involve
a journal of mathematics

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mathematical sciences publishers

2009 Vol. 2, No. 1

INVOLVE 2:1(2009)

Zero-divisor ideals and realizable zero-divisor graphs

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(Communicated by Scott Chapman)

We seek to classify the sets of zero-divisors that form ideals based on their zero-divisor graphs. We offer full classification of these ideals within finite commutative rings with identity. We also provide various results concerning the realizability of a graph as a zero-divisor graph.

1. Definitions and notation

We will begin by introducing the necessary definitions and notation that will be used throughout the paper. In Section 2, we will determine when the zero-divisors form an ideal in a finite commutative ring with identity, and Section 3 partially generalizes these results to the cases where R is infinite or lacks identity. Section 4 is concerned with the realizability of graphs as zero-divisor graphs.

Given a commutative ring R, an element $x \in R$ is a *zero-divisor* if there exists a nonzero $y \in R$ such that $xy = 0$. We denote the set of zero-divisors as $Z(R)$, and the set of nonzero zero-divisors denoted by $Z(R)^*$. For $x \in R$, the *annihilator* of x, denoted $\operatorname{ann}(x)$, is $\{y \in R \mid xy = 0\}$. It can be shown that the annihilator of any element in a ring is an ideal. An element x is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. The set of all units in R is denoted $U(R)$. If $x, y \in R$ where R is integral domain, we say x and y are *associates* if $x = uy$, where $u \in U(R)$. A ring R is a *local ring* if and only if R has a unique maximal ideal.

For a graph G, we define $V(G)$ and $E(G)$ to be the sets of vertices and edges of G, respectively. Two elements $x, y \in V(G)$ are defined to be *adjacent*, denoted by

MSC2000: 13A99.

Keywords: zero-divisor, graph, commutative ring.

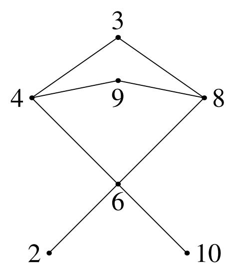
Wallace Trampbachls is a pseudonym which serves to represent a group of student participants at the Wabash Summer Institute in Mathematics in Crawfordsville, Indiana. The first name, Wallace, commemorates Crawfordsville's famous son, Lew Wallace, who authored *Ben Hur*, served as the U.S. Ambassador to the Ottoman Empire, and was a general for the Union Forces in the Civil War. The surname 'Trampbachls' was formed from the first letter of the last name of each student participant. A complete list of the participants is given in the *Acknowledgments*.

x–y, if there exists an edge between them. A *path* between two elements

$$a_1, a_n \in V(G)$$

is an ordered sequence of distinct vertices of G, $\{a_1, a_2, \ldots, a_n\}$, such that a_{i-1}–a_i. The *length* of a path between x and y is the number of edges crossed to get from x to y in the path. The *distance* between $x, y \in G$, denoted $d(x, y)$, is the length of a shortest path between x and y, if such a path exists; otherwise, $d(x, y) = \infty$. For the purposes of this paper, we define $d(x, x) = 0$. The *diameter* of a graph is $\mathrm{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}$. An element $x \in V(G)$ is said to be *looped* if there exists an edge from x to itself. A graph G is called *complete bipartite* if there exist disjoint subsets A, B of $V(G)$ such that $A \cup B = V(G)$, $x \not- y$ for any distinct $x, y \in A$ or $x, y \in B$, and x–y for any $x \in A$ and $y \in B$. Finite complete bipartite graphs are denoted as $K^{m,n}$, where $|A| = m$ and $|B| = n$. A graph G is said to be *complete bipartite reducible* if and only if there exists a complete bipartite graph G' such that $V(G') = V(G)$ and $E(G') \subsetneq E(G)$. A graph G is a *star graph* if $G = K^{1,n}$. A graph G is said to be *star-shaped reducible* if and only if there exists a $g \in V(G)$ such that g is adjacent to all other vertices in G and g is looped. More information about graph theory may be found in [Wilson 1972]. We define the *zero-divisor graph* of R, denoted $\Gamma(R)$, as follows: $x \in V(\Gamma(R))$ if and only if $x \in Z(R)^*$, and x–y if and only if $xy = 0$. We will allow loops in $\Gamma(R)$, which is a change from other definitions of zero-divisor graphs as in [Anderson and Livingston 1999; Axtell et al. 2006; Lucas 2006; Redmond 2007].

As an illustration of zero-divisor graphs, we show $\Gamma(\mathbb{Z}_{12})$:



2. Finite rings with identity

In this section, we will ascertain when $Z(R)$ is an ideal in a finite commutative ring with identity by using $\Gamma(R)$, and we will determine the nature of loops in $\Gamma(R)$. Note that to show $Z(R)$ is an ideal, we need only show it is closed under addition.

The following lemma is well known.

Lemma 2.1. *In a finite commutative ring with identity, every element is either a unit or a zero divisor.*

Lemma 2.2. *Let R be a commutative ring. Given any finite set*

$$\{p_1, p_2, \dots, p_n\} \subset R,$$

where all p_i are nilpotent, there exists a nonzero $a \in R$ such that $ap_i = 0$ for all $1 \le i \le n$.

Proof. Since p_1 is nilpotent, there exists a minimal m_1 such that $p_1^{m_1} = 0$. Let $a_1 = p_1^{m_1-1}$. Clearly, $a_1 p_1 = 0$, and $a_1 \neq 0$. Inductively, since p_i is nilpotent, and $a_{i-1} \neq 0$, there exists $m_i - 1$ (possibly zero, in which case, $a_i = a_{i-1}$), such that $a_i = a_{i-1} p_i^{m_i-1} \neq 0$, but $a_i p_i = 0$. Let $a = a_n$. By construction, a annihilates every p_i. □

Theorem 2.3. *Let R be a finite commutative ring with identity. Then the following are equivalent*:

(1) *$Z(R)$ is an ideal*;
(2) *$Z(R)$ is a maximal ideal*;
(3) *R is local*;
(4) *Every $x \in Z(R)$ is nilpotent*;
(5) *There exists $b \in Z(R)$ such that $bZ(R) = 0$, and hence $\Gamma(R)$ is star-shaped reducible.*

Proof. Since R is finite, every element is either a zero-divisor or a unit by Lemma 2.1. Hence, whenever $Z(R)$ is an ideal, it must be maximal, since any ideal that properly contains $Z(R)$ must contain a unit and must therefore contain all of R. Also, whenever $Z(R)$ is a maximal ideal, it must be the only one; for consider another ideal I of R. Then either $I \subsetneq Z(R)$, in which case it is not maximal, or else it contains a unit and is not proper. If R is local, then it has a single maximal ideal M. By Lemma 2.1, every element is either a unit or a zero-divisor. In addition, every nonunit is in M, since M is maximal, and every unit is not in M, so M is the set of zero-divisors. Hence the zero-divisors form an ideal. Thus, (1), (2), and (3) are all equivalent.

($1 \Rightarrow 4$) Assume $Z(R)$ is an ideal. Suppose $x \in Z(R)$. Then there exist minimal $i > j > 0$ such that $x^i = x^j$. Then, $x^i - x^j = 0$ implies $x^j(x^{i-j} - 1) = 0$. Thus $x^j = 0$ or $x^{i-j} - 1 \in Z(R)$. In the latter case, since $Z(R)$ is an ideal and $x^{i-j} \in Z(R)$, we get $(x^{i-j} - 1) - x^{i-j} = -1 \in Z(R)$, which implies that $Z(R) = R$, a contradiction. Thus $x^j = 0$.

($4 \Rightarrow 5$) Assume every element of $Z(R)$ is nilpotent. If $Z(R) = \{0\}$, then condition (5) holds vacuously. Otherwise, by Lemma 2.2 there exists a $b \in Z(R)$ such that $bZ(R) = 0$.

($5 \Rightarrow 1$) Assume $\Gamma(R)$ is star-shaped reducible and let b be the center of $\Gamma(R)$. Let x, y be any two elements of $Z(R)$. Then, $b(x-y) = bx - by = 0$, so $x - y \in Z(R)$. Thus, $Z(R)$ is an ideal. □

Corollary 2.4. *If R is a finite commutative ring with identity and* $\operatorname{diam}\Gamma(R) = 3$, *then the zero-divisors do not form an ideal.*

Proof. If the $\operatorname{diam}(\Gamma(R)) = 3$, then $\Gamma(R)$ is not star-shaped reducible. Thus, by Theorem 2.3, $Z(R)$ is not an ideal. □

Theorem 2.5. *For any commutative ring R, if $Z(R)$ is an ideal, then*

$$\Gamma(R) \neq K^{m,n}, m, n > 1.$$

Proof. Assume $Z(R)$ is an ideal. Suppose $\Gamma(R) = K^{m,n}$, $m, n > 1$. Let the partition of $\Gamma(R)$ be $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. Since $Z(R)$ is an ideal, $a_1 + b_1 \in Z(R)$. If $a_1 + b_1 = 0$, then $a_1 = -b_1$, and hence $b_1 b_2 = 0$, a contradiction. Thus, without loss of generality, we may assume that $a_1 + b_1 \in A$. So, $a_1 + b_1 = a_i$ for some $i \geq 2$. Since $\Gamma(R)$ is complete bipartite,

$$0 = a_i b_2 = (a_1 + b_1) b_2 = a_1 b_2 + b_1 b_2 = b_1 b_2,$$

a contradiction. Thus $\Gamma(R) \neq K^{m,n}$, $m, n > 1$. □

For the remainder of this section we assume that $\Gamma(R)$ is a star graph $K^{1,n}$ with center a. First, we show that the center element of $\Gamma(R)$ is almost always not looped.

Lemma 2.6. *If $\Gamma(R) = K^{1,n}$, then the center element of the star graph a is looped if and only if*

$$R \cong \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_3[x]/(x^2),\ \textit{or}\ \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proof. ($\Rightarrow$) By [Redmond 2007], observe that for $n = 0, 1, 2$, we have star graphs with looped centers:

$$\mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_3[x]/(x^2),\ \text{and}\ \mathbb{Z}_4[x]/(2x, x^2 - 2),$$

respectively. By [Anderson and Livingston 1999, Corollary 2.6], $|\Gamma(R)| > 3$, and $\Gamma(R) = K^{1,n}, n > 2$ if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a field. This implies that the center element of the star graph is $(1, 0)$, and $(1, 0)$ is not looped.

($\Leftarrow$) Trivial. □

The previous lemma is useful for determining whether a given graph is a potential zero-divisor graph: if it is a star graph with more than 3 vertices and the center element is looped, then it cannot be a zero-divisor graph.

The next lemma determines when $Z(R)$ is an ideal if $\Gamma(R)$ is a star graph.

Lemma 2.7. *If $\Gamma(R) = K^{1,n}$, then $Z(R)$ is an ideal if and only if*

$$R \cong \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_3[x]/(x^2),\ \textit{or}\ \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proof. ($\Rightarrow$) Let a be the center of $\Gamma(R)$. Let $b \in Z(R)^*$. Then $a+b \in Z(R)$. Thus, $0 = a(a+b) = a^2 + ab = a^2$. Thus a is looped. By Lemma 2.6,

$$R \cong \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_3[x]/(x^2),\ \textit{or}\ \mathbb{Z}_4[x]/(2x, x^2-2)$$

($\Leftarrow$) Trivial. □

Theorem 2.8. *Let R be a finite commutative ring with identity such that*

$$\Gamma(R) = K^{1,n}$$

with center a. Then the following are equivalent:

(1) *$Z(R)$ is an ideal*;
(2) $a^2 = 0$;
(3) $R \cong \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_3[x]/(x^2)$, *or* $\mathbb{Z}_4[x]/(2x, x^2-2)$.

3. General commutative rings

In this section, we will examine the structure of $Z(R)$ with respect to $\Gamma(R)$ in the general case where R does not necessarily have identity or is infinite. By [Anderson and Livingston 1999, Theorem 2.3], we know that $\operatorname{diam}(\Gamma(R)) \leq 3$ for any zero-divisor graph $\Gamma(R)$. We consider each possible diameter of $\Gamma(R)$ separately.

The diameter 0 and 1 cases have already been investigated thoroughly in [Axtell et al. 2006]. In particular, $\operatorname{diam}(\Gamma(R)) = 0$ if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. In each case, $Z(R)$ forms an ideal. Also, if $\operatorname{diam}(\Gamma(R)) = 1$, then $Z(R)$ is an ideal if and only if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

We now expand on the existing results regarding the diameter 2 case and present new results in the diameter 3 case.

The following lemma adds the reverse direction to Lemma 2.3 in [Axtell et al. 2006]; the proof of the forward direction is taken from the same source.

Lemma 3.1. *Let R be a commutative ring such that* $\operatorname{diam}(\Gamma(R)) = 2$. *Then $Z(R)$ is an ideal if and only if for all $x, y \in Z(R)$, there exists a nonzero z such that $xz = yz = 0$.*

Proof. ($\Rightarrow$) Let $x, y \in Z(R)$. If $x = 0$, $y = 0$, or $x = y$, the choice of z to satisfy the statement is clear. Therefore, assume x and y are distinct and nonzero. Since $\operatorname{diam}(\Gamma(R)) = 2$, whenever $xy \neq 0$, there exists $z \in Z(R)^*$ such that $xz = yz = 0$. Thus, assume $xy = 0$. If $x^2 = 0$, then $z = x$ yields the desired element, and likewise if $y^2 = 0$. Suppose $x^2, y^2 \neq 0$. Let $X' = \{x' \in Z(R)^* \mid xx' = 0\}$ and $Y' = \{y' \in Z(R)^* \mid yy' = 0\}$. Observe that $x \in Y'$ and $y \in X'$, so X' and Y' are nonempty. If $X' \cap Y' \neq \varnothing$, choose $z \in X' \cap Y'$. We show $X' \cap Y' \neq \varnothing$. Consider $x+y$. Clearly $x+y \neq x$ and $x+y \neq y$. Also if $x+y = 0$, then $x^2 = 0$ and we are done. If $x+y \neq 0$, since $Z(R)$ is an ideal and thus a subring, we have $x+y \in Z(R)^*$. As $x^2, y^2 \neq 0$, we see that $x+y \notin X'$ and $x+y \notin Y'$. Because $\operatorname{diam}(\Gamma(R)) = 2$, there exists $w \in X'$ such that the following path exists: $x - w - (x+y)$. Then $0 = w(x+y) = wx + wy = wy$ and so $w \in Y'$. Thus, there exists a nonzero z such that $xz = yz = 0$.

($\Leftarrow$) Let $x, y \in Z(R)$. By hypothesis, there exists $z \in Z(R)^*$ such that $xz = yz = 0$. Thus, $(x+y)z = xz + yz = 0$, and $x+y \in Z(R)$. Therefore $Z(R)$ is an ideal. □

Recall Corollary 2.4, which states that there are no finite rings with identity and zero-divisor graph of diameter 3 where $Z(R)$ forms an ideal. This however does not hold for the infinite case. In [Lucas 2006], an example has been given of an infinite ring R in which $Z(R)$ forms an ideal and $\operatorname{diam}(\Gamma(R)) = 3$. We present what we consider to be a more constructive example.

Before we present this example, some notation, definitions, and lemmas are needed. The following definitions are for an integral domain R. An *irreducible element* p is a nonzero, nonunit element that cannot be divided, that is, if $p = qr$, then q or r is a unit. A *unique factorization domain* is an integral domain in which each nonzero nonunit can be factored uniquely, up to associates, as a product of irreducible elements.

Consider the ring $R = \mathbb{Z}_2[x, y, z_1, z_2, \ldots]$. Note that R is a unique factorization domain. Define the set $A = \{p \in R \mid p \in \mathbb{Z}_2[x, y]$ and p is irreducible with zero constant term$\}$. Notice that there are infinitely many such irreducible polynomials in x and y. Indeed, the polynomials $x, x+y, x+y^2, \ldots$ are all irreducible. To see this, consider $\mathbb{Z}_2[x, y]$ in the equivalent form $(\mathbb{Z}_2[y])[x]$. Since $x + y^n$ has degree 1 in x, it can only be factored into something of the form $(f(y)x + g(y)) \cdot (h(y))$. Since the coefficient of x in $x + y^n$ is 1, we must have $f(y), h(y) = \pm 1$. Thus, one of the factors of $x + y^n$, namely $h(y)$, has to be a unit, and thus, $x + y^n$ is irreducible.

Since $\{z_i\}$ and A are countably infinite, there exists a bijection between them, that is, $z_i \to p_i$. Now consider the ideal $Q = (X_1, X_2)$ where $X_1 = \{z_i z_j \mid i, j \in \mathbb{N}\}$ and $X_2 = \{z_i p_i(x, y) \mid i \in \mathbb{N}\}$.

Lemma 3.2. *$f(x, y, z_{i_1}, \ldots, z_{i_n}) + Q \in Z(R/Q)$ if and only if $f(x, y, z_{i_1}, \ldots, z_{i_n})$ has a zero constant term.*

Proof. ($\Leftarrow$) If $f(x, y, z_{i_1}, \ldots, z_{i_n})$ has a zero constant term, then

$$f(x, y, z_{i_1}, \ldots, z_{i_n}) + Q$$

can be written in the form

$$f_{xy} + z_{i_1} f_1 + \cdots + z_{i_n} f_n + Q,$$

where for every k, f_{xy} and f_k are functions in x and y only. Notice that f_{xy} is either irreducible with zero constant term or can be factored into irreducibles, at least one of which has zero constant term, since $\mathbb{Z}_2[x, y]$ is a unique factorization domain. So, there is a z_j such that $z_j f_{xy} + Q = 0 + Q$. Thus,

$$z_j f(x, y, z_{i_1}, \ldots, z_{i_n}) + Q = 0 + Q,$$

and hence $f + Q$ is a zero-divisor in R/Q.

($\Rightarrow$) Consider the contrapositive, and assume f has a nonzero constant term. Thus, $f + Q$ cannot be a zero-divisor, since R is an integral domain and no element of Q has a nonzero constant term. □

Proposition 3.3. *In R/Q, $Z(R/Q)$ is an ideal.*

Proof. Since it suffices to show closure under addition, let $f(x, y, z_{i_1}, \ldots, z_{i_n}) + Q$, $g(x, y, z_{j_1}, \ldots, z_{j_m}) + Q \in Z(R/Q)$. Then

$$(f + g) + Q = h(x, y, z_{i_1}, \ldots, z_{i_n}, z_{j_1}, \ldots, z_{j_m}) + Q,$$

where h is a polynomial with a zero constant term since f and g both have zero constant terms by Lemma 3.2. □

Theorem 3.4. *In R/Q,* $\operatorname{diam}(\Gamma(R/Q)) = 3$.

Proof. Consider the polynomials $\bar{x} = x + Q$, $\bar{y} = y + Q \in R/Q$. Clearly $\bar{x}$ and $\bar{y}$ $\in Z(R/Q)$, and $\overline{xy} \neq \bar{0}$. Therefore the $d(\bar{x}, \bar{y}) \geq 2$. Suppose $d(\bar{x}, \bar{y}) = 2$. Then there exists $\bar{g} = g + Q \in R/Q$ such that $\bar{x}\bar{g} = \bar{y}\bar{g} = \bar{0}$. By Lemma 3.2, $\bar{g}$ can be written in the form $g_{xy} + z_{i_1} g_1 + z_{i_2} g_2 + \ldots + z_{1_n} g_n + Q$ for some $n \in \mathbb{N}$. Thus, $\bar{x}\bar{g} = x g_{xy} + x z_{i_1} g_1 + x z_{i_2} g_2 + \cdots + x z_{i_n} g_n + Q$. Clearly $x g_{xy} \notin Q$ unless $g_{xy} \in Q$. Thus $\bar{g} = z_{i_1} g_1 + z_{i_2} g_2 + \cdots + z_{i_n} g_n + Q$. However, by construction, there is a unique $\bar{z}_x$ term such that $\bar{x}\bar{z}_x = \bar{0}$. Therefore, $\bar{g} = \bar{g}_{z_x}$, since for any $\bar{z}_i \neq \bar{z}_x$, we have $\bar{x}\bar{z}_i \neq \bar{0}$. An analogous argument holds for $\bar{y}$. Hence, $\bar{g} = \bar{g}_{z_y}$. Therefore, $\bar{z}_x = \bar{z}_y$, a contradiction, since R is a unique factorization domain, and we have a bijection between the indeterminates and the irreducible polynomials. Therefore, $d(\bar{x}, \bar{y}) = 3$ by [Anderson and Livingston 1999]. Thus, $\operatorname{diam}(\Gamma(R/Q)) = 3$. □

Categorizing infinite graphs of diameter 3 for which $Z(R)$ is an ideal is still unresolved.

4. Realizable zero-divisor graphs

In this section, we will analyze the realizability of graphs as zero-divisor graphs of commutative rings with identity through endpoint and cut vertex analysis. We define an *endpoint* to be a vertex that is adjacent to only one other vertex.

Observe that if Γ is a graph on two vertices, it is realizable as a zero-divisor graph of a commutative ring if both endpoints are looped, as can be seen in $\mathbb{Z}_9$ or

$\mathbb{Z}_3[x]/(x^2)$. A two-vertex graph where neither endpoint is looped can be realized as the graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$. If Γ is a graph on two vertices and only one endpoint is looped, then it is not realizable as a zero-divisor graph, as shown by [Redmond 2007].

Theorem 4.1. *Let G be a graph such that $|G| > 2$. If G has at least one looped endpoint, then G is not realizable as the zero-divisor graph of a commutative ring.*

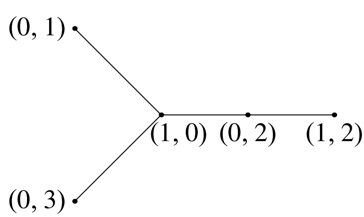
Proof. Assume $G = \Gamma(R)$ for some commutative ring R with identity. Suppose a is a looped endpoint adjacent to a vertex b, and c is a vertex adjacent to b distinct from a in $\Gamma(R)$. Since $a(a+b) = a^2 + ab = 0$, we must have $a+b = a$, $a+b = b$, or $a+b = 0$. If $a+b = a$, then $b = 0$, a contradiction. If $a+b = b$, then $a = 0$, another contradiction. If $a+b = 0$, then $a = -b$ which means any c adjacent to b is adjacent to a, a contradiction. □

A vertex a of a connected graph G is a *cut vertex* if G can be expressed as a union of two subgraphs X and Y such that $E(X) \neq \varnothing$, $E(Y) \neq \varnothing$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $X \backslash \{a\} \neq \varnothing$, and $Y \backslash \{a\} \neq \varnothing$. In other words, the removal of a cut vertex and its incident edges results in an increase in the number of connected components.

Theorem 4.2. *If $\Gamma(R)$ is partitioned into two subgraphs X and Y with cut vertex a such that $X \backslash \{a\}$ is a complete subgraph, then $I = V(X) \cup \{0\}$ is an ideal.*

Proof. Choose $b \in X \backslash \{a\}$ such that $a - b$. Since $X \backslash \{a\}$ is a complete subgraph and $ab = 0$, we have $bx = 0 = by$ for all $x, y \in V(X) \cup \{0\}$. So, $b(x+y) = 0$, and hence, $x + y \in V(X)$. Similarly, if $r \in R$, we have $b(rx) = r(bx) = 0$, and so $rx \in V(X) \cup \{0\}$. □

The converse of Theorem 4.2 is false. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $(1, 0)$ is a cut vertex. The set $\{(0, 0), (1, 0), (0, 2), (1, 2)\}$ forms an ideal of R; however, their corresponding subgraph is not complete:



Theorem 4.3. *Let R be a commutative ring with identity such that $\Gamma(R)$ is partitioned into two subgraphs X and Y with cut vertex a and $|X| > 2$. If X is a complete subgraph, then every vertex of X is looped.*

Proof. Assume X is a complete subgraph with cut vertex a. Let $b \in X$ such that $b \neq a$. Suppose $b^2 \neq 0$. If $b^2 = b$, then $b(b-1) = 0$, implying $b - 1 \in Z(R) \cap V(X)$. Let $c \in V(X)$. Thus, $0 = c(b-1) = cb - c$ implies $c = 0$, a contradiction. If $b^2 \neq b$, then $\operatorname{ann}(b) \subseteq \operatorname{ann}(b^2)$, implying $b^2 \in V(X)$. Thus, $b(b^2) = 0$ since X is complete. So, $b^2(b^2 - b) = 0$, implies $b^2 - b \in Z(R)$. By assumption, $b^2 - b \neq 0$, so $b^2 - b \in X$. Now, $0 = b(b^2 - b) = b^3 - b^2 = -b^2$ yielding $b^2 = 0$.

Now consider the zero-divisor $a + b$. Clearly $a + b \neq a, b$ and since $b(a+b) = 0$, $a + b \in V(X) \cup \{0\}$. Since X is complete, $0 = a(a+b) = a^2$. □

Theorem 4.4. *Let $\Gamma(R)$ have partitions X and Y with cut vertex a. Then $\{0, a\}$ is an ideal.*

Proof. Let $e \in X \backslash \{a\}$ such that $ea = 0$, and let $c \in Y \backslash \{a\}$ such that $ac = 0$. Clearly, $a + a \neq a$. If $a + a = b$ for some $b \in X \backslash \{a\}$, then $c(a+a) = cb = 0$, a contradiction. Similarly, $a + a \notin Y \backslash \{a\}$. Thus, $a + a = 0$. Let $r \in Z(R)$. If $ra \in X \backslash \{a\}$, then $c(ra) = r(ac) = 0$, a contradiction. Similarly, $ra \notin Y \backslash \{a\}$. Thus, $ra \in \{a, 0\}$. □

Theorem 4.5. *If Γ is realizable as a zero-divisor graph of a finite commutative ring with identity, then it is star-shaped reducible, complete bipartite, complete bipartite reducible, or diameter 3.*

Proof. Any finite ring R can be written as $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$, where each R_i is local and F_i is a field [Dummit and Foote 2004, p. 752]. If $n + m = 1$, then either R is local or R is a field. If R is local, then zero-divisors form an ideal, and the graph is star-shaped reducible by Theorem 2.3. If R is a field, then $\Gamma(R) = \varnothing$. Now suppose $n + m = 2$. If $R \cong R_1 \times F$, then $\Gamma(R)$ is complete bipartite reducible. If $R \cong F_1 \times F_2$, then $\Gamma(R)$ is complete bipartite. If $R = R_1 \times R_2$, where R_1 and R_2 are local, then let $z \in Z(R_1)^*$, $w \in Z(R_2)^*$. Consider the zero-divisors $z_1 = (z, 1)$ and $z_2 = (1, w)$. The shortest path between z_1 and z_2 must then be of length 3, and hence $\operatorname{diam}(\Gamma(R)) = 3$. If $n + m \geq 3$, $z_1 = (0, 1, 1, \ldots, 1)$ is only attached to $(1, 0, 0, \ldots, 0)$, and $z_2 = (1, 0, 1, \ldots, 1)$ is only attached to $(0, 1, 0, \ldots, 0)$. Since z_1 and z_2 do not have a common annihilator, $\operatorname{diam}(\Gamma(R)) = 3$. □

Corollary 4.6. *A finite graph with no looped vertices is realizable as $\Gamma(R)$ for some commutative ring with identity R if and only if it is the graph of a ring which is a direct product of finite fields.*

Proof. ($\Rightarrow$) If in the decomposition of R, we have that R_1 is local, then by Theorem 2.3, there exists a $k \in R_1$ such that $k^2 = 0$. So, $(k, 0, \ldots, 0)^2 = 0$, and thus $\Gamma(R)$ contains a looped vertex, a contradiction.

($\Leftarrow$) A direct product of fields contains no nonzero nilpotent elements, and hence, $\Gamma(R)$ has no looped vertices. □

Corollary 4.7. *A finite complete bipartite graph* Γ *with partitions* P *and* Q *is realizable as the zero-divisor graph of some commutative ring with identity* R *if and only if* $|P| = p^n - 1$ *and* $|Q| = q^m - 1$ *for some* $m, n \in \mathbb{N}$ *and primes* p, q.

Proof. ($\Rightarrow$) By Theorem 4.5, complete bipartite zero-divisor graphs only arise when $R \cong F_1 \times F_2$. These rings always produce graphs with partitions P and Q such that $|P| = p^n - 1$, and $|Q| = q^m - 1$ for $m, n \in \mathbb{N}$ and primes p, q.

($\Leftarrow$) The ring $R = \mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$ suffices. □

The following two theorems concern the properties of minimal paths in $\Gamma(R)$.

Theorem 4.8. *Let* R *be a commutative ring. If* a–b–c–d *is a minimal path from* a *to* d *in* $\Gamma(R)$, *then* $\operatorname{ann}(a) \subsetneq \operatorname{ann}(c)$. *Furthermore*, $\operatorname{ann}(a)\operatorname{ann}(d) = 0$.

Proof. Since $ad \neq 0$, $\operatorname{ann}(a) \neq \operatorname{ann}(c)$. Suppose there exists $e \in R$ such that $ae = 0$, but $ce \neq 0$. Then $a(ce) = (ae)c = 0$, and $d(ce) = (dc)e = 0$, so $a - ce - d$ is a path of length 2, a contradiction. Thus, $\operatorname{ann}(a) \subsetneq \operatorname{ann}(c)$. Furthermore, suppose there exists $z_a \in \operatorname{ann}(a)$ and $z_d \in \operatorname{ann}(d)$ so that $z_a z_d \neq 0$. Then $a - (z_a z_d) - d$ is also a path of length at most 2, a contradiction. □

Theorem 4.9. *Let* R *be a commutative ring, and* $a, d \in Z(R)^*$. *If* a–b–c–d *is a minimal path from* a *to* d, *then* a *and* d *are not nilpotent.*

Proof. Without loss of generality, suppose $a^n = 0$ for some $n \in \mathbb{N}$. Consider the sequence $c, ac, a^2c, a^3c, \ldots, a^nc$. By assumption, $c \neq 0$ and $a^nc = 0$. So, there exists a minimal i such that $a^ic \neq 0$, but $a^{i+1}c = 0$. Thus a^ic is adjacent to both a and d. So $a - a^ic - d$ is a path of length 2, a contradiction. □

5. Acknowledgement

This paper is the result of three summers' worth of undergraduate effort at the Wabash College mathematics REU in collaboration with Professors Axtell and Stickles. The undergraduates involved are Luanne Benson-Lender, University of Evansville (2007), Abigail Bishop, Appalachian State University (2008), Thomas Cuchta, Marshall University (2008), John Holmes, Wabash College (2008), Kathryn Lokken, University of Wisconsin (2008), Michael Martinez, University of Colorado (2006), Oliver Pechenik, Oberlin College (2008), Josiah Roberts, Wabash College (2007), Adam Shull, Valparaiso University (2008), Michael Skalak, Northwestern University (2006), Hilary Smallwood, Fort Lewis College (2008), Drew Swartz, IPFW (2008), and Monty Taylor, University of Tennessee (2007). We would like to thank the referee whose comments proved very useful.

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Received: 2008-07-10 Revised: 2008-07-16 Accepted: 2008-10-04

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