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Patch and crossover planar dyadic wavelet sets

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A single dyadic orthonormal wavelet on the plane $\mathbb{R}^2$ is a measurable square integrable function $\psi(x, y)$ whose images under translation along the coordinate axes followed by dilation by positive and negative integral powers of 2 generate an orthonormal basis for $\mathscr{L}^2(\mathbb{R}^2)$. A planar dyadic wavelet set E is a measurable subset of $\mathbb{R}^2$ with the property that the inverse Fourier transform of the normalized characteristic function $\frac{1}{2\pi}\chi(E)$ of E is a single dyadic orthonormal wavelet. While constructive characterizations are known, no algorithm is known for constructing all of them. The purpose of this paper is to construct two new distinct uncountably infinite families of dyadic orthonormal wavelet sets in $\mathbb{R}^2$. We call these the crossover and patch families. Concrete algorithms are given for both constructions.

Introduction

Wavelet theory is interesting to mathematicians both for its applications to signal analysis and image analysis and also because of the rich mathematical structure underlying the theory of wavelets. Wavelet sets are measurable sets whose normalized characteristic functions are the Fourier transforms of wavelets. Planar dyadic wavelet sets are interesting mathematically because they are fractal-like, and there are hands-on methods for working with them and constructing new examples. They are also interesting because while constructive characterizations are known, no algorithm is known for constructing all planar dyadic wavelets. There are open problems associated with their classification. Algorithms for constructing new examples or classes of examples can provide useful counterexamples to conjectures as well as be appreciated for their intrinsic mathematical beauty.

A single dyadic *orthonormal wavelet* on the plane $\mathbb{R}^2$ is a (Lebesgue) measurable square-integrable function $\psi(x, y)$ whose translations along the coordinate axes followed by dilations by positive and negative integral powers of 2 generate an orthonormal basis for $L^2(\mathbb{R}^2)$. A planar dyadic *wavelet* set E is a measurable

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subset of $\mathbb{R}^2$ with the property that the inverse Fourier transform of the normalized characteristic function $1/2\pi\chi(E)$ of E is a single dyadic orthonormal wavelet. As usual, the Fourier transform on $\mathscr{L}^2(\mathbb{R}^2)$ is the tensor product of two copies of the Fourier transform on $\mathscr{L}^2(\mathbb{R})$. In this paper we discuss two algorithms which generate two distinct uncountably infinite classes of dyadic orthonormal wavelet sets in $\mathbb{R}^2$. We denote these classes the *crossover* and *patch* classes and denote the algorithms for these constructions the crossover and patch algorithms. A free parameter in both of the algorithms is a partition of the so-called inner square, $[-\pi/2, \pi/2) \times [-\pi/2, \pi/2)$, into four measurable subsets $X_\ominus$, $X_\oplus$, $Y_\ominus$, $Y_\oplus$, with the property that $X_\ominus$ is contained in the left half of the inner square, $X_\oplus$ is contained in the right half, $Y_\ominus$ is contained in the bottom half, and $Y_\oplus$ is contained in the top half. Notice that if the boundary of any two of these sets is the same, and if this boundary has Lebesgue measure 0, then these two sets are still essentially disjoint although their boundaries are the same.

Wavelets for dilations other than 2 (the dyadic case) on the line $\mathbb{R}$ and in $\mathbb{R}^n$ have been investigated by many authors. In higher dimensions both scalar dilations and matrix dilations have been studied. However, much of the interesting work in the literature has been for the dyadic case, which is the case we focus on.

For completeness, we give the form used for abstract matrix dilations: A dilation A wavelet is a function on $\mathbb{R}^n$ whose dilations by integral powers of A and translations along the coordinate axes (or, more generally, translations along a full-rank lattice) form an orthonormal basis for the space of all square-integrable measurable functions over $\mathbb{R}^n$ with respect to Lebesgue measure. In precise terms, a function f on $\mathbb{R}^n$ is a dilation A wavelet if and only if it is measurable with respect to product Lebesgue measure, and

$$\{|\det A|^{\frac{m}{2}} f\left(A^m t - l\right) : m \in \mathbb{Z}, l \in \mathbb{Z}^n\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. A dilation A wavelet set is a measurable set W for which the inverse Fourier transform of the normalized characteristic function is a dilation A orthonormal wavelet. The dyadic case is where $A := 2I_2$, where I_2 is the identity matrix in two dimensions.

Existence of wavelet sets in $\mathbb{R}^n$ was first demonstrated in 1994 [Dai et al. 1997]. The proof used an algorithmic approach which generated wavelet sets that were unbounded and had 0 as a limit point, rendering them difficult to visualize [Dai et al. 1997; Zhang and Larson $\geq$ 2008]. It showed that there are uncountably many such sets in $\mathbb{R}^2$ for many matrix dilations, including the dyadic case. Subsequently, several authors [Soardi and Weiland 1998; Dai et al. 1998; Benedetto and Leon 1999; 2001; Baggett et al. 1999; Gu and Han 2000] constructed wavelet sets in $\mathbb{R}^2$ which were more easily pictured due to their qualities of being bounded and bounded away from 0, and had other interesting structural properties. Two such sets were included in the final remarks section of [Dai and Larson 1998]. Recent papers that construct new planar wavelet sets with reasonable graphics and interesting properties can be found in [Zhang and Larson $\geq$ 2008] and [Merrill $\geq$ 2008]. A brief history of wavelet sets can be found in [Zhang and Larson $\geq$ 2008, Section 5].

In the summer of 2007, the first three authors were undergraduate student participants in the Texas A&M Mathematics REU on *matrix analysis and wavelets* (funded by the NSF), which was mentored by D. Larson. They set out to classify multiple categories of wavelet sets in $\mathbb{R}^2$ using an algorithmic approach. The present paper is the upshot of that project. Two algorithms were obtained. The wavelet sets in Figures 1 and 2 are called crossover wavelet sets because in their generation, regions are added to or subtracted from alternating sides of the inner square. Alternatively, patch wavelet sets are created by adding regions to the same side of the square for each translation; see, for example, the set in Figure 3. This category of sets is so named because in computer networking a patch cable is the opposite of a crossover cable.

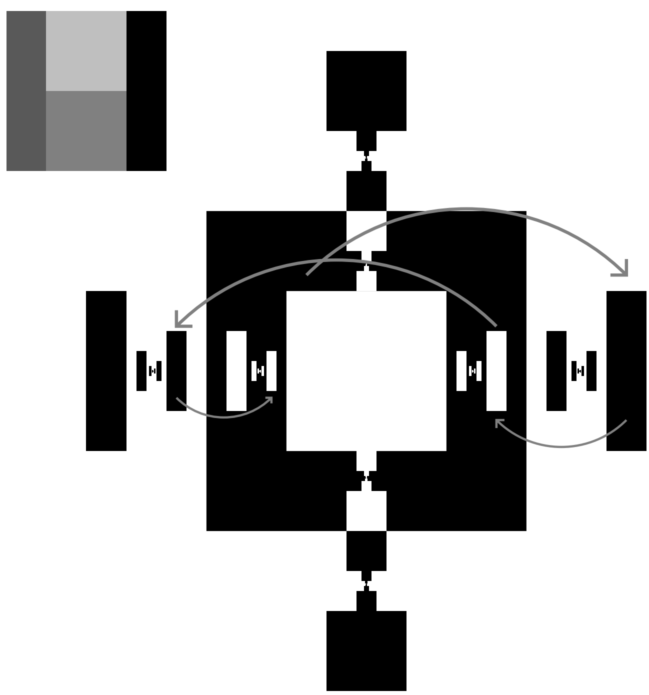

Figure 1. The two-dimensional wavelet set formed in Example 1.

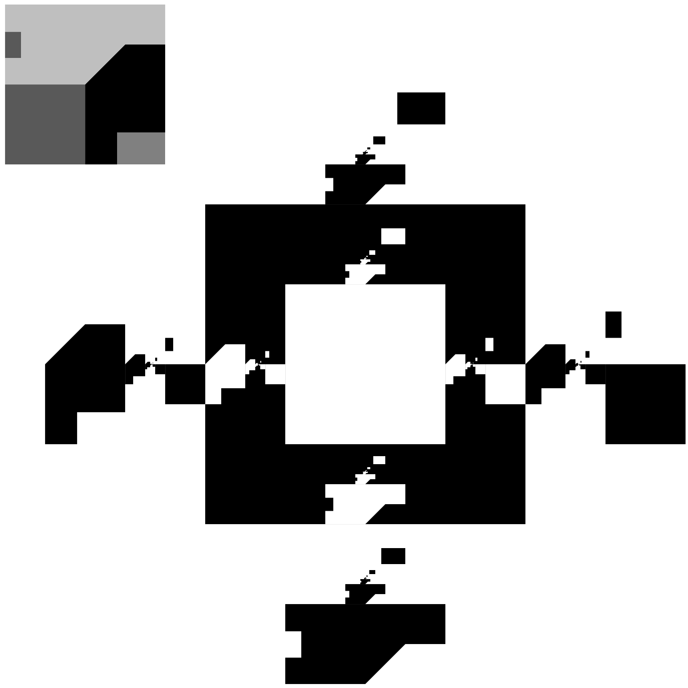

Figure 2. An arbitrary (conforming) partition, with wavelet set.

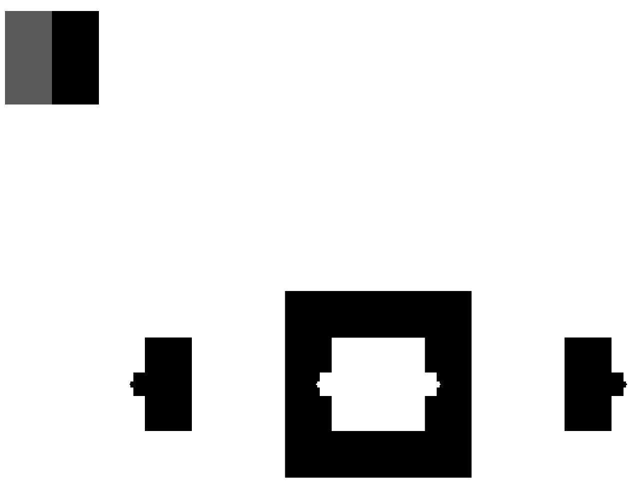

Figure 3. A patch wavelet set: the *wedding cake set*.

1. Preliminaries

We begin with some basic formal definitions.

A *single dyadic orthonormal wavelet* is a function $\psi \in \mathcal{L}^2(\mathbb{R})$ (Lebesgue measure) with the property that the set $\{ 2^{\frac{n}{2}} \psi(2^n \cdot -l) \mid n, l \in \mathbb{Z} \}$ forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$ [Consortium 1998]; see also [Larson 2007a, pp. 6–7]. More generally, if A is any real invertible $n \times n$ matrix, then a single function $\psi \in L^2(\mathbb{R}^n)$ is a multivariate orthonormal wavelet for A if

$$\{|\det A|^{\frac{n}{2}} \psi(A^n \cdot -l) \mid n \in \mathbb{Z}, l \in \mathbb{Z}^{(n)}\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. It was shown in [Dai et al. 1997] that if A is *expansive* (equivalently, all eigenvalues of A are required to have absolute value strictly greater than 1) then orthonormal wavelets for A always exist. The dyadic case is the case where $A := 2I_2$ (two times the identity matrix on R^n). This is the simplest (and most investigated) case.

We let $\mathcal{F}$ denote the n-dimensional Fourier transform on $\mathcal{L}^2(\mathbb{R}^n)$ defined by

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-s \circ t} f(t) dm,$$

for all $f \in L^2(\mathbb{R}^n)$. Here, $s \circ t$ denotes the real inner product. A measurable set $E \subseteq \mathbb{R}^n$ is defined to be a wavelet set for a dilation matrix A if

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\mu(E)}} \chi_E\right)$$

is an orthonormal wavelet for A, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

In this paper, we will not explicitly use properties of $\mathcal{F}$ and $\mathcal{F}^{-1}$; however we state the formal definition of Fourier transform because it is needed to give the proper definition of a wavelet set. The Fourier transform is a unitary transform from $\mathcal{L}(\mathbb{R}^n)$, where $\mathbb{R}^n$ is usually considered as a multivariate time domain, to another copy of $\mathcal{L}(\mathbb{R}^n)$, where $\mathbb{R}^n$ is considered a multivariate frequency domain. We will do our work with wavelet sets entirely in the frequency domain. We can do this because there is a set-theoretic characterization of wavelet sets, Proposition 1.1, which allows one to construct and otherwise work with wavelet sets without using the Fourier transform.

A sequence of measurable subsets $\{E_n\}$ of a measurable set E is called a *measurable partition* of E if the relative complement of $\bigcup_n E_n$ in E is a null set (that is, has measure zero) and $E_n \cap E_m$ is a null set whenever $n \neq m$.

Measurable subsets E and F of $\mathbb{R}$ are called 2π*-translation congruent* to each other, denoted by $E \sim_{2\pi} F$, if there exists a measurable partition $\{E_n\}$ of E, such that $\{E_n + 2n\pi\}$ is a measurable partition of F. Similarly, E and F are called *2-dilation congruent* to each other, denoted by $E\,{}_2\!\sim F$, if there is a measurable partition $\{E_n\}$ of E, such that $\{2^n E_n\}$ is a measurable partition of F. A measurable set E is called a 2π*-translation generator of a measurable partition* of $\mathbb{R}$ if $\{E+2n\pi\}_{n\in\mathbb{Z}}$ forms a measurable partition of $\mathbb{R}$. Similarly, a measurable set F is called a 2*-dilation generator of a measurable partition of* $\mathbb{R}$ if $\{2^n F\}_{n\in\mathbb{Z}}$ forms a measurable partition of $\mathbb{R}$.

Lemma 4.3 in [Dai and Larson 1998] gives the following characterization of dyadic wavelet sets in $\mathbb{R}$, which was also obtained independently in [Fang and Wang 1996] using different techniques. *Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a dyadic wavelet set if and only if E is both a 2π-translation generator of a measurable partition of $\mathbb{R}$ and a 2-dilation generator of a measurable partition of $\mathbb{R}$.*

In [Dai et al. 1997], this criterion was generalized to the multivariate case. We will consider only the dyadic planar case in this paper because we will only use the criterion for that case, although the criterion actually applies for the arbitrary expansive case [Dai et al. 1997; 1998]. So from [Dai et al. 1997] we have that E is a dyadic wavelet set in $\mathbb{R}^n$ if and only if E is both a 2π-translation generator of a measurable partition of $\mathbb{R}^n$ and a 2-dilation generator of a measurable partition of $\mathbb{R}^n$. Here, to achieve a translation partition one translates by all integral multiples of 2π separately in each coordinate direction. To achieve a dilation partition, one dilates by all integral powers of 2 simultaneously in all coordinates. For example, it is clear that the set $[-\pi,\pi)\times[-\pi,\pi)$ is a 2π-translation generator of a measurable partition of $\mathbb{R}^2$, and

$$G_{TO}\backslash\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

is a 2-dilation generator of a measurable partition of $\mathbb{R}^2$.

Properly generalizing the one dimensional definition to the planar case, we say that two Lebesgue measurable sets $A, B\subset\mathbb{R}^2$ are 2π-translation congruent if there is a measurable partition $\{A_{k,l} : k, l\in\mathbb{Z}\}$ of A such that

$$\{A_{k,l}+\begin{bmatrix}2k\pi\\2l\pi\end{bmatrix} : k, l\in\mathbb{Z}\}$$

is a measurable partition of B, and they are 2-dilation congruent if there is a measurable partition $\{A_n : n\in\mathbb{Z}\}$ of A such that

$$\{2^n A_n : n\in\mathbb{Z}\}$$

is a measurable partition of B.

Translation congruence and dilation congruence are both equivalence relations on the class of all measurable subsets. If a set A is 2π-translation congruent to a 2π-translation generator of a measurable partition of $\mathbb{R}^2$, it is clear that A itself is a 2π-translation generator of a measurable partition of $\mathbb{R}^2$. Moreover, sets A

and B which are both 2π-translation generators of measurable partitions of $\mathbb{R}^2$ are necessarily translation congruent to each other. All this is in [Dai et al. 1997], and other expositions can be found in [Dai and Larson 1998; Larson 2007b; Zhang and Larson $\geq$ 2008]. This yields a useful criterion.

In the following proposition (and in the rest of the paper), let

$$G_{TO} := [-\pi,\pi)\times[-\pi,\pi)\,,\qquad \text{and}\qquad G_{SO} := G_{TO}\backslash\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right)\right).$$

Proposition 1.1 (Working Principle Criterion). *A measurable set $W\subseteq\mathbb{R}^2$ is a dyadic wavelet set if and only if W is 2π-translation congruent to G_{TO} and 2-dilation congruent to G_{SO}.*

2. The crossover algorithm

We first consider a special case of a wavelet set to illustrate the *crossover algorithm*. We will then generalize this to obtain Theorem 2.1.

Example 2.1. Let

$$X_\ominus=\left[-\frac{\pi}{2},-\frac{\pi}{4}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right),\qquad X_\oplus=\left[\frac{\pi}{4},\frac{\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right),$$
$$Y_\ominus=\left[-\frac{\pi}{4},\frac{\pi}{4}\right)\times\left[-\frac{\pi}{2},0\right),\qquad Y_\oplus=\left[-\frac{\pi}{4},\frac{\pi}{4}\right)\times\left[0,\frac{\pi}{2}\right).$$

We can generate a wavelet set in the plane using the above partition of the inner square using an algorithm (the crossover algorithm) which we will illustrate with the following example.

Let $X_{\ominus 1} := X_\ominus$. Start by adding the vector $\begin{bmatrix}2\pi\\0\end{bmatrix}$ to the set $X_\ominus$, translating it (that is, crossing it over) to the right half-plane. The set formed is

$$\left[\frac{3\pi}{2},\frac{7\pi}{4}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Call this $X_{\ominus 2}$. Secondly, scale $X_{\ominus 2}$ by $\frac{1}{2}$ to obtain

$$\left[\frac{3\pi}{4},\frac{7\pi}{8}\right)\times\left[-\frac{\pi}{4},\frac{\pi}{4}\right).$$

Call this $X_{\ominus 3}$. Thirdly, translate $X_{\ominus 3}$ to the opposite half-plane by adding $\begin{bmatrix}-2\pi\\0\end{bmatrix}$ to obtain

$$\left[-\frac{5\pi}{4},-\frac{9\pi}{8}\right)\times\left[-\frac{\pi}{4},\frac{\pi}{4}\right),$$

and call this set $X_{\ominus 4}$. Notice that $X_{\ominus 4}$ is on the same side half-plane as $X_{\ominus}$. Finally, scale $X_{\ominus 4}$ by $\frac{1}{2}$ to form the set

$$\left[-\frac{5\pi}{8},-\frac{9\pi}{16}\right)\times\left[-\frac{\pi}{8},\frac{\pi}{8}\right)$$

and call it $X_{\ominus 5}$. Continue these four steps inductively for $X_{\ominus}$.

We perform four similar steps on the set $X_{\oplus}$; however, we translate by $\begin{bmatrix}-2\pi\\0\end{bmatrix}$ for the first step (instead of $\begin{bmatrix}2\pi\\0\end{bmatrix}$) and translate by $\begin{bmatrix}2\pi\\0\end{bmatrix}$ for the third step (instead of $\begin{bmatrix}-2\pi\\0\end{bmatrix}$). We obtain the following from the first four steps:

$$X_{\oplus 2}=\left[-\frac{7\pi}{4},-\frac{3\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right),\qquad X_{\oplus 3}=\left[-\frac{7\pi}{8},-\frac{3\pi}{4}\right)\times\left[-\frac{\pi}{4},\frac{\pi}{4}\right),$$
$$X_{\oplus 4}=\left[\frac{9\pi}{8},\frac{5\pi}{4}\right)\times\left[-\frac{\pi}{4},\frac{\pi}{4}\right),\qquad X_{\oplus 5}=\left[\frac{9\pi}{16},\frac{5\pi}{8}\right)\times\left[-\frac{\pi}{8},\frac{\pi}{8}\right).$$

Continue this process inductively for $X_{\oplus}$ as well. Perform similar steps for $Y_{\oplus}$ and $Y_{\ominus}$, translating by $\begin{bmatrix}0\\\pm 2\pi\end{bmatrix}$ instead of $\begin{bmatrix}\pm 2\pi\\0\end{bmatrix}$, beginning with a translation of $\begin{bmatrix}0\\2\pi\end{bmatrix}$ for $Y_{\ominus}$ and a translation of $\begin{bmatrix}0\\-2\pi\end{bmatrix}$ for $Y_{\oplus}$.

Let W be the set

$$\begin{aligned}&\Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i}\cup X_{\oplus 2i}\cup Y_{\ominus 2i}\cup Y_{\oplus 2i}]\Big)\\&\qquad\qquad\cup\Big(G_{TO}\setminus\Big[\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}\cup Y_{\ominus 2i-1}\cup Y_{\oplus 2i-1}]\Big]\Big)\\&=\Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i}\cup X_{\oplus 2i}\cup Y_{\ominus 2i}\cup Y_{\oplus 2i}]\Big)\\&\qquad\qquad\cup\Big(G_{SO}\setminus\Big[\bigcup_{i=2}^{\infty}[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}\cup Y_{\ominus 2i-1}\cup Y_{\oplus 2i-1}]\Big]\Big).\end{aligned}$$

We can think of W as being the union of G_{TO} and the exterior *black pieces* of the form $X_{\oplus 2n}$, $X_{\ominus 2n}$, $Y_{\oplus 2n}$, $Y_{\ominus 2n}$, with *white spaces* of the form $X_{\oplus 2n+1}$, $X_{\ominus 2n+1}$, $Y_{\oplus 2n+1}$, $Y_{\ominus 2n+1}$ removed from G_{TO}.

This set W (see Figure 1) is a wavelet set. To see this, let

$$G(X_{\ominus\text{odd}}):=\bigcup_{i=1}^{\infty}X_{\ominus 2i-1},\qquad\text{and}\qquad G(X_{\ominus\text{even}}):=\bigcup_{i=1}^{\infty}X_{\ominus 2i}.$$

Observe that

$$G(X_{\ominus\mathrm{odd}})\backslash X_{\ominus} \subset G_{SO}, \qquad \text{and} \qquad G(X_{\ominus\mathrm{even}}) \not\subseteq G_{SO}.$$

Similarly, define sets for $X_{\oplus}$, $Y_{\ominus}$, and $Y_{\oplus}$ with analogous characteristics. Observe that W is translation congruent to G_{TO} modulo 2π because

$$\Big[\Big(\bigcup_{i=0}^{\infty} X_{\ominus 4i+2}\Big) \cup \Big(\bigcup_{i=1}^{\infty} X_{\oplus 4i}\Big)\Big] - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \Big[\Big(\bigcup_{i=0}^{\infty} X_{\ominus 4i+1}\Big) \cup \Big(\bigcup_{i=0}^{\infty} X_{\oplus 4i+3}\Big)\Big],$$

$$\Big[\Big(\bigcup_{i=0}^{\infty} X_{\oplus 4i+2}\Big) \cup \Big(\bigcup_{i=1}^{\infty} X_{\ominus 4i}\Big)\Big] + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \Big[\Big(\bigcup_{i=0}^{\infty} X_{\oplus 4i+1}\Big) \cup \Big(\bigcup_{i=0}^{\infty} X_{\ominus 4i+3}\Big)\Big],$$

and

$$\Big[\Big(\bigcup_{i=0}^{\infty} X_{\oplus 4i+1}\Big) \cup \Big(\bigcup_{i=0}^{\infty} X_{\ominus 4i+3}\Big)\Big] \cup \Big[\Big(\bigcup_{i=0}^{\infty} X_{\ominus 4i+1}\Big) \cup \Big(\bigcup_{i=0}^{\infty} X_{\oplus 4i+3}\Big)\Big] = G(X_{\ominus\mathrm{odd}}) \cup G(X_{\oplus\mathrm{odd}}),$$

so that all gaps in the set G_{TO} due to the crossover algorithm applied to the sets $X_{\ominus 1}$ and $X_{\oplus 1}$ are filled when we translate the black sets formed by the crossover algorithm applied to the sets $X_{\ominus 1}$ and $X_{\oplus 1}$ by multiples of $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$. Similar results will apply for $Y_{\ominus}$ and $Y_{\oplus}$.

Moreover, W is dilation congruent to G_{SO} because

$$\tfrac{1}{2}G(X_{\ominus\mathrm{even}}) = G(X_{\ominus\mathrm{odd}}) \in G_{SO}$$

(that is, the even pieces of the form $X_{\ominus n}$ scale into the odd pieces of the form $X_{\ominus n}$), with similar results for $G(X_{\oplus\mathrm{even}})$, $G(Y_{\ominus\mathrm{even}})$, and $G(Y_{\oplus\mathrm{even}})$.

The set of steps indicated above, applied to all four pieces of the partition of the inner square, is a special case of the crossover algorithm. An uncountably infinite family of wavelet sets in $\mathbb{R}^2$ can be similarly constructed using a natural generalization of this algorithm. The generalized crossover algorithm will be presented rigorously in the later sections of this paper in the context of the proof of Theorem 2.1 and the constructions involved in the proof.

A brief description of the general crossover algorithm is the following:

(i) Partition the inner square into a maximum of four subsets satisfying the conditions given in the statement of Theorem 2.1 below. (These conditions are necessary because not all partitions of the inner square will lead to a wavelet set in this way.)

(ii) Translate one piece of the partition by $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$, moving it out of the inner square to the opposite side of the x- or y-axis.

(iii) Dilate the set formed in step 2 into G_{SO} by $\frac{1}{2}$.

(iv) Translate the set formed in step 3 in the opposite direction (compared to the first translation), that is, by $\begin{bmatrix} \mp 2\pi \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \mp 2\pi \end{bmatrix}$.

(v) Dilate the set formed in step 4 into G_{SO} by $\frac{1}{2}$.

(vi) Repeat these steps inductively for this piece, and perform steps 2–5 on the other pieces of the partition inductively as well.

Theorem 2.1 (Crossover Algorithm). *Let $\{X_\ominus, X_\oplus, Y_\ominus, Y_\oplus\}$ be a partition of the set*

$$\Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big) \times \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

such that $X_\ominus$ is contained in the left half of the inner square (*that is*, *the maximum possible x-coordinate of any point in the set $X_\ominus$ is* 0), *$X_\oplus$ is contained in the right half of the inner square* (*that is*, *the minimum possible x-coordinate of any point in the set $X_\oplus$ is* 0), *$Y_\ominus$ is contained in the bottom half of the inner square* (*that is*, *the maximum possible y-coordinate of the set $Y_\ominus$ is* 0), *and $Y_\oplus$ is contained in the top half of the inner square* (*that is*, *the minimum possible y-coordinate of the set $Y_\oplus$ is* 0). *Then the set W generated by this partition*, *under translation by*

$$\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$$

and dilation by powers of 2 *using steps* (1)–(6) *above*, *defined as*

$$\Big[\Big(\bigcup_{i=1}^{\infty}\Big[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\Big]\Big) \cup G_{TO}\Big] \setminus \Big[\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\Big],$$

is a dyadic wavelet set in $\mathbb{R}^2$.

Remark 2.1. If either both $X_\oplus$ and $X_\ominus$ are defined, or both $Y_\oplus$ and $Y_\ominus$ are defined, then the other two sets are automatically determined due to our constraints.

3. Expressions for $X_{\ominus n}$, $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$

Before proving our main result, Theorem 2.1, we first give rigorous expressions for the sets $X_{\ominus n}$, $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$. We begin with the sets of the form $X_{\ominus n}$. Suppose first that n is odd and $n \geq 3$. We can derive the formula for $X_{\ominus n}$ in terms of n by looking at the first few terms. Let $X_{\ominus 1} := X_\ominus$, which has the above constraints listed according to the theorem. We can then find the next few odd terms using the crossover algorithm described in the example:

$$X_{\ominus 3} = \frac{1}{2}\left(X_{\ominus} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right),$$

$$X_{\ominus 5} = \frac{1}{2}\left(X_{\ominus 3} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right) = \left(\frac{X_{\ominus}}{4} + \frac{1}{4}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right),$$

$$X_{\ominus 7} = \frac{1}{2}\left(X_{\ominus 5} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right) = \left(\frac{X_{\ominus}}{8} + \frac{1}{8}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right),$$

and, in general, we find that $X_{\ominus n+2} = \frac{1}{2}\left(X_{\ominus n} + (-1)^{\frac{n-1}{2}}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right)$. Solving this recurrence relation, we find that

$$\begin{aligned}
X_{\ominus n} &= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\Big(\frac{1}{2^{\frac{n-1}{2}}} - \ldots + \frac{(-1)^{\frac{n-3}{2}}}{2}\Big) \\
&= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n-3}{2}}\Big(\frac{(-1)^{\frac{n-3}{2}}}{2^{\frac{n-1}{2}}} + \frac{(-1)^{\frac{n-3}{2}-1}}{2^{\frac{n-1}{2}-1}} + \ldots - \frac{1}{4} + \frac{1}{2}\Big) \\
&= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n-3}{2}}\sum_{i=1}^{\frac{n-1}{2}}\frac{(-1)^{i-1}}{2^i} \\
&= \frac{X_{\ominus}}{2^{\frac{n-1}{2}}} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\frac{1}{3}(-1)^{\frac{n-3}{2}}\Big(1 - (-\frac{1}{2})^{\frac{n-1}{2}}\Big),
\end{aligned} \tag{1}$$

using the formula for a geometric series summation. In order to formally verify that our formula for $X_{\ominus n}$ solves the recurrence relation, we merely plug in the expressions for $X_{\ominus n+2}$ and $X_{\ominus n}$ and carry out basic computations.

Now suppose n' is even and $n' > 2$. We can derive the formula for $X_{\ominus n'}$ in terms of n' similarly:

$$X_{\ominus 2} = X_{\ominus} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

$$X_{\ominus 4} = \frac{X_{\ominus 2}}{2} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{2} + \frac{1}{2}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

$$X_{\ominus 6} = \frac{X_{\ominus 4}}{2} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{4} + \frac{1}{4}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

$$X_{\ominus 8} = \frac{X_{\ominus 6}}{2} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{X_{\ominus}}{8} + \frac{1}{8}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

and, in general, we find that

$$X_{\ominus n'+2} = \frac{1}{2} X_{\ominus n'} + (-1)^{\frac{n'}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}.$$

Observe that $\frac{X_{\ominus n'}}{2} = X_{\ominus n'+1}$ is, in general, true based on our construction using the crossover algorithm since n' is even. Since $n'+1$ is odd, we plug into our formula the values for $X_{\ominus n}$ (see the bottom of page 69), where n is odd, to find $X_{\ominus n'}$.

$$\begin{aligned} X_{\ominus n'} = 2X_{\ominus n'+1} &= 2\left[\frac{X_\ominus}{2^{\frac{n'}{2}}} + \frac{1}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-2}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right)\right] \\ &= \left[\frac{X_\ominus}{2^{\frac{n'-2}{2}}} + \frac{2}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-2}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right)\right]. \end{aligned}$$

Once again, the proof that the recurrence relation is satisfied involves plugging in the expressions for $X_{\ominus n'+2}$ and $X_{\ominus n'}$ and performing basic computations.

Notice that when n' is even,

$$X_{\ominus n'} = X_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

consistent with the crossover algorithm, because of the following:

$$\begin{aligned} X_{\ominus n'} &= X_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}} \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \\ &= \frac{X_\ominus}{2^{\frac{n'-2}{2}}} + \frac{1}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-4}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n'-2}{2}}\right) + (-1)^{\frac{n'-2}{2}}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \\ &= \frac{X_\ominus}{2^{\frac{n'-2}{2}}} + \frac{2}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-2}{2}}\left(-\frac{1}{2} - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right) + (-1)^{\frac{n'-2}{2}}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \\ &= \frac{X_\ominus}{2^{\frac{n'-2}{2}}} + \frac{2}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-2}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right). \end{aligned}$$

Thus, we can say in general that

$$X_{\ominus n} = \begin{cases} \frac{X_\ominus}{2^{\frac{n-1}{2}}} + \frac{1}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n-3}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd}, \\ \frac{X_\ominus}{2^{\frac{n'-2}{2}}} + \frac{2}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}(-1)^{\frac{n'-2}{2}}\left(1 - \left(-\frac{1}{2}\right)^{\frac{n'}{2}}\right), & \text{for } n \text{ even}, \end{cases}$$

but then clearly

$$X_{\oplus n} = \begin{cases} \frac{X_\oplus}{2^{\frac{n-1}{2}}} - \frac{1}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n-3}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n-1}{2}}\big), & \text{for } n \text{ odd,} \\ \frac{X_\oplus}{2^{\frac{n'-2}{2}}} - \frac{2}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} (-1)^{\frac{n'-2}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n'}{2}}\big), & \text{for } n \text{ even.} \end{cases}$$

Analogously, we find

$$Y_{\ominus n} = \begin{cases} \frac{Y_\ominus}{2^{\frac{n-1}{2}}} + \frac{1}{3}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n-3}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n-1}{2}}\big), & \text{for } n \text{ odd,} \\ \frac{Y_\ominus}{2^{\frac{n'-2}{2}}} + \frac{2}{3}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n'-2}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n'}{2}}\big), & \text{for } n \text{ even,} \end{cases}$$

and

$$Y_{\oplus n} = \begin{cases} \frac{Y_\oplus}{2^{\frac{n-1}{2}}} - \frac{1}{3}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n-3}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n-1}{2}}\big), & \text{for } n \text{ odd,} \\ \frac{Y_\oplus}{2^{\frac{n'-2}{2}}} - \frac{2}{3}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix} (-1)^{\frac{n'-2}{2}}\big(1-(-\tfrac{1}{2})^{\frac{n'}{2}}\big), & \text{for } n \text{ even.} \end{cases}$$

Comment: By our construction we have (analogous to the properties for $X_{\ominus n}$) that for n' even,

$$\frac{X_{\oplus n'}}{2} = X_{\oplus n'+1}, \quad \frac{Y_{\ominus n'}}{2} = Y_{\ominus n'+1}, \quad \frac{Y_{\oplus n'}}{2} = Y_{\oplus n'+1}.$$

Moreover,

$$X_{\oplus n'} = X_{\oplus n'-1} - (-1)^{\frac{n'-2}{2}}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \quad Y_{\ominus n'} = Y_{\ominus n'-1} + (-1)^{\frac{n'-2}{2}}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix},$$
$$Y_{\oplus n'} = Y_{\oplus n'-1} - (-1)^{\frac{n'-2}{2}}\begin{bmatrix} 0 \\ 2\pi \end{bmatrix}.$$

4. Proof of Theorem 2.1

For the proof of Theorem 2.1 we will require three technical lemmas.

Lemma 4.1. *For all odd $n \geq 3$, $X_{\ominus n} \subseteq G_{SO}$.*

Proof. Since all such

$$X_\ominus \subseteq \Big[-\frac{\pi}{2}, 0\Big) \times \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

by definition, for all functions f,

$$f(X_\ominus) \subseteq f\bigg(\Big[-\frac{\pi}{2}, 0\Big) \times \Big[\frac{\pi}{2}, \frac{\pi}{2}\Big)\bigg).$$

Therefore, we only need to prove that for all odd $n \geq 3$,

$$\left[\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n} \subseteq G_{SO}.$$

Let

$$S_{\ominus n} := \left[\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n}$$

represent the result of the n^{th} step of the crossover algorithm applied to

$$\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Notice that $S_{\ominus n}$ is of the form of the more general set $X_{\ominus n}$, and, therefore, we can use our derived bounds (see above on page 71) for $X_{\ominus n}$ in terms of n to determine the bounds for $S_{\ominus n}$.

We begin by showing that $S_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$. That this is satisfied for the vertical bounds of $S_{\ominus n}$ is clear, so we will only consider the horizontal bounds. Note that when we use the phrase "vertical bound," we refer to both upper and lower bounds. By a vertical upper bound, we mean to say that such a number is greater than or equal to all y-coordinates of the points in that set. When we use the phrase "horizontal bound of a set," we refer to both left and right hand bounds of a set. By left hand bound, we mean to signify a number that is less than or equal to all of the horizontal coordinates of the points in that set.

Case 1. $n = 4k + 1$ for some $k \in \mathbb{Z}$. Then $S_{\ominus n}$ is on the left side of the origin. Thus, the horizontal left hand bound (LHB) for $S_{\ominus n}$ is

$$-\frac{1}{2^{\frac{n-1}{2}}}\begin{bmatrix}\pi\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\pi\\0\end{bmatrix}(-1)^{\frac{n-3}{2}}\left(1-\left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right),$$

and we must show it is bounded below by $\begin{bmatrix}-\pi\\0\end{bmatrix}$. In the x-coordinate,

$$\begin{aligned}
-\pi \leq -\frac{\pi}{2^{\frac{n-1}{2}}} + \frac{2\pi}{3}(-1)^{\frac{n-3}{2}}\left(1-\left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) &\iff 1 \geq \frac{1}{2^{2k}} - \frac{2}{3}(-1)^{2k-1}\left(1-\left(-\frac{1}{2}\right)^{2k}\right)\\
&\iff 1 \geq \frac{1}{2^{2k}} + \frac{2}{3}\left(1-\frac{1}{2^{2k}}\right)\\
&\iff \frac{1}{3} \geq \frac{1}{2^{2k}}\left(1-\frac{2}{3}\right)\\
&\iff 1 \geq \frac{1}{2^{2k}},
\end{aligned}$$

which is clearly true because $3 \le n = 4k+1 \Rightarrow \frac{1}{2} \le k$, and $k \in \mathbb{Z}$, so $1 \le k$. Moreover, since the LHB on the x-coordinates of $S_{\ominus n}$ is less than the horizontal right hand bound (RHB), we know that $-\pi <$ RHB.

Case 2. $n = 4k+3$ for some $k \in \mathbb{Z}$. Then $S_{\ominus n}$ is on the right hand side of the origin. Therefore, we want to show that the horizontal RHB on $S_{\ominus n}$ is less than or equal to π, that is, that

$$\begin{aligned}\frac{2\pi}{3}(-1)^{\frac{n-3}{2}}\Big(1-\big(-\frac{1}{2}\big)^{\frac{n-1}{2}}\Big) \le \pi &\iff \frac{2}{3}(-1)^{2k}\Big(1-\big(-\frac{1}{2}\big)^{2k+1}\Big) \le 1\\ &\iff \frac{2}{3}\Big(1+\big(\frac{1}{2}\big)^{2k+1}\Big) \le 1\\ &\iff \Big(1+\big(\frac{1}{2}\big)^{2k+1}\Big) \le \frac{3}{2}\\ &\iff \Big(\frac{1}{2}\Big)^{2k} \le 1,\end{aligned}$$

which is trivially true since n is odd and $n \ge 3 \Rightarrow 3 \le 4k+3 \Rightarrow 0 \le k$. We know that in this case, LHB $\le$ RHB $\le \pi$, as needed. But then in all possible cases, it is true that $S_{\ominus n} \subseteq [-\pi,\pi) \times [-\pi,\pi)$, and therefore that $X_{\ominus n} \subseteq [-\pi,\pi) \times [-\pi,\pi)$.

Now we want to show that

$$X_{\ominus n} \nsubseteq \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big) \times \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big),$$

for all $n \ge 3$. Recall from our earlier discussion that this will follow from the fact that

$$S_{\ominus n} \nsubseteq \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big) \times \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big).$$

We can prove this fact by merely showing that the horizontal bounds on the set $S_{\ominus n}$ are not contained in the set $\big[-\frac{\pi}{2},\frac{\pi}{2}\big]$. Thus, the vertical bounds on the set $S_{\ominus n}$ are irrelevant.

Case 1. $n = 4k+1$ for some $k \in \mathbb{Z}$. Recall $S_{\ominus n}$ is on the left hand side of the origin, so we must show that the horizontal RHB on the x-coordinates of the set $S_{\ominus n}$ is less than or equal to $-\frac{\pi}{2}$.

$$\begin{aligned}-\frac{\pi}{2} \ge \frac{2\pi}{3}(-1)^{\frac{n-3}{2}}\left(1-\left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) &\iff -\frac{1}{2} \ge \frac{2}{3}(-1)^{2k-1}\left(1-\left(-\frac{1}{2}\right)^{2k}\right)\\ &\iff -\frac{1}{2} \ge \frac{2}{3}\left(\frac{1}{2^{2k}}-1\right)\\ &\iff \frac{1}{4} \ge \frac{1}{2^{2k}},\end{aligned}$$

which is true since $3 \le n = 4k+1 \Rightarrow \frac{1}{2} \le k$ and $k \in \mathbb{Z}$ (so $\Rightarrow 1 \le k$).

Case 2. $n = 4k+3$ for some $k \in \mathbb{Z}$. Then $S_{\ominus n}$ is on the right hand side of the origin, and therefore we want to show that the horizontal LHB on $S_{\ominus n}$ is greater than or equal to $\frac{\pi}{2}$.

$$\begin{aligned}
\frac{\pi}{2} \le -\frac{\pi}{2^{\frac{n+1}{2}}} + \frac{2\pi}{3}(-1)^{\frac{n-3}{2}}\left(1-\left(-\frac{1}{2}\right)^{\frac{n-1}{2}}\right) \\
\iff \frac{1}{2} \le -\frac{1}{2^{2k+2}} + \frac{2}{3}(-1)^{2k}\left(1-\left(-\frac{1}{2}\right)^{2k+1}\right) \\
\iff -\frac{1}{6} \le \frac{1}{2^{k+1}}\left(\frac{2}{3}-\frac{1}{2}\right) \\
\iff -1 \le \frac{1}{2^{k+1}},
\end{aligned}$$

which is trivially true since $0 < \frac{1}{2^{k+1}}\ \forall k$.

Thus, $\forall n$, the horizontal bounds of $S_{\ominus n}$ are not contained in the set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ but are contained in the set $[-\pi, \pi]$, and the vertical bounds are contained in the set $[-\pi, \pi]$. Thus $S_{\ominus n} \subseteq G_{SO}$ for all odd $n \ge 3$. Recall from our earlier discussion that it therefore follows that $X_{\ominus n} \subseteq G_{SO}$ for all odd $n \ge 3$, as needed. □

Analogously, for all odd $n \ge 3$, $X_{\oplus n} \subseteq G_{SO}$, $Y_{\ominus n} \subseteq G_{SO}$, and $Y_{\oplus n} \subseteq G_{SO}$.

Lemma 4.2. *$X_{\ominus n+4}$ and $X_{\ominus n}$ are disjoint for all $n > 1 \in \mathbb{Z}$.*

Proof. Let

$$S_{\ominus n} := \left[\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n}$$

represent the result of the n^{th} step of the crossover algorithm applied to

$$\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Since all $X_{\ominus} \subseteq \left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for all functions f,

$$f(X_{\ominus n}) \subseteq f\left(\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)\right),$$

and therefore $X_{\ominus n} \subseteq S_{\ominus n}$. Therefore, we will prove that all $S_{\ominus n}$ and $S_{\ominus(n+4)}$ are disjoint, from which our lemma follows immediately.

Consider odd n ($n = 2k+1$ for some $k \in \mathbb{Z}$). The horizontal LHB of $S_{\ominus(2k+1)}$ is

$$-\frac{\pi}{2^{k+1}} + \frac{2\pi}{3}(-1)^{k-1}\left(1-\left(-\frac{1}{2}\right)^{k}\right),$$

as follows from the rigorous definition of the set $X_{\ominus n}$, and the RHB of $S_{\ominus(2k+5)}$ is

$$\frac{2\pi}{3}(-1)^{k+1}\left(1-\left(-\frac{1}{2}\right)^{k+2}\right).$$

Therefore, the RHB of $S_{\ominus(2k+5)}$ equals the LHB of $S_{\ominus(2k+1)}$ if and only if

$$\begin{aligned}
-\frac{\pi}{2^{k+1}}+\frac{2\pi}{3}(-1)^{k-1}\left(1-\left(-\frac{1}{2}\right)^{k}\right)&=\frac{2\pi}{3}(-1)^{k+1}\left(1-\left(-\frac{1}{2}\right)^{k+2}\right)\\
\iff -\frac{1}{2^{k+1}}+\frac{2}{3}(-1)^{k-1}+\frac{2}{3}\frac{1}{(2)^k}&=\frac{2}{3}(-1)^{k+1}+\frac{2}{3}\frac{1}{(2)^{k+2}}\\
\iff -\frac{1}{2}\frac{1}{2^k}&=\frac{2}{3}\frac{1}{2^k}\left(\frac{1}{4}-1\right),
\end{aligned}$$

which is clearly true. So for all odd n, the LHB of the set $S_{\ominus n}$ is equivalent to the RHB of the set $S_{\ominus(n+4)}$. Recall that for n' even,

$$\tfrac{1}{2}X_{\ominus n'}=X_{\ominus n'+1},\qquad \text{which implies}\qquad \tfrac{1}{2}S_{\ominus n'}=S_{\ominus(n'+1)},$$

that is,

$$\tfrac{1}{2}S_{\ominus(n-1)}=S_{\ominus n}\qquad\text{and}\qquad \tfrac{1}{2}S_{\ominus(n+3)}=S_{\ominus(n+4)}.$$

Thus, the LHB of $S_{\ominus(n-1)}$ is equivalent to the RHB of the set $S_{\ominus(n+3)}$. But since $n-1\in\mathbb{Z}^+$ is even, both even and odd cases are satisfied. Thus, we can say that for all $n''>1\in\mathbb{Z}$, the LHB of the set $S_{\ominus n''}$ is equivalent to the RHB of the set $S_{\ominus(n''+4)}$. Nonetheless, these two sets are still "essentially disjoint" because their intersection has measure 0 using Lebesgue Measure. Therefore, by our earlier argument, our lemma follows. □

Lemma 4.3. *All $X_{\oplus n}$, $X_{\ominus n'}$, $Y_{\oplus n''}$, $Y_{\ominus n'''}$ are disjoint for all natural numbers n, n', n'', and n'''. Moreover, $X_{\ominus i}$ and $X_{\ominus j}$ are disjoint when $i\neq j$, with analogous properties following for sets of the form $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$.*

Proof. First we show that all $X_{\oplus n}$, $X_{\ominus n}$ are disjoint (a similar argument shows that all $Y_{\oplus n}$, $Y_{\ominus n}$ are disjoint). Consider the maximal case for $X_{\oplus 1}$ and $X_{\ominus 1}$, namely,

$$\begin{aligned}
X_{\oplus 1}&=\left[0,\frac{\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right),\\
X_{\ominus 1}&=\left[-\frac{\pi}{2},0\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right).
\end{aligned}$$

Because all other $X_{\oplus n}$, $X_{\ominus n}$ are copies of $X_{\oplus 1}$ and $X_{\ominus 1}$ that have been translated along the x-axis and scaled, we will consider only their x-coordinates. We note that because sets of the form $X_{\oplus n}$, $X_{\ominus n}$ are never scaled by factors α, for $|\alpha|>1$, they are all contained in $[-\infty,\infty)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right)$, that is, their vertical bounds are contained in the set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$. From the crossover algorithm, we have that the following hold for all m (except for $4m+1=1$) in the x-coordinate:

$$
\begin{aligned}
X_{\oplus 4m+1} &= \tfrac{1}{2} X_{\oplus 4m}, & Xm4m+1 &= \tfrac{1}{2} X_{\ominus 4m},\\
X_{\oplus 4m+2} &= X_{\oplus 4m+1} - 2\pi, & X_{\ominus 4m+2} &= X_{\ominus 4m+1} + 2\pi,\\
X_{\oplus 4m+3} &= \tfrac{1}{2} X_{\oplus 4m+2}, & X_{\ominus 4m+3} &= \tfrac{1}{2} X_{\ominus 4m+2},\\
X_{\oplus 4m+4} &= X_{\oplus 4m+3} + 2\pi, & X_{\ominus 4m+4} &= X_{\ominus 4m+3} - 2\pi,\\
\Rightarrow X_{\oplus 4(m+1)+1} &= \tfrac{1}{4} X_{\oplus 4m+1} + \tfrac{\pi}{2}, & \Rightarrow X_{\ominus 4(m+1)+1} &= \tfrac{1}{4} X_{\ominus 4m+1} - \tfrac{\pi}{2}.
\end{aligned}
$$

Solving the recurrence relation for $X_{\oplus 4m+1}$ and $X_{\ominus 4m+1}$, and using the solution to obtain the other cases, we obtain the following:

$$
\begin{aligned}
X_{\oplus 4m+1} &= \frac{1}{4^m} X_{\oplus 1} + \frac{2}{3}\left(1 - \frac{1}{4^m}\right)\pi, & X_{\ominus 4m+1} &= \frac{1}{4^m} X_{\ominus 1} + \frac{2}{3}\left(\frac{1}{4^m} - 1\right)\pi,\\
X_{\oplus 4m+2} &= \frac{1}{4^m} X_{\oplus 1} - \frac{2}{3}\left(2 + \frac{1}{4^m}\right)\pi, & X_{\ominus 4m+2} &= \frac{1}{4^m} X_{\ominus 1} + \frac{2}{3}\left(\frac{1}{4^m} + 2\right)\pi,\\
X_{\oplus 4m+3} &= \frac{1}{2}\frac{1}{4^m} X_{\oplus 1} - \frac{1}{3}\left(2 + \frac{1}{4^m}\right)\pi, & X_{\ominus 4m+3} &= \frac{1}{2}\frac{1}{4^m} X_{\ominus 1} + \frac{1}{3}\left(\frac{1}{4^m} + 2\right)\pi,\\
X_{\oplus 4m+4} &= \frac{1}{2}\frac{1}{4^m} X_{\oplus 1} + \frac{1}{3}\left(4 - \frac{1}{4^m}\right)\pi, & X_{\ominus 4m+4} &= \frac{1}{2}\frac{1}{4^m} X_{\ominus 1} + \frac{1}{3}\left(\frac{1}{4^m} - 4\right)\pi.
\end{aligned}
$$

Using our maximal $X_{\oplus 1}$ and $X_{\ominus 1}$, we find that

$$
\begin{aligned}
X_{\oplus 4m+1} &= \left[\left(\tfrac{2}{3} - \tfrac{2}{3}\left(\tfrac{1}{4}\right)^m\right)\pi, \left(\tfrac{2}{3} - \tfrac{1}{6}\left(\tfrac{1}{4}\right)^m\right)\pi\right) \subset \left[0, \tfrac{2}{3}\pi\right),\\
X_{\oplus 4m+2} &= \left[\left(-\tfrac{4}{3} - \tfrac{2}{3}\left(\tfrac{1}{4}\right)^m\right)\pi, \left(-\tfrac{4}{3} - \tfrac{1}{6}\left(\tfrac{1}{4}\right)^m\right)\pi\right) \subset \left[-2\pi, -\tfrac{4}{3}\pi\right),\\
X_{\oplus 4m+3} &= \left[\left(-\tfrac{2}{3} - \tfrac{1}{3}\left(\tfrac{1}{4}\right)^m\right)\pi, \left(-\tfrac{2}{3} - \tfrac{1}{12}\left(\tfrac{1}{4}\right)^m\right)\pi\right) \subset \left[-\pi, -\tfrac{2}{3}\pi\right),\\
X_{\oplus 4m+4} &= \left[\left(\tfrac{4}{3} - \tfrac{1}{3}\left(\tfrac{1}{4}\right)^m\right)\pi, \left(\tfrac{4}{3} - \tfrac{1}{12}\left(\tfrac{1}{4}\right)^m\right)\pi\right) \subset \left[\pi, \tfrac{4}{3}\pi\right),\\
X_{\ominus 4m+1} &= \left[\left(\tfrac{1}{6}\left(\tfrac{1}{4}\right)^m - \tfrac{2}{3}\right)\pi, \left(\tfrac{2}{3}\left(\tfrac{1}{4}\right)^m - \tfrac{2}{3}\right)\pi\right) \subset \left[-\tfrac{2}{3}\pi, 0\right),\\
X_{\ominus 4m+2} &= \left[\left(\tfrac{1}{6}\left(\tfrac{1}{4}\right)^m + \tfrac{4}{3}\right)\pi, \left(\tfrac{2}{3}\left(\tfrac{1}{4}\right)^m + \tfrac{4}{3}\right)\pi\right) \subset \left[\tfrac{4}{3}\pi, 2\pi\right),\\
X_{\ominus 4m+3} &= \left[\left(\tfrac{1}{12}\left(\tfrac{1}{4}\right)^m + \tfrac{2}{3}\right)\pi, \left(\tfrac{1}{3}\left(\tfrac{1}{4}\right)^m + \tfrac{2}{3}\right)\pi\right) \subset \left[\tfrac{2}{3}\pi, \pi\right),\\
X_{\ominus 4m+4} &= \left[\left(\tfrac{1}{12}\left(\tfrac{1}{4}\right)^m - \tfrac{4}{3}\right)\pi, \left(\tfrac{1}{3}\left(\tfrac{1}{4}\right)^m - \tfrac{4}{3}\right)\pi\right) \subset \left[-\tfrac{4}{3}\pi, -\pi\right).
\end{aligned}
$$

Trivially, we conclude that the eight different sets of intervals are disjoint. Within each set of intervals, note that both endpoints of the intervals either monotonically increase (for $X_{\oplus n}$) or monotonically decrease (for $X_{\ominus n}$) as n increases; we also find that the right endpoint of $X_{\oplus 4m+k}$ is equal to the left endpoint of $X_{\oplus 4(m+1)+k}$ for all possible values of k, and the left endpoint of $X_{\ominus 4m+k}$ is equal to the right endpoint of $X_{\ominus 4(m+1)+k}$ for all possible values of k (See the proof of Lemma 4.2.) Thus, all the $X_{\oplus 4m+k}$ and $X_{\ominus 4m+k}$ are disjoint for all k, meaning we have proved that all $X_{\oplus n}$, $X_{\ominus n}$ are disjoint. Moreover, all $X_{\ominus i}$ and $X_{\ominus j}$ and all $X_{\oplus i}$ and $X_{\ominus j}$ are disjoint when i does not equal j. Analogously, all $Y_{\ominus i}$ and $Y_{\ominus j}$ and all $Y_{\oplus i}$ and $Y_{\oplus j}$ are disjoint when i does not equal j.

To show that the sets of the form $X_{\oplus n}$, $X_{\ominus n}$, $Y_{\oplus n}$ and $Y_{\ominus n}$ are disjoint, consider the following: All the sets of the form $X_{\oplus n}$, $X_{\ominus n}$ are contained in the region

$$[-2\pi, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Similarly, all the sets of the form $Y_{\oplus n}$, $Y_{\ominus n}$ are contained the region

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [-2\pi, 2\pi).$$

The intersection between these two regions is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

but the only sets in this region are $X_{\oplus 1}$, $X_{\ominus 1}$, $Y_{\oplus 1}$, and $Y_{\ominus 1}$, and by definition, these are disjoint, completing the proof. □

Remark 4.1. The proof of Lemma 4.3 shows that all sets of the form $X_{\oplus n}$, $X_{\ominus n}$ for odd $n \geq 3$ are in the area

$$[-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \in G_{SO}.$$

Exclusion from the inner square is due to the fact that $X_{\oplus 1}$, $X_{\ominus 1}$, $Y_{\oplus 1}$, and $Y_{\ominus 1}$ occupy that space, and all $X_{\ominus i}$ and $X_{\ominus j}$ are disjoint if $i \neq j$ (with analogous results for $X_{\oplus}$, $Y_{\ominus}$, and $Y_{\oplus}$), implying no other sets of the form $X_{\ominus n}$, $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$ can occupy that space, providing a short proof of Lemma 4.1.

We can now prove our main result.

Proof of Theorem 2.1. Let W be defined as in Example 1 to be

$$\Big(\bigcup_{i=1}^{\infty}\big[X_{\ominus 2i}\cup X_{\oplus 2i}\cup Y_{\ominus 2i}\cup Y_{\oplus 2i}\big]\Big)$$
$$\cup\Big(G_{TO}\backslash\Big[\bigcup_{i=1}^{\infty}\big[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}\cup Y_{\ominus 2i-1}\cup Y_{\oplus 2i-1}\big]\Big]\Big)$$
$$=\Big(\bigcup_{i=1}^{\infty}\big[X_{\ominus 2i}\cup X_{\oplus 2i}\cup Y_{\ominus 2i}\cup Y_{\oplus 2i}\big]\Big)$$
$$\cup\Big(G_{SO}\backslash\Big[\bigcup_{i=2}^{\infty}[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}\cup Y_{\ominus 2i-1}\cup Y_{\oplus 2i-1}]\Big]\Big).$$

Part I. We will first show that W is dilation congruent to G_{SO}. Let

$$\Phi_{X_\ominus}:\{X_{\ominus i}: i \text{ is even} \geq 2\}\to\{X_{\ominus j}: j \text{ is odd} \geq 3\},$$

be such that for all even n,

$$\Phi_{X_\ominus}(X_{\ominus n}):=\tfrac{1}{2}X_{\ominus n}=X_{\ominus n+1}\in G_{SO},$$

from Lemma 4.1 and the discussion on the top of Section 3 of our paper.

Claim 1. $\Phi_{X_\ominus}$ is surjective. Take an arbitrary $X_{\ominus j}$ such that $j\geq 3$ is odd. Then $\Phi_{X_\ominus}(X_{\ominus j-1})=X_{\ominus j}$.

Claim 2. $\Phi_{X_\ominus}$ is injective. Suppose

$$\Phi_{X_\ominus}(X_{\ominus j})=\Phi_{X_\ominus}(X_{\ominus i}),$$

for some even i and j. Then $X_{\ominus j+1}=X_{\ominus i+1}$, and therefore

$$X_{\ominus j}=2X_{\ominus j+1}=2X_{\ominus i+1}=X_{\ominus i},$$

as needed.

Therefore, $\Phi_{X_\ominus}$ is a bijection. Similarly, define $\Phi_{X_\oplus}$, $\Phi_{Y_\ominus}$ and $\Phi_{Y_\oplus}$, which are all bijections by analogous arguments. But then

$$\Phi_{X_\ominus}\Big(\bigcup_{i=1}^{\infty}X_{\ominus 2i}\Big)=\bigcup_{i=1}^{\infty}\Phi_{X_\ominus}(X_{\ominus 2i})=\bigcup_{i=1}^{\infty}X_{\ominus 2i+1}\in G_{SO},$$

by Lemma 4.1, all of the white spaces ($X_{\ominus k}$ for $k\geq 3$ and odd) in G_{SO} are filled because $\Phi_{X_\ominus}$ is surjective, and all of the black pieces ($X_{\ominus n}$ for n even) have been mapped into G_{SO} injectively so that no two distinct black pieces map to the same white space. Analogous properties follow for $\Phi_{X_\oplus}$, $\Phi_{Y_\ominus}$, and $\Phi_{Y_\oplus}$.

Let

$$\Phi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even} \geq 2\} \to \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd} \geq 3\}$$

be such that

$$\begin{aligned}\Phi\left(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}\right) &= \Phi_{X_\ominus}\left(X_{\ominus i}\right) \cup \Phi_{X_\oplus}\left(X_{\oplus i}\right) \cup \Phi_{Y_\ominus}\left(Y_{\ominus i}\right) \cup \Phi_{Y_\oplus}\left(Y_{\oplus i}\right) \\ &= \left(X_{\ominus i+1} \cup X_{\oplus i+1} \cup Y_{\ominus i+1} \cup Y_{\oplus i+1}\right).\end{aligned}$$

Then

$$\begin{aligned}&\Phi\Big(\bigcup_{i=1}^{\infty}\big[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\big]\Big) \\ &\qquad = \bigcup_{i=1}^{\infty}\big[\Phi_{X_\ominus}\left(X_{\ominus 2i}\right) \cup \Phi_{X_\oplus}\left(X_{\oplus 2i}\right) \cup \Phi_{Y_\ominus}\left(Y_{\ominus 2i}\right) \cup \Phi_{Y_\oplus}\left(Y_{\oplus 2i}\right)\big] \\ &\qquad = \bigcup_{i=1}^{\infty}\big[X_{\ominus 2i+1} \cup X_{\oplus 2i+1} \cup Y_{\ominus 2i+1} \cup Y_{\oplus 2i+1}\big] \in G_{SO}.\end{aligned}$$

Φ is clearly a bijection, so that Φ maps all exterior black pieces into all interior white pieces such that no distinct black pieces map to the same white piece.

Thus,

$$\begin{aligned}&\Phi\Big(\bigcup_{i=1}^{\infty}\big[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\big]\Big) \\ &\qquad \bigcup\Big[G_{SO} \setminus \Big(\bigcup_{i=2}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\Big)\Big] = G_{SO},\end{aligned}$$

as needed. Therefore, W is dilation congruent to G_{SO}.

Part II. We will now prove that W is translation congruent to G_{TO}. Let

$$\Psi_{X_\ominus} : \{X_{\ominus i} : i \text{ is even} \geq 2\} \to \{X_{\ominus j} : j \text{ is odd} \geq 1\}$$

be such that for all even n,

$$\Psi_{X_\ominus}(X_{\ominus n}) := X_{\ominus n} - (-1)^{\frac{n-2}{2}}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{1}{2}X_{\ominus(n-2)} = X_{\ominus n-1} \in G_{TO},$$

using the discussion on the top of page 70, Lemma 4.1, and the fact that

$$X_{\ominus 1} \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big) \times \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

Claim 1. $\Psi_{X_\ominus}$ is surjective. Take an arbitrary $X_{\ominus j}$ such that $j \geq 1$ is odd. Then $\Phi_{X_\ominus}(X_{\ominus j+1}) = X_{\ominus j}$.

Claim 2. $\Psi_{X_\ominus}$ is injective. Suppose $\Psi_{X_\ominus}(X_{\ominus j}) = \Psi_{X_\ominus}(X_{\ominus i})$, for some even i and j. Then $X_{\ominus j-1} = X_{\ominus i-1}$. Suppose that $(j-1) \neq (i-1)$. Then by Lemma 4.3, $X_{\ominus j-1} \cap X_{\ominus i-1} = \varnothing$, which contradicts $X_{\ominus j-1} = X_{\ominus i-1}$. Thus it must be true that $(j-1) = (i-1)$ and thus $X_{\ominus i} = X_{\ominus j}$, as needed.

Therefore, $\Psi_{X_\ominus}$ is a bijection. But then

$$\Psi_{X_\ominus}\Big(\bigcup_{i=1}^{\infty} X_{\ominus 2i}\Big) = \bigcup_{i=1}^{\infty} \Psi_{X_\ominus}\left(X_{\ominus 2i}\right) = \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}.$$

Therefore, all blank spaces in G_{TO} of the form $X_{\ominus n}$ are filled when $\Psi_{X_\ominus}$ acts on $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$ since $\Psi_{X_\ominus}$ is onto. Moreover, all black pieces of the form $X_{\ominus n}$ are contained in the set $\bigcup_{i=1}^{\infty} X_{\ominus 2i}$, and therefore have been mapped into G_{TO}. Since $\Psi_{X_\ominus}$ is injective, no two distinct black pieces will map to the same white piece.

Similarly, define $\Psi_{X_\oplus}$, $\Psi_{Y_\ominus}$, and $\Psi_{Y_\oplus}$, which are all bijections by analogous arguments.

Define

$$\Psi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \to \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd} \geq 1\},$$

to be such that

$$\begin{aligned}\Psi\left(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}\right) &= \Psi_{X_\ominus}\left(X_{\ominus i}\right) \cup \Psi_{X_\oplus}\left(X_{\oplus i}\right) \cup \Psi_{Y_\ominus}\left(Y_{\ominus i}\right) \cup \Psi_{Y_\oplus}\left(Y_{\oplus i}\right) \\ &= \left[X_{\ominus i-1} \cup X_{\oplus i-1} \cup Y_{\ominus i-1} \cup Y_{\oplus i-1}\right] \in G_{TO}.\end{aligned}$$

Ψ is clearly a bijection, and therefore when Ψ acts on the entire domain, that is,

$$\begin{aligned}\Psi\Big(\bigcup_{i=1}^{\infty}\left[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\right]\Big) &= \bigcup_{i=1}^{\infty} \Psi\Big(X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\Big) \\ &= \bigcup_{i=1}^{\infty}\left[X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}\right] \in G_{TO},\end{aligned}$$

every white space (of the form $X_{\ominus n}$ for odd n) in G_{TO} is filled by some black piece (of the form $X_{\ominus n'}$ for some even n') from the exterior of G_{TO} since Ψ is surjective. No two distinct black pieces map to the same white piece since Ψ is injective. Moreover, every black piece outside G_{TO} is contained in the domain of Ψ, and therefore every black piece outside G_{TO} is mapped into G_{TO}. Thus,

$$\begin{aligned}&\Psi\Big(\bigcup_{i=1}^{\infty}\left[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}\right]\Big) \\ &\qquad \bigcup\Big[G_{TO} \backslash \Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\Big)\Big] = G_{TO},\end{aligned}$$

and thus by definition, W is dilation congruent modulo 2π to G_{TO}. By definition W is a wavelet set. □

A different example of a partition of the inner square conforming to the requirements of Theorem 2.1 is shown in Figure 2 with the resulting wavelet set.

5. Patch wavelet sets

All of the wavelet sets we have considered thus far are crossover wavelet sets. In this class, regions are added to or subtracted from alternating sides of the inner square. Alternatively, we could add or subtract regions to the same side of the square for each translation. Such wavelet sets are called patch wavelet sets. To illustrate the patch algorithm, we give an example. The reader will note that this example is actually a well known wavelet set: the *wedding cake* wavelet set (Figure 3); see [Dai and Larson 1998, Example 6.6.1] and also [Dai et al. 1998].

Patch Example 1. Let

$$X_\ominus = \left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad X_\oplus = \left[0, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$Y_\ominus = \varnothing, \qquad Y_\oplus = \varnothing.$$

Consider the piece $X_\ominus$. Start by translating $X_\ominus$ by $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$ (keeping it on the same side of the origin) to obtain $X_{\ominus 2}$. We find that

$$X_{\ominus 2} = \left[-\frac{5\pi}{2}, -2\pi\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Secondly, scale $X_{\ominus 2}$ by $\frac{1}{4}$ to obtain

$$X_{\ominus 3} = \left[-\frac{5\pi}{8}, -\frac{\pi}{2}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right).$$

Thirdly, translate $X_{\ominus 3}$ in the same direction as that of the first translation (that is, by $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$) to obtain

$$X_{\ominus 4} = \left[-\frac{21\pi}{8}, -\frac{5\pi}{2}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right).$$

Finally, scale $X_{\ominus 4}$ by $\frac{1}{4}$ to form the set

$$X_{\ominus 5} = \left[-\frac{21\pi}{32}, -\frac{5\pi}{32}\right) \times \left[-\frac{\pi}{32}, \frac{\pi}{32}\right).$$

Continue these two steps inductively for $X_\ominus$.

We perform two similar steps on the set $X_\oplus$ inductively as well; however, we translate by $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ (instead of $\begin{bmatrix} -2\pi \\ 0 \end{bmatrix}$). We obtain the following as a result from the first four steps of the patch algorithm:

$$X_{\oplus 2} = \left[2\pi, \frac{5\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad X_{\oplus 3} = \left[\frac{\pi}{2}, \frac{5\pi}{8}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right),$$

$$X_{\oplus 4} = \left[\frac{5\pi}{2}, \frac{21\pi}{8}\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right), \qquad X_{\oplus 5} = \left[\frac{5\pi}{32}, \frac{21\pi}{32}\right) \times \left[-\frac{\pi}{32}, \frac{\pi}{32}\right).$$

Continue this process inductively for $X_\oplus$ as well. In theory, we would perform similar steps for $Y_\oplus$ and $Y_\ominus$, but in this example both are the null set, and thus we have no computations to carry out for the sets $Y_\ominus$ and $Y_\oplus$.

Let W' be the set

$$\begin{aligned}&\Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i} \cup X_{\oplus 2i}]\Big) \cup \Big(G_{TO} \setminus \Big[\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1}]\Big]\Big)\\ &\qquad = \Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i} \cup X_{\oplus 2i}]\Big) \cup \Big(G_{SO} \setminus \Big[\bigcup_{i=2}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1}]\Big]\Big),\end{aligned}$$

see Figure 3. Similarly to the crossover case, we can think of the set W' as being the union of G_{TO} combined with the sets on the exterior of G_{TO} of the form $X_{\oplus n}$, $X_{\ominus n}$ where n is even and with subsets of G_{TO} of the form $X_{\oplus n}$, $X_{\ominus n}$ where n is odd erased from G_{TO}. The reader should check that this set W' is indeed a wavelet set.

This algorithm can be generalized as follows:

(i) Partition the inner square into a maximum of four pieces. The conditions on this partition are identical to those on the partition of the inner square using the crossover algorithm as given in Theorem 2.1, and the proof for the case of the patch algorithm is similar to the proof given for the crossover algorithm.

(ii) Translate one piece of the partition by $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$ so that the piece is translated out of the inner square and onto the half of the plane in which the original piece of the partition previously lay.

(iii) Dilate the set formed in step 2 into G_{SO} by $\frac{1}{4}$.

(iv) Translate the set formed in step 3 out of G_{SO} in the same direction as the translation in step 2 (that is, by $\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$).

(v) Dilate the set formed in step 4 into G_{SO} by $\frac{1}{4}$.

(vi) Repeat steps 2 and 3 inductively for this piece of the partition, and perform the same steps inductively on the other three pieces of the partition of the inner square.

Theorem 5.1 (Patch Algorithm). *Let $\{X_\ominus, X_\oplus, Y_\ominus, Y_\oplus\}$ be a partition of the set*

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

such that $X_\ominus$ is contained in the left half of the inner square, $X_\oplus$ is contained in the right half of the inner square, $Y_\ominus$ is contained in the bottom half of the inner square, and $Y_\oplus$ is contained in the top half of the inner square. Then the set W, defined as

$$\left[\left(\bigcup_{i=1}^{\infty}[X_{\ominus 2i} \cup X_{\oplus 2i} \cup Y_{\ominus 2i} \cup Y_{\oplus 2i}]\right) \cup G_{TO}\right] \setminus \left[\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1} \cup X_{\oplus 2i-1} \cup Y_{\ominus 2i-1} \cup Y_{\oplus 2i-1}]\right],$$

generated by this partition under translation by

$$\begin{bmatrix} \pm 2\pi \\ 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 \\ \pm 2\pi \end{bmatrix}$$

and dilation by powers of 2 using steps (i)–(vi) above, is a dyadic wavelet set in $\mathbb{R}^2$.

Proof. Begin by showing the following for natural numbers n odd and n' even:

$$X_{\ominus n+2} = \frac{1}{4}\left(X_{\ominus n} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right), \qquad X_{\ominus n'+2} = \frac{1}{4}X_{\ominus n'} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix},$$

$$\frac{X_{\ominus n'}}{4} = X_{\ominus n'+1}, \qquad X_{\ominus n'} = X_{\ominus n'-1} - \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}.$$

First, we solve the recurrence relation for n odd, and use this and the fact that $\frac{X_{\ominus n'}}{4} = X_{\ominus n'+1}$ to obtain a form for n' odd. From this point forward let n be an arbitrary odd or even natural number. We find that

$$X_{\ominus n} = \begin{cases} \frac{X_\ominus}{4^{\frac{n-1}{2}}} - \frac{1}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\left(1 - (\frac{1}{4})^{\frac{n-1}{2}}\right), & \text{for } n \text{ odd} \\ \frac{X_\ominus}{4^{\frac{n'+1}{2}}} - \frac{4}{3}\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\left(1 - \left(\frac{1}{4}\right)^{\frac{n'-1}{2}}\right), & \text{for } n \text{ even.} \end{cases}$$

We derive similar expressions for $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$.

An analogous property to that of Lemma 4.1 can be seen for the patch algorithm. Once again, we use the maximal possible $X_{\ominus n}$, that is,

$$S_{\ominus n} := \left[\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]_{\ominus n},$$

the result of the n^{th} step of the patch algorithm applied to

$$\left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We use our derived bounds for $X_{\ominus n}$ in terms of n to determine the bounds for $S_{\ominus n}$. We begin by showing that $S_{\ominus n} \subseteq [-\pi, \pi) \times [-\pi, \pi)$. That this is satisfied for the vertical bounds of $S_{\ominus n}$ is clear, so we will only consider the horizontal bounds. There is only one case to consider for the patch algorithm, the case that $n = 2k + 1$ for some nonnegative integer k. (The patch algorithm requires only one case because the algorithm always translates the odd pieces out to the same side of the inner square rather than to alternating sides, as in the crossover algorithm, leading to two cases for the crossover algorithm.) Second, show that

$$X_{\ominus n} \nsubseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

for all $n \geq 3$, by showing that

$$S_{\ominus n} \nsubseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

This follows from the fact that the horizontal bounds on the set $S_{\ominus n}$ are not contained in the set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, the vertical bounds on the set $S_{\ominus n}$ are irrelevant. Once again, here we find that there is only one case to consider (the case that $n = 2k + 1$ for some nonnegative integer k). We conclude that $S_{\ominus n} \subseteq G_{SO}$ for all odd $n \geq 3$, and therefore that $X_{\ominus n} \subseteq G_{SO}$ for all odd $n \geq 3$. Analogously, for all odd $n \geq 3$, $X_{\oplus n} \subseteq G_{SO}$, $Y_{\ominus n} \subseteq G_{SO}$, and $Y_{\oplus n} \subseteq G_{SO}$.

An analogous property is true for the patch case to Lemma 4.2 for the crossover algorithm, that $X_{\ominus n+2}$ and $X_{\ominus n}$ are disjoint for all $n > 0 \in \mathbb{Z}$. We modify the argument that was used for the crossover case by showing that the left hand bound of $S_{\ominus 2k+1}$ equals the right hand bound of $S_{\ominus 2k+3}$.

Next, an analogous property is true for the patch case to that of Lemma 4.3 for crossover sets, namely, that all $X_{\oplus n}, X_{\ominus n'}, Y_{\oplus n''}, Y_{\ominus n'''}$ are disjoint for all natural numbers n, n', n'', and n'''. Moreover, $X_{\ominus i}$ and $X_{\ominus j}$ are disjoint when $i \neq j$, with analogous properties following for sets of the form $X_{\oplus n}$, $Y_{\ominus n}$, and $Y_{\oplus n}$.

First we show that all $X_{\oplus n}, X_{\ominus n}$ are disjoint. Once again, consider the maximal case for $X_{\oplus 1}$ and $X_{\ominus 1}$. Because all other $X_{\oplus n}, X_{\ominus n}$ are copies of $X_{\oplus 1}$ and $X_{\ominus 1}$ that have been translated along the x-axis and scaled, consider only the x-coordinates.

Because sets of the form $X_{\oplus n}$, $X_{\ominus n}$ are never scaled by factors α, for $|\alpha| > 1$, they are all contained in

$$[-\infty, \infty) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

From the Patch Algorithm, observe that, where $2m{+}1{>}1$, the following hold for all m in the x-coordinate:

$$\begin{aligned}
& X_{\ominus 2m+1} = \tfrac{1}{4} X_{\ominus 2m}, \\
& X_{\ominus 2m+2} = X_{\ominus 2m+1} - 2\pi, \\
\Rightarrow\ & X_{\ominus 2(m+1)+1} = \tfrac{1}{16} X_{\ominus 2m} - \tfrac{\pi}{2}, \\
& X_{\oplus 2m+1} = \tfrac{1}{4} X_{\oplus 2m}, \\
& X_{\oplus 2m+2} = X_{\oplus 2m+1} + 2\pi, \\
\Rightarrow\ & X_{\oplus 2(m+1)+1} = \tfrac{1}{16} X_{\oplus 2m} + \tfrac{\pi}{2}.
\end{aligned}$$

Solving these recurrence relations, we find a collection of disjoint sets, each of which contains one of the following as a subset: $X_{\ominus 2m+1}$, $X_{\ominus 2m+2}$, $X_{\oplus 2m+1}$, and $X_{\oplus 2m+2}$. Trivially, we conclude that the four different sets of intervals are disjoint. Within each set of intervals, note that both endpoints of the intervals either monotonically increase (for $X_{\oplus n}$) or monotonically decrease (for $X_{\ominus n}$) as n increases. Recall from our argument for the property similar to Lemma 4.2 (but for the patch case) that the left hand bound of $S_{\ominus 2k+1}$ equals the right hand bound of $S_{\ominus 2k+3}$. We will also find that the right hand bound of $S_{\ominus 2k+1}$ equals the left hand bound of $S_{\ominus 2k+3}$. Thereby we conclude that all $X_{\oplus n}$, $X_{\ominus n}$ are disjoint along with all $X_{\ominus i}$, $X_{\ominus j}$ and all $X_{\oplus i}$, $X_{\oplus j}$ when $i \neq j$. Analogously, all $Y_{\oplus n}$ and $Y_{\ominus n}$ are disjoint along with all $Y_{\ominus i}$ and $Y_{\ominus j}$ and all $Y_{\oplus i}$ and $Y_{\oplus j}$ when $i \neq j$.

To show that the sets of the form $X_{\oplus n}$, $X_{\ominus n}$, $Y_{\oplus n}$ and $Y_{\ominus n}$ are disjoint, consider the following: All the sets of the form $X_{\oplus n}$, $X_{\ominus n}$ are contained in the region

$$[-\infty, \infty) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Similarly, all the sets of the form $Y_{\oplus n}$, $Y_{\ominus n}$ are contained the region

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [-\infty, \infty).$$

The intersection between these two regions is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

but the only sets in this region are $X_{\oplus 1}$, $X_{\ominus 1}$, $Y_{\oplus 1}$, and $Y_{\ominus 1}$, and by definition, these are disjoint.

Define the set W in the same way it was defined in the proof of the theorem for the crossover case. To show that W is dilation congruent to G_{SO}, define the bijection

$$\Phi_{X_\ominus} : \{X_{\ominus i} : i \text{ is even} \geq 2\} \to \{X_{\ominus j} : j \text{ is odd} \geq 3\},$$

such that for all even n,

$$\Phi_{X_\ominus}(X_{\ominus n}) := \frac{1}{4} X_{\ominus n} = X_{\ominus n+1} \in G_{SO}.$$

Observe that

$$\Phi_{X_\ominus}\Big(\bigcup_{i=1}^{\infty} X_{\ominus 2i}\Big) = \bigcup_{i=1}^{\infty} \Phi_{X_\ominus}\Big(X_{\ominus 2i}\Big) = \bigcup_{i=1}^{\infty} X_{\ominus 2i+1} \in G_{SO},$$

using the property analogous to Lemma 4.1 Lemma 1 but applied to the patch case. All of the white spaces ($X_{\ominus k}$ for $k \geq 3$ and odd) in G_{SO} are filled, and all of the black pieces ($X_{\ominus n}$ for n even) have been mapped into G_{SO} injectively.

Similarly, define the bijections $\Phi_{X_\oplus}$, $\Phi_{Y_\ominus}$, and $\Phi_{Y_\oplus}$. Analogous properties follow for $\Phi_{X_\oplus}$, $\Phi_{Y_\ominus}$, and $\Phi_{Y_\oplus}$. Let

$$\Phi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even} \geq 2\} \\ \to \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd} \geq 3\}$$

be such that

$$\begin{aligned}\Phi\left(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}\right) &= \Phi_{X_\ominus}\left(X_{\ominus i}\right) \cup \Phi_{X_\oplus}\left(X_{\oplus i}\right) \cup \Phi_{Y_\ominus}\left(Y_{\ominus i}\right) \cup \Phi_{Y_\oplus}\left(Y_{\oplus i}\right) \\ &= \left(X_{\ominus i+1} \cup X_{\oplus i+1} \cup Y_{\ominus i+1} \cup Y_{\oplus i+1}\right).\end{aligned}$$

Using Φ, we show that W is dilation congruent to G_{SO}.

To show that W is translation congruent to G_{TO}, let

$$\Psi_{X_\ominus} : \{X_{\ominus i} : i \text{ is even} \geq 2\} \to \{X_{\ominus j} : j \text{ is odd} \geq 1\}$$

be such that for all even n,

$$\Psi_{X_\ominus}(X_{\ominus n}) := X_{\ominus n} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} = \frac{1}{2} X_{\ominus (n-2)} = X_{\ominus n-1} \in G_{TO}.$$

$\Psi_{X_\ominus}$ is a bijection. Observe that

$$\Psi_{X_\ominus}\Big(\bigcup_{i=1}^{\infty} X_{\ominus 2i}\Big) = \bigcup_{i=1}^{\infty} \Psi_{X_\ominus}\Big(X_{\ominus 2i}\Big) = \bigcup_{i=1}^{\infty} X_{\ominus 2i-1}.$$

Therefore, all blank spaces in G_{TO} of the form $X_{\ominus n}$ are filled when $\Psi_{X_\ominus}$ acts on

$$\bigcup_{i=1}^{\infty} X_{\ominus 2i},$$

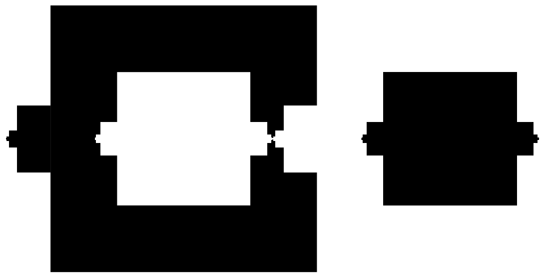

Figure 4. A wavelet set which has characteristics of both patch and crossover wavelet sets.

since $\Psi_{X_\ominus}$ is onto. Moreover, all black pieces of the form $X_{\ominus n}$ are contained in the set

$$\bigcup_{i=1}^{\infty} X_{\ominus 2i},$$

and therefore have been mapped into G_{TO}. Define similarly $\Psi_{X_\oplus}$, $\Psi_{Y_\ominus}$, and $\Psi_{Y_\oplus}$ which are all bijections by analogous arguments.

Define the bijection

$$\Psi : \{X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i} : i \text{ is even } \geq 2\} \to \{X_{\ominus j} \cup X_{\oplus j} \cup Y_{\ominus j} \cup Y_{\oplus j} : j \text{ is odd} \geq 1\}$$

to be such that

$$\begin{aligned}\Psi\,(X_{\ominus i} \cup X_{\oplus i} \cup Y_{\ominus i} \cup Y_{\oplus i}) &= \Psi_{X_\ominus}\,(X_{\ominus i}) \cup \Psi_{X_\oplus}\,(X_{\oplus i}) \cup \Psi_{Y_\ominus}\,(Y_{\ominus i}) \cup \Psi_{Y_\oplus}\,(Y_{\oplus i}) \\ &= \left[X_{\ominus i-1} \cup X_{\oplus i-1} \cup Y_{\ominus i-1} \cup Y_{\oplus i-1}\right] \in G_{TO}.\end{aligned}$$

Using Φ, we show W is dilation congruent modulo 2π to G_{TO}. We conclude now that W is a wavelet set. □

6. Concluding remarks

In Figure 4, we partition the inner square in the following way:

$$X_\ominus = \left[-\frac{\pi}{2}, 0\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad X_\oplus = \left[0, \frac{\pi}{2}\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$Y_\ominus = \varnothing, \qquad Y_\oplus = \varnothing.$$

To the piece $X_\ominus$ we apply the crossover algorithm. We obtain the following:

$$X_{\ominus 2} = \left[\frac{3\pi}{2}, 2\pi\right) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad X_{\ominus 3} = \left[\frac{3\pi}{4}, \pi\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right),$$
$$X_{\ominus 4} = \left[-\frac{5\pi}{4}, -2\pi\right) \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right), \qquad X_{\ominus 5} = \left[-\frac{5\pi}{8}, -\pi\right) \times \left[-\frac{\pi}{8}, \frac{\pi}{8}\right).$$

To the piece $X_{\oplus}$, we apply the patch algorithm and obtain the following as a result from the first four steps of the algorithm:

$$X_{\oplus 2}=\left[2\pi,\frac{5\pi}{2}\right)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right),\qquad X_{\oplus 3}=\left[\frac{\pi}{2},\frac{5\pi}{8}\right)\times\left[-\frac{\pi}{8},\frac{\pi}{8}\right),$$
$$X_{\oplus 4}=\left[\frac{5\pi}{2},\frac{21\pi}{8}\right)\times\left[-\frac{\pi}{8},\frac{\pi}{8}\right),\qquad X_{\oplus 5}=\left[\frac{5\pi}{32},\frac{21\pi}{32}\right)\times\left[-\frac{\pi}{32},\frac{\pi}{32}\right).$$

We continue application of the patch algorithm to the piece $X_{\oplus}$ and application of the crossover algorithm to the piece $X_{\ominus}$ inductively. Once again we let W be the set

$$\begin{aligned}&\Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i}\cup X_{\oplus 2i}]\Big)\cup\Big(G_{TO}\backslash\Big[\bigcup_{i=1}^{\infty}[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}]\Big]\Big)\\ &\qquad=\Big(\bigcup_{i=1}^{\infty}[X_{\ominus 2i}\cup X_{\oplus 2i}]\Big)\cup\Big(G_{SO}\backslash\Big[\bigcup_{i=2}^{\infty}[X_{\ominus 2i-1}\cup X_{\oplus 2i-1}]\Big]\Big),\end{aligned}$$

where $X_{\ominus n}$ is defined according to our definition for a set of this form operated on by the crossover algorithm (see page 65), and $X_{\oplus n}$ is defined according to our definition given for a set of this form operated on by the Patch Algorithm.

This set W (see Figure 4) is a wavelet set. To see this, let

$$G(X_{\ominus\text{odd}}):=\bigcup_{i=1}^{\infty}X_{\ominus 2i-1},\qquad\text{and}\qquad G(X_{\ominus\text{even}}):=\bigcup_{i=1}^{\infty}X_{\ominus 2i}.$$

Similarly, define sets for $X_{\oplus}$, $Y_{\ominus}$, $Y_{\oplus}$ with analogous characteristics. Observe that W is translation congruent to G_{TO} modulo 2π because

$$\begin{aligned}\bigcup_{i=1}^{\infty}X_{\ominus 4i}+\begin{bmatrix}2\pi\\0\end{bmatrix}&=\bigcup_{i=0}^{\infty}X_{\ominus 4i+3},\\ \bigcup_{i=0}^{\infty}X_{\ominus 4i+2}-\begin{bmatrix}2\pi\\0\end{bmatrix}&=\bigcup_{i=0}^{\infty}X_{\ominus 4i+1},\\ \bigcup_{i=1}^{\infty}X_{\oplus 2i}-\begin{bmatrix}2\pi\\0\end{bmatrix}&=\bigcup_{i=0}^{\infty}X_{\oplus 2i+1}.\end{aligned}$$

Notice

$$\bigcup_{i=0}^{\infty}X_{\oplus 2i+1}\cup\bigcup_{i=0}^{\infty}X_{\ominus 4i+1}\cup\bigcup_{i=0}^{\infty}X_{\ominus 4i+3}=G(X_{\ominus\text{odd}})\cup G(X_{\oplus\text{odd}}),$$

and thus we observe that all of the white spaces in the set G_{TO} are filled when we translate the black sets on the exterior of G_{TO} by multiples of $\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$. Moreover, W is dilation congruent to G_{SO} because

$$\tfrac{1}{2}G(X_{\ominus\text{even}}) = G(X_{\ominus\text{odd}}) \in G_{SO},$$

that is, the even pieces of the form $X_{\ominus n}$ scale into the odd pieces of the form $X_{\ominus n}$, and

$$\tfrac{1}{4}G(X_{\oplus\text{even}}) = G(X_{\oplus\text{odd}}).$$

Thus, W is a wavelet set by definition.

Thanks to this example, we see that a wavelet set may demonstrate characteristics of both patch and crossover wavelet sets, and thereby not be classified as either type. The set contains both a patch region and a crossover region. Therefore, we have not made a complete classification of all two dimensional wavelet sets, but note that crossover wavelet sets seem to be maximally nonpatch. Finding a broader algorithm which encompasses both the patch and crossover algorithms would be an interesting problem to consider.

As a final comment, we remark that crossover and patch wavelet sets make perfect sense in one-dimension (that is, in $\mathbb{R}^1$). The reader can easily prove that all dyadic one-dimensional wavelet sets of two or three intervals are necessarily crossover wavelet sets. (Here crossover would mean through the origin.) On the other hand, the well known Journe wavelet set of 4 intervals (see [Dai and Larson 1998], Example 4.5 (i)), is easily seen to be a patch wavelet set. A characterization is not known at this time of all finite interval patch wavelet sets.

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