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Douglas R. Anderson and William R. Hall

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Oscillation criteria for two-dimensional systems of first-order linear dynamic equations on time scales

Douglas R. Anderson and William R. Hall

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Oscillation criteria for two-dimensional difference systems of first-order linear difference equations are generalized and extended to arbitrary dynamic equations on time scales. This unifies under one theory corresponding results from differential systems, and includes second-order self-adjoint differential, difference, and q-difference equations within its scope. Examples are given illustrating a key theorem.

1. Prelude

Jiang and Tang [2007] have established sufficient conditions for the oscillation of the linear two-dimensional difference system

$$\Delta x_n = p_n y_n, \quad \Delta y_{n-1} = -q_n x_n, \quad n \in \mathbb{Z}, \tag{1-1}$$

where $\{p_n\}$, $\{q_n\}$ are nonnegative real sequences and Δ is the forward difference operator given via $\Delta x_n = x_{n+1} - x_n$; see also [Li 2001]. The system (1-1) may be viewed as a discrete analogue of the differential system

$$x'(t) = p(t)y(t), \quad y'(t) = -q(t)x(t), \quad t \in \mathbb{R}, \tag{1-2}$$

investigated in [Lomtatidze and Partsvania 1999].

Oscillation questions in difference and differential equations are an interesting and important area of study in modern mathematics. Furthermore, within the past two decades, these two related but distinct areas have begun to be combined under a powerful, more robust and general theory titled dynamic equations on time scales, a theory introduced by Hilger [1990]. For example, equations (1-1) and (1-2) would take the form

$$x^{\Delta}(t) = p(t)y(t), \quad y^{\nabla}(t) = -q(t)x(t), \quad t \in \mathbb{T}, \tag{1-3}$$

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where $\mathbb{T}$ is an arbitrary time scale (any nonempty closed set of real numbers) unbounded above, with the special cases of $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ yielding systems (1-1) and (1-2), respectively, as important corollaries. In this general time-scale setting, Δ represents the delta (or Hilger) derivative [Bohner and Peterson 2001, Definition 1.10], and ∇ represents the nabla derivative, introduced in [Atici and Guseinov 2002, Section 2]:

$$x^{\Delta}(t) := \lim_{s\to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}, \qquad x^{\nabla}(t) := \lim_{s\to t} \frac{x(\rho(t)) - x(s)}{\rho(t) - s},$$

where $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ is the backward jump operator. Moreover, $\mu(t) := \sigma(t) - t$ is the forward graininess function, and $\nu(t) := t - \rho(t)$ is the backward graininess function. In particular, if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t = \rho(t)$ and $x^{\Delta} = x' = x^{\nabla}$, while if $\mathbb{T} = h\mathbb{Z}$ for any $h > 0$, then $\sigma(t) = t + h$ and $\rho(t) = t - h$, so that

$$x^{\Delta}(t) = \frac{x(t+h) - x(t)}{h} \quad \text{and} \quad x^{\nabla}(t) = \frac{x(t) - x(t-h)}{h},$$

respectively. A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at each right-dense point $t \in \mathbb{T}$ (a point where $\sigma(t) = t$) and has a left-sided limit at each left-dense point $t \in \mathbb{T}$ (a point where $\rho(t) = t$). The set of right-dense continuous functions on $\mathbb{T}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T})$. It can be shown that any right-dense continuous function f has an antiderivative (a function $F : \mathbb{T} \to \mathbb{R}$ with the property $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$). The Cauchy delta integral of f is defined by

$$\int_{t_0}^{t_1} f(t)\Delta t = F(t_1) - F(t_0),$$

where F is an antiderivative of f on $\mathbb{T}$. Similar notions hold for left-dense continuous functions and the Cauchy nabla integral. For example, if $\mathbb{T} = \mathbb{Z}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \sum_{t=t_0}^{t_1-1} f(t) \quad \text{and} \quad \int_{t_0}^{t_1} f(t)\nabla t = \sum_{t=t_0+1}^{t_1} f(t),$$

and if $\mathbb{T} = \mathbb{R}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \int_{t_0}^{t_1} f(t)dt = \int_{t_0}^{t_1} f(t)\nabla t.$$

Throughout we assume that $t_0 < t_1$ are points in $\mathbb{T}$, and define the time-scale interval $[t_0, t_1]_{\mathbb{T}} = \{t \in \mathbb{T} : t_0 \le t \le t_1\}$. Other time-scale intervals are defined similarly. For convenience, the composition $x \circ \sigma$ is denoted x^{σ}, and $x \circ \rho$ is denoted x^{ρ}. For more on time scales and time-scale notation, see the fundamental texts [Bohner and Peterson 2001; 2003].

System (1-3) is a generalization of a key second-order linear dynamic equation. To see this, suppose the potential p is nabla differentiable and strictly positive. Then we have

$$x^{\Delta\nabla}(t)=[p(t)y(t)]^{\nabla}=p^{\rho}(t)y^{\nabla}(t)+p^{\nabla}(t)y(t)=-p^{\rho}(t)q(t)x(t)+\frac{p^{\nabla}(t)}{p(t)}x^{\Delta}(t),$$

which we can rewrite in the (formally) self-adjoint form

$$\Big(\frac{1}{p}x^{\Delta}\Big)^{\nabla}(t)+q(t)x(t)=0, \quad t\in\mathbb{T}; \tag{1-4}$$

see [Bohner and Peterson 2003, Section 4.3]. Thus the system (1-3) is an extension of the second-order self-adjoint Equation (1-4), and many important equations are included under the rubric of our discussion below, including the second-order self-adjoint differential equation

$$\Big(\frac{1}{p}x'\Big)'(t)+q(t)x(t)=0, \quad t\in\mathbb{R},$$

the second-order self-adjoint difference equation

$$\Delta\Big(\frac{1}{p_{n-1}}\Delta x_{n-1}\Big)+q_n x_n=0, \quad n\in\mathbb{Z},$$

and the second-order self-adjoint q-difference (quantum) equation ($q>1$)

$$D^q\Big(\frac{1}{p(t)}D_q x(t)\Big)+q(t)x(t)=0, \quad t\in q^{\mathbb{Z}},$$

where

$$D^q x(t)=\frac{x(t)-x(t/q)}{t-t/q} \quad \text{and} \quad D_q x(t)=\frac{x(qt)-x(t)}{qt-t} \tag{1-5}$$

are the quantum backward and forward derivatives, respectively.

2. Preliminary results on oscillation

Let $\mathbb{T}$ be a time scale that is unbounded above, and let $t_0\in\mathbb{T}$. In (1-3), assume $p:\mathbb{T}\to\mathbb{R}$ is right-dense continuous with $p>0$ on $[t_0,\infty)_{\mathbb{T}}$, and $q:\mathbb{T}\to\mathbb{R}$ is continuous with $q\ge 0$ on $[t_0,\infty)_{\mathbb{T}}$; then p is delta integrable and q is integrable. Note the stronger continuity condition on the potential q; from the right-hand equation in (1-3), we then have that y^{∇} is continuous, so that y is delta differentiable as well, with $y^{\Delta}=y^{\nabla\sigma}=-q^{\sigma}x^{\sigma}$. An alternative approach would be to use only delta derivatives in (1-3), with p and q both right-dense continuous functions. The results in the sequel would be analogous to those derived below, but would not incorporate the self-adjoint form (1-4), nor directly extend (1-1). Our techniques are modelled after those found in [Jiang and Tang 2007; Lomtatidze and Partsvania 1999] and the references therein.

A solution (x, y) of (1-3) is oscillatory if both component functions x and y are oscillatory, that is to say neither eventually positive nor eventually negative; otherwise, the solution is nonoscillatory. The dynamic system (1-3) is oscillatory if all its solutions are oscillatory.

Lemma 2.1. *The component functions x and y of a nonoscillatory solution (x, y) of* (1-3) *are themselves nonoscillatory.*

Proof. Assume to the contrary that x oscillates but y is eventually positive. Then $x^\Delta = py > 0$ eventually, so that $x(t) > 0$ or $x(t) < 0$ for all large $t \in \mathbb{T}$, a contradiction. The case where y is eventually negative is similar. Likewise, assuming that y oscillates while x is eventually positive or eventually negative leads to comparable contradictions. $\square$

Lemma 2.2. *If*

$$\int_{t_0}^{\infty} p(r)\Delta r = \infty \quad \textit{and} \quad \int_{\rho(t_0)}^{\infty} q(s)\nabla s = \infty, \tag{2-1}$$

then each solution of (1-3) *is oscillatory.*

Proof. Let (x, y) be a nonoscillatory solution of (1-3). Without loss of generality, we may assume that $x > 0$; then $y^\nabla = -qx \le 0$, and in view of Lemma 2.1, y must be of constant sign eventually. If $y(t_1) < 0$ for some $t_1 \in [t_0, \infty)_\mathbb{T}$, then $y < 0$ on $[t_1, \infty)_\mathbb{T}$ and $x^\Delta = py < 0$ on $[t_1, \infty)_\mathbb{T}$; after delta integrating from t_1 to t, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} p(r)y(r)\Delta r.$$

Since y is negative and nonincreasing, by the first assumption in (2-1) the right-hand side tends to $-\infty$, in contradiction with $x > 0$. Consequently, $y > 0$ with $y^\nabla \le 0$ on $[t_0, \infty)_\mathbb{T}$, and $x^\Delta > 0$ on $[t_0, \infty)_\mathbb{T}$ by the first equation of (1-3). Thus there exists a constant $c > 0$ and $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $x(t) \ge c$ for $t \in [t_1, \infty)_\mathbb{T}$. Nabla integrating the second equation of (1-3), we obtain

$$c \int_{t_1}^{\infty} q(s)\nabla s \le y(t_1) < \infty,$$

and this contradicts the second assumption in (2-1). $\square$

Lemma 2.3. *If*

$$\int_{t_0}^{\infty} p(r)\Delta r < \infty \quad \textit{and} \quad \int_{\rho(t_0)}^{\infty} q(s)\nabla s < \infty, \tag{2-2}$$

then the system (1-3) *is nonoscillatory.*

Proof. Suppose that (2-2) holds. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\int_{t_1}^{\infty} p(r) \left(1 + 2 \int_r^{\infty} q(s) \nabla s\right) \Delta r < 1. \tag{2-3}$$

Let $\mathcal{B} = \mathrm{C_{rd}}(\mathbb{T})$ be the Banach space of right-dense continuous functions on $\mathbb{T}$, with norm $\|x\| = \sup_{t \geq t_1, t \in \mathbb{T}} |x(t)|$ and the usual pointwise ordering $\leq$. Define a subset $\mathcal{S}$ of $\mathcal{B}$ by

$$\mathcal{S} = \big\{x \in \mathcal{B} : 1 \leq x(t) \leq 2,\ t \in [t_1, \infty)_{\mathbb{T}}\big\}.$$

For any subset $\mathcal{Q}$ of $\mathcal{S}$, we have that $\inf \mathcal{Q} \in \mathcal{S}$ and $\sup \mathcal{Q} \in \mathcal{S}$. Let $L : \mathcal{S} \to \mathcal{B}$ be the functional given via

$$(Lx)(t) = 1 + \int_{t_1}^{t} p(r) \left(1 + \int_r^{\infty} q(s) x(s) \nabla s\right) \Delta r, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

By the assumptions on $x \in \mathcal{S}$ and p and q, $(Lx)(t) \geq 1$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, and

$$(Lx)(t) \leq 1 + \int_{t_1}^{t} p(r) \left(1 + \int_r^{\infty} 2q(s) \nabla s\right) \Delta r \leq 2$$

by (2-3). Moreover,

$$(Lx)^{\Delta}(t) = p(t) \left(1 + \int_t^{\infty} q(s) x(s) \nabla s\right) > 0, \tag{2-4}$$

ensuring that $L : \mathcal{S} \to \mathcal{S}$ is increasing. By Knaster's fixed-point theorem [Knaster 1928], we can conclude that there exists an $x \in \mathcal{S}$ such that $x = Lx$. If we let

$$y(t) = 1 + \int_t^{\infty} q(s) x(s) \nabla s, \quad t \in [t_1, \infty)_{\mathbb{T}}$$

using the fixed point $x \in \mathcal{S}$, then we have

$$x^{\Delta}(t) = (Lx)^{\Delta}(t) = p(t) y(t) \quad \text{and} \quad y^{\nabla}(t) = -q(t) x(t)$$

for $t \in [t_1, \infty)_{\mathbb{T}}$ by using (2-4). Thus (x, y) is a nonoscillatory solution of (1-3). □

In light of Lemmas 2.2 and 2.3, respectively, we could assume that either

$$\int_{t_0}^{\infty} p(r) \Delta r = \infty \quad \text{and} \quad \int_{\rho(t_0)}^{\infty} q(s) \nabla s < \infty \tag{2-5}$$

or

$$\int_{t_0}^{\infty} p(r) \Delta r < \infty \quad \text{and} \quad \int_{\rho(t_0)}^{\infty} q(s) \nabla s = \infty;$$

in fact, we will focus on (2-5). Moreover, in preparation for what follows, we introduce the following notation. Let

$$P(t) := \int_{t_0}^{t} p(r)\Delta r. \tag{2-6}$$

Lemma 2.4. *Assume that* (2-5) *holds*, P *is given by* (2-6), *and* $\lambda \in [0, 1)$ *is a real number. If*

$$\lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0, \quad \left(\text{equivalently}, \quad \lim_{t\to\infty} \frac{P^{\sigma}(t)}{P(t)} = 1\right) \tag{2-7}$$

then given $\varepsilon > 0$ *there exists a* $t_1 \equiv t_1(\varepsilon) \in (t_0, \infty)_{\mathbb{T}}$ *such that for any* $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\int_t^{\infty} \frac{\left[(P^{\lambda})^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \Delta r \le \frac{\lambda^2}{1-\lambda} (1+\varepsilon)^{2-\lambda} P^{\lambda-1}(t), \quad \text{and} \tag{2-8}$$

$$\int_t^{\infty} \frac{p(r)}{P^{2-\lambda}(r)} \Delta r \le \frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda} P^{\lambda-1}(t). \tag{2-9}$$

Proof. For $r \in (t_0, \infty)_{\mathbb{T}}$, by the chain rule [Bohner and Peterson 2001, Theorem 1.90] we have

$$(P^{\lambda})^{\Delta}(r) = \begin{cases} \dfrac{P^{\lambda}(\sigma(r)) - P^{\lambda}(r)}{\mu(r)} & \text{if } \mu(r) > 0, \\ \lambda p(r) P^{\lambda-1}(r) & \text{if } \mu(r) = 0. \end{cases}$$

By [Bohner and Peterson 2001, Theorem 1.16 (iv)], $\mu P^{\Delta} = P^{\sigma} - P$, so that $\mu p = P^{\sigma} - P$ on $\mathbb{T}$. If $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-scattered point, then $\mu(r) > 0$ and, suppressing the r,

$$\begin{aligned} \frac{\left[(P^{\lambda})^{\Delta}\right]^2}{pP^{\lambda}} &= \frac{p}{\mu^2 p^2 P^{\lambda}} \left((P^{\sigma})^{\lambda} - P^{\lambda}\right)^2 = \frac{p}{P^{\lambda}} \left(\frac{(P^{\sigma})^{\lambda} - P^{\lambda}}{P^{\sigma} - P}\right)^2 \\ &\overset{\text{MVT}}{=} \frac{p}{P^{\lambda}} \left(\lambda \xi^{\lambda-1}\right)^2, \quad \xi \in \left(P(r), P^{\sigma}(r)\right)_{\mathbb{R}} \\ &\le \frac{p\lambda^2}{P^{\lambda}} P^{2\lambda-2}, \quad \lambda - 1 < 0 \\ &= \lambda^2 p P^{\lambda-2}. \end{aligned}$$

If $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-dense point, then $\mu(r) = 0$ and

$$\frac{\left[(P^{\lambda})^{\Delta}\right]^2}{pP^{\lambda}} = \frac{\left[\lambda p P^{\lambda-1}\right]^2}{pP^{\lambda}} = \lambda^2 p P^{\lambda-2}.$$

It follows that in either case,

$$\frac{\left[(P^{\lambda})^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \leq \lambda^2 p(r) P^{\lambda-2}(r), \quad r \in (t_0, \infty)_{\mathbb{T}}. \tag{2-10}$$

Similarly, if $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-scattered point, then once again $\mu(r) > 0$ and, suppressing the r,

$$\begin{aligned} -(P^{\lambda-1})^{\Delta} &= \frac{-p}{\mu p}\left((P^{\sigma})^{\lambda-1} - P^{\lambda-1}\right) = -p\left(\frac{(P^{\sigma})^{\lambda-1} - P^{\lambda-1}}{P^{\sigma} - P}\right) \\ &\overset{\text{MVT}}{=} p(1-\lambda)\eta^{\lambda-2}, \quad \eta \in \left(P(r), P^{\sigma}(r)\right)_{\mathbb{R}} \\ &\geq p(1-\lambda)(P^{\sigma})^{\lambda-2}. \end{aligned}$$

If r is a right-dense point, then $P^{\sigma} = P$, $\mu(r) = 0$, and $p(1-\lambda)P^{\lambda-2} = -(P^{\lambda-1})^{\Delta}$. Summarizing, in either case we have

$$-(P^{\lambda-1})^{\Delta} \geq p(1-\lambda)(P^{\sigma})^{\lambda-2}, \quad r \in (t_0, \infty)_{\mathbb{T}}. \tag{2-11}$$

Combining (2-10) and (2-11), we see that

$$\frac{\left[(P^{\lambda})^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \leq \frac{\lambda^2}{1-\lambda}\left(\frac{P(r)}{P^{\sigma}(r)}\right)^{\lambda-2}\left[-(P^{\lambda-1})^{\Delta}(r)\right].$$

By (2-7), given $\varepsilon > 0$ there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $P^{\sigma}/P \leq (1+\varepsilon)$ on $[t_1, \infty)_{\mathbb{T}}$. Consequently, for any $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \int_t^{\infty} \frac{\left[(P^{\lambda})^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \Delta r \quad &\leq \quad \frac{\lambda^2}{1-\lambda}(1+\varepsilon)^{2-\lambda} \int_t^{\infty} \left[-(P^{\lambda-1})^{\Delta}(r)\right] \Delta r \\ &\overset{(2\text{-}5),(2\text{-}6)}{=} \frac{\lambda^2}{1-\lambda}(1+\varepsilon)^{2-\lambda} P^{\lambda-1}(t), \end{aligned}$$

which is (2-8). Moreover, again for any $r \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \frac{p(r)}{P^{2-\lambda}(r)} \quad &= \quad \frac{p(r)}{P^{2-\lambda}(\sigma(r))}\frac{P^{2-\lambda}(\sigma(r))}{P^{2-\lambda}(r)} \leq (1+\varepsilon)^{2-\lambda}\frac{p(r)}{P^{2-\lambda}(\sigma(r))} \\ &\overset{(2\text{-}11)}{\leq} \frac{(1+\varepsilon)^{2-\lambda}}{\lambda-1}(P^{\lambda-1})^{\Delta}(r). \end{aligned} \tag{2-12}$$

Delta integrating (2-12) from t to infinity, we obtain

$$\int_t^{\infty} \frac{p(r)}{P^{2-\lambda}(r)} \Delta r \leq \frac{(1+\varepsilon)^{2-\lambda}}{\lambda-1} \int_t^{\infty} (P^{\lambda-1})^{\Delta}(r) \Delta r \overset{(2\text{-}5),(2\text{-}6)}{=} \frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda} P^{\lambda-1}(t),$$

which is (2-9). $\square$

Note that if $\mathbb{T} = \mathbb{R}$, then (2-7) is automatically satisfied, as $\mu(t) \equiv 0$.

Lemma 2.5. *Assume that* (2-5) *holds, that P is given by* (2-6), *and that* (2-7) *holds. If for some real number $\lambda < 1$ we have*

$$\int_{t_1}^{\infty} q^{\sigma}(r) P^{\lambda}(r) \Delta r = \infty \quad \textit{for} \quad t_1 > \sigma(t_0), \tag{2-13}$$

then the system (1-3) *is oscillatory.*

Proof. By Lemma 2.3, we can focus on $\lambda \in (0, 1)$. Assume that (x, y) is a nonoscillatory solution of the system (1-3); without loss of generality, assume that $x > 0$ on $[t_0, \infty)_{\mathbb{T}}$. As in the proof of Lemma 2.2, $y > 0$ with $y^{\nabla} \le 0$ and $x^{\Delta} > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Let $w := y/x$. Then $w > 0$, and suppressing the argument, we have by the delta quotient rule and (1-3) that on $[t_0, \infty)_{\mathbb{T}}$,

$$w^{\Delta} = \frac{x y^{\Delta} - y x^{\Delta}}{x x^{\sigma}} = -q^{\sigma} - p w y / x^{\sigma} \le -q^{\sigma} - p w w^{\sigma} < 0. \tag{2-14}$$

In fact this gives us

$$w^{\Delta} \le -q^{\sigma} - p (w^{\sigma})^2, \tag{2-15}$$

and from the previous line we obtain on $[t_0, \infty)_{\mathbb{T}}$ that

$$\left(\frac{1}{w}\right)^{\Delta} = \frac{-w^{\Delta}}{w w^{\sigma}} \ge \frac{q^{\sigma} + p w w^{\sigma}}{w w^{\sigma}} \ge p;$$

delta integrating from t_0 to t we see that

$$1 > 1 - \frac{w(t)}{w(t_0)} \ge w(t) \int_{t_0}^{t} p(r) \Delta r = w(t) P(t) \ge 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{2-16}$$

Again by the mean value theorem, $(P^{\lambda})^{\Delta} \le \lambda p P^{\lambda - 1}$ for $\lambda \in (0, 1)$. Recall that q is assumed to be continuous, so q^{σ} is right-dense continuous, and thus delta integrable. Multiplying (2-15) by P^{λ} and delta integrating from $t_1 > \sigma(t_0)$ to t gives

$$\begin{aligned}
&\int_{t_1}^{t} q^{\sigma}(r) P^{\lambda}(r) \Delta r \\
&\le -\int_{t_1}^{t} P^{\lambda}(r) w^{\Delta}(r) \Delta r - \int_{t_1}^{t} p(r) P^{\lambda}(r) (w^{\sigma})^2(r) \Delta r \\
&\overset{\text{parts}}{=} -P^{\lambda}(t) w(t) + P^{\lambda}(t_1) w(t_1) + \int_{t_1}^{t} (P^{\lambda})^{\Delta}(r) w^{\sigma}(r) \Delta r - \int_{t_1}^{t} p(r) P^{\lambda}(r) (w^{\sigma})^2(r) \Delta r \\
&\le -P^{\lambda}(t) w(t) + P^{\lambda}(t_1) w(t_1) \\
&\qquad + \int_{t_1}^{t} \lambda p(r) P^{\lambda-1}(r) w^{\sigma}(r) \Delta r - \int_{t_1}^{t} p(r) P^{\lambda}(r) (w^{\sigma})^2(r) \Delta r \\
&= -P^{\lambda}(t) w(t) + P^{\lambda}(t_1) w(t_1) + \int_{t_1}^{t} p(r) P^{\lambda-2}(r) \left[P(r) w^{\sigma}(r) \big(\lambda - P(r) w^{\sigma}(r) \big) \right] \Delta r.
\end{aligned}$$

Since by (2-16) we have

$$0 < P(t)w^\sigma(t) \le P(t)w(t) < 1, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{2-17}$$

there exists a positive real number k such that

$$\left|P(r)w^\sigma(r)\big(\lambda - P(r)w^\sigma(r)\big)\right| < k.$$

As a result we have $\lim_{t\to\infty} -P^\lambda(t)w(t) = 0$ by (2-16) for $0 < \lambda < 1$, and

$$\begin{aligned}\left|\int_{t_1}^{t} p(r)P^{\lambda-2}(r)\left[P(r)w^\sigma(r)\big(\lambda - P(r)w^\sigma(r)\big)\right]\Delta r\right| &< k\int_{t_1}^{\infty} p(r)P^{\lambda-2}(r)\Delta r \\ &\overset{(2\text{-}9)}{\le} k\frac{(1+\varepsilon)^{2-\lambda}}{1-\lambda}P^{\lambda-1}(t_1)\end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore we get $\displaystyle\int_{t_1}^{\infty} q^\sigma(r)P^\lambda(r)\Delta r < \infty$, in contradiction with (2-13). □

Due to (2-5) and the establishment of Lemma 2.5, we will henceforth restrict our analysis to the case where

$$\int_{t_0}^{\infty} p(r)\Delta r = \infty, \qquad \int_{t_1}^{\infty} q^\sigma(r)P^\lambda(r)\Delta r < \infty \quad \text{for } \lambda < 1,\ t_1 > \sigma(t_0). \tag{2-18}$$

We also adopt the following notation. Set

$$g(t,\lambda) := \begin{cases} P^{1-\lambda}(t)\int_t^\infty q^\sigma(r)P^\lambda(r)\Delta r & \text{if } \lambda < 1, \\ P^{1-\lambda}(t)\int_{t_0}^t q^\sigma(r)P^\lambda(r)\Delta r & \text{if } \lambda > 1. \end{cases}$$

Then take

$$g_*(\lambda) := \liminf_{t\to\infty} g(t,\lambda) \quad \text{and} \quad g^*(\lambda) := \limsup_{t\to\infty} g(t,\lambda).$$

Lemma 2.6. *Assume that* (2-18) *holds, that P is given by* (2-6), *and that* (2-7) *holds. If (x, y) is a nonoscillatory solution of the system* (1-3), *then*

$$\liminf_{t\to\infty} w(t)P(t) \ge \tfrac{1}{2}\big(1 - \sqrt{1-4g_*(0)}\big), \tag{2-19}$$

$$\limsup_{t\to\infty} w(t)P(t) \le \tfrac{1}{2}\big(1 + \sqrt{1-4g_*(2)}\big), \tag{2-20}$$

where again $w := y/x$.

Proof. By (2-16), we can introduce the constants

$$r := \liminf_{t\to\infty} w(t)P(t), \qquad R := \limsup_{t\to\infty} w(t)P(t), \tag{2-21}$$

and by (2-18), we must have

$$\lim_{t\to\infty} w(t) = 0. \tag{2-22}$$

From (2-14) we have $w^\Delta \le -q^\sigma - pww^\sigma$; delta integrate this from t to ∞, use (2-22), and multiply by P to see that

$$w(t)P(t) \ge P(t)\int_t^\infty q^\sigma(\tau)\Delta\tau + P(t)\int_t^\infty p(\tau)w(\tau)w^\sigma(\tau)\Delta\tau \tag{2-23}$$

holds for $t \in [t_1, \infty)_{\mathbb{T}}$. From (2-21) this yields

$$r \ge g_*(0). \tag{2-24}$$

This time multiply (2-15) by P^2 and delta integrate from t_1 to t to get

$$\begin{aligned}
&\int_{t_1}^t q^\sigma(\tau)P^2(\tau)\Delta\tau \\
&\le -\int_{t_1}^t P^2(\tau)w^\Delta(\tau)\Delta\tau - \int_{t_1}^t p(\tau)P^2(\tau)\big(w^\sigma\big)^2(\tau)\Delta\tau \\
&= -P^2(t)w(t) + P^2(t_1)w(t_1) + \int_{t_1}^t (P^2)^\Delta(\tau)w^\sigma(\tau)\Delta\tau - \int_{t_1}^t p(\tau)P^2(\tau)(w^\sigma)^2(\tau)\Delta\tau \\
&= -P^2(t)w(t) + P^2(t_1)w(t_1) \\
&\qquad + \int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau + \int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)\big[2 - P(\tau)w^\sigma(\tau)\big]\Delta\tau
\end{aligned}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$, which leads to

$$\begin{aligned}
w(t)P(t) \le -P^{-1}(t)&\int_{t_1}^t q^\sigma(\tau)P^2(\tau)\Delta\tau \\
&+ P^{-1}(t)\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau + P^{-1}(t)P^2(t_1)w(t_1) \\
&+ P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)\big[2 - P(\tau)w^\sigma(\tau)\big]\Delta\tau. \qquad (2\text{-}25)
\end{aligned}$$

Using (2-17), we obtain $0 < (1 - Pw^\sigma)^2$, leading to $Pw^\sigma[2 - Pw^\sigma] < 1$. Thus, for large $t \in \mathbb{T}$,

$$P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)\big[2 - P(\tau)w^\sigma(\tau)\big]\Delta\tau \le 1.$$

Applying l'Hôpital's rule [Bohner and Peterson 2001, Theorem 1.120], (2-17) again, and (2-7) we have

$$0 \le \lim_{t\to\infty} \frac{\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau}{P(t)} = \lim_{t\to\infty} \mu(t)p(t)w^\sigma(t) \le \lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0.$$

Altogether then, inequality (2-25) implies that

$$R \le 1 - g_*(2). \tag{2-26}$$

If $g_*(0) = 0 = g_*(2)$, then estimates (2-19) and (2-20) follow directly from (2-24) and (2-26), respectively. Thus we pick a real number $\varepsilon \in \big(0, \min\{g_*(0), g_*(2)\}\big)$ and $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$r - \varepsilon < w(t)P(t) < R + \varepsilon, \qquad w(t)P(t) \ge P(t)\int_t^\infty q^\sigma(\tau)\Delta\tau > g_*(0) - \varepsilon,$$

$$P^{-1}(t)\int_{t_0}^t q^\sigma(\tau)P^2(\tau)\Delta\tau > g_*(2) - \varepsilon.$$

From (2-23) and l'Hôpital's rule we have $w(t)P(t) \ge g_*(0) - \varepsilon + (r-\varepsilon)^2$ for $t \in [t_2, \infty)_{\mathbb{T}}$, Multiply (2-14) by P^2 and delta integrate from t_1 to t to see that this leads to

$$\begin{aligned} w(t)P(t) \le -P^{-1}(t)&\int_{t_1}^t q^\sigma(\tau)P^2(\tau)\Delta\tau \\ &+ P^{-1}(t)\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau + P^{-1}(t)P^2(t_1)w(t_1) \\ &\quad + P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)\,[2 - w(\tau)P(\tau)]\,\Delta\tau. \end{aligned} \tag{2-27}$$

From (2-27) we have, for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$w(t)P(t) \le \frac{P^2(t_1)w(t_1) + \int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau}{P(t)} - g_*(2) + \varepsilon + (R+\varepsilon)(2 - R - \varepsilon),$$

since $w^\sigma P \le wP < 1$. These two inequalities lead to

$$r \ge g_*(0) + r^2, \qquad R \le R(2-R) - g_*(2). \tag{2-28}$$

Consequently we have $r \ge \frac{1}{2}\big(1 - \sqrt{1 - 4g_*(0)}\big)$ and $R \le \frac{1}{2}\big(1 + \sqrt{1 - 4g_*(2)}\big)$, and the lemma is proven. □

3. Main oscillation results

We use the lemmas obtained previously to prove our main results.

Theorem 3.1. *Assume that* (2-18) *holds, that P is given by* (2-6), *and that* (2-7) *holds. If*

$$g_*(0) = \liminf_{t\to\infty} P(t)\int_t^\infty q^\sigma(\tau)\Delta\tau > \frac{1}{4} \tag{3-1}$$

or

$$g_*(2) = \liminf_{t\to\infty} \frac{1}{P(t)}\int_{t_0}^t q^\sigma(\tau)P^2(\tau)\Delta\tau > \frac{1}{4}, \tag{3-2}$$

then every solution of the system (1-3) *is oscillatory.*

Proof. Suppose to the contrary that (x, y) is a nonoscillatory solution of (1-3) with $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Let

$$r := \liminf_{t\to\infty} w(t)P(t), \qquad R := \limsup_{t\to\infty} w(t)P(t),$$

where $w = y/x$. By Lemma 2.6 and its proof (in particular (2-28)) and simple calculus, we have

$$g_*(0) \le r - r^2 \le \tfrac{1}{4} \quad \text{and} \quad g_*(2) \le R - R^2 \le \tfrac{1}{4},$$

in contradiction with both (3-1) and (3-2). $\square$

Theorem 3.2. *Assume that* (2-18) *holds, that P is given by* (2-6), *and that* (2-7) *holds. Let $g_*(2) \le \frac{1}{4}$, and assume there exists a real number $\lambda \in [0, 1)$ such that*

$$g^*(\lambda) > \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}\left(1 + \sqrt{1 - 4g_*(2)}\right). \tag{3-3}$$

Then every solution of the system (1-3) *is oscillatory.*

Proof. Suppose to the contrary that (x, y) is a nonoscillatory solution of (1-3) with $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. By (2-15) we have

$$q^\sigma(t) \le -w^\Delta(t) - p(t)(w^\sigma)^2(t), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

where $w = y/x$; multiply this by P^λ and delta integrate from t to infinity to get

$$\begin{aligned}
&\int_t^\infty q^\sigma(\tau)P^\lambda(\tau)\Delta\tau \\
&\quad \le -\int_t^\infty w^\Delta(\tau)P^\lambda(\tau)\Delta\tau - \int_t^\infty p(\tau)(w^\sigma)^2(\tau)P^\lambda(\tau)\Delta\tau \\
&\quad = P^\lambda(t)w(t) + \int_t^\infty (P^\lambda)^\Delta(\tau)w^\sigma(\tau)\Delta\tau - \int_t^\infty p(\tau)P^\lambda(\tau)(w^\sigma)^2(\tau)\Delta\tau \\
&\quad = P^\lambda(t)w(t) + \frac{1}{4}\int_t^\infty \frac{\left((P^\lambda)^\Delta\right)^2(\tau)}{p(\tau)P^\lambda(\tau)}\Delta\tau \\
&\qquad\qquad - \int_t^\infty \left(\sqrt{p(\tau)}P^{\lambda/2}(\tau)w^\sigma(\tau) - \frac{(P^\lambda)^\Delta(\tau)}{2\sqrt{p(\tau)}P^{\lambda/2}(\tau)}\right)^2 \Delta\tau \\
&\quad \le P^\lambda(t)w(t) + \frac{1}{4}\int_t^\infty \frac{\left((P^\lambda)^\Delta\right)^2(\tau)}{p(\tau)P^\lambda(\tau)}\Delta\tau.
\end{aligned}$$

It follows that

$$P^{1-\lambda}(t)\int_t^\infty q^\sigma(\tau)P^\lambda(\tau)\Delta\tau \, < \, P(t)w(t) + \frac{P^{1-\lambda}(t)}{4}\int_t^\infty \frac{\left((P^\lambda)^\Delta\right)^2(\tau)}{p(\tau)P^\lambda(\tau)}\Delta\tau. \tag{3-4}$$

By (2-8), (2-20), and (3-4),

$$g^*(\lambda) \le \tfrac{1}{2}\big(1+\sqrt{1-4g_*(2)}\,\big) + \frac{\lambda^2}{4(1-\lambda)},$$

in contradiction with (3-3). □

Corollary 3.3. *Assume that* (2-18) *holds*, *that* P *is given by* (2-6), *and that* (2-7) *holds. If* $g_*(2) \le \frac{1}{4}$ *and* $g^*(0) > \frac{1}{2}\big(1+\sqrt{1-4g_*(2)}\,\big)$, *then every solution of the system* (1-3) *is oscillatory.*

Theorem 3.4. *Assume that* (2-18) *holds*, *that* P *is given by* (2-6), *and that* (2-7) *holds. Let* $g_*(0), g_*(2) \le \frac{1}{4}$, *and assume there exists a real number* $\lambda \in [0, 1)$ *such that*

$$g_*(0) > \frac{\lambda(2-\lambda)}{4} \quad \textit{and} \quad g^*(\lambda) > \frac{g_*(0)}{1-\lambda} + \tfrac{1}{2}\big(\sqrt{1-4g_*(0)}+\sqrt{1-4g_*(2)}\,\big). \tag{3-5}$$

Then every solution of the system (1-3) *is oscillatory.*

Proof. Suppose to the contrary that (x, y) is a nonoscillatory solution of (1-3) with $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Set

$$r = \liminf_{t\to\infty} w(t)P(t) \quad \text{and } R = \limsup_{t\to\infty} w(t)P(t),$$

where $w = y/x$. By (2-19) and (2-20),

$$r \ge m := \tfrac{1}{2}\big(1-\sqrt{1-4g_*(0)}\,\big) \quad \text{and} \quad R \le M := \tfrac{1}{2}\big(1+\sqrt{1-4g_*(2)}\,\big). \tag{3-6}$$

Using this and the first inequality in (3-5) we find that $m > \lambda/2$, whence given $\varepsilon \in (0, m-\lambda/2)$, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$m-\varepsilon < w(t)P(t) < M+\varepsilon, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3-7}$$

Similar to what we did before (bottom of page 8), we multiply (2-15) by P^λ and delta integrate from t to infinity to get

$$\int_t^\infty q^\sigma(\tau)P^\lambda(\tau)\Delta\tau \le w(t)P^\lambda(t) + \int_t^\infty p(\tau)P^{\lambda-2}(\tau)\big[\lambda w^\sigma(\tau)P(\tau) - \big(P(\tau)w^\sigma(\tau)\big)^2\big]\Delta\tau;$$

this leads to

$$P^{1-\lambda}(t)\int_t^\infty q^\sigma(\tau)P^\lambda(\tau)\Delta\tau \le w(t)P(t) + P^{1-\lambda}(t)\int_t^\infty p(\tau)P^{\lambda-2}(\tau)\big[\lambda w^\sigma(\tau)P(\tau) - \big(P(\tau)w^\sigma(\tau)\big)^2\big]\Delta\tau.$$

Since the function $f(x) := \lambda x - x^2$ is decreasing over the real interval $[\lambda/2, \infty)$, it follows from the preceding inequality together with (3-7) and Lemma 2.4 that

$$\begin{aligned}P^{1-\lambda}(t)&\int_t^\infty q^\sigma(\tau)P^\lambda(\tau)\Delta\tau\\ &< M+\varepsilon+(m-\varepsilon)(\lambda-m+\varepsilon)P^{1-\lambda}(t)\int_t^\infty p(\tau)P^{\lambda-2}(\tau)\Delta\tau\\ &< M+\varepsilon+\frac{(m-\varepsilon)(\lambda-m+\varepsilon)(1+\varepsilon)^{2-\lambda}}{1-\lambda}.\end{aligned}$$

This in tandem with (3-6) yields

$$g^*(\lambda) \le M + \frac{m(\lambda-m)}{1-\lambda} = \frac{g_*(0)}{1-\lambda} + \tfrac{1}{2}\big(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\big),$$

in contradiction with the second inequality in (3-5). $\square$

Corollary 3.5. *Assume that* (2-18) *holds, that P is given by* (2-6), *and that* (2-7) *holds. Let* $0 < g^*(0) \le \frac{1}{4}$ *and* $g_*(2) \le \frac{1}{4}$. *If*

$$g^*(0) > g_*(0) + \tfrac{1}{2}\big(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\big),$$

then every solution of the system (1-3) *is oscillatory.*

4. Examples

We illustrate Theorem 3.1 with the following examples.

Example 4.1. Let $\mathbb{T} = \mathbb{R}$ and $\varepsilon > 0$. Then the continuous linear system

$$x'(t) = \frac{1}{2}\big(\varepsilon + \cos^2 t\big)y(t), \qquad y'(t) = -\frac{1}{t^2}x(t) \tag{4-1}$$

is oscillatory on $[1, \infty)$.

Since $p(t) = \frac{1}{2}(\varepsilon + \cos^2 t)$, we have $P(t) = \frac{1}{8}\big(-2-4\varepsilon+t(2+4\varepsilon)-\sin 2+\sin 2t\big)$. Thus

$$g_*(0) = \liminf_{t\to\infty} P(t)\int_t^\infty q(r)dr = \liminf_{t\to\infty} P(t)/t = \tfrac{1}{4}(1+2\varepsilon) > \tfrac{1}{4}.$$

By Theorem 3.1, any solution pair (x, y) oscillates. Let $x(1) = 0$, $x'(1) = 1$. Numerically generated data for the solutions to (4-1) show a decreasing frequency in oscillations as ε goes to 0, as one might expect. The table shows the value of ε and the estimated value of the first zero of x after $t = 1$.

ε	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
next zero of x	100	128	171	244	379	666	1424	4321	27498

The results led us to wonder whether $\frac{1}{4}$ is sharp for $\mathbb{T} = \mathbb{R}$, and to the next example.

Example 4.2. Let $\mathbb{T} = \mathbb{R}$ and let p be a positive constant. The continuous linear system

$$x'(t) = py(t), \qquad y'(t) = -\frac{1}{t^2}x(t), \qquad t \in [1, \infty) \tag{4-2}$$

is nonoscillatory for $0 < p \le \frac{1}{4}$ and oscillatory for $p > \frac{1}{4}$.

Since $p(t) \equiv p$ we have $P(t) = p(t-1)$. Thus

$$g_*(0) = \liminf_{t\to\infty} P(t) \int_t^\infty q(\tau)d\tau = \liminf_{t\to\infty} \frac{p(t-1)}{t} = p.$$

By Theorem 3.1 and (3-1), any solution (x, y) of (4-2) oscillates if $p > \frac{1}{4}$. Converting (4-2) to a second-order equation for x, we arrive at a Cauchy–Euler equation with general solution

$$x(t) = t^{(1-\sqrt{1-4p})/2}\big(A + Bt^{\sqrt{1-4p}}\big).$$

From elementary analysis we know that x is nonoscillatory for $p \le \frac{1}{4}$ and oscillatory for $p > \frac{1}{4}$, showing in particular that the $\frac{1}{4}$ in (3-1) is sharp when $\mathbb{T} = \mathbb{R}$.

5. Future directions: half-linear systems

Let $\varphi_p(x) = |x|^{p-2}x$ for $p > 1$ be the one-dimensional p-Laplacian, and consider the following half-linear equation

$$\big[u(t)\varphi_r\big(x^\Delta(t)\big)\big]^\nabla + w(t)\varphi_q\big(x(t)\big) = 0, \tag{5-1}$$

where $r, q > 1$ and $u, w : \mathbb{T} \to \mathbb{R}$ satisfy $u(t) > 0$ and $w(t) \ge 0$, respectively. Let $y = u\varphi_r(x^\Delta)$. Then $y^\nabla = -w\varphi_q(x)$ and $x^\Delta = \varphi_r^{-1}(1/u)\varphi_r^{-1}(y)$, which can be generalized to the half-linear, p-Laplacian system

$$x^\Delta(t) = v(t)\varphi_p\big(y(t)\big), \quad y^\nabla(t) = -w(t)\varphi_q\big(x(t)\big), \quad t \in \mathbb{T}. \tag{5-2}$$

Here we assume $v : \mathbb{T} \to \mathbb{R}$ is right-dense continuous with $v > 0$ on $[t_0, \infty)_\mathbb{T}$, and $w : \mathbb{T} \to \mathbb{R}$ is continuous with $w \ge 0$ on $[t_0, \infty)_\mathbb{T}$. Future research would take the earlier results shown valid for (1-3) and try to modify them to cover (5-2) and thus (5-1) as well.

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andersod@cord.edu *Department of Mathematics and Computer Science, Concordia College, 901 South 8th Street, Moorhead, MN 56562, United States*
http://www.cord.edu/faculty/andersod/

wrhall@cord.edu *1624 King Cove, Grand Forks, ND 58201, United States*