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An asymptotic for the representation of integers as sums of triangular numbers

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Motivated by the result of Rankin for representations of integers as sums of squares, we use a decomposition of a modular form into a particular Eisenstein series and a cusp form to show that the number of ways of representing a positive integer n as the sum of k triangular numbers is asymptotically equivalent to the modified divisor function $\sigma^{\sharp}_{2k-1}(2n+k)$.

1. Introduction

1A. *General problem.* We wish to study $\delta_k(n)$, the number of ways to write n as the sum of k triangular numbers. This problem dates back to Gauss, who discovered that every nonnegative integer can be represented as a sum of three triangular numbers. The basic problem is similar to questions about representations of integers as sums of squares, and some of the basic techniques for attacking that problem carry over. We define the function

$$\Theta(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

so that

$$\Theta^k(q) = \sum_{n\geq 0} r_k(n) q^n$$

where $r_k(n)$ is the number of representations of n as the sum of k squares. It was exploited in [Rankin 1965] the fact that $\Theta(1)$ is a modular form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ to study the functions $r_k(n)$. Ono, Robins, and Wahl [1995] defined an analogous modular form to study triangular numbers.

We begin by defining triangular numbers.

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Definition 1.1. The *n-th triangular number ($n \geq 0$)* is

$$T_n := \frac{n(n+1)}{2}.$$

These numbers may be geometrically interpreted as the number of dots in a grid with the shape of an equilateral triangle of side length n. We also introduce the generating functions

$$\Psi(q) := \sum_{n=0}^{\infty} q^{T_n} = 1 + q + q^3 + q^6 + \cdots$$

and

$$\Psi^k(q) = \sum_{n=0}^{\infty} \delta_k(n) q^n.$$

1B. ***Modular group and congruence subgroups.*** Before we formally define modular forms, we need to define the *modular group* and its subgroups.

Definition 1.2. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The *modular group* Γ is

$$\mathrm{SL}_2(\mathbb{Z}) = \{A \mid a, b, c, d \in \mathbb{Z} \text{ and } \det A = \pm 1\}.$$

It is well known that Γ is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 1.3. The *congruence subgroups of level $N \in \mathbb{N}$* are defined as follows:

(1) $\Gamma_0(N) := \{A \in \Gamma \mid c \equiv 0 \bmod N\}$;

(2) $\Gamma_1(N) := \{A \in \Gamma \mid c \equiv 0 \bmod N \text{ and } a \equiv d \equiv 1 \bmod N\}$;

(3) $\Gamma(N) := \{A \in \Gamma \mid c \equiv b \equiv 0 \bmod N \text{ and } a \equiv d \equiv 0 \bmod N\}$.

It is easy to check that they are, in fact, subgroups.

It is clear that for every level $N \in \mathbb{N}$, $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \leq \Gamma$. More precisely, the following identities hold [Koblitz 1993, p. 231]:

$$[\Gamma : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$

$$[\Gamma : \Gamma_1(N)] = N^2 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right),$$

$$[\Gamma : \Gamma(N)] = N^3 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right).$$

We will make use of $\Gamma_0(4)$, which is generated by T and $ST^{-4}S$. In particular, $[\Gamma : \Gamma_0(4)] = 6$, with coset representatives

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S^{-1}T^{-2}S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad ST^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad ST^3 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

Any group of 2×2 matrices gives rise to an action on the complex plane, namely the linear fractional transformation

$$Az := \frac{az+b}{cz+d},$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. In particular, Γ and its subgroups act on the upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

Considering the geometric meaning of orbits and equivalence classes under this action on $\mathcal{H}$ leads to the idea of a fundamental domain. This is a subset of $\mathcal{H}$ which possesses both convenient topological and geometric properties and is also algebraically related to some Γ or one of its subgroups.

Definition 1.4. A closed, simply connected region F in $\mathcal{H}$ is called a *fundamental domain* for a subgroup Γ' of Γ if every point in the plane is equivalent under Γ' to a point in F and no two points in the interior of F are equivalent under Γ'.

For example, a fundamental domain for Γ is the set

$$R_\Gamma = \{z \in \mathbb{C} \mid -\tfrac{1}{2} \le \operatorname{Re}(z) \le \tfrac{1}{2}, |z| \ge 1\}.$$

Figure 1 shows this fundamental domain, as well as the fundamental domain for $\Gamma_0(4)$

For the sake of consistency, we will use $R_{\Gamma'}$ to denote a fundamental domain for Γ'. The following lemma provides an algorithm to compute $R_{\Gamma'}$ using R_Γ, and coset representation of Γ' in Γ.

Lemma 1.5. *Let* $\Gamma' \le \Gamma$ *be of finite index* n *in* Γ. *If* $\Gamma = \bigcup_{i=1}^n \gamma_i \Gamma'$ *is its coset representation*, *then*

$$R_{\Gamma'} = \bigcup_{i=1}^n \gamma_i^{-1} R_\Gamma.$$

Proof. This is proved in [Koblitz 1993, p.105]. □

Definition 1.6. Let $\Gamma' \le \Gamma$, and fix a fundamental domain $R_{\Gamma'}$. The points where $R_{\Gamma'}$ intersects the boundary $\partial\mathcal{H} = \{i\infty\} \cup \mathbb{R}$ are called the *cusps of* Γ'.

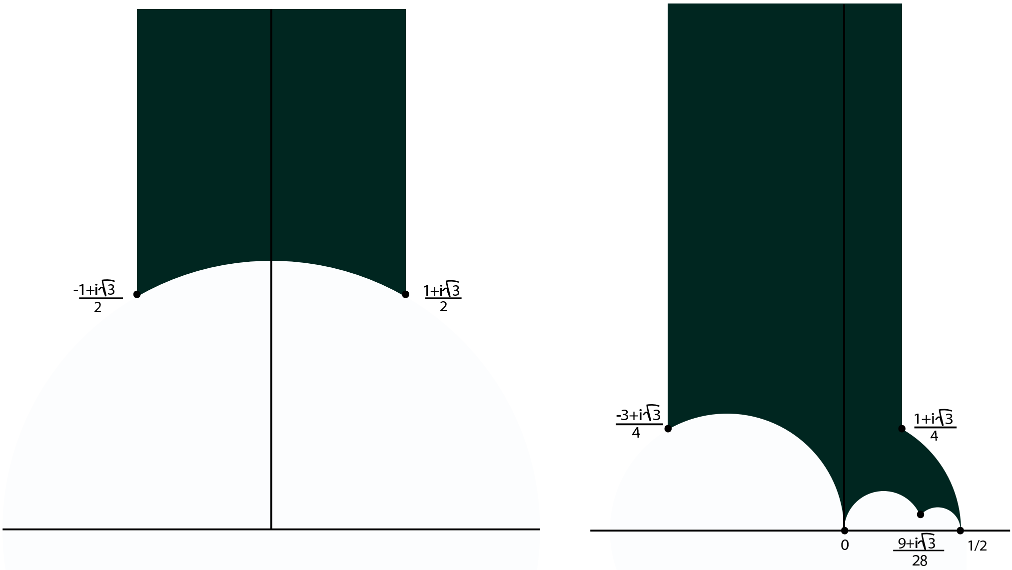

Figure 1. The fundamental domains for Γ (left) and for $\Gamma_0(4)$ (right).

The full modular group has a single cusp at $i\infty$. From the fundamental domain for $\Gamma_0(4)$ shown in Figure 1, we see that $\Gamma_0(4)$ has three cusps, namely 0, $\frac{1}{2}$ and $i\infty$.

1C. ***Modular forms.*** Modular forms are holomorphic functions on $\mathcal{H}$ which have nice symmetry properties under the action of Γ or one of its subgroups. Specifically, we say

Definition 1.7. $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight* $k \in \mathbb{N}$ *over* $\Gamma_0(N)$ if

(i) f is holomorphic on $\mathcal{H}$;

(ii) f is holomorphic at the cusps of $\Gamma_0(N)$;

(iii) for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, the equation $f(Az) = (cz+d)^k f(z)$ holds for all $z \in \mathcal{H}$.

Definition 1.8. A modular form f over Γ' is called a *cusp form* if it vanishes at all cusps of Γ'.

If $T \in \Gamma'$, it follows that a modular form over Γ' always has period 1. In other words, $f(z) = f(z+1)$ for all $z \in \mathcal{H}$. Therefore f has a *Fourier expansion* (also called q*-expansion*) in $q = e^{2\pi i z}$:

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n.$$

Modular forms of a given congruence subgroup and of fixed weight form a vector space. This structure can be of great help when trying to study a particular modular form. Using a suitable basis, we can decompose elements of a given space of modular forms in terms of basis vectors. This technique often produces expressions that are easy to work with.

Definition 1.9. The vector space of modular forms of weight k over the congruence subgroup Γ' of Γ is denoted $M_k(\Gamma')$. The subspace of cusp forms is denoted $S_k(\Gamma')$.

For all $\Gamma' \leq \Gamma$ which contain $-I$, both $M_{2k+1}(\Gamma')$ and $S_{2k+1}(\Gamma')$ are trivial. This follows by applying the transformation $-I$ to a modular form $f(z)$, implying

$$f(z) = (-1)^{2k+1} f(z) = -f(z),$$

and hence $f(z) = 0$ for all $z \in \mathcal{H}$. Thus, we may consider only modular forms of even weight. Since $\Gamma_0(4)$ will be important in our work, we state without proof a characterization of its spaces of even-weight modular forms. Recall the definition of $\Theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$.

Definition 1.10. Let $F(z)$ be the following modular form of weight 2 over $\Gamma_0(4)$:

$$F(z) = \sum_{n=1}^{\infty} \sigma_1(2n+1) q^{2n+1}.$$

Lemma 1.11. *$M_{2k}(\Gamma_0(4))$ is a $(k+1)$-dimensional vector space with basis*

$$\{F^k, F^{k-1}\Theta^4, \ldots, F\Theta^{4(k-1)}, \Theta^{2k}\}.$$

Furthermore, $S_{2k}(\Gamma_0(4))$ consists of all polynomials divisible by

$$\Theta^4 F(\Theta^4 - 16F) = \eta^{12}(2z).$$

Therefore, there exists an isomorphism between $S_{2k}(\Gamma_0(4))$ and $M_{2k-6}(\Gamma_0(4))$.

Proof. This is Exercise III.3.17 in [Koblitz 1993], proved on pp. 235–6. □

1D. *Representations as sums of triangular numbers.* In our study of $\delta_k(n)$, we focus on the expressions for $\delta_{4k}(n)$. The generating function $q^k\Psi^{4k}(q^2)$ is in $M_{2k}(\Gamma_0(4))$, which is a well-understood space of small dimension. By decomposing elements of $M_{2k}(\Gamma_0(4))$ for some k into basis vectors, it is possible to find identities between $q^k\Psi^{4k}(q^2)$ and other forms in the same space with accessible coefficients. This is the method used by Ono et al. [1995] for $\delta_k(n)$, $k = 2, 3, 4, 6, 8, 10, 12$ and 24. Their results for $\delta_{4k}(n)$ are summarized in the following lemma.

Lemma 1.12. *For $n \geq 0$,*

$$\begin{aligned}
\delta_4(n) &= \sigma_1(2n+1),\\
\delta_8(n) &= -\tfrac{1}{8}\sigma_3^{\sharp}(n+1),\\
\delta_{12}(n) &= \tfrac{1}{256}(\sigma_5(2n+3) - a(2n+3)), \ \textit{and}\\
\delta_{24}(n) &= \frac{1}{17689}\Big(\sigma_{11}^{\sharp}(n+3) - \tau(n+3) - 2072\,\tau\Big(\frac{n+3}{2}\Big)\Big),
\end{aligned}$$

where $a(n)$ is defined by $\eta^{12}(2z) = \sum_{n=1}^{\infty} a(n)q^n$ and $\tau(n)$ is the n-th Fourier coefficient of $\Delta(z) = (2\pi)^{12}\eta^{24}(z)$.

Looking at the case δ_4, this lemma states that

$$q\Psi^4(q^2) = \sum_{n=0}^{\infty} \delta_4(n)q^{2n+1} = F(z),$$

where F is as defined previously, so we have the following useful corollary:

Corollary 1.13.

$$q^k\Psi^{4k}(q^2) = F^k.$$

2. δ_{4k} as an Eisenstein series plus a cusp form

The generating function $\Theta^k(z)$ for $r_k(n)$ can be decomposed into a cusp form and a particular Eisenstein series. In the same vein as the work by Rankin [1965] on $\Theta^k(z)$, we would like to similarly decompose $q^k\Psi^{4k}(q^2)$.

Definition 2.1. Let $k \in \mathbb{N}$. Then let H_{2k} be the Eisenstein series of weight $2k$ on $\Gamma_0(4)$ defined by

$$H_{2k}(z) = \begin{cases} \displaystyle\sum_{\substack{n>0\\ n \text{ odd}}}^{\infty} \sigma_{2k-1}^{\sharp}(n)q^n & \text{if } k \text{ is odd},\\ \displaystyle\sum_{\substack{n>0\\ n \text{ even}}}^{\infty} \sigma_{2k-1}^{\sharp}(n)q^n & \text{if } k \text{ is even}. \end{cases}$$

Definition 2.2. We define the *partial zeta function $\zeta^i(s)$ for i modulo N* to be

$$\zeta^i(s) := \sum_{n \equiv i \mod N} \frac{1}{n^s}.$$

Proposition 2.3. *For a given congruence subgroup $\Gamma_0(N)$, let $G_k^{(a_1,a_2)}(z)$ be the Eisenstein series*

$$\sum_{\substack{m_1 \equiv a_1 (N)\\ m_2 \equiv a_2 (N)}} \frac{1}{(m_1 z + m_2)^k},$$

and let $B_{2k} \in \mathbb{Q}$ be the $2k$-th Bernoulli number. Then

(1) *as a modular form for $\Gamma_0(2)$,*

$$\sum_{n=1}^{\infty} \sigma^{\sharp}_{2k-1}(n) q^n = -\frac{B_{2k}}{8k\zeta(2k)} \big(G^{(1,0)}_{2k}(z) + G^{(1,1)}_{2k}(z) \big);$$

(2) *as a modular form for $\Gamma_0(4)$,*

$$\sum_{n=1}^{\infty} \sigma^{\sharp}_{2k-1}(n) q^n = -\frac{2^{2k} B_{2k}}{8k\zeta(2k)} \big(G^{(2,0)}_{2k}(z) + G^{(2,2)}_{2k}(z) \big).$$

Proof. Koblitz [1993, p. 133] shows that for $\Gamma_0(N)$,

$$G^{(a_1,a_2)}_k(z) = b^{(a_1,a_2)}_0 + \frac{(-1)^{k-1} 2k\zeta(k)}{N^k B_k} \cdot \Bigg(\sum_{\substack{m_1 \equiv a_1 \bmod N \\ m_1 > 0}} \sum_{j=1}^{\infty} j^{k-1} \xi^{j a_2} q_N^{j m_1} + (-1)^k \sum_{\substack{m_1 \equiv -a_1 \bmod N \\ m_1 > 0}} \sum_{j=1}^{\infty} j^{k-1} \xi^{-j a_2} q_N^{j m_1} \Bigg),$$

where

$$\xi := e^{2\pi i/N}, \ q_N := e^{2\pi i z/N}, \ b^{(a_1,a_2)}_0 = \begin{cases} 0 & \text{if } a_1 \neq 0, \\ \zeta^{a_1}(k) + (-1)^k \zeta^{-a_2}(k) & \text{if } a_1 = 0. \end{cases}$$

We can collect terms with $jm_1 = n$ to find explicit expansions of some particular $G^{(a_1,a_2)}_k(z)$. From the above expression, we have two assertions:

(i)
$$G^{(1,0)}_{2k}(z) = 2c_{2k} \sum_{n=1}^{\infty} \Bigg(\sum_{j \mid n, n/j \text{ odd}} j^{2k-1} \Bigg) q_2^n,$$

$$G^{(1,1)}_{2k}(z) = 2c_{2k} \sum_{n=1}^{\infty} \Bigg(\sum_{j \mid n, n/j \text{ odd}} j^{2k-1} (-1)^j \Bigg) q_2^n.$$

Adding these two series, we get

$$G^{(1,0)}_{2k}(z) + G^{(1,1)}_{2k}(z) = 2^{2k+1} c_{2k} \sum_{n=1}^{\infty} \sigma^{\sharp}_{2k-1}(n) q^n = -\frac{8k\zeta(2k)}{B_{2k}} \sum_{n=1}^{\infty} \sigma^{\sharp}_{2k-1}(n) q^n,$$

which is the first assertion.

(ii) The second assertion follows similarly, except that $c_{2k} = -\dfrac{4k\zeta(2k)}{2^{4k} B_{2k}}$ for the Eisenstein series of $\Gamma_0(4)$. □

If we take the first identity from Proposition 2.3 and substitute in $2z$, we obtain

$$\sum_{n=1}^{\infty} \sigma^{\sharp}_{2k-1}(n) q^{2n} = -\frac{B_{2k}}{8k\zeta(2k)} \big(G^{(1,0)}_{2k}(2z) + G^{(1,1)}_{2k}(2z) \big).$$

This is a modular form for $\Gamma_0(4)$. By the definition of $\sigma^{\sharp}_{2k-1}(n)$ and the definition of $G_k^{(a_1,a_2)}(z)$,

$$\sum_{n=1}^{\infty}\sigma^{\sharp}_{2k-1}(2n)q^{2n} = -\frac{2^{2k}B_{2k}}{16\zeta(2k)}\big(G_{2k}^{(2,0)}(z)+G_{2k}^{(2,1)}(z)+G_{2k}^{(2,2)}(z)+G_{2k}^{(2,3)}(z)\big). \quad (2\text{-}1)$$

We have proven

Corollary 2.4. $\sum_{\substack{n>0\\ \text{even}}}\sigma^{\sharp}_{2k-1}(n)q^n$ *and* $\sum_{\substack{n>0\\ \text{odd}}}\sigma^{\sharp}_{2k-1}(n)q^n$ *are both modular forms for* $\Gamma_0(4)$. *The former is given by Equation (2-1), and the latter is equal to*

$$-\frac{2^{2k}B_{2k}}{16\zeta(2k)}\big(G_{2k}^{(2,0)}(z)-G_{2k}^{(2,1)}(z)+G_{2k}^{(2,2)}(z)-G_{2k}^{(2,3)}(z)\big).$$

We can now compute the desired values at the cusps. From [Koblitz 1993], we have

$$G_{2k}^{(2,i)}(z) = -\frac{4k\zeta(2k)}{4^{2k}B_{2k}}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{j\,|\,n\\ n/j\equiv i\,(4)}} j^{2k-1}+\sum_{\substack{j\,|\,n\\ n/j\equiv -i\,(4)}} j^{2k-1}\Bigg)q_4^n$$

so it follows that $\sum_{\substack{n>0\\ \text{even}}}\sigma^{\sharp}_{2k-1}(n)q^n$ and $\sum_{\substack{n>0\\ \text{odd}}}\sigma^{\sharp}_{2k-1}(n)q^n$ are both 0 at $i\infty$.

To find the values at the cusp 0, we use the transformation S. We have

$$G_{2k}^{(2,0)}(z)|[S]_{2k} = G^{(0,2)}(z);\quad G_{2k}^{(2,1)}(z)|[S]_{2k} = G^{(1,2)}(z);$$
$$G_{2k}^{(2,2)}(z)|[S]_{2k} = G^{(2,2)}(z);\quad G_{2k}^{(2,3)}(z)|[S]_{2k} = G^{(3,2)}(z).$$

Additionally, $G^{(1,2)}(i\infty)=G^{(2,2)}(i\infty)=G^{(3,2)}(i\infty)=0$ (from [Koblitz 1993] again) and

$$G^{(0,2)}(i\infty) = 2\zeta^2(2k) = 2\sum_{\substack{n>0\\ n\equiv 2(4)}}\frac{1}{n^{2k}} = 2\Big(\frac{1}{2^{2k}}-\frac{1}{2^{4k}}\Big)\zeta(2k).$$

Hence, $\sum_{\substack{n>0\\ \text{even}}}\sigma^{\sharp}_{2k-1}(n)q^n$ and $\sum_{\substack{n>0\\ \text{odd}}}\sigma^{\sharp}_{2k-1}(n)q^n$ both equal

$$-\frac{B_{2k}}{8k}\Big(1-\frac{1}{4^k}\Big)$$

at 0.

To find the values at the cusp $\frac{1}{2}$, we use the transformation $ST^{-2}S$. We have

$$G_{2k}^{(2,0)}(z)|[ST^{-2}S]_{2k} = G^{(2,0)}(z); \quad G_{2k}^{(2,1)}(z)|[ST^{-2}S]_{2k} = G^{(0,1)}(z);$$
$$G_{2k}^{(2,2)}(z)|[ST^{-2}S]_{2k} = G^{(2,2)}(z); \quad G_{2k}^{(2,3)}(z)|[ST^{-2}S]_{2k} = G^{(0,3)}(z).$$

We know $G_{2k}^{(2,0)}(i\infty) = G_{2k}^{(2,2)}(i\infty) = 0$, and

$$G_{2k}^{(0,1)}(i\infty) = G_{2k}^{(0,3)}(i\infty) = \zeta^1(2k) + \zeta^3(2k) = \sum_{\substack{n>0 \\ n \text{ odd}}} \frac{1}{n^{2k}} = \Big(1 - \frac{1}{2^{2k}}\Big)\zeta(2k).$$

Hence, at the cusp $\frac{1}{2}$,

$$\sum_{n>0,\text{even}} \sigma_{2k-1}^{\sharp}(n)q^n = -\frac{4^k B_{2k}}{8k}\Big(1 - \frac{1}{4^k}\Big),$$
$$\sum_{n>0,\text{odd}} \sigma_{2k-1}^{\sharp}(n)q^n = \frac{4^k B_{2k}}{8k}\Big(1 - \frac{1}{4^k}\Big).$$

Theorem 2.5. *Let $k \in \mathbb{N}$. Then*

$$q^k \Psi^{4k}(q^2) = \frac{1}{d_k}(H_{2k}(z) - T_{2k}(z)), \tag{2-2}$$

where

$$d_k = -\frac{(-16)^k B_{2k}(4^k - 1)}{8k} \in \mathbb{Q}$$

and $T_{2k}(z) \in S_{2k}(\Gamma_0(4))$.

Proof. We know that $F^k(z)$ is 0 at $i\infty$, $\left(-\frac{1}{64}\right)^k$ at 0, and $\left(\frac{1}{16}\right)^k$ at $\frac{1}{2}$, as is $\frac{1}{d_k}H_{2k}(z)$. Hence,

$$q^k \Psi^{4k}(q^2) - \frac{1}{d_k}H_{2k}(z)$$

is a cusp form. □

Corollary 2.6.

$$\delta_{4k}(n) = \frac{1}{d_k}(\sigma_{2k-1}^{\sharp}(2n+k) - a(2n+k)), \tag{2-3}$$

where

$$T_{2k}(z) = \sum_n a(n)q^n \in S_{2k}(\Gamma_0(4)).$$

Proof. This follows from equating the coefficients of the Fourier series in (2-2). □

k	$\sum a(n)q^n$
1	0
2	0
3	$\Theta^8 F - 16\Theta^4 F^2$
4	$2^7(\Theta^8 F^2 - 16\Theta^4 F^3)$
5	$\Theta^{16} F - 32\Theta^{12} F^2 + 19968\Theta^8 F^3 - 3159\Theta^4 F^4$
6	$2^{11}(\Theta^{16} F^2 - 32\Theta^{12} F^3 + 2328\Theta^8 F^4 - 33152\Theta^4 F^5)$
7	$\Theta^{24} F - 48\Theta^{20} F^2 + 1595136\Theta^{16} F^3$ $-510238722\Theta^{12} F^4 + 1660747776\Theta^8 F^5 - 20041433088\Theta^4 F^6$
8	$2^{15}\big(\Theta^{24} F^2 - 48\Theta^{20} F^3 + 33576\Theta^{16} F^4$ $-1053952\Theta^{12} F^5 + 23271936\Theta^8 F^6 - 237969408\Theta^4 F^7\big)$

Table 1

Corollary 2.7. $\delta_{4k}(n) \sim \sigma^{\sharp}_{2k-1}(2n+k)$.

Proof. The cusp form coefficients in (2-3) $a(2n+k) \in O(n^k)$ [Apostol 1990]. The $\sigma^{\sharp}_{2k-1}(2n+k)$ term has lower bound n^{2k-1}, and thus this term is asymptotically dominant. Therefore

$$\lim_{n\to\infty} \frac{\delta_{4k}(n)}{\sigma^{\sharp}_{2k-1}(2n+k)} = 1\,. \qquad \square$$

For particular k, we can compute the value of c_{2k}, and then, by equating finitely many coefficients, compute the remaining cusp form $\sum a(n)q^n$ as a homogeneous polynomial in F and Θ^4. We list the result of this computation for several values of k in Table 1.

We can rewrite (2-3) using

$$\sigma^{\sharp}_k(n) = \begin{cases} \sigma_k(n) & \text{if } n \text{ is odd,} \\ 2^k \sigma^{\sharp}_k(\frac{n}{2}) & \text{if } n \text{ is even,} \end{cases}$$

and the values in Table 1. The resulting formulae for $k = 1$, 2, 3, and 6 agree with those in Lemma 1.12.

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References

[Apostol 1990] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, 2nd ed., Graduate Texts in Mathematics **41**, Springer, New York, 1990. MR 90j:11001 Zbl 0697.10023

[Koblitz 1993] N. Koblitz, *Introduction to elliptic curves and modular forms*, 2nd ed., Graduate Texts in Mathematics **97**, Springer, New York, 1993. MR 94a:11078 Zbl 0804.11039

[Ono et al. 1995] K. Ono, S. Robins, and P. T. Wahl, "On the representation of integers as sums of triangular numbers", *Aequationes Math.* **50**:1-2 (1995), 73–94. MR 96i:11044 Zbl 0828.11057

[Rankin 1965] R. A. Rankin, "Sums of squares and cusp forms", *Amer. J. Math.* **87** (1965), 857–860. MR 32 #5605 Zbl 0132.30802

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