Open book decompositions
versus prime factorizations
of closed, oriented 3–manifolds

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Let M be a closed, oriented, connected 3–manifold and (B, π) an open book decomposition on M with page Σ and monodromy φ. It is easy to see that the first Betti number of Σ is bounded below by the number of $S^2 \times S^1$–factors in the prime factorization of M. Our main result is that equality is realized if and only if φ is trivial and M is a connected sum of copies of $S^2 \times S^1$. We also give some applications of our main result, such as a new proof of the fact that if the closure of a braid with n strands is the unlink with n components then the braid is trivial.

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1 Introduction

An *abstract open book* is a pair (Σ, φ), where Σ is a connected, oriented surface with $\partial\Sigma \neq \varnothing$ and the *monodromy* φ is an element of the group $\mathrm{Diff}^+(\Sigma, \partial\Sigma)$ of orientation-preserving diffeomorphisms of Σ which restrict to the identity on a neighborhood of the boundary. We say that the monodromy φ is *trivial* if it is isotopic to the identity of Σ via diffeomorphisms which fix $\partial\Sigma$ pointwise. Let N_φ denote the mapping torus

$$N_\varphi = \Sigma \times [0, 1]/(p, 1) \sim (\varphi(p), 0).$$

To the open book (Σ, φ) one can associate a closed, oriented, connected 3–manifold $M_{(\Sigma,\varphi)}$ by using the natural identification of $\partial N_\varphi = \partial\Sigma \times S^1$ with the boundary of $\partial\Sigma \times D^2$:

$$M_{(\Sigma,\varphi)} := N_\varphi \cup_\partial \partial\Sigma \times D^2.$$

The link $B := \partial\Sigma \times \{0\} \subset M_{(\Sigma,\varphi)}$ is fibered, with fibration $\pi\colon M_{(\Sigma,\varphi)} \setminus B \to S^1$ given by the obvious extension of the natural projection

$$N_\varphi = \Sigma \times [0, 1]/(p, 1) \sim (\varphi(p), 0) \to S^1 = [0, 1]/1 \sim 0$$

and monodromy equal to φ. In other words, the pair (B, π) is an *open book decomposition* of $M = M_{(\Sigma, \varphi)}$ with *binding* B, pages $\Sigma_\theta := \pi^{-1}(\theta)$, $\theta \in S^1$, and monodromy φ. We will always identify N_φ with the complement of a tubular neighborhood of B in M.

If (B, π) is an open book decomposition of M with page Σ, it is easy to see that M has a Heegaard splitting of genus $b_1(\Sigma)$. Since M is obtained from each handlebody of the splitting by attaching 2–disks and 3–balls, this immediately implies the inequality

$$(1) \qquad\qquad\qquad b_1(M) \leq b_1(\Sigma).$$

We will provide a refinement of Inequality (1) in Proposition 2.2.

The following theorem is our main result. Its proof is based on well-known results due to Reidemeister [14], Singer [15] and Haken [8] (see Section 3). Recall that each closed, oriented, connected 3–manifold M has a prime factorization, unique up to order of the factors, of the form

$$(2) \qquad\qquad M = M_1 \# \cdots \# M_h \# S^2 \times S^1 \# \overset{(k)}{\cdots} \# S^2 \times S^1,$$

where each M_i is irreducible (see eg [10]).

Theorem 1.1 *Let (B, π) be an open book decomposition of a closed, oriented, connected 3–manifold M with page Σ and monodromy φ. Then $b_1(\Sigma)$ is equal to the number of $S^2 \times S^1$–factors in the prime factorization of M if and only if φ is trivial and M is a connected sum of copies of $S^2 \times S^1$.*

Theorem 1.1 immediately implies the following corollary, which is also proved in Ni [12, Proof of Theorem 1.3] and Grigsby and Wehrli [7, Theorem 2] using the fact that finitely generated free groups are not isomorphic to any of their nontrivial quotients.

Corollary 1.2 *Any open book decomposition of $\#^k S^2 \times S^1$ whose page Σ satisfies $b_1(\Sigma) = k$ must have trivial monodromy.*

Corollary 1.2 implies Corollary 1.3, which was obtained previously by Cochran [3] using the fact that finitely generated free groups are not isomorphic to any of their nontrivial quotients, and by Birman and Menasco [2] as an application of their braid foliation techniques. Grigsby and Wehrli [7] gave another proof of Corollary 1.3 using Khovanov homology.

Corollary 1.3 *Let $b \in B_n$ be a braid on n strands such that its closure $\hat{b}$ is the trivial link U_n with n components. Then b is the identity.*

Proof Put $\hat{b}$ in braid form with respect to the binding of the trivial open book decomposition of S^3 and consider the two-fold branched cover $\Sigma(\hat{b})$ along $\hat{b}$. Then

$$\Sigma(\hat{b}) = \Sigma(U_n) = \overset{n-1}{\#} S^2 \times S^1.$$

Pulling back the trivial open book of S^3 to $\Sigma(\hat{b})$ we obtain an open book decomposition of $\#^{n-1} S^2 \times S^1$, whose page is a surface Σ with $b_1(\Sigma) = n-1$, which we view as a 2–fold branched cover of the disk with n branch points. Under the identification of B_n with the subgroup of the mapping class group of Σ given by the elements commuting with the covering involution [1], the monodromy of the open book is equal to b. By Corollary 1.2, the braid b must be the identity in B_n. □

Let Σ and Σ' be two orientable surfaces. By performing a boundary connected sum between them we obtain a surface $\Sigma \natural \Sigma'$. If φ is a diffeomorphism of Σ, ψ is a diffeomorphism of Σ' and both φ and ψ are the identity on a neighborhood of the boundary, we can form a diffeomorphism $\varphi \natural \psi$ of $\Sigma \natural \Sigma'$. This geometric operation yields a homomorphism

$$\Gamma_\Sigma \times \Gamma_{\Sigma'} \to \Gamma_{\Sigma \natural \Sigma'},$$

which we will call *boundary connected sum homomorphism.* A combination of Inequality (1) with Corollary 1.2 yields the following Corollary 1.4, which can also be proved by applying [13, Corollary 4.2 (iii)].

Corollary 1.4 *Let Γ_Σ be the mapping class group of the orientable surface Σ. Then the boundary connected sum homomorphism*

$$\Gamma_\Sigma \times \Gamma_{\Sigma'} \to \Gamma_{\Sigma \natural \Sigma'}$$

is injective.

Proof Under the map $(\Sigma, \varphi) \to M_{(\Sigma, \varphi)}$ described above, boundary connected sum of abstract open books corresponds to connected sum of 3–manifolds:

$$M_{(\Sigma \natural \Sigma', \varphi \natural \psi)} = M_{(\Sigma, \varphi)} \# M_{(\Sigma', \psi)}.$$

Observe that $b_1(\Sigma \natural \Sigma') = b_1(\Sigma) + b_1(\Sigma')$. Therefore, if $\varphi \natural \psi$ is isotopic to the identity relative to the boundary then $M_{(\Sigma \natural \Sigma', \varphi \natural \psi)}$ is diffeomorphic to $\#^{b_1(\Sigma)+b_1(\Sigma')} S^2 \times S^1$. The uniqueness of the prime factorization for 3–manifolds [10] implies that

$$M_{(\Sigma, \varphi)} = \overset{k}{\#} S^2 \times S^1 \quad \text{and} \quad M_{(\Sigma', \psi)} = \overset{l}{\#} S^2 \times S^1$$

for some nonnegative integers k, l such that $k + l = b_1(\Sigma) + b_1(\Sigma')$. By Inequality (1) we have $k \leq b_1(\Sigma)$ and $l \leq b_1(\Sigma')$, which forces $k = b_1(\Sigma)$ and $l = b_1(\Sigma')$ as the only possibility. Corollary 1.2 implies that φ and ψ are isotopic to the identity. □

The rest of the paper is organized as follows. In Section 2 we recall two well-known results independent of Theorem 1.1, ie Propositions 2.1 and 2.2. Proposition 2.1 shows that any embedded 2–sphere disjoint from the binding of an open book decomposition bounds an embedded ball. Proposition 2.2 is a refinement of Inequality (1) and can be viewed as saying that the homology of a closed, oriented, connected 3–manifold M puts homological constrains on the monodromy of any open book decomposition of M. In Section 3 we prove Theorem 1.1.

2 Nonseparating 2–spheres and refinement of Inequality (1)

Given a closed, oriented, connected 3–manifold M endowed with an open book decomposition (B, π) and having a prime factorization as in (2), one of the first questions one could ask is how a nonseparating 2–sphere S in M can be positioned with respect to the binding B. Since B is homologically trivial in M, the following proposition implies that, possibly after a small isotopy, each such S must intersect B transversally at least twice.

Proposition 2.1 *Let (B, π) be an open book decomposition with page Σ and monodromy φ of a closed, oriented, connected 3–manifold M. Then each embedded 2–sphere $S \subset M \setminus B$ bounds an embedded ball in $M \setminus B$ and, in particular, is homologically trivial in M.*

Proof Recall that $M = N_\varphi \cup V$, where V is a tubular neighborhood of the binding. Up to an isotopy of S, we can assume $S \subset N_\varphi$. The universal cover of N_φ is homeomorphic to $\mathbb{R}^3$ and from this the triviality of $[S]$ in $H_2(M \setminus B)$, and therefore in $H_2(M)$, follows immediately.

In order to prove that S bounds a ball in $M \setminus B$ we need to use some basic results in three-dimensional topology. In fact $\mathbb{R}^3$ is irreducible [9, Theorem 1.1] and this

implies [9, Proposition 1.6] that N_φ is also irreducible, therefore S bounds an embedded ball in N_φ. □

We now establish a result which refines Inequality (1). Proposition 2.2 below can be viewed as saying that the homology of a closed, oriented, connected 3–manifold M puts homological constraints on the monodromy of any open book decomposition of M.

For the rest of this section all homology groups will be taken with coefficients in the field $\mathbb{Q}$ of rational numbers unless specified otherwise. Let $H_1(\Sigma, \partial\Sigma)^\varphi$ denote the subspace of $H_1(\Sigma, \partial\Sigma)$ consisting of the elements fixed by the map

$$\varphi_* \colon H_1(\Sigma, \partial\Sigma) \to H_1(\Sigma, \partial\Sigma)$$

induced by the monodromy $\varphi \colon \Sigma \to \Sigma$.

Proposition 2.2 *Let* (B, π) *be an open book decomposition with page* Σ *and monodromy* φ *of a closed, oriented, connected 3–manifold* M. *Then*

$$b_1(M) = \dim_{\mathbb{Q}} H_1(\Sigma, \partial\Sigma)^\varphi.$$

More precisely, there is an isomorphism $H_2(M) \cong H_1(\Sigma, \partial\Sigma)^\varphi$ *induced by a well-defined map* $H_2(M; \mathbb{Z}) \to H_1(\Sigma, \partial\Sigma; \mathbb{Z})^\varphi$ *given by* $\alpha \mapsto [F \cap \Sigma]$, *where* $F \subset M$ *is any closed, oriented and properly embedded surface which represents* α *and intersects the page* $\Sigma \times \{0\}$ *transversally.*

Proof We can view N_φ as the union of $\Sigma \times [0, \frac{1}{2}]$ and $\Sigma \times [\frac{1}{2}, 1]$ with $(x, 1)$ identified to $(\varphi(x), 0)$. Using the fact that Σ times an interval is homotopically equivalent to Σ, the (relative) Mayer–Vietoris sequence for this splitting gives the exact sequence

$$H_2(\Sigma, \partial\Sigma)^2 \xrightarrow{f_1} H_2(N_\varphi, \partial N_\varphi) \xrightarrow{f_2} H_1(\Sigma, \partial\Sigma)^2 \xrightarrow{f_3} H_1(\Sigma, \partial\Sigma)^2.$$

The map f_3 is given by the matrix

$$\begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ \varphi_* & \mathrm{Id} \end{pmatrix} \in M_2(\mathrm{End}(H_1(\Sigma, \partial\Sigma))).$$

This immediately implies that the image of f_2 is isomorphic to $H_1(\Sigma, \partial\Sigma)^\varphi$.

Recall the decomposition $M = N_\varphi \cup V$, where V is a tubular neighborhood of the binding. Since $H_2(V) = \{0\}$, the homology exact sequence for the pair (M, V) implies that the map $g \colon H_2(M) \to H_2(M, V)$ induced by the inclusion map is injective. On the other hand, by excision the inclusion $N_\varphi \subset M$ induces an isomorphism

$$\psi \colon H_2(N_\varphi, \partial N_\varphi) \xrightarrow{\cong} H_2(M, V).$$

Moreover, it is easy to see that the image of the map $\psi \circ f_1$ maps injectively to $H_1(V)$ under the next map $\delta \colon H_2(M, V) \to H_1(V)$ in the exact sequence of the pair, while the image of g maps trivially. This shows that the images of f_1 and of $\psi^{-1} \circ g$ have trivial intersection. Therefore the composition $f_2 \circ \psi^{-1} \circ g$ sends $H_2(M)$ injectively into the image of the map f_2, which, as we have just shown, is isomorphic to $H_2(\Sigma, \partial\Sigma)^\varphi$.

We claim that $f_2 \circ \psi^{-1} \circ g$ sends $H_2(M)$ also surjectively onto the image of f_2. In order to verify this, we argue by induction. Assume first that $\partial\Sigma$ is connected. In this situation the map $\delta \circ \psi \circ f_1$ is clearly surjective. Therefore, if $x \in H_2(N_\varphi, \partial N_\varphi)$ with $f_2(x) \neq 0$, there exists $y \in H_2(\Sigma, \partial\Sigma)^2$ with $\delta \circ \psi \circ f_1(y) = \delta \circ \psi(x)$. It follows that setting $x' = x - f_1(y)$ we have $f_2(x') = f_2(x)$ and $\delta \circ \psi(x') = 0$; therefore x' is in the image of $\psi^{-1} \circ g$, and the claim is proved when $\partial\Sigma$ is connected.

Now assume $\partial\Sigma$ is disconnected and denote by $|\partial\Sigma|$ the number of its connected components. By the inductive hypothesis we assume that the claim holds for open books with $|\partial\Sigma| - 1$ binding components. Let $(\widehat{\Sigma}, \widehat{\varphi})$ be another abstract open book, constructed as follows. The connected, oriented surface $\widehat{\Sigma}$ is obtained by attaching a 2–dimensional 1–handle h to $\partial\Sigma$ so that $|\partial\widehat{\Sigma}| = |\partial\Sigma| - 1$, while $\widehat{\varphi}$ is defined by first extending φ as the identity over h, and then composing with a (positive or negative) Dehn twist along a simple closed curve in $\widehat{\Sigma}$ which intersects the cocore c of h transversely once. It is a well-known fact that the open book decomposition $(\widehat{B}, \widehat{\pi})$ associated to $(\widehat{\Sigma}, \widehat{\varphi})$ is obtained from the open book decomposition (B, π) associated to (Σ, φ) by plumbing with a Hopf band, and that $M_{(\widehat{\Sigma}, \widehat{\varphi})}$ is diffeomorphic to M (see [6]). We can choose a basis $[c_1], \dots, [c_{b_1(\Sigma)}]$ of $H_1(\Sigma, \partial\Sigma)$ such that each $c_i \subset \Sigma$ is a properly embedded arc disjoint from $\gamma \cap \Sigma$, and so that, viewing the classes $[c_i]$ in $H_1(\widehat{\Sigma}, \partial\widehat{\Sigma})$, when we add $[c]$ we obtain a basis of $H_1(\widehat{\Sigma}, \partial\widehat{\Sigma})$. Using this basis one can easily check that the natural inclusion map $H_1(\Sigma, \partial\Sigma) \to H_1(\widehat{\Sigma}, \partial\widehat{\Sigma})$ restricts to an isomorphism

$$H_1(\Sigma, \partial\Sigma)^\varphi \cong H_1(\widehat{\Sigma}, \partial\widehat{\Sigma})^{\widehat{\varphi}}.$$

Since $|\partial\widehat{\Sigma}| = |\partial\Sigma| - 1$, by the inductive assumption we have

$$b_1(M) = \dim_{\mathbb{Q}} H_1(\widehat{\Sigma}, \partial\widehat{\Sigma})^{\widehat{\varphi}}.$$

This proves the claim in full generality. Finally, observe that the maps f_2 and $f_2 \circ \psi^{-1} \circ g$ are well-defined over the integers. If we represent homology classes in $H_2(N_\varphi, \partial N_\varphi; \mathbb{Z})$ and $H_2(M; \mathbb{Z})$ by oriented, properly embedded surfaces intersecting the page $\Sigma \times \{0\}$ transversally and we follow the construction of the connecting homomorphism, we see that the maps f_2 and $f_2 \circ \psi^{-1} \circ g$ are both realized geometrically by intersecting with $\Sigma \times \{0\}$. This concludes the proof. $\square$

3 The proof of Theorem 1.1

We start by recalling a basic result of Reidemeister and Singer about collections of compressing disks in a handlebody. We refer to [11] for a modern presentation of this material. Let H_g be a 3–dimensional handlebody of genus g. A properly embedded disk $D \subset H_g$ is *essential* if ∂D does not bound a disk in ∂H_g.

Definition 3.1 A collection $\{D_1, \ldots, D_g\} \subset H_g$ of g properly embedded, pairwise disjoint essential disks is a *minimal system of disks* for H_g if the complement of a regular neighborhood of $\bigcup_i D_i$ in H_g is homeomorphic to a 3–dimensional ball.

Let $D_1, D_2 \subset H$ be properly embedded, essential disks in the handlebody H_g. Let $a \subset \partial H$ be an embedded arc with one endpoint on ∂D_1 and the other endpoint on ∂D_2. Let N be the closure of a regular neighborhood of $D_1 \cup D_2 \cup a$ in H. Then N is homeomorphic to a closed 3–ball, and it intersects ∂H_g in a subset of ∂N homeomorphic to a three-punctured 2–sphere. The complement $\partial N \setminus \partial H_g$ of this subset consists of the disjoint union of three disks, two of which are isotopic to D_1 and D_2 respectively, and the third one is denoted by $D_1 *_a D_2$. See Figure 1.
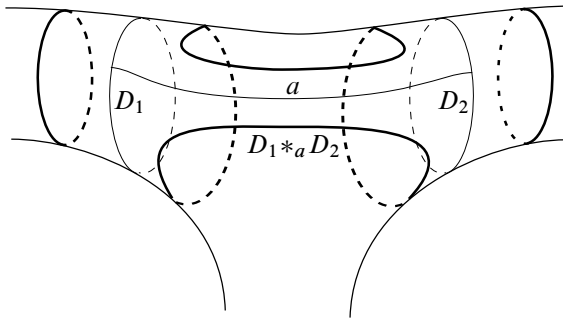

Figure 1: A disk slide

Let $\boldsymbol{D} = \{D_1, \ldots, D_g\}$ be a minimal system of disks for a handlebody H_g, $a \subset \partial H_g$ an embedded arc with one endpoint on ∂D_i, the other endpoint on ∂D_j, with $i \neq j$, and the interior of a disjoint from $\bigcup_i \partial D_i$. Then, removing either D_i or D_j from $\boldsymbol{D}$ and adding $D_i *_a D_j$ yields a new minimal system of disks $\boldsymbol{D}'$ for H_g, well-defined up to isotopy [11, Corollary 2.11]. In this situation we say that $\boldsymbol{D}'$ is obtained from $\boldsymbol{D}$ by a *disk slide*.

Definition 3.2 Two minimal systems of disks for H_g are *slide equivalent* if they are connected by a finite sequence $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_m$ such that $\boldsymbol{D}_{i+1}$ is obtained from $\boldsymbol{D}_i$ by a disk slide for each i.

To prove Theorem 1.1 we need the following result (see [11, Theorem 2.13] for a modern exposition).

Theorem 3.3 [14; 15] *Any two minimal systems of disks for a handlebody are slide equivalent.*

We can now start the formal proof of Theorem 1.1. The first step is to normalize the position of certain nonseparating 2–spheres with respect to a Heegaard splitting. This will be done in the following lemma.

Lemma 3.4 Let $M = H \cup H'$ *be a Heegaard splitting of a* 3*–manifold* M *which admits a prime factorization*

$$(3) \qquad\qquad M = M_1 \# \cdots \# M_h \# S^2 \times S^1 \# \overset{(k)}{\cdots} \# S^2 \times S^1$$

with $b_1(M) = k$. *Then there are pairwise disjoint, embedded* 2*–spheres* $S_1, \ldots, S_k$ *in* M *such that each* S_i *intersects the Heegaard surface* ∂H *in a single circle* C_i. *Moreover, after choosing an orientation of each* S_i, *the corresponding* 2*–homology classes* $[S_i]$ *generate* $H_2(M; \mathbb{Q})$.

Proof Suppose that $M' = M_1 \# \cdots \# M_h$, where each M_i is irreducible. By definition any embedded 2–sphere $S \subset M_i$ bounds a 3–ball. Therefore, if we denote by $S'_1, \ldots, S'_{h-1} \subset M'$ the separating spheres along which the connected sums are performed and $S'_h \subset M'$ is any smoothly embedded 2–sphere disjoint from $S'_1, \ldots, S'_{h-1}$, then the closure of some component of $M' \setminus \bigcup_{i=1}^{h} S'_i$ is a punctured 3–ball.

In the terminology of Haken [8], a collection of pairwise disjoint, embedded 2–spheres with such a property is called a *complete system of spheres*. Thus, the collection $S'_1, \ldots, S'_{h-1}$ is a complete system of spheres for M'. If we view each sphere S'_i as contained in M and denote by $S'_{h-1+i} \subset M$, for $i = 1, \ldots, k$, the embedded 2–sphere corresponding to $S^2 \times \{1\}$ in the ith $S^2 \times S^1$–factor of the factorization (3), the whole collection $S'_1, \ldots, S'_{h-1}, S'_h, \ldots, S'_{h-1+k}$ is a complete system of spheres for M.

Observe that, since $b_1(M) = k$, $b_1(M') = 0$. Then, after choosing orientations, the homology classes $[S'_{h-1+i}] \in H_2(M; \mathbb{Q})$ generate $H_2(M; \mathbb{Q})$ as a $\mathbb{Q}$–vector space, and *a fortiori* the same is true for the classes $[S'_1], \ldots, [S'_{h-1+k}]$.

Now, according to the lemma on page 84 of [8], the system of spheres $S'_1, \ldots, S'_{h-1+k}$ may be transformed by a finite sequence of isotopies and "ρ–operations" (see [8] for the definition) into a collection of pairwise disjoint, incompressible 2–spheres $S_1, \ldots, S_t$, $t \geq h - 1 + k$, such that each S_i intersects the Heegaard surface ∂H in a single

circle $C_i = S_i \cap \partial H$, and moreover the classes $[S_i]$ still generate $H_2(M;\mathbb{Q})$. Since $\dim_{\mathbb{Q}} H_2(M;\mathbb{Q}) = k$, up to renaming the spheres we may assume that $[S_1], \ldots, [S_k]$ are generators of $H_2(M;\mathbb{Q})$. This finishes the proof of the lemma. $\qquad\square$

Proof of Theorem 1.1 Let (B, π) be an open book decomposition of a closed, oriented, connected 3–manifold M with page Σ and monodromy φ. If φ is trivial then it is easy to check that M is homeomorphic to the connected sum of $b_1(\Sigma)$ copies of $S^2 \times S^1$. This proves one direction of the statement. For the other direction, suppose that M factorizes as in (2). In view of Proposition 2.2 or Inequality (1) we have

$$b_1(\Sigma) \geq b_1(M) \geq k.$$

If $b_1(\Sigma) = k$, the above inequality implies $b_1(M) = k$ and therefore if we set

$$M' := M_1 \# \cdots \# M_h$$

we have $b_1(M') = 0$.

Denote by $H_{b_1(\Sigma)} \subset M$ the handlebody of genus $b_1(\Sigma)$ consisting of a regular neighborhood of Σ in M. Since Σ is the fiber of a fibration, the closure of the complement $M \setminus H_{b_1(\Sigma)}$ is a handlebody as well, which we denote by $H'_{b_1(\Sigma)}$. It follows that M admits the Heegaard splitting

$$(4) \qquad\qquad M = H_{b_1(\Sigma)} \cup H'_{b_1(\Sigma)}.$$

By Lemma 3.4 there are pairwise disjoint embedded spheres $S_1, \ldots, S_k \subset M$ which generate $H_2(M;\mathbb{Q})$ and such that each S_i intersects the Heegaard surface $\partial H_{b_1(\Sigma)}$ in a single circle C_i.

Observe that each circle C_i bounds the disk $D_i = S_i \cap H_{b_1(\Sigma)}$ inside $H_{b_1(\Sigma)}$ and the disk $S_i \cap H'_{b_1(\Sigma)}$ inside $H'_{b_1(\Sigma)}$. Since the map

$$H_2(M;\mathbb{Q}) \to H_1(\partial H_{b_1(\Sigma)};\mathbb{Q})$$

appearing in the Mayer–Vietoris sequence associated with the decomposition (4) is injective, after choosing orientations we see that the induced homology classes $[C_i]$ generate a half-dimensional subspace of $H_1(\partial H_{b_1(\Sigma)};\mathbb{Q})$ which is Lagrangian for the intersection form on $H_1(\partial H_{b_1(\Sigma)};\mathbb{Q})$ because the C_i are pairwise disjoint.

We now claim that the D_i are a minimal system of compressing disks for $H_{b_1(\Sigma)}$. To see this we can argue by induction on $b_1(\Sigma)$. If $b_1(\Sigma) = 0$ there is nothing to prove, so we may assume $b_1(\Sigma) > 0$. Let N be an open regular neighborhood of D_1. Since $[C_1] \neq 0$, $H_{b_1(\Sigma)} \setminus N$ is connected and therefore by [11, Proposition 5.18] it is a handlebody. Moreover, the remaining homology classes $[C_i]$, $i \geq 2$, generate a

Lagrangian subspace in the first homology group of the boundary of $H_{b_1(\Sigma)} \setminus N$. By the inductive assumption the disks D_i for $i \geq 2$ are a minimal system of compressing disks for $H_{b_1(\Sigma)} \setminus N$, which proves the claim.

Recall that, by construction, the curves $C_i = \partial D_i$ bound compressing disks in $H'_{b_1(\Sigma)}$. Arguing as for $H_{b_1(\Sigma)}$ shows that such disks constitute a minimal system for $H'_{b_1(\Sigma)}$. Thus, surgering M along the spheres $S_1, \ldots, S_k$ yields a 3–manifold having a genus-0 Heegaard splitting, ie S^3. This implies that M is a connected sum of k copies of $S^2 \times S^1$, and we are left to show that the monodromy φ is trivial.

Now we choose a system of arcs for Σ, ie a collection of properly embedded, pairwise disjoint oriented arcs $a_1, \ldots, a_{b_1(\Sigma)} \subset \Sigma$ whose associated homology classes $[a_i] \in H_1(\Sigma, \partial\Sigma; \mathbb{Q})$ generate the $\mathbb{Q}$–vector space $H_1(\Sigma, \partial\Sigma; \mathbb{Q})$. Then, after fixing an identification $H_{b_1(\Sigma)} = \Sigma \times I$, the disks $a_i \times I \subset \Sigma \times I$ yield another minimal system of disks $\{D'_i\}_{i=1}^g$ for $H_{b_1(\Sigma)}$. Thus, according to Theorem 3.3, the system $\{D_i\}_{i=1}^g$ is slide equivalent to the system $\{D'_i\}_{i=1}^g$. But recall that, by construction, each curve $C_i = \partial D_i$ bounds a compressing disk in $H'_{b_1(\Sigma)}$, and a moment's reflection shows that any disk slide among the D_i gives rise to a disk $D_i *_a D_j$ whose boundary also bounds a compressing disk in $H'_{b_1(\Sigma)}$. By induction we conclude that any minimal system of disks $\{\widetilde{D}_i\}_{i=1}^g$ obtained from $\{D_i\}_{i=1}^g$ by a finite sequence of isotopies and disk slides still has the property that each curve $\partial\widetilde{D}_i$ bounds a compressing disk in $H'_{b_1(\Sigma)}$.

In particular, this conclusion applies to the system $\{D'_i\}_{i=1}^g$, showing that each of the circles $\partial D'_i$ bounds a compressing disk in $H'_{b_1(\Sigma)}$. Since the splitting (4) is induced by the open book decomposition (B, π), we can choose an identification $H'_{b_1(\Sigma)} = \Sigma \times [0, 1]$ such that each $\partial D'_i$ is of the form

$$a_i \times \{0\} \bigcup \varphi(a_i) \times \{1\},$$

where φ is the monodromy of (B, π). The fact that $\partial D'_i$ bounds a disk in $H'_{b_1(\Sigma)}$ says that there is a family of arcs in $\Sigma \times I$ interpolating between $a_i \times \{0\}$ and $\varphi(a_i) \times \{1\}$. Mapping such family to Σ via the projection $\Sigma \times I \to \Sigma$ shows that each a_i is homotopic to $\varphi(a_i)$ (with fixed endpoints), and therefore by [4] each a_i is isotopic to $\varphi(a_i)$ via an isotopy which keeps the endpoints fixed. Since $\{a_i\}$ is a system of arcs for Σ, a standard argument based on the Alexander lemma [5, Lemma 2.1] implies that φ is isotopic to the identity of Σ via diffeomorphisms which fix $\partial\Sigma$ pointwise. This concludes the proof of Theorem 1.1. $\square$

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