 9.2 *Let S be hyperbolic and $h \in \text{Homeo}(S, \partial S)$. If conclusion (1) or conclusion (2) of Theorem 9.1 is satisfied, call θ the Thurston representative of h . If instead conclusion (3) holds, let $\theta: (S, \partial S) \rightarrow (S, \partial S)$ denote the piecewise continuous function uniquely determined by the following constraints:*

- (3.1) θ restricted to each component of the complement of the union $\bigcup_i C_i$ is freely isotopic to the restriction of h' to this component and is either periodic or pseudo-Anosov.
- (3.2) θ restricted to each simple closed geodesic C_i is freely isotopic to the restriction of h' to C_i and is a periodic isometry.

Again, we refer to θ as the Thurston representative of h .

Now consider again the open book decomposition $M = (S, h)$ and let θ denote the Thurston representative of h . When θ is periodic or pseudo-Anosov, the link complement $M_0 = M \setminus \text{int } N(L)$ is also homeomorphic to the mapping torus $S \times [0, 1]/(x, 1) \sim (\theta(x), 0)$ of θ , and in the discussions which follow, we will typically view M_0 as the mapping torus of θ . When θ is reducible and so only piecewise continuous, we will typically view M_0 as the union along essential tori of the mapping tori of the extension to $\bigcup S_i$ of the restriction of θ to the complement of the union $\bigcup C_i$.

Let Θ_0 be the flow obtained by integrating the vector field $\partial/\partial t$, where points of M_0 are given by $[(x, t)]$, $x \in S$, $t \in [0, 1]$. We will refer to this flow as either the *suspension flow of θ* or the *Thurston flow (associated to h)*.

Since θ is area-preserving with respect to some metric on the fiber, Θ_0 is volume-preserving with respect to some choice of metric on M_0 .

Notice that the suspension flow Θ_0 is pseudo-Anosov (respectively, periodic) when θ is pseudo-Anosov (respectively, periodic). In particular, when θ is periodic, all orbits of Θ_0 are closed. When θ is pseudo-Anosov, there are an even number of alternately attracting and repelling closed orbits of $\Theta_0 \mid \partial N(I_i)$ for each i . When θ is periodic or pseudo-Anosov, this flow is continuous and the orbits are smoothly embedded. When θ is reducible, this flow is not continuous, but orbits of the flow are smoothly embedded.

Since many closed manifolds, together with corresponding open book decompositions, can be realized by Dehn filling M_0 , it is useful to have canonical framings on the boundary components of M_0 which are defined independently from M . As in [49], we will use the Thurston flow Θ_0 to define these canonical coordinate systems on ∂M_0 .

9.3 The Thurston flow framing on surface bundles over S^1

Definition 9.3 Let $\partial_i M_0$ denote the i^{th} boundary component of M_0 . Choose an oriented identification $\partial_i M_0 \sim \mathbb{R}^2/\mathbb{Z}^2$ by choosing oriented curves λ_i and μ_i , so that λ_i has slope 0 and μ_i has slope ∞ , as follows. Let $\lambda_i = \partial(S \times \{0\})$, with orientation induced by the orientation on S . Let γ_i be a closed orbit of the flow Θ_0 restricted to $\partial_i M_0$. Choose μ_i to be an oriented simple closed curve which has algebraic intersection number $\langle \lambda_i, \mu_i \rangle = 1$ and which minimizes the geometric intersection number $|\gamma_i \cap \mu_i|$. This choice is unique except in the case that the geometric intersection number $|\gamma_i \cap \lambda_i| = 2$. In this case we choose μ_i so that γ_i has slope $+2$. Call the resulting framing the *Thurston flow framing* on $\partial_i M_0$. Slopes expressed in terms of the flow framing will be said to be given in *Thurston flow coordinates*. This was originally called the *natural framing* or *natural coordinates* by Roberts in [49]. In this section, we are beginning with a fixed fibering and hence the associated Thurston flow coordinates are well defined. In general, different choices of fibering can lead to nonisotopic closed orbits γ_i , and hence to different Thurston flow coordinates. Notice also that in these coordinates, the slope of γ_i always satisfies

$$1/(\text{slope } \gamma_i) \in (-\frac{1}{2}, \frac{1}{2}].$$

Now let's consider the relationship between the flow framing and the fractional Dehn twist coefficient defined by Honda, Kazez, and Matic in [31]. First recall the definition of fractional Dehn twist coefficient.

Definition 9.4 [31] Fixing a component C_i of ∂S and restricting the flow Θ_0 to the component of $\partial N(L)$ corresponding to C_i , Θ_0 necessarily has periodic orbits. Let γ_i be one such, and write

$$\gamma_i = p_i \lambda_i + q_i \nu_i,$$

where $\lambda_i = C_i$, ν_i is the meridian (oriented so that $\langle \lambda_i, \nu_i \rangle = 1$), and p_i and q_i are relatively prime integers with $q_i > 0$. The fractional Dehn twist coefficient of h with respect to the component C_i of ∂S is given by

$$c_i(h) = p_i/q_i.$$

In particular, when $\gamma_i = \nu_i$, $(p_i, q_i) = (0, 1)$ and $c_i(h) = 0$.

Recall that M is obtained from M_0 by (ν_1, \dots, ν_b) Dehn filling along the boundary components of M_0 . Beginning with the open book decomposition of M and the associated fractional Dehn twist coefficients $c_i = p_i/q_i$, $1 \leq i \leq b$, (as above, expressed in (λ_i, ν_i) coordinates), it is sometimes useful to express the slopes of ν_i and γ_i in terms of the Thurston flow coordinates, (λ_i, μ_i) . We now describe how to do this. In flow coordinates, λ_i has slope 0. Since $|\lambda_i \cap \nu_i| = 1$, it follows that, in flow coordinates, ν_i has slope $1/k_i$ for some integer k_i . As noted in [32], the integer k_i is uniquely determined by the fractional Dehn twist coefficient $c_i(h)$ for each i , $1 \leq i \leq b$. This relationship can be very simply stated:

Proposition 9.5 (Coordinate translation I) *Let $c_i = c_i(h)$ and let n_i be the integer determined by the condition*

$$c_i \in (n_i - \frac{1}{2}, n_i + \frac{1}{2}].$$

In other words,

$$n_i = \lceil c_i - \frac{1}{2} \rceil,$$

the integer nearest to c_i , with ties in the case $c_i \in \mathbb{Z} + \frac{1}{2}$ broken by rounding down.

Then $k_i = -n_i$ and so ν_i has slope $-1/n_i$. Moreover, γ_i has slope $1/(c_i - n_i)$ and so:

- (1) *If $c_i = n_i$, then $\gamma_i = \mu_i$ has slope $1/0$.*
- (2) *If $c_i > n_i$, then γ_i has positive slope.*
- (3) *If $c_i < n_i$, then γ_i has negative slope.*

Proof The meridian ν_i has slope $1/k_i$ and so $\nu_i = \mu_i + k_i \lambda_i$. So

$$\gamma_i = p_i \lambda_i + q_i \nu_i = p_i \lambda_i + q_i (\mu_i + k_i \lambda_i) = q_i \mu_i + (p_i + k_i q_i) \lambda_i$$

has slope $q_i/(p_i + k_i q_i) = 1/(c_i + k_i)$ and $|\gamma_i \cap \mu_i| = |p_i + k_i q_i|$. By definition of flow coordinates, k_i is chosen to minimize $|\gamma_i \cap \mu_i| = |p_i + k_i q_i|$, and hence to minimize $|c_i + k_i|$. There is a unique such minimizing k_i unless $c_i \in \mathbb{Z} + \frac{1}{2}$. In this case, k_i is chosen so that γ_i has slope $\frac{2}{1}$; namely, so that $c_i = -k_i + \frac{1}{2}$. So k_i is the unique integer satisfying $c_i \in (-k_i - \frac{1}{2}, -k_i + \frac{1}{2}]$. \square

Conversely, given the fibered 3-manifold M_0 and meridional Dehn filling slopes $v_i, 1 \leq i \leq b$, in terms of the Thurston flow coordinates (λ_i, μ_i) , it is often useful to express the slopes v_i and γ_i in terms of the associated open book coordinates (λ_i, v_i) . We have the following.

Proposition 9.6 (Coordinate translation II) *Suppose M_0 is fibered, with the boundary components $\partial_i M_0$ given the Thurston flow framing (λ_i, μ_i) for each i . As above, let v_i be a meridional slope and let γ_i be a closed orbit of the Thurston flow on $\partial_i M_0$. In terms of the Thurston framing, $v_i = -1/n_i$ and $\gamma_i = r_i/s_i$, for some integers n_i, r_i and s_i . Again as above, let M be the manifold obtained by (v_1, \dots, v_b) filling M_0 and let (S, h) be the open book decomposition of M determined by the fibering of M_0 . Then, in terms of the open book framing (λ_i, v_i) on $\partial_i M_0 = \partial N(l_i)$,*

$$v_i = 1/0, \quad \mu_i = 1/n_i \quad \text{and} \quad \gamma_i = r_i/(n_i r_i + s_i).$$

In particular, the fractional Dehn twist coefficient along or $S \cap \partial_i M_0$ is given by

$$c_i(h) = n_i + s_i/r_i,$$

where, as noted in Definition 9.3, $s_i/r_i \in (-\frac{1}{2}, \frac{1}{2}]$.

Proof To eliminate the ugliness of subscripts, focus on a particular boundary component $\partial_i M_0$ and drop all reference to i .

Two right handed framings are related by a unique transformation in $SL_2(\mathbb{Z})$. Notice that

$$A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

is the element in $SL_2(\mathbb{Z})$ which maps the pair $((\frac{1}{0}), (\frac{-n}{1}))$ to the pair $((\frac{1}{0}), (\frac{0}{1}))$.

Hence, with the correspondence $(\frac{a}{b}) \mapsto b/a$, A describes the translation from (λ, μ) coordinates to (λ, v) coordinates. The slope computations follow immediately. \square

9.4 Dehn filling the Thurston flow

Dehn surgery on Anosov flows is defined by Goodman in [27]. (See also [16].) This definition generalizes naturally to the setting of pseudo-Anosov flows and permits us to consider the effect of Dehn filling of Thurston flows. We define Dehn filling of a Thurston flow as follows. Let Y be any closed 3-manifold obtained by Dehn filling M_0 . For each $i, 1 \leq i \leq b$, let X_i denote the solid torus in Y bounded by $\partial_i M_0$ and let κ_i denote the core of X_i . As long as the surgery coefficient along $\partial_i M_0$ is not γ_i , it is possible to blow down X_i to its core and obtain a flow Θ defined on Y . Notice that the cores κ_i are closed orbits of Θ . Also, either Θ_0 is periodic in a neighborhood of $\partial_i M_0$ and therefore so is Θ in a neighborhood of κ_i , or else Θ_0 is pseudo-Anosov (with possibly a single prong pair along κ_i) in a neighborhood of $\partial_i M_0$ and therefore so is Θ in a neighborhood of κ_i . We shall refer to this flow Θ as the *surgered Thurston flow*. Notice that since Θ_0 is volume-preserving for some metric, so is Θ .

Lemma 9.7 *Suppose M has an open book decomposition (S, h) with binding L and corresponding Dehn filling description $M = M_0(v_1, \dots, v_b)$. Let Θ_0 denote the Thurston flow on the complement of L . Then Θ_0 extends to a flow Θ on M if and only if all fractional Dehn twist coefficients c_i are nonzero.*

Proof Notice that

$$c_i(h) = 0 \iff p_i = 0, q_i = 1 \iff \gamma_i = v_i \implies v_i = \mu_i = \gamma_i.$$

In particular, if at every component C_i of ∂S the fractional Dehn twist coefficient $c_i(h) \neq 0$, then $v_i \neq \gamma_i$ and so it is possible to blow down the flow Θ_0 to the surgered Thurston flow Θ on M . Otherwise it is not. □

Notice that the binding $L = \bigcup I_i$ inherits orientations both from the open book structure and from the flow Θ .

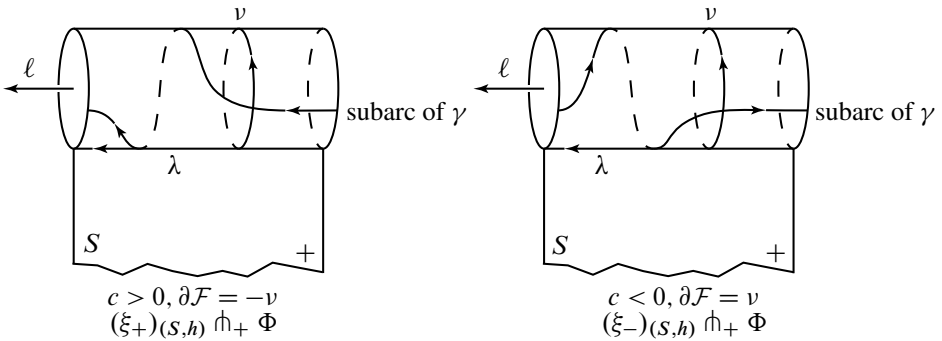


Figure 5: Curves on a boundary component of M_0

Lemma 9.8 When $c_i > 0$, these orientations agree on l_i . When $c_i < 0$, these orientations do not agree on l_i .

Proof The orientation of Θ restricted to the binding is determined by the sign of the slope of γ_i as expressed in (λ_i, ν_i) coordinates. This is illustrated in [Figure 5](#). \square

9.5 Contact structures supported by an open book

In [\[26\]](#), Giroux defined the notion of contact structure supported by an open book: a positive (respectively, negative) contact structure ξ on M is *supported by, or compatible with, an open book decomposition* (S, h) of M if ξ can be isotoped through contact structures so that there is a contact 1-form α for ξ such that:

- (1) $d\alpha$ is a positive area form on each page S_t of the open book.
- (2) $\alpha > 0$ (respectively, $\alpha < 0$) on the binding l .

Proposition 9.9 [\[26\]](#) Two contact structures supported by the same open book are contact isotopic.

We may therefore abuse language and refer to *the* contact structure compatible with the open book decomposition (S, h) .

Notation 9.10 Let $(\xi_+)_{(S,h)}$ denote the positive contact structure compatible with the open book decomposition (S, h) . Let $(\xi_-)_{(S,h)}$ denote the negative contact structure compatible with the open book decomposition (S, h) .

Lemma 9.11 Suppose all fractional Dehn twist coefficients are nonzero and let Θ denote the surgered Thurston flow.

- (1) $(\xi_+)_{(S,h)}$ is positively transverse to Θ if and only if all fractional Dehn twist coefficients are positive.
- (2) $(\xi_-)_{(S,h)}$ is positively transverse to Θ if and only if all fractional Dehn twist coefficients are negative.

Proof This follows immediately from [Lemma 9.8](#). \square

Any contact structure (M, ξ) is supported by infinitely many open book decompositions. Honda, Kazez, and Matic proved in [\[32\]](#) that if there is a compatible open book decomposition with nonpositive fractional Dehn twist coefficient c_i for some i , then necessarily ξ is overtwisted. In fact, they show the following.

Theorem 9.12 [31, Theorem 1.1] *A contact structure (M, ξ) is tight if and only if all of its compatible open book decompositions (S, h) have fractional Dehn twist coefficients $c_i \geq 0$ for $1 \leq i \leq |\partial S|$.*

Let S' be the smallest invariant subsurface of S for the Thurston representative of h ; thus $S' = S$ if and only if h is not reducible. If $c_i = 0$ for some common boundary component of S and S' , there are two possibilities for $h|_{S'}$. The first is that it is periodic, and it follows that it is equal to the identity map. The second possibility is that $h|_{S'}$ is pseudo-Anosov, but this immediately implies ξ is overtwisted [31]. Since our primary focus is studying tightness for pseudo-Anosov maps it is enough to consider open book decompositions for which all fractional Dehn twist coefficients c_i are positive (or, in the case that the contact structure is negative, to a consideration of open book decompositions for which all fractional Dehn twist coefficients c_i are negative).

9.6 Foliations compatible with an open book decomposition

Definition 9.13 Let \mathcal{F} be an oriented foliation of M . Let (S, h) be an open book decomposition of M , with binding L . Let Φ be the surgered Thurston flow associated to h . If \mathcal{F} is everywhere transverse to Φ , then we say that \mathcal{F} is *compatible with the open book (S, h)* .

Proposition 9.14 *Let (S, h) be an open book decomposition of M , with binding L , and surgered Thurston flow Φ . Set $M_0 = M \setminus \text{int } N(L)$ and Φ_0 to be the Thurston flow associated to h .*

- (1) *Suppose all fractional Dehn twist coefficients are positive. If there exists a foliation \mathcal{F}_- in M_0 which is L -taut, is transverse to Φ_0 and satisfies $m_{\mathcal{F}_-}^T < 0$ with respect to $\langle v^T, \Phi|_T \rangle$ for all components T of ∂M_0 , then $(M, (\xi_+)_{(S,h)})$ is weakly symplectically fillable and universally tight.*
- (2) *Suppose all fractional Dehn twist coefficients are negative. If there exists a foliation \mathcal{F}_+ in M_0 which is L -taut, is transverse to Φ_0 and satisfies $m_{\mathcal{F}_+}^T > 0$ with respect to $\langle v^T, \Phi|_T \rangle$ for all components T of ∂M_0 , then $(-M, (\xi_-)_{(S,h)})$ is weakly symplectically fillable and universally tight.*

Proof Consider case (1). Since $(\xi_+)_{(S,h)}$ is transverse to Φ , it suffices to show that there is a negative contact structure ξ_- which is transverse to Φ . This follows immediately from the assumptions by [Theorem 6.10](#).

Case (2) follows similarly. □

Set \mathcal{G}_0 to be the fibration $S \times [0, 1]/h$ in the complement of L . Note that $\mathcal{G}_0 \cap T$ has either positive or negative slope with respect to $\langle v^T, \Phi|_T \rangle$, on each component T of $\partial N(L)$. For completeness, we note the following:

Lemma 9.15 *Let T_i be a component of $\partial N(L)$. Then $\mathcal{G}_0 \cap T_i$ has positive (respectively, negative) slope with respect to $\langle v^{T_i}, \Phi|_{T_i} \rangle$ if the corresponding fractional Dehn twist c_i is positive (respectively, negative).*

Proof Consider the relative slope values of γ, λ , and ν on any boundary component T_i of ∂M_0 . This is captured in Figure 5. Notice that λ represents the slope of $\partial \mathcal{G}_0$ and γ represents the slope of Φ . Recalling the slope convention, illustrated in Figure 4, we see that $\mathcal{G}_0 \cap T_i$ has positive (respectively, negative) slope with respect to $\langle v^{T_i}, \Phi|_{T_i} \rangle$ if the corresponding fractional Dehn twist c_i is positive (respectively, negative). \square

So \mathcal{G}_0 extends to a positive confoliation transverse to Φ when $c_i > 0$ for all i and to a negative confoliation transverse to Φ when $c_i < 0$ for all i . In other words, and unsurprisingly, \mathcal{G}_0 as a foliation playing the role of \mathcal{F}_+ (respectively, \mathcal{F}_-) gives a second way of establishing the existence of ξ_+ (respectively ξ_-).

Finally, we use Proposition 9.5 to rephrase Proposition 9.14 in terms of Thurston flow coordinates. For simplicity of exposition, we restrict attention to the case that all fractional Dehn twist coefficients are positive. There is a symmetric statement in the case that all fractional Dehn twist coefficients are negative.

Proposition 9.16 *Let (S, h) be an open book decomposition of M , with binding L . Suppose all fractional Dehn twist coefficients $c_i, 1 \leq i \leq b$, are positive. For each i , let n_i be the integer nearest to c_i , with ties in the case $c_i \in \mathbb{Z} + \frac{1}{2}$ broken by rounding down. Set $M_0 = M \setminus \text{int } N(L)$. Let Φ_0 denote the Thurston flow associated to h and let Φ denote the surgered Thurston flow on M .*

Suppose there exists a foliation \mathcal{F}_- in M_0 which is L -taut, is transverse to Φ , and such that, for each component T_i of ∂M_0 , c_i, n_i and

$$m_i = m_{\mathcal{F}_-}^{T_i}$$

satisfy one of the following:

- (1) $c_i = n_i$ and $m_i \in (-\infty, -\frac{1}{n_i})$.
- (2) $c_i > n_i$ and $m_i \in (\frac{1}{c_i - n_i}, \infty] \cup [-\infty, -\frac{1}{n_i})$.
- (3) $c_i < n_i$ and $m_i \in (\frac{1}{c_i - n_i}, -\frac{1}{n_i})$.

Thus $m_{\mathcal{F}_-}^T < 0$ with respect to $\langle v^T, \Phi|_T \rangle$ for all components T of ∂M_0 , and consequently, $(\xi_+)_{(S,h)}$ is weakly symplectically fillable and universally tight.

Proof The boundary slope of \mathcal{F}_- on the i^{th} boundary component lies between Φ and v_i as shown in Figure 4. By Proposition 9.5 the slope of Φ is $1/(c_i - n_i)$ while the slope of v_i is $-1/n_i$. The form of the intervals given, depends on a case by case analysis of whether or not they contain slope ∞ . Weak symplectic fillability and universal tightness of $(\xi_+)_{(S,h)}$ follow from Proposition 9.14. \square

One can translate the results of [32; 49] into the current context as follows.

Theorem 9.17 [49] *When the binding L is connected, $c > 0$, and the monodromy h has pseudo-Anosov representative, there are ∂M_0 -taut foliations in M_0 transverse to Φ_0 which realize all slopes in an interval J as follows:*

- (1) $c = n$ and $J = (-\infty, \infty)$.
- (2) $c > n$ and $J = (-\infty, 1)$.
- (3) $c < n$ and $J = (-1, \infty)$.

The next corollary follows by intersecting the intervals where foliations exist in Theorem 9.17 with the intervals where they are required in Proposition 9.16.

Corollary 9.18 *There exists \mathcal{F}_- as described in Proposition 9.16, if one of the following is true:*

- (1) $c = n$ and $(-\infty, -\frac{1}{n}) \cap (-\infty, \infty) \neq \emptyset$.
- (2) $c > n$ and $((\frac{1}{c-n}, \infty] \cup [-\infty, -\frac{1}{n})) \cap (-\infty, 1) \neq \emptyset$.
- (3) $c < n$ and $(-\frac{1}{n-c}, -\frac{1}{n}) \cap (-1, \infty) \neq \emptyset$.

The case when the fractional Dehn twist coefficient is greater than or equal to 1 is of particular interest since

Corollary 9.19 *If $c \geq 1$ there exists \mathcal{F}_- as described in Proposition 9.16.*

Proof If $c > 0$ then $n \geq 0$, with $n = 0$ only when $c \in (0, \frac{1}{2}]$. Thus $c \geq 1$ implies $n \geq 1$, and it follows that the intersections in Cases (1) and (2) are nonempty. In Case (3), $1 \leq c < n$, and the intersection is again nonempty. \square

This corollary is exactly what is needed to complete the proof of a theorem of Honda, Kazez, and Matić in [32] that was one of the original motivations for this work.

Theorem 9.20 *If (S, h) is an open book decomposition such that S has connected boundary, h is isotopic to a pseudo-Anosov homeomorphism, and the fractional Dehn twist coefficient of h is greater than or equal to 1, then the contact structure canonically associated to the open book decomposition $\xi(S, h)$ is weakly symplectically fillable and universally tight.*

The proof strategy of [32] used a single foliation \mathcal{F} defined on all of (S, h) as constructed by Roberts, [48; 49] with boundary slope related to open book data. Next they wanted to apply the Eliashberg–Thurston theorem to produce a weakly symplectically fillable and universally tight contact structure ξ . Finally they argued that the two contact structures $\xi(S, h)$ and ξ were necessarily equivalent.

There is a somewhat surprising aspect that arises in addressing the issue of lack of smoothness of \mathcal{F} . It is that we do not use the same foliation. To complete their proof with our strategy, we require the existence of two foliations \mathcal{F}^+ and \mathcal{F}^- , both of which exist on the complement of the binding of (S, h) , by the work of Roberts, and have boundary slopes on either side of the boundary slope of \mathcal{F} . One foliation is used to produce a positive contact structure, the other a negative contact structure, and both are necessary to conclude that the approximating contact structure is weakly symplectically fillable and universally tight.

The notion of Φ -approximating contact structure that we produce is sufficient to conclude that $\xi(S, h)$ and ξ are equivalent using the argument of [32]. \square

When $M = (S, h)$ has binding which is not connected, our results can be applied to the Kalelkar–Roberts constructions of ∂M_0 -taut foliations in M_0 transverse to Φ_0 .

Theorem 9.21 [33, Theorem 1.1] *There are ∂M_0 -taut oriented foliations in M_0 transverse to Φ_0 and realizing a neighborhood of rational boundary multislopes about the boundary multislope of the fibration.*

Corollary 9.22 *Suppose M has open book decomposition (S, h) and fractional Dehn twist coefficients c_i , $1 \leq i \leq b$. There are constants $A_i = A_i(M_0)$, $1 \leq i \leq b$, dependent on M_0 such that if $c_i > A_i$, then $\xi(S, h)$ is weakly symplectically fillable and universally tight.*

Work of Baldwin and Etnyre [1] implies that any such constants A_i must depend on M_0 , at least in the case that the page S has genus one.

Theorem 9.23 [1, Theorem 1.16] *There exist open books whose fractional Dehn twist coefficients are arbitrarily large, but whose compatible contact structures are not C^0 close to smooth orientable taut foliations.*

Corollary 9.24 *There exist open books whose fractional Dehn twist coefficients are arbitrarily large, but whose compatible contact structures are not Φ -close to taut oriented bracketed $C^{1,0}$ -foliations for any volume-preserving flow Φ .*

Appendix: Some symplectic topology

This section contains an overview of the relationship between foliations, volume-preserving flows, symplectic topology, and contact topology that is summarized in [Theorem 2.2](#).

Let \mathcal{F} be a transversely oriented, taut $C^{1,0}$ -foliation in M . Fix a metric on M , and let Φ be a volume-preserving flow transverse to \mathcal{F} . The starting point for the interconnections we will describe is a carefully chosen 2-form.

Definition A.1 [[12](#), Section 3.2] Let ξ be a cooriented C^k 2-plane field on a smooth 3-manifold M with $k \geq 0$. A smooth closed 2-form ω on M is said to *dominate* ξ if $\omega|\xi$ does not vanish (ie if $p \in M$ and X_p, Y_p is a basis for ξ_p , then $\omega_p(X_p, Y_p) \neq 0$). A smooth closed 2-form ω on M is said to *positively dominate* ξ if $\omega|\xi$ is positive (ie for all $p \in M$, if X_p, Y_p is a positively oriented basis for ξ_p , then $\omega_p(X_p, Y_p) > 0$).

To produce such a dominating closed 2-form, let Ω be the volume form on M preserved by the smooth flow Φ , and let X be the vector field which generates Φ . Define

$$\omega = X \lrcorner \Omega.$$

Recall that Φ is volume-preserving if and only if $\mathcal{L}_X \Omega = 0$, where \mathcal{L}_X denotes the Lie derivative with respect to X . (See for example [[39](#), Proposition 18.16].) By Cartan's formula (see for example [[39](#), Proposition 18.13]),

$$\mathcal{L}_X \Omega = X \lrcorner (d\Omega) + d(X \lrcorner \Omega) = d(X \lrcorner \Omega) = d\omega.$$

It follows that ω is closed. From its definition, ω is killed by the flow direction X , thus a cooriented 2-plane field ξ is positively dominated by ω if and only if it is everywhere positively transverse to Φ .

A closed 2-form dominating $T\mathcal{F}$ can be produced directly from a taut foliation [[29](#); [51](#)] thereby eliminating the need to choose a metric, a volume form, and a flow-preserving the volume form. We have chosen to emphasize the volume-preserving flow since it clarifies the local nature of our foliations in flow boxes.

Specialize now to the case of [Theorem 2.2](#) in which ξ is a contact structure positively transverse to Φ . Choose a 1-form α such that $\ker \alpha = \xi$ and $\alpha \wedge \omega > 0$. Define a 2-form $\tilde{\omega}$ on $M \times [-1, 1]$ using the projection map p and the formula

$$(1) \quad \tilde{\omega} = p^*(\omega) + \epsilon d(t\alpha).$$

Direct computation shows that if ϵ is positive and small enough, $(M \times [-1, 1], \tilde{\omega})$ is a symplectic manifold with boundary, that is, $\tilde{\omega}$ is a nondegenerate 2-form. The important role of the positive and negative contact structures ξ_+ and ξ_- of [Theorem 2.2](#) will be described after the next definition.

Definition A.2 A boundary component Y of a symplectic manifold $(W, \tilde{\omega})$ is called *weakly convex* if Y admits a positive contact structure dominated by $\tilde{\omega}|_Y$.

This is precisely the structure that the boundary components of $(M \times [-1, 1], \tilde{\omega})$ have. The restriction of $\tilde{\omega}$ positively dominates ξ_+ on $M \times \{1\}$. Because of boundary orientations, ξ_- defines a positive contact structure on $M \times \{-1\}$ that is positively dominated by the restriction of $\tilde{\omega}$. Moreover, $\tilde{\omega}$ restricts to ω on $M \times \{0\}$, thus $M \times [-1, 0]$ also has weakly convex boundary.

Since both boundary components of either $(M \times [-1, 1], \tilde{\omega})$ or the restriction of $\tilde{\omega}$ to $M \times [-1, 0]$ are weakly convex, they give examples of weak symplectic fillings.

Definition A.3 A *weak symplectic filling* of a contact manifold (M, ξ) is a symplectic manifold $(W, \tilde{\omega})$ with $\partial W = M$ (as oriented manifolds) such that $\tilde{\omega}|_\xi > 0$. A contact manifold (M, ξ) which admits a weak symplectic filling is called *weakly symplectically fillable*.

Definition A.4 A *strong symplectic filling* of a contact manifold (M, ξ) is a symplectic manifold $(W, \tilde{\omega})$ with $\partial W = M$ (as oriented manifolds), where $\xi = \ker \alpha$ for a 1-form α satisfying $d\alpha = \tilde{\omega}|_M$. A contact manifold (M, ξ) which admits a strong symplectic filling is called *strongly symplectically fillable*.

In general, strong symplectic fillability is a stronger condition than weak symplectic fillability [10]. However, by [44, Lemma 1.1], when M is a rational homology sphere, (M, ξ) is weakly symplectically fillable if and only if it is strongly symplectically fillable.

Definition A.5 A contact manifold is said to be weakly (or strongly) *semifillable* if it is one boundary component of a weak (or strong) symplectic filling.

A weakly (or strongly) semifillable contact manifold is weakly (or strongly) fillable [11; 13].

The following fundamental theorem gives an example of the importance of weak symplectic fillability in contact topology.

Theorem A.6 [28; 9; 12] *Weakly symplectically fillable contact structures are tight. Weakly symplectically fillable contact structures that are Φ -close to taut foliations are universally tight.*

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