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For a connected graph G of order n, the detour distance $D(u, v)$ between two vertices u and v in G is the length of a longest $u-v$ path in G. A Hamiltonian labeling of G is a function $c : V(G) \to \mathbb{N}$ such that $|c(u) - c(v)| + D(u, v) \geq n$ for every two distinct vertices u and v of G. The value $\mathrm{hn}(c)$ of a Hamiltonian labeling c of G is the maximum label (functional value) assigned to a vertex of G by c; while the Hamiltonian labeling number $\mathrm{hn}(G)$ of G is the minimum value of Hamiltonian labelings of G. Hamiltonian labeling numbers of some well-known classes of graphs are determined. Sharp upper and lower bounds are established for the Hamiltonian labeling number of a connected graph. The corona $\mathrm{cor}(F)$ of a graph F is the graph obtained from F by adding exactly one pendant edge at each vertex of F. For each integer $k \geq 3$, let $\mathscr{H}_k$ be the set of connected graphs G for which there exists a Hamiltonian graph H of order k such that $H \subset G \subseteq \mathrm{cor}(H)$. It is shown that $2k - 1 \leq \mathrm{hn}(G) \leq k(2k - 1)$ for each $G \in \mathscr{H}_k$ and that both bounds are sharp.

1. Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest path between these two vertices. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v to a vertex of G. The *radius* $\mathrm{rad}(G)$ of G is the minimum eccentricity among the vertices of G, while the *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity among the vertices of G. A vertex v with $e(v) = \mathrm{rad}(G)$ is called a *central vertex* of G. If $d(u, v) = \mathrm{diam}(G)$, then u and v are *antipodal vertices* of G.

For a connected graph G with diameter d, an *antipodal coloring* of a connected graph G is defined in [Chartrand et al. 2002a] as an assignment $c : V(G) \to \mathbb{N}$ of colors to the vertices of G such that

$$|c(u) - c(v)| + d(u, v) \geq d,$$

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for every two distinct vertices u and v of G. In the case of paths of order $n \geq 2$, this gives

$$|c(u) - c(v)| + d(u, v) \geq n - 1.$$

Antipodal colorings of paths gave rise to the more general Hamiltonian colorings of graphs defined in terms of another distance parameter.

The *detour distance* $D(u, v)$ between two vertices u and v in a connected graph G is the length of a longest path between these two vertices. A $u - v$ path of length $D(u, v)$ is a $u - v$ *detour*. Thus if G is a connected graph of order n, then

$$d(u, v) \leq D(u, v) \leq n - 1,$$

for every two vertices u and v in G, and

$$D(u, v) = n - 1,$$

if and only if G contains a Hamiltonian $u - v$ path. Furthermore $d(u, v) = D(u, v)$ for every two vertices u and v in G if and only if G is a tree. As with standard distance, the detour distance is a metric on the vertex set of a connected graph.

A *Hamiltonian coloring* of a connected graph G of order n is a coloring

$$c : V(G) \to \mathbb{N}$$

of G such that

$$|c(u) - c(v)| + D(u, v) \geq n - 1,$$

for every two distinct vertices u and v of G. Consequently, if u and v are distinct vertices such that $|c(u) - c(v)| = k$ for some Hamiltonian coloring c of G, then there is a $u - v$ path in G missing at most k vertices of G. The *value* $\mathrm{hc}(c)$ of a Hamiltonian coloring c of G is the maximum color assigned to a vertex of G. The *Hamiltonian chromatic number* of G is the minimum value of Hamiltonian colorings of G. Hamiltonian colorings of graphs have been studied in [Chartrand et al. 2002b; 2005a; 2005b; Nebeský 2003; 2006].

For a connected graph G with diameter d, a *radio labeling* of G is defined in [Chartrand et al. 2001] as an assignment $c : V(G) \to \mathbb{N}$ of labels to the vertices of G such that

$$|c(u) - c(v)| + d(u, v) \geq d + 1,$$

for every two distinct vertices u and v of G. Thus for a radio labeling of a graph, colors assigned to adjacent vertices of G must differ by at least d, colors assigned to two vertices at distance 2 must differ by at least $d - 1$, and so on, up to two vertices at distance d (that is, antipodal vertices), whose colors are only required to differ. The *value* $\mathrm{rn}(c)$ of a radio labeling c of G is the maximum color assigned to a vertex of G. The *radio number* of G is the minimum value of a radio labeling of G. In the case of paths of order $n \geq 2$, this gives

$$|c(u) - c(v)| + d(u, v) \geq n.$$

In a similar manner, radio labelings of paths and detour distance in graphs give rise to a related labeling, which we introduce in this work.

A *Hamiltonian labeling* of a connected graph G of order n is an assignment $c : V(G) \to \mathbb{N}$ of labels to the vertices of G such that

$$|c(u) - c(v)| + D(u, v) \geq n,$$

for every two distinct vertices u and v of G. Therefore, in a Hamiltonian labeling of G, every two vertices are assigned distinct labels and two vertices u and v can be assigned consecutive labels in G only if G contains a Hamiltonian $u - v$ path. We can assume that every Hamiltonian labeling of a graph uses the integer 1 as one of its labels. The *value* $\mathrm{hn}(c)$ of a Hamiltonian labeling c of G is the maximum label assigned to a vertex of G by c, that is, $\mathrm{hn}(c) = \max\{c(v) : v \in V(G)\}$. The *Hamiltonian labeling number* $\mathrm{hn}(G)$ of G is the minimum value of Hamiltonian labelings of G, that is, $\mathrm{hn}(G) = \min\{\mathrm{hn}(c)\}$, where the minimum is taken over all Hamiltonian labelings c of G. A Hamiltonian labeling c of G with value $\mathrm{hn}(c) = \mathrm{hn}(G)$ is called a *minimum Hamiltonian labeling* of G. Therefore,

$$\mathrm{hn}(G) \geq n. \tag{1}$$

for every connected graph G of order n.

To illustrate these concepts, we consider the Petersen graph P. It is known that $\chi(P) = \mathrm{hc}(P) = 3$. In fact, it is observed in [Chartrand et al. 2005a] that every proper coloring of P is also a Hamiltonian coloring. On the other hand, since the order of P is 10, it follows that $\mathrm{hn}(P) \geq 10$. Observe that $D(u, v) = 8$ if $uv \in E(G)$ and $D(u, v) = 9$ if $uv \notin E(G)$. Thus if c is a Hamiltonian labeling of P, then $|c(u) - c(v)| \geq 2$ if $uv \in E(G)$ and $|c(u) - c(v)| \geq 1$ if $uv \notin E(G)$. Therefore, the labeling shown in Figure 1 is a Hamiltonian labeling and so $\mathrm{hn}(P) = 10$.

2. Bounds for Hamiltonian labeling numbers of graphs

It is convenient to introduce some notation. For a Hamiltonian labeling c of a graph G, an ordering $u_1, u_2, \ldots, u_n$ of the vertices of G is called the *c-ordering* of G if

$$1 = c(u_1) < c(u_2) < \ldots < c(u_n) = \mathrm{hn}(c).$$

We refer to [Chartrand and Zhang 2008] for graph theory notation and terminology not described in this paper. In order to establish a relationship between the

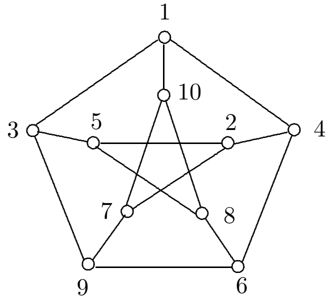

Figure 1. A Hamiltonian labeling of the Petersen graph.

Hamiltonian chromatic number and Hamiltonian labeling number of a connected graph, we first present a lemma.

Lemma 2.1. *Every connected graph of order $n \geq 3$ with Hamiltonian labeling number n is 2-connected.*

Proof. Assume, to the contrary, that there exists a connected graph G of order $n \geq 3$ with $\operatorname{hn}(G) = n$ such that G is not 2-connected. Then G contains a cut-vertex v. Let c be a minimum Hamiltonian labeling of G and let $v_1, v_2, \ldots, v_n$ be the c-ordering of the vertices of G, where then $1 = c(v_1) < c(v_2) < \ldots < c(v_n) = n$. Thus $c(v_i) = i$ for $1 \leq i \leq n$. Let $u \in V(G)$ such that u and v are consecutive in the c-ordering. Thus $\{u, v\} = \{v_j, v_{j+1}\}$ for some integer j with $1 \leq j \leq n-1$. Hence $D(v_j, v_{j+1}) \leq n-2$. However then,

$$|c(v_j) - c(v_{j+1})| + D(v_j, v_{j+1}) \leq n-1,$$

which contradicts the fact that c is a Hamiltonian labeling of G. $\square$

The corollary below now follows immediately.

Corollary 2.2. *No connected graph of order $n \geq 3$ with Hamiltonian labeling number n contains a bridge.*

While $\operatorname{hc}(K_1) = \operatorname{hn}(K_1) = 1$ and $\operatorname{hc}(K_2) = 1$ and $\operatorname{hn}(K_2) = 2$, $\operatorname{hc}(G)$ and $\operatorname{hn}(G)$ must differ by at least 2 for every connected graph G of order 3 or more. In fact, the following result provides upper and lower bounds for the Hamiltonian labeling number of a connected graph in terms of its order and Hamiltonian chromatic number.

Theorem 2.3. *For every connected graph G of order $n \geq 3$,*

$$\operatorname{hc}(G) + 2 \leq \operatorname{hn}(G) \leq \operatorname{hc}(G) + (n-1).$$

Proof. We first show that $\mathrm{hn}(G) \geq \mathrm{hc}(G) + 2$. Let c be a minimum Hamiltonian labeling of G and let $v_1, v_2, \ldots, v_n$ be the c-ordering of the vertices of G, where then $1 = c(v_1) < c(v_2) < \ldots < c(v_n) = \mathrm{hn}(c)$. Define a coloring c^* of G by

$$c^*(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ c(v_i) - 1 & \text{if } 2 \leq i \leq n-1, \\ c(v_i) - 2 & \text{if } i = n. \end{cases}$$

We show that c^* is a Hamiltonian coloring of G. Let $v_i, v_j \in V(G)$, where

$$1 \leq i < j \leq n.$$

We consider two cases.

Case 1. $i = 1$. Suppose first that $2 \leq j \leq n-2$. Then

$$|c^*(v_j) - c^*(v_1)| + D(v_j, v_1) = c(v_j) - c(v_1) - 1 + D(v_j, v_1) \geq n - 1.$$

Next suppose that $j = n$. Then

$$\begin{aligned} |c^*(v_n) - c^*(v_1)| + D(v_n, v_1) &= c(v_n) - c(v_1) - 2 + D(v_n, v_1) \\ &= c(v_n) - 3 + D(v_n, v_1). \end{aligned}$$

If $c(v_n) \geq n+1$, then $c(v_n) - 3 + D(v_n, v_1) \geq n - 1$. If $c(v_n) = n$, then $v_1 v_n$ is not a bridge by Corollary 2.2 and so $D(v_n, v_1) \geq 2$. Thus $c(v_n) - 3 + D(v_n, v_1) \geq n - 1$.

Case 2. $i \geq 2$. In this case,

$$|c^*(v_j) - c^*(v_i)| + D(v_j, v_i) = \begin{cases} c(v_j) - c(v_i) + D(v_j, v_i), & \text{if } j \leq n-1, \\ c(v_j) - c(v_i) - 1 + D(v_j, v_i), & \text{if } j = n, \end{cases} \tag{2}$$

which is greater than or equal to $c(v_j) - c(v_i) - 1 + D(v_j, v_i) \geq n - 1$. Thus c^* is a Hamiltonian coloring of G, as claimed. Therefore,

$$\mathrm{hc}(G) \leq \mathrm{hc}(c^*) = \mathrm{hn}(c) - 2 = \mathrm{hn}(G) - 2,$$

and so $\mathrm{hn}(G) \geq \mathrm{hc}(G) + 2$.

Next, we show that $\mathrm{hn}(G) \leq \mathrm{hc}(G) + (n-1)$. Let c' be a Hamiltonian coloring of G such that $\mathrm{hc}(c') = \mathrm{hc}(G)$. We may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that

$$1 = c'(v_1) \leq c'(v_2) \leq \ldots \leq c'(v_n) = \mathrm{hc}(c').$$

Define a labeling c'' of G by $c''(v_i) = c'(v_i) + (i-1)$ for $1 \leq i \leq n$. Let v_j and v_k be two distinct vertices of G. Then

$$\begin{aligned} |c''(v_j) - c''(v_k)| + D(v_j, v_k) &= |c'(v_j) - c'(v_k)| + |j - k| + D(v_j, v_k) \\ &\geq (n-1) + |j - k| \geq n, \end{aligned}$$

and so c'' is a Hamiltonian labeling of G. Since $\mathrm{hn}(c'') = \mathrm{hc}(c) + (n-1)$, it follows that $\mathrm{hn}(G) \le \mathrm{hc}(G) + (n-1)$. □

While the upper and lower bounds in Theorem 2.3 are sharp (as we will see later), both inequalities in Theorem 2.3 can be strict. For example, consider the Petersen graph P of order $n = 10$ and $\mathrm{hn}(P) = 10$. Thus

$$5 = \mathrm{hc}(P) + 2 < \mathrm{hn}(P) < \mathrm{hc}(P) + (n-1) = 12.$$

In fact, more can be said. The following result was established in [Chartrand et al. 2005a].

Theorem 2.4 [Chartrand et al. 2005a]. *If G is a Hamiltonian graph of order $n \ge 3$, then $\mathrm{hc}(G) \le n-2$. Furthermore, for each pair k, n of integers with $1 \le k \le n-2$, there is a Hamiltonian graph of order n with Hamiltonian chromatic number k.*

On the other hand, every Hamiltonian graph of order n has Hamiltonian labeling number n, as we show next.

Proposition 2.5. *If G is a Hamiltonian graph of order $n \ge 3$, then $\mathrm{hn}(G) = n$.*

Proof. Let $C : v_1, v_2, \ldots, v_{n+1} = v_1$ be a Hamiltonian cycle of G. Define the labeling c of G by $c(v_i) = i$ for $1 \le i \le n$. Let i, j be two integers with $1 \le i < j \le n$. If $j - i \le n/2$, then $D(v_i, v_j) \ge n - (j-i)$; while if $j - i > n/2$, then $D(v_i, v_j) \ge j - i$. In either case, $|c(v_i) - c(v_j)| + D(v_i, v_j) \ge n$. Thus c is a Hamiltonian labeling and so $\mathrm{hn}(G) = n$ by Equation (1). □

The converse of Proposition 2.5 is not true. For example, it is well known that the Petersen graph P is a nonHamiltonian graph of order 10 but $\mathrm{hn}(P) = 10$. Whether there exists a connected graph G of order $n \ge 3$ with $\mathrm{hn}(G) = n$ that is neither a Hamiltonian graph nor the Petersen graph is not known. The following realization result is a consequence of Theorem 2.4 and Proposition 2.5.

Corollary 2.6. *For each pair k, n of integers with $2 \le k \le n-1$, there exists a Hamiltonian graph G of order n such that $\mathrm{hn}(G) = \mathrm{hc}(G) + k$.*

In the remainder of this section, we consider the complete bipartite graphs $K_{r,s}$ of order $n = r + s \ge 3$, where $1 \le r \le s$. The Hamiltonian chromatic number of a complete bipartite graph has been determined in [Chartrand et al. 2005a]. For positive integers r and s with $r \le s$ and $r + s \ge 3$,

$$\mathrm{hc}(K_{r,s}) = \begin{cases} r & \text{if } r = s, \\ (s-1)^2 + 1 & \text{if } 1 = r < s, \\ (s-1)^2 - (r-1)^2 & \text{if } 2 \le r < s. \end{cases} \tag{3}$$

If $r \ge 2$, then $K_{r,r}$ is Hamiltonian and so $\mathrm{hn}(K_{r,r}) = n = 2r$ by Proposition 2.5. Thus, we may assume that $r < s$, beginning with $r = 1$.

Theorem 2.7. *For each integer $n \geq 3$,*

$$\mathrm{hn}(K_{1,n-1}) = n + (n-2)^2.$$

Proof. Let $G = K_{1,n-1}$ with vertex set $\{v, v_1, v_2, \ldots, v_{n-1}\}$, where v is the central vertex of G. By Equation (3) and Theorem 2.3, it suffices to show that

$$\mathrm{hn}(G) \geq n + (n-2)^2.$$

Let c be a minimum Hamiltonian labeling of G. Since no two vertices of G can be labeled the same, we may assume that

$$c(v_1) < c(v_2) < \ldots < c(v_{n-1}).$$

We consider three cases.

Case 1. $c(v) = 1$. Since $D(v_1, v) = 1$ and $D(v_i, v_{i+1}) = 2$ for $1 \leq i \leq n-2$, it follows that $c(v_1) \geq n$ and

$$c(v_{i+1}) \geq c(v_i) + (n-2) \geq c(v_1) + i(n-2) \geq n + i(n-2)$$

for all $1 \leq i \leq n-2$. This implies that

$$c(v_{n-1}) \geq n + (n-2)(n-2) = n + (n-2)^2.$$

Therefore, $\mathrm{hn}(G) = \mathrm{hn}(c) \geq n + (n-2)^2$.

Case 2. $c(v) = \mathrm{hn}(c)$. Then $1 = c(v_1) < c(v_2) < \ldots < c(v_{n-1}) < c(v)$. For each i with $2 \leq i \leq n-1$, it follows that

$$c(v_i) \geq c(v_1) + (i-1)(n-2) = 1 + (i-1)(n-2).$$

In particular, $c(v_{n-1}) \geq 1 + (n-2)^2$. Thus

$$c(v) \geq c(v_{n-1}) + n - 1 = n + (n-2)^2.$$

Therefore, $\mathrm{hn}(G) = \mathrm{hn}(c) \geq n + (n-2)^2$.

Case 3. $c(v_j) < c(v) < c(v_{j+1})$ for some j with $1 \leq j \leq n-2$. Thus

$$\begin{aligned} c(v_j) &\geq 1 + (j-1)(n-2), \\ c(v) &\geq c(v_j) + n - 1 \geq n + (j-1)(n-2), \\ c(v_{j+1}) &\geq c(v) + n - 1 \geq 2n - 1 + (j-1)(n-2). \end{aligned}$$

This implies that

$$\begin{aligned} c(v_{n-1}) &\geq (n-j-2)(n-2) + c(v_{j+1}) \\ &\geq (n-j-2)(n-2) + (2n-1) + (j-1)(n-2) \\ &= 2n - 1 + (n-3)(n-2) = n + 1 + (n-2)^2 > n + (n-2)^2. \end{aligned}$$

In each case, we have $\mathrm{hn}(G) \geq n + (n-2)^2$. □

We now consider $K_{r,s}$, where $2 \le r < s$, with partite sets V_1 and V_2 such that $|V_1| = r$ and $|V_2| = s$. Then

$$D(u, v) = \begin{cases} 2r - 2 = n - s + r - 2 & \text{if } u, v \in V_1, \\ 2r - 1 = n - s + r - 1 & \text{if } uv \in E(K_{r,s}), \\ 2r = n - s + r & \text{if } u, v \in V_2. \end{cases}$$

Consequently, if c is a Hamiltonian labeling of $K_{r,s}$ $(r < s)$, then

$$|c(u) - c(v)| \ge \begin{cases} s - r + 2 & \text{if } u, v \in V_1, \\ s - r + 1 & \text{if } uv \in E(K_{r,s}), \\ s - r & \text{if } u, v \in V_2. \end{cases}$$

Theorem 2.8. *For integers r and s with $2 \le r < s$,*

$$\operatorname{hn}(K_{r,s}) = (s-1)^2 - (r-1)^2 + s + r - 1.$$

Proof. By Equation (3) and Theorem 2.3, it suffices to show that

$$\operatorname{hn}(K_{r,s}) \ge (s-1)^2 - (r-1)^2 + s + r - 1.$$

Let $V_1 = \{u_1, u_2, \dots, u_r\}$ and $V_2 = \{v_1, v_2, \dots, v_s\}$ be the partite sets of $K_{r,s}$, and let c be a Hamiltonian labeling of $K_{r,s}$ and let $w_1, w_2, \dots, w_{r+s}$ be the c-ordering of the vertices of $K_{r,s}$. We define a V_1-*block* of $K_{r,s}$ to be a set

$$A = \{w_\alpha, w_{\alpha+1}, \dots, w_\beta\},$$

where $1 \le \alpha \le \beta \le r + s$, such that $A \subseteq V_1$, $w_{\alpha-1} \in V_2$ if $\alpha > 1$, and $w_{\beta+1} \in V_2$ if $\beta < r + s$. A V_2-*block* of $K_{r,s}$ is defined similarly. Let

$$A_1, A_2, \dots, A_p \quad (p \ge 1)$$

be the distinct V_1-blocks of $K_{r,s}$ such that if

$$w' \in A_i, \quad w'' \in A_j,$$

where $1 \le i < j \le p$, then $c(w') < c(w'')$. If $p \ge 2$, then $K_{r,s}$ contains V_2-blocks $B_1, B_2, \dots, B_{p-1}$ such that for each integer i $(1 \le i \le p - 1)$ and for $w' \in A_i$, $w \in B_i$, $w'' \in A_{i+1}$, it follows that

$$c(w') < c(w) < c(w'').$$

The graph $K_{r,s}$ may contain up to two additional V_2-blocks, namely B_0 and B_p such that if $y \in B_0$ and $y' \in A_1$, then $c(y) < c(y')$; while if $z \in A_p$ and $z' \in B_p$, then $c(z) < c(z')$. If $p = 1$, then at least one of B_0 and B_1 must exist. Hence $K_{r,s}$ contains p V_1-blocks and $p - 1 + t$ V_2-blocks, where $t \in \{0, 1, 2\}$. Consequently, there are exactly

(a) $r - p$ distinct pairs $\{w_i, w_{i+1}\}$ of vertices, both of which belong to V_1;

(b) $2p-2+t$ distinct pairs $\{w_i, w_{i+1}\}$ of vertices, exactly one of which belongs to V_1;

(c) $s-(p-1+t)$ distinct pairs $\{w_i, w_{i+1}\}$ of vertices, both of which belong to V_2.

Since (1) the colors of every two vertices w_i and w_{i+1}, both of which belong to V_1, must differ by at least $s-r+2$, (2) the colors of every two vertices w_i and w_{i+1}, exactly one of which belongs to V_1, must differ by at least $s-r+1$, and (3) the colors of every two vertices w_i and w_{i+1}, both of which belong to V_2, must differ by at least $s-r$, it follows that

$$\begin{aligned} c(w_{r+s}) &\geq 1+(r-p)(s-r+2)+(2p-2+t)(s-r+1)+(s-(p-1+t))(s-r) \\ &= (s-1)^2-(r-1)^2+s+r-1+t. \end{aligned} \tag{4}$$

Since $\mathrm{hn}(K_{r,s}) \leq (s-1)^2-(r-1)^2+s+r-1$ and $t \geq 0$, it follows that $t=0$ and that $\mathrm{hn}(K_{r,s}) = (s-1)^2-(r-1)^2+s+r-1$. □

Combining Proposition 2.5 and Theorems 2.7 and 2.8, we obtain the following.

Corollary 2.9. *For integers r and s with $1 \leq r \leq s$,*

$$\mathrm{hn}(K_{r,s}) = \begin{cases} r+s & \text{if } r=s, \\ (s-1)^2+s+1 & \text{if } r=1 \text{ and } s \geq 2, \\ (s-1)^2-(r-1)^2+r+s-1 & \text{if } 2 \leq r < s. \end{cases}$$

3. Hamiltonian labeling numbers of subgraphs of coronas of Hamiltonian graphs

A common question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. If G is a Hamiltonian graph and u and v are two nonadjacent vertices of G, then $G+uv$ is also Hamiltonian and so $\mathrm{hn}(G) = \mathrm{hn}(G+uv)$. On the other hand, if we add a pendant edge to a Hamiltonian graph G producing a nonHamiltonian graph H, then the Hamiltonian labeling number of H can be significantly larger than that of G, as we show in this section. We begin with those graphs obtained from a cycle or a complete graph by adding a single pendant edge.

Theorem 3.1. *If G is the graph of order $n \geq 5$ obtained from C_{n-1} by adding a pendant edge, then* $\mathrm{hn}(G) = 2n-2$.

Proof. Let $C: v_1, v_2, \ldots, v_{n-1}, v_1$ and let $v_{n-1}v_n$ be the pendant edge of G. We first show that $\mathrm{hn}(G) \leq 2n-2$. Define a labeling c_0 of G by

$$c_0(v_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq n-1, \\ 1 & \text{if } i=n. \end{cases}$$

We show that c_0 is a Hamiltonian labeling. First let

$$v_i, v_j \in V(C),$$

where $1 \le i < j \le n-1$. If $j-i \ge \frac{n-1}{2}$, then $D(v_i, v_j) = j-i$ and so

$$\begin{aligned} |c_0(v_i) - c_0(v_j)| + D(v_i, v_j) &= |2i-2j| + (j-i) = 3(j-i) \\ &\ge 3\Big(\frac{n-1}{2}\Big) = \frac{3n}{2} - \frac{3}{2} \ge n, \end{aligned}$$

since $n \ge 3$. If $j-i \le \frac{n-1}{2}$, then $D(v_i, v_j) = (n-1)-(j-i)$ and so

$$\begin{aligned} |c_0(v_i) - c_0(v_j)| + D(v_i, v_j) &= 2(j-i) + [(n-1)-(j-i)] \\ &= n-1+(j-i) \ge n. \end{aligned}$$

Next, we consider each pair v_i, v_n where $1 \le i \le n-1$. Since $D(v_i, v_n) \ge n-i$ and $|c_0(v_i) - c_0(v_n)| \ge 2i-1$, it follows that

$$|c_0(v_i) - c_0(v_n)| + D(v_i, v_n) \ge n+i-1 \ge n.$$

Therefore, c_0 is a Hamiltonian labeling, as claimed.

Next, we show that $\mathrm{hn}(G) \ge 2n-2$. Let c be a minimum Hamiltonian labeling of G. First, we make some observations.

(a) For each pair i, j with $1 \le i \ne j \le n-1$, $D(v_i, v_j) \le n-2$ and so $|c(v_i) - c(v_j)| \ge 2$.

(b) For each i with $i \in \{1, n-2\}$, $D(v_n, v_i) = n-1$ and so $|c(v_n) - c(v_i)| \ge 1$.

(c) For each i with $1 \le i \le n-1$ and $i \notin \{1, n-2\}$, $D(v_n, v_i) \le n-2$ and so $|c(v_n) - c(v_i)| \ge 2$.

Let $u_1, u_2, \dots, u_n$ be the c-ordering of the vertices of G and let

$$X = \{c(u_{i+1}) - c(u_i) : 1 \le i \le n-1\}.$$

By observations (a)–(c), at most two terms in X are 1. If at most one term in X is 1, then $\mathrm{hn}(c) = c(u_n) \ge 1+1+2(n-2) = 2n-2$. If at least one term in X is 3 or more, then $\mathrm{hn}(c) = c(u_n) \ge 1+1+1+3+2(n-4) = 2n-2$. Thus we may assume that exactly two terms in X are 1 and the remaining terms in X are 2. Then $v_n = u_i$ for some i with $2 \le i \le n-1$ and $\{v_1, v_{n-2}\} = \{u_{i-1}, u_{i+1}\}$, where $c(u_i) - c(u_{i-1}) = c(u_{i+1}) - c(u_i) = 1$. This implies that $v_{n-1} = u_j$ for some j with $1 \le j \le n$ and $j \ne i$. If $2 \le j \le n-1$, then $\{u_{j-1}, u_{j+1}\} \ne \{v_1, v_{n-2}\}$; if $j = 1$, then $u_2 \notin \{v_1, v_{n-2}\}$, for otherwise

$$\begin{aligned} |c(v_{n-1}) - c(v_n)| + D(v_{n-1}, v_n) &\le |c(v_{n-1}) - c(u_2)| + |c(u_2) - c(v_n)| + 1 \\ &\le 2+1+1 = 4 < n, \end{aligned}$$

which is impossible; if $j = n$, then $u_{n-1} \notin \{v_1, v_{n-2}\}$, for otherwise

$$\begin{aligned}|c(v_{n-1}) - c(v_n)| + D(v_{n-1}, v_n) &\le |c(v_{n-1}) - c(u_{n-1})| + |c(u_{n-1}) - c(v_n)| + 1\\ &\le 2 + 1 + 1 = 4 < n,\end{aligned}$$

again, which is impossible. Therefore, for each j with $1 \le j \le n$, there exists $k \in \{j-1, j+1\}$ such that $u_k \notin \{v_1, v_{n-2}\}$. Assume, without loss of generality, that $u_{j-1} \notin \{v_1, v_{n-2}\}$. Since $D(u_{j-1}, u_j) \le n-3$, it follows that $c(u_j) - c(u_{j-1}) \ge 3$, which is impossible since each term in X is at most 2. Thus, $\mathrm{hn}(G) \ge 2n - 2$. □

Theorem 3.2. *If G is the graph of order $n \ge 4$ obtained from K_{n-1} by adding a pendant edge, then* $\mathrm{hn}(G) = 2n - 3$.

Proof. Let $V(K_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$ and let G be obtained from K_{n-1} by adding the pendant edge $v_{n-1}v_n$. We first show that $\mathrm{hn}(G) \le 2n - 3$. Define a labeling c_0 of G by

$$c_0(v) = \begin{cases} 2i - 1 & \text{if } v = v_i \text{ for } 1 \le i \le n-1,\\ 2 & \text{if } v = v_n. \end{cases}$$

For each pair i, j of integers with $1 \le i \ne j \le n-1$,

$$D(v_i, v_j) = n - 2 \quad \text{and} \quad |c_0(v_i) - c_0(v_j)| \ge 2.$$

For each i with $1 \le i \le n-2$,

$$D(v_n, v_i) = n - 1 \quad \text{and} \quad |c_0(v_n) - c_0(v_i)| \ge 1.$$

Furthermore, $D(v_n, v_{n-1}) = 1$ and

$$|c_0(v_n) - c_0(v_{n-1})| \ge (2n-3) - 2 = 2n - 5 \ge n - 1$$

for $n \ge 4$. In each case,

$$D(v_i, v_j) + |c_0(v_i) - c_0(v_j)| \ge n,$$

for all i, j with $1 \le i \ne j \le n$. Therefore, c_0 is a Hamiltonian labeling and so $\mathrm{hn}(G) \le \mathrm{hn}(c_0) = c_0(v_{n-1}) = 2n - 3$.

Next, we show that $\mathrm{hn}(G) \ge 2n - 3$. Let c be a minimum Hamiltonian labeling of G. Suppose that the vertices of K_{n-1} in G can be ordered as $u_1, u_2, \ldots, u_{n-1}$ such that $c(u_1) < c(u_2) < \ldots < c(u_{n-1})$. Since

$$D(u_i, u_j) = n - 2$$

for $1 \le i < j \le n-1$, it follows that

$$|c(u_i) - c(u_j)| = c(u_j) - c(u_i) \ge 2.$$

This implies that

$$\operatorname{hn}(c) \geq c(u_{n-1}) \geq 1+2(n-2) = 2n-3.$$

Therefore, $\operatorname{hn}(G) \geq 2n-3$. □

Let G be a connected graph containing an edge e that is not a bridge. Then $G-e$ is connected. For every two distinct vertices u and v in $G-e$, the length of a longest $u-v$ path in $G-e$ does not exceed the length of a longest $u-v$ path in G. Thus every Hamiltonian labeling of $G-e$ is a Hamiltonian labeling of G. This observation yields the following useful lemma.

Lemma 3.3. *If F is a connected subgraph of a connected graph G, then*

$$\operatorname{hn}(G) \leq \operatorname{hn}(F).$$

The following is a consequence of Theorems 3.1 and 3.2 and Lemma 3.3.

Corollary 3.4. *Let H be a Hamiltonian graph of order $n-1 \geq 3$. If G is a graph obtained from H by adding a pendant edge, then*

$$2n-3 \leq \operatorname{hn}(G) \leq 2n-2.$$

Proof. Let C be a Hamiltonian cycle in H. If $H = C_{n-1}$, then $\operatorname{hn}(G) = 2n-2$ by Theorem 3.1; while if $H = K_{n-1}$, then $\operatorname{hn}(G) = 2n-3$ by Theorem 3.2. Thus, we may assume that $H \neq C_{n-1}$ and $H \neq K_{n-1}$. Let F be the graph obtained from K_{n-1} by adding a pendant edge and F' be the graph obtained from C_{n-1} by adding a pendant edge. Then G can be obtained from F by deleting nonbridge edges and F' can be obtained from G by deleting nonbridge edges. It then follows by Lemma 3.3 that $\operatorname{hn}(F) \leq \operatorname{hn}(G) \leq \operatorname{hn}(F')$ and so $2n-3 \leq \operatorname{hn}(G) \leq 2n-2$. □

In fact, there exists a Hamiltonian graph H of order $n-1$ such that adding a pendant edge at a vertex x of H produces a graph G with $\operatorname{hn}(G) = 2n-3$ but adding a pendant edge at a different vertex y of H produces a graph F with $\operatorname{hn}(F) = 2n-2$. For example, let H be the Hamiltonian graph obtained from the cycle $C : v_1, v_2, \ldots, v_{n-1}, v_1$ of order $n-1 \geq 4$ by adding the edge v_1v_{n-2}. If G is formed from H by adding a pendant edge at v_{n-1}, then $\operatorname{hn}(G) = 2n-3$; while if F is formed from H by adding the pendant edge v_1, then $\operatorname{hn}(F) = 2n-2$.

In order to study graphs obtained from a Hamiltonian graph by adding pendant edges, we first establish some additional definitions and notation. For a graph F, the *corona* $\operatorname{cor}(F)$ of F is that graph obtained from F by adding exactly one pendant edge at each vertex of F. For a connected graph G, the *core* $C(G)$ of G is obtained from G by successively deleting vertices of degree 1 until none remain. Thus, if G is a tree, then its core is K_1; while if G is not a tree, then the core of G is the induced subgraph F of maximum order with $\delta(F) \geq 2$. For each integer $k \geq 3$, let $\mathcal{H}_k$ be the set of nonHamiltonian graphs that can be obtained from a

Hamiltonian graph of order k by adding pendant edges to this graph in such a way that at most one pendant edge is added to each vertex of the graph. Thus if $G \in \mathcal{H}_k$, then there is a Hamiltonian graph H of order k such that G is a connected subgraph of $\mathrm{cor}(H)$ whose core is H. We now establish lower and upper bounds for the Hamiltonian labeling number of a graph in $\mathcal{H}_k$ in terms of the integer k and the order of the graph, beginning with a lower bound.

Theorem 3.5. *Let $G \in \mathcal{H}_k$ be a graph of order n and $k+1 \le n \le 2k$. Then*

$$\mathrm{hn}\,(G) \ge (n-1)(n-k) + (2k-n).$$

Proof. Suppose that H is a Hamiltonian graph of order $k \ge 3$ and that $H \cong C(G)$. If $H \not\cong K_k$, then G can be obtained from some graph $F \in \mathcal{H}_k$ by deleting nonbridge edges from F, where $C(F) \cong K_k$, and $V(G-H) = V(F-K_k)$. That is, G and F possess the same end-vertices. It then follows by Lemma 3.3 that

$$\mathrm{hn}\,(F) \le \mathrm{hn}(G).$$

Therefore, it suffices to show that

$$\mathrm{hn}\,(F) \ge (n-1)(n-k) + (2k-n).$$

Let $V(F) = U \cup W$, where $U = V(K_k)$ and $W = V(F) - U$. First we make some observations:

(a) If $x, y \in U$, then $D(x, y) = k-1$.

(b) If $x, y \in W$, then $D(x, y) = k+1$.

(c) If $x \in U$ and $y \in W$, then $D(x, y) = 1$ if $xy \in E(F)$ and $D(x, y) = k$ otherwise.

Let c be a minimum Hamiltonian labeling of F and let $v_1, v_2, \ldots, v_n$ be the c-ordering of the vertices of F. We define the four subsets S_u, S_w, $S_{u,w}$, and $S_{w,u}$ of $V(F)$ as follows:

$$\begin{aligned} S_u &= \{v_i : v_{i-1}, v_i \in U \text{ for } 2 \le i \le n\}, \\ S_w &= \{v_i : v_{i-1}, v_i \in W \text{ for } 2 \le i \le n\}, \\ S_{u,w} &= \{v_i : v_{i-1} \in U \text{ and } v_i \in W \text{ for } 2 \le i \le n\}, \\ S_{w,u} &= \{v_i : v_{i-1} \in W \text{ and } v_i \in U \text{ for } 2 \le i \le n\}. \end{aligned}$$

Let $|S_u| = n_u$, $|S_w| = n_w$, $|S_{u,w}| = n_{u,w}$, $|S_{w,u}| = n_{w,u}$. Since

$$S_u \cup S_w \cup S_{u,w} \cup S_{w,u} = V(F) - \{v_1\},$$

it follows that

$$n_u + n_w + n_{u,w} + n_{w,u} = n - 1. \tag{5}$$

For each integer i with $2 \le i \le n$,

(A) if $v_i \in S_u$, then $c(v_i) - c(v_{i-1}) \geq n-k+1$ by (a);

(B) if $v_i \in S_w$, then $c(v_i) - c(v_{i-1}) \geq n-k-1$ by (b);

(C) if $v_i \in S_u \cup S_w$, then either $c(v_i) - c(v_{i-1}) \geq n-1$ or $c(v_i) - c(v_{i-1}) \geq n-k$ by (iii), and so $c(v_i) - c(v_{i-1}) \geq n-k$ in this case.

It then follows by (A)–(C) and (5) that

$$\begin{aligned}\mathrm{hn}(c) = c(v_n) &\geq 1 + n_u(n-k+1) + n_w(n-k-1) + (n_{u,w} + n_{w,u})(n-k)\\ &= 1 + (n_u + n_w + n_{u,w} + n_{w,u})(n-k) + (n_u - n_w)\\ &= 1 + (n-1)(n-k) + (n_u - n_w).\end{aligned}$$

We claim that $n_u - n_w \geq 2k - n - 1$. Since

$$S_u \cup S_{u,w} = \{v_i : v_{i-1} \in U \text{ for } 2 \leq i \leq n\},$$

it follows that

$$|S_u \cup S_{u,w}| = \begin{cases} |U| - 1 & \text{if } v_n \in U,\\ |U| & \text{otherwise;}\end{cases}$$

and so

$$n_u + n_{u,w} = k \text{ or } n_u + n_{u,w} = k-1. \tag{6}$$

Since

$$\begin{aligned}S_w \cup S_{u,w} &= \{v_i : \quad v_i \in W \text{ for } 2 \leq i \leq n\}\\ &= \begin{cases} W - \{v_1\} & \text{if } v_1 \in W,\\ W & \text{otherwise,}\end{cases}\end{aligned}$$

it follows that

$$n_w + n_{u,w} = n-k \text{ or } n_w + n_{u,w} = n-k-1. \tag{7}$$

By Equations (6) and (7), we obtain

$$n_u - n_w = (n_u + n_{u,w}) - (n_w + n_{u,w}) \geq (k-1) - (n-k) = 2k - n - 1,$$

as claimed. Therefore,

$$\mathrm{hn}(G) = \mathrm{hn}(c) \geq 1 + (n-1)(n-k) + (n_u - n_w) \geq (n-1)(n-k) + (2k-n).$$

This completes the proof. □

Theorem 3.6. *Let $G \in \mathcal{H}_k$ be a graph of order n and $k+2 \leq n \leq 2k$. Then*

$$\mathrm{hn}\,(G) \leq 1 + n + (n-k-1)^2 + (k-2)(n-k+1).$$

Proof. Suppose that H is a Hamiltonian graph of order $k \geq 3$ and that $H \cong C(G)$. If $H \not\cong C_k$, then C_k can be obtained from H by deleting edges. Thus there exists $F \in \mathscr{H}_k$ such that $C(F) \cong C_k$ and F can be obtained from G by deleting edges that are not bridges. It then follows by Lemma 3.3 that

$$\mathrm{hn}\,(G) \leq \mathrm{hn}(F).$$

Therefore, we may assume that $H \cong C_k : x_1, x_2, \ldots, x_k, x_1$. Now let

$$X = \{x_1, x_2, \ldots, x_k\} \quad \text{and} \quad Y = V(G) - X = \{y_1, y_2, \ldots, y_{n-k}\}$$

such that y_i is adjacent to x_{j_i}, for $1 \leq i \leq n-k$, and $1 = j_1 < j_2 < \ldots < j_{n-k} \leq k$. For each i with $1 \leq i \leq n-k$, let

$$g_i = j_{i+1} - j_i - 1, \tag{8}$$

where $j_{n-k+1} = j_1$; that is, g_i is the number of vertices of degree 2 between x_{j_i} and $x_{j_{i+1}}$ on C_k. Thus if $x_{j_i} y_i \in E(G)$, then $x_{j_i+g_i+1} y_{i+1} \in E(G)$, for $1 \leq i \leq n-k$, and

$$\sum_{i=1}^{n-k} g_i = 2k - n.$$

Now define the labeling c of G by

$$c(v) = \begin{cases} 1 & \text{if } v = x_k, \\ 1 + n - k & \text{if } v = y_1, \\ c(y_{i-1}) + (n-k-1) + g_{i-1} & \text{if } v = y_i \text{ and } 2 \leq i \leq n-k, \\ c(y_{n-k}) + n - k + g_{n-k} & \text{if } v = x_1, \\ c(x_{j-1}) + (n-k+1) & \text{if } v = x_j \text{ and } 2 \leq j \leq k-1. \end{cases} \tag{9}$$

Thus the c-ordering of the vertices of G is

$$x_k, y_1, y_2, \ldots, y_{n-k}, x_1, x_2, \ldots, x_{k-1},$$

and by Equation (9)

$$\begin{aligned} &c(x_k) = 1, \\ &c(y_i) = 1 + n - k + (i-1)(n-k-1) + \sum_{\ell=1}^{i-1} g_\ell \text{ for } 1 \leq i \leq n-k, \\ &c(x_1) = 1 + n + (n-k-1)^2, \\ &c(x_j) = 1 + n + (n-k-1)^2 + (j-1)(n-k+1) \text{ for } 2 \leq j \leq k-1. \end{aligned} \tag{10}$$

Therefore, the value of c is

$$\mathrm{hn}(c) = c(x_{k-1}) = 1 + n + (n-k-1)^2 + (k-2)(n-k+1).$$

Thus it remains to show that c is a Hamiltonian labeling of G. First, we make some observations. Let $u, v \in V(G)$, where $u \neq v$.

(α) If $u = x_i$ and $v = x_j$ where $1 \le i \neq j \le k$, then $D(u, v) = \max\{|i-j|, k-|i-j|\}$.

(β) If $u = y_i$ and $v = y_j$ where $1 \le i < j \le n-k$, then

$$D(u, v) = 2 + \max\left\{ j - i + \sum_{\ell=i}^{j-1} g_\ell, k - \left(j - i + \sum_{\ell=i}^{j-1} g_\ell \right) \right\}.$$

(γ) If $u = x_i$, $v \in Y$, and $vx_j \in E(G)$ where $1 \le i, j \le k$ (possibly $i = j$), then $D(u, v) = 1$ if $i = j$ and $D(u, v) = 1 + \max\{|i - j|, k - |i - j|\}$ if $i \neq j$.

We show that

$$D(u, v) + |c(u) - c(v)| \ge n, \tag{11}$$

for every pair u, v of distinct vertices of G. We consider three cases.

Case 1. $u, v \in X$. Let $u = x_i$ and $v = x_j$, where $1 \le i, j \le k$. We may assume, without loss of generality, that $i < j$. If $j = k$, then

$$\begin{aligned} |c(x_i) - c(x_j)| &= c(x_i) - c(x_k) \\ &= [1 + n + (n-k-1)^2 + (i-1)(n-k+1)] - 1 \ge n, \end{aligned}$$

and so condition (11) is satisfied. Thus we may assume that $j \neq k$.

If $j - i = 1$, then $D(x_i, x_j) = k - 1$ and $|c(x_i) - c(x_j)| = n - k + 1$. Thus (11) holds in this case. If $j - i \ge \frac{k}{2}$, then

$$\begin{aligned} D(x_i, x_j) + |c(x_i) - c(x_j)| &= c(x_j) - c(x_i) + D(x_i, x_j) \\ &= (j-i)(n-k+1) + (j-i) = (j-i)(n-k+2) \\ &\ge \frac{k}{2}(n-k+2) = k\left(\frac{n-k}{2} + 1\right) \ge 2k \ge n. \end{aligned}$$

If $2 \le j - i \le \frac{k}{2}$, then

$$\begin{aligned} D(x_i, x_j) + |c(x_i) - c(x_j)| &= c(x_j) - c(x_i) + D(x_i, x_j) \\ &= (j-i)(n-k+1) + (k - (j-i)) \\ &= (j-i)(n-k) + k \\ &\ge 2(n-k) + k = 2n - k \ge n. \end{aligned}$$

Case 2. $u, v \in Y$. Let $u = y_i$ and $v = y_j$, where $1 \le i, j \le n - k$. We may assume, without loss of generality, that $i < j$. Then

$$|c(y_i) - c(y_j)| = c(y_j) - c(y_i) = (j-i)(n-k-1) + \sum_{\ell=i}^{j-1} g_\ell.$$

If $j-i+\sum_{\ell=i}^{j-1} g_\ell \geq \frac{k}{2}$, then

$$D(y_i, y_j) = 2 + j - i + \sum_{\ell=i}^{j-1} g_\ell$$

by (β), and so

$$\begin{aligned} D(y_i, y_j) + |c(y_i) - c(y_j)| &= (j-i)(n-k-1) + \left(\sum_{\ell=i}^{j-1} g_\ell\right) + 2 + j - i + \left(\sum_{\ell=i}^{j-1} g_\ell\right) \\ &\geq (j-i)(n-k-1) + 2 + (j-i) + [k - 2(j-i)] \\ &= (j-i)(n-k-2) + k + 2 \geq n. \end{aligned}$$

If $1 \leq j-i+\sum_{\ell=i}^{j-1} g_\ell \leq \frac{k}{2}$, then

$$D(y_i, y_j) = 2 + k - \left(j - i + \sum_{\ell=i}^{j-1} g_\ell\right)$$

by (β), and so

$$\begin{aligned} D(y_i, y_j) + |c(y_i) - c(y_j)| &= (j-i)(n-k-1) + \left(\sum_{\ell=i}^{j-1} g_\ell\right) + 2 + k - \left(j - i + \sum_{\ell=i}^{j-1} g_\ell\right) \\ &= (j-i)(n-k-2) + k + 2 \\ &\geq n - k - 2 + k + 2 = n. \end{aligned}$$

Case 3. *One of u and v is in X and the other is in Y, say $u \in X$ and $v \in Y$.* Let $u = x_i$ and $v = y_j$, where $1 \leq i \leq k$ and $1 \leq j \leq n-k$. We consider two subcases, according to whether $x_i y_j \in E(G)$ or $x_i y_j \notin E(G)$.

Subcase 3.1. $x_i y_j \in E(G)$. We proceed by induction to show that

$$c(x_i) - c(y_j) \geq n - 1$$

when $x_i y_j \in E(G)$. For $i = j = 1$,

$$\begin{aligned} |c(x_1) - c(y_1)| = c(x_1) - c(y_1) &= [1 + n + (n-k-1)^2] - (1 + n - k) \\ &= (n-k-1)^2 + k \geq n - 1 \text{ for } n \geq k + 2. \end{aligned}$$

Assume that $c(x_i) - c(y_j) \geq n - 1$. Since $x_{i+1+g_j} y_{j+1} \in E(G)$ by (8), we show that $c(x_{i+1+g_j}) - c(y_{j+1}) \geq n - 1$. Observe that

$$c(x_{i+1+g_j}) = c(x_i) + (g_j + 1)(n-k+1) \quad \text{and} \quad c(y_{j+1}) = c(y_j) + (n-k-1) + g_j.$$

It then follows by the induction hypothesis that

$$\begin{aligned} c(x_{i+1+g_j}) - c(y_{j+1}) &\geq n-1+(g_j+1)(n-k+1)-(n-k-1)-g_j \\ &= n+1+g_j(n-k) \geq n-1. \end{aligned}$$

Therefore if $x_i y_j \in E(G)$, then $|c(x_i)-c(y_j)| + D(x_i, y_j) \geq n-1+1 = n$. Thus condition (11) is satisfied.

Subcase 3.2. $x_i y_j \notin E(G)$. Then $i \neq j$. By (8), if $y_j x_m \in E(G)$, then

$$\sum_{\ell=1}^{j-1} g_\ell = m - j,$$

and

$$\begin{aligned} &D(x_i, y_j) + |c(x_i)-c(y_j)| = c(x_i)-c(y_j)+D(x_i,x_m)+1 \\ &= (n-k-1)^2+(i-j)(n-k)+i+j-(m-j)+k-1+D(x_i,x_m) \\ &= [(n-k-1)^2+(i-j)(n-k)+k+2j-1]+(i-m)+D(x_i,x_m). \end{aligned} \tag{12}$$

Now observe, if $i > j$, then $(i-j)(n-k)+k \geq n$; whereas if $1 \leq i < j \leq n-k$, then

$$\begin{aligned} &(n-k-1)^2+(i-j)(n-k)+k+2j-1 \\ &\qquad = [(n-k)^2-2(n-k)]+i(n-k)-j(n-k-2)+k \\ &\qquad \geq [(n-k)^2-2(n-k)]+i(n-k)-[(n-k)^2-2(n-k)]+k \geq n. \end{aligned}$$

Therefore, by Equation (12)

$$D(x_i, y_j) + |c(x_i)-c(y_j)| \geq n+(i-m)+D(x_i,x_m). \tag{13}$$

We then have three possible situations. If $i > m$, then $i-m > 0$ and so by condition (13), (11) is satisfied. If $m > i$ and $m-i \geq k/2$, then $D(x_i, x_m) = m-i$ and so by (13)

$$D(x_i, y_j) + |c(x_i)-c(y_j)| \geq n+(i-m)+(m-i) = n.$$

Finally, if $m > i$ and $m-i \leq k/2$, then $D(x_i, x_m) = k-(m-i)$ and so from (13)

$$\begin{aligned} D(x_i, y_j) + |c(x_i)-c(y_j)| &\geq n+(i-m)+[k-(m-i)] \\ &= n+k-2(m-i) \geq n+k-k = n. \end{aligned}$$

For each situation, condition (11) is satisfied. Therefore c is a Hamiltonian labeling of G. □

We now present two corollaries of Theorems 3.5 and 3.6.

Corollary 3.7. *If G is a graph of order n that is the corona of a Hamiltonian graph, then*

$$\mathrm{hn}(G) = \binom{n}{2}.$$

Proof. Suppose that H is a Hamiltonian graph of order $k \geq 3$ and that $G = \mathrm{cor}(H)$. Then the order of G is $n = 2k$. We show that

$$\mathrm{hn}(G) = \binom{n}{2} = k(2k-1).$$

If $H \neq C_k$ and $H \neq K_k$, then G can be obtained from $\mathrm{cor}(K_k)$ by deleting nonbridge edges and $\mathrm{cor}(C_k)$ can be obtained from G by deleting edges that are not bridges. It then follows by Lemma 3.3 that

$$\mathrm{hn}(\mathrm{cor}(K_k)) \leq \mathrm{hn}(G) \leq \mathrm{hn}\big(\mathrm{cor}(C_k)\big).$$

Therefore, it suffices to show that

$$k(2k-1) \leq \mathrm{hn}\big(\mathrm{cor}(K_k)\big) \text{ and } \mathrm{hn}\big(\mathrm{cor}(C_k)\big) \leq k(2k-1).$$

From Theorems 3.5 and 3.6, we find that

$$\mathrm{hn}(\mathrm{cor}(K_k)) \geq (2k-1)(2k-k) + (2k-2k) = k(2k-1)$$

and

$$\begin{aligned}\mathrm{hn}(\mathrm{cor}(C_k)) &\leq 1 + 2k + (2k-k-1)^2 + (k-2)(2k-k+1)\\ &= 1 + 2k + k^2 - 2k + 1 + k^2 - k - 2 = k(2k-1).\end{aligned}$$

Therefore, $\mathrm{hn}(G) = k(2k-1)$. □

Corollary 3.8. *For each graph $G \in \mathcal{H}_k$,*

$$2k-1 \leq \mathrm{hn}\,(G) \leq k(2k-1).$$

Proof. Let

$$f(x) = (x-1)(x-k) + (2k-x),$$

for $k+1 \leq x \leq 2k$ and let

$$g(x) = 1 + x + (x-k-1)^2 + (k-2)(x-k+1),$$

for $k+2 \leq x \leq 2k$. Let $G \in \mathcal{H}_k$ be a graph of order n where $k+1 \leq n \leq 2k$. Then by Corollary 3.4 and Theorems 3.5 and 3.6,

$$f(n) \leq \mathrm{hn}\,(G) \leq g(n).$$

Since each $f(x)$ and $g(x)$ is an increasing function in its domain, it follows that $f(x) \geq f(k+1) = 2k-1$ and $g(x) \leq g(2k) = k(2k-1)$, implying the desired result. □

Both lower and upper bound in Corollary 3.8 are sharp. For example, if $G' \in \mathcal{H}_k$ is a graph of order $k+1$ whose core is K_k, then $\mathrm{hn}(G') = 2n-3 = 2k-1$ by Theorem 3.2; while if $G'' \in \mathcal{H}_k$ is a graph of order $2k$ whose core is K_k, then

$$\mathrm{hn}(G'') = \binom{n}{2} = k(2k-1)$$

by Corollary 3.7.

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References

[Chartrand and Zhang 2008] G. Chartrand and P. Zhang, *Chromatic graph theory*, Chapman & Hall/CRC, Boca Raton, FL, 2008.

[Chartrand et al. 2001] G. Chartrand, D. Erwin, P. Zhang, and F. Harary, "Radio labelings of graphs", *Bull. Inst. Combin. Appl.* **33** (2001), 77–85. MR 2003d:05185 Zbl 0989.05102

[Chartrand et al. 2002a] G. Chartrand, D. Erwin, and P. Zhang, "Radio antipodal colorings of graphs", *Math. Bohem.* **127**:1 (2002), 57–69. MR 2003b:05065 Zbl 0995.05056

[Chartrand et al. 2002b] G. Chartrand, L. Nebeský, and P. Zhang, "Bounds for the Hamiltonian chromatic number of a graph", pp. 113–125 in *Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2002)*, vol. 157, 2002. MR 2004e:05062 Zbl 1029.05059

[Chartrand et al. 2005a] G. Chartrand, L. Nebeský, and P. Zhang, "Hamiltonian colorings of graphs", *Discrete Appl. Math.* **146**:3 (2005), 257–272. MR 2005i:05061 Zbl 1056.05054

[Chartrand et al. 2005b] G. Chartrand, L. Nebesky, and P. Zhang, "On Hamiltonian colorings of graphs", *Discrete Math.* **290**:2-3 (2005), 133–143. MR 2006a:05058 Zbl 1059.05046

[Nebeský 2003] L. Nebeský, "Hamiltonian colorings of graphs with long cycles", *Math. Bohem.* **128**:3 (2003), 263–275. MR 2004h:05051 Zbl 1050.05055

[Nebeský 2006] L. Nebeský, "The Hamiltonian chromatic number of a connected graph without large Hamiltonian-connected subgraphs", *Czech. Math. J.* **56**:2 (2006), 317–338. MR 2007k:05125

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