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Boundary data smoothness for solutions of nonlocal boundary value problems for n-th order differential equations

Johnny Henderson, Britney Hopkins, Eugenie Kim and Jeffrey Lyons

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Under certain conditions, solutions of the boundary value problem

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

$y^{(i-1)}(x_1) = y_i$ for $1 \le i \le n-1$, and $y(x_2) - \sum_{i=1}^{m} r_i y(\eta_i) = y_n$, are differentiated with respect to boundary conditions, where $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and $r_1, \dots, r_m, y_1, \dots, y_n \in \mathbb{R}$.

1. Introduction

In this paper, we will be concerned with differentiating solutions of certain nonlocal boundary value problems with respect to boundary data for the n-th order ordinary differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad a < x < b, \tag{1}$$

satisfying

$$y^{(i-1)}(x_1) = y_i,\ 1 \le i \le n-1, \qquad y(x_2) - \sum_{k=1}^{m} r_k y(\eta_k) = y_n, \tag{2}$$

where $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and $y_1, \dots, y_n, r_1, \dots, r_m \in \mathbb{R}$, and where we assume

(i) $f(x, u_1, \dots, u_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\partial f / \partial u_i (x, u_1, \dots, u_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, 2, \dots, n$, and

(iii) solutions of initial value problems for (1) extend to (a, b).

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We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1) and (2), we will establish analogues of a result that Hartman [1964] attributes to Peano concerning differentiation of solutions of (1) with respect to initial conditions. For our differentiation with respect to the boundary conditions results, given a solution $y(x)$ of (1), we will give much attention to the *variational equation for* (1) *along* $y(x)$, which is defined by

$$z^{(n)} = \sum_{k=1}^{n} \frac{\partial f}{\partial u_k}(x, y(x),\ y'(x), \dots, y^{(n-1)}(x)) z^{(k-1)}. \tag{3}$$

There has long been interest in multipoint nonlocal boundary value problems for ordinary differential equations, with much attention given to positive solutions. To see only a few of these papers, we refer the reader to [Bai and Fang 2003; Gupta and Trofimchuk 1998; Ma 1997; 2002; Yang 2002].

Likewise, many papers have been devoted to smoothness of solutions of boundary value problems with respect to boundary data. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, functional differential equations versions and dynamic equations on time scales versions, we suggest results from among the many papers [Datta 1998; Ehme 1993; Ehme et al. 1993; Ehme and Henderson 1996; Ehme and Lawrence 2000; Hartman 1964; Henderson 1984; 1987; Henderson et al. 2005; Henderson and Lawrence 1996; Lawrence 2002; Peterson 1976; 1978; 1981; 1987; Spencer 1975]. In fact, smoothness results have been given some consideration for (1) and (2) when $n = 2$ and for specific and general values of m [Ehrke et al. 2007; Henderson and Tisdell 2004].

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1) as follows

Theorem 1.1. [Peano] *Assume that*, *with respect to* (1), *conditions* (i)–(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2, \dots, c_n)$ *denote the solution of* (1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i,\ 1 \le i \le n$. *Then*,

(i) *For each* $1 \le i \le n$, $\partial y/\partial c_i$ *exists on* (a, b) *and* $\alpha_i \equiv \partial y/\partial c_i$ *is a solution of the variational equation* (3) *along* $y(x)$ *and satisfies the initial condition*,

$$\alpha_i^{(j-1)}(x_0) = \delta_{ij}, \quad 1 \le i, j \le n.$$

(ii) $\partial y/\partial x_0$ *exists on* (a, b), *and* $\beta \equiv \partial y/\partial x_0$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions*,

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0), \quad 1 \le i \le n.$$

(iii) $\partial y/\partial x_0(x) = -\sum_{k=1}^{n} y^{(k)}(x_0)\partial y/\partial c_k(x)$.

In addition, our analogue of Theorem 1.1 depends on uniqueness of solutions of (1) and (2), a condition we list as an assumption.

(iv) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, if $y^{(i-1)}(x_1) = z^{(i-1)}(x_1)$ for each $1 \le i \le n-1$, and $y(x_2) - \sum_{k=1}^{m} r_k y(\eta_k) = z(x_2) - \sum_{k=1}^{m} r_k z(\eta_k)$, where $y(x)$ and $z(x)$ are solutions of (1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (3) along solutions $y(x)$ of (1).

(v) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and a solution $y(x)$ of (1), if $u^{(i-1)}(x_1) = 0$, $1 \le i \le n-1$, and $u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) = 0$, where $u(x)$ is a solution of (3) along $y(x)$, then $u(x) \equiv 0$.

2. An analogue of Peano's Theorem for Equations (1) and (2)

In this section, we derive our analogue of Theorem 1.1 for boundary value problem (1), (2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. The arguments for this continuous dependence follow much along the lines of those in [Henderson and Tisdell 2004], when (1) is of second order. For that reason, we omit the details of the proof.

Theorem 2.1. *Assume* (i)–(iv) *are satisfied with respect to* (1). *Let $u(x)$ be a solution of* (1) *on* (a, b), *and let* $a < c < x_1 < \eta_1 < \cdots < \eta_m < x_2 < d < b$ *be given. Then, there exists a* $\delta > 0$ *such that, for*

$$\begin{aligned} |x_i - t_i| &< \delta, && i = 1, 2, \\ |\eta_i - \tau_i| < \delta \quad \textit{and} \quad |r_i - \rho_i| &< \delta, && 1 \le i \le m, \\ |u^{(i-1)}(x_1) - y_i| &< \delta, && 1 \le i \le n-1 \\ \Big| u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) - y_n \Big| &< \delta, \end{aligned}$$

there exists a unique solution $u_\delta(x)$ of (1) *such that*

$$u_\delta^{(i-1)}(t_1) = y_i, \quad 1 \le i \le n-1,$$
$$u_\delta(t_2) - \sum_{k=1}^{m} \rho_k u_\delta(\tau_k) = y_n$$

and $\{u_\delta^{(j-1)}(x)\}$ converges uniformly to $u^{(j-1)}(x)$, as $\delta \to 0$, on $[c, d]$, for $1 \le j \le n$.

We now present the result of the paper.

Theorem 2.2. *Assume conditions* (i)–(v) *are satisfied. Let* $u(x)$ *be a solution* (1) *on* (a, b). *Let* $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$ *be given, so that*

$$u(x) = u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where $u^{(i-1)}(x_1) = u_i$, $1 \le i \le n-1$, *and* $u(x_2) - \sum_{k=1}^{m} r_k u(\eta_k) = u_n$. *Then,*

(i) *For each* $1 \le i \le n$, $\partial u/\partial u_i$ *exists on* (a, b). *Moreover, for each* $1 \le j \le n-1$, $y_j \equiv \partial u/\partial u_j$ *solves Equation* (3) *along* $u(x)$ *and satisfies the boundary conditions,*

$$y_j^{(i-1)}(x_1) = \delta_{ij},\ 1 \le i \le n-1, \quad y_j(x_2) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $y_n \equiv \partial u/\partial u_n$ *solves* (3) *along* $u(x)$ *and satisfies the boundary conditions,*

$$y_n^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad y_n(x_2) - \sum_{k=1}^{m} r_k y_n(\eta_k) = 1.$$

(ii) $\partial u/\partial x_1$ *and* $\partial u/\partial x_2$ *exist on* (a, b), *and* $z_i \equiv \partial u/\partial x_i$, $i = 1, 2$, *are solutions of* (3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$z_1^{(i-1)}(x_1) = -u^{(i)}(x_1),\ 1 \le i \le n-1, \qquad z_1(x_2) - \sum_{k=1}^{m} r_k z_1(\eta_k) = 0,$$

$$z_2^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \qquad z_2(x_2) - \sum_{k=1}^{m} r_k z_2(\eta_k) = -u'(x_2).$$

(iii) *For* $1 \le j \le m$, $\partial u/\partial \eta_j$ *exists on* (a, b), *and* $w_j \equiv \partial u/\partial \eta_j$, $j = 1, \ldots, m$, *is a solution of* (3) *along* $u(x)$ *and satisfies*

$$w_j^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad w_j(x_2) - \sum_{k=1}^{m} r_k w_j(\eta_k) = r_j u'(\eta_j).$$

(iv) *For* $1 \le j \le m$, $\partial u/\partial r_j$ *exists on* (a, b), *and* $v_j \equiv \partial u/\partial r_j$, $j = 1, \ldots, m$, *is a solution of* (3) *along* $u(x)$ *and satisfies,*

$$v_j^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1, \quad v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

Proof. For part (i), let $1 \le j \le n-1$, and consider $\partial u/\partial u_j$, since the argument for $\partial u/\partial u_n$ is similar. In this case we designate, for brevity, $u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, u_j)$.

Let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$y_{jh}(x) = \frac{1}{h}\big[u(x, u_j + h) - u(x, u_j)\big].$$

Note that $u^{(j-1)}(x_1, u_j + h) = u_j + h$, and $u^{(j-1)}(x_1, u_j) = u_j$, so that, for every $h \neq 0$,

$$y_{jh}^{(j-1)}(x_1) = \frac{1}{h}[u_j + h - u_j] = 1.$$

Also, for every $h \neq 0,\ 1 \leq i \leq n-1,\ i \neq j$,

$$y_{jh}^{(i-1)}(x_1) = \frac{1}{h}\big[u^{(i-1)}(x_1, u_j + h) - u^{(i-1)}(x_1, u_j)\big] = \frac{1}{h}[u_i - u_i] = 0,$$

and

$$\begin{aligned} y_{jh}(x_2) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) = \frac{1}{h}\big[u(x_2, u_j + h) - u(x_2, u_j)\big] \\ - \sum_{k=1}^{m} \frac{r_k}{h}\big[u(\eta_k, u_j + h) - u(\eta_k, u_j)\big] = \frac{1}{h}[u_n - u_n] = 0. \end{aligned}$$

Let $\beta = u^{(n-1)}(x_1, u_j)$, and $\epsilon = \epsilon(h) = u^{(n-1)}(x_1, u_j + h) - \beta$. By Theorem 2.1, $\epsilon = \epsilon(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1.1 for solutions of initial value problems for Equation (1) and viewing the solutions u as solutions of initial value problems and denoting $y(x, x_1, u_1, \ldots, u_j, \ldots, u_{n-1}, \beta)$ by $y(x, x_1, u_j, \beta)$, we have

$$y_{jh}(x) = \frac{1}{h}\big[y(x, x_1, u_j + h, \beta + \epsilon) - y(x, x_1, u_j, \beta)\big].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{jh}(x) = \frac{1}{h}\big[\{y(x, x_1, u_j + h, \beta + \epsilon) - y(x, x_1, u_j, \beta + \epsilon)\} \\ + \{y(x, x_1, u_j, \beta + \epsilon) - y(x, x_1, u_j, \beta)\}\big]. \end{aligned}$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{jh}(x) = \frac{1}{h}\alpha_j\big(x, y(x, x_1, u_j + \bar{h},\ \beta + \epsilon)\big)(u_j + h - u_j) \\ + \frac{1}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon})\big)(\beta + \epsilon - \beta), \end{aligned}$$

where $\alpha_k(x, y(\cdot)),\ k \in \{j, n\}$, is the solution of the variational Equation (3) along $y(\cdot)$ and satisfies, in each case,

$$\alpha_j^{(i-1)}(x_1) = \delta_{ij} \quad \alpha_n^{(i-1)}(x_1) = \delta_{in}, \qquad 1 \leq i \leq n,$$

respectively. Furthermore, $u_j + \bar{h}$ is between u_j and $u_j + h$, and $\beta + \bar{\epsilon}$ is between β and $\beta + \epsilon$. Now simplifying,

$$y_{jh}(x) = \alpha_j\big(x, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big) + \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon})\big).$$

Thus, to show $\lim_{h\to 0} y_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon/h$ exists.

Now $\alpha_n(x, y(\cdot))$ is a nontrivial solution of Equation (3) along $y(\cdot)$, and

$$\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0, \qquad 1 \le i \le n-1.$$

So, by assumption (v), $\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^m r_k \alpha_n(\eta_k, y(\cdot)) \ne 0$. However, we observed that $y_{jh}(x_2) - \sum_{k=1}^m r_k y_{jh}(\eta_k) = 0$, from which we obtain

$$\frac{\epsilon}{h} = \frac{\sum_{k=1}^m r_k \alpha_j\big(\eta_k, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big) - \alpha_j\big(x_2, y(x, x_1, u_j + \bar{h}, \beta + \epsilon)\big)}{\alpha_n\big(x_2, y(x, x_1, u_j, \beta + \bar{\epsilon})\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_j, \beta + \bar{\epsilon})\big)}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\begin{aligned}\lim_{h\to 0} \frac{\epsilon}{h} &= -\frac{\alpha_j\big(x_2, y(x, x_1, u_j, \beta_2)\big) - \sum_{k=1}^m r_k \alpha_j\big(\eta_k, y(x, x_1, u_j, \beta)\big)}{\alpha_n\big(x_2, y(x, x_1, u_j, \beta)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_j, \beta)\big)} \\ &= -\frac{\alpha_j(x_2, u(x)) - \sum_{k=1}^m r_k \alpha_j(\eta_k, u(x))}{\alpha_n(x_2, u(x)) - \sum_{k=1}^m r_k \alpha_n(\eta_k, u(x))} =: D.\end{aligned}$$

Let $y_j(x) = \lim_{h\to 0} y_{jh}(x)$, and note by construction of $y_{jh}(x)$ that

$$y_j(x) = \frac{\partial u}{\partial u_j}(x).$$

Furthermore,

$$y_j(x) = \lim_{h\to 0} y_{jh}(x) = \alpha_j(x, y(x, x_1, u_j, \beta)) + D\alpha_n(x, (u(x)),$$

which is a solution of the variational Equation (3) along $u(x)$. In addition because of the boundary conditions satisfied by $y_{jh}(x)$, we also have

$$y_j^{(i-1)}(x_1) = \delta_{ij}, \ 1 \le i \le n-1, \quad y_j(x_2) - \sum_{k=1}^m r_k y_j(\eta_k) = 0.$$

This completes the argument for $\partial u/\partial u_j$.

In part (ii) of the theorem, we will produce the details for $\partial u/\partial x_1$, with the arguments for $\partial u/\partial x_2$ being similar. This time, we designate

$$u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$$

by $u(x, x_1)$.

So, let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[u(x, x_1 + h) - u(x, x_1)].$$

Note that, for $1 \le i \le n-1$,

$$\begin{aligned}
z_{1h}^{(i-1)}(x_1) &= \frac{1}{h}[u^{(i-1)}(x_1, x_1+h) - u^{(i-1)}(x_1, x_1)] \\
&= \frac{1}{h}[u^{(i-1)}(x_1, x_1+h) - u^{(i-1)}(x_1+h, x_1+h)] \\
&= -\frac{1}{h}[u^{(i)}(c_{x_1,h}, x_1+h)\cdot h] \\
&= -u^{(i)}(c_{x_1,h}, x_1+h),
\end{aligned}$$

where $c_{x_1,h}$ lies between x_1 and x_1+h. In addition, we note that, for every $h \neq 0$,

$$\begin{aligned}
z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k) = \frac{1}{h}[u(x_2, x_1+h) - \sum_{k=1}^{m} r_k u(\eta_k, x_1+h) \\
- \{u(x_2, x_1) - \sum_{k=1}^{m} r_k u(\eta_k, x_1)\}] = \frac{1}{h}[u_n - u_n] = 0.
\end{aligned}$$

Next, let

$$\begin{aligned}
\beta &= u^{(n-1)}(x_1, x_1), \\
\epsilon_j &= \epsilon_j(h) = u^{(j-1)}(x_1, x_1+h) - u_j, \\
\epsilon_\beta &= \epsilon_\beta(h) = u^{(n-1)}(x_1, x_1+h) - \beta.
\end{aligned}$$

Let us note at this point that

$$\frac{\epsilon_j}{h} = z_{1h}^{(j-1)}(x_1) = -u^{(j)}(c_{x_1,h}, x_1+h).$$

By Theorem 2.1, both $\epsilon_j \to 0$ and $\epsilon_\beta \to 0$, as $h \to 0$. As in part (i), we employ the notation of Theorem 1.1 for solutions of initial value problems for (1). Viewing the solutions u as solutions of initial value problems, and denoting

$$y(x, x_1, u_1, \ldots, u_j, \ldots, u_n - 1, \beta)$$

by $y(x, x_1, u_j, \beta)$, we have

$$\begin{aligned}
z_{1h}(x) &= \frac{1}{h}[y(x, x_1, u_j+\epsilon_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta)] \\
&= \frac{1}{h}[y(x, x_1, u_j+\epsilon_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta+\epsilon_\beta) \\
&\qquad + y(x, x_1, u_j, \beta+\epsilon_\beta) - y(x, x_1, u_j, \beta)].
\end{aligned}$$

By the Mean Value Theorem,

$$z_{1h}(x) = \frac{1}{h}\big[\epsilon_j \alpha_j\big(x, y(x, x_1, u_j+\bar{\epsilon}_j, \beta+\epsilon_\beta)\big) + \epsilon_\beta \alpha_n\big(x, y(x, x_1, u_j, \beta+\bar{\epsilon}_\beta)\big)\big],$$

where $u_j+\bar{\epsilon}_j$ lies between u_j and $u_j+\epsilon_j$, $\beta+\bar{\epsilon}_\beta$ lies between β and $\beta+\epsilon_\beta$, and $\alpha_j(x, y(\cdot))$ and $\alpha_n(x, y(\cdot))$ are the solutions of Equation (3) along $y(\cdot)$ and satisfy, respectively,

$$\alpha_j^{(i-1)}(x_1)=\delta_{ij}, \quad 1\le i\le n,$$
$$\alpha_n^{(i-1)}(x_1)=\delta_{in}, \quad 1\le i\le n.$$

As before, to show that $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show that

$$\lim_{h\to 0}\frac{\epsilon_j}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{\epsilon_\beta}{h}$$

exist. Now, from above,

$$\lim_{h\to 0}\frac{\epsilon_j}{h}=\lim_{h\to 0} z_{1h}^{(j-1)}(x_1)=\lim_{h\to 0} u^{(j)}(c_{x_1,h}, x_1+h)=-u^{(j)}(x_1).$$

Since $\alpha_n(x,y(\cdot))$ is a nontrivial solution of (3) along $y(\cdot)$ and

$$\alpha_n^{(i-1)}(x_1,y(\cdot))=0, \qquad 1\le i\le n-1,$$

it follows from assumption (v) that

$$\alpha_n(x_2,y(\cdot))-\sum_{k=1}^{m} r_k\alpha_n(\eta_k,y(\cdot))\neq 0.$$

Since

$$z_{1h}(x_2)-\sum_{k=1}^{m} r_k z_{1h}(\eta_k)=0,$$

we have

$$\frac{\epsilon_\beta}{h}=\left(\frac{-\epsilon_j}{h}\right)\frac{A}{\alpha_n\big(x_2, y(x,x_1,u_j,\beta+\bar{\epsilon}_\beta)\big)-\sum_{k=1}^{m} r_k\alpha_n\big(\eta_k, y(x,x_1,u_j,\beta+\bar{\epsilon}_\beta)\big)},$$

where

$$A=\alpha_j\big(x_2, y(x,x_1,u_j+\bar{\epsilon}_j,\beta+\epsilon_\beta)\big)-\sum_{k=1}^{m} r_k\alpha_j\big(\eta_k, y(x,x_1,u_j+\bar{\epsilon}_j,\beta+\epsilon_\beta)\big).$$

And so,

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon_\beta}{h}&=\frac{u^{(j)}(x_1)\big[\alpha_j\big(x_2, y(x,x_1,u_j,\beta)\big)-\sum_{i=1}^{m} r_i\alpha_j\big(\eta_i, y(x,x_1,u_j,\beta)\big)\big]}{\alpha_n\big(x_2, y(x,x_1,u_j,\beta)\big)-\sum_{i=1}^{m} r_i\alpha_n\big(\eta_i, y(x,x_1,u_j,\beta)\big)}\\&=\frac{u^{(j)}(x_1)\big[\alpha_j\big(x_2,u(x)\big)-\sum_{i=1}^{m} r_i\alpha_j\big(\eta_i,u(x)\big)\big]}{\alpha_n\big(x_2,u(x)\big)-\sum_{i=1}^{m} r_i\alpha_n\big(\eta_i,u(x)\big)}=:E.\end{aligned}$$

From the above expression,

$$z_{1h}(x) = \frac{\epsilon_j}{h}\alpha_j\big(x, y(x, x_1, u_j + \bar{\epsilon}_j, \beta + \epsilon_\beta)\big) + \frac{\epsilon_\beta}{h}\alpha_n\big(x, y(x, x_1, u_j, \beta + \bar{\epsilon}_\beta)\big),$$

and we can evaluate the limit as $h \to 0$. If we let $z_1(x) = \lim_{h\to 0} z_{1h}(x)$, then $z_1(x) = \partial u/\partial x_1$, and

$$\begin{aligned} z_1(x) &= \lim_{h\to 0} z_{1h}(x) \\ &= -u^{(j)}(x_1)\alpha_j\big(x, y(x, x_1, u_j, \beta)\big) + E\alpha_n\big(x, y(x, x_1, u_j, \beta)\big) \\ &= -u^{(j)}(x_1)\alpha_j\big(x, u(x)\big) + E\alpha_n\big(x, u(x)\big), \end{aligned}$$

which is a solution of Equation (3) along $u(x)$. In addition, from above observations, $z_1(x)$ satisfies the boundary conditions,

$$z_1^{(i-1)}(x_1) = \lim_{h\to 0} z_{1h}^{(i-1)}(x_1) = -u^{(i)}(x_1),\ 1 \le i \le n-1,$$

and

$$z_1(x_2) - \sum_{k=1}^{m} r_k z_1(\eta_k)) = \lim_{h\to 0}(z_{1h}(x_2) - \sum_{k=1}^{m} r_k z_{1h}(\eta_k)) = 0.$$

This completes the proof for $\partial u/\partial x_1$.

The proofs of (iii) and (iv) are in very much the same spirit. For (iii), we fix $1 \le j \le m$, and this time we designate

$$u(x,x_1,x_2,u_1,\ldots,u_n,\eta_1,\ldots,\eta_m,r_1,\ldots,r_m)$$

by $u(x, \eta_j)$. Let $\delta > 0$ be as in Theorem 2.1 and $0 < |h| < \delta$ be given. Define

$$w_{jh}(x) = \frac{1}{h}[u(x, \eta_j + h) - u(x, \eta_j)].$$

Note that for every $h \ne 0$,

$$w_{jh}^{(i-1)}(x_1) = 0,\ 1 \le i \le n-1.$$

Next, let $\beta = u^{(n-1)}(x_1, \eta_j)$, and

$$\epsilon = \epsilon(h) = u^{(n-1)}(x_1, \eta_j + h) - \beta.$$

By Theorem 2.1, $\epsilon \to 0$, as $h \to 0$. Again, we use the notation of Theorem 1.1 for solutions of initial value problems for (1); viewing the solutions u as solutions of initial value problems and denoting

$$y(x, x_1, u_1, \ldots, u_{n-1}, \beta)$$

by $y(x, x_1, \beta)$, we have

$$w_{jh}(x) = \frac{1}{h}[y(x, x_1, \beta+\epsilon) - y(x, x_1, \beta)].$$

By the Mean Value Theorem,

$$w_{jh}(x) = \frac{\epsilon}{h}\alpha_n(x, y(x, x_1, \beta+\bar{\epsilon})),$$

where $\alpha_n(x, y(\cdot))$ is the solution of Equation (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1) = \delta_{in}, \quad 1 \le i \le n-1,$$

and $\beta+\bar{\epsilon}$ lies between β and $\beta+\epsilon$. Once again, to show $\lim_{h\to 0} w_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon/h$ exists.

Since $\alpha_n(x,y(\cdot))$ is a nontrivial solution of (3) along $y(\cdot)$ and

$$\alpha_n^{(i-1)}(x_1,y(\cdot)) = 0, \quad 1 \le i \le n-1,$$

it follows from assumption (v) that

$$\alpha_n(x_2,y(\cdot)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k,y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_i)}{\alpha_n\big(x_2, y(x, x_1, \beta_2+\bar{\epsilon})\big) - \sum_{k=1}^{m} r_k\alpha_n\big(\eta_k, y(x, x_1, \beta+\bar{\epsilon})\big)}.$$

We look in more detail at the numerator of this quotient. Consider

$$\begin{aligned}
&w_{jh}(x_2) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k)\\
&\quad= \frac{1}{h}\Big[u(x_2, \eta_j+h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j+h) - \big[u(x_2, \eta_j) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j)\big]\Big]\\
&\quad= \frac{1}{h}\Big[u(x_2, \eta_j+h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, \eta_j+h)\\
&\qquad\quad - r_j u(\eta_j+h, \eta_j+h) + r_j u(\eta_j+h, \eta_j+h)\\
&\qquad\qquad - r_j u(\eta_j, \eta_j+h)\Big] - \frac{u_n}{h}\\
&\quad= \frac{u_n}{h} - \frac{u_n}{h} + \frac{r_j u(\eta_j+h, \eta_j+h) - r_j u(\eta_j, \eta_j+h)}{h}\\
&\quad= \frac{r_j}{h}\big[u(\eta_j+h, \eta_j+h) - u(\eta_j, \eta_j+h)\big]
\end{aligned}$$

$$= \frac{r_j}{h} \int_{\eta_j}^{\eta_j+h} u'(s, \eta_j + h) ds$$

$$= \frac{r_j}{h} u'(c_{j,h}, \eta_j + h)(\eta_j + h - \eta_j) = r_j u'(c_{j,h}, \eta_j + h),$$

where $c_{j,h}$ is between η_j and $\eta_j + h$. So, as $h \to 0$ we obtain

$$r_j u'(c_h, \eta_j + h) \to r_j u'(\eta_j, \eta_j) = r_j u'(\eta_j).$$

When we return to the quotient defining ϵ/h, we compute the limit,

$$\lim_{h\to 0} \frac{\epsilon}{h} = \frac{r_j u'(\eta_j)}{\alpha_n\big(x_2, y(x, x_1, u_1, \beta)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, y(x, x_1, u_1, \beta)\big)}$$
$$= \frac{r_j u'(\eta_j)}{\alpha_n\big(x_2, u(x)\big) - \sum_{k=1}^m r_k \alpha_n\big(\eta_k, u(x)\big)} =: E_j.$$

From

$$w_{jh}(x) = \frac{\epsilon}{h} \alpha_n\big(x, y(x, x_1, u_1, \beta + \bar{\epsilon})\big),$$

if we let $w_j(x) = \lim_{h\to 0} w_{jh}(x)$, then $w_j(x) = \partial u/\partial \eta_j$, and

$$w_j(x) = \lim_{h\to 0} w_{jh}(x) = E_j \alpha_n(x, y(x, x_1, u_1, \beta)) = E_j \alpha_n(x, u(x)),$$

which is a solution of Equation (3) along $u(x)$. In addition, from above observations, $w_j(x)$ satisfies the boundary conditions,

$$w_j^{(i-1)}(x_1) = \lim_{h\to 0} w_{jh}^{(i-1)}(x_1) = 0, \quad 1 \le i \le n-1,$$
$$w_j(x_2) - \sum_{k=1}^m r_k w_j(\eta_k) = r_j u'(\eta_j).$$

This concludes the proof of (iii). It remains to verify part (iv).

Fix $1 \le j \le m$ as before and consider $\partial u/\partial r_j$. Again, let $\delta > 0$ be as in Theorem 2.1 and $0 < |h| < \delta$. Define

$$v_{jh}(x) = \frac{1}{h}[u(x, r_j + h) - u(x, r_j)],$$

where, for brevity, we designate

$$u(x, x_1, x_2, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$$

by $u(x, r_j)$. Note that

$$v_{jh}^{(i-1)}(x_1) = \frac{1}{h}(u_i - u_i) = 0,$$

for every $h \neq 0$ and $1 \leq i \leq n-1$. Also, we see that

$$
\begin{aligned}
&v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{k=1}^{m} r_k\big(u(\eta_k, r_j + h) - u(\eta_k, r_j)\big)\Big] \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) + \sum_{k=1}^{m} r_k u(\eta_k, r_j)\Big] \\
&\quad = \frac{1}{h}u(x_2, r_j + h) - \frac{1}{h}\sum_{k=1}^{m} r_k u(\eta_k, r_j + h) - \frac{u_n}{h} \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\qquad\qquad - r_j u(\eta_j, r_j + h) - hu(\eta_j, r_j + h) + hu(\eta_j, r_j + h)\Big] - \frac{u_n}{h} \\
&\quad = \frac{1}{h}\Big[u(x_2, r_j + h) - \sum_{k\in\{1,\dots,m\}\setminus\{j\}} r_k u(\eta_k, r_j + h) \\
&\qquad\qquad - (r_j + h)u(\eta_j, r_j + h)\Big] + u(\eta_j, r_j + h) - \frac{u_n}{h} \\
&\quad = \frac{u_n}{h} + u(\eta_j, r_j + h) - \frac{u_n}{h} \\
&\quad = u(\eta_j, r_j + h).
\end{aligned}
$$

And so by Theorem 2.1,

$$\lim_{h\to 0} v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) = u(\eta_j, r_j).$$

Now recall that $u^{(n-2)}(x_1, r_j) = u_{n-1}$, and define

$$\beta = u^{(n-1)}(x_1, r_j), \quad \text{and} \quad \epsilon = \epsilon(h) = u^{(n-1)}(x_1, r_j + h) - \beta.$$

As usual, $\epsilon \to 0$ as $h \to 0$. Once again, using the notation for solutions of initial value problems for (1) and denoting $y(x, x_1, u_1, \dots, u_{n-1}, \beta)$ by $y(x, x_1, \beta)$, we have

$$v_{jh}(x) = \frac{1}{h}\big[y(x, x_1, \beta + \epsilon) - y(x, x_1, \beta)\big].$$

By the Mean Value Theorem,

$$
\begin{aligned}
v_{jh}(x) &= \frac{1}{h}\alpha_n\big(x, y(x, x_1, \beta + \bar{\epsilon})\big)(\beta + \epsilon - \beta) \\
&= \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, \beta + \bar{\epsilon})\big),
\end{aligned}
$$

where $\alpha_n(x, y(\cdot))$ is the solution of Equation (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1, y(\cdot)) = 0, \qquad 1 \le i \le n-1,$$
$$\alpha_n^{(n-1)}(x_1, y(\cdot)) = 1,$$

and $\beta+\bar{\epsilon}$ lies between β and $\beta+\epsilon$. As in previous cases, it follows from assumption (v) that

$$\alpha_n(x_2, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon}{h} = \frac{v_{jh}(x_2) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k)}{\alpha_n\big(x_2, y(x, x_1, \beta+\bar{\epsilon})\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, y(x, x_1, \beta+\bar{\epsilon})\big)},$$

and so from above,

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon}{h} &= \frac{r_j u(\eta_j)}{\alpha_n\big(x_2, y(x, x_1, \beta)\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, y(x, x_1, \beta)\big)}\\ &= \frac{r_j u(\eta_j)}{\alpha_n\big(x_2, u(x)\big) - \sum_{k=1}^{m} r_k \alpha_n\big(\eta_k, u(x)\big)} =: E_j.\end{aligned}$$

From

$$v_{jh}(x) = \frac{\epsilon}{h}\alpha_n\big(x, y(x, x_1, \beta+\bar{\epsilon})\big),$$

if we set $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, we obtain $v_j(x) = \partial u/\partial r_j$. In particular,

$$v_j(x) = \lim_{h\to 0} v_{jh}(x) = E_j \alpha_n(x, y(x, x_1, \beta)) = E_j \alpha_n(x, u(x)),$$

which is a solution of (3) along $u(x)$. In addition, $v_j(x)$ satisfies the boundary conditions,

$$v_j(x_1) = \lim_{h\to 0} v_{jh}^{(i-1)}(x_1) = 0, \quad 1 \le i \le n-1,$$
$$v_j(x_2) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

This completes case (iv), which in turn completes the proof of the theorem. □

We conclude the paper with a corollary to Theorem 2.2, whose verification is a consequence of the n-dimensionality of the solution space for the variational Equation (3). In addition, this corollary establishes an analogue of part (iii) of Theorem 1.1.

Corollary 2.2.1. *Assume the conditions of Theorem 2.2. Then*,

$$\frac{\partial u}{\partial x_1} = -\sum_{k=1}^{n-1} u^{(k)}(x_1)\frac{\partial u}{\partial u_i} \qquad \text{and} \qquad \frac{\partial u}{\partial x_2} = -u'(x_2)\frac{\partial u}{\partial u_n},$$

and for $1 \leq j \leq m$,

$$\frac{\partial u}{\partial \eta_j} = r_j \frac{u'(\eta_j)}{u(\eta_j)} \frac{\partial u}{\partial r_j}.$$

References

[Bai and Fang 2003] C. Bai and J. Fang, "Existence of multiple positive solutions for nonlinear m-point boundary value problems", *J. Math. Anal. Appl.* **281**:1 (2003), 76–85. MR 2004b:34035

[Datta 1998] A. Datta, "Differences with respect to boundary points for right focal boundary conditions", *J. Differ. Equations Appl.* **4**:6 (1998), 571–578. MR 99k:39007

[Ehme 1993] J. A. Ehme, "Differentiation of solutions of boundary value problems with respect to nonlinear boundary conditions", *J. Differential Equations* **101**:1 (1993), 139–147. MR 93k:34047

[Ehme and Henderson 1996] J. Ehme and J. Henderson, "Functional boundary value problems and smoothness of solutions", *Nonlinear Anal.* **26**:1 (1996), 139–148. MR 96h:34130

[Ehme and Lawrence 2000] J. Ehme and B. A. Lawrence, "Linearized problems and continuous dependence for finite difference equations", *Panamer. Math. J.* **10**:2 (2000), 13–24. MR 2001g:39013

[Ehme et al. 1993] J. Ehme, P. W. Eloe, and J. Henderson, "Differentiability with respect to boundary conditions and deviating argument for functional-differential systems", *Differential Equations Dynam. Systems* **1**:1 (1993), 59–71. MR 97f:34049

[Ehrke et al. 2007] J. Ehrke, J. Henderson, C. Kunkel, and Q. Sheng, "Boundary data smoothness for solutions of nonlocal boundary value problems for second order differential equations", *J. Math. Anal. Appl.* **333**:1 (2007), 191–203. MR 2008f:34008

[Gupta and Trofimchuk 1998] C. P. Gupta and S. I. Trofimchuk, "Solvability of a multi-point boundary value problem and related a priori estimates", *Canad. Appl. Math. Quart.* **6**:1 (1998), 45–60. Geoffrey J. Butler Memorial Conference in Differential Equations and Mathematical Biology (Edmonton, AB, 1996). MR 99f:34020

[Hartman 1964] P. Hartman, *Ordinary differential equations*, John Wiley & Sons Inc., New York, 1964. MR 30 #1270

[Henderson 1984] J. Henderson, "Right focal point boundary value problems for ordinary differential equations and variational equations", *J. Math. Anal. Appl.* **98** (1984), 363–377. MR 85m:34029

[Henderson 1987] J. Henderson, "Disconjugacy, disfocality, and differentiation with respect to boundary conditions", *J. Math. Anal. Appl.* **121**:1 (1987), 1–9. MR 88c:34043

[Henderson and Lawrence 1996] J. Henderson and B. A. Lawrence, "Smooth dependence on boundary matrices", *J. Differ. Equations Appl.* **2**:2 (1996), 161–166. Difference equations: theory and applications (San Francisco, CA, 1995). MR 97d:39003

[Henderson and Tisdell 2004] J. Henderson and C. C. Tisdell, "Boundary data smoothness for solutions of three point boundary value problems for second order ordinary differential equations", *Z. Anal. Anwendungen* **23**:3 (2004), 631–640. MR 2005h:34036

[Henderson et al. 2005] J. Henderson, B. Karna, and C. C. Tisdell, "Existence of solutions for three-point boundary value problems for second order equations", *Proc. Amer. Math. Soc.* **133**:5 (2005), 1365–1369 (electronic). MR 2005j:34026

[Lawrence 2002] B. A. Lawrence, "A variety of differentiability results for a multi-point boundary value problem", *J. Comput. Appl. Math.* **141**:1-2 (2002), 237–248. Dynamic equations on time scales. MR 2003d:34063

[Ma 1997] R. Ma, "Existence theorems for a second order three-point boundary value problem", *J. Math. Anal. Appl.* **212**:2 (1997), 430–442. MR 98h:34041

[Ma 2002] R. Ma, "Existence and uniqueness of solutions to first-order three-point boundary value problems", *Appl. Math. Lett.* **15**:2 (2002), 211–216. MR 2002k:34032

[Peterson 1976] A. C. Peterson, "Comparison theorems and existence theorems for ordinary differential equations", *J. Math. Anal. Appl.* **55**:3 (1976), 773–784. MR 55 #5956

[Peterson 1978] A. C. Peterson, "Existence-uniqueness for ordinary differential equations", *J. Math. Anal. Appl.* **64**:1 (1978), 166–172. MR 58 #1383

[Peterson 1981] A. C. Peterson, "Existence-uniqueness for focal-point boundary value problems", *SIAM J. Math. Anal.* **12**:2 (1981), 173–185. MR 83h:34017

[Peterson 1987] A. C. Peterson, "Existence and uniqueness theorems for nonlinear difference equations", *J. Math. Anal. Appl.* **125**:1 (1987), 185–191. MR 88f:39003

[Spencer 1975] J. D. Spencer, "Relations between boundary value functions for a nonlinear differential equation and its variational equations", *Canad. Math. Bull.* **18**:2 (1975), 269–276. MR 53 #3402

[Yang 2002] B. Yang, *Boundary Value Problems for Ordinary Differential Equations*, Ph.D. thesis, Mississippi State University, 2002.

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