involve
a journal of mathematics

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mathematical sciences publishers

2008 Vol. 1, No. 2

INVOLVE 1:2(2008)

On graphs for which every planar immersion lifts to a knotted spatial embedding

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(Communicated by Ann Trenk)

We call a graph G intrinsically linkable if there is a way to assign over/under information to any planar immersion of G such that the associated spatial embedding contains a pair of nonsplittably linked cycles. We define intrinsically knottable graphs analogously. We show there exist intrinsically linkable graphs that are not intrinsically linked. (Recall a graph is intrinsically linked if it contains a pair of nonsplittably linked cycles in every spatial embedding.) We also show there are intrinsically knottable graphs that are not intrinsically knotted. In addition, we demonstrate that the property of being intrinsically linkable (knottable) is not preserved by vertex expansion.

1. Introduction

We start with a quick review of some definitions. A *graph* G consists of a finite nonempty set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of (usually distinct) vertices, called *edges*. If $x = (u, v) \in E(G)$, for $u, v \in V(G)$, we say that u and v are *adjacent* vertices, and that vertex u and edge x are incident with each other, as are v and x.

A *walk* in a graph G is an alternating sequence of vertices and edges

$$v_0, x_1, v_1, \dots, v_{n-1}, x_n, v_n$$

beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A *cycle* is a walk with $n \geq 2$ vertices and with all vertices distinct except $v_0 = v_n$. We say such a cycle has *length* n.

MSC2000: 57M25, 57M15.

Keywords: spatially embedded graph, intrinsically linked, intrinsically knotted, regular projection.

These results were obtained during an NSF (DMS 0353050) and NSA-sponsored Summer Research Experience for Undergraduates. The second author was faculty advisor, and the other authors were student participants.

Let G be a graph with

$$V(G)=\{v_1,v_2,\dots,v_n\} \qquad \text{and} \qquad E(G)=\{x_1,x_2,\dots,x_m\}.$$

A *spatial embedding* of G is a map f of G to a subspace $G(M)$ of $\mathbb{R}^3$ such that

$$G(M)=\left(\bigcup_{i=1}^{n} v_i(M)\right)\cup\left(\bigcup_{j=1}^{m} x_j(M)\right),$$

where

(i) $v_1(M), v_2(M),\dots,v_n(M)$ are n distinct points of $\mathbb{R}^3$ with $f(v_i)=v_i(M)$;

(ii) $x_1(M),\dots,x_m(M)$ are m mutually disjoint open arcs in $\mathbb{R}^3$ with

$$f(x_i)=x_i(M);$$

(iii) $x_j(M)\cap v_i(M)=\varnothing$, $i=1,\dots,n$, $j=1,\dots,m$;

(iv) if $x_j=(v_{j_1},v_{j_2})$, then the open arc $x_j(M)$ has $v_{j_1}(M)$ and $v_{j_2}(M)$ as end points for $j=1,\dots,m$.

In the above definition, an *arc* in $\mathbb{R}^3$ is a homeomorphic image of [0, 1]; an *open arc* is an arc less its two end points, the images of 0 and 1. More informally, a spatial embedding is a way to place a given graph in space.

We define a *planar immersion* of a graph G similar to a spatial embedding of G, except the codomain is $\mathbb{R}^2$ instead of $\mathbb{R}^3$, and we allow the image of edges of G to intersect, though we require that no three edges can intersect at the same point and we require the image of our edges to intersect transversely (they intersect locally in only one point, and they are not tangent to each other). We will assume that all embeddings and immersions are *tame*, that is, can be approximated by a finite collection of line segments. We will often simply use the term *immersion* instead of planar immersion. We use $\hat{G}$ to denote the image of an immersion of G under the map $\hat{f}$. If H is a subgraph of G, we similarly denote by $\hat{H}$ the image of H under $\hat{f}$.

Given an immersion $\hat{f}$ of a graph G with image $\hat{G}$, one can, by assigning over/under information to its double points, lift the immersion into 3-space, thereby creating a well-defined spatial embedding $\tilde{f}$ with image $\tilde{G}$. If π is the standard projection $\pi(x,y,z)=(x,y)$, and $\hat{f}=\pi\circ\tilde{f}$, we have the commutative diagram

$$\begin{tikzcd} & \tilde{G} \arrow[d, "\pi"] \\ G \arrow[ur, "\tilde{f}"] \arrow[r, "\hat{f}"'] & \hat{G} \end{tikzcd}$$

If there exists a lift of the immersion $\hat{f}$ whose image contains a pair of nonsplittably linked cycles (in other words, cannot be deformed to have a planar projection with no crossings between strands from two different components), then we say the immersion is *linkable*. We define the graph G to be *intrinsically linkable* if every immersion of G is linkable. We define *knottable* and *intrinsically knottable* analogously.

The study of intrinsically linkable graphs was inspired by two different ideas: intrinsically linked graphs, and graphs with a knot inevitable projection. The property of having a knot inevitable projection was introduced by Taniyama [1995] and studied by others (for example, Sugiura and Suzuki [2000], and Tamura [2004]). A (planar) graph has a *knot inevitable projection* if there exists a regular projection (that is, a planar immersion) of the graph such that every choice of over/under-crossings induces a spatial embedding that is knotted (in other words, cannot be deformed to a spatial embedding that has a planar projection without crossings).

The first results concerning intrinsically linked graphs were written up by Conway and Gordon [1983], and by Sachs [1983], who independently showed that every spatial embedding of K_6 (the graph on 6 vertices that contains all 15 possible edges between vertices) contains a pair of disjoint cycles that form a nonsplittable link, that is, K_6 is *intrinsically linked*. (See [Adams 2004] for a good background on knot theory in general, and on intrinsically linked and knotted graphs in particular.)

Conway and Gordon [1983] also showed that every spatial embedding of K_7 contains a cycle that forms a nontrivial knot, that is, K_7 is *intrinsically knotted*. Robertson et al. [1995] later showed that the collection of minor-minimal intrinsically linked graphs is exactly the *Petersen family*, that is, the seven graphs obtainable from the classic Petersen graph by repeated Δ-Y and Y-Δ exchanges. No one has yet classified the minor-minimal intrinsically knotted graphs, though they are known to be finite in number [Robertson and Seymour 2004].

Recall that a graph H is a *minor* of a graph G if H can be obtained from G by a sequence of deletions and/or contractions of edges and/or deletions of vertices. A graph G is *minor minimal* with respect to a given property if it has the property, but no minor of G has the property. Let a, b, and c be vertices of a graph G such that edges (a, b), (a, c), and (b, c) exist. Then a *Δ-Y exchange* on a triangle (a, b, c) of graph G is as follows. Vertex v is added to G, edges (a, b), (a, c), and (b, c) are deleted, and edges (a, v), (b, v), and (c, v) are added. A Y-Δ exchange is the reverse operation.

Clearly, an intrinsically linked (knotted) graph is also intrinsically linkable (knottable), but the converse is not true. In this paper, we present several intrinsically linkable graphs, each of which is a proper minor of some graph in the Petersen family (and hence not intrinsically linked), and several intrinsically knottable graphs, which are all in the Petersen family (and not intrinsically knotted).

Recall that a *vertex expansion* of a vertex v in a graph G is achieved by replacing v with two vertices v' and v'', adding the edge (v', v'') and connecting a subset of the edges that were incident to v to v' and the rest of the edges that were incident to v to v''. A graph G is considered to be an *expansion* of a graph H if G can be obtained by vertex expansions of H. It is well known that vertex expansions preserve intrinsic linking and intrinsic knotting; see [Nešetřil and Thomas 1985; Fellows and Langston 1988]. We demonstrate several intrinsically linkable (knottable) graphs for which vertex expansion destroys intrinsic linkability (knottability). We thus conjecture that vertex expansion preserves intrinsic linkability (knottability) only for those graphs that are intrinsically linked (knotted).

2. Intrinsically linkable graphs

We start this section with a quick introduction to the linking number. Recall that given a link of two components, L_1 and L_2 (two disjoint circles embedded in space), one computes the linking number of the link by examining a projection (with over and under-crossing information) of the link. Choose an orientation for each component of the link. At each crossing between two components, one of the pictures in Figure 1 will hold. We count $+1$ for each crossing of the first type (where you can rotate the over-strand counterclockwise to line up with the under-strand) and -1 for each crossing of the second type. To get the linking number, $lk(L_1, L_2)$, take the sum of $+1$'s and -1's and divide by 2. One can show that the absolute value of the linking number is independent of projection, and of chosen orientations (see [Adams 2004] for further explanation). Note that if $lk(L_1, L_2) \neq 0$, then the associated link is nonsplit. The converse does not hold. That is, there are nonsplit links with linking number 0 (the Whitehead link is a famous example, see again [Adams 2004]).

Lemma 2.1. *Let a graph G consist of two disjoint cycles A and B. A planar immersion $\hat{f}$ of G is linkable if and only if $\hat{A}$ and $\hat{B}$ intersect.*

Proof. Suppose there is a planar immersion $\hat{f}$ with disjoint cycles $\hat{A}$ and $\hat{B}$ that intersect. We will construct from $\hat{f}$ a spatial embedding $\tilde{f}$ in which the linking number $lk(\tilde{A}, \tilde{B})$ is nonzero. Arbitrarily choose orientations for $\hat{A}$ and $\hat{B}$, and then choose each crossing in $\hat{G}$ to be positive. It is assumed that $\hat{A}$ and $\hat{B}$ intersect, so there exists at least one crossing between them. We now have an induced spatial embedding $\tilde{f}$ in which $lk(\tilde{A}, \tilde{B}) > 0$.

The other implication is trivial to prove. □

Here, we provide a sufficient condition for a graph to be intrinsically linkable:

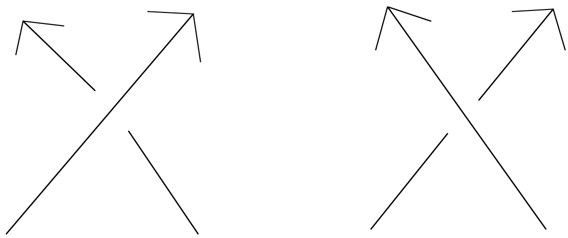

Figure 1. Computing the linking number.

Lemma 2.2. *A graph G is intrinsically linkable if it contains a nonplanar subgraph H such that for any pair $\{e_1, e_2\}$ of nonadjacent edges in H, e_1 and e_2 belong to disjoint cycles in G.*

Proof. Let G be any graph that satisfies the above condition and let $\hat{f}$ be any immersion of G. Since H is nonplanar, there exists in $\hat{H}$ at least one pair $\{e_1, e_2\}$ of nonadjacent edges that intersect. By hypothesis there are disjoint cycles, C_1 and C_2, that contain e_1 and e_2 respectively. Since $\hat{C}_1$ and $\hat{C}_2$ intersect, by Lemma 2.1 $\hat{f}$ is linkable. □

Remark (A remark on notation). We use the notation $G - e_{m,n}$ to denote the subgraph of G obtained by removing an edge connecting a vertex of degree m to a vertex of degree n. This notation is used only when the edge classes of G are uniquely determined by the degree of the incident vertices. If no subscript is present on e, then all edges of G belong to the same class. (Recall that the *degree* of a vertex is the number of edges incident to that vertex.)

We denote the graph in the Petersen family obtained from K_6 by a single Δ-Y exchange by P_7, and we denote the graph in the Petersen family obtained from $K_{3,3,1}$ by a single Δ-Y exchange by P_8. Finally, we denote the graph in the Petersen family obtained from P_8 by a single Δ-Y exchange by P_9. (Recall that $K_{3,3,1}$ is the graph of 7 vertices with vertices in three classes: $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7\}$ and edges between two vertices if and only if they lie in different classes. The graph $K_{4,4}$ is defined similarly on 8 vertices with two vertex classes of size 4.)

Theorem 2.3. *The following graphs are intrinsically linkable*: $K_6 - e$, $K_{3,3,1} - e_{4,6}$, $P_7 - e_{4,5}$, $P_7 - e_{5,5}$, $(K_{4,4} - e) - e_{4,4}$, *and* $P_8 - e_{4,5}$.

Proof. We will show that $G = K_{3,3,1} - e_{4,6}$ is intrinsically linkable. Proofs for the remaining graphs are similar.

Label the vertices as in Figure 2. Notice that in this labeling scheme the vertex classes are $S = \{s_1\}$, $U = \{u_1\}$, $V = \{v_1, v_2, v_3\}$, and $W = \{w_1, w_2\}$. We say that

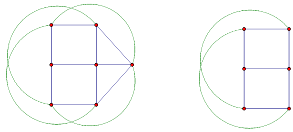

Figure 2. Vertex classes of $K_{3,3,1} - e_{4,6}$ and the subgraph H.

an edge is in the class SV if it connects a vertex in S with a vertex in V. Naming the other edge classes similarly, we have four edge classes in total: SV, SW, UV, and VW.

Take any immersion $\hat{f}$ of G. Let H be the subgraph induced by

$$\{u_1, v_1, v_2, v_3, w_1, w_2\}.$$

Since H is isomorphic to $K_{3,3}$, H is nonplanar and thus $\hat{H}$ has a pair of nonadjacent intersecting edges. There are two cases.

Case 1: Suppose one edge belongs to UV and the other to VW. We may assume the two edges to be (u_1, v_2) and (v_1, w_1). Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Case 2: Suppose both edges belong to VW. We may assume the two edges to be (v_1, w_1) and (v_2, w_2). Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Thus in either case we have a pair of disjoint cycles that intersect in $\hat{G}$. By Lemma 2.2, G is intrinsically linkable. $\square$

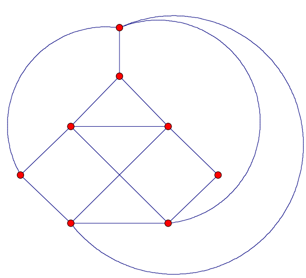

Figure 3. An immersion of $P_8 - e_{3,3}$ with only one crossing.

Since vertex expansion, Δ-Y exchange, and Y-Δ exchange preserve intrinsic linking [Nešetřil and Thomas 1985; Fellows and Langston 1988; Motwani et al. 1988; Robertson et al. 1995], it is natural to ask if these same graph operations preserve intrinsic linkability. In general, this is not the case. For example, $P_8 - e_{4,4}$ can be obtained from $P_7 - e_{4,5}$ by Δ-Y exchange, but $P_8 - e_{4,4}$ is not intrinsically linkable (See Figure 3).

In addition, certain expansions of $K_6 - e$ and $K_{3,3,1} - e_{4,6}$, which are exhibited in Figure 4 (notice that the expanded immersions contain only one crossing), are not intrinsically linkable. Any intrinsically linkable graph for which vertex expansion does preserve linkability, we call *strongly linkable*. Having found many examples in which expansion kills intrinsic linkability, we conjecture the following:

Conjecture 2.4. *A graph is strongly linkable if and only if it is intrinsically linked.*

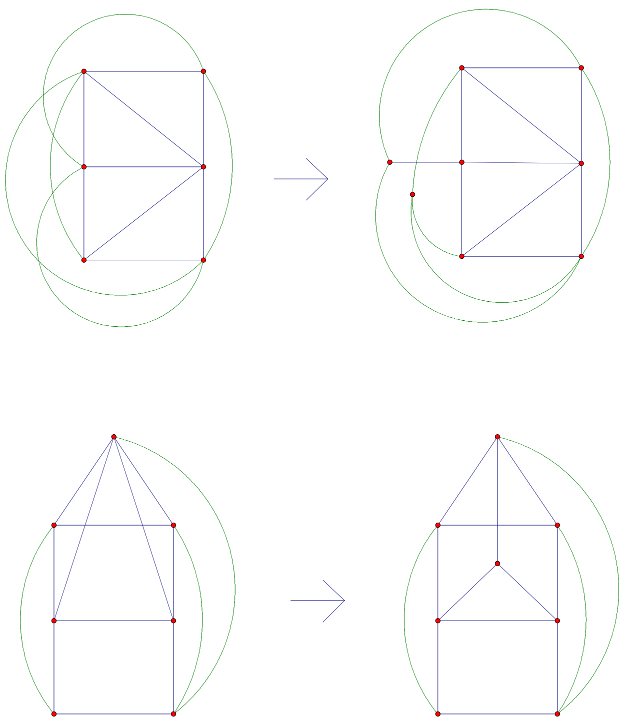

Figure 4. Two graphs for which vertex expansion destroys intrinsic linkability.

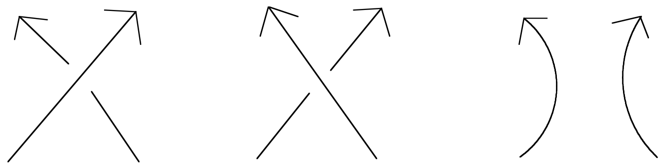

Figure 5. The neighborhoods involved in Lemma 3.1.

3. Intrinsically knottable graphs

3A. ***Introduction.*** The following lemma about knots is from [Kauffman 1983]. Note that we use $lk_2(L_1, L_2)$ to denote the mod 2 linking number for link components L_1 and L_2. Recall that a *knot* is a tame embedding of S^1 into $\mathbb{R}^3$.

Lemma 3.1. *For a knot K, the Arf invariant $\alpha(K)$ is the second coefficient of the Conway polynomial* (*mod* 2)*. It satisfies the following Skein relation* (*see Figure 5*):

$$\alpha(K_+) = \alpha(K_-) + lk_2(L_1, L_2).$$

Note that if $\alpha(K) \neq 0$, then K is nontrivial. (There are, however, many nontrivial knots with vanishing Arf invariant).

We use the following lemma from [Taniyama and Yasuhara 2001] (see also [Foisy 2002]). This lemma uses the second coefficient of the Conway polynomial of a knot, which is denoted by $a_2(K)$, for a knot K (again, if $a_2(K) \neq 0$, then K is nontrivial). Recall that a Hamiltonian cycle in a graph is a cycle that uses every vertex of the graph.

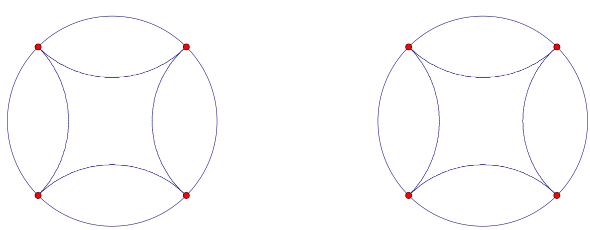

Figure 6. A planar embedding of D_4.

Lemma 3.2. *Consider the graph D_4, labeled as in Figure 6. Let f be a function embedding D_4 in space. Let S_0 and S_1 be sets of Hamiltonian cycles where*

$$S_0 = \{\, (a_i b_j c_k d_l) \mid i+j+k+l \text{ is even}\,\},$$
$$S_1 = \{\, (a_i b_j c_k d_l) \mid i+j+k+l \text{ is odd}\,\}.$$

Let

$$\lambda(f) = \sum_{C \in S_0} a_2(f(C)) - \sum_{C \in S_1} a_2(f(C)).$$

Then

$$\lambda(f) = \big|\, lk(C_1, C_3) \cdot lk(C_2, C_4) \big|.$$

In particular, if $\lambda(f)$ is nonzero, one of the Hamiltonian cycles must be knotted.

The following corollary is an immediate consequence; see [Taniyama and Yasuhara 2001; Foisy 2002].

Corollary 3.3. *If for a given embedding of G, there is an expansion of D_4 contained as an embedded subgraph with*

$$lk(C_1, C_3) \cdot lk(C_2, C_4) > 0,$$

then the embedded G contains a knotted cycle.

3B. ***Nontrivial examples of intrinsically knottable graphs.*** We explore the connection between intrinsic linking and intrinsic knottability by looking at the Petersen graphs. We originally conjectured that an intrinsically linked graph would necessarily be intrinsically knottable, but we quickly found counterexamples. It is easy to see that an immersion must have at least three crossings in order to be knottable. There are immersions of P_9, PG, and P_8 that have only two crossings (see, for example Figure 7), so clearly these graphs are not intrinsically knottable.

Theorem 3.4. *The graph K_6 is intrinsically knottable.*

Our proof of this theorem relies heavily on the following lemma which is similar to Lemma 3.2.

Lemma 3.5. *Let D_4' be a graph with four vertices, two nonadjacent 2-cycles C_1 and C_2, and two nonadjacent edges A_1 and A_2 that connect C_1 and C_2 (see Figure 8). Given any immersion of D_4', if C_1 and C_2 cross and A_1 and A_2 cross, then the immersion is knottable.*

Proof. Take any immersion of D_4' such that C_1 and C_2 cross and A_1 and A_2 are crossed. Assign over/under information to the crossings of C_1 and C_2 such that $lk_2(C_1, C_2) = 1$. We will show that there is a way to assign over/under information to the crossings on A_1 and A_2 such that the resulting embedding contains a knot.

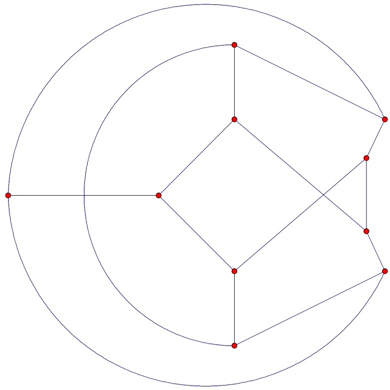

Figure 7. An immersion of the classic Petersen graph with only two crossings.

Let S be the set of all Hamiltonian cycles of D_4'. Given any embedding of D_4', we can define σ as follows:

$$\sigma = \sum_{C \in S} \alpha(C).$$

For disjoint arcs a_1 and a_2 in an embedding of D_4', define $\omega(a_1, a_2) \in \mathbb{Z}_2$ to be the number of times mod 2 that a_1 crosses over a_2. Note that by definition, for any embedding of D_4',

$$\omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) = lk_2(C_1, C_2).$$

Assign arbitrarily all crossings of A_1 with A_2 but one. Consider the crossing that has not been assigned. Let D_+ denote the embedding of D_4' in which A_1 crosses over A_2 at that crossing and D_- denote the embedding of D_4' in which A_2 crosses over A_1. Consider the change $\Delta\sigma$ in σ that will result from changing the crossing on A_1 and A_2.

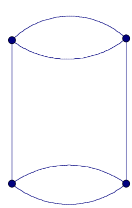

Figure 8. The graph D_4'.

Let C be a Hamiltonian cycle containing A_1 and A_2 and $\epsilon(C)$ be the change in $\alpha(C)$ induced by the crossing change. Now by Lemma 3.1 above,

$$\epsilon(C) = \alpha(C_+) + \alpha(C_-) = lk_2(L_1, L_2) = \sum_{E_1 \in L_1, E_2 \in L_2} \omega(E_1, E_2).$$

Now, summing up $\epsilon(C)$ over all Hamiltonian cycles C gives the change in σ. Fortunately most of the terms cancel out and we are left with

$$\begin{aligned}\Delta\sigma &= \sum_{C \in S} \epsilon(C) \\ &= \omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) \\ &= lk_2(C_1, C_2) = 1.\end{aligned}$$

This means that either D_+ or D_- contains a knot. □

Proof of Theorem 3.4. Take any lift of any immersion of K_6. Since K_6 is intrinsically linked, there is a pair of linked triangles, C_1 and C_2, in the resulting embedding.

Suppose that we temporarily ignore the edges of C_1 and C_2. We are left with $K_{3,3}$, which has a crossing in nonadjacent edges, say A_1 and A_2. Notice that A_1 and A_2 connect the cycles C_1 and C_2. The cycles C_1 and C_2, along with the edges A_1 and A_2, make up a subgraph of K_6 that is D_4' (with some extra degree 2 vertices). Since C_1 and C_2 are linkable and A_1 and A_2 cross, this subgraph immersion is knottable, by Lemma 3.5. Thus K_6 is knottable. □

Now we show that $K_{4,4} - e$ is intrinsically knottable. First we need the following lemma.

Lemma 3.6. *Suppose G is a graph that contains in every immersion two pairs of linkable cycles, C_1 and C_2, C_3 and C_4. Suppose the union of the cycles is an expansion of D_4 with C_1 and C_2 opposite each other and C_3 and C_4 opposite each other (so C_1 and C_2 are disjoint, C_3 and C_4 are disjoint, and all other pairs of C_i and C_j, for $i \neq j$, intersect in either a vertex, an edge, or a simple path). If there is a way to orient the cycles consistently, then G is intrinsically knottable.*

Proof. Orient the cycles in a consistent way, and assign all crossings to be positive. Then $lk(C_1, C_2)$ and $lk(C_3, C_4)$ are both positive. Since the cycles C_1, C_2, C_3, and C_4 form a subgraph of G that is an expansion of D_4 with the desired linking properties, we can apply Corollary 3.3 and conclude that the resulting embedding contains a knot. □

Theorem 3.7. *The graph $K_{4,4} - e$ is intrinsically knottable.*

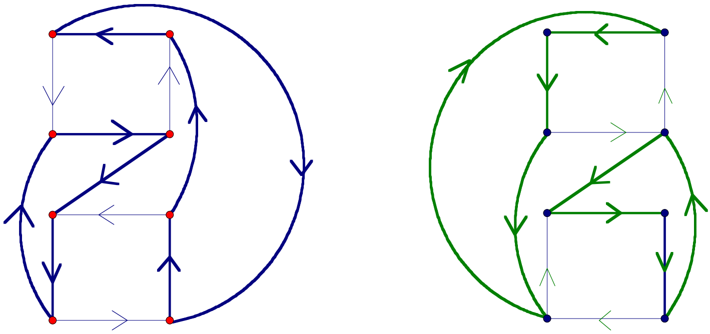

Figure 9. Case 1 (left): C_3 shares exactly one edge with C_1 and one edge with C_2. Case 2 (right): C_3 shares exactly one edge with C_1 and one edge with C_2.

Proof. We first label the vertices of $K_{4,4}-e$ as $v_1, \dots, v_4, w_1, \dots, w_4$, where every v_i belongs to one partition and every w_i belongs to the other partition. Let (v_1, w_3) be the missing edge.

Take any lift of any immersion of $K_{4,4}-e$. Since $K_{4,4}-e$ is intrinsically linked, there is a pair of nonsplittably linked (thus linkable) 4-cycles in the lift embedding. We again denote these 4-cycles as C_1 and C_2 where C_1 is (v_1, w_1, v_2, w_2) and C_2 is (v_3, w_3, v_4, w_4). (Up to symmetry this is the only way to get disjoint 4-cycles.)

Now the subgraph of $K_{4,4}-e$ resulting from the removal of (v_1, w_1) is intrinsically linkable by Theorem 2.3 above. So there is a pair of linkable cycles, C_3 and C_4 in the subgraph. There are two ways in which C_3 and C_4 can be related to C_1 and C_2: C_3 shares exactly one edge with C_1 and one edge with C_2, or C_3 shares exactly one edge with C_1 and one edge with C_2.

In each case, there is a way to orient the cycles C_1, C_2, C_3, and C_4 consistently. (See Figure 9.) Since the cycles C_1, C_2, C_3, and C_4 form a subgraph of $K_{4,4}-e$ that is an expansion of D_4 with the desired linkability properties, we can apply Lemma 3.6 and conclude that $K_{4,4}-e$ is intrinsically knottable. □

The techniques of this proof can also be applied to prove that K_6, P_7 and $K_{3,3,1}$ are intrinsically knottable.

3C. *Strongly knottable graphs.* We say that a graph G is *strongly knottable* if every expansion of G is intrinsically knottable.

Proposition 3.8. *The graphs K_6, $K_{3,3,1}$, $K_{4,4}$, and P_7 are not strongly knottable.*

Proof. In Figures 10 and 11, we exhibit immersions of expansions of K_6 and $K_{4,4}$, such that each immersion has only two crossings, and thus certainly is not knottable. Similar immersions for P_7 and $K_{3,3,1}$ exist. □

This leads us to the following conjecture.

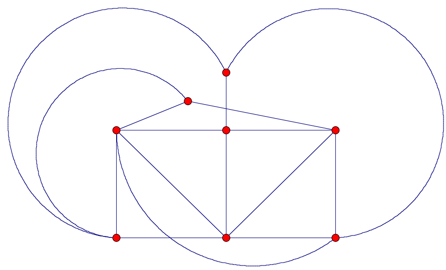

Figure 10. An immersion of an expansion of K_6 with only two crossings.

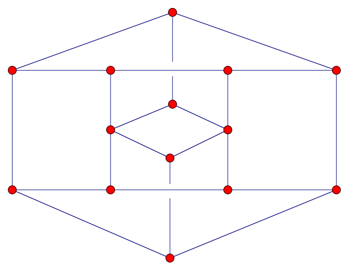

Figure 11. An immersion of an expansion of $K_{4,4}$ with only two crossings.

Conjecture 3.9. *A graph is strongly knottable if and only if it is intrinsically knotted.*

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Received: 2007-06-10 Accepted: 2007-12-01

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