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Sally Wolfe

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Sally Wolfe

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We relate the parity of the partition function to the parity of the q-series coefficients of certain powers of the modular discriminant using their generating functions. This allows us to make statements about the parity of the initial values of the partition function and to obtain a modified Euler recurrence for its parity.

1. Introduction and statement of results

We begin by defining two power series in q, the power series of the modular discriminant, and the generating function of the partition function, $p(n)$. The q-series expansion of the modular discriminant $\Delta(q)$ defines the Ramanujan τ-function. Namely, we have that

$$\begin{aligned}\Delta(q) &= q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=0}^{\infty}\tau(n)q^n \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 \cdots .\end{aligned} \tag{1.1}$$

Ramanujan investigated $\tau(n)$ and observed that $\tau(nm) = \tau(n)\tau(m)$ for $(n, m) = 1$, as well as congruences like $\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}$.

The partition function counts the number of distinct partitions of integers n. Like $\Delta(q)$,the generating function for $p(n)$ is an infinite product. More precisely, we have

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 \cdots . \tag{1.2}$$

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Ramanujan proved that for all nonnegative integers n

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.3}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.4}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.5}$$

However, much less is known about $p(n) \pmod{2}$. For example, it is conjectured that as x approaches infinity, the number of even and odd values of $p(n)$ with $n \le x$ approaches $\frac{1}{2}x$. Nicolas et al. [1998] prove that as $x \to \infty$,

$$\#\{n \le x : p(n) \equiv 0 \pmod{2}\} \gg \sqrt{x}$$
$$\#\{n \le x : p(n) \equiv 1 \pmod{2}\} \gg \sqrt{x} \cdot e^{\frac{-(\log 2+\epsilon)\log x}{\log\log x}}.$$

Ahlgren [1999] proves a slightly better bound for the number of odd values of $p(n)$: for sufficiently large x,

$$\#\{n \le x : p(n) \equiv 1 \pmod{2}\} \gg \frac{\sqrt{x}}{\log x}.$$

Nicolas [2006] proves that there exists a constant $\kappa > 0$ such that for sufficiently large x,

$$\#\{n \le x : p(n) \equiv 1 \pmod{2}\} \gg \frac{\sqrt{x}(\log\log x)^{\kappa}}{\log x}. \tag{1.6}$$

He proves this bound for all $\kappa > 0$ and sufficiently large x [Nicolas 2008], as well as proving a bound for the number of even values of $p(n)$ up to x:

$$\#\{n \le x : p(n) \equiv 0 \pmod{2}\} \gg 0.28\sqrt{x \log\log x} \tag{1.7}$$

The purpose of this paper is to investigate the parity of $p(n)$. We first recall Euler's recurrence for $p(n)$ [Andrews 1971]. If n is a positive integer, then

$$p(n) = \sum_{k\ge 1}(-1)^{k+1} p\Big(n - \frac{3k^2+k}{2}\Big) + \sum_{k\ge 1}(-1)^{k+1} p\Big(n - \frac{3k^2-k}{2}\Big).$$

We deform this to obtain many recurrences for $p(n) \pmod{2}$.

Theorem 1.1. *For integers $s \ge 2$, we have*:

$$\Delta(q)^{\frac{4^s-1}{3}} \equiv \Big(\sum_{n=0}^{\infty} p(n) q^{8n+\frac{4^s-1}{3}}\Big)\Big(\sum_{n=-\infty}^{\infty} q^{4^{s+1}(3n^2-n)}\Big) \pmod{2}.$$

To state the next theorem, we let $\tau_m(n)$ denote the n^{th} coefficient of $\Delta(q)^m$.

Theorem 1.2. *If $s \geq 2$ is an integer, then for any positive integer n we have*

$$p(n) \equiv \tau_{\frac{4^s-1}{3}}\left(8n + \frac{4^s - 1}{3}\right) + \sum_{m=1}^{\lfloor \frac{1}{6} + \frac{1}{3 \cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 - m))$$
$$+ \sum_{m=1}^{\lfloor -\frac{1}{6} + \frac{1}{3 \cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 + m)) \pmod 2.$$

Remark 1. For n such that $\tau_{(4^s-1)/3}(n) \equiv 0 \pmod 2$, this gives an *Euler-type* recurrence. We note that it is known [Serre 1974] that

$$\lim_{x \to \infty} \frac{\#\{n \leq x : \tau_{(4^s-1)/3}(n) \equiv 0 \pmod 2\}}{x} = 1.$$

Therefore, for almost all n, we have

$$p(n) \equiv \sum_{m=1}^{\lfloor \frac{1}{6} + \frac{1}{3 \cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 - m)) + \sum_{m=1}^{\lfloor -\frac{1}{6} + \frac{1}{3 \cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 + m)) \pmod 2.$$

In order to state the next theorem, we define a function which counts the number of representations of an integer n by certain t-ary quadratic forms:

$$r_t(n) = \#\{n = x_1^2 + 4x_2^2 + \cdots 4^{t-1}x_t^2 : x_i \text{ are positive odd integers}\}.$$

Theorem 1.3. *If n is a positive integer, then for $s \geq 2$, we have*

$$\tau_{(4^s-1)/3}(n) \equiv r_s(n) \pmod 2.$$

Now we turn to some applications of Theorem 1.1. In particular, we study the case of $s = 2$ where we can determine $\tau_5(8n+5) \pmod 2$.

Theorem 1.4. *If n is an integer, then*

$$\tau_5(8n+5) \equiv \begin{cases} 1 \pmod 2 & \text{if } 8n+5 = k \cdot l^2, \text{ where } k \equiv 5 \pmod 8 \text{ is prime and } l \equiv 1 \pmod 2, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Corollary 1.5. *If $8n+5 = k \cdot l^2$, where $k \equiv 5 \pmod 8$ is prime and $l \equiv 1 \pmod 2$, then*

$$p(n) \equiv 1 + \sum_{m=1}^{\lfloor \frac{1}{6} + \frac{1}{12}\sqrt{4+6n} \rfloor} p(n - 8(3m^2 - m)) + \sum_{m=1}^{\lfloor -\frac{1}{6} + \frac{1}{12}\sqrt{4+6n} \rfloor} p(n - 8(3m^2 + m)) \pmod 2.$$

If $8n+5$ *cannot be written in such a form, then*

$$p(n) \equiv \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m)) \pmod 2.$$

Using these results, we obtain estimates for the parity of $p(n)$ which fall just short of (1.7) and (1.6).

Corollary 1.6. *For all sufficiently large positive integers* x, *we have*

$$\#\{n \le x : p(n) \equiv 1 \pmod 2\} \gg \frac{\sqrt{x}}{\log x}.$$

Corollary 1.7. *For all sufficiently large positive integers* x, *we have*

$$\#\{n \le x : p(n) \equiv 0 \pmod 2\} \gg \sqrt{x}.$$

2. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We recall the definition of $\Delta(q)$ as in (1.1),

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{2.8}$$

Raising the series to the $\frac{4^s-1}{3}$ power, we find

$$\begin{aligned} \Delta(q)^{\frac{4^s-1}{3}} &= \left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)^{\frac{4^s-1}{3}} \\ &\equiv q^{\frac{4^s-1}{3}}\prod_{n=1}^{\infty}(1-q^{8n})^{4^s-1} \\ &\equiv q^{\frac{4^s-1}{3}}\prod_{n=1}^{\infty}(1-q^{8n\cdot 4^s})\frac{1}{\prod_{n=1}^{\infty}(1-q^{8n})} \pmod 2. \end{aligned} \tag{2.9}$$

Using the fact that $P(q) = \dfrac{1}{\prod_{k=1}^{\infty}(1-q^k)}$, and replacing q by q^8, we have

$$\Delta(q)^{\frac{4^s-1}{3}} \equiv q^{\frac{4^s-1}{3}}\left(\sum_{n=0}^{\infty}p(n)q^{8n}\right)\left(\prod_{n=1}^{\infty}(1-q^{8n\cdot 4^s})\right) \pmod 2.$$

Using Euler's identity,

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}},$$

and replacing q by $q^{8\cdot 4^s}$, we find

$$\Delta(q)^{\frac{4^s-1}{3}} \equiv q^{\frac{4^s-1}{3}}\Big(\sum_{k=0}^{\infty} p(n)q^{8n}\Big)\Big(\sum_{n=-\infty}^{\infty} q^{\frac{8\cdot 4^s(3n^2-n)}{2}}\Big)$$

$$\equiv \Big(\sum_{n=0}^{\infty} p(n)q^{8n+\frac{4^s-1}{3}}\Big)\Big(\sum_{n=-\infty}^{\infty} q^{4^{s+1}(3n^2-n)}\Big) \pmod 2. \qquad \square$$

Proof of Theorem 1.2.. By Theorem 1.1, we have

$$\begin{aligned}
\Delta(q)^{\frac{4^s-1}{3}} &\equiv \Big(\sum_{n=0}^{\infty} p(n)q^{8n+\frac{4^s-1}{3}}\Big)\Big(\sum_{n=-\infty}^{\infty} q^{4^{s+1}(3n^2-n)}\Big) \pmod 2 \\
&\equiv \Big(\sum_{k=0}^{\infty} p(k)q^{8k+\frac{4^s-1}{3}}\Big)\Big(1+\sum_{m=1}^{\infty} q^{4^{s+1}(3m^2+m)}+\sum_{m=1}^{\infty} q^{4^{s+1}(3m^2-m)}\Big) \\
&\equiv \sum_{k=0}^{\infty} p(k)q^{8k+\frac{4^s-1}{3}} + \sum_{m=1}^{\infty}\Big(\sum_{k=0}^{\infty} p(k)q^{8k+\frac{4^s-1}{3}+4^{s+1}(3m^2+m)}\Big) \\
&\qquad + \sum_{m=1}^{\infty}\Big(\sum_{k=0}^{\infty} p(k)q^{8k+\frac{4^s-1}{3}+4^{s+1}(3m^2-m)}\Big) \pmod 2.
\end{aligned} \tag{2.10}$$

We now examine the coefficient of q^r, where r is of the form $8n+\frac{4^s-1}{3}$. The left side of (2.10) becomes $\tau_{(4^s-1)/3}(8n+\frac{4^s-1}{3})$. The right side becomes the sum of $p(k)$ for all k such that there exists an integral m such that

$$8n+\frac{4^s-1}{3} = 8k+\frac{4^s-1}{3}+4^{s+1}(3m^2-m)$$

or

$$8n+\frac{4^s-1}{3} = 8k+\frac{4^s-1}{3}+4^{s+1}(3m^2+m).$$

Solving for k, we obtain

$$k = n-2^{2s-1}(3m^2\pm m).$$

Because $k\geq 0$, the limits on the sums must be chosen so that $n-2^{2s-1}(3m^2\pm m)\geq 0$. Thus, we have

$$\begin{aligned}
\tau_{\frac{4^s-1}{3}}(8n+\frac{4^s-1}{3}) \equiv\ & p(n) + \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{3\cdot 2^s}\sqrt{4^{s-1}+6n}\rfloor} p(n-2^{2s-1}(3m^2-m)) \\
&+ \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{3\cdot 2^s}\sqrt{4^{s-1}+6n}\rfloor} p(n-2^{2s-1}(3m^2+m))) \pmod 2.
\end{aligned}$$

Solving for $p(n)$, we obtain a recurrence formula,

$$p(n) \equiv \tau_{\frac{4^s-1}{3}}(8n + \frac{4^s-1}{3}) + \sum_{m=1}^{\lfloor \frac{1}{6} + \frac{1}{3\cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 - m))$$
$$+ \sum_{m=1}^{\lfloor -\frac{1}{6} + \frac{1}{3\cdot 2^s}\sqrt{4^{s-1}+6n} \rfloor} p(n - 2^{2s-1}(3m^2 + m)) \pmod 2. \qquad \square$$

3. Proof of Theorems 1.3 and 1.4 and Corollary 1.5

Lemma 3.1. *If n is a positive integer, then*

$$\tau(n) \equiv \begin{cases} 1 & \text{if } n = (2k+1)^2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of $\Delta(q)$, we have

$$\begin{aligned} \Delta(q) &= q \prod_{n=1}^{\infty} (1-q^n)^{24} \\ &= q\Big(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} \Big)^8 \\ &\equiv \sum_{k=0}^{\infty} q\big(q^{4k(k+1)}\big) \\ &\equiv \sum_{k=0}^{\infty} q^{(2k+1)^2} \pmod 2. \end{aligned} \qquad \square$$

Lemma 3.2. *For integers $s \geq 2$, we have*

$$\Delta(q)^{\frac{4^s-1}{3}} \equiv \Delta(q)\Delta(4q)\cdots\Delta(4^{s-1}q) \pmod 2.$$

Proof. We can write $\frac{4^s-1}{3}$ as $1 + 4 + \cdots 4^{s-1}$. Substituting this into the expression $\Delta(q)^{\frac{4^s-1}{3}}$, we find

$$\begin{aligned} \Delta(q)^{\frac{4^s-1}{3}} &= \Delta(q)^{1+4+\cdots 4^{s-1}} \\ &= \Delta(q)\Delta(q)^4 \cdots \Delta(q)^{4^{s-1}} \\ &\equiv \Delta(q)\Delta(4q)\cdots\Delta(4^{s-1}q) \pmod 2. \end{aligned} \qquad \square$$

Proof of Theorem 1.3. Combining Lemmas 3.1 and 3.2, we find that $\tau_{(4^s-1)/3}(n)$ (mod 2) is equivalent to the number of representations of n as

$$x_1^2 + 4x_2^2 + \cdots 4^{s-1}x_{s-1}^2,$$

where x_i are positive odd integers. We can write this as

$$r_s(n) = \#\{x_1^2 + 4x_2^2 + \cdots 4^{s-1}x_s : x_i \text{ are positive odd integers}\}.$$

Thus, we have $\tau_{(4^s-1)/3}(n) \equiv r_s(n) \pmod 2$. □

We examine the number of representations of n as $x^2 + y^2$ for any integers x, y in order to find a formula for the number of representations of the form $k^2 + 4l^2$ for positive, odd integers k, l.

We define $F(q)$, a power series in q whose coefficients give the number of representations of n as the sum x^2+y^2 for integers x, y. This function is generated by summing $q^{x^2+y^2}$ over all integers x and y:

$$F(q) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} q^{x^2+y^2} = \sum_{n=0}^{\infty} f(n)q^n. \tag{3.11}$$

We find a factorization for the coefficients of $F(q)$.

Theorem 3.3. *Let n be a positive integer such that the factorization of n contains no odd powers of primes which are* $3 \pmod 4$. *Then* $f(n)$ *has the factorization*

$$f(n) = \Big(4 \cdot \prod (m_p - 1)\Big),$$

where the product is taken over all primes $p \equiv 1 \pmod 4$ *which divide n and where* m_p *is the largest integer such that* $p^{m_p} | n$. *If the factorization of n contains an odd power of a prime which is* $3 \pmod 4$, *then* $f(n) = 0$.

This follows from the unique factorization of n in $\mathbb{Z}[i]$ [Hardy and Wright 1979].

If we restrict our function to count only the representations of n of the form k^2+4l^2 for positive odd k, l, we can create a similar power series in q, denoted by $G(q)$, such that the coefficients of $G(q)$ give the number of these representations. We write

$$G(q) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} q^{(2x+1)^2+4(2y+1)^2} = \sum_{n=0}^{\infty} g(n)q^n. \tag{3.12}$$

We again find a factorization for these coefficients.

Theorem 3.4. *For integers* $n \equiv 5 \pmod 8$, *we have*

$$g(n) = \tfrac{1}{8} f(n).$$

Proof. Because the only quadratic residues of 8 are 0, 1 and 4, and $n \equiv 5 \pmod 8$, the only representations of n as the sum of two squares are of the form $k^2 + (2l)^2$, where k, l are positive odd integers. Therefore, the theorem states that for every representation of n as k^2+4l^2 for positive, odd k, l, there are 8 representations of n as $x^2 + y^2$ for integers x, y. For each k, l, we can choose x and y to be either positive or negative. This gives us four new representations. Additionally, although switching l and k produces a different n, switching x and y yields two different representations of n.

Combining both above methods of generating multiple representations in x and y, we find that for each representation of n as the k^2+4l^2 for k and l nonnegative odd integers, there exists 8 representations of n as x^2+y^2, for integers x, y. □

Proof of Theorem 1.4. We now investigate the parity of $\tau_5(8n+5)$.

By Theorem 1.3,

$$\tau_5(8n+5) \equiv r_2(8n+5),$$

and by Theorem 3.4,

$$r_2(8n+5) = g(8n+5) = \tfrac{1}{8}(8n+5).$$

Combining these facts with the formula for $f(n)$ from Theorem 3.3, we have

$$\tau_5(8n+5) \equiv \tfrac{1}{2}\prod(m_p+1) \pmod 2.$$

The odd values of $\tau_5(8n+5)$ are those for which the factorization of $\prod(m_p+1)$ has exactly one power of 2. This occurs when exactly one m_{p_1} is odd, in which case we can write

$$8n+5 = p_1^{m_{p_1}}(p_2^{m_{p_2}}\cdots p_n^{m_{p_n}})(r),$$

where r is the product of even powers of primes which are $3 \pmod 4$. In the factorization of $8n+5$, there are an even number of factors of every prime except p_1, so we can write

$$8n+5 = p_1^{m_{p_1}}s^2$$

where s is odd. Because $p_1^{m_{p_1}}s^2 \equiv 5 \pmod 8$, and the only quadratic residues of 8 are 0, 1 and 4, $p_1 \equiv 5 \pmod 8$.

If we cannot write $8n+5$ in this form, then

$$\tfrac{1}{2}\prod(m_p+1) \equiv 0 \pmod 2,$$

so $\tau_{(4^2-1)/3}(8n+5)$ is even. □

Remark 2. We have proven the additional result that $m_{p_1} = 4m+1$ for some nonnegative integer m, so $8n+5 = p_1^{4m+1}s^2$ with $p_1 \equiv 5 \pmod 8$ prime, $m \geq 0$, s odd, and $p_1 \nmid s$. This stronger version of Theorem 1.4 first appeared as Exercise 6.7 in [Serre 1976], and also appears in [Nicolas 2006].

Proof of Corollary 1.5. By Theorem 1.2 we have

$$p(n) \equiv \tau_5(8n+5) + \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m)) \pmod 2.$$

By Theorem 1.4, we find for n such that $8n+5 = k \cdot l^2$, where $k \equiv 5 \pmod 8$ is prime and $l \equiv 1 \pmod 2$,

$$p(n) \equiv 1 + \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m)) \pmod 2.$$

If $8n+5$ cannot be written in this form, then

$$p(n) \equiv \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m)) \pmod 2. \qquad \square$$

Lemma 3.5. *For all sufficiently large positive integers x,*

$$\#\{n \le x : \tau_5(n) \equiv 1 \pmod 2\} \gg \frac{x}{\log x}.$$

Proof. By Theorem 1.4, $\tau_5(n) \equiv 1 \pmod 2$ if n can be written in the form kl^2, where $k \equiv 5 \pmod 8$ is prime, and $l \equiv 1 \pmod 2$. We look at the case where $n \equiv 5 \pmod 8$ is prime, and $k = n$ and $l = 1$. For sufficiently large x, we have

$$\frac{x}{4\log x}.$$

such that $n \le x$ [Apostol 1976]. This gives us a lower bound for the number of odd values of $\tau_5(n)$ where $n \le x$. $\square$

Proof of Corollary 1.6. We rewrite Theorem 1.2 with $s = 2$:

$$\tau_5(8n+5) \equiv p(n) + \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m))) \pmod 2. \tag{3.13}$$

By the proof of Lemma 3.5, we have

$$\#\{n \le x : \tau_5(8n+5) \equiv 1 \pmod 2\} \gg \frac{x}{\log x}. \tag{3.14}$$

For each of these n, there exists a nonnegative integer r such that

$$p(n-8(3r^2-r)) \equiv 1 \pmod 2$$

or

$$p(n-8(3r^2+r)) \equiv 1 \pmod 2.$$

Because the number of possible r is

$$\left\lfloor \tfrac{1}{6}+\tfrac{1}{12}\sqrt{4+6x} \right\rfloor + \left\lfloor -\tfrac{1}{6}+\tfrac{1}{12}\sqrt{4+6x} \right\rfloor + 1 \sim \sqrt{x},$$

there must be at least

$$c \cdot \tfrac{x}{\log x}\left(\tfrac{1}{\sqrt{x}}\right)$$

distinct values of $n \le x$ such that $p(n)$ is odd.

Therefore, we have

$$\#\{n \le x : p(n) \equiv 1 \pmod 2\} \gg \frac{\sqrt{x}}{\log x}. \qquad \square$$

Proof of Corollary 1.7. We rewrite Theorem 1.2 in the case of $s = 2$:

$$\tau_5(8n+5) \equiv p(n) + \sum_{m=1}^{\lfloor \frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2-m)) + \sum_{m=1}^{\lfloor -\frac{1}{6}+\frac{1}{12}\sqrt{4+6n}\rfloor} p(n-8(3m^2+m)) \pmod 2. \tag{3.15}$$

We note that the number of terms on the right hand side of (3.15) is

$$1 + \left\lfloor \tfrac{1}{6} + \tfrac{1}{12}\sqrt{4+6n} \right\rfloor + \left\lfloor -\tfrac{1}{6} + \tfrac{1}{12}\sqrt{4+6n} \right\rfloor,$$

which is odd only if, for some positive integer z,

$$24z^2 + 8z \le n < 24z^2 + 40z + 16. \tag{3.16}$$

We also note, by the remark following Theorem 1.2, that

$$\lim_{x\to\infty} \frac{\#\{n \le x : \tau_5(n) \equiv 0 \pmod 2\}}{x} = 1. \tag{3.17}$$

When an odd number of integers add up to an even number, at least one of the integers must be even. Thus, when $\tau_5(8n+5)$ is even, and (3.16) is satisfied, one of the terms on the right side of (3.15) must be even. We now count the number of intervals such that (3.16) holds and all values in the interval are $\le x$. This yields

$$\left\lfloor -\tfrac{1}{6} + \tfrac{1}{12}\sqrt{4+6n} \right\rfloor$$

invervals, each of which contains $32z+16$ integers. Therefore, the number of $n \le x$ for which the right side of (3.15) has an odd number of terms is at least

$$\tfrac{2}{3}n + c_1\sqrt{4+6n} + c_2 \tag{3.18}$$

for some constants $c_1, c_2 > 0$.

Combining (3.17) and (3.18), we find that, as $x \to \infty$, the number of $n \le x$ for which $\tau_5(8n+5)$ is even and there are an odd number of terms on the right side of (3.15) approaches

$$\tfrac{2}{3}x. \tag{3.19}$$

For each of these n, there must be an even term on the right hand side of (3.15). However, (3.19) does not give the total number distinct n for which $p(n)$ is even;

we may be counting an integer w for each n, m such that $n - 8(3m^2 - m)$ or $n - 8(3m^2 + m) = w$.

We can put an upper bound on the number of m for which we are counting w because there are only $c\sqrt{x}$ values of m for which $n - 8(3m \pm m)$ is positive for some n, for some constant $c > 0$. We divide (3.19) by the number of m in order to compensate for the possibility of counting any w multiple times. Thus, we have, as $x \to \infty$,

$$\#\{n \leq x : p(n) \equiv 0 \pmod 2\} \gg \sqrt{x} \qquad \square$$

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swolfe2@wisc.edu

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, United States