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An automorphism θ of a spherical building Δ is called *capped* if it satisfies the following property: if there exist both type J_1 and J_2 simplices of Δ mapped onto opposite simplices by θ then there exists a type $J_1 \cup J_2$ simplex of Δ mapped onto an opposite simplex by θ. In previous work we showed that if Δ is a thick irreducible spherical building of rank at least 3 with no Fano plane residues then every automorphism of Δ is capped. In the present work we consider the spherical buildings with Fano plane residues (the *small buildings*). We show that uncapped automorphisms exist in these buildings and develop an enhanced notion of "opposition diagrams" to capture the structure of these automorphisms. Moreover we provide applications to the theory of "domesticity" in spherical buildings, including the complete classification of domestic automorphisms of small buildings of types F_4 and E_6.

Introduction

Let θ be an automorphism of a thick irreducible spherical building Δ of type (W, S). The *opposite geometry* of θ is the set $\mathrm{Opp}(\theta)$ of all simplices σ of Δ such that σ and σ^θ are opposite in Δ. This geometry forms a natural counterpart to the more familiar fixed element geometry $\mathrm{Fix}(\theta)$, however by comparison very little is known about $\mathrm{Opp}(\theta)$.

This paper is the continuation of [Parkinson and Van Maldeghem 2019], where we initiated a systematic study of $\mathrm{Opp}(\theta)$ for automorphisms of spherical buildings. In particular in [Parkinson and Van Maldeghem 2019] we showed that if Δ is a thick irreducible spherical building of rank at least 3 containing no Fano plane residues then $\mathrm{Opp}(\theta)$ has the following weak closure property: if there exist both type J_1 and J_2 simplices in $\mathrm{Opp}(\theta)$ then there exists a type $J_1 \cup J_2$ simplex in $\mathrm{Opp}(\theta)$.

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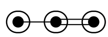
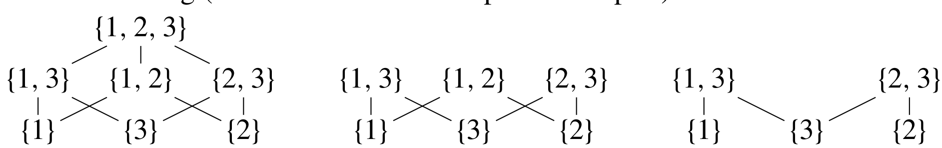
Automorphisms with this property are called *capped*, and the thick irreducible spherical buildings of rank at least 3 with no Fano plane residues are called *large buildings*. Thus every automorphism of a large building is capped.

In the present paper we investigate $\mathrm{Opp}(\theta)$ for the thick irreducible spherical buildings of rank at least 3 containing a Fano plane residue. These are called the *small buildings*. In particular we show that, in contrast to the case of large buildings, uncapped automorphisms exist for all small buildings (with the possible exception of $\mathsf{E}_8(2)$ where we provide conjectural examples).

A key tool in [Parkinson and Van Maldeghem 2019] was the notion of the *opposition diagram* of an automorphism θ, consisting of the triple (Γ, J, π), where Γ is the Coxeter graph of (W, S), J is the union of all $J' \subseteq S$ such that there exists a type J' simplex in $\mathrm{Opp}(\theta)$, and π is the automorphism of Γ induced by θ (less formally, the opposition diagram is drawn by encircling the nodes J of Γ). If θ is capped then this diagram turns out to encode a lot of information about the automorphism, essentially because it completely determines the partially ordered set $\mathcal{T}(\theta)$ of all types of simplices mapped onto opposite simplices by θ. However for an uncapped automorphism the opposition diagram does not necessarily determine $\mathcal{T}(\theta)$. For example in the polar space $\Delta = \mathsf{B}_3(2)$ there are collineations θ_1, θ_2 and θ_3 each with opposition diagram  (that is, each θ_i maps a vertex of each type to an opposite vertex) whose partially ordered sets $\mathcal{T}(\theta_i)$, for $i = 1, 2, 3$, are the following (see Theorem 3.7 for explicit examples):



Note that only θ_1 is capped (hence, in particular, analogues of θ_2 and θ_3 cannot exist for polar spaces $\mathsf{B}_3(\mathbb{F})$ with $|\mathbb{F}| > 2$ by the main result of [Parkinson and Van Maldeghem 2019]).

Thus the opposition diagram of an uncapped automorphism needs to be enhanced to properly understand these automorphisms. We achieve this by defining the *decorated opposition diagram* of an uncapped automorphism.

The full definition is given in Section 1, however for the purpose of this introduction consider the following simplified situation. Suppose that θ is an automorphism with the property that the induced automorphism π of the Coxeter graph Γ is the opposition automorphism w_0. Then the *decorated opposition diagram* of θ is the quadruple (Γ, J, K, π) where (Γ, J, π) is the opposition diagram, and

$$K = \{ j \in J \mid \text{there exists a type } J \backslash \{j\} \text{ simplex mapped onto an opposite simplex by } \theta \}.$$

Less formally, the decorated opposition diagram is drawn by encircling the nodes of J, and then shading those nodes of K. Thus, for example, the decorated

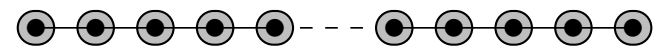
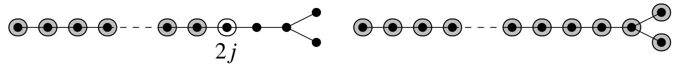
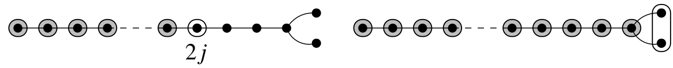
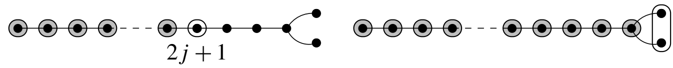
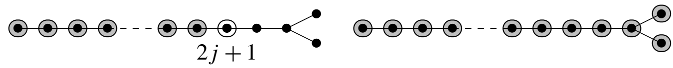
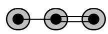
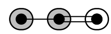
Δ	*diagrams*
$\mathsf{A}_n(2)$	
$\mathsf{B}_n(2)$ *or* $\mathsf{B}_n(2,4)$, $(3 \leq j \leq n)$	
$\mathsf{D}_n(2)$, $n \geq 4$ *even* $(4 \leq 2j \leq n-2)$	
$\mathsf{D}_n(2)$, $n \geq 4$ *odd* $(4 \leq 2j \leq n-3)$	
$\mathsf{D}_n(2)$, $n \geq 4$ *even* $(3 \leq 2j+1 \leq n-3)$	
$\mathsf{D}_n(2)$, $n \geq 4$ *odd* $(3 \leq 2j+1 \leq n-2)$	

Table 1. Decorated opposition diagrams of uncapped automorphisms (classical types).

opposition diagrams of the two uncapped automorphisms of $\mathsf{B}_3(2)$ given above are

 and .

At an intuitive level, the more encircled nodes that are shaded on the decorated opposition diagram of an uncapped automorphism, the "closer" the automorphism is to being capped.

The main theorem of this paper is Theorem 1 below. Part (a) of the theorem shows that the decorated opposition diagram of an uncapped automorphism lies in a small list of diagrams, hence severely restricting the structure of uncapped automorphisms. Part (b) deals with the existence of uncapped automorphisms, showing that the list provided in part (a) has no redundancies, with only the $\mathsf{E}_8(2)$ case remaining open due to the size of the building rendering our computational techniques inadequate. We strongly believe that the two $\mathsf{E}_8(2)$ diagrams are indeed realised as opposition diagrams; see Conjecture 4.8 for details.

Theorem 1.

(a) *Let θ be an uncapped automorphism of a thick irreducible spherical building Δ of rank at least* 3*. Then the decorated opposition diagram of θ appears in Table 1 or Table 2.*

(b) *Let Δ be a small building. Each diagram appearing in the respective row of Table 1 or Table 2 can be realised as the decorated opposition diagram of some uncapped automorphism of Δ, with the exception perhaps of the two* $\mathsf{E}_8(2)$ *diagrams.*

Let us briefly describe corollaries to Theorem 1(a) (see Section 2B for details and precise statements). Recall that the *displacement* $\mathrm{disp}(\theta)$ of an automorphism θ is the maximum length of $\delta(C, C^\theta)$, with C a chamber.

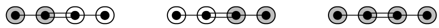
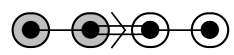
Δ	*diagrams*
$\mathsf{E}_6(2)$	
$\mathsf{E}_7(2)$	
$\mathsf{E}_8(2)$	
$\mathsf{F}_4(2)$	
$\mathsf{F}_4(2,4)$	

Table 2. Decorated opposition diagrams of uncapped automorphisms (exceptional types). The arrow in the $\mathsf{F}_4(2,4)$ diagram indicates that the residues of type $\{1,2\}$ are projective planes of order 2.

Corollary 2. *Let θ be an automorphism of a thick irreducible spherical building Δ.*

(a) *If θ is an involution, θ is capped.*

(b) *If θ is uncapped, $\mathcal{T}(\theta)$ is determined by the decorated opposition diagram of θ.*

(c) *If θ is uncapped,* $\mathrm{disp}(\theta)$ *is determined by the decorated opposition diagram of θ.*

In particular, if Δ has type (W, S) and $J = \mathrm{Typ}(\theta)$ then Corollary 2(c) implies that (see Corollary 2.29)

$$\mathrm{disp}(\theta) = \begin{cases} \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) & \text{if } \theta \text{ is capped,} \\ \mathrm{diam}(W) - \mathrm{diam}(W_{S\backslash J}) - 1 & \text{if } \theta \text{ is uncapped.} \end{cases}$$

To illustrate this in an example, it follows that if θ is a nontrivial automorphism of a thick E_8 building then $\mathrm{disp}(\theta) \in \{57, 90, 107, 108, 119, 120\}$, which is a surprisingly restricted list of possibilities (see Remark 2.30). Moreover, displacements of 107 or 119 can only occur for uncapped automorphisms of the small building $\mathsf{E}_8(2)$.

We also provide applications of Theorem 1(a) to the study of *domesticity* in spherical buildings (recall that an automorphism is called *domestic* if it maps no chamber to an opposite chamber). These automorphisms have recently enjoyed extensive investigation, including the series [Temmermans et al. 2011; 2012a; 2012b] where domesticity in projective spaces, polar spaces, and generalised quadrangles is studied, [Van Maldeghem 2012] where symplectic polarities of large E_6 buildings are classified in terms of domesticity, [Van Maldeghem 2014] where domestic trialities of D_4 buildings are classified, and [Parkinson et al. 2015] where domesticity in generalised polygons is studied.

To give one example of our applications to domesticity, suppose that Δ is a simply laced spherical building, and that θ is a domestic automorphism inducing opposition on the type set with the property that θ maps at least one vertex of each type onto an opposite vertex (such automorphisms are called "exceptional domestic”). Then we show that in fact θ maps simplices of each type $J \subsetneq S$ onto opposite simplices (such automorphisms are called “strongly exceptional domestic”). In particular, this implies that $\mathrm{disp}(\theta) = \mathrm{diam}(\Delta) - 1$ for exceptional domestic automorphisms.

Theorem 1(b) provides the first known examples of exceptional domestic automorphisms of spherical buildings of rank at least 3 (examples were previously only known for generalised polygons; see [Parkinson et al. 2015]). In fact Theorem 1(b) shows that, with the possible exception of $\mathsf{E}_8(2)$, every small building admits a strongly exceptional domestic automorphism.

The proof of Theorem 1(b) for the small buildings of exceptional type involves computations using [Magma], and in particular the groups of Lie type package [Cohen et al. 2004]. In fact for the small buildings of type F_4 and E_6 we are able to prove a much stronger result and completely classify the domestic automorphisms of these buildings. To perform these calculations we implemented the minimal faithful permutation representations of the ATLAS groups $\mathsf{F}_4(2)$, $\mathsf{F}_4(2).2$, $\mathsf{E}_6(2)$, $\mathsf{E}_6(2).2$, ${}^2\mathsf{E}_6(2^2)$, and ${}^2\mathsf{E}_6(2^2).2$ (respective permutation degrees 69615, 139230, 139503, 279006, 3968055 and 3968055) into the Magma system. At the time of writing these representations were not readily available in either Magma or GAP, and therefore they are provided on Parkinson’s webpage.

We conclude this introduction with an outline of the structure of the paper. In Section 1 we provide definitions and background. The proofs of Theorem 1(a) and its corollaries are contained in Section 2. The proof of Theorem 1(b) is divided across Section 3 for the classical types and Section 4 for the exceptional types. Moreover, Section 4 contains the complete classification of domestic automorphisms of the small buildings of types F_4 and E_6.

1. Definitions and background

We refer to [Abramenko and Brown 2008] for the general theory of buildings. In this section we will briefly recall some notation, mainly from [Parkinson and Van Maldeghem 2019, Section 1]. Let Δ be a spherical building of type (W, S), typically considered as a simplicial complex with type map $\tau : \Delta \to 2^S$. Let $\mathcal{C}$ be the set of chambers (maximal simplices) of Δ, and let $\delta : \mathcal{C} \times \mathcal{C} \to W$ be the Weyl distance function.

Chambers C and D of Δ are *opposite* if and only if they are at maximal distance in the chamber graph (with adjacency given by the union of the s-adjacency relations: $C \sim_s D$ if and only $\delta(C, D) = s$). Equivalently, chambers $C, D \in \mathcal{C}$ are opposite if and only if $\delta(C, D) = w_0$ where w_0 is the longest element of W.

If $J \subseteq S$ we write $J^{\mathrm{op}} = J^{w_0} = w_0^{-1} J w_0$ (the “opposite type” to J). The definition of opposition for chambers extends naturally to arbitrary simplices as follows (see [Abramenko and Brown 2008, Lemma 5.107]).

Definition 1.1. Simplices α, β of Δ are *opposite* if $\tau(\beta) = \tau(\alpha)^{\mathrm{op}}$ and there exists a chamber A containing α and a chamber B containing β such that A and B are opposite.

An *automorphism* of Δ is a simplicial complex automorphism $\theta : \Delta \to \Delta$. Note that θ does not necessarily preserve types. Indeed each automorphism $\theta : \Delta \to \Delta$ induces a permutation π_θ of the type set S, given by $\delta(C, D) = s$ if and only if $\delta(C^\theta, D^\theta) = s^{\pi_\theta}$, and this permutation is a diagram automorphism of the Coxeter graph Γ of (W, S). If Δ is irreducible, then from the classification of irreducible spherical Coxeter systems we see that $\pi_\theta : S \to S$ either

(1) is the identity, in which case θ is called a *collineation* (or *type-preserving*),

(2) has order 2, in which case θ is called a *duality*, or

(3) has order 3, in which case θ is called a *triality*; this only occurs in type D_4.

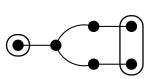
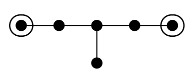
Automorphisms $\theta : \Delta \to \Delta$ that induce opposition on the type set (that is, $\pi_\theta = w_0$, where w_0 is the diagram automorphism given by $s^{w_0} = w_0^{-1} s w_0$) are called *oppomorphisms*. For example, oppomorphisms of an E_6 building are dualities, and oppomorphisms of an E_7 building are collineations (see, for example, [Abramenko and Brown 2008, Section 5.7.4]).

Let θ be an automorphism of Δ. The *opposite geometry* of θ is

$$\mathrm{Opp}(\theta) = \{\sigma \in \Delta \mid \sigma \text{ is opposite } \sigma^\theta\}.$$

A fundamental result of Leeb [2000, Section 5] and Abramenko and Brown [2009, Proposition 4.2] states that if θ is a nontrivial automorphism of a thick spherical building then $\mathrm{Opp}(\theta)$ is necessarily nonempty (this result has been generalised to the setting of twin buildings; see [Devillers et al. 2013]).

The *type* $\mathrm{Typ}(\theta)$ of an automorphism θ is the union of all subsets $J \subseteq S$ such that there exists a type J simplex in $\mathrm{Opp}(\theta)$. The *opposition diagram* of θ is the triple $(\Gamma, \mathrm{Typ}(\theta), \pi_\theta)$. Less formally, the opposition diagram of θ is depicted by drawing Γ and encircling the nodes of $\mathrm{Typ}(\theta)$, where we encircle nodes in minimal subsets invariant under $w_0 \circ \pi_\theta$. We draw the diagram "bent" (in the standard way) if $w_0 \circ \pi_\theta \neq 1$. For example, consider the following diagrams:

The diagram on the left represents a collineation θ of an E_6 building with $\mathrm{Typ}(\theta) = \{1, 2, 6\}$, and the diagram on the right represents a duality θ of an E_6 building with $\mathrm{Typ}(\theta) = \{1, 6\}$.

We call an opposition diagram *empty* if no nodes are encircled (i.e., $\mathrm{Typ}(\theta) = \varnothing$), and *full* if all nodes are encircled (i.e., $\mathrm{Typ}(\theta) = S$).

Definition 1.2. Let Δ be a spherical building of type (W, S). Let θ be a nontrivial automorphism of Δ, and let $J \subseteq S$. Then θ is called:

(a) *Capped* if there exists a type $\mathrm{Typ}(\theta)$ simplex in $\mathrm{Opp}(\theta)$, and *uncapped* otherwise.

(b) *Domestic* if $\mathrm{Opp}(\theta)$ contains no chamber.

(c) *J-domestic* if $\mathrm{Opp}(\theta)$ contains no type J simplex (this terminology is reserved for subsets J which are stable under $w_0 \circ \pi_\theta$).

(d) *Exceptional domestic* if θ is domestic with full opposition diagram.

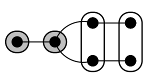
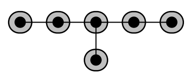
(e) *Strongly exceptional domestic* if θ is domestic, but not J-domestic for any strict subset J of S invariant under $w_0 \circ \pi_\theta$.

Note that if θ is a domestic automorphism with $w_0 \circ \pi_\theta = 1$ then θ is exceptional domestic if and only if there exists a vertex of each type mapped to an opposite vertex, and θ is strongly exceptional domestic if and only if there exists a panel of each cotype mapped to an opposite panel (recall that a *panel* is a codimension 1 simplex).

To study uncapped automorphisms θ we introduce the decorated opposition diagram. Let $\mathcal{J}_\theta$ denote the set of subsets $I \subseteq S$ which are minimal with respect to the condition $I^{\pi_\theta w_0} = I$. For example, if θ induces opposition on Γ then $\mathcal{J}_\theta = \{\{s\} \mid s \in S\}$ is the set of all singleton subsets of S.

Definition 1.3. The *decorated opposition diagram* of an uncapped automorphism θ is the quadruple $(\Gamma, J, K_\theta, \pi_\theta)$ where $J = \mathrm{Typ}(\theta)$ and $K_\theta \subseteq J$ is the union of all $J' \in \mathcal{J}_\theta$ such that there exists a type $J \backslash J'$ simplex mapped onto an opposite simplex.

Less formally, the decorated opposition diagram is drawn by shading the nodes of K_θ on the opposition diagram. For example, consider the following.

The decorated opposition diagram on the left represents an uncapped collineation of $\mathsf{E}_6(2)$ with the property that there are simplices of types $S\backslash\{2\}$ and $S\backslash\{4\}$ mapped onto opposite simplices, and no simplices of types $S\backslash\{3, 5\}$ nor $S\backslash\{1, 6\}$ mapped onto opposite simplices — this automorphism is exceptional domestic, but it is not strongly exceptional domestic. The diagram on the right represents an uncapped duality of $\mathsf{E}_6(2)$ with the property that there are panels of each cotype mapped onto opposite panels — this automorphism is strongly exceptional domestic.

Residue arguments are used extensively in the proof of Theorem 1(a), and so we conclude this section with a summary of the techniques. We first briefly define residues and projections (see [Abramenko and Brown 2008] for details). The *residue* $\mathrm{Res}(\alpha)$ of a simplex $\alpha \in \Delta$ is the set of all simplices of Δ which contain α, together with the order relation induced by that of Δ. Then $\mathrm{Res}(\alpha)$ is a building whose diagram is obtained from the diagram of Δ by removing all nodes which belong to $\tau(\alpha)$. The *projection onto* α is the map $\mathrm{proj}_\alpha : \Delta \to \mathrm{Res}(\alpha)$ defined as follows. Firstly, if B is a chamber of Δ then there is a unique chamber $A \in \mathrm{Res}(\alpha)$ such that $\ell(\delta(A, B)) < \ell(\delta(A', B))$ for all chambers $A' \in \mathrm{Res}(\alpha)$ with $A' \neq A$, and we define $\mathrm{proj}_\alpha(B) = A$. In other words, $\mathrm{proj}_\alpha(B)$ is the unique chamber A of $\mathrm{Res}(\alpha)$ with the property that every minimal length gallery from B to $\mathrm{Res}(\alpha)$ ends with the chamber A. Now, if β is an arbitrary simplex we define

$$\mathrm{proj}_\alpha(\beta) = \bigcap_B \mathrm{proj}_\alpha(B),$$

where the intersection is over all chambers B in $\mathrm{Res}(\beta)$. In other words, $\mathrm{proj}_\alpha(\beta)$ is the unique simplex γ of $\mathrm{Res}(\alpha)$ which is maximal subject to the property that every minimal length gallery from a chamber of $\mathrm{Res}(\beta)$ to $\mathrm{Res}(\alpha)$ ends in a chamber containing γ.

Let θ be an automorphism of Δ, and suppose that $\sigma \in \mathrm{Opp}(\theta)$. It follows from [Tits 1974, Theorem 3.28] that the projection map $\mathrm{proj}_\sigma : \mathrm{Res}(\sigma^\theta) \to \mathrm{Res}(\sigma)$ is an isomorphism. Define

$$\theta_\sigma : \mathrm{Res}(\sigma) \xrightarrow{\sim} \mathrm{Res}(\sigma) \quad \text{by} \quad \theta_\sigma = \mathrm{proj}_\sigma \circ \theta.$$

The type map induced by θ_σ is as follows.

Proposition 1.4. *Let θ be an automorphism of a spherical building Δ of type (W, S). Suppose that $\sigma \in \mathrm{Opp}(\theta)$ and let $J = \tau(\sigma)$. Then the type map on $S\backslash J$ induced by θ_σ is $w_{S\backslash J} \circ w_0 \circ \pi_\theta$.*

Proof. This follows easily from [Abramenko and Brown 2008, Corollary 5.116]. □

Example 1.5. We will use Proposition 1.4 many times in our residue arguments. For example, consider a duality θ of an D_n building, and suppose that $v \in \mathrm{Opp}(\theta)$ is a type i vertex, with $i \le n-2$. The residue of v is a building of type $\mathsf{A}_{i-1} \times \mathsf{D}_{n-i}$, and the induced automorphism θ_v of $\mathrm{Res}(v)$ is a duality on the A_{i-1} component, and a duality (respectively, collineation) on the D_{n-i} component if i is even (respectively, odd).

It is useful to note that if θ is an oppomorphism, and if $\sigma \in \mathrm{Opp}(\theta)$, then θ_σ is a oppomorphism of $\mathrm{Res}(\sigma)$ (this follows immediately from Proposition 1.4).

From [Tits 1974, Proposition 3.29] we have:

Proposition 1.6. *Let θ be an automorphism of a spherical building Δ and let $\alpha \in \mathrm{Opp}(\theta)$. If $\beta \in \mathrm{Res}(\alpha)$ then β is opposite β^θ in the building Δ if and only if β is opposite β^{θ_α} in the building* $\mathrm{Res}(\alpha)$.

The following corollary facilitates inductive residue arguments.

Corollary 1.7. *Let $\theta : \Delta \to \Delta$ be a domestic automorphism and let $\sigma \in \mathrm{Opp}(\theta)$. Then $\theta_\sigma : \mathrm{Res}(\sigma) \to \mathrm{Res}(\sigma)$ is a domestic automorphism of the building $\mathrm{Res}(\sigma)$.*

Proof. Let $J = \tau(\sigma)$. If θ_σ is not domestic then there is a chamber σ' of $\mathrm{Res}(\sigma)$ mapped onto an opposite chamber by θ_σ. Then $\sigma \cup \sigma'$ is a chamber of Δ, and from Proposition 1.6 this chamber is mapped onto an opposite chamber, which is a contradiction. □

2. Theorem 1(a) and its corollaries

In this section we prove Theorem 1(a) and give applications to determining the partially ordered set $\mathcal{T}(\theta)$, domesticity, cappedness of involutions, and calculating displacement.

2A. ***Proof of Theorem 1(a).*** By [Parkinson and Van Maldeghem 2019, Theorem 1] if θ is an uncapped automorphism of a thick irreducible spherical building Δ of rank at least 3 then Δ is a small building. These are precisely the buildings listed in the first column of Tables 1 and 2. Moreover, the following proposition from [Parkinson and Van Maldeghem 2019] explains why collineations of A_n, trialities of D_4, and dualities of F_4 do not appear in Tables 1 and 2.

Proposition 2.1. *Every collineation of a thick A_n building is capped, every triality of a thick D_4 building is capped, and every duality of a thick F_4 building is capped.*

Proof. See [Parkinson and Van Maldeghem 2019, Corollary 3.9, Theorem 3.17, Lemma 4.1]. □

Buildings of type A_n play an important role in our proof techniques owing to their prevalence as residues of spherical buildings of arbitrary type. Every thick building of type A_n with $n > 2$ is a projective space $\mathsf{PG}(n, \mathbb{K})$ over a division ring $\mathbb{K}$, where the type i vertices of the building are the $(i-1)$-spaces of the projective space. Thus points have type 1, lines have type 2, and so on.

Definition 2.2. Let $\mathbb{F}$ be a field. A duality of $\mathsf{A}_{2n-1}(\mathbb{F})$ with $U^\theta = \{v \mid (u, v) = 0$ for all $u \in U\}$ for some nondegenerate symplectic form $(\cdot\,,\cdot)$ on $\mathbb{F}^{2n}$ is called a *symplectic polarity*.

Let us recall some useful facts concerning dualities of type A buildings.

Lemma 2.3 [Temmermans et al. 2011, Lemma 3.2]. *If the projective space $\Delta = \mathsf{PG}(n, \mathbb{K})$ admits a duality θ for which all points are absolute (equivalently no type 1 vertex is mapped to an opposite), then n is odd, $\mathbb{K}$ is a field, and θ is a symplectic polarity.*

Lemma 2.4 [Parkinson and Van Maldeghem 2019, Lemma 3.4]. *If θ is a symplectic polarity of an A_{2n-1} building then θ is $\{i\}$-domestic for each odd i, and* *each vertex mapped to an opposite vertex is contained in a type* $\{2, 4, \ldots, 2n-2\}$ *simplex mapped to an opposite simplex. Hence symplectic polarities are capped.*

Theorem 2.5 [Parkinson and Van Maldeghem 2019, Theorems 3.10 and 3.11]. *Let* θ *be a domestic duality of the small building* $\Delta = \mathsf{A}_n(2)$ *with* $n \geq 2$. *Then either* θ *is a strongly exceptional domestic duality or* n *is odd and* θ *is a symplectic polarity.*

The following proposition shows that the diagrams for uncapped dualities of A_n buildings are as claimed in the first row of Table 1.

Proposition 2.6. *Every uncapped duality of* $\mathsf{A}_n(2)$ *is a strongly exceptional domestic duality.*

Proof. If θ is uncapped then necessarily θ is domestic, and so by Theorem 2.5 θ is either a symplectic polarity or is strongly exceptional domestic. The first case is eliminated by Lemma 2.4. □

We now consider the small buildings of types B_n and D_n. We first require some preliminary results. It is convenient at times to use terminology like "x is domestic for θ" and "x is nondomestic for θ" as short hand for "θ does not map x to an opposite" and "x is mapped to an opposite by θ". If the automorphism θ is clear from context we will simply say "x is domestic" or "x is nondomestic".

Lemma 2.7. *Let* $n \geq 4$ *and let* Δ *be a building of type* B_n *or* D_{n+2} *with thick projective plane residues. Let* θ *be an automorphism and let* $J = \mathrm{Typ}(\theta)$. *If there exists* $j \in J$ *odd with* $j \leq n$, *then* $\{1, 2, \ldots, j\} \subseteq J$.

Proof. Let v be a nondomestic type j vertex. Then θ_v acts as a duality on the A_{j-1} component of the residue of v (by Proposition 1.4). Since j is odd, this duality is either nondomestic or is exceptional domestic (see Theorem 2.5), and in either case $1, 2, \ldots, j-1 \in J$, and hence the result. □

Lemma 2.8. *Let* Δ *be a building of type* B_n *or* D_{n+2} *with* $n \geq 4$ *and thick projective plane residues, and let* θ *be a collineation. Let* $J = \mathrm{Typ}(\theta)$. *Suppose that* $3 \leq j < n$, *and that* $\{j-1, j\} \subseteq J$ *and* $j+1 \notin J$. *Then there exists a type* $\{1, j\}$*-simplex mapped onto an opposite simplex by* θ.

Proof. We first show that θ is not $\{j-1, j\}$-domestic. For if θ is $\{j-1, j\}$-domestic, then since θ is also $\{j-1, j+1\}$-domestic it follows from [Parkinson and Van Maldeghem 2019, Lemma 3.25] that either θ is $\{j-1\}$-domestic or $\{j\}$-domestic, a contradiction. Thus there exists a type $\{j-1, j\}$ simplex σ mapped onto an opposite. If v is the type j vertex of this simplex then θ_v acts as a duality on the A_{j-1} component (Proposition 1.4) mapping a hyperplane to an opposite (by Proposition 1.6). Thus θ_v is either nondomestic or strongly exceptional domestic on the A_{j-1} component, and in either case there exists a nondomestic type $\{1, j\}$ simplex (note that $j-1 \geq 2$). □

Lemma 2.9. *Let Δ be a small building of type B_n or D_{n+1}, and let $j < n$. Suppose that θ is an uncapped collineation of type $J = \{1, 2, 3, \dots, j\}$. Then θ is $\{1, 2, 3, \dots, j-1\}$-domestic.*

Proof. Suppose that there is a nondomestic type $\{1, 2, \dots, j-1\}$ simplex, and let v be the type $j-1$ vertex of this simplex. If θ is uncapped then necessarily θ_v acts as the identity on the "upper" residue of type B_{n-j+1} or D_{n-j+2} (by Proposition 1.6). Thus [Parkinson and Van Maldeghem 2019, Lemma 3.28] with $i = j-2$ and $\ell = j-3$ (note the index shift due to the fact that we used projective dimension in [Parkinson and Van Maldeghem 2019]) implies that every $(j-1)$-space in the polar space of Δ has a fixed point. Thus no type j vertex of Δ is mapped onto an opposite vertex, contradicting the fact that $j \in J$. □

We can now complete the proof of Theorem 1(a) for buildings of type B_n. We allow the additional generality of thin cotype n panels to facilitate our later arguments for type D_n.

Proposition 2.10. *Let Δ be a (possibly nonthick) building of type B_n with Fano plane residues and $n \geq 3$, and let θ be a collineation of Δ. If θ is uncapped, then the decorated opposition diagram of θ is one of the diagrams in Table 1.*

Proof. Suppose that θ is uncapped. Let $J = \mathrm{Typ}(\theta)$, and let $j = \max J$. Then $j \geq 3$, for if $j = 1$ then θ is capped, and if $j = 2$ then either $J = \{2\}$ and θ is capped, or $J = \{1, 2\}$ in which case [Parkinson and Van Maldeghem 2019, Fact 3.21] implies that θ is capped.

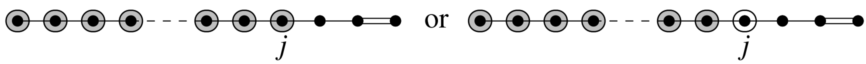
We claim that J contains an odd element. For if every element of J is even then for each nondomestic type j-vertex v the induced automorphism θ_v is a point domestic duality of an A_{j-1} building (by Propositions 1.4 and 1.6). Thus θ_v is a symplectic polarity (Lemma 2.3), and so there exists a type $\{2, 4, \dots, j-2\}$ simplex of the residue mapped to an opposite (Lemma 2.4). Hence by Proposition 1.6 there is a type $\{2, 4, \dots, j-2, j\} = J$ simplex of Δ mapped onto an opposite and so θ is capped, a contradiction.

Let $k \in J$ be the maximal odd node. By Lemma 2.7 we have $\{1, 2, \dots, k\} \subseteq J$. Consider the following cases.

(1) If $j = n$ then by [Parkinson and Van Maldeghem 2019, Proposition 3.12(2)] there is a nondomestic type $\{1, n\}$ simplex. In the A_{n-1} residue of the type n vertex of this simplex we have a strongly exceptional domestic duality of A_{n-1} (since it is domestic and maps a point to an opposite), and hence there are panels of each cotype $1, 2, \dots, n-1$ mapped onto opposites in Δ. Thus θ has either the first diagram listed in Table 1 (with $j = n$) or the second diagram listed in Table 1 (strongly exceptional domestic).

(2) If $k = j < n$ then $J = \{1, 2, \dots, j\}$, and by Lemma 2.8 there exists a nondomestic

type $\{1, j\}$ simplex. Considering the type A_{j-1} residue of the type j vertex of this simplex, and noting that $j-1$ is even, we see that in Δ there are nondomestic simplices of each type $J\backslash\{j'\}$ with $j' = 1, 2, \ldots, j-1$ (using Theorem 2.5), and hence the diagram of θ is either



(2-1)

The first digram is eliminated by Lemma 2.9.

(3) If $k < j < n$ then j is even, and as above we have $\{2, 4, \ldots, j-2, j\} \subseteq J$. In particular $\{k, k+1\} \subseteq J$ and $k+2 \notin J$ (as k is the maximum odd node of J, and note that $k+2 \leq n$). Lemma 2.8 implies that there is a nondomestic type $\{1, k+1\}$ simplex. If $k+1 = j$ then as above we have the diagrams (2-1) and Lemma 2.9 eliminates the first of the diagrams. If $k+1 < j$ then $k+3 \leq j < n$. If θ is $\{1, k+3\}$-domestic, then since θ is not $\{k{+}3\}$-domestic, [Parkinson and Van Maldeghem 2019, Lemma 3.29] implies that θ is $\{1, k+1\}$-domestic, a contradiction. Hence there exists a type $\{1, k+3\}$ simplex mapped onto an opposite. However, considering the A_{k+2} residue of the type $k+3$ vertex of this simplex we see that θ is not $\{k{+}2\}$-domestic, contradicting the maximality of k.

Hence the result. □

Corollary 2.11. *Let Δ be a building of type B_n with thick projective spaces, and let θ be a collineation and $n \geq i \geq 3$. If θ is $\{1, i\}$-domestic then θ is either $\{1\}$-domestic or $\{i\}$-domestic.*

Proof. If θ is capped then the result is true by definition. If θ is uncapped then the result follows directly from the classification of uncapped diagrams given above. □

Remark 2.12. The assumption $i \geq 3$ cannot be removed from Corollary 2.11. For example, consider the exceptional domestic collineation of the generalised quadrangle $\mathsf{B}_2(2)$ (see [Temmermans et al. 2012b, Section 4]). More generally, for each $n \geq 2$ there exists an uncapped collineation of $\mathsf{B}_n(2)$ with $\mathrm{Typ}(\theta) = \{1, 2\}$ (see Theorem 3.7).

We now continue with the analysis of buildings of type D_n. Recall that each building of type D_n can be realised as the oriflamme geometry of the space $\mathbb{F}^{2n}$ equipped with an orthogonal form of Witt index n, for some field $\mathbb{F}$. The vertices of type j for $j \in \{1, \ldots, n-2\}$ are the totally isotropic spaces of dimension j, and the vertices of type $n-1$ and n are the totally isotropic subspaces of dimension n (corresponding to the orbits of the action of the associated simple orthogonal group). To each such building Δ of type D_n there is an associated (nonthick) building Δ' of type B_n. The type j vertices of Δ', for $1 \leq j \leq n$, are the totally isotropic subspaces of dimension j. Each type $n-1$ vertex of Δ' determines a type $\{n-1, n\}$

simplex of Δ, and vice versa, as follows. A type $n-1$ vertex of Δ' is an $(n-1)$-dimensional totally isotropic space W, and there are precisely two totally isotropic n-dimensional subspaces U, V containing W and (U, V) is an $\{n-1, n\}$-simplex of Δ. Conversely, if (U, V) is a type $\{n-1, n\}$ simplex of Δ then $W = U \cap V$ is a type $n-1$ vertex of Δ'.

We first recall two facts from [Parkinson and Van Maldeghem 2019].

Lemma 2.13 [Parkinson and Van Maldeghem 2019, Lemma 3.32]. *Let Δ be a thick building of type D_n with n odd, and let Δ' be the associated nonthick B_n building. A collineation θ maps a type $\{n-1, n\}$ simplex of Δ to an opposite simplex if and only if it maps the associated type $n-1$ vertex of Δ' to an opposite vertex.*

Lemma 2.14 [Parkinson and Van Maldeghem 2019, Proposition 3.16]. *No duality of a thick building of type D_n is $\{1\}$-domestic.*

Lemma 2.15. *Let Δ be a thick building of type D_n with $n \geq 5$ odd, and let θ be a collineation. If θ is $\{1, n-1, n\}$-domestic then θ is either $\{1\}$-domestic or $\{n-1, n\}$-domestic.*

Proof. Suppose that θ is neither $\{1\}$-domestic nor $\{n-1, n\}$-domestic. Since θ maps a type $\{n-1, n\}$-simplex to an opposite, by familiar residue arguments there are vertices of types $2, 4, \ldots, n-3$ mapped onto opposite vertices. These vertex types are therefore also mapped onto opposites in the associated nonthick B_n building Δ'. If there are no type $n-2$ or $n-1$ vertices of Δ' mapped onto opposite vertices, then θ is $\{n-3, n-2\}$-domestic and $\{n-3, n-1\}$-domestic (on Δ') and thus since θ is not $\{n-3\}$-domestic it follows from [Parkinson and Van Maldeghem 2019, Lemma 3.25] that every space of vector space dimension at least $n-2$ contains a fixed point. However by Lemma 2.13 there are $n-1$ dimensional spaces mapped onto opposites, a contradiction. Thus either (i) θ is not $\{n-3, n-2\}$-domestic, or (ii) θ is not $\{n-3, n-1\}$-domestic (on Δ').

Consider case (i). Let v be the type $n-2$ vertex of a nondomestic type $\{n-3, n-2\}$ simplex. Then θ_v acts on the upper type $\mathsf{A}_1 \times \mathsf{A}_1$ residue by permuting the components, and thus θ_v is nondomestic on this upper residue (see [Parkinson and Van Maldeghem 2019, Lemma 3.7]). Moreover θ_v is a duality on the lower type A_{n-3} residue mapping a hyperplane (a type $n-3$ vertex) of this residue onto an opposite, and thus θ_v also maps a point (a type 1 vertex) to an opposite. Thus θ maps a type $\{1, n-1, n\}$ simplex to an opposite, a contradiction.

Consider case (ii). Since θ is neither $\{1\}$-domestic nor $\{n-1\}$-domestic on Δ', and since $n-1 \leq 4$, Corollary 2.11 implies that there exists a type $\{1, n-1\}$ simplex of Δ' mapped to an opposite. Now Lemma 2.13 implies that θ is not $\{1, n-1, n\}$-domestic on Δ. This contradiction establishes the result. ☐

Proposition 2.16. *Let Δ be the building $\mathsf{D}_n(2)$, $n \geq 4$, and let θ be a collineation of Δ. If θ is uncapped then the decorated opposition diagram of θ is contained in Table 1.*

Proof. Let θ be an uncapped collineation of $\mathsf{D}_n(2)$, and let $J = \mathrm{Typ}(\theta)$. Let $j = \max J$.

Case 1: $j \in \{n-1, n\}$ with n odd. Then necessarily $\{n-1, n\} \subseteq J$. If $J \backslash \{n-1, n\}$ contains no odd types, then the induced automorphism in every residue of a non-domestic $\{n-1, n\}$-simplex is a symplectic polarity, and hence θ is capped, a contradiction. Thus $J \backslash \{n-1, n\}$ contains an odd node, and so by Lemma 2.7 we have $1 \in J$. Thus by Lemma 2.15 there exists a type $\{1, n-1, n\}$ simplex mapped onto an opposite simplex, and it easily follows that θ maps simplices of each type $S \backslash \{i\}$ with $i = 1, 2, \ldots, n-2$ to opposite. Hence the claimed diagram.

Case 2: $j \in \{n-1, n\}$ with n even. By duality symmetry we may assume that $j = n$. If $n-1 \in J$, then by [Parkinson and Van Maldeghem 2019, Proposition 3.12(3)(b)] there is a type $\{n-1, n\}$-simplex mapped onto an opposite, and then considering the type A_{n-2} residue we easily deduce that there are simplices of each cotype $S \backslash \{i\}$ with $i = 1, 2, \ldots, n-2$ mapped onto opposites. It then easily follows that there are also simplices of each type $S \backslash \{n-1\}$ and $S \backslash \{n\}$ mapped onto opposite. So suppose that $n-1 \notin J$. If $J \backslash \{n-1, n\}$ contains no odd indices, then as above we deduce that θ is capped. Thus $J \backslash \{n-1, n\}$ contains an odd node, and so $1 \in J$ by Lemma 2.7, and by [Parkinson and Van Maldeghem 2019, Proposition 3.12(3)(a)] there is a type $\{1, n\}$ simplex mapped onto an opposite. It now easily follows that θ is strongly exceptional domestic.

Case 3: $j \notin \{n-1, n\}$. If j is odd, then considering the upper residue of a type j nondomestic we obtain a duality of a D_{n-j}, and since every duality of a D_{n-j} maps a point to an opposite point (Lemma 2.14) we have $j+1 \in J$, a contradiction. Thus j is even. If $j = 2$ then θ is capped (see [Parkinson and Van Maldeghem 2019, Fact 3.22]). So $j \geq 4$ (and hence $n \geq 6$). If J has only even types then clearly θ is capped. Thus J contains an odd node, and hence by Lemma 2.7 we have $1 \in J$. Applying Corollary 2.11 in the nonthick B_n building it follows that there is a type $\{1, j\}$-simplex mapped onto an opposite, and the result easily follows, using Lemma 2.9 to show that the last node is not shaded. □

Proposition 2.17. *Let θ be a duality of the $\mathsf{D}_n(2)$ building. If θ is uncapped then the decorated opposition diagram of θ is contained in Table 1.*

Proof. Let θ be an uncapped duality of $\mathsf{D}_n(2)$, and let $J = \mathrm{Typ}(\theta)$. Let $j = \max J$.

Case 1: $j \in \{n-1, n\}$ with n even. Then necessarily $\{n-1, n\} \subseteq J$. In the residue of such a simplex we have an exceptional domestic duality of $\mathsf{A}_{n-2}(2)$, and the result easily follows.

Case 2: $j \in \{n-1, n\}$ with n odd. In the residue of a nondomestic type j vertex we obtain an exceptional domestic duality of $\mathsf{A}_{n-1}(2)$, and again the result easily follows.

Case 3: $j \notin \{n-1, n\}$. If j is even, then considering the upper residue of a nondomestic type j vertex we obtain a duality of $\mathsf{D}_{n-j}(2)$, and since every duality of $\mathsf{D}_{n-j}(2)$ maps a point to an opposite point we have $j+1 \in J$, a contradiction. Thus j is odd. If $j = 1$ then θ is obviously capped. So $j \geq 3$ (and hence $n \geq 5$). In the lower residue of a nondomestic type j vertex we obtain an exceptional domestic duality of $\mathsf{A}_{j-1}(2)$, and hence the result, using Lemma 2.9 to see that the last node is not shaded. □

Propositions 2.16 and 2.17 establish Theorem 1(a) for buildings of type D_n. We now consider the exceptional types.

Lemma 2.18. *Let Δ be the building $\mathsf{F}_4(2)$, and let θ be a collineation. If* $\mathrm{Typ}(\theta) = \{1, 2, 3, 4\}$ *then there exists either a nondomestic type* $\{1, 2\}$ *simplex, or a nondomestic type* $\{3, 4\}$ *simplex.*

Proof. This follows from the classification given in Theorem 4.3. We note that no circular logic is introduced by postponing the proof until Section 4. □

We are now ready to prove Theorem 1(a) for the small exceptional buildings. Before doing so we would like to correct [Van Maldeghem 2012, Main result 2.2], where it is asserted that every domestic duality of an E_6 building is a symplectic polarity. In fact this result only holds for large E_6 buildings. The oversight in the proof of [Van Maldeghem 2012, Main result 2.2] is in the proof of [Van Maldeghem 2012, Lemma 5.2], where the existence of exceptional domestic automorphisms of $\mathsf{A}_4(2)$ is overlooked.

Proposition 2.19. *If θ is an uncapped automorphism of a building of exceptional type then the decorated opposition diagram of θ is contained in Table 2.*

Proof. (1) Let θ be an uncapped collineation of $\mathsf{E}_6(2)$ and let $J = \mathrm{Typ}(\theta)$. Suppose that $J = S$, and so the opposition diagram has the subsets $\{2\}$, $\{4\}$, $\{3, 5\}$ and $\{1, 6\}$ encircled. Let σ be a nondomestic type $\{3, 5\}$ simplex. Then θ_σ is an automorphism of an $\mathsf{A}_2 \times \mathsf{A}_1 \times \mathsf{A}_1$ building acting as a duality on the A_2 component and interchanging the two A_1 components (by Proposition 1.4). Thus θ_σ is not domestic on the $\mathsf{A}_1 \times \mathsf{A}_1$ component (see [Parkinson and Van Maldeghem 2019, Lemma 3.7]) and must be exceptional domestic on the A_2 component (for otherwise θ is capped). Hence there are nondomestic simplices of types $S\backslash\{2\}$ and $S\backslash\{4\}$, and so the encircled nodes 2 and 4 are shaded. Suppose that there is a nondomestic simplex σ' either of type $S\backslash\{3, 5\}$ or $S\backslash\{1, 6\}$. Then $\theta_{\sigma'}$ is an automorphism of an $\mathsf{A}_1 \times \mathsf{A}_1$ building interchanging the two components (again by Proposition 1.4), and hence is not domestic, and hence θ is capped, a contradiction. Thus the encircled subsets $\{3, 5\}$ and $\{1, 6\}$ are not shaded.

Suppose that $J \neq S$. Then the first argument of the previous paragraph shows that $\{3, 5\} \cap J = \varnothing$. A similar argument shows that $4 \notin J$. Thus if $J \neq S$ we have $\{3, 4, 5\} \cap J = \varnothing$. If $\{1, 6\} \subseteq J$ then $2 \in J$ (for in the residue of a nondomestic type $\{1, 6\}$ simplex we obtain a duality of D_4, and no duality of D_n is point domestic; see [Parkinson and Van Maldeghem 2019, Proposition 3.16]), and θ is capped. If $J = \{2\}$ then θ is obviously capped. Thus there are no uncapped collineations of E_6 with $\mathrm{Typ}(\theta) \neq S$.

(2) Let θ be an uncapped duality of an E_6 building and let $J = \mathrm{Typ}(\theta)$. We claim that $J = S$. If $1 \in J$ then $6 \in J$, and vice versa (since no duality of D_n is point domestic), and this argument shows that if $J = \{1, 6\}$ then θ is capped, a contradiction. So $\{2, 3, 4, 5\} \cap J \neq \varnothing$. If $3 \in J$ then $\{2, 3, 4, 5, 6\} \subseteq J$ (considering the A_4 component of the residue of a nondomestic type 3 vertex) and similarly if $5 \in J$ then $\{1, 2, 3, 4, 5\} \subseteq J$. Thus if either $3 \in J$ or $5 \in J$ then $J = S$. If $2 \in J$ then $\{2, 3, 5\} \subseteq J$ (considering the A_5 residue of a nondomestic type 2 vertex), and thus again $J = S$. If $4 \in J$ then $\{1, 3, 4, 5, 6\} \subseteq J$ (considering the $\mathsf{A}_2 \times \mathsf{A}_2$ component of the residue of a nondomestic type 4 vertex), and so once more $J = S$.

Thus all nodes are encircled. We claim that θ is strongly exceptional domestic, and so all nodes are shaded. To prove that there exist cotype j panels mapped onto opposite panels for each $j \in \{1, 3, 4, 5, 6\}$, note first that there exists a nondomestic type $\{2, 4\}$ simplex (by considering the A_4 component of the residue of a nondomestic type 3 vertex). If v is the type 2 vertex of such a simplex, then θ_v is a domestic duality of A_5 mapping a plane of this projective space onto an opposite, and thus θ_v is strongly exceptional domestic, and hence the result. Finally, to see that there is a nondomestic cotype 2 panel, let v be the type 1 vertex of a nondomestic cotype 4 panel. Using the classification of uncapped D_5 diagrams we see that θ_v is strongly exceptional domestic, and it follows that there exists a cotype 2 panel of E_6 mapped onto an opposite.

(3) Let θ be an uncapped collineation of an E_7 building and let $J = \mathrm{Typ}(\theta)$. If $J = S$ then θ is strongly exceptional domestic (considering the A_6 residue of a nondomestic type 2 vertex shows that θ maps simplices of each type $S\backslash\{j\}$ onto opposites for $j = 1, 3, 4, 5, 6, 7$, and considering the E_6 residue of the type 7 vertex of a nondomestic type $\{2, 7\}$ simplex, and using (2), shows that there is a simplex of type $S\backslash\{2\}$ mapped onto an opposite).

Suppose that $J \neq S$. Then $2 \notin J$ (for otherwise the induced duality of the A_6 residue is strongly exceptional domestic) and $5 \notin J$ (for otherwise the induced dualities of the A_4 and A_2 residues are both strongly exceptional domestic). We note the following: if $3 \in J$ then $\{3, 4, 6\} \subseteq J$ (considering the A_5 component of the residue) and if $4 \in J$ then $\{1, 3, 4, 6\} \subseteq J$ (considering the A_2 and A_3 components of the residue). Thus if either $3 \in J$ or $4 \in J$ then $\{1, 3, 4, 6\} \subseteq J$. If $6 \in J$ then $\{1, 6\} \subseteq J$ (since no duality of the D_5 component of the residue is point domestic). If $7 \in J$ then $\{1, 6, 7\} \subseteq J$ (since every duality of E_6 maps both type 1 and type 6 vertices to opposites). It follows that either $J = \{1\}$, $J = \{1, 6\}$, $J = \{1, 6, 7\}$, $J = \{1, 3, 4, 6\}$, or $J = \{1, 3, 4, 6, 7\}$. In the first, second, and third cases it is clear using the above arguments that θ is capped, a contradiction. We claim that $J = \{1, 3, 4, 6, 7\}$ is impossible (for any collineation, capped or uncapped), for if so, then by [Parkinson and Van Maldeghem 2019, Proposition 4.3(2)] there exists a type $\{3, 7\}$ simplex σ mapped to an opposite simplex, and if v is the type 7 vertex of σ then θ_v is a duality of an E_6 building mapping a type 3 vertex to an opposite, thus forcing $2, 5 \in J$, a contradiction.

The previous paragraph shows that if θ is uncapped and $J \neq S$ then $J = \{1, 3, 4, 6\}$. Considering the $\mathsf{A}_2 \times \mathsf{A}_3$ component of the residue of a nondomestic type 4 vertex shows that there are simplices of types $\{3, 4, 6\}$ and $\{1, 4, 6\}$ mapped onto opposites, thus the nodes 1 and 3 are shaded. If there exist either type $\{1, 3, 6\}$ or $\{1, 3, 4\}$ simplices mapped onto opposite simplices then considering the residue of the type 1 vertex of such a simplex we deduce that θ is capped, a contradiction. Thus the nodes 4 and 6 are not shaded.

(4) Let θ be an uncapped (hence nontrivial) collineation of an E_8 building and let $J = \mathrm{Typ}(\theta)$. If $J = S$ then easy residue arguments show that θ is strongly exceptional domestic.

We claim that if $J \neq S$ then $J \subseteq \{1, 6, 7, 8\}$. To see this, note that if $2 \in J$ then $\{3, 5, 7\} \in J$ (considering an A_7 residue), if $3 \in J$ then $\{2, 4, 5, 6, 7, 8\} \subseteq J$ (considering the A_6 component of the residue), if $4 \in J$ then $\{1, 3, 5, 6, 7, 8\} \subseteq J$ (considering the $\mathsf{A}_2 \times \mathsf{A}_4$ component of the residue), and if $5 \in J$ then $\{1, 2, 3, 4, 7\} \subseteq J$ (considering the $\mathsf{A}_4 \times \mathsf{A}_3$ residue). Combining these statements it follows that if $\{2, 3, 4, 5\} \cap J \neq \varnothing$ then $J = S$, and hence the claim.

Suppose $J \neq S$, and so $J \subseteq \{1, 6, 7, 8\}$. We claim $J = \{1, 6, 7, 8\}$. For if $1 \in J$ then $8 \in J$ (since no duality of D_7 is point domestic), if $6 \in J$ then $J = \{1, 6, 7, 8\}$ (considering the $\mathsf{D}_5 \times \mathsf{A}_2$ residue and recalling that no duality of D_5 is point domestic), and if $7 \in J$ then $6 \in J$ (considering the duality of E_6 and using (2) above) and so again $J = \{1, 6, 7, 8\}$. Thus $J = \{8\}, \{1, 8\}$ or $\{1, 6, 7, 8\}$. The first two cases are clearly capped, hence the claim. Now considering the residue of a type 6 nondomestic vertex we see that there are simplices of types $\{1, 6, 7\}$ and $\{1, 6, 8\}$ mapped onto opposite simplices (hence the nodes 7 and 8 are shaded). If there exists a simplex of type $\{6, 7, 8\}$ or $\{1, 7, 8\}$ mapped onto an opposite then considering the D_5 residue we deduce that θ is capped, and so the nodes 1 and 6 are not shaded.

(5) Let θ be an uncapped collineation of an F_4 building and let $J = \mathrm{Typ}(\theta)$. If $2 \in J$

then $3, 4 \in J$ (by the duality in the A_2 component of the residue) and similarly if $3 \in J$ then $1, 2 \in J$. Thus either $J = \{1\}$, $J = \{4\}$, $J = \{1, 4\}$, or $J = \{1, 2, 3, 4\}$. The first and second cases are trivially capped. The third case is capped by [Parkinson and Van Maldeghem 2019, Lemma 4.5]. Thus $J = \{1, 2, 3, 4\}$.

If $\Delta = \mathsf{F}_4(2)$ then by Lemma 2.18 there is either a type $\{1, 2\}$ or $\{3, 4\}$ simplex mapped onto an opposite simplex. In the first case, by considering the residue of the type 2 vertex, we see that there are panels of cotype 3 and 4 mapped onto opposites, and hence the nodes 3 and 4 are shaded. The second case is symmetric, with the nodes 1 and 2 shaded. Of course both cases may occur simultaneously, and then all nodes are shaded. Finally, note that if either nodes 1 or 2 are shaded then both are shaded (if the i node is shaded and $i \in \{1, 2\}$ then consider the residue of the type 3 vertex of a nondomestic cotype i panel). Similarly, if either nodes 3 or 4 are shaded then both are shaded. Hence the result for $\mathsf{F}_4(2)$.

If $\Delta = \mathsf{F}_4(2, 4)$ then considering the $\mathsf{A}_2(4)$ component of a type 2 nondomestic vertex we deduce that there are simplices of type $\{2, 3, 4\}$ mapped onto opposites. Then considering the $\mathsf{A}_2(2)$ residue of a type $\{3, 4\}$ nondomestic simplex we deduce that there are also simplices of type $\{1, 3, 4\}$ mapped onto opposites. Thus the nodes $1, 2$ are shaded. If there exists a simplex of type $\{1, 2, 4\}$ or $\{1, 2, 3\}$ mapped onto an opposite, then considering the type $\mathsf{A}_2(4)$ residue of the $\{1, 2\}$ subsimplex we deduce that θ is nondomestic, and hence capped, a contradiction. Thus the nodes 3 and 4 are not shaded. □

Theorem 1(a) now follows from Propositions 2.1, 2.6, 2.10, 2.16, 2.17, and 2.19.

2B. ***Applications.*** This section contains applications and corollaries of Theorem 1(a).

Corollary 2.20. *Let θ be a an exceptional domestic automorphism of a thick irreducible spherical building Δ.*

(a) *If θ is an oppomorphism and Δ is simply laced, then θ is strongly exceptional domestic.*

(b) *If θ is not an oppomorphism then θ is not strongly exceptional domestic.*

Proof. The first statement follows by noting that in Tables 1 and 2, if θ is an oppomorphism and Δ is simply laced, then whenever all nodes are encircled they are all shaded (see the first, third, sixth rows of Table 1 and the first, second, and third rows of Table 2). The second statement follows by inspecting the third and fourth rows of Table 1 and the first row of Table 2. □

The following lemma is in preparation for our next corollary to Theorem 1(a).

Lemma 2.21. *Let θ be an involution of a thick spherical building, and suppose that the simplex σ is mapped onto an opposite simplex. Then the induced automorphism θ_σ of* $\mathrm{Res}(\sigma)$ *is either the identity or it is an involution.*

Proof. Let α be a simplex of $\mathrm{Res}(\sigma)$. If $\alpha^\theta = \mathrm{proj}_{\sigma^\theta}(\alpha)$ then $\alpha^{\theta_\sigma} = \alpha$ (because the projection maps $\mathrm{proj}_\sigma : \mathrm{Res}(\sigma^\theta) \to \mathrm{Res}(\sigma)$ and $\mathrm{proj}_{\sigma^\theta} : \mathrm{Res}(\sigma) \to \mathrm{Res}(\sigma^\theta)$ are mutually inverse bijections). If $\alpha^\theta = \mathrm{proj}_{\sigma^\theta}(\alpha)$ then $\alpha^{\theta_\sigma} = \alpha$. If $\alpha^\theta \neq \mathrm{proj}_{\sigma^\theta}(\alpha)$ then, since θ maps α^θ onto α, the projection $\mathrm{proj}_\sigma(\alpha^\theta)$ is mapped onto $\mathrm{proj}_{\sigma^\theta}(\alpha)$. Thus $\theta_\sigma^2 = 1$. □

Corollary 2.22. *Every involution of a thick irreducible spherical building is capped.*

Proof. The result is of course true for large buildings of rank at least 3 (where all automorphisms are capped, by [Parkinson and Van Maldeghem 2019]), and thus it remains to show that involutions of small buildings and of arbitrary generalised polygons are capped. Let us begin with the former. We use the decorated opposition diagrams in Tables 1 and 2 to show that every uncapped automorphism has order strictly greater than 2. Consider type A_n, and let θ be uncapped. By Theorem 1(a) there exists a nondomestic type $\{3, 4, \dots, n\}$ simplex σ. Then θ_σ is a domestic duality of the Fano plane. However by [Parkinson et al. 2015] the only domestic duality of the Fano plane is the unique exceptional domestic duality, and this has order 8. Thus, by Lemma 2.21 θ has order strictly greater than 2.

The arguments are similar for all other uncapped diagrams. The key fact is that in some residue one finds a domestic duality of the Fano plane. For example, in the first $\mathsf{E}_6(2)$ diagram in Table 2 we have a nondomestic type $\{1, 3, 5, 6\}$ simplex σ (because, for example, the node 2 is shaded), and θ_σ is a domestic duality of the Fano plane residue.

We now show that every involution of an arbitrary generalised m-gon, $m \geq 2$, is capped. Recall that a generalised m-gon Δ is a bipartite graph with diameter m and girth $2m$. A chamber is a pair of vertices connected by an edge. If $\{x, y\}$ is a chamber we write $x \sim y$ and call x and y adjacent. In particular, if $x \sim y$ then the vertices x and y have different types. Vertices x and y of Δ are opposite if and only if the distance between them is m, and this in turn is equivalent to the existence of a path $x = x_0 \sim x_1 \sim \cdots \sim x_m = y$ with $x_j \neq x_{j+2}$ for all $j = 0, \dots, m-2$. If the distance between vertices x, y is $k < m$ then there is a unique geodesic from x to y. In this case, writing $x = z_0 \sim z_1 \sim \cdots \sim z_k = y$ the vertex z_1 (resp. z_{k-1}) is the projection of y onto x (resp. x onto y).

Claim 1: Every involutory collineation of a thick generalised $2n$-gon Δ, $n \geq 1$, is capped.

Proof of Claim 1. The case $n = 1$ is trivial, and so suppose that θ is an uncapped involutory collineation of a generalised $2n$-gon with $n \geq 2$. Thus θ is domestic (on chambers), and maps at least one vertex of each type onto an opposite vertex. Let Δ' denote the fixed elements of θ. Let x_0 be a type 1 vertex mapped onto an opposite vertex $x_{2n} = x_0^\theta$, and consider any geodesic path $x_0 \sim x_1 \sim \cdots \sim x_{2n-1} \sim x_{2n}$. If $x_1^\theta \neq x_{2n-1}$ then the chamber $\{x_0, x_1\}$ is mapped onto an opposite chamber and θ is capped. Hence $x_1^\theta = x_{2n-1}$, and it follows that $x_i^\theta = x_{2n-i}$, for all $i \in \{0, 1, 2, \ldots, 2n\}$. In particular $x_n^\theta = x_n$ is fixed. Consider another geodesic $x_0 \sim y_1 \sim \cdots \sim y_{2n-1} \sim x_{2n}$ with $y_1 \neq x_1$. Then $y_n^\theta = y_n$. By considering the path from x_n to x_0 to y_n we see that x_n and y_n are opposite, and thus there is a pair of opposite vertices $x_n, y_n \in \Delta'$.

Similarly, by considering a type 2 vertex x_0' that is mapped onto an opposite vertex we deduce the existence of a pair of opposite vertices $x_n', y_n' \in \Delta'$. Since the vertices x_n', y_n' have different type to the vertices x_n, y_n we conclude that for each type $j \in \{1, 2\}$ there are pairs of opposite vertices of type j in Δ'. It follows that Δ' is a sub-$2n$-gon (because the fixed structure of a collineation of a $2n$-gon is either empty, consists of pairwise opposite elements, is a tree of diameter at most $2n$, or is a sub-$2n$-gon, and the first three options are impossible from the above considerations).

Now, the distance from x_n' to x_n is at most $2n-1$ (by types and diameter) and hence the unique geodesic from x_n' to x_n is fixed by θ. In particular the chamber $\{z, x_n\}$ is fixed, where $z \sim x_n$ is the projection of x_n' onto x_n. Note that $z \neq x_{n-1}, x_{n+1}$ because $x_{n-1}^\theta = x_{n+1}$ is not fixed. We claim that every vertex $z_1 \sim z$ is fixed. With y_j as above, note that z and y_{n-1} are opposite (consider the path from z to x_0 to y_{n-1}). Hence the distance from z_1 to y_{n-1} is $2n-1$, and so there is a unique geodesic $z_1 \sim z_2 \sim \cdots z_{2n-1} = y_{n-1}$. If $z_1^\theta \neq z_1$ then z_n and z_n^θ are opposite (consider the path from z_n to z_0 to z_n^θ). Similarly, since $y_{n-1}^\theta = y_{n+1}$ we have $y_{n-1} \neq y_{n-1}^\theta$ and so z_{n+1} and z_{n+1}^θ are opposite. Hence the chamber $\{z_n, z_{n+1}\}$ is mapped onto an opposite chamber, a contradiction.

It now follows from [Van Maldeghem 1998, Proposition 1.8.1] that the sub-$2n$-gon Δ' has the property that whenever $x \in \Delta'$ has the same type as z, then all neighbours of x are fixed (and hence are in Δ'). But x_n' has the same type as z, contradicting the fact that the projection of x_0' onto x_n' is mapped onto the projection of $x_0'^\theta$ onto x_n' and that these projections are distinct. This contradiction completes the proof of Claim 1. □

Claim 2: Every involutary duality of a thick generalised $(2n-1)$-gon Δ, $n \geq 2$, is capped.

Proof of Claim 2. Let θ be a polarity of a generalised $(2n-1)$-gon and suppose that θ maps some element x_0 to an opposite element x_{2n-1}. Suppose that θ is not capped, i.e., θ does not map any chamber to an opposite chamber. Let $x_1 \sim x_0$ be arbitrary. Consider the path $x_0 \sim x_1 \sim \cdots \sim x_{2n-1}$. Using a similar approach to the one in the previous proof, we deduce that $x_i^\theta = x_{2n-1-i}$ for all $i \in \{0, 1, 2, \ldots, 2n-1\}$. Hence $x_n^\theta = x_{n-1}$. Consider a second path $x_0 \sim y_1 \sim \cdots \sim y_{2n-2} \sim x_{2n-1}$ with $y_1 \neq x_1$. Then also $y_{n-1}^\theta = y_n$. Let $z_0 \sim x_n$ be arbitrary but distinct from x_{n-1} and x_{n+1} (using thickness). There is a unique path $z_0 \sim z_1 \sim \cdots \sim z_{2n-2} = y_{n-1}$ from

z_0 to y_{n-1}. By considering the path $z_{n-2} \sim \cdots \sim z_0 \sim x_n \sim x_n^\theta \sim z_0^\theta \sim \cdots \sim z_{n-2}^\theta$ we see that z_{n-2} is mapped onto an opposite vertex. Similarly, since $y_{n-1}^\theta = y_n$ we see that z_{n-1} is mapped onto an opposite vertex (consider the path $z_{n-1} \sim \cdots \sim y_{n-1} \sim y_{n-1}^\theta \sim \cdots \sim z_{n-1}^\theta$). Hence the chamber $\{z_{n-2}, z_{n-1}\}$ is mapped onto an opposite chamber, a contradiction. This completes the proof of Claim 2. □

Finally, we note that no duality of a thick generalised $2n$-gon is domestic and no collineation of a thick generalised $(2n-1)$-gon is domestic (see [Parkinson et al. 2015, Lemmas 3.1 and 3.2]), completing the proof of the corollary. □

Corollary 2.22 shows that every uncapped automorphism has order at least 3. Since every known example of an uncapped automorphism has order at least 4 (see the examples in Sections 3 and 4, and also the rank 2 classification in [Parkinson et al. 2015]) we are led to make the following conjecture.

Conjecture 2.23. *If θ is an automorphism of a thick irreducible spherical building, and if θ has order* 3, *then θ is capped.*

Note that if we remove the shading from the diagrams in Tables 1 and 2 then the diagrams we obtain are contained in [Parkinson and Van Maldeghem 2019, Tables 1–5]. Thus Theorem 1(a) has the following immediate corollary.

Corollary 2.24. *The* (*undecorated*) *opposition diagram of any automorphism of a thick irreducible spherical building is contained in* [*Parkinson and Van Maldeghem 2019, Tables* 1–5].

We now use Theorem 1(a) to determine the partially ordered set $\mathcal{T}(\theta)$ for all automorphisms θ. We first note that, by the proposition below, it is sufficient to determine the maximal elements of $\mathcal{T}(\theta)$.

Proposition 2.25. *Let $\mathcal{M}(\theta)$ be the set of maximal elements of $\mathcal{T}(\theta)$. Then*

$$\mathcal{T}(\theta) = \{J \subseteq S \mid J^{\pi_\theta w_0} = J \text{ and } J \subseteq M \text{ for some } M \in \mathcal{M}(\theta)\}.$$

Proof. This follows immediately from the facts that if σ is a nondomestic type K simplex then (i) K is preserved by $w_0 \circ \pi_\theta$, and (ii) if $J \subseteq K$ is preserved under $w_0 \circ \pi_\theta$ then the type J subsimplex of σ is also nondomestic (see [Parkinson and Van Maldeghem 2019, Lemma 1.3]). □

Thus it remains to compute the set $\mathcal{M}(\theta)$ of maximal elements of $\mathcal{T}(\theta)$. We do this in the corollary below. Recall that if θ is uncapped then the decorated opposition diagram of θ is $(\Gamma, \mathrm{Typ}(\theta), K_\theta, \pi_\theta)$ where, in particular, K_θ is the set of shaded nodes.

Corollary 2.26. *Let θ be an automorphism of a spherical building Δ.*

(a) *If θ is capped then $\mathcal{M}(\theta) = \{\mathrm{Typ}(\theta)\}$.*

(b) *If θ is uncapped then $\mathcal{M}(\theta) = \{\mathrm{Typ}(\theta)\backslash\{k\} \mid k \in K_\theta\}$.*

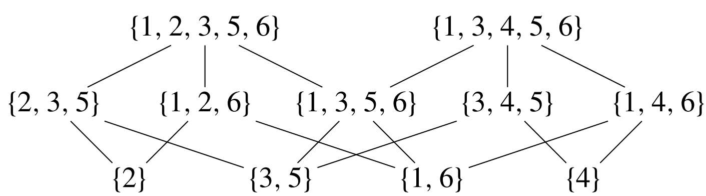
Proof. The first statement is obvious, so consider the second statement. Let (Γ, J, K, π) be the decorated opposition diagram, and so $J = \mathrm{Typ}(\theta)$. If $J = K$ then there are nondomestic simplices of each type $\mathrm{Typ}(\theta)\backslash\{k\}$ with $k \in J$, and these are clearly the maximal types mapped to opposite (otherwise θ is capped). Suppose now that $J\backslash K$ consists of a single minimal $w_0 \circ \pi$ invariant subset J' (thus J' is either a singleton, or J' consists of a pair, as in the second $\mathsf{D}_{2n}(2)$ diagram in Table 1). In this case the only $w_0 \circ \pi$ stable strict subset of J that is not contained in an element of $\{J\backslash\{k\} \mid k \in K\}$ is $J\backslash J'$, and since J' is not shaded all simplicies of this type are domestic. Hence the result in this case.

By Theorem 1(a) the only remaining cases are the six diagrams where $J\backslash K$ consists of precisely two minimal $w_0 \circ \pi$ invariant sets. Specifically, these examples are the $\mathsf{E}_6(2)$ collineation diagram, the first $\mathsf{E}_7(2)$ and $\mathsf{E}_8(2)$ diagrams, the first two $\mathsf{F}_4(2)$ diagrams (these are dual to one another), and the $\mathsf{F}_4(2,4)$ diagram. In these cases the result is implied by the following claim.

Claim: Suppose that the decorated opposition diagram of θ is one of the six diagrams listed above. Then θ is $\{i, j\}$-domestic where i and j are the two shaded nodes.

Proof of Claim. Consider the E_6 diagram. If there is a nondomestic type $\{2, 4\}$ simplex then with v the type 4 vertex of this simplex the map θ_v acts on the $\mathsf{A}_2 \times \mathsf{A}_2$ component of the residue swapping the components (by Proposition 1.4). It follows that θ is not domestic, a contradiction. Similar arguments apply for E_7 and E_8, using an A_5 and E_6 residue, respectively. For the first $\mathsf{F}_4(2)$ diagram, suppose there is a nondomestic type $\{1, 2\}$ simplex σ. Then θ_σ is a domestic duality of $\mathsf{A}_2(2)$, and hence is the exceptional domestic duality of the Fano plane. It follows that there is a nondomestic type $\{1, 2, 3\}$ simplex, contradicting the node 4 being unshaded. A dual argument applies to the second $\mathsf{F}_4(2)$ diagram. The $\mathsf{F}_4(2, 4)$ diagram is similar. Hence the proof of the claim is complete, and the corollary follows. □

Example 2.27. Suppose that θ has the $\mathsf{E}_6(2)$ collineation diagram in Table 2. Then the partially ordered set $\mathcal{T}(\theta)$ is (using Proposition 2.25 and Corollary 2.26):



As a final application we will compute the displacement of an arbitrary automorphism θ in Corollary 2.29. Recall that $\mathrm{disp}(\theta) = \max\{d(C, C^\theta) \mid C \in \mathcal{C}\}$, where $\mathcal{C}$ is the set of chambers of Δ, and $d(C, D) = \ell(\delta(C, D))$ for chambers $C, D \in \mathcal{C}$.

Proposition 2.28. *Let θ be any automorphism of a thick irreducible spherical building of type (W, S). Then*

$$\mathrm{disp}(\theta) = \mathrm{diam}(W) - \min\{\mathrm{diam}(W_{S\setminus J}) \mid J \in \mathcal{M}(\theta)\}.$$

Proof. Let $N = \min\{\mathrm{diam}(W_{S\setminus J}) \mid J \in \mathcal{M}(\theta)\}$. We note first that

$$N = \min\{\mathrm{diam}(W_{S\setminus J}) \mid \text{there exists a type } J \text{ simplex in } \mathrm{Opp}(\theta)\} \tag{2-2}$$

because the minimum is obviously attained at a maximal element of $\mathcal{T}(\theta)$.

Let $J \subseteq \mathrm{Typ}(\theta)$ be any subset for which there exists a nondomestic type J simplex. Then for all chambers C containing this simplex we have $\delta(C, C^\theta) \in W_{S\setminus J} w_0$ (see [Parkinson and Van Maldeghem 2019, Lemma 2.5]) and thus

$$\mathrm{disp}(\theta) \geq \ell(\delta(C, C^\theta)) \geq \ell(w_0) - \ell(w_{S\setminus J}) = \mathrm{diam}(W) - \mathrm{diam}(W_{S\setminus J}).$$

Since this inequality holds for all J such that there exists a type J simplex in $\mathrm{Opp}(\theta)$ the formula (2-2) gives

$$\mathrm{disp}(\theta) \geq \mathrm{diam}(W) - N.$$

On the other hand, let C be any chamber with $\ell(\delta(C, C^\theta))$ maximal. By the arguments of [Abramenko and Brown 2009, Lemma 2.4 and Theorem 4.2] we have $\delta(C, C^\theta) = w_I w_0$ for some $I \subseteq S$ with $I^{\pi_\theta} = I^{w_0}$. Hence the type $J = S\setminus I$ simplex of C is mapped onto an opposite simplex. Thus

$$\mathrm{disp}(\theta) = \ell(\delta(C, C^\theta)) = \ell(w_0) - \ell(w_{S\setminus J}) = \mathrm{diam}(W) - \mathrm{diam}(W_{S\setminus J}) \leq \mathrm{diam}(W) - N,$$

and hence the result. □

Corollary 2.29. *Let θ be an automorphism of a thick irreducible spherical building and let $J = \mathrm{Typ}(\theta)$. Then*

$$\mathrm{disp}(\theta) = \begin{cases} \mathrm{diam}(W) - \mathrm{diam}(W_{S\setminus J}) & \text{if } \theta \text{ is capped,} \\ \mathrm{diam}(W) - \mathrm{diam}(W_{S\setminus J}) - 1 & \text{if } \theta \text{ is uncapped.} \end{cases}$$

In particular, if θ is exceptional domestic then $\mathrm{disp}(\theta) = \mathrm{diam}(\Delta) - 1$.

Proof. The case of capped automorphisms is [Parkinson and Van Maldeghem 2019, Theorem 5]. In the case of an uncapped automorphism we note that by Corollary 2.26 the maximal elements of $\mathcal{T}(\theta)$ are of the form $\mathrm{Typ}(\theta)\setminus\{j\}$ for some $j \in \mathrm{Typ}(\theta)$, and then the result follows from Proposition 2.28. □

Remark 2.30. Corollary 2.29 shows that the set of possible displacements is extremely restricted. For example, consider an E_8 building Δ, where a priori there are $\ell(w_0) = 120$ potential displacements. However, by Corollary 2.29, [Parkinson and Van Maldeghem 2019, Theorem 3], and Theorem 1(a) the only possible displacements of an automorphism θ are:

$$
\begin{aligned}
0 &= \mathrm{diam}(\mathsf{E}_8)-\mathrm{diam}(\mathsf{E}_8) && \text{if } \theta \text{ is the identity,}\\
57 &= \mathrm{diam}(\mathsf{E}_8)-\mathrm{diam}(\mathsf{E}_7) && \text{if } \theta \text{ is capped with type } \{8\},\\
90 &= \mathrm{diam}(\mathsf{E}_8)-\mathrm{diam}(\mathsf{D}_6) && \text{if } \theta \text{ is capped with type } \{1,8\},\\
107 &= \mathrm{diam}(\mathsf{E}_8)-\mathrm{diam}(\mathsf{D}_4)-1 && \text{if } \theta \text{ is uncapped with type } \{1,6,7,8\},\\
108 &= \mathrm{diam}(\mathsf{E}_8)-\mathrm{diam}(\mathsf{D}_4) && \text{if } \theta \text{ is capped with type } \{1,6,7,8\},\\
119 &= \mathrm{diam}(\mathsf{E}_8)-1 && \text{if } \theta \text{ is uncapped with type } S,\\
120 &= \mathrm{diam}(\mathsf{E}_8) && \text{if } \theta \text{ is nondomestic.}
\end{aligned}
$$

In particular, note that for E_8 buildings the displacement determines the (decorated) opposition diagram of the automorphism. This phenomenon is not true for all types; for example in $\mathsf{B}_7(\mathbb{F})$ displacement 45 is obtained by both capped automorphisms with $\mathrm{Typ}(\theta)=\{1,2,3,4,5\}$ and capped automorphisms with $\mathrm{Typ}(\theta)=\{2,4,6\}$.

3. Uncapped automorphisms for classical types

In this section we prove Theorem 1(b) for classical types. Thus our aim is to construct uncapped automorphisms with each of the diagrams listed in Tables 1 and 2 for the buildings $\mathsf{A}_n(2)$, $\mathsf{B}_n(2)$, $\mathsf{B}_n(2,4)$, and $\mathsf{D}_n(2)$.

3A. *The buildings* $\mathsf{A}_n(2)$. In this section we work with the concrete model $\mathsf{A}_n(2)=\mathsf{PG}(n,\mathbb{F}_2)$ for the small building of type A_n. Thus an i-space of $\mathsf{A}_n(2)$ means a subspace of $\mathbb{F}_2^n$ of (projective) dimension i, and this corresponds to a type $i+1$ vertex of the building. Let θ be a duality of $\mathsf{A}_n(2)$. Recall that a point p of $\mathsf{A}_n(2)$ is called *absolute* with respect to θ if $p\in p^\theta$ (that is, p is not mapped to an opposite hyperplane). Dually, a hyperplane π is absolute if $\pi^\theta\in\pi$ (that is, π is not mapped to an opposite point).

Lemma 3.1. *Let θ be a duality of a projective space. Suppose that U is an m-space consisting of absolute points of θ, and let $k=\mathrm{dim}(U\cap U^\theta)$. Then $m-k$ is even.*

Proof. The hyperplanes through $\langle U^\theta, U\rangle$ form a dual space of (projective) dimension k, and the inverse image is a k-space contained in U. Choose a complementary $(m-k-1)$-space H in U, and so H intersects neither U^θ nor $U^{\theta^{-1}}$. Then for each $x\in H$ we have that $x^\theta\cap H$ is a hyperplane of H through x, and hence is absolute. Thus θ is a symplectic polarity on H, and so $m-k$ is even (see Lemma 2.3). □

Theorem 3.2. *For each $n\geq 2$ there exists a unique duality θ of $\mathsf{A}_n(2)$ (up to conjugation) with the property that the set of absolute points of θ is the union of two distinct hyperplanes. This duality is strongly exceptional domestic, with order 8 if n is even and 4 if n is odd.*

Proof. We first demonstrate the existence of a duality whose absolute points form the union of two hyperplanes. Let J_1, J_2, and J_3 be the matrices

$$J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

and let A be the $(n+1)\times(n+1)$ matrix in block diagonal form,

$$A = \mathrm{diag}(J, J_1, J_1, \dots, J_1) \quad \text{with } J = J_2 \text{ for even } n \text{ and } J = J_3 \text{ for odd } n.$$

Let θ be the duality of $\mathsf{A}_n(2)$ with matrix A. That is, $X^\theta = (AX)^\perp$ where X is written as a column vector. Then X is absolute if and only if $X \in (AX)^\perp$, and hence by direct calculation X is absolute if and only if $X_0X_1 = 0$. The matrix for the collineation θ^2 is given by $A^{-t}A$, and it follows by calculation that θ has order 8 if n is even, and order 4 if n is odd.

We now prove that there is at most one duality θ up to conjugation with the given property, and that such a duality is necessarily strongly exceptional domestic. We proceed by induction on n, the case $n = 2$ being contained in [Parkinson et al. 2015].

So let θ be a duality of $\mathsf{A}_n(2)$ such that $\alpha_1 \cup \alpha_2$ is the set of absolute points for θ with $\alpha_1 \neq \alpha_2$ two hyperplanes of $\mathsf{A}_n(2)$. Let β be the hyperplane containing $\alpha_1 \cap \alpha_2$ and different from both α_1 and α_2. Note that $\alpha_1 \cup \alpha_2 \cup \beta$ is the entire point set. Let $p_i = \alpha_i^\theta$, $i = 1, 2$ and $q = \beta^\theta$; then $L = \{p_1, p_2, q\}$ is a line.

Note that q is absolute (for if $q \in \beta$ we have $q^\theta \ni \beta^\theta = q$). Thus $q \in \alpha_1 \cup \alpha_2$. In fact we claim that $q \in \alpha_1 \cap \alpha_2$. For if not we have $\beta^\theta = q \notin \beta$ and so β is not absolute, contradicting the fact that $\beta = q^{\theta^{-1}}$ is absolute (since q is absolute).

Since $L = \{p_1, p_2, q\}$ is a line and $q \in \alpha_1 \cap \alpha_2$ we either have $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$ or $p_1, p_2 \in \alpha_1 \cup \alpha_2$. We treat these two cases below. Before doing this, we observe that in the first case n is necessarily even, and in the second case n is necessarily odd. To see this, note that if $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$ then the point p_1 is nonabsolute and the mapping $\rho_1 : z \mapsto z^\theta \cap \alpha_1$, $z \in \alpha_1$, is a duality on α_1 every point of which is absolute, forcing n to be even (see Lemma 2.3). On the other hand, if $p_1, p_2 \in \alpha_1 \cup \alpha_2$ then we have $(\alpha_1 \cap \alpha_2)^\theta = \langle p_1, p_2 \rangle \subseteq \alpha_1 \cap \alpha_2$ and so Lemma 3.1 implies $(n-2)-1 = n-3$ is even, and so n is odd. We also observe that since α_1 and α_2 are the only two hyperplanes all of whose points are absolute, every even power of θ preserves the set $\{\alpha_1, \alpha_2\}$, and hence also the set $\{p_1, p_2\}$. It follows that $p_i^\theta \in \{\alpha_1, \alpha_2\}$ for $i = 1, 2$.

Case 1: $p_1, p_2 \in \beta \backslash (\alpha_1 \cup \alpha_2)$. As noted above n is even, and so we may assume $n \geq 4$. Let $\sigma = \{x, \xi\}$ be any nondomestic (point-hyperplane)-flag for θ (that is, a nondomestic type $\{1, n\}$-simplex of the building). We note that such simplices exist, and indeed they obviously all arise as follows: Since the absolute hyperplanes for θ are precisely the hyperplanes through one of the points p_1 or p_2, if we select any point $x \in \beta \setminus (\alpha_1 \cup \alpha_2)$ and any hyperplane ξ through x not containing p_1 or p_2, then $\sigma = \{x, \xi\}$ is nondomestic.

We claim that the mapping $\theta_\sigma : z \mapsto z^\theta \cap \xi \cap x^\theta$ for $z \in \xi \cap x^\theta$ has exactly two hyperplanes consisting entirely of absolute points. Note that $q \in \xi$ and also $q \in x^\theta$. Note also that, since p_i^θ contains the absolute point $q_i := \langle p_i, x\rangle \cap (\alpha_1 \cap \alpha_2)$, also x^θ contains q_i, $i = 1, 2$. Since ξ does not contain p_i, but it does contain x, it does not contain q_i, $i = 1, 2$. Consequently $x^\theta \cap \alpha_1 \cap \alpha_2$ is not contained in ξ and the claim follows.

Thus for every nondomestic (point-hyperplane)-pair $\sigma = \{x, \xi\}$ the induced duality θ_σ on the $\mathsf{A}_{n-2}(2)$ residue has precisely two hyperplanes of absolute points. Since $n - 2$ is even this duality again satisfies the condition of Case 1, and so by induction θ is domestic. Since θ has nondomestic points necessarily θ is strongly exceptional domestic by Theorem 2.5.

We now show that θ is unique, up to a projectivity (and under the assumptions of Case 1). Let ρ_1 be the symplectic polarity on α_1 introduced in the paragraph before Case 1. Noting that $q^{\rho_1} = \alpha_1 \cap \alpha_2$, we see that the data α_1, α_2 and ρ_1 are projectively unique. This determines q. All choices of p_1 outside $\alpha_1 \cup \alpha_2$ are projectively equivalent, and then p_2 is the third point on the line determined by p_1 and q. We then know the image of an arbitrary point x_1 of $\alpha_1 \setminus \alpha_2$, as $x_1^\theta = \langle x^{\rho_1}, p_1\rangle$. This determines the images of all points of α_1. Since $p_1^\theta = \alpha_1$, we know the images of a basis, which suffices to determine the whole duality.

Case 2: $p_1, p_2 \in \alpha_1 \cup \alpha_2$. As noted above, n is odd. Take an arbitrary point $z \in \beta \setminus (\alpha_1 \cup \alpha_2)$ and set $H := z^\theta$. Then $\varphi : x \mapsto x^\theta \cap H$ is a duality in the $(n-1)$-dimensional projective space H such that its absolute points form two hyperplanes $H \cap \alpha_i$, $i = 1, 2$. Hence by the previous case φ is domestic, and since z was arbitrary amongst the nondomestic points for θ we conclude that θ is domestic. Thus by Theorem 2.5 θ is strongly exceptional domestic.

It remains to show that θ is unique up to conjugation with a projectivity. Let $D_i = H \cap \alpha_i$, $i = 1, 2$. Set $\{i, j\} = \{1, 2\}$ and $D_i^{\varphi^{-1}} = p_i'$. Then $\{q, p_1', p_2'\}$ is a line in $H \cap \beta$ (since $p_i'^\varphi = D_i$ it suffices to see that $q^\varphi = \beta \cap H$, and this follows from the definition of φ as $\beta = q^\theta$). It also follows that $D_i^{\theta^{-1}} = \langle p_i', z\rangle$. Since $D_i \subseteq \alpha_i$, we conclude $\alpha_i^{\theta^{-1}} \in \langle p_i', z\rangle$. But $\alpha_i^{\theta^{-1}} \in \{p_1, p_2\}$. We claim that $\alpha_i^{\theta^{-1}} = p_i$. Suppose not. Then $\alpha_i^{\theta^{-1}} = p_j$. Now from $z^\theta = H$ and $p_i^\theta = \alpha_j$ follows that $t_i^\theta = \langle D_j, z\rangle$, with $\{t_i, p_i, z\}$ a line. But $p_j'^\theta$ is a hyperplane through D_j distinct from α_j and H (as $p_j \in H$ and is not absolute); hence $p_j'^\theta = \langle D_j, z\rangle$ and so $t_i = p_j'$. Now $p_j'^{\theta^{-1}} = \langle D_i, z\rangle$ and $p_i^{\theta^{-1}} = \alpha_i$. It follows that $z^{\theta^{-1}} = H$. Hence $z^{\theta^2} = z$, for all $z \in \beta \setminus (\alpha_1 \cup \alpha_2)$. It follows that $p_i^{\theta^2} = p_i$, contradicting $p_i^{\theta^2} = \alpha_j^\theta = p_j$. Our claim follows.

But now, just like in the proof of our previous claim, we have that $\{p_i, p_i', z\}$ is a line and $p_i'^{\theta} = \langle D_j, z\rangle$. It follows that $p_i^{\theta^2} = p_j$ and so $z^{\theta^2} = z'$, with $\{z, z', q\}$ a line.

Now, α_1, α_2, H, z and φ are unique up to conjugation with a projectivity. But then, given $z^\theta = H$, the duality θ is completely determined, since q is determined and hence also z' (with the above notation). This determines the image x^θ of an arbitrary point in H as $x^\theta = \langle x^\varphi, z'\rangle$. Furthermore, we also have $z^\theta = H$, and so θ is determined. □

3B. ***The buildings*** $\mathsf{B}_n(2)$***,*** $\mathsf{B}_n(2,4)$***, and*** $\mathsf{D}_n(2)$**.** It will be more convenient for us to regard $\mathsf{B}_n(2) \cong \mathsf{C}_n(2)$ as a symplectic polar space. We begin by recalling the standard models of the $\mathsf{C}_n(2)$, $\mathsf{D}_n(2)$, and $\mathsf{B}_{n-1}(2,4)$ buildings in the ambient projective space $\mathsf{PG}(2n-1, 2)$. Let $V = \mathbb{F}_2^{2n}$, and let $(\cdot\,,\cdot)$ be the (symplectic and symmetric) bilinear form on $V = \mathbb{F}_2^{2n}$ given by

$$(X, Y) = X_1Y_{2n} + X_2Y_{2n-1} + \cdots + X_{2n}Y_1. \tag{3-1}$$

The points of the polar space $\mathsf{C}_n(2)$ are the 0-spaces of $\mathsf{PG}(2n-1, 2)$, and points $p = \langle X\rangle$ and $q = \langle Y\rangle$ are collinear (including the case $p = q$) if and only if $(X, Y) = 0$. A subspace U of V is *totally isotropic* if $(X, Y) = 0$ for all $X, Y \in U$. The totally isotropic subspaces of maximal dimension have projective dimension $n-1$, and for each $0 \le k \le n-1$ the k-spaces of the polar space $\mathsf{C}_n(2)$ are the totally isotropic subspaces of V with projective dimension k. To obtain the building of $\mathsf{C}_n(2)$ as a labelled simplicial complex one takes the totally isotropic $(k-1)$-spaces to be the type k vertices of the building for $1 \le k \le n$, with incidence of vertices given by symmetrised containment of the corresponding spaces. The full collineation group of $\mathsf{C}_n(2)$ is the symplectic group $\mathsf{Sp}_{2n}(2)$ consisting of all matrices $g \in \mathsf{GL}_{2n}(2)$ satisfying $g^TJg = J$, where J is the matrix of the symplectic form $(\cdot\,,\cdot)$ (see [Tits 1974, Corollary 5.9]).

Let F^+ and F^- be quadratic forms on V with Witt indices n and $n-1$, respectively. We will fix the specific choices

$$\begin{aligned} F^+(X) &= X_1X_{2n} + X_2X_{2n-1} + \cdots + X_nX_{n+1}, \\ F^-(X) &= X_1X_{2n} + X_2X_{2n-1} + \cdots + X_nX_{n+1} + X_n^2 + X_{n+1}^2. \end{aligned}$$

For $\epsilon \in \{-,+\}$, a subspace $U \subseteq V$ is *singular* with respect to F^ϵ if $F^\epsilon(X) = 0$ for all $X \in U$. The maximal dimensional singular subspaces of V with respect to F^ϵ have vector space dimension equal to the Witt index of F^ϵ. The points of $\mathsf{D}_n(2)$, (respectively, the polar space $\mathsf{B}_{n-1}(2,4)$), are those points of $\mathsf{PG}(2n-1, 2)$ that are singular with respect to F^+, (respectively, F^-). In both cases points $p = \langle X\rangle$ and $q = \langle Y\rangle$ are collinear (including the case $p = q$) if and only if $(X, Y) = 0$, where $(\cdot\,,\cdot)$ is as in (3-1).

Let $\mathsf{GO}^{\epsilon}_{2n}(2)$ be the group of all matrices of $\mathsf{GL}_{2n}(2)$ preserving the quadratic form F^{ϵ}, and let $\mathsf{O}^{\epsilon}_{2n}(2)$ be the corresponding index 2 simple subgroup of $\mathsf{GO}^{\epsilon}_{2n}(2)$ (see [Conway et al. 1985, §2.4]). Since $\mathsf{GO}^{\epsilon}_{2n}(2)$ preserves collinearity, the group $\mathsf{GO}^{+}_{2n}(2)$ acts on $\mathsf{D}_n(2)$ and the group $\mathsf{GO}^{-}_{2n}(2)$ acts on $\mathsf{B}_{n-1}(2,4)$. In fact the group $\mathsf{GO}^{-}_{2n}(2)$ is the full automorphism group of $\mathsf{B}_{n-1}(2,4)$ (see [Tits 1974]). In the case of $\mathsf{D}_n(2)$ the maximal singular subspaces are partitioned into two sets of equal cardinality by the action of $\mathsf{O}^{+}_{2n}(2)$, and an automorphism θ of $\mathsf{D}_n(2)$ mapping points to points is called a *collineation* if this partition of maximal singular subspaces is preserved by θ, and a *duality* otherwise. Then $\mathsf{O}^{+}_{2n}(2)$ is the group of all collineations of $\mathsf{D}_n(2)$, and $\mathsf{GO}^{+}_{2n}(2)\backslash\mathsf{O}^{+}_{2n}(2)$ is the set of all dualities of $\mathsf{D}_n(2)$ (see [Tits 1974]).

To obtain the building of $\mathsf{B}_{n-1}(2,4)$ as a labelled simplicial complex one takes the singular $(k-1)$-spaces to be the type k vertices of the building for $1\leq k\leq n-1$, with incidence of vertices given by symmetrised containment of the corresponding spaces. The situation for $\mathsf{D}_n(2)$ is slightly different: For $1\leq k\leq n-2$ the singular $(k-1)$-spaces are taken to be the type k vertices of the building, and the singular $(n-1)$-spaces in one part of the partition mentioned above are taken to be the type $n-1$ vertices of the building, and those in the other part of the partition are taken to be the type n vertices of the building. A type $n-1$ vertex is declared to be incident with a type n vertex if the corresponding $(n-1)$-spaces meet in an $(n-2)$-space. For all other types incidence is given by symmetrised containment of the corresponding spaces.

Note the index shifts that occur (for example an $\{n\}$-domestic collineation of a $\mathsf{C}_n(2)$ building is a collineation that is domestic on the totally isotropic $(n-1)$-spaces). A point p of a polar space is an *absolute point* of an automorphism θ if p^{θ} is collinear with p (including $p^{\theta}=p$).

Lemma 3.3. *Let θ be a collineation of $\mathsf{C}_n(2)$.*

(a) *If θ fixes a subspace of $\mathsf{PG}(2n-1,2)$ of projective dimension $k\geq n$ then θ is $\{j\}$-domestic for each $2n-k\leq j\leq n$.*

(b) *If the set of absolute points of θ strictly contains the union of two distinct hyperplanes of $\mathsf{PG}(2n-1,2)$ then θ is $\{1\}$-domestic.*

Proof. (a) By considering dimensions, each $(j-1)$-space of $\mathsf{PG}(2n-1,2)$ with $j\geq 2n-k$ intersects the subspace of fixed points. In particular, no totally isotropic $(j-1)$-space is mapped onto an opposite and so θ is $\{j\}$-domestic for all $2n-k\leq j\leq n$.

(b) A point X is an absolute point of $\theta\in\mathsf{Sp}_4(2)$ if and only if $(X,\theta X)=X^TJ\theta X=0$, where J is the matrix of the symplectic form $(\cdot\,,\cdot)$. Thus the set of absolute points of θ is a quadric, and so if it strictly contains the union of two distinct hyperplanes then all points are absolute. □

In the following proofs we use the standard notations $p \perp q$ if points p and q are collinear (including the case $p = q$), and $p^\perp$ for the set of all points collinear to p.

Lemma 3.4. *Let $\Delta = \mathsf{C}_n(2)$ with $n \geq 2$ and let θ be a collineation.*

(a) *If the fixed points of θ form a $(2n-3)$-space W, then the absolute points form a subspace containing W.*

(b) *If the fixed points of θ form a $(2n-2)$-space W, then every absolute point is fixed.*

Proof. (a) Let p be a point not contained in W and suppose p is absolute. Let $q \in \langle W, p\rangle \setminus W$. We claim that q is absolute. Indeed, let $r := \langle p, q\rangle \cap W$. If $p \perp q$, then the plane $\pi = \langle p, q, p^\theta\rangle$ contains the triangle $\{p, p^\theta, r\}$ of points collinear in $\mathsf{C}_n(2)$ and so $q \perp q^\theta$, as both points belong to π. If $p \notin q^\perp$, then π contains the line $\langle p, p^\perp\rangle$, which belongs to $\mathsf{C}_n(2)$, but also contains the line $\langle p, r\rangle$, which does not belong to $\mathsf{C}_n(2)$. Also $\langle p^\theta, r\rangle$ does not belong to $\mathsf{C}_n(2)$, and it follows that the line $\langle r, s\rangle$, where $\{p, p^\theta, s\}$ is the line of $\mathsf{C}_n(2)$ through p and p^θ, belongs to $\mathsf{C}_n(2)$. Hence also the line $\{s, q, q^\theta\}$ belongs to $\mathsf{C}_n(2)$, which proves our claim.

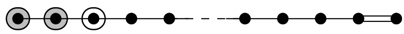
So, if there are no absolute points besides those in W, then (a) holds. If some absolute point $p \notin W$ exists, then there are three possibilities. Either exactly one hyperplane through W consists of absolute points (and then (a) holds), or all three hyperplanes through W consist of absolute points (and then, again, (a) holds), or exactly two hyperplanes H_1 and H_2 through W consist of absolute points. In this final case, let H be the third hyperplane through W. Let t, t_1, t_2 be points such that $t^\perp = H$ and $t_i^\perp = H_i$, $i = 1, 2$. Then, since θ fixes H, we have $t \in W$. Since $t_i \in t_i^\perp = H_i$, $i = 1, 2$, we deduce $t_i \in W$, $i = 1, 2$. Hence θ induces collineations in H, H_1, H_2 having a hyperplane W as fixed points. Consequently, these collineations are central involutions. Since all points of W are fixed, all subspaces through $\{t, t_1, t_2\}$ are fixed. Hence the centres of the above collineations are t, t_1, t_2. Since the collineations in H_i, $i = 1, 2$, map points to a collinear point, the centers are t_i. But then the centre of the collineation in H is t and hence it also maps points to collinear points, a contradiction. This shows (a).

(b) If the fixed points of θ form a $(2n-2)$-space W, then θ is a central elation in $\mathsf{PG}(2n-1, 2)$, and the centre is necessarily $W^\perp$ since every point of W is fixed, and hence every hyperplane through $W^\perp$ is fixed. No line through $W^\perp$ not contained in W is a line of $\mathsf{C}_n(2)$, whence (b). □

Lemma 3.5. *A collineation θ of the generalised quadrangle $\mathsf{C}_2(2)$ is exceptional domestic if and only if the set of absolute points of θ equals the union of two distinct hyperplanes in $\mathsf{PG}(3, 2)$.*

Proof. It is known that $\mathsf{C}_2(2)$ admits a unique exceptional domestic collineation (see [Temmermans et al. 2012b]), and direct inspection shows that the set of absolute points of this collineation forms the union of two distinct hyperplanes in $\mathsf{PG}(3,2)$. It remains to show that no other collineation of $\mathsf{C}_2(2)$ has such a structure of absolute points. This can be done, for example, using the character tables in the ATLAS; see [Conway et al. 1985, page 5]. We omit the details. □

Lemma 3.6. *Let $\Delta = \mathsf{C}_n(2)$ with $n \geq 3$ and let θ be a collineation. If the absolute points of θ lie on a union of two hyperplanes, and if the fixed points of θ form a $(2n-4)$-space W, then θ has the following decorated opposition diagram*:



Proof. The hypothesis implies that every 3-space contains a fixed point, and thus θ is $\{i\}$-domestic for all $4 \leq i \leq n$.

By the hypothesis on the structure of the absolute points of θ there exist points in $\mathrm{Opp}(\theta)$. Let p be an arbitrary point in $\mathrm{Opp}(\theta)$. We will show below that the induced collineation θ_p of $\mathsf{C}_{n-1}(2)$ is $\{2\}$-domestic (in the inherited labelling). Hence θ is $\{1,2\}$-domestic. So if θ is capped then θ is $\{2\}$-domestic, however by [Temmermans et al. 2012a, Theorem 5.1] every such collineation fixes a geometric hyperplane pointwise, contrary to our hypothesis that the fixed points form a $(2n-4)$-space. Thus θ is uncapped, and then by Theorem 1(a) the decorated opposition diagram of θ is forced to be as claimed.

Therefore it only remains to show that θ_p is $\{2\}$-domestic (that is, point-domestic on $\mathsf{C}_{n-1}(2)$). We fix some notation. Let H_i, $i = 1, 2$, be the two hyperplanes all points of which are absolute. Set $S = H_1 \cap H_2$ and let H be the hyperplane distinct from H_i, $i = 1, 2$, and containing S. Note that all points of $\mathrm{Opp}(\theta)$ are contained in H (more precisely they form the set $H \setminus S$).

First we claim that any line in $\mathrm{Opp}(\theta)$ incident to p must necessarily be contained in the hyperplane H. Suppose the such a line L is not contained in H. Then $L = \{p, q_1, q_2\}$, with $q_i \in H_i$ and hence $q_i^\theta \perp q_i$, $i = 1, 2$. Since p is not collinear to p^θ, it must be collinear to q_i^θ for some $i \in \{1, 2\}$. But then q_i^θ is collinear to all points of L, and so the line $L^\theta \ni q_i^\theta$ is not opposite the line L. Hence the claim.

Consider the subspace $\xi := p^\perp \cap (p^\theta)^\perp$ of dimension $2n-3$. Then clearly ξ contains the subspace $p^\perp \cap W$. We claim that $\dim(p^\perp \cap W) = 2n-5$. Indeed, if not, then W is a hyperplane of ξ. By Lemma 3.4(b) and our previous claim, all lines of $\mathsf{C}_n(2)$ through p are contained in H, implying $p^\perp = H$. But since H is fixed by θ we deduce that $p \in W$, a contradiction. Our claim follows.

Hence $\dim(p^\perp \cap W) = 2n-5$. It follows that $\dim(\xi \cap W) = 2n-5$ as well, since $p^\perp \cap W = (p^\theta)^\perp \cap W$. Now let $q \in \xi \setminus W$. Suppose $q \notin H$. Then the line $\langle p, q\rangle$ is not mapped to an opposite, as we showed above. Suppose $q \in S \setminus W$. Then $q^\theta \perp q$, and since $p^\theta \perp q$, we deduce that q is collinear to $\langle p, q\rangle^\theta$, implying that

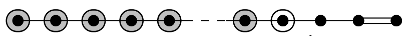
$\langle p, q\rangle \notin \mathrm{Opp}(\theta)$. Hence, if θ_p is not $\{2\}$-domestic, then $\xi \cap (H \setminus S) \neq \varnothing$. Under that conditon, if ξ is not contained in H, then $\xi \cap H_i$ is a hyperplane of ξ, $i = 1, 2$, and this contradicts Lemma 3.4(a).

Hence we deduce that if θ_p is not $\{2\}$-domestic, then $\xi \subseteq H$. In this case, since both p and p^θ are in H, we have $p^\perp = \langle p, \xi\rangle = H$ and $(p^\theta)^\perp = \langle p^\theta, \xi\rangle = H$. However $\perp$ is a symplectic polarity and so $p^\perp = H = (p^\theta)^\perp$ forces $p = p^\theta$, a contradiction. The lemma is proved. □

Theorem 3.7. *Let θ be a collineation of $\mathsf{C}_n(2)$. Suppose that the set of absolute points of θ equals the union of two distinct hyperplanes of $\mathsf{PG}(2n-1, 2)$. Then θ is domestic. Moreover, if k is the projective dimension of the subspace of points of $\mathsf{PG}(2n-1, 2)$ fixed by θ, then*

(a) *if $k = n-2$ then θ is strongly exceptional domestic, and*

(b) *if $k = n-1+j$ for some $0 \le j \le n-3$ then θ is uncapped with the following decorated opposition diagram:*



Moreover examples exist for each $n-2 \le k \le 2n-4$.

Proof. Suppose that θ is a collineation of $\mathsf{C}_n(2)$ such that the set of absolute points of θ is the union of two distinct hyperplanes H_1 and H_2 of $\mathsf{PG}(2n-1, 2)$. We show by induction on $n-j$ that θ is domestic, with Lemma 3.5 providing the base case $n-j = 3$.

Let p be any point not in $H_1 \cup H_2$. Thus p is mapped to an opposite point by θ. Let $\mathrm{Res}(p)$ be the set of totally isotropic subspaces containing p. Thus $\mathrm{Res}(p)$ is a $\mathsf{C}_{n-1}(2)$ building, whose points are the lines through p, whose lines are the planes through p, and so forth. Let $\theta_p = \mathrm{proj}_{\mathrm{Res}(p)} \circ \theta$, regarded as a collineation of $\mathsf{C}_{n-1}(2)$. Since $p^\perp$ and $(p^\theta)^\perp$ are hyperplanes of $\mathsf{PG}(2n-1, 2)$ the spaces $H_i' = p^\perp \cap (p^\theta)^\perp \cap H_i$ are $(2n-4)$-spaces for $i = 1, 2$ (as in the proof of Lemma 3.5). Let $q \in p^\perp \cap (p^\theta)^\perp \cap (H_1 \cup H_1)$, and let $L = \langle p, q\rangle$. Similar arguments as those in Lemma 3.5 show that

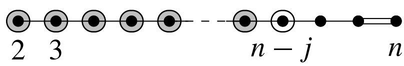
(i) if q is fixed by θ, then L is fixed by θ_p, and

(ii) if q is mapped to a distinct collinear point by θ then L is either fixed by θ_p, or is mapped to a distinct coplanar line by θ_p.

Thus for all nondomestic points p the induced collineation θ_p of the $\mathsf{C}_{n-1}(2)$ building $\mathrm{Res}(p)$ has the property that the set of points mapped to collinear points (including fixed points) contains the union of two distinct hyperplanes in $\mathsf{PG}(2n-3, 2)$. Thus by Lemma 3.3 and the induction hypothesis the collineation θ_p is domestic, and hence θ is domestic.

Now suppose that the absolute points of θ form a union of two hyperplanes, and that the fixed point set F of θ is an $(n-2)$-space of $\mathsf{PG}(2n-1,2)$. We prove by induction on n that θ is strongly exceptional domestic, with Lemma 3.5 providing the base case. The above argument shows that θ is necessarily domestic, and so it remains to show that there are nondomestic panels of each cotype $1, 2, \ldots, n$. We claim that for $n \geq 3$ there exists a nondomestic point p such that the hyperplane $p^{\perp}$ intersects F in an $(n-3)$-space F'. To see this it suffices to show that there is a point p with $p \notin H_1 \cup H_2$ and $p \notin F^{\perp}$. The number of points in $H_1 \cup H_2$ is $3 \cdot 2^{2n-2} - 1$ and the number of points in $F^{\perp}$ is $2^{n+1} - 1$. Thus for $n \geq 3$ there is a point $p \notin H_1 \cup H_2$ and $p \notin F^{\perp}$. By the induction hypothesis, there are panels of cotypes $2, 3, \ldots, n$ of $\mathrm{Res}(p)$ mapped to an opposite panels by θ_p, and thus there are panels of each cotype $2, 3, \ldots, n$ of $\mathsf{C}_n(2)$ mapped to an opposite by θ. It is then easy to see that there is also a nondomestic cotype 1 panel (by a residue argument) and hence θ is strongly exceptional domestic.

Now suppose the absolute points of θ form a union of two hyperplanes, and that the fixed point set F of θ is a k-space with $k = n-1+j$ for some $0 \leq j \leq n-3$. An argument as in the previous paragraph shows there is a nondomestic point p such that $p^{\perp}$ intersects F in an $(n-2+j)$-space. By induction, with Lemma 3.6 as the base case, the collineation θ_p of the $\mathsf{C}_{n-1}(2)$ building $\mathrm{Res}(p)$ has the following diagram:



Moreover, for any other nondomestic point p we have that either θ_p has the above diagram, or θ_p is domestic on type $n-1-j$ vertices. Thus no simplex $\mathsf{C}_n(2)$ of type

$$\{1, 2, \ldots, n-j-1\}$$

is mapped to an opposite by θ, hence the result.

To conclude we prove existence of collineations with each diagram. Recursively define elements $g_n \in \mathsf{Sp}_{2n}(2)$, for $n \geq 2$, by

$$g_2 = \begin{bmatrix} 0&1&0&0\\ 1&0&0&0\\ 1&0&0&1\\ 0&0&1&0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&1\\ 1&0&0&0&1&0\\ 0&0&0&1&0&0 \end{bmatrix}, \quad g_n = \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 0&0&g_{n-2}&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{bmatrix}.$$

Moreover, for each $j \geq 0$ define $g_n^{(j)} \in \mathsf{Sp}_{2n}(2)$ in block diagonal form by

$$g_n^{(j)} = \mathrm{diag}(I_j, g_{n-j}, I_j) \quad \text{where } I_j \text{ is the } j \times j \text{ identity matrix.}$$

By direct calculation, the absolute points of g_{2n} and $g_{2n}^{(j)}$ are given by

$$X_{2n-1}X_{2n} = 0$$

and the collinear points of g_{2n+1} and $g_{2n+1}^{(j)}$ are given by

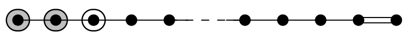
$$X_{n-1}(X_{n-2} + X_n) = 0.$$

Moreover, the fixed points of g_n form an $(n-2)$-space of $\mathsf{PG}(2n-1, 2)$, and the fixed points of $g_n^{(j)}$ form an $(n-2+j)$-space of $\mathsf{PG}(2n-1, 2)$. Thus, by the arguments above, g_n is a strongly exceptional domestic collineation of $\mathsf{C}_n(2)$ for each $n \geq 2$, and $g_n^{(j+1)}$ diagram as in (b). □

Similar theorems hold, with similar proofs, for the $\mathsf{B}_n(2, 4)$ and $\mathsf{D}_n(2)$ buildings. We will only sketch the details below. Consider first the case $\mathsf{B}_n(2, 4)$. The following lemmas are similar to the $\mathsf{C}_n(2)$ case.

Lemma 3.8. *A collineation θ of the generalised quadrangle $\mathsf{B}_2(2, 4)$ is exceptional domestic if and only if the set of absolute points of θ is the set of points of $\mathsf{B}_2(2, 4)$ lying on the union of two distinct hyperplanes in $\mathsf{PG}(5, 2)$.*

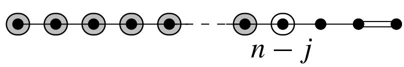
Lemma 3.9. *Let $\Delta = \mathsf{B}_n(2, 4)$ with $n \geq 3$ and let θ be a collineation. If the absolute points of θ lie on a union of two hyperplanes, and if the fixed points of θ are the isotropic points of a $(2n-3)$-space in $\mathsf{PG}(2n+1, 2)$, then θ has the following decorated opposition diagram:*



Theorem 3.10. *Let θ be a collineation of $\mathsf{B}_n(2, 4)$. Suppose that the set of absolute points of θ is the set of isotropic points lying on the union of two hyperplanes of $\mathsf{PG}(2n+1, 2)$. Let k be the projective dimension of the subspace of points of $\mathsf{PG}(2n+1, 2)$ fixed by θ. Then θ is domestic, and*

(a) *if $k = n$ then θ is strongly exceptional domestic, and*

(b) *if $k = n + 1 + j$ for some $0 \leq j \leq n - 3$ then θ is uncapped with the following decorated diagram:*



Moreover examples exist for each $n \leq k \leq 2n - 2$.

Proof. The proofs are very similar to Theorem 3.7, with the base cases given by Lemma 3.8 and 3.9, and we omit the details. Thus it only remains to exhibit the existence of collineations of $\mathsf{B}_n(2, 4)$ with the desired properties. To this end, define matrices g_n, $n \geq 3$ by

$$g_2 = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&1&0 \end{bmatrix}, \quad g_n = \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 0&0&g_{n-2}&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{bmatrix}.$$

Moreover, for each $j \geq 1$ define $g_n^{(j)}$ in block diagonal form by

$$g_n^{(j)} = \mathrm{diag}(I_j, g_{n-j}, I_j).$$

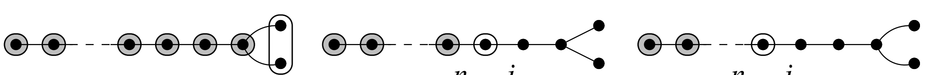
Since $g_n, g_n^{(j)} \in \mathsf{GO}_{2n+2}^-(2)$ these matrices induce collineations of $\mathsf{B}_n(2,4)$. It is straightforward to check that g_n satisfies the condition (a) and $g_n^{(j+1)}$ satisfies the condition (b). □

Consider now the case $\mathsf{D}_n(2)$.

Theorem 3.11. *Let θ be an automorphism of $\mathsf{D}_n(2)$. Suppose that the set of absolute points of θ is the set of points of $\mathsf{D}_n(2)$ lying on the union of two hyperplanes of $\mathsf{PG}(2n-1,2)$. Let k be the projective dimension of the subspace of points of $\mathsf{PG}(2n-1,2)$ fixed by θ. Then θ is domestic, and*

(a) *if $k = n-1$ and θ is an oppomorphism then θ is strongly exceptional domestic, and*

(b) *if $k = n-1+j$ for some $1 \leq j \leq n-3$ and θ is a nonoppomorphism (for odd j) and an oppomorphism (for even j) then θ has the following diagram:*



(*if* $j = 1$) (*if* j *is even*) (*if* $j > 1$ *is odd*)

Moreover examples exist for all $n-1 \leq k \leq 2n-4$.

Proof. The proofs of the statements (a) and (b) are again analogous to those in Theorem 3.7, with an appropriate start to the induction. We omit the details.

To prove existence, note that the matrices g_{n-1}, $n \geq 3$, from the proof of Theorem 3.10 are also elements of $\mathsf{GO}_{2n}^+(2)$. Let $h_3 = g_2$ and $h_4 = g_3$. Then h_3 induces a duality of $\mathsf{D}_3(2)$ and h_4 induces a collineation of $\mathsf{D}_4(2)$. Let $h_n = g_{n-1}$, and for each $1 \leq j \leq n-3$ let $h_n^{(j)} = g_{n-1}^{(j)}$. It is easy to check that h_n satisfies conditions (a), and $h_n^{(j)}$ satisfies conditions (b). □

4. Uncapped automorphisms for exceptional types

In this section we prove Theorem 1(b) for the small buildings of exceptional type. Moreover we completely classify the domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$. We begin, in Section 4A, by developing a (computationally feasible) method of detecting whether a given automorphism is domestic. In Section 4B we briefly describe the implementation of the minimal faithful permutation representations of the relevant ATLAS groups into Magma, and then in Section 4C we give the classification of domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$ making use of these permutation representations. We provide examples of uncapped automorphisms in $\mathsf{E}_7(2)$, and give conjectures for $\mathsf{E}_8(2)$ in Section 4D.

Throughout this section we use standard notation for Chevalley and twisted Chevalley groups G, and we refer to Carter [1989] for details. In particular, the symbols B, H, N, U, W, S, R, $x_\alpha(a)$, $n_\alpha(a)$, etc., have their usual meanings. However we note that in the twisted case we use these symbols for the objects in the twisted group (rather than the untwisted group). Then the quadruple (B, N, W, S) forms a Tits system in G, and thus (Δ, δ) is a building of type (W, S) where $\Delta = G/B$ and $\delta(gB, hB) = w$ if and only if $g^{-1}h \in BwB$. In the case of graph automorphisms of a simply laced Dynkin diagram we assume the Chevalley generators are chosen so that [Carter 1989, Proposition 12.2.3] holds (in particular $x_\alpha(a)^\sigma = x_{\sigma(\alpha)}(\pm a)$).

4A. ***Detecting domesticity.*** The following lemma shows that under certain hypotheses, to verify domesticity it is sufficient to show that no chamber opposite a given chamber is mapped onto an opposite.

Lemma 4.1. *Let θ be an automorphism of a thick spherical building Δ, and let $L = \operatorname{disp}(\theta)$. Let C be any chamber. Suppose that either*

(i) *each panel of Δ has at least 4 chambers, or*

(ii) *θ is an involution, or*

(iii) *θ induces opposition on the type set and $L = \ell(w_0)$.*

Then there exists a chamber D with $\delta(C, D) = w_0$ and $\ell(\delta(D, D^\theta)) = L$.

Proof. Let E be a chamber with $\ell(\delta(E, E^\theta)) = L$, and write $v = \delta(E, E^\theta)$. Let $w = \delta(C, E)$, and suppose that $w \neq w_0$. Then there exists $s \in S$ with $\ell(ws) > \ell(w)$. We show that there is a chamber D with $\delta(E, D) = s$ such that $\ell(\delta(D, D^\theta)) = L$. Consider each case.

(1) $\ell(sv) < \ell(v)$. Then either:

  (a) $\ell(svs^\theta) = \ell(v)$, in which case we choose the unique D with $\delta(E, D) = s$ such that $\delta(D, E^\theta) = sv$. Since $\delta(E^\theta, D^\theta) = s^\theta$ and $\ell(svs^\theta) > \ell(sv)$ we have $\delta(D, D^\theta) = svs^\theta$ and so $\ell(\delta(D, D^\theta)) = L$.

(b) $\ell(svs^\theta) < \ell(v)$, in which case necessarily $\ell(vs^\theta) < \ell(v)$, and it follows that there exists a reduced expression for v starting with s and ending with s^θ. Thus there exists a minimal length gallery

$$E = E_0 \sim_{s_1} E_1 \sim_{s_2} \cdots \sim_{s_{\ell-1}} E_{\ell-1} \sim_{s_\ell} E_\ell = E^\theta$$

with $s_1 = s$ and $s_\ell = s^\theta$.

(i) If every panel of Δ has at least 4 chambers then there exists a chamber D with $\delta(E, D) = s$ such that $D \notin \{E_1, E_{\ell-1}^{\theta^{-1}}\}$. Then there is a gallery $D \sim_{s_1} E_1 \sim_{s_2} \cdots \sim_{s_{\ell-1}} E_{\ell-1} \sim_{s_\ell} D^\theta$, and hence $\delta(D, D^\theta) = v$ has length L.

(ii) If θ is an involution then θ maps every minimal length gallery from E to E^θ to a minimal length gallery from E^θ to E, and it follows by considering types of first and last steps that $E_1^\theta = E_{\ell-1}$. Thus for any D with $\delta(E, D) = s$ and $D \neq E_1$ we again have $\delta(D, D^\theta) = v$.

(iii) If θ induces opposition and $L = \ell(w_0)$ then $v = w_0$, and $svs^\theta = sw_0s^\theta = w_0s^\theta s^\theta = w_0$, and so case (1)(b) cannot occur.

(2) $\ell(sv) > \ell(v)$. Then either:

(a) $\ell(svs^\theta) > \ell(v)$, in which case every chamber D with $\delta(E, D) = s$ has $\delta(D, D^\theta) = svs^\theta$, contradicting $\ell(v) = \mathrm{disp}(\theta)$. Thus this case cannot occur.

(b) $\ell(svs^\theta) = \ell(v)$, in which case we can choose D to be any chamber with $\delta(E, D) = s$. Then $\delta(D, E^\theta) = sv$ (since $\ell(sv) > \ell(v)$), and thus $\delta(D, D^\theta) = sv$ or $\delta(D, D^\theta) = svs^\theta$. The first case is impossible by the definition of displacement, and thus $\delta(D, D^\theta) = svs^\theta$ has length L.

Hence the result. □

Remark 4.2. The following examples illustrate that the conclusion of Lemma 4.1 may fail if the hypotheses of the lemma are not satisfied.

(1) The collineation θ of the Fano plane $\mathsf{PG}(2, \mathbb{F}_2)$ given by the upper triangular 3×3 matrix with all upper triangular entries equal to 1 maps no chamber opposite the base chamber $C = (\langle e_1\rangle, \langle e_1 + e_2\rangle)$ to an opposite chamber. However this collineation has displacement $\ell(w_0) = 3$, since no nontrivial collineation of a projective plane is domestic. Note that this collineation has order 4, and so none of the conditions of Lemma 4.1 are satisfied.

(2) The exceptional domestic collineation of the generalised quadrangle $\mathsf{GQ}(2, 2) = \mathsf{C}_2(2)$ (see [Temmermans et al. 2012b, Section 4]) is given by $\theta = x_1(1)x_2(1)$ in Chevalley generators. The chambers opposite the base chamber B of G/B are mapped to distances s_1s_2 or s_2s_1, however θ has displacement 3 (by both $s_1s_2s_1$ and $s_2s_1s_2$). Note that this collineation has order 4, and so again none of the conditions of Lemma 4.1 are satisfied.

4B. ***Minimal faithful permutation representations.*** Let $\mathcal{G}$ be the following set of ATLAS groups:

$$\mathcal{G} = \{\mathsf{F}_4(2), \mathsf{F}_4(2).2, {}^2\mathsf{E}_6(2^2), {}^2\mathsf{E}_6(2^2).2, \mathsf{E}_6(2), \mathsf{E}_6(2).2\}.$$

These groups are, respectively, the collineation group of $\mathsf{F}_4(2)$, the full automorphism group of $\mathsf{F}_4(2)$ (including dualities), the "inner" automorphism group of $\mathsf{F}_4(2,4)$, the full automorphism group of $\mathsf{F}_4(2,4)$, the collineation group of $\mathsf{E}_6(2)$, and the full automorphism group of $\mathsf{E}_6(2)$. In the following section we will need an explicit set of conjugacy class representatives for the groups in $\mathcal{G}$. With the exception of perhaps $\mathsf{F}_4(2)$, these groups appear to be too large for the conjugacy class algorithms in Magma (or GAP) when input as matrix groups using the adjoint representation. For example $\mathsf{E}_6(2).2$ has order 429683151044011150540800, and in any case it is not an entirely trivial task to construct such extensions as matrix groups. However the available algorithms in both Magma and GAP for permutation groups turn out to be considerably more efficient, and therefore we require faithful permutation representations of the groups in $\mathcal{G}$.

The degrees $\deg(G)$ of the minimal faithful permutation representations of the groups in $\mathcal{G}$ are well known (see for example [Vasilev 1996; 1997; 1998]):

$$\deg(\mathsf{F}_4(2)) = 69615, \quad \deg(\mathsf{F}_4(2).2) = 139230,$$
$$\deg({}^2\mathsf{E}_6(2^2)) = \deg({}^2\mathsf{E}_6(2^2).2) = 3968055,$$
$$\deg(\mathsf{E}_6(2)) = 139503, \quad \deg(\mathsf{E}_6(2).2) = 279006.$$

In each case the permutation representation can naturally be realised by the action of G on certain maximal parabolic coset spaces (equivalently, on certain vertices of the building). For example, for $G = \mathsf{E}_6(2).2$ we consider the action on $G/P_1 \cup G/P_6$ (the set of type 1 and type 6 vertices of the $\mathsf{E}_6(2)$ building), and for $G = {}^2\mathsf{E}_6(2^2).2$ we consider the action on ${}^2\mathsf{E}_6(2^2)/P_1$ (the set of type 1 vertices of the $\mathsf{F}_4(2,4)$ building), where P_i denotes the maximal parabolic subgroup of type $S\backslash\{s_i\}$.

To our knowledge, at the time of writing these minimal faithful permutation representations were not available in either GAP or Magma. Therefore we have implemented these permutation representations in Magma, using the above action on vertices of the building, and making use of the "Groups of Lie Type" package [Cohen et al. 2004]. The resulting permutation representations are available on Parkinson's webpage, where we also provide lists of conjugacy class representatives and code relevant to the computations in the following sections. We would like to thank Bill Unger from the Magma team at Sydney University for helping us generate the conjugacy class representatives from the permutation representations.

4C. ***Domestic automorphism of small buildings of types*** F_4 ***and*** E_6**.** In this section we classify domestic automorphisms of the buildings $\mathsf{F}_4(2)$, $\mathsf{F}_4(2,4)$, and $\mathsf{E}_6(2)$.

This requires two main steps. We first exhibit a list of n examples of pairwise nonconjugate domestic automorphisms for each building (for some integer n). Next, using an explicit set of conjugacy class representatives, we show that all but n of these representatives map some chamber to an opposite and are hence nondomestic. Thus we conclude that our list of n examples is complete.

We make frequent use of both commutator relations, and the formula

$$n_\alpha(a) = x_\alpha(a)x_{-\alpha}(-a^{-1})x_\alpha(a). \tag{4-1}$$

We will also use the following observation: for the buildings $\mathsf{E}_n(2)$, $n = 6, 7, 8$, the displacement of an automorphism θ determines the (decorated) opposition diagram of θ (see Remark 2.30). For the buildings $\mathsf{F}_4(2)$ and $\mathsf{F}_4(2,4)$ the (capped) automorphisms with types $\{1\}$ and $\{4\}$ are not distinguished by displacement, and furthermore in $\mathsf{F}_4(2)$ the three uncapped diagrams all have displacement 23.

Before beginning we outline a useful technique. Suppose that $\theta \in G$ induces an automorphism of $\Delta = G/B$ such that the hypothesis of Lemma 4.1 holds. Then there exists $gB \in Bw_0B/B$ such that $\mathrm{disp}(\theta) = \ell(\delta(gB, \theta gB))$. Each $gB \in Bw_0B/B$ can be written as $gB = uw_0B$ with $u \in U$, and $\delta(gB, \theta gB)$ is the unique $w \in W$ such that

$$w_0^{-1}u^{-1}\theta uw_0 \in BwB. \tag{4-2}$$

Thus to determine $\mathrm{disp}(\theta)$ it is sufficient to analyse the terms $w_0^{-1}u^{-1}\theta uw_0$ with $u \in U$. However $|U| = |\mathbb{F}|^{\ell(w_0)}$, and so even for relatively small buildings it is not computationally feasible to check each $u \in U$ (for example, in $\mathsf{E}_6(2)$ we have $|U| = 2^{36}$).

The following idea often provides a considerable computational efficiency. Note that each $u \in U$ can be written as $\prod_{\alpha \in R^+} x_\alpha(a_\alpha)$ with $a_\alpha \in \mathbb{F}$ and the product taken in any order (see [Steinberg 2016, Lemma 17]; of course the a_α depend on the order chosen). Writing

$$A = \{\alpha \in R^+ \mid x_\alpha(a)\theta \neq \theta x_\alpha(a) \text{ for all } a \in \mathbb{F}\}$$

we can write $u = u'_A u_A$ where u_A is a product over terms $\alpha \in A$, and u'_A is a product over the remaining positive roots. Then u'_A commutes with θ, and so

$$w_0^{-1}u^{-1}\theta uw_0 = w_0^{-1}u_A^{-1}\theta u_A w_0. \tag{4-3}$$

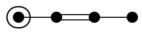
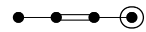
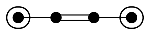
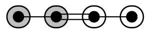
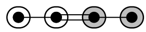
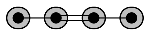
There are $|\mathbb{F}|^{|A|}$ such elements, and so the technique works best if a conjugacy class representative for θ is chosen with the property that it commutes with as many elements $x_\alpha(a)$, $\alpha \in R^+$, as possible.

The residue of the type J simplex of the chamber gB is the coset $gP_{S\backslash J}$, and this residue is nondomestic for θ if and only if $g^{-1}\theta g \in P_{S\backslash J}w_0P_{S\backslash J}$, and thus if and only if

$$g^{-1}\theta g \in BwB \quad \text{for some } w \in w_0W_{S\backslash J} \tag{4-4}$$

In the following we write $g_1 \sim g_2$ to mean that g_1 and g_2 are conjugate in G.

Theorem 4.3. *Let $G = \mathsf{F}_4(2)$, and let $\Delta = G/B$ be the associated building. Let $\varphi = (2342)$ and $\varphi' = (1232)$ be the highest root and highest short root (respectively) of the F_4 root system. There are precisely six conjugacy classes of domestic collineations of Δ, as follows*:

θ	capped	diagram	fixed type 1/4 vertices	ATLAS
$\theta_1 = x_\varphi(1)$	yes		2287/5103	$2B$
$\theta_2 = x_{\varphi'}(1)$	yes		5103/2287	$2A$
$\theta_3 = x_\varphi(1)x_{\varphi'}(1)$	yes		1263/1263	$2C$
$\theta_4 = x_1(1)x_2(1)$	no		127/399	$4D$
$\theta_5 = x_4(1)x_3(1)$	no		399/127	$4C$
$\theta_6 = x_2(1)x_3(1)$	no		151/151	$4E$

Moreover, $\theta_{3+i}^2 \sim \theta_i$ for $i = 1, 2, 3$, and $\theta_2 = \sigma(\theta_1)$, $\theta_3 = \sigma(\theta_3)$, $\theta_5 = \sigma(\theta_4)$, and $\theta_6 = \sigma(\theta_6)$.

Proof. We first show that the automorphisms have the claimed diagrams. Note that θ_1, θ_2, and θ_3 are involutions, and hence the hypothesis of Lemma 4.1 applies. Consider θ_1. Following the strategy of (4-2) we notice that $\theta_1 = x_\varphi(1)$ is central in U (by the commutator formulae), and hence, for all $u \in U$, using (4-1) we have

$$w_0^{-1}u^{-1}\theta_1 u w_0 = w_0^{-1}x_\varphi(1)w_0 = x_{-\varphi}(1) = x_\varphi(1)n_\varphi(1)x_\varphi(1) \in Bs_\varphi B.$$

Thus $\delta(gB, \theta_1 gB) = s_\varphi$ for all $gB \in Bw_0B/B$, and so $\mathrm{disp}(\theta) = \ell(s_\varphi) = 15$ (using Lemma 4.1). Moreover, note that $s_\varphi = w_0 w_{\{2,3,4\}}$ (for example, by computing inversion sets), and so there exists a nondomestic type 1 vertex. All type 2 or 3 vertices are domestic, for if, for example, there is a nondomestic type 2 vertex then there is $g \in G$ with $\delta(gB, \theta gB) \in w_0 W_{\{1,3,4\}}$ and hence $\mathrm{disp}(\theta) \geq 24 - 4 > 15$. If there exists a nondomestic type 4 vertex then by [Parkinson and Van Maldeghem 2019, Lemma 4.5] there exists a nondomestic type $\{1, 4\}$ simplex, which again contradicts the displacement calculation. Thus the diagram for θ_1 is as claimed, and since $\theta_2 = \sigma(\theta_1)$ (with σ the graph automorphism) the result for θ_2 also follows.

Consider θ_3. Since $x_{\varphi'}(1)$ is also central in U (this special feature of characteristic 2 follows from the commutator relations) we see that θ_3 is central in U. Thus, using commutator relations and (4-1) we have

$$\begin{aligned} w_0^{-1}u^{-1}\theta_3 u w_0 \\ &= x_{-\varphi'}(1)x_{-\varphi}(1) = x_{-\varphi'}(1)x_\varphi(1)n_\varphi(1)x_\varphi(1) \\ &= x_\varphi(1)x_{(1110)}(1)x_{-(0122)}(1)x_{-\varphi'}(1)n_\varphi(1)x_\varphi(1) \in Bx_{-(0122)}(1)x_{-\varphi'}(1)s_\varphi B \\ &= Bs_\varphi x_{-(0122)}(1)x_{(1110)}(1)B = Bs_\varphi s_{(0122)}B. \end{aligned}$$

We have $s_\varphi s_{(0122)} = w_0 w_{\{2,3\}}$ (for example, by computing the inversion sets), and hence there exists a nondomestic type $\{1, 4\}$ simplex; see (4-4). By Lemma 4.1 the above calculation also shows that $\mathrm{disp}(\theta) = \ell(w_0 w_{\{2,3\}}) = 20$, and the diagram of θ_3 follows.

Consider θ_4. We first show that θ_4 is domestic. We will work with the conjugate

$$\theta_4' = x_{(1220)}(1)x_{1122}(1) = w^{-1}\theta_4 w \quad \text{where } w = s_{(0110)}s_{(1242)}$$

because this representative commutes with more elements $x_\alpha(1)$ with $\alpha \in R^+$, making (4-2) more effective. Indeed θ_4' commutes with all $x_\alpha(1)$ with $\alpha \in R^+\backslash A$, where

$$A = \{(0100), (0001), (0110), (0011), (0120), (1220), (0122), (1122)\}.$$

Then, as in (4-3), we have $w_0^{-1}u^{-1}\theta_4' u w_0 = w_0^{-1}u_A^{-1}\theta_4' u_A w_0$. There are 2^8 distinct elements u_A, and using the groups of Lie type package in Magma we can easily verify that $w_0^{-1}u_A^{-1}\theta_4' u_A w_0 \notin Bw_0B$ for all u_A (see Parkinson's webpage for the code). This implies that θ_4' is domestic, for if θ_4' were not domestic then the third hypothesis of Lemma 4.1 would hold and hence there would exist an element u_A with $w_0^{-1}u_A^{-1}\theta_4' u_A w_0 \in Bw_0B$.

One may see that θ_4' maps panels of cotypes 1 and 2 to opposites by simply exhibiting such panels (the groups of Lie type package is helpful here). Checking that there are no cotype 3 or 4 panels mapped to opposite panels is more complicated, and we have resorted to exhaustively verifying this by computation. However some efficiencies must be found to make the search feasible. Firstly, it is sufficient to check that there are no nondomestic type $\{1, 2\}$ simplices (by a simple residue argument). Writing $P = P_{\{3,4\}}$, the (residues of the) type $\{1, 2\}$ simplices of Δ are the cosets gP, $g \in G$. Let $T \subseteq W$ denote a transversal of minimal length representatives for cosets in $W/W_{\{3,4\}}$. A complete set of representatives for P cosets in G (and hence type $\{1, 2\}$ simplices in Δ) is

$$\{u_w(a)w \mid w \in T,\ a \in \mathbb{F}_2^{\ell(w)}\} \quad \text{where } u_w(a) = x_{\beta_1}(a_1)\cdots x_{\beta_k}(a_k),$$

where $R(w) = \{\beta_1, \dots, \beta_k\}$ is the inversion set of w. Thus, using (4-4), it is sufficient to check that $\delta(g, \theta_4' g) \notin w_0 W_{\{3,4\}}$ for all $g = u_w(a)w$ with $w \in T$. However there are 4385745 such elements g (the cardinality of G/P) and this would be computationally expensive. Considerable efficiency can be gained by using the fact that the product $u_w(a)$ can be taken in any order (again, see [Steinberg 2016, Lemma 17]). Thus, applying the technique (4-3), we only need to consider terms $u_w'(a) = x_{\gamma_1}(a_1)\cdots x_{\gamma_\ell}(a_\ell)$ with $\{\gamma_1, \dots, \gamma_\ell\} = R(w) \cap A$. This drastically reduces the number of cases needing checking. In fact it turns out that there are only 3885 elements to check, and these are very quickly checked by the computer.

Since $\theta_5 = \sigma(\theta_4)$ the result for θ_5 follows.

Consider θ_6. Again we use a different conjugate $\theta_6 \sim \theta_6' = x_{(1110)}(1)x_{(0122)}(1)$. This element commutes with all $x_\alpha(1)$ with $\alpha \in R^+ \backslash A$, where

$$A = \{(0001), (0011), (0122), (0111), (0121), (1120), (1220), (1110), (1100), (1000)\}.$$

A similar argument to before, this time checking 2^{10} cases, verifies that θ_6' (and hence θ_6) is domestic. It is then straightforward to provide panels of each cotype mapped onto opposites, and hence θ_6 has the claimed diagram.

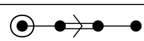
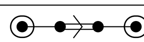
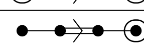
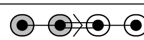
There are 95 conjugacy classes in the group $\mathsf{F}_4(2)$ (computed using the permutation representation), and for 88 of these classes a quick search finds nondomestic chambers. The seven remaining classes must therefore be domestic, because the six examples given above are clearly nonconjugate (they have distinct decorated opposition diagrams), and the identity is also trivially domestic.

The number of fixed type 1 vertices for each example is easily computed using the permutation representation, and the number of fixed type 4 vertices is obtained by considering the dual. Finally the ATLAS classes can be determined by the orders and fixed structures. □

Since no duality of a thick F_4 building is domestic the classification of domestic automorphisms of $\mathsf{F}_4(2)$ is complete (see [Parkinson and Van Maldeghem 2019, Lemma 4.1]). We also note that Lemma 2.18 follows from the above classification.

We now consider the building $\mathsf{F}_4(2,4)$. The full automorphism group of this building is ${}^2\mathsf{E}_6(2^2).2$ (that is, ${}^2\mathsf{E}_6(2^2)$ extended by the diagram automorphism σ of E_6; see [Tits 1974, Section 10.4] and [Conway et al. 1985, page 191]). We write $x_\alpha(a)$ for the Chevalley generators in the twisted group ${}^2\mathsf{E}_6(2^2)$. Thus $a \in \mathbb{F}_2$ (respectively, $a \in \mathbb{F}_4$) if α is a long root (respectively, short root) of the twisted root system.

Theorem 4.4. *Let $G = {}^2\mathsf{E}_6(2^2)$, and let $\Delta = G/B$ be the associated building of type $\mathsf{F}_4(2,4)$. Let φ (respectively, φ') be the highest root (respectively, highest short root) of the F_4 root system. There are precisely four classes of nontrivial domestic collineations, as follows (where σ is the graph automorphism of E_6):*

θ	capped	diagram	fixed points	ATLAS
$\theta_1 = x_\varphi(1)$	yes		46135	$2A$
$\theta_2 = x_{\varphi'}(1)$	yes		20279	$2B$
$\theta_3 = \sigma$	yes		69615	$2E$
$\theta_4 = x_1(1)x_2(1)$	no		855	$4A$

Here $x_\alpha(a)$ denote the Chevalley generators in the twisted group. Further, $\theta_4^2 \sim \theta_1$.

Proof. The analysis for θ_1 is similar to the analysis of θ_1 for $\mathsf{F}_4(2)$. Specifically, this element commutes with all terms $x_\alpha(a)$, and the result easily follows.

Consider θ_2. This element commutes with all terms $x_\alpha(a)$ with $\alpha \in R^+$ except for $x_{(0010)}(a)$, $x_{(0110)}(a)$ and $x_{(1110)}(a)$ with $a \in \{\xi, \xi^2\}$ (where ξ is a generator of $\mathbb{F}_4^*$). By commutator relations, if $a \in \{\xi, \xi^2\}$ we have

$$x_{(0010)}(-a)\theta_2 x_{(0010)}(a) = \theta_2 x_{\varphi-\alpha_1-\alpha_2}(1)$$
$$x_{(0110)}(-a)\theta_2 x_{(0110)}(a) = \theta_2 x_{\varphi-\alpha_1}(1)$$
$$x_{(1110)}(-a)\theta_2 x_{(1110)}(a) = \theta_2 x_{\varphi}(1),$$

and it follows that for all $u \in U$ we have

$$w_0^{-1}u^{-1}\theta_2 u w_0 = x_{-\varphi'}(1)x_{-\varphi+\alpha_1+\alpha_2}(a_1)x_{-\varphi+\alpha_1}(a_2)x_{-\varphi}(a_3) \text{ with } a_1, a_2, a_3 \in \{0, 1\}.$$

Considering each of the eight possibilities for the triple $(a_1, a_2, a_3) \in \mathbb{F}_2^3$ we see that the maximum length of $w = \delta(uw_0B, \theta_2 uw_0B)$ is 20 with $w = s_\varphi s_{(0122)}$, and the result follows.

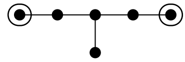
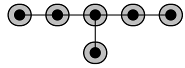
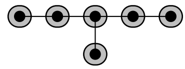
Consider θ_4. This element is conjugate to $\theta_4' = x_{(1220)}(1)x_{(1122)}(1)$, and then an analysis very similar to the case of θ_4 for $\mathsf{F}_4(2)$ applies. In particular, with A as in the $\mathsf{F}_4(2)$ case, we need to check each of the elements $\delta(u_A w_0 B, \theta_4' u_A w_0 B)$. This time there are $2048 = 4^3 \times 2^5$ elements u_A to check (since there are three roots in A whose root subgroup is isomorphic to $\mathbb{F}_4$ and the remaining five root subgroups are isomorphic to $\mathbb{F}_2$). A quick check with the computer shows that the maximum length of $\delta(u_A w_0 B, \theta_4' u_A w_0 B)$ is 23, and hence $\theta_4' \sim \theta_4$ is domestic. Then necessarily θ_4 maps no panels of cotypes 3 or 4 to opposite (by a simple residue argument), and then since $\mathrm{disp}(\theta_4) = 23$ it is forced that there are panels of cotypes both 1 and 2 mapped onto opposites.

Consider $\theta_3 = \sigma$. This element acts on the untwisted group $\mathsf{E}_6(4)$ as a symplectic polarity, and thus is $\{i\}$-domestic for $i \in \{2, 3, 4, 5\}$ (see [Van Maldeghem 2012]). It follows that σ is $\{i\}$-domestic for $i \in \{1, 2, 3\}$ on the building $\mathsf{F}_4(2, 4)$, hence the result.

Thus the diagrams of the four automorphisms are as claimed. Next, as in the $\mathsf{F}_4(2)$ example, we use the permutation representation of ${}^2\mathsf{E}_6(2^2).2$ to compute a complete list of conjugacy class representatives of this group. It turns out that there are 189 conjugacy classes, and for 184 of these classes one can exhibit a chamber mapped onto an opposite chamber. Thus there are at most 4 classes of nontrivial domestic collineations, and since the examples exhibited above are pairwise nonconjugate (by decorated opposition diagrams) the list is complete.

Finally, the calculation of the numbers of fixed points is immediate from the permutation representation, and the ATLAS classes can be determined by the orders and fixed structures. □

Theorem 4.5. *Let $G = \mathsf{E}_6(2).2$, and let $\Delta = \mathsf{E}_6(2)/B$ be the associated building of type $\mathsf{E}_6(2)$. There are precisely three classes of domestic dualities* (*up to* *conjugation in the full automorphism group*), *as follows*:

θ	capped	diagram	order
$\theta_1 = \sigma$	yes		2
$\theta_2 = x_1(1)\sigma$	no		4
$\theta_3 = x_1(1)x_3(1)\sigma$	no		8

Proof. As noted in Theorem 4.4, the element $\theta_1 = \sigma$ acts as a symplectic polarity on $\mathsf{E}_6(2)$, and thus has the diagram claimed (see [Van Maldeghem 2012]). For the remaining cases θ_2 and θ_3 we note that it is easy to find vertices of each type mapped onto opposite vertices. Thus it remains to show that these dualities are domestic. The working here is slightly more complicated than the case of collineations of the F_4 buildings. Writing $\theta = \tilde{\theta}\sigma$ with $\tilde{\theta} \in G$, we must show $w_0^{-1}u^{-1}\tilde{\theta}u^{\sigma}w_0 \notin Bw_0B$ for all $u \in U$ (here we are applying Lemma 4.1).

Consider θ_2. We use the conjugate $\theta_2' = x_\beta(1)\sigma$ with $\beta = (111221)$. It turns out, by commutator relations, that if $u \in U$ is arbitrary then $u^{-1}x_\beta(1)u^{\sigma}$ can be written in the following form (where we use Magma's built-in lexicographic order on the positive roots $\alpha_1, \ldots, \alpha_{32}$):

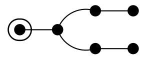
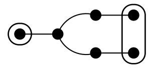
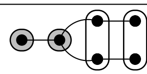
$$
\begin{aligned}
&x_1(a_1)x_7(a_2)x_{12}(a_3)x_{18}(a_4)x_{23}(0)x_{17}(a_5)x_{22}(a_6)x_{27}(0)x_{26}(a_7)x_{30}(0)\\
&\quad x_{29}(a_8)x_{32}(a_9)x_{33}(a_9+1)x_{34}(a_{10})x_{35}(a_{11})x_{36}(a_{12})x_3(a_{13})x_9(a_{14})\\
&\qquad x_{13}(a_{15})x_{15}(0)x_{19}(0)x_{21}(a_4)x_{25}(a_6)x_{24}(0)x_{28}(a_7)x_{31}(a_{16})x_4(0)\\
&\qquad\quad x_{10}(a_{14})x_8(0)x_{14}(a_{15})x_{16}(a_3)x_{20}(a_5)x_5(a_{13})x_{11}(a_2)x_2(0)x_6(a_1),
\end{aligned}
$$

where $a_1, \ldots, a_{16} \in \mathbb{F}_2$. The point is that there are only 2^{16} such terms, rather than $2^{36} = |U|$ terms. It is then a quick check on the computer to verify that θ_2 is domestic (and hence strongly exceptional domestic by Corollary 2.20).

The analysis of θ_3 is slightly more challenging. Using the conjugate $\theta_3' = x_\beta(1)x_{\beta'}(1)\sigma$ with $\beta = (010111)$ and $\beta' = (001111)$ we see that $u^{-1}x_\beta(1)x_{\beta'}(1)u^{\sigma}$ can be written in a similar way to the θ_2 case above, this time with 2^{22} degrees of freedom. The verification that θ_3 is domestic is then a long search with the computer. The details are on Parkinson's webpage.

To verify that our list of domestic examples is complete we again use explicit conjugacy class representatives computed from the minimal faithful permutation representation, as in the previous theorems. See Parkinson's webpage for the relevant code. Note that the character table of $\mathsf{E}_6(2)$ is not printed in ATLAS, and therefore it is not possible to provide the ATLAS conjugacy class names. □

Theorem 4.6. *Let $G = \mathsf{E}_6(2)$, and let $\Delta = G/B$ be the associated building of type* $\mathsf{E}_6(2)$. *There are precisely* 3 *classes of domestic collineations, as follows*:

θ	capped	diagram	fixed points	order
$\theta_1 = x_1(1)$	yes		10479	2
$\theta_2 = x_1(1)x_2(1)$	yes		2543	2
$\theta_3 = x_1(1)x_3(1)$	no		847	4

Proof. To analyse θ_1 we work with the conjugate $\theta_1 \sim x_\varphi(1)$, where φ is the highest root. Then an analysis very similar to the $\mathsf{F}_4(2)$ case shows that θ_1 has the diagram claimed.

The analysis for θ_2 can be done by hand. We work with the conjugate $\theta_2' = x_\varphi(1)x_{\varphi'}(1)$ where φ is the highest root and $\varphi' = (101111)$ is the highest root of the A_5 subsystem. Let $u \in U$. By commutator relations and a simple induction we see that $u^{-1}\theta_2' u$ is a product of terms $x_\alpha(a)$ with $\alpha \geq \varphi'$ (with $\geq$ being the natural dominance order). In particular, each such α is in $R^+\backslash\mathsf{D}_5$, where D_5 is the subsystem generated by $\alpha_2, \dots, \alpha_6$. Let $v = w_0 w_{\mathsf{D}_5}$, where w_{D_5} is the longest element of the parabolic subgroup $\langle s_2, \dots, s_6\rangle$. Then $R^+\backslash\mathsf{D}_5 = \{\alpha \in R^+ \mid v^{-1}\alpha \in -R^+\}$. It follows that $v^{-1}(w_0^{-1}u^{-1}\theta_2' u w_0)v \in B$ for all $u \in U$, and therefore

$$w_0^{-1}u^{-1}\theta_2' u w_0 \in vBv^{-1} \subseteq BvB \cdot Bv^{-1}B.$$

Hence $w_0^{-1}u^{-1}\theta_2' u w_0 \in BwB$ for some w with $\ell(w) \leq 2\ell(v) = 2(\ell(w_0) - \ell(w_{\mathsf{D}_5})) = 32$ (in fact we necessarily have strict inequality here by double coset combinatorics). Thus $\mathrm{disp}(\theta) \leq 32$, and it then follows from the classification of diagrams (and hence of possible displacements) that $\mathrm{disp}(\theta) \leq 30$. On the other hand, a quick calculation shows that $w_0^{-1}\theta_2' w_0 \in Bs_\varphi s_{\varphi'}B$, and by computing inversion sets we have $s_\varphi s_{\varphi'} = w_0 w_{\mathsf{A}_3}$ (where A_3 is the subsystem generated by $\alpha_3, \alpha_4, \alpha_5$). Thus θ_2' maps the type $\{1, 2, 6\}$ simplex of the chamber $w_0 B$ to an opposite simplex, hence the result.

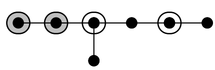
The working for θ_3 is more involved. Here Lemma 4.1 cannot be applied, and it is not practical to directly check every chamber for domesticity (there are 3126356394525 of them). Instead we argue in a similar fashion as we did for the collineation θ_4 in Theorem 4.3. First replace θ_3 by the conjugate $\theta_3 \sim \theta_3' = x_{(111210)}(1)x_{(011111)}(1)$. Then θ_3' commutes with all $x_\alpha(a)$ with $\alpha \in R^+\backslash A$ where

$$A = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, (000110), (000011), (101100), (101110), (001111), (0111111), (111210)\}.$$

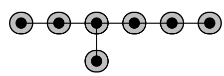
By a residue argument it is sufficient to show that there are no nondomestic type $\{2,4\}$ simplices (see the claim in the proof of Corollary 2.26). Again one cannot feasibly check all type $\{2,4\}$ simplices (there are 7089243525 of them). However, as in Theorem 4.3, with T a transversal of minimal length representatives for the cosets in $W/W_{\{1,3,5,6\}}$, it is sufficient to check that $\delta(g,\theta_3' g) \notin w_0 W_{\{1,3,5,6\}}$ for all $g = u_w'(a)w$ with $w \in T$ and $u_w'(a) = x_{\gamma_1}(a_1)\cdots x_{\gamma_\ell}(a_\ell)$ with $\{\gamma_1,\ldots,\gamma_\ell\} = R(w)\cap A$. It turns out that there are only 64158 such elements g, and they are readily checked by computer in under an hour. □

4D. ***Automorphisms of small buildings of types E_7 and E_8.*** Consider the E_7 root system R. Fix the ordering $\alpha_1,\ldots,\alpha_{63}$ of the positive roots according to increasing height, using the natural lexicographic order for roots of the same height (for example, $(1122100) < (1112110)$). Note that this is the inbuilt order in Magma. With this order, the roots $\alpha_{44} = (1112111)$, $\alpha_{45} = (0112211)$, and $\alpha_{46} = (1122210)$ play a special role below.

Theorem 4.7. *Let $\theta_1 = x_{44}(1)x_{46}(1)$ and $\theta_2 = x_{44}(1)x_{45}(1)x_{46}(1)$ in $\mathsf{E}_7(2)$. Then θ_1 and θ_2 are uncapped with the following respective decorated opposition diagrams*:



and



Moreover $\theta_1^2 = \theta_2^2 = x_\varphi(1)$ where φ is the highest root, and hence θ_1 and θ_2 have order 4.

Proof. Consider θ_2 first. We show that θ_2 is domestic using Lemma 4.1. Applying (4-3) verbatim requires us to check 2^{26} elements. The following modification of the theme is more efficient. It follows from commutator relations that

$$w_0^{-1}u^{-1}\theta_2 u w_0 = \prod_{\beta\in B} x_{-\beta}(a_\beta),$$

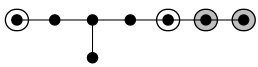
where $B = \{\beta \in R^+ \mid \beta \geq \alpha_{44} \text{ or } \beta \geq \alpha_{45} \text{ or } \beta \geq \alpha_{46}\}$ (where here $\alpha \geq \beta$ if and only if $\alpha - \beta$ is a nonnegative combination of simple roots). There are 20 roots in B. Moreover $a_{44} = a_{45} = a_{46} = 1$ (by commutator relations), and so there remain only 2^{17} elements to consider. It is then readily checked by computer that θ_2 is domestic, and we easily find vertices of each type mapped onto opposite vertices. Finally, commutator relations show that $\theta_2^2 = x_\varphi(1)$.

For θ_1 we do a similar search to the above to show that θ_1 is domestic. The remaining difficultly is showing that θ_1 is $\{1,3\}$-domestic. Arguing as we did for θ_4 in Theorem 4.3 it turns out that there are 1141419 elements to check, and this can be done in an overnight run on the computer. □

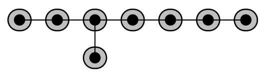
Thus the proof of Theorem 1(b) is complete. Our computational techniques are not efficient enough to handle the two diagrams for $\mathsf{E}_8(2)$ due to the formidable size of the group. Thus for these diagrams we provide conjectural examples. For each of these conjectures we have randomly selected 10^5 chambers and verified that restricted to this subset of the chamber set the structure of the automorphism is as claimed.

Fix the ordering $\alpha_1, \dots, \alpha_{120}$ of the positive roots of E_8 according to increasing height, using the natural lexicographic order for roots of the same height. Then the roots $\alpha_{88} = (11232221)$, $\alpha_{89} = (12243210)$ and $\alpha_{90} = (12233211)$ play a special role below.

Conjecture 4.8. *Let $\theta_1 = x_{88}(1)x_{90}(1)$ and $\theta_2 = x_{88}(1)x_{89}(1)x_{90}(1)$ in $\mathsf{E}_8(2)$. Then θ_1 and θ_2 are uncapped with the following respective decorated opposition diagrams*:



and



We note that $\theta_1^2 = \theta_2^2 = x_\varphi(1)$ where φ is the highest root, and hence θ_1 and θ_2 have order 4. It is not difficult to verify that $\mathrm{Typ}(\theta_1) = \{1, 6, 7, 8\}$ and $\mathrm{Typ}(\theta_2) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Thus the difficulty in the above conjecture is to show that θ_1 is $\{7, 8\}$-domestic, and that θ_2 is domestic. In principle the approach taken for $\mathsf{E}_7(2)$ is applicable, however in practice the enormous size of the group $\mathsf{E}_8(2)$ makes the search impractical. For example, applying the technique of Theorem 4.7 to θ_2 amounts to checking $2^{30} = 1073741824$ elements. Each of these checks requires a sequence of commutator relations in the group $\mathsf{E}_8(2)$, and while Magma has remarkably efficient algorithms implemented for this, the number of cases renders this computational approach unfeasible.

Remark 4.9. The examples of uncapped automorphisms that we have constructed thus far fix a chamber of the building. This is clear for the examples in exceptional types because the representatives are either in the Borel subgroup B, or are a composition of an element of B with a standard graph automorphism. For the examples constructed in classical types we note that all examples have either order 4 or 8. It follows that they lie in a Sylow 2-group of the automorphism group, and hence are conjugate to an element of B (or $\langle B, \sigma \rangle$ in the case of an order 2 graph automorphism). However there do exist uncapped automorphisms that do not fix a chamber. For example, in $\mathsf{C}_3(2) = \mathsf{Sp}_6(2)$ the element

$$\theta = x_2(1)x_3(1)n_2 = E_{11} + E_{23} + E_{24} + E_{25} + E_{32} + E_{33} + E_{45} + E_{54} + E_{55} + E_{66}$$

is exceptional domestic (in fact strongly exceptional domestic), with order 6. Thus θ does not lie in any conjugate of B, and hence θ fixes no chamber of $\mathsf{C}_3(2)$. In fact the fixed structure of θ consists of three points p_1, p_2, p_3, a line L, and three planes π_1, π_2, and π_3 such that π_1, π_2 and π_3 intersect in L, $p_i \in \pi_i$ for $i = 1, 2, 3$, and $p_i \notin L$ for $i = 1, 2, 3$.

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