Charts, signatures, and stabilizations
of Lefschetz fibrations

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We employ a certain labeled finite graph, called a chart, in a closed oriented surface to describe the monodromy of a(n achiral) Lefschetz fibration over the surface. Applying charts and their moves with respect to Wajnryb's presentation of mapping class groups, we first generalize a signature formula for Lefschetz fibrations over the 2–sphere obtained by Endo and Nagami to that for Lefschetz fibrations over arbitrary closed oriented surface. We then prove two theorems on stabilization of Lefschetz fibrations under fiber summing with copies of a typical Lefschetz fibration as generalizations of a theorem of Auroux.

1 Introduction

Matsumoto [31] proved that every Lefschetz fibration of genus one over a closed oriented surface is isomorphic to a fiber sum of copies of a holomorphic elliptic fibration on $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ and a trivial torus bundle over the surface if it has at least one critical point. This result played a crucial role in completing the classification of diffeomorphism types of elliptic surfaces (see Gompf and Stipsicz [13, Section 8.3]). Although such a classification has not been established for Lefschetz fibrations of higher genus, Auroux [1] proved a stabilization theorem for Lefschetz fibrations of genus two, which states that every Lefschetz fibration of genus two over the 2–sphere becomes isomorphic to a fiber sum of copies of three typical fibrations after fiber summing with a holomorphic fibration on $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Auroux [2] gave a generalization of this theorem for Lefschetz fibrations of higher genus, which states that two Lefschetz fibrations of the same genus over the 2–sphere which have the same signature, the same numbers of singular fibers of each type, and admit sections of the same self-intersection number become isomorphic after fiber summing the same number of copies of a "universal" Lefschetz fibration.

Kamada [17; 18] introduced charts, which are labeled finite graphs in a disk, to describe monodromies of surface braids (see also a textbook of Kamada [19]). Kamada, Matsumoto, Matumoto, and Waki [22] considered a variant of charts for Lefschetz fibrations of genus one to give a remarkably simple proof of the above result of Matsumoto. Furthermore, Kamada [21] and Endo and Kamada [6; 5] made use of generalized charts to give a simple proof of the above theorem of Auroux for Lefschetz fibrations of genus two, and to investigate a stabilization theorem and an invariant of hyperelliptic Lefschetz fibrations of arbitrary genus. See also Baykur and Kamada [4], and Hayano [16] for applications of charts to broken Lefschetz fibrations.

In this paper we introduce a chart description for Lefschetz fibrations of genus greater than two over closed oriented surfaces of arbitrary genus to show a signature formula and two theorems on stabilization for such fibrations. In Section 2 we introduce charts and chart moves with respect to Wajnryb's presentation of mapping class groups to examine monodromies of Lefschetz fibrations. After a short survey of Meyer's signature cocycle, we generalize a signature formula of Endo and Nagami [8] for Lefschetz fibrations over the 2–sphere to that for Lefschetz fibrations over a closed oriented surface of arbitrary genus in Section 3. Section 4 is devoted to proofs of two theorems on stabilization of Lefschetz fibrations under fiber summing with copies of a "universal" Lefschetz fibration. In particular the first of our stabilization theorems is a generalization of the theorem of Auroux [2]. We make several comments on variations of chart description and propose some possible directions for future research in Section 5.

2 Chart description for Lefschetz fibrations

In this section we review a definition and properties of Lefschetz fibrations and introduce a chart description for Lefschetz fibrations of genus greater than two.

2.1 Lefschetz fibrations and their monodromies

In this subsection we review a precise definition and basic properties of Lefschetz fibrations. More details can be found in Matsumoto [32] and Gompf and Stipsicz [13].

Let Σ_g be a connected closed oriented surface of genus g.

Definition 2.1 Let M and B be connected closed oriented smooth 4–manifold and 2–manifold, respectively. A smooth map $f\colon M \to B$ is called a *Lefschetz fibration* of genus g if it satisfies the following conditions:

   (i)   The set $\Delta \subset B$ of critical values of f is finite and f is a smooth fiber bundle over $B - \Delta$ with fiber Σ_g.

(ii) For each $b \in \Delta$, there exists a unique critical point p in the *singular fiber* $F_b := f^{-1}(b)$ such that f is locally written as $f(z_1, z_2) = z_1 z_2$ or $\overline{z}_1 z_2$ with respect to some local complex coordinates around p and b which are compatible with the orientations of M and B.

(iii) No fiber contains a ± 1–sphere.

We call M the *total space*, B the *base space*, and f the *projection*. We call p a critical point of *positive type* (resp. of *negative type*) and F_b a singular fiber of *positive type* (resp. of *negative type*) if f is locally written as $f(z_1, z_2) = z_1 z_2$ (resp. $f(z_1, z_2) = \overline{z}_1 z_2$) in (ii). For a regular value $b \in B$ of f, $f^{-1}(b)$ is often called a *general fiber*.

Remark 2.2 A Lefschetz fibration in this paper is called an *achiral* Lefschetz fibration in many other papers.

Let $f: M \to B$ and $f': M' \to B$ be Lefschetz fibrations of genus g over the same base space B. We say that f is *isomorphic* to f' if there exist orientation-preserving diffeomorphisms $H: M \to M'$ and $h: B \to B$ which satisfy $f' \circ H = h \circ f$. If we can choose such an h isotopic to the identity relative to a given base point $b_0 \in B$, we say that f is *strictly isomorphic* to f'.

Let $\mathcal{M}_g$ be the mapping class group of Σ_g, namely, the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g. We assume that $\mathcal{M}_g$ acts on the *right*: the symbol $\varphi\psi$ means that we apply φ first and then ψ for $\varphi, \psi \in \mathcal{M}_g$. We denote the mapping class group of Σ_g acting on the left by $\mathcal{M}_g^*$. Hence the identity map $\mathcal{M}_g \to \mathcal{M}_g^*$ is an anti-isomorphism.

Let $f: M \to B$ be a Lefschetz fibration of genus g as in Definition 2.1. Take a base point $b_0 \in B$ and an orientation-preserving diffeomorphism $\Phi: \Sigma_g \to F_0 := f^{-1}(b_0)$. Since f restricted over $B - \Delta$ is a smooth fiber bundle with fiber Σ_g, we can define a homomorphism

$$\rho: \pi_1(B - \Delta, b_0) \to \mathcal{M}_g,$$

called the *monodromy representation* of f with respect to Φ. Let γ be the loop based at b_0 consisting of the boundary circle of a small disk neighborhood of $b \in \Delta$ oriented counterclockwise and a simple path connecting a point on the circle to b_0 in $B - \Delta$. It is known that $\rho([\gamma])$ is a Dehn twist along some essential simple closed curve c on Σ_g, which is called the *vanishing cycle* of the critical point p on $f^{-1}(b)$. If p is of positive type (resp. of negative type), then the Dehn twist is right-handed (resp. left-handed).

A singular fiber is said to be of *type* I if the vanishing cycle is non-separating and of *type* II_h for $h = 1, \ldots, [g/2]$ if the vanishing cycle is separating and it bounds a genus-h subsurface of Σ_g. A singular fiber is said to be of *type* I^+ (resp. *type* I^- and *type* II_h^+, *type* II_h^-) if it is of type I and of positive type (resp. of type I and of negative type, of type II_h and of positive type, of type II_h and of negative type). We denote by $n_0^+(f)$, $n_0^-(f)$, $n_h^+(f)$, and $n_h^-(f)$, the numbers of singular fibers of f of type I^+, I^-, II_h^+, and II_h^-, respectively. A Lefschetz fibration is called *irreducible* if every singular fiber is of type I. A Lefschetz fibration is called *chiral* if every singular fiber is of positive type.

Suppose that the cardinality of Δ is equal to n. A system $\mathcal{A} = (A_1, \ldots, A_n)$ of arcs on B is called a *Hurwitz arc system* for Δ with base point b_0 if each A_i is an embedded arc connecting b_0 with a point of Δ in B such that $A_i \cap A_j = \{b_0\}$ for $i \neq j$, and they appear in this order around b_0 (see Kamada [19]). When B is a 2–sphere, the system $\mathcal{A}$ determines a system of generators of $\pi_1(B - \Delta, b_0)$, say $(a_1, \ldots, a_n)$. We call $(\rho(a_1), \ldots, \rho(a_n))$ a *Hurwitz system* of f.

2.2 Chart description and Wajnryb's presentation

In this subsection we introduce a chart description for Lefschetz fibrations of genus greater than two by employing Wajnryb's finite presentation [40] of mapping class groups. General theories of charts for presentations of groups were developed independently by Kamada [20] and Hasegawa [15]. We use the terminology of chart description in Kamada [20].
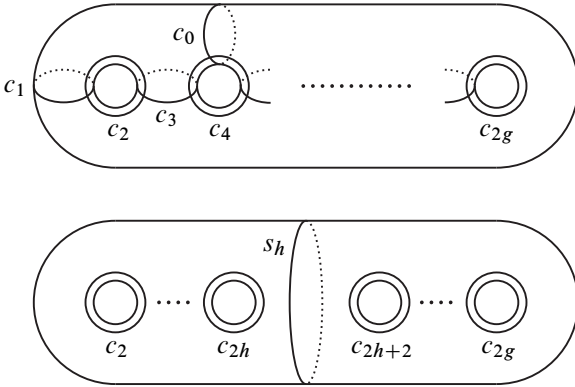

Figure 1: Simple closed curves on Σ_g

We first review a finite presentation of the mapping class group of a closed oriented surface due to Wajnryb. For $i = 0, 1, \ldots, 2g$, let ζ_i be a right-handed Dehn twist along the simple closed curve c_i on Σ_g depicted in Figure 1.

Theorem 2.3 (Wajnryb [40; 41]) *Suppose that g is greater than two. The mapping class group $\mathcal{M}_g$ is generated by the elements $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{2g}$ and has the following defining relations:*

- *(Far commutation)*

$$\zeta_i \zeta_j = \zeta_j \zeta_i \quad (1 \leq i < j - 1 \leq 2g - 1),$$
$$\zeta_0 \zeta_j = \zeta_j \zeta_0 \quad (j = 1, 2, 3, 5, \ldots, 2g).$$

- *(Braid relation)*

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1} \quad (i = 1, \ldots, 2g - 1),$$
$$\zeta_0 \zeta_4 \zeta_0 = \zeta_4 \zeta_0 \zeta_4.$$

- *(3–chain relation)*

$$(\zeta_3 \zeta_2 \zeta_1)^4 = \zeta_0 \zeta_4^{-1} \zeta_3^{-1} \zeta_2^{-1} \zeta_1^{-2} \zeta_2^{-1} \zeta_3^{-1} \zeta_4^{-1} \zeta_0 \zeta_4 \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \zeta_4.$$

- *(Lantern relation)*

$$\delta_3 \zeta_1 \zeta_3 \zeta_5 = \zeta_0 \tau_2 \zeta_0 \tau_2^{-1} \tau_1 \tau_2 \zeta_0 \tau_2^{-1} \tau_1^{-1},$$

where

$$\tau_1 := \zeta_2 \zeta_3 \zeta_1 \zeta_2, \quad \tau_2 := \zeta_4 \zeta_5 \zeta_3 \zeta_4, \quad \mu := \zeta_5 \zeta_6 \tau_2 \zeta_0 \tau_2^{-1} \zeta_6^{-1} \zeta_5^{-1},$$
$$\nu := \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_0 \zeta_4^{-1} \zeta_3^{-1} \zeta_2^{-1} \zeta_1^{-1}, \quad \delta_3 := \zeta_6^{-1} \zeta_5^{-1} \zeta_4^{-1} \zeta_3^{-1} \zeta_2^{-1} \mu^{-1} \nu \mu \zeta_2 \zeta_3 \zeta_4 \zeta_5 \zeta_6.$$

- *(Hyperelliptic relation)*

$$\zeta_{2g} \cdots \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \cdots \zeta_{2g} \delta_g = \delta_g \zeta_{2g} \cdots \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \cdots \zeta_{2g},$$

where

$$\tau_1 := \zeta_2 \zeta_3 \zeta_1 \zeta_2, \qquad\qquad\qquad \tau_i := \zeta_{2i} \zeta_{2i-1} \zeta_{2i+1} \zeta_{2i},$$
$$\nu_1 := \zeta_4^{-1} \zeta_3^{-1} \zeta_2^{-1} \zeta_1^{-2} \zeta_2^{-1} \zeta_3^{-1} \zeta_4^{-1} \zeta_0 \zeta_4 \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \zeta_4, \quad \nu_i := \tau_{i-1} \tau_i \nu_{i-1} \tau_i^{-1} \tau_{i-1}^{-1},$$
$$\mu_1 := \zeta_2 \zeta_3 \zeta_4 \nu_1 \zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-1} \zeta_4^{-1},$$
$$\mu_i := \zeta_{2i} \zeta_{2i+1} \zeta_{2i+2} \nu_i \zeta_{2i-1}^{-1} \zeta_{2i}^{-1} \zeta_{2i+1}^{-1} \zeta_{2i+2}^{-1},$$
$$\delta_g := \mu_{g-1}^{-1} \cdots \mu_2^{-1} \mu_1^{-1} \zeta_1 \mu_1 \mu_2 \cdots \mu_{g-1},$$

for $i = 2, \ldots, g - 1$.

We make use of the presentation above to introduce a notion of chart which gives a graphic description of monodromy representations of Lefschetz fibrations. We set

$$\mathcal{X} := \{\zeta_0, \zeta_1, \ldots, \zeta_{2g}\},$$
$$\mathcal{R} := \{r_F(i, j) \mid 1 \leq i < j - 1 \leq 2g - 1\} \cup \{r_F(0, j) \mid j = 1, 2, 3, 5, \ldots, 2g\}$$
$$\cup \{r_B(i) \mid i = 0, 1, \ldots, 2g - 1\} \cup \{r_C, r_L, r_H\},$$
$$\mathcal{S} := \{\ell_0(i)^{\pm 1} \mid i = 0, 1, \ldots, 2g\} \cup \{\ell_h^{\pm 1} \mid h = 1, \ldots, [g/2]\},$$

for $g \geq 3$, where

$$r_F(i, j) := \zeta_i \zeta_j \zeta_i^{-1} \zeta_j^{-1},$$
$$r_B(0) := \zeta_0 \zeta_4 \zeta_0 \zeta_4^{-1} \zeta_0^{-1} \zeta_4^{-1},$$
$$r_B(i) := \zeta_i \zeta_{i+1} \zeta_i \zeta_{i+1}^{-1} \zeta_i^{-1} \zeta_{i+1}^{-1} \quad (i = 1, \ldots, 2g - 1),$$
$$r_C := (\zeta_3 \zeta_2 \zeta_1)^4 \zeta_4^{-1} \zeta_3^{-1} \zeta_2^{-1} \zeta_1^{-2} \zeta_2^{-1} \zeta_3^{-1} \zeta_4^{-1} \zeta_0^{-1} \zeta_4 \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \zeta_4 \zeta_0^{-1},$$
$$r_L := \delta_3 \zeta_1 \zeta_3 \zeta_5 \tau_1 \tau_2 \zeta_0^{-1} \tau_2^{-1} \tau_1^{-1} \tau_2 \zeta_0^{-1} \tau_2^{-1} \zeta_0^{-1},$$
$$r_H := \zeta_{2g} \cdots \zeta_3 \zeta_2 \zeta_1^2 \zeta_2 \zeta_3 \cdots \zeta_{2g} \delta_g \zeta_{2g}^{-1} \cdots \zeta_3^{-1} \zeta_2^{-1} \zeta_1^{-2} \zeta_2^{-1} \zeta_3^{-1} \cdots \zeta_{2g}^{-1} \delta_g^{-1},$$
$$\ell_0(i) := \zeta_i \quad (i = 0, 1, \ldots, 2g),$$
$$\ell_h := (\zeta_1 \zeta_2 \cdots \zeta_{2h})^{4h+2} \quad (h = 1, \ldots, [g/2]),$$

and $\delta_3, \tau_1, \tau_2, \delta_g$ are defined as in Theorem 2.3.

Let B be a connected closed oriented surface and Γ a finite graph in B such that each edge of Γ is oriented and labeled with an element of $\mathcal{X}$. We denote the label ζ_i by i for short. Choose a simple path γ which intersects with edges of Γ transversely and does not intersect with vertices of Γ. For such a path γ, we obtain a word $w_\Gamma(\gamma)$ in $\mathcal{X} \cup \mathcal{X}^{-1}$ by reading off the labels of intersecting edges along γ with exponents as in Figure 2(a). We call the word $w_\Gamma(\gamma)$ the *intersection word* of γ with respect to Γ. Conversely, we can specify the number, orientations, and labels of consecutive edges in Γ by indicating a (dashed) arrow intersecting the edges transversely together with the intersection word of the arrow with respect to Γ (see Figure 2(b) and (c)).
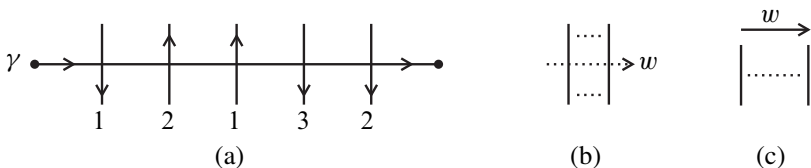

Figure 2: Intersection word $w_\Gamma(\gamma) = w = \zeta_1 \zeta_2^{-1} \zeta_1^{-1} \zeta_3 \zeta_2$

For a vertex v of Γ, a small simple closed curve surrounding v in the counterclockwise direction is called a *meridian loop* of v and denoted by m_v. The vertex v is said to be *marked* if one of the regions around v is specified by an asterisk. If v is marked, the intersection word $w_\Gamma(m_v)$ of m_v with respect to Γ is well-defined. If not, it is determined up to cyclic permutation. See Kamada [20] for details.

Definition 2.4 A *chart* in B is a finite graph Γ in B (possibly being empty or having *hoops* that are closed edges without vertices) whose edges are labeled with an element of $\mathcal{X}$, and oriented so that the following conditions are satisfied (see Figures 3–5):

(1) The vertices of Γ are classified into two families: *white vertices* and *black vertices*.

(2) If v is a white vertex (resp. a black vertex), the word $w_\Gamma(m_v)$ is a cyclic permutation of an element of $\mathcal{R} \cup \mathcal{R}^{-1}$ (resp. of $\mathcal{S}$).

A white vertex v is said to be of *type r* (resp. of *type r^{-1}*) if $w_\Gamma(m_v)^{-1}$ is a cyclic permutation of $r \in \mathcal{R}$ (resp. of $r^{-1} \in \mathcal{R}^{-1}$). A black vertex v is said to be of *type s* if $w_\Gamma(m_v)$ is a cyclic permutation of $s \in \mathcal{S}$. A chart Γ is said to be *marked* if each white vertex (resp. black vertex) v is marked and $w_\Gamma(m_v)$ is exactly an element of $\mathcal{R} \cup \mathcal{R}^{-1}$ (resp. of $\mathcal{S}$). If a base point b_0 of B is specified, we always assume that a chart Γ is disjoint from b_0. A chart consisting of two black vertices and one edge connecting them is called a *free edge*.
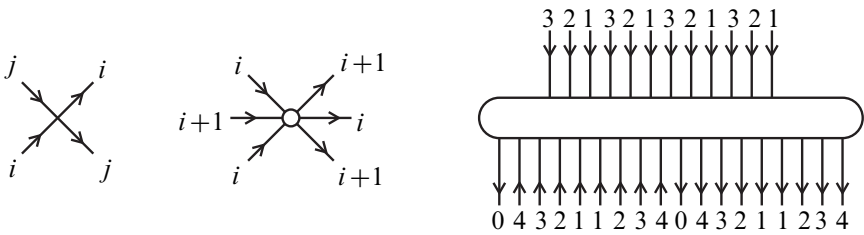

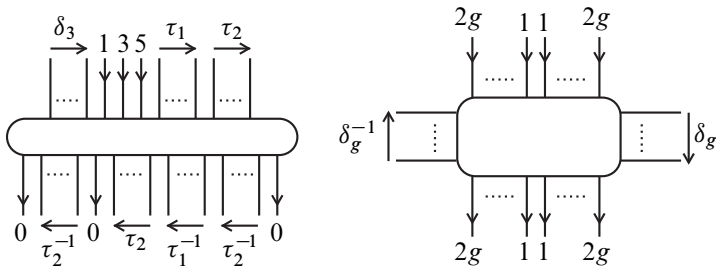Figure 3: Vertices of type $r_F(i, j)$, $r_B(i)$ $(i \neq 0)$, r_C
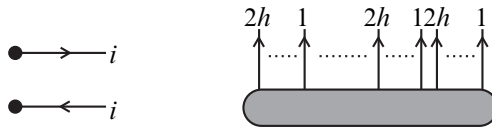

Figure 4: Vertices of type r_L and r_H

Figure 5: Vertices of type $\ell_0(i)^{\pm 1}$ and ℓ_h

Remark 2.5 It is worth noting that the intersection word of a "clockwise" meridian of a white vertex of type r is equal to r, while that of a "counterclockwise" meridian of a black vertex of type s is equal to s in this paper. This notation is different from those of Kamada [20] and Hasegawa [15], who always consider "counterclockwise" meridians for both white and black vertices.

We next introduce several moves for charts. Let Γ and Γ' be two charts on B and b_0 a base point of B.

Let D be a disk embedded in $B - \{b_0\}$. Suppose that the boundary ∂D of D intersects Γ and Γ' transversely.

Definition 2.6 We say that Γ' is obtained from Γ by a *chart move of type W* if $\Gamma \cap (B - \text{Int } D) = \Gamma' \cap (B - \text{Int } D)$ and both $\Gamma \cap D$ and $\Gamma' \cap D$ have no black vertices. We call chart moves of type W shown in Figure 6(a)–(c) a *channel change*, a *birth/death of a hoop*, and a *birth/death of a pair of white vertices*, respectively.
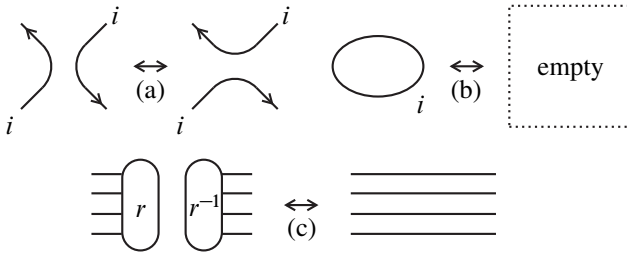

Figure 6: Chart moves of type W

Let s and s' be elements of $\mathcal{S}$. Suppose that there exists a word w in $\mathcal{X} \cup \mathcal{X}^{-1}$ such that two words s' and wsw^{-1} determine the same element of $\mathcal{M}_g$.

Definition 2.7 If a chart Γ contains a black vertex of type s, then we can change a part of Γ near the vertex by using a local replacement depicted in Figure 7 to obtain another chart Γ'. We say that Γ' is obtained from Γ by a *chart move of transition*. Note that the blank labeled with T can be filled only with edges and white vertices.
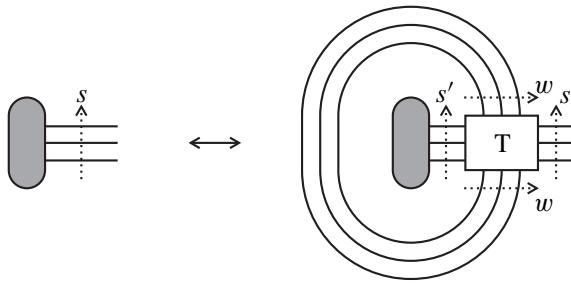
Figure 7: Chart move of transition

Definition 2.8 We say that Γ' is obtained from Γ by a *chart move of conjugacy type* if Γ' is obtained from Γ by a local replacement depicted in Figure 8.

Figure 8: Chart moves of conjugacy type

Let Γ be a chart in B with base point b_0 and Δ_Γ the set of black vertices of Γ. For a loop γ in $B - \Delta_\Gamma$ based at b_0, the element of $\mathcal{M}_g$ determined by the intersection word $w_\Gamma(\gamma)$ of γ with respect to Γ does not depend on a choice of representative of the homotopy class of γ. Thus we obtain a homomorphism $\rho_\Gamma \colon \pi_1(B - \Delta_\Gamma, b_0) \to \mathcal{M}_g$, which is called the *homomorphism determined by* Γ.

We now state a classification of Lefschetz fibrations in terms of charts and chart moves. Let B be a connected closed oriented surface.

Proposition 2.9 *Suppose that g is greater than two.*

(1) *Let f be a Lefschetz fibration of genus g over B and ρ a monodromy representation of f. Then there exists a chart Γ in B such that the homomorphism ρ_Γ determined by Γ is equal to ρ.*

(2) *For every chart Γ in B, there exists a Lefschetz fibration f of genus g over B such that a monodromy representation of f is equal to the homomorphism ρ_Γ determined by Γ.*

We call such Γ as in Proposition 2.9(1) a chart *corresponding to* f, and such f as in Proposition 2.9(2) a Lefschetz fibration *described by* Γ.

Instead of giving a proof of Proposition 2.9, we show an example of a chart and describe the correspondence of the chart to a Hurwitz system of a Lefschetz fibration.

Example 2.10 Let B be a 2–sphere. We consider a chart Γ in B with base point b_0 and a system $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of loops based at b_0, which is determined by a Hurwitz arc system $\mathcal{A}$ for the set Δ_Γ of black vertices of Γ, as in Figure 9. The intersection words of the loops with respect to Γ are

$$w_\Gamma(\gamma_1) = \zeta_1^{-1}\zeta_2^{-1}\zeta_1\zeta_2\zeta_1, \qquad w_\Gamma(\gamma_2) = \zeta_1^{-1}\zeta_3\zeta_1,$$
$$w_\Gamma(\gamma_3) = \zeta_2^{-1}\zeta_3^{-1}\zeta_2^{-1}\zeta_3\zeta_2, \qquad w_\Gamma(\gamma_4) = \zeta_2^{-1},$$

each of which represents the image $\rho_\Gamma(a_i)$ of the homotopy class a_i of γ_i under the homomorphism $\rho_\Gamma : \pi_1(B - \Delta_\Gamma, b_0) \to \mathcal{M}_g$. Since the group $\pi_1(B - \Delta_\Gamma, b_0)$ has a presentation $\langle a_1, a_2, a_3, a_4 \mid a_1a_2a_3a_4 = 1 \rangle$, ρ_Γ is determined by the system $(\rho_\Gamma(a_1), \rho_\Gamma(a_2), \rho_\Gamma(a_3), \rho_\Gamma(a_4))$, which is a Hurwitz system of a certain Lefschetz fibration of genus g over B because each $\rho_\Gamma(a_i)$ is a Dehn twist. Note that the product $w_\Gamma(\gamma_1)w_\Gamma(\gamma_2)w_\Gamma(\gamma_3)w_\Gamma(\gamma_4)$ of the intersection words represents the identity of $\mathcal{M}_g$.
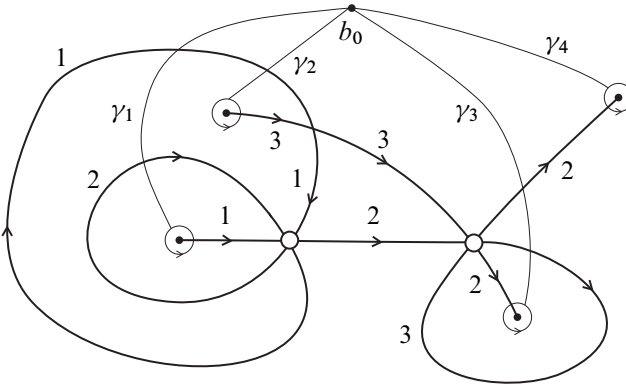

Figure 9: Monodromy of a chart Γ

Theorem 2.11 *Suppose that g is greater than two. Let f and f' be Lefschetz fibrations of genus g over B, and Γ and Γ' charts corresponding to f and f', respectively. Then f is strictly isomorphic to f' if and only if Γ is transformed to Γ' by a finite sequence of chart moves of type W, chart moves of transitions, chart moves of conjugacy type, and ambient isotopies of B relative to b_0.*

Proposition 2.9 and Theorem 2.11 follow from a classification theorem of Lefschetz fibrations due to Kas [23] and Matsumoto [32] together with fundamental theorems on charts and chart moves by Kamada [20, Sections 4–8].

We end this subsection with a definition and chart description of fiber sums of Lefschetz fibrations. Let $f: M \to B$ and $f': M' \to B'$ be Lefschetz fibrations of genus g. Take

regular values $b_0 \in B$ and $b_0' \in B'$ of f and f', and small disks $D_0 \subset B - \Delta$ and $D_0' \subset B - \Delta'$ near b_0 and b_0', respectively. Consider general fibers $F_0 := f^{-1}(b_0)$ and $F_0' := f'^{-1}(b_0')$ and orientation-preserving diffeomorphisms $\Phi: \Sigma_g \to F_0$ and $\Phi': \Sigma_g \to F_0'$, respectively.

Definition 2.12 Let $\Psi: \Sigma_g \to \Sigma_g$ be an orientation-preserving diffeomorphism and $r: \partial D_0 \to \partial D_0'$ an orientation-reversing diffeomorphism. The new manifold $M \#_F M'$ obtained by gluing $M - f^{-1}(\mathrm{Int}\, D_0)$ and $M' - f'^{-1}(\mathrm{Int}\, D_0')$ by $(\Phi' \circ \Psi \circ \Phi^{-1}) \times r$ admits a Lefschetz fibration $f \#_\Psi f': M \#_F M' \to B \# B'$ of genus g. We call $f \#_\Psi f'$ the *fiber sum* of f and f' with respect to Ψ. Although the diffeomorphism type of $M \#_F M'$ and the isomorphism type of $f \#_\Psi f'$ depend on a choice of the diffeomorphism Ψ in general, we often abbreviate $f \#_\Psi f'$ as $f \# f'$.

Let Γ and Γ' be charts corresponding to f and f', and D_0 and D_0' small disks near b_0 and b_0' disjoint from Γ and Γ', respectively. Connecting $B - \mathrm{Int}\, D_0$ with $B' - \mathrm{Int}\, D_0'$ by a tube, we have a connected sum $B \# B'$ of B and B'. Let w be a word in $\mathcal{X} \cup \mathcal{X}^{-1}$ which represents the mapping class of Ψ in $\mathcal{M}_g$. Let $\Gamma \#_w \Gamma'$ be the union of Γ, Γ', and hoops on the tube representing w (see Figure 10). Then the fiber sum $f \#_\Psi f'$ is described by this new chart $\Gamma \#_w \Gamma'$ in $B \# B'$ with base point b_0. If the word w is trivial, then the chart $\Gamma \#_w \Gamma'$ is denoted also by $\Gamma \oplus \Gamma'$, which is called a *product* of Γ and Γ'.
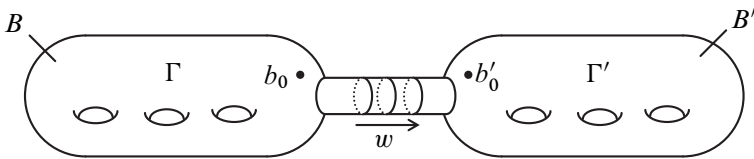

Figure 10: Chart $\Gamma \#_w \Gamma'$ in $B \# B'$

3 Signature of Lefschetz fibrations

In this section we review the signature cocycle discovered by Meyer and prove a signature theorem for Lefschetz fibrations.

3.1 Meyer's signature cocycle

In this subsection we give a brief survey on Meyer's signature cocycle. We begin with the definition of the signature cocycle. Let g be a positive integer.

Definition 3.1 (Meyer [33]) For $A, B \in \mathrm{Sp}(2g, \mathbb{Z})$, we consider the vector space

$$V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}$$

and the bilinear form $\langle \, , \, \rangle_{A,B}: V_{A,B} \times V_{A,B} \to \mathbb{R}$ defined by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := (x_1 + y_1) \cdot J(I_{2g} - B)y_2,$$

where $\cdot$ is the standard inner product of $\mathbb{R}^{2g}$ and $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. Since $\langle \, , \, \rangle_{A,B}$ is symmetric, we can define an integer $\tau_g(A, B)$ to be the signature of $(V_{A,B}, \langle \, , \, \rangle_{A,B})$. The map $\tau_g: \mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g, \mathbb{Z}) \to \mathbb{Z}$ is called the *signature cocycle*.

Let P be a compact connected oriented surface of genus 0 with three boundary components and $\pi: E \to P$ a fiber bundle over P with fiber Σ_g and structure group $\mathrm{Diff}_+ \Sigma_g$. The fundamental group $\pi_1(P, *)$ of P with base point $*$ is a free group generated by two loops a and b depicted in Figure 11. If we take an orientation-preserving diffeomorphism $\Sigma_g \to \pi^{-1}(*)$, we obtain the monodromy representation $\pi_1(P, *) \to \mathcal{M}_g$ which sends a to α and b to β. Since $\mathcal{M}_g^*$ acts on $H := H_1(\Sigma_g; \mathbb{Z})$ and preserves the intersection form, we have a representation $\mathcal{M}_g^* \to \mathrm{Sp}(2g, \mathbb{Z})$ by fixing a symplectic basis on H. Let A and B denote matrices corresponding to α and β, respectively.
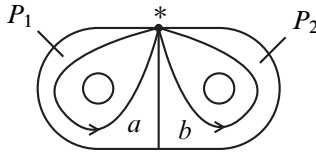

Figure 11: Pair of pants P

Meyer closely studied the signature of the total space E to obtain the following theorem.

Theorem 3.2 (Meyer [33]) *The signature $\sigma(E)$ of E is equal to $-\tau_g(A, B)$.*

Theorem 3.2 and Novikov's additivity implies that τ_g is a 2–cocycle of $\mathrm{Sp}(2g, \mathbb{Z})$.

We recall the Maslov index of a triple of Lagrangian subspaces and Wall's non-additivity theorem, which are used in the proof of Theorem 3.2.

Let V be a real vector space of dimension $2n$, $\omega \in \Lambda^2 V^*$ a symplectic form on V, and $\Lambda(V, \omega)$ the Lagrangian Grassmannian of (V, ω), which is the set of Lagrangian subspaces of (V, ω). For $L_1, L_2, L_3 \in \Lambda(V, \omega)$, the bilinear form

$$\Psi: (L_3 + L_1) \cap L_2 \times (L_3 + L_1) \cap L_2 \to \mathbb{R}, \quad (v, w) \mapsto \omega(v, w_3),$$

where $v, w \in (L_3 + L_1) \cap L_2$ and $w = w_1 + w_3$ $(w_1 \in L_1, w_3 \in L_3)$, is symmetric. We define an integer $i(L_1, L_2, L_3)$ to be the signature of $((L_3 + L_1) \cap L_2, \Psi)$, which is called the *ternary Maslov index* of the triple (L_1, L_2, L_3).

Let M_1, M_2 be compact oriented smooth 4–manifolds, X_1, X_2, X_3 compact oriented smooth 3–manifolds, and Σ a closed oriented smooth 2–manifold. We assume that $M = M_1 \cup M_2, \partial M_1 = X_1 \cup X_2, \partial M_2 = X_2 \cup X_3, \partial X_1 = \partial X_2 = \partial X_3 = \Sigma$, and the orientations of these manifolds satisfy

$$[M] = [M_1] + [M_2], \quad \partial_*[M_1] = [X_2] - [X_1], \quad \partial_*[M_2] = [X_3] - [X_2],$$
$$\partial_*[X_1] = \partial_*[X_2] = \partial_*[X_3] = [\Sigma].$$

Let $\omega \colon V \times V \to \mathbb{R}$ be the intersection form on $V := H_1(\Sigma; \mathbb{R})$ and L_i the kernel of the homomorphism $V \to H_1(X_i; \mathbb{R})$ induced by the inclusion $\Sigma \to X_i$ for $i = 1, 2, 3$. Since $L_i \in \Lambda(V, \omega)$ for $i = 1, 2, 3$, we can define the ternary Maslov index $i(L_1, L_2, L_3)$ of the triple (L_1, L_2, L_3).

Theorem 3.3 (Wall [42]) $\sigma(M) = \sigma(M_1) + \sigma(M_2) - i(L_1, L_2, L_3)$.

Gambaudo and Ghys [10] (and independently the first author) made use of Theorem 3.3 to give the following proof of Theorem 3.2. See also Gilmer and Masbaum [12].

Proof of Theorem 3.2 Consider P to be a boundary sum of two annuli P_1 and P_2 (see Figure 11). We set $M := E$, $M_i := \pi^{-1}(P_i)(i = 1, 2)$, $X_2 := M_1 \cap M_2$, $X_1 := \partial M_1 - \text{Int } X_2$, $X_3 := \partial M_3 - \text{Int } X_2$, and $\Sigma := \partial X_2$. Applying Theorem 3.3 to these manifolds, we have

$$\sigma(E) = \sigma(M_1) + \sigma(M_2) - i(L_1, L_2, L_3) = -i(L_1, L_2, L_3)$$

because each of M_1 and M_2 is a product of a mapping torus with an interval, which has signature zero. Since the bordered component of X_i is diffeomorphic to $I \times \Sigma_g$ for $i = 1, 2, 3$, we put $V := H \oplus H$, $\omega := \mu \oplus (-\mu)$, and obtain

$$L_1 = \{(-\xi, \alpha_*^{-1}(\xi)) \in V \mid \xi \in H\},$$
$$L_2 = \{(-\xi, \xi) \in V \mid \xi \in H\},$$
$$L_3 = \{(-\xi, \beta_*(\xi)) \in V \mid \xi \in H\},$$

where H is the first homology $H_1(\Sigma_g; \mathbb{R})$ of Σ_g and $\mu \colon H \times H \to \mathbb{R}$ is the intersection form of Σ_g. It is easily seen that the subspace $(L_1 + L_3) \cap L_2$ is written as

$$(L_1 + L_3) \cap L_2 = \{(-\xi - \eta, \alpha_*^{-1}(\xi) + \beta_*(\eta)) \in V \mid \xi + \eta = \alpha_*^{-1}(\xi) + \beta_*(\eta)(\xi, \eta \in H)\}$$

and the symmetric bilinear form Ψ on $(L_1 + L_3) \cap L_2$ is written as

$$\Psi\big((-\xi - \eta, \alpha_*^{-1}(\xi) + \beta_*(\eta)), (-\xi' - \eta', \alpha_*^{-1}(\xi') + \beta_*(\eta'))\big) = \mu\big(\xi + \eta, (\mathrm{id} - \beta_*)(\eta')\big).$$

We consider the vector space

$$U_{\alpha,\beta} := \{(\xi, \eta) \in V \mid (\alpha_*^{-1} - \mathrm{id})(\xi) + (\beta_* - \mathrm{id})(\eta) = 0\}$$

and the symmetric bilinear form $\langle \ , \ \rangle_{\alpha,\beta}$ on $U_{\alpha,\beta}$ defined by

$$\langle (\xi, \eta), (\xi', \eta') \rangle_{\alpha,\beta} := \mu(\xi + \eta, (\mathrm{id} - \beta_*)(\eta')) \quad ((\xi, \eta), (\xi', \eta') \in U_{\alpha,\beta}).$$

Since the linear map $U_{\alpha,\beta} \to (L_1 + L_3) \cap L_2 \colon (\xi, \eta) \mapsto (-\xi - \eta, \xi + \eta)$ is compatible with the bilinear forms, the signature of $((L_1 + L_3) \cap L_2, \Psi)$ is equal to that of $(U_{\alpha,\beta}, \langle \ , \ \rangle_{\alpha,\beta})$, which is isomorphic to $(V_{A,B}, \langle \ , \ \rangle_{A,B})$ under a choice of a symplectic basis of H. Therefore we conclude that $i(L_1, L_2, L_3) = \tau_g(A, B)$. $\qquad\square$

Remark 3.4 It is known that τ_g is a normalized, symmetric 2–cocycle of $\mathrm{Sp}(2g, \mathbb{Z})$ and invariant under conjugation. The cohomology class $[\tau_g] \in H^2(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Z})$ corresponds to $-4c_1$ under the homomorphisms

$$H^2(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Z}) \leftarrow H^2(B\,\mathrm{Sp}(2g, \mathbb{R}); \mathbb{Z}) \cong H^2(BU(g); \mathbb{Z}) \cong \mathbb{Z}.$$

For more details, see Meyer [33], Turaev [39], Barge and Ghys [3], and Kuno [27].

3.2 A signature formula

In this subsection we describe the signature of a Lefschetz fibration of genus greater than two in terms of charts. Let g be an integer greater than two.

Let B be a connected closed oriented surface and Γ a chart in B. We denote the number of white vertices of type $r_F(i, j)$ (resp. $r_B(i), r_C, r_L, r_H$) minus the number of white vertices of type $r_F(i, j)^{-1}$ (resp. $r_B(i)^{-1}, r_C^{-1}, r_L^{-1}, r_H^{-1}$) included in Γ by $n_F(i, j)(\Gamma)$ (resp. $n_B(i)(\Gamma), n_C(\Gamma), n_L(\Gamma), n_H(\Gamma)$). Similarly, we denote the number of black vertices of type $\ell_0(i)^{\pm 1}$ (resp. $\ell_h^{\pm 1}$) included in Γ by $n_0^{\pm}(i)(\Gamma)$ (resp. $n_h^{\pm}(\Gamma)$), and set $n_0(i)(\Gamma) := n_0^+(i)(\Gamma) - n_0^-(i)(\Gamma)$ (resp. $n_h(\Gamma) := n_h^+(\Gamma) - n_h^-(\Gamma)$) and $n_0^{\pm}(\Gamma) := \sum_{i=0}^{2g} n_0^{\pm}(i)(\Gamma)$.

Definition 3.5 The number

$$\sigma(\Gamma) := -6n_C(\Gamma) - n_L(\Gamma) + \sum_{h=1}^{[g/2]} (4h(h+1) - 1)n_h(\Gamma)$$

is called the *signature* of Γ.

Let $f\colon M \to B$ be a Lefschetz fibration of genus g and Γ a chart in B corresponding to f. The purpose of this subsection is to show the following theorem.

Theorem 3.6 *The signature $\sigma(M)$ of M is equal to $\sigma(\Gamma)$.*

Remark 3.7 It immediately follows from Theorem 3.6 that $\sigma(\Gamma)$ is invariant under chart moves of type W and chart moves of transition. Although any combinatorial proof of this fact does not seem to be known, Hasegawa [15] proved that $\sigma(\Gamma)$ is invariant under chart moves of transitions by a purely combinatorial method on the assumption that it is invariant under chart moves of type W.

Let $\widetilde{\mathcal{X}}$ be the set of right-handed Dehn twists along simple closed curves in Σ_g and $\widetilde{\mathcal{R}}$ the set of words in $\widetilde{\mathcal{X}} \cup \widetilde{\mathcal{X}}^{-1}$ representing an element of the kernel of the natural epimorphism from the free group generated by $\widetilde{\mathcal{X}}$ to $\mathcal{M}_g$.

Definition 3.8 For a word $w = \alpha_1 \cdots \alpha_n \in \widetilde{\mathcal{R}}$, we define an integer

$$I_g(w) := -\sum_{j=1}^{n-1} \tau_g(\overline{\alpha}_{n-j}, \overline{\alpha}_{n-j+1} \cdots \overline{\alpha}_n) - s(w),$$

where τ_g is the signature cocycle (Definition 3.1), $\overline{\alpha}$ is the image of $\alpha \in \widetilde{\mathcal{X}} \cup \widetilde{\mathcal{X}}^{-1}$ under the composition of the natural map $\widetilde{\mathcal{X}} \cup \widetilde{\mathcal{X}}^{-1} \to \mathcal{M}_g$ and a natural epimorphism $\mathcal{M}_g^* \to \mathrm{Sp}(2g, \mathbb{Z})$, and $s(w)$ is the number of Dehn twists along separating simple closed curves included in w.

Suppose that B is a 2–sphere. If we choose a monodromy representation ρ and a Hurwitz arc system $\mathcal{A}$ for Δ with base point b_0, we have a Hurwitz system $(\alpha_1, \ldots, \alpha_n) \in (\mathcal{M}_g)^n$ of f. Since $\alpha_1, \ldots, \alpha_n$ are Dehn twists and $\alpha_1 \cdots \alpha_n = 1$ in $\mathcal{M}_g$, we think $(\alpha_1, \ldots, \alpha_n)$ as a word $w := \alpha_1 \cdots \alpha_n$ in $\widetilde{\mathcal{R}}$. Theorem 3.2 and Novikov's additivity for signature imply the next theorem.

Theorem 3.9 (Endo and Nagami [8]) *The signature $\sigma(M)$ of M is equal to $I_g(w)$.*

We are now ready to prove Theorem 3.6.

Proof of Theorem 3.6 Choose a base point $b_0 \in B - \Gamma$ and a disk D in $B - \Gamma$ centered at b_0. We denote the set of edges of Γ by $E(\Gamma)$. For each $e \in E(\Gamma)$, we choose a point b_e in a region of $B - \Gamma$ adjacent to e, and a simple path γ_e from b_e to b_0 which intersects with edges of Γ transversely and does not intersect with vertices of Γ. Let w_e be the intersection word of γ_e with respect to Γ and $i_e \in \{0, 1, \ldots, 2g\}$
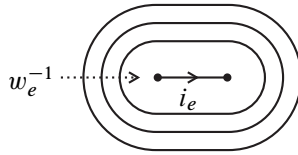
Figure 12: Chart Γ_e

the label of e. We choose a family $\{D_e\}_{e \in E(\Gamma)}$ of mutually disjoint disks included in D and put the chart Γ_e depicted in Figure 12 in D_e for each e.

Taking the union of Γ with Γ_e for all $e \in E(\Gamma)$, we obtain a new chart Γ_1 in B, which describes a fiber sum $f_1 \colon M_1 \to B$ of f with Lefschetz fibrations over S^2 described by a free edge. For each $e \in E(\Gamma)$, we apply channel changes as in Figure 13 to let a free edge pass through the edges intersecting with γ_e. We then apply a channel change as in Figure 14 to "cut" e into two edges. Thus we obtain a new chart Γ_2 in B.
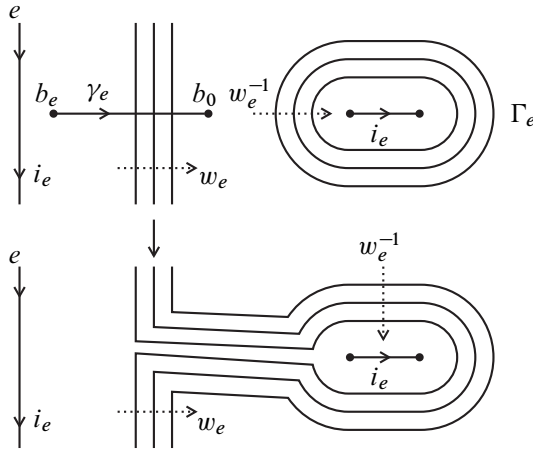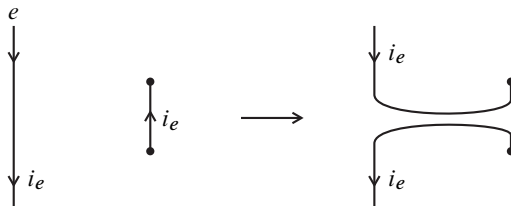

Figure 13: Channel change

Figure 14: Channel change

Since each component of Γ_2 is a tree, a Lefschetz fibration $f_2 \colon M_2 \to B$ corresponding to Γ_2 is a fiber sum of a Lefschetz fibration $f_3 \colon M_3 \to S^2$ with a trivial Σ_g–bundle

over B. Drawing a copy of Γ_2 in S^2, we have a chart Γ_3 corresponding to f_3. The signature of a Lefschetz fibration over S^2 described by a free edge is equal to zero because $\tau_g(A, A^{-1}) = 0$ for any $A \in \mathrm{Sp}(2g, \mathbb{Z})$ (see Meyer [33, Section 2]). Hence we have

$$\sigma(M) = \sigma(M_1) = \sigma(M_2) = \sigma(M_3) + \sigma(\Sigma_g \times B) = \sigma(M_3)$$

by Theorem 2.11 and Novikov's additivity. Since we did not change the numbers of white vertices and black vertices of type $\ell_h^{\pm}$ to make Γ_3 from Γ, we see $\sigma(\Gamma_3) = \sigma(\Gamma)$. Hence we only have to show $\sigma(M_3) = \sigma(\Gamma_3)$ in order to conclude $\sigma(M) = \sigma(\Gamma)$.

Applying chart moves of transition to each component of Γ_3 as in Figure 15, we remove white vertices of type $r_F(i, j)^{\pm 1}, r_B(i)^{\pm 1}, r_H^{\pm 1}$ to obtain a union of copies of $L_0(i), L_h, L_h^*, R_C, R_C^*, R_L, R_L^*$, where $L_0(i), L_h, R_C, R_L$ are charts depicted in Figure 16 and Figure 17, and L_h^* (resp. R_C^*, R_L^*) is the mirror image of L_h (resp. R_C, R_L) with edges orientation reversed. For the proof of $\sigma(M_3) = \sigma(\Gamma_3)$, it is enough to show that the signature of a Lefschetz fibration described by each of these charts coincides with the signature of the chart.
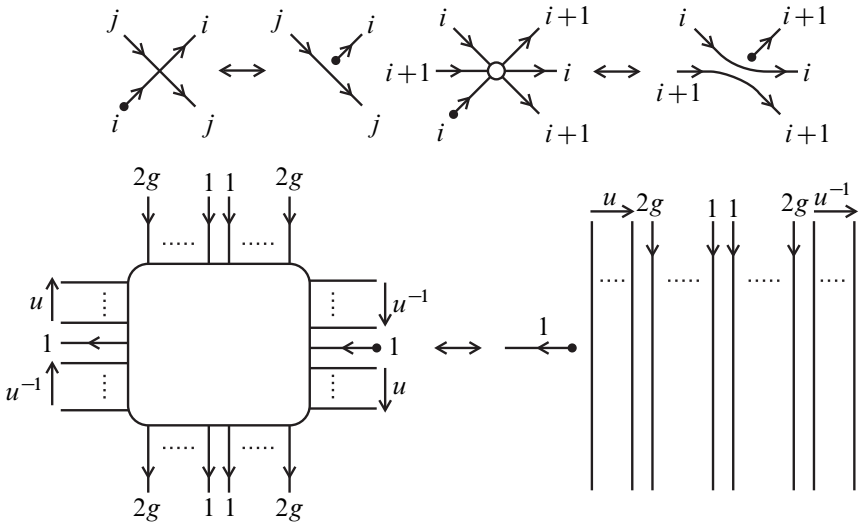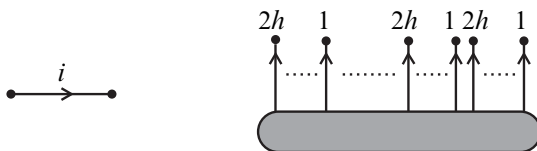

Figure 15: Chart moves of transition
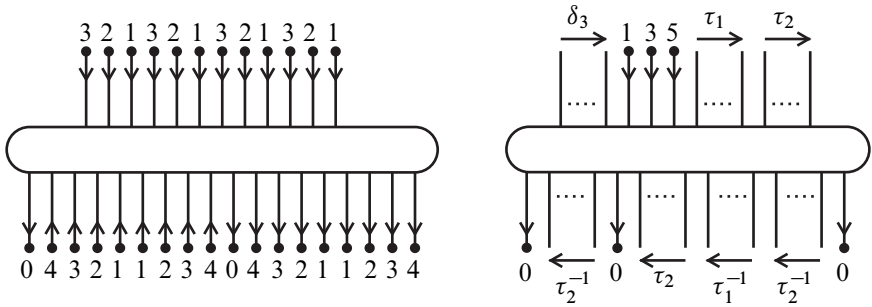

Figure 16: Charts $L_0(i)$ and L_h

Figure 17: Charts R_C and R_L

Let Γ_4 be one of $L_0(i), L_h, L_h^*, R_C, R_C^*, R_L, R_L^*$ drawn in S^2 and $f_4 \colon M_4 \to S^2$ a Lefschetz fibration described by Γ_4. If Γ_4 is equal to $L_0(i)$, it is easily seen that $\sigma(M_4) = \sigma(\Gamma_4)$. If Γ_4 is equal to L_h, the word $\ell_h^{-1}\sigma_h$ corresponds to a Hurwitz system of f_4 (see Figure 16), where σ_h is a right-handed Dehn twist along the curve s_h depicted in Figure 1. Thus we have

$$\sigma(M_4) = I_g(\ell_h^{-1}\sigma_h) = 4h(h+1) - 1 = \sigma(\Gamma_4)$$

from Definition 3.5, Theorem 3.9, and explicit computations for I_g due to Endo and Nagami [8, Lemma 3.5, Proposition 3.9]. If Γ_4 is equal to R_C (resp. R_L), the word r_C (resp. r_L) corresponds to a Hurwitz system of f_4 (see Figure 17). Thus we have

$$\sigma(M_4) = I_g(r_C) = -6 = \sigma(\Gamma_4) \quad (\text{resp. } \sigma(M_4) = I_g(r_L) = -1 = \sigma(\Gamma_4))$$

from Definition 3.5, Theorem 3.9, and formulas of Endo and Nagami [8, Lemma 3.5, Remark 3.7, Propositions 3.9 and 3.10]. Suppose that Γ_4 is equal to one of L_h^*, R_C^*, R_L^*. The mirror image Γ_4^* of Γ_4 with edges orientation reversed corresponds to the Lefschetz fibration $f_4 \colon -M_4 \to S^2$ with total space orientation reversed. Hence we have

$$\sigma(M_4) = -\sigma(-M_4) = -\sigma(\Gamma_4^*) = \sigma(\Gamma_4)$$

because we have already shown that $\sigma(M_4) = \sigma(\Gamma_4)$ is valid for $\Gamma_4 = L_h, R_C, R_L$. This completes the proof of Theorem 3.6.   □

4   Stabilization theorems

In this section we prove two theorems on stabilization of Lefschetz fibrations under taking fiber sums with copies of a fixed Lefschetz fibration.

Following Auroux [2], we first introduce a notion of universality for Lefschetz fibrations. Suppose that g is greater than two.

Definition 4.1   A Lefschetz fibration of genus g over S^2 is called *universal* if it is irreducible, chiral, and it contains $2g + 1$ singular fibers of type I^+ whose vanishing cycles $a_0, a_1, \ldots, a_{2g} \subset \Sigma_g$ satisfy the following conditions:

   (i) a_i and a_{i+1} intersect transversely at one point for every $i \in \{1, \ldots, 2g - 1\}$.

   (ii) a_0 and a_4 intersect transversely at one point.

   (iii) a_i and a_j does not intersect for other pairs (i, j).

A Lefschetz fibration over S^2 is universal if and only if it is described by a chart Γ_0 depicted in Figure 18 by virtue of Proposition 2.9, where the blank labeled by T_0 is filled only with edges, white vertices, and black vertices of type $\ell_0(i)$.
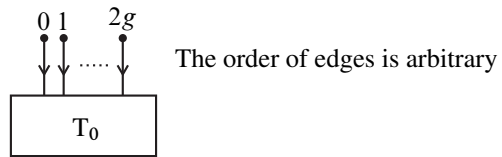

Figure 18:   Universal chart Γ_0

Remark 4.2   A universal Lefschetz fibration exists for every g greater than two. For example, the Lefschetz fibrations $f_g^0, f_g^A, f_g^B, f_g^C, f_g^D$ constructed by Auroux [2] are universal except f_g^D for $g = 3$. There are many universal Lefschetz fibrations of genus g for a fixed g.

We now state the first of our main theorems. Let B be a connected closed oriented surface and $f_0 \colon M_0 \to S^2$ a universal Lefschetz fibration of genus g.

Theorem 4.3   Let $f \colon M \to B$ and $f' \colon M' \to B$ be Lefschetz fibrations of genus g. There exists a non-negative integer N such that $f \# N f_0$ is isomorphic to $f' \# N f_0$ if and only if the following conditions hold:

   (i) $n_0^\pm(f) = n_0^\pm(f')$.

   (ii) $n_h^\pm(f) = n_h^\pm(f')$ for every $h = 1, \ldots, [g/2]$.

   (iii) $\sigma(M) = \sigma(M')$.

Remark 4.4   Auroux [2] proved the "if" part of Theorem 4.3 for chiral Lefschetz fibrations over S^2 under the assumption that f and f' have sections with the same self-intersection number. Hasegawa [15] gave another proof of Auroux's theorem by using chart descriptions. Moreover, he removed the assumption on existence and self-intersection number of sections in Auroux's theorem.

Remark 4.5 The isomorphism class of a fiber sum $f \#_\Psi f_0$ of a Lefschetz fibration f with a universal Lefschetz fibration f_0 does not depend on a choice of an orientation-preserving diffeomorphism Ψ (see the proof of Theorem 4.3).

Proof of Theorem 4.3 We first prove the "if" part. Assume that f and f' satisfy the conditions (i), (ii), and (iii). Let Γ and Γ' be charts in B corresponding to f and f', respectively. We suppose that f_0 is described by a chart Γ_0 depicted in Figure 18. Since every edge has two adjacent vertices, the sum of the signed numbers of adjacent edges for all vertices of Γ is equal to zero:

$$10 n_C(\Gamma) + n_L(\Gamma) - \sum_{i=0}^{2g} n_0(i)(\Gamma) - 4 \sum_{h=1}^{[g/2]} h(2h+1) \cdot n_h(\Gamma) = 0.$$

A similar equality for Γ' also holds. Interpreting the conditions (i) and (ii) as conditions on Γ and Γ', we have $\sum_{i=0}^{2g} n_0(i)(\Gamma) = \sum_{i=0}^{2g} n_0(i)(\Gamma')$ and $n_h(\Gamma) = n_h(\Gamma')$ for $h = 1, \ldots, [g/2]$. Thus we obtain

$$10 n_C(\Gamma) + n_L(\Gamma) = 10 n_C(\Gamma') + n_L(\Gamma').$$

On the other hand, we have

$$-6 n_C(\Gamma) - n_L(\Gamma) = -6 n_C(\Gamma') - n_L(\Gamma')$$

by the condition (iii), Theorem 3.6, and $n_h(\Gamma) = n_h(\Gamma')$ for $h = 1, \ldots, [g/2]$. Hence $n_C(\Gamma) = n_C(\Gamma')$ and $n_L(\Gamma) = n_L(\Gamma')$.

Let N be an integer larger than both of the number of edges of Γ and that of Γ'. Choose a base point $b_0 \in B - (\Gamma \cup \Gamma')$. The fiber sum $f \# N f_0$ is described by a chart $(\cdots((\Gamma \#_{w_1} \Gamma_0) \#_{w_2} \Gamma_0) \cdots) \#_{w_N} \Gamma_0$ for some words $w_1, \ldots, w_N$ in $\mathcal{X} \cup \mathcal{X}^{-1}$. Since hoops surrounding Γ_0 can be removed by use of the edges of Γ_0 as in Figure 19, the chart is transformed into a product $\Gamma \oplus N \Gamma_0$ by channel changes. Similarly, the fiber sum $f' \# N f_0$ is described by a product $\Gamma' \oplus N \Gamma_0$.
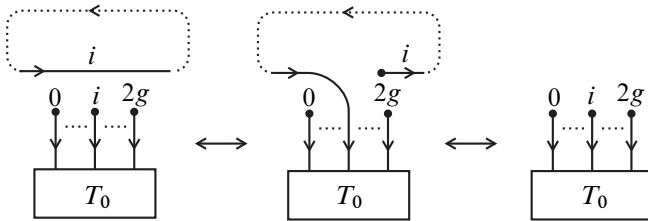

Figure 19: Removing a hoop

We choose and fix $2g+1$ edges of Γ_0 which are labeled with $0, 1, \ldots, 2g$ and adjacent to black vertices. We apply chart moves only to these edges in the following. Since Γ_0 can pass through any edge of Γ as shown in Figure 20, we can move Γ_0 to any region of $B - \Gamma$ by channel changes.
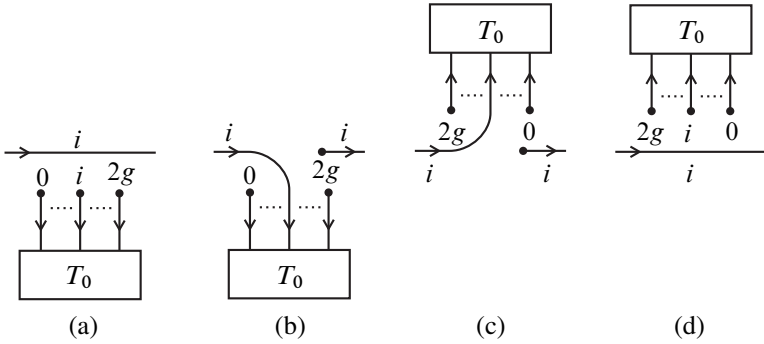

Figure 20: Passing through an edge

For each edge of Γ, we move a copy of Γ_0 to a region adjacent to the edge and apply a channel change to the edge and Γ_0 as in Figure 20(a) and (b). Applying chart moves of transition to each component of the chart as in Figure 15, we remove white vertices of type $r_F(i, j)^{\pm 1}, r_B(i)^{\pm 1}, r_H^{\pm 1}$ to obtain a union of copies of $L_0(i), \widetilde{L}_h, L_h^*, \widetilde{R}_C, \widehat{R}_C, \widetilde{R}_L, \widehat{R}_L, \Gamma_0$ shown in Figures 21–23, where we use a simplification of diagrams as in Figure 24.
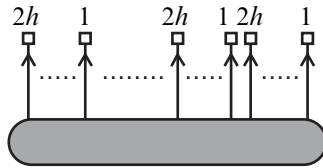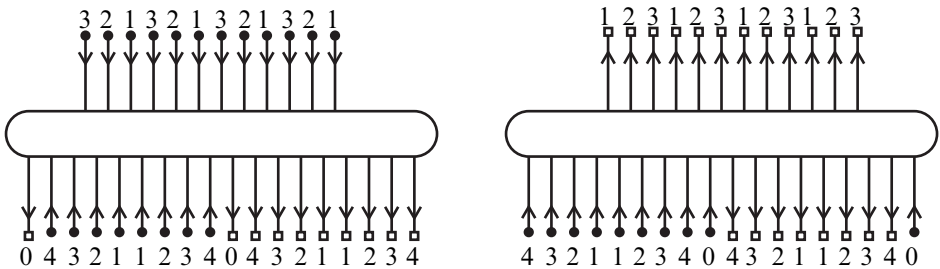

Figure 21: Chart $\widetilde{L}_h$
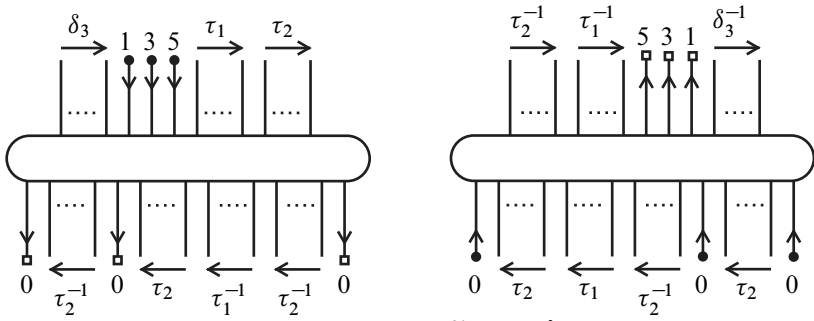

Figure 22: Charts $\widetilde{R}_C$ and $\widehat{R}_C$

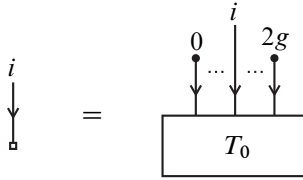Figure 23: Charts $\widetilde{R}_L$ and $\widehat{R}_L$

Figure 24: Simplification of diagram

If there is a pair of $\widetilde{R}_C$ and $\widehat{R}_C$, we remove them by a death of a pair of white vertices to obtain many copies of Γ_0. Similarly, we remove a pair of $\widetilde{R}_L$ and $\widehat{R}_L$. Since there is at least one Γ_0, any copy of $L_0(i)$ can be transformed into $L_0(1)$ as in Figure 25.
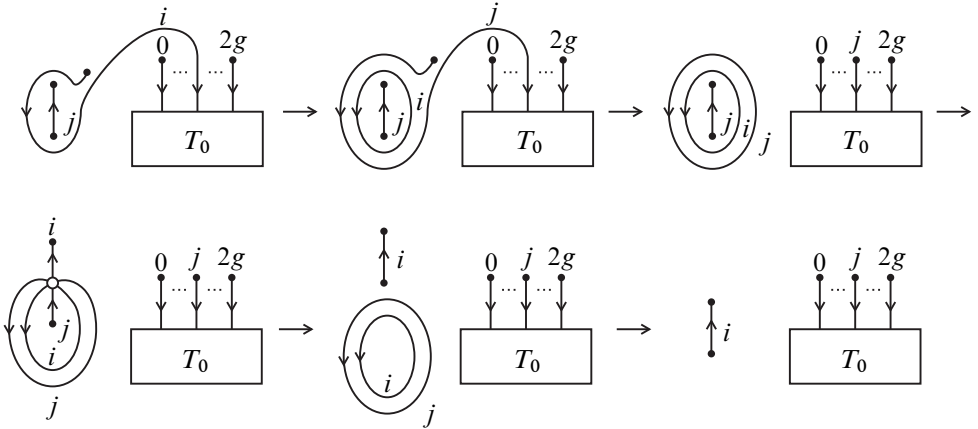

Figure 25: Changing a label ($j = i + 1$ or $(i, j) = (4, 0)$)

Thus we have a union Γ_1 of $n_0^-(\Gamma)$ copies of $L_0(1)$, $n_h^+(\Gamma)$ copies of $\widetilde{L}_h$, $n_h^-(\Gamma)$ copies of L_h^*, $|n_C(\Gamma)|$ copies of $\widetilde{R}_C$ (or $\widehat{R}_C$), $|n_L(\Gamma)|$ copies of $\widetilde{R}_L$ (or $\widehat{R}_L$), and k copies of Γ_0 for some k. A similar argument implies that $\Gamma' \oplus N\Gamma_0$ is transformed into a union Γ_1' of $n_0^-(\Gamma')$ copies of $L_0(1)$, $n_h^+(\Gamma')$ copies of $\widetilde{L}_h$, $n_h^-(\Gamma')$ copies of L_h^*, $|n_C(\Gamma')|$ copies of $\widetilde{R}_C$ (or $\widehat{R}_C$), $|n_L(\Gamma')|$ copies of $\widetilde{R}_L$ (or $\widehat{R}_L$), and k'

copies of Γ_0 for some k' by chart moves of type W and chart moves of transition. By virtue of the conditions (i) and (ii) together with $n_C(\Gamma) = n_C(\Gamma')$, $n_L(\Gamma) = n_L(\Gamma')$, $n_0^+(\Gamma \oplus N\Gamma_0) = n_0^+(\Gamma_1)$, and $n_0^+(\Gamma' \oplus N\Gamma_0) = n_0^+(\Gamma_1')$, we conclude that $k = k'$ because of $n_0^+(\Gamma_0) \neq 0$. Hence Γ_1 is transformed into Γ_1' by an ambient isotopy of B relative to b_0, which means that $\Gamma \oplus N\Gamma_0$ is transformed into $\Gamma' \oplus N\Gamma_0$ by chart moves of type W, chart moves of transition, and ambient isotopies of B relative to b_0. Therefore $f \# Nf_0$ is (strictly) isomorphic to $f' \# Nf_0$ by Theorem 2.11.

We next prove the "only if" part. Take a non-negative integer N so that $f \# Nf_0$ is isomorphic to $f' \# Nf_0$. Since an isomorphism preserves numbers and types of vanishing cycles and signatures, we have

$$n_0^\pm(f \# Nf_0) = n_0^\pm(f' \# Nf_0), \quad n_h^\pm(f \# Nf_0) = n_h^\pm(f' \# Nf_0)$$

for every $h = 1, \ldots, [g/2]$, and

$$\sigma(M \#_F NM_0) = \sigma(M' \#_F NM_0).$$

The conditions (i), (ii), (iii) follow from additivity of $n_0^\pm, n_h^\pm, \sigma$ under fiber sum. $\square$

Definition 4.6 A Lefschetz fibration of genus g over S^2 is called *elementary* if it contains exactly two singular fibers of type I^+ and of type I^- which have the same vanishing cycles. A chart $L_0(i)$ in S^2 corresponds to an elementary Lefschetz fibration.

Remark 4.7 Two elementary Lefschetz fibrations of genus g are isomorphic to each other. The total space of an elementary Lefschetz fibration of genus g is diffeomorphic to $\Sigma_{g-1} \times S^2 \# S^1 \times S^3$.

We state the second of our main theorems. Let B be a connected closed oriented surface and $f_\star \colon M_\star \to S^2$ an elementary Lefschetz fibration of genus g.

Theorem 4.8 *Let $f \colon M \to B$ and $f' \colon M' \to B$ be Lefschetz fibrations of genus g. There exists a non-negative integer N such that a fiber sum $f \# Nf_\star$ is isomorphic to a fiber sum $f' \# Nf_\star$ if and only if the following conditions hold:*

  (i) $n_0^\pm(f) = n_0^\pm(f')$.
  (ii) $n_h^\pm(f) = n_h^\pm(f')$ *for every* $h = 1, \ldots, [g/2]$.
  (iii) $\sigma(M) = \sigma(M')$.

Remark 4.9 In contrast to Theorem 4.3, the isomorphism class of a fiber sum $f \#_\Psi f_\star$ of a Lefschetz fibration f with an elementary Lefschetz fibration $f_\star$ depends on a choice of an orientation-preserving diffeomorphism Ψ in general.

Proof of Theorem 4.8   We only show the "if" part. The "only if" part is the same as that of the proof of Theorem 4.3.

Assume that f and f' satisfy the conditions (i), (ii), and (iii). Let Γ and Γ' be charts in B corresponding to f and f', respectively. It follows from the same argument as in the proof of Theorem 4.3 that $n_C(\Gamma) = n_C(\Gamma')$ and $n_L(\Gamma) = n_L(\Gamma')$. Let N be an integer larger than both of the number of edges of Γ and that of Γ'. We construct the chart Γ_e in B for each $e \in E(\Gamma)$ as in the proof of Theorem 3.6. Taking the union of Γ with Γ_e for all $e \in E(\Gamma)$ and with $N - \#E(\Gamma)$ copies of $L_1(1)$, we obtain a new chart Γ_1 in B, which describes a fiber sum $f \# N f_\star$. Applying channel changes as in the proof of Theorem 3.6 and deaths of pairs of white vertices appropriately, we obtain a union Γ_2 of $n_h^+(\Gamma)$ copies of L_h, $n_h^-(\Gamma)$ copies of L_h^*, $|n_C(\Gamma)|$ copies of R_C (or R_C^*), $|n_L(\Gamma)|$ copies of R_L (or R_L^*), and k_i copies of $L_0(i)$ for some k_i, Similarly, Γ' is transformed into Γ_1', which describes a fiber sum $f' \# N f_\star$, and then a union Γ_2' of $n_h^+(\Gamma')$ copies of L_h, $n_h^-(\Gamma')$ copies of L_h^*, $|n_C(\Gamma')|$ copies of R_C (or R_C^*), $|n_L(\Gamma')|$ copies of R_L (or R_L^*), and k_i' copies of $L_0(i)$ for some k_i'.

A similar argument on the number n_0^+ as in the proof of Theorem 4.3 implies that $k_0 + k_1 + \cdots + k_{2g} = k_0' + k_1' + \cdots + k_{2g}'$. Adding $|k_i - k_i'|$ copies of $L_0(i)$ to either Γ_2 or Γ_2' if necessary, we may assume that $k_i = k_i'$ for every $i \in \{0, 1, \ldots, 2g\}$. Hence Γ_2 is transformed into Γ_2' by an ambient isotopy of B relative to b_0, which means that $f \# N f_\star$ is (strictly) isomorphic to $f' \# N f_\star$ by Theorem 2.11. $\square$

Let g be an integer greater than two and $B_1, \ldots, B_r$ connected closed oriented surfaces. We consider a Lefschetz fibration $f_i \colon M_i \to B_i$ of genus g for each $i \in \{1, \ldots, r\}$, and a universal Lefschetz fibration $f_0 \colon M_0 \to S^2$ of genus g.

Proposition 4.10   *For (possibly different) fiber sums f and f' of $f_1, \ldots, f_r$, fiber sums $f \# f_0$ and $f' \# f_0$ are isomorphic to each other.*

Proof   Let Γ and Γ' be charts corresponding to f and f'. Since hoops surrounding a component of Γ (and Γ') can be removed by use of the edges of Γ_0 as in Figure 19, $\Gamma \# \Gamma_0$ and $\Gamma' \# \Gamma_0$ are transformed into the same chart. $\square$

Remark 4.11   Proposition 4.10 implies that there are many examples of non-isomorphic Lefschetz fibrations with the same base, the same fiber, and the same numbers of singular fibers of each type which become isomorphic after one stabilization. For example, the Lefschetz fibration on $E(n)_K$ constructed by Fintushel and Stern [9, Theorem 14] (see also Park and Yun [37]) for a fibered knot K becomes isomorphic to that on $E(n)_{K'}$ for another fibered knot K' of the same genus after one stabilization.

Similar results hold for Lefschetz fibrations on $Y(n; K_1, K_2)$ constructed by Fintushel and Stern [9, Section 7] (see also Park and Yun [38]) as well as fiber sums of (generalizations of) Matsumoto's fibration studied by Ozbagci and Stipsicz [36], Korkmaz [24; 25], and Okamori [35].

5 Variations and problems

In this section we discuss possible variations of chart description for Lefschetz fibrations.

If we replace the triple $(\mathcal{X}, \mathcal{R}, \mathcal{S})$ defined in Section 2 with other triples, we obtain various chart descriptions for Lefschetz fibrations (see Kamada [20] and Hasegawa [15]).

We first choose large $\mathcal{X}, \mathcal{R}$, and $\mathcal{S}$. Let $\mathcal{X}$ be the set of right-handed Dehn twists along simple closed curves in Σ_g and $\mathcal{S}$ the set of Dehn twists along non-trivial simple closed curves in Σ_g. By virtue of a theorem of Luo [29], $\langle \mathcal{X} \mid \mathcal{R} \rangle$ gives an infinite presentation of $\mathcal{M}_g$ for the set $\mathcal{R}$ of the following four kinds of words:

(0) Trivial relator $r_T := a$, where a is the Dehn twist along a trivial simple closed curve on Σ_g.

(1) Primitive braid relator $r_P := b^{-1}abc^{-1}$, where $a, b, c \in \mathcal{X}$ and the curve for c is the image of the curve for a by b.

(2) 2–chain relator $r_C := (c_2 c_1)^6 d^{-1}$, where $c_1, c_2, d \in \mathcal{X}$ and the curves for c_1 and c_2 intersect transversely at one point and the curve for d is the boundary curve of a regular neighborhood of the union of the curves for c_1 and c_2.

(3) Lantern relator $r_L := cbad_4^{-1}d_3^{-1}d_2^{-1}d_1^{-1}$, where $a, b, c, d_1, d_2, d_3, d_4 \in \mathcal{X}$ and the curves for a and b intersect transversely at two points with algebraic intersection number zero, the curve for c is obtained by resolving the intersections of these two curves, and the curves for d_1, d_2, d_3, d_4 are the boundary curves of a regular neighborhood of those for a, b, c.

Let B be a connected closed oriented surface. Charts in B for the triple $(\mathcal{X}, \mathcal{R}, \mathcal{S})$ defined above have white vertices of type $r_T^{\pm 1}, r_P^{\pm 1}, r_C^{\pm 1}, r_L^{\pm 1}$ (see Figure 26). For a chart Γ in B, we denote the number of white vertices of type r_X minus the number of white vertices of type r_X^{-1} included in Γ by $n_X(\Gamma)$, where $X = T, P, C, L$.

Proposition 5.1 *The signature $\sigma(M)$ of the total space M of a Lefschetz fibration $f: M \to B$ described by Γ is equal to $-n_T(\Gamma) - 7n_C(\Gamma) + n_L(\Gamma)$.*

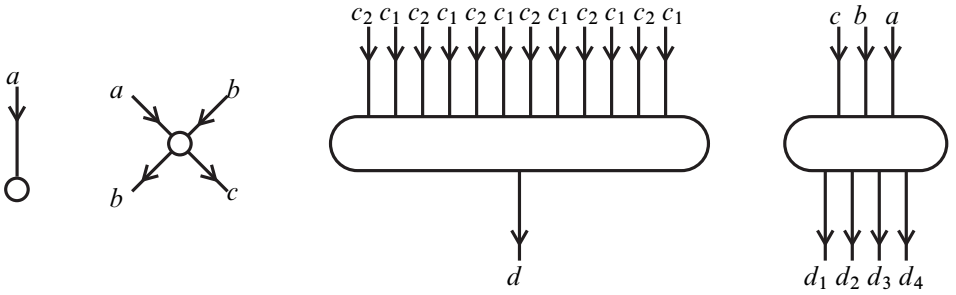**Proof** It is seen by a similar argument to the proof of Theorem 3.6. $\qquad \square$

Figure 26: Vertices of type r_T, r_P, r_C, r_L

Example 5.2 Let B be a connected closed oriented surface of genus 2 and $(\mathcal{X}, \mathcal{R}, \mathcal{S})$ the triple defined above for $g = 3$. Let $a, b, c, d_1, d_2, d_3, d_4, c_1, c_2, c_3$ be right-handed Dehn twists along simple closed curves of the same names on Σ_3 depicted in Figure 27. We present B as an octagon with opposite sides identified and consider a chart Γ and loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ based at b_0 in B as in Figure 28. We use a simplification of diagrams as in Figure 29(a) if the curves for $x, y \in \mathcal{X}$ intersect transversely at one point, and that as in Figure 29(b) if the curves for x and y are disjoint.

Since the intersection words of $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ with respect to Γ are

$$w_\Gamma(\gamma_1) = d_2^{-1}, i \qquad\qquad w_\Gamma(\gamma_2) = c_3^{-1} c^{-1} d_2^{-1} c_3^{-1},$$
$$w_\Gamma(\gamma_3) = c_1^{-1} b^{-1} c_2^{-1} d_3^{-1} a^{-1} c_2^{-1} d_4^{-1} c_1^{-1},$$
$$w_\Gamma(\gamma_4) = d_4 a^{-1}, \qquad\qquad w_\Gamma(\gamma_5) = d_1,$$

a Lefschetz fibration $f : M \to B$ of genus 3 described by Γ is isomorphic to the Lefschetz fibration constructed by Korkmaz and Ozbagci [26, Theorem 1.2]. f has only one singular fiber and it is of type I^+. We can compute the signature $\sigma(M)$ of the total space M by Proposition 5.1 as

$$\sigma(M) = n_L(\Gamma) = -1,$$

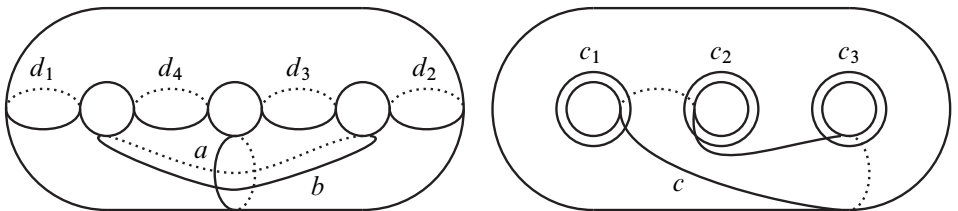which coincides with the value computed in [7, Proposition 14].

Figure 27: Simple closed curves on Σ_3
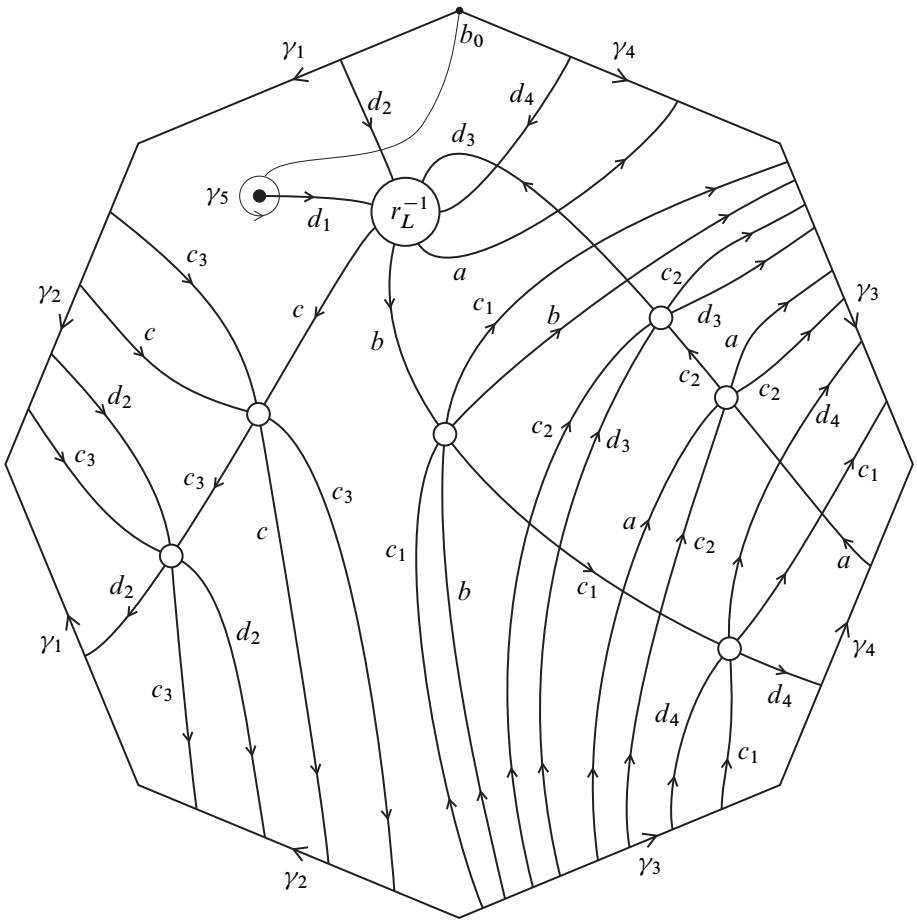
Figure 28: Chart for Lefschetz fibration of Korkmaz and Ozbagci

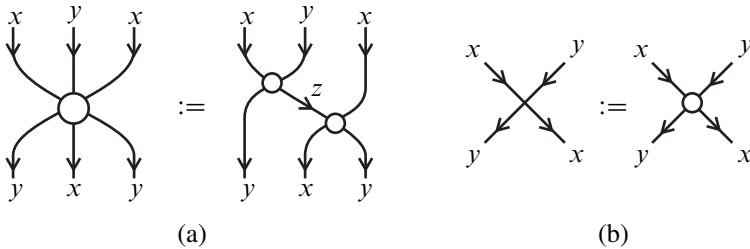(a)   (b)

Figure 29: Simplification of vertices

Problem 5.3 Study various properties of Lefschetz fibrations by using chart descriptions of the triple $(\mathcal{X}, \mathcal{R}, \mathcal{S})$ defined above.

We next mention chart descriptions for Lefschetz fibrations with bordered base and fiber. Kamada [20] gave a general theory for charts in a compact oriented surface with boundary. Various presentations of mapping class groups of surfaces with boundary have been investigated by researchers including Gervais [11], Labruère and Paris [28], Margalit and McCammond [30]. Combining these two kinds of studies, one can immediately obtain a chart description for Lefschetz fibrations with bordered base and fiber.

Problem 5.4 Make use of chart descriptions to study PALFs and Stein surfaces.

It would be worth considering compositions of monodromy representations with appropriate homomorphisms and charts corresponding to the compositions. For example, Hasegawa [14; 15] adopted a homomorphism from the m–string braid group B_m to the semi-direct product $(\mathbb{Z}_2)^m \rtimes S_m$, while Endo and Kamada [6] used a standard epimorphism from the hyperelliptic mapping class group of a closed oriented surface of genus g to the mapping class group of a sphere with $2g + 2$ marked points.

Problem 5.5 Consider chart descriptions for "nice" representations of mapping class groups to study invariants and classifications of Lefschetz fibrations.

Theorem 4.3 and Theorem 4.8 tell us that the numbers of singular fibers of all types and the signature of the total space completely determine the stable isomorphism class of a Lefschetz fibration with given base and fiber. Thus any numerical invariant of Lefschetz fibrations which is additive under fiber sum is determined by these invariants in principle.

Problem 5.6 Construct numerical invariants of Lefschetz fibrations which are *not* additive under fiber sum.

Nosaka [34] has recently defined an invariant which is not additive under fiber sum. Non-numerical invariants such as monodromy group would be also useful (see Matsumoto [32] and Park and Yun [37; 38]).

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