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Some characterizations of type-3 slant helices in Minkowski space-time

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In this work, the concept of a slant helix is extended to Minkowski space-time. In an analogous way, we define type-3 slant helices whose trinormal lines make a constant angle with a fixed direction. Moreover, some characterizations of such curves are presented.

1. Introduction

Many important results in the theory of curves in E^3 were initiated by G. Monge; G. Darboux pioneered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations, which play an important role in mechanics and kinematics as well as in differential geometry (for more details see [Boyer 1968]). At the beginning of the twentieth century, Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold.

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal, and trinormal) was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in the space E_1^4 [O'Neill 1983]. Helices (inclined curves) are a well-known concept in classical differential geometry [Millman and Parker 1977].

The notion of a slant helix is due to Izumiya and Takeuchi [2004], who defined a slant helix in E^3 as a curve $\varphi = \varphi(s)$ with nonvanishing first curvature if the principal lines of φ make a constant angle with a fixed direction. In the same space, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves are presented in [Kula and Yayli 2005].

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to

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Lorentz manifolds. For instance, Duggal and Bejancu [1996] studied null curves in Lorentz manifolds and determined Frenet frame for lightlike curves. Ferrández et al. [2002] studied null generalized helices in Lorentz–Minkowski space. In the light of degenerate submanifold theory, Karadag and Karadag [2008] defined null slant helices and wrote some characterizations in E_1^3 and also proved that there does not exist a null slant helix in E_1^4. Some characterizations of the Cartan framed null generalized helix and the null slant helix having a nonnull axis in Lorentz–Minkowski space were given in [Erdogan and Yilmaz 2008].

In the literature, all works adopt the definition of a slant helix in [Izumiya and Takeuchi 2004] as one whose principal lines make a constant angle with a fixed direction. Some of them deal with null curves in Lorentz–Minkowski spaces. In this paper, we define a special slant helix whose *trinormal* lines make a constant angle with a fixed direction and call such curves *type-3 slant helices*. Additionally, we present some characterizations of curves in the space E_1^4.

2. Preliminaries

The basic elements of the theory of curves in the space E_1^4 are briefly presented here (a more complete elementary treatment can be found in [O'Neill 1983]).

Minkowski space-time E_1^4 is a Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characterizations:

(i) it can be space-like if $g(v, v) > 0$ or $v = 0$;

(ii) it can be time-like if $g(v, v) < 0$;

(iii) and it can be null (light-like) if $g(v, v)=0$ and $v \neq 0$.

Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like, or null. Also, recall that the norm of a vector v is given by

$$\|v\| = \sqrt{|g(v, v)|}.$$

Therefore, v is a unit vector if $g(v, v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Let a and b be two space-like vectors in E_1^4, then there is a unique real number $0 \leq \delta \leq \pi$, called the angle between a and b, such that $g(a, b) = \|a\|.\|b\| \cos\delta$. Let $\vartheta = \vartheta(s)$ be a curve in E_1^4. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a helix or an inclined curve. Recall that $\vartheta = \vartheta(s)$ is a slant helix if its principal lines make a constant angle with a fixed direction.

Denote by $\{T(s), N(s), B(s), E(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B, E are, respectively, the tangent, the principal normal, the binormal, and the trinormal vector fields. A space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a space-like curve in the space-time E_1^4, parametrized by arclength function s. Then, for the unit speed curve α with nonnull frame vectors, the following Frenet equations are given in [Walrave 1995]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix},$$

where T, N, B, and E are mutually orthogonal vectors satisfying $g(B, B) = -1$ and $g(T, T) = g(N, N) = g(E, E) = 1$, where, κ, τ and σ are first, second and third curvature of the curve α, respectively.

3. Type–3 slant helix in Minkowski space-time

Definition 3.1. *A curve $\psi = \psi(s)$ is called a type-3 slant helix if the trinormal lines of ψ make a constant angle with a fixed direction in E_1^4.*

Recall that an arbitrary curve is called a *W*-curve if it has constant Frenet curvatures [İlarslan and Boyacıoğlu 2007].

Lemma 3.2. *A type-3 slant helix with curvatures $\kappa \neq 0$, $\tau \neq 0$, and $\sigma \neq 0$ cannot be a W-curve in E_1^4.*

Proof. Let $\psi = \psi(s)$ be a type-3 slant helix with curvatures $\kappa \neq 0$, $\tau \neq 0$, and $\sigma \neq 0$ and also be a *W*-curve in E_1^4. From the definition of type-3 slant helix, we have

$$g(E, U) = \cos\delta, \tag{1}$$

where U is a fixed direction (a constant space-like vector) and δ is a constant angle. Differentiating Equation (1) with respect to s, we easily get $\sigma g(B, U) = 0$, which yields B perpendicular to U. Therefore, we may express U as

$$U = u_1 T + u_2 N + u_3 E. \tag{2}$$

Differentiating Equation (2) with respect to s and considering the Frenet equations, we have the following system of equations

$$\frac{du_1}{ds} - u_2\kappa = 0, \qquad \frac{du_2}{ds} + u_1\kappa = 0, \qquad u_2\tau + u_3\sigma = 0, \qquad \frac{du_3}{ds} = 0. \tag{3}$$

From $(3)_4$ we easily obtain $u_3 = \cos\delta = \text{constant} \neq 0$. Substituting this into $(3)_3$, (owing to $\sigma = \text{constant}$ and $\tau = \text{constant}$) we have $u_2 = -\sigma/\tau \cos\delta = \text{constant}$. Using this in $(3)_2$, we get $u_1 = 0$. This result and Equation $(3)_1$ imply that $u_2 = 0$, which is a contradiction. Therefore, type-3 slant helix cannot be a *W*-curve in E_1^4. □

Theorem 3.3. *Let $\psi = \psi(s)$ be a space-like curve with curvatures $\kappa \neq 0$, $\tau \neq 0$ and $\sigma \neq 0$ in E_1^4. Then $\psi = \psi(s)$ is a type-3 slant helix if and only if*

$$\left(\frac{\sigma}{\tau}\right)^2 + \left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\right]^2 = \textit{constant}. \tag{4}$$

Proof. Let $\psi = \psi(s)$ be a type-3 slant helix in E_1^4. Then the equations in (3) hold. Thus, we easily have $u_3 = \cos\delta = \text{constant} \neq 0$ and

$$u_2 = -\frac{\sigma}{\tau}\cos\delta. \tag{5}$$

If we consider $(3)_1$ and $(3)_2$, we obtain a second order differential equation with respect to u_2 as follows:

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{du_2}{ds}\right] + u_2\kappa = 0. \tag{6}$$

Using an exchange variable $t = \int_0^s \kappa\, ds$ in Equation (6),

$$\frac{d^2u_2}{dt^2} + u_2 = 0 \tag{7}$$

is obtained. Solution of Equations (7) and (5) gives us

$$A\cos\int_0^s \kappa\, ds + B\sin\int_0^s \kappa\, ds = -\frac{\sigma}{\tau}\cos\delta, \tag{8}$$

where A and B are real numbers. Differentiating Equation (8) with respect to s, we obtain

$$-A\kappa\sin\int_0^s \kappa\, ds + B\kappa\cos\int_0^s \kappa ds = -\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\cos\delta. \tag{9}$$

In terms of (8) and (9), coefficients A and B can be calculated by the Cramer method. They are obtained as

$$\begin{aligned} A &= -\left(\frac{\sigma}{\tau}\cos\delta\right)\cos\int_0^s \kappa\, ds + \frac{\cos\delta}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\sin\int_0^s \kappa\, ds,\\ B &= -\frac{\cos\delta}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\cos\int_0^s \kappa ds - \left(\frac{\sigma}{\tau}\cos\delta\right)\sin\int_0^s \kappa\, ds. \end{aligned} \tag{10}$$

If we form $A^2 + B^2$, we get

$$\left(\frac{\sigma}{\tau}\right)^2 + \left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\right]^2 = \frac{A^2 + B^2}{(\cos\delta)^2} = \text{constant}. \tag{11}$$

Conversely, let us consider a vector given by

$$U = \left\{\frac{\sigma}{\tau}T + \frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)N + E\right\}\cos\delta. \tag{12}$$

Differentiating vector U and considering the differential of (11), we get

$$\frac{dU}{ds} = 0, \tag{13}$$

where δ is a constant angle. Equation (13) shows that U is a constant vector. And then considering a space-like curve $\psi = \psi(s)$ with nonvanishing curvatures, we have

$$g(E, U) = \cos\delta, \tag{14}$$

and it follows that $\psi = \psi(s)$ is a type-3 slant helix in E_1^4. □

Now, considering the differential of (11) and (12), we give the following result and remark.

Corollary 3.4. *Let $\psi = \psi(s)$ be a space-like curve with curvatures $\kappa \neq 0$, $\tau \neq 0$, and $\sigma \neq 0$. Then ψ is a type-3 slant helix in E_1^4 if and only if*

$$\frac{\kappa\sigma}{\tau} + \frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma}{\tau}\right)\right] = 0. \tag{15}$$

Remark 3.5. *The fixed direction in the definition of type-3 slant helix can be taken as Equation (12).*

Let us solve Equation (15) with respect to σ/τ. Using the exchange variable $t = \int_0^s \kappa\, ds$ in (15), we get

$$\frac{d^2}{dt^2}\left(\frac{\sigma}{\tau}\right) + \left(\frac{\sigma}{\tau}\right) = 0, \tag{16}$$

and so

$$\frac{\sigma}{\tau} = L_1 \cos\int_0^s \kappa\, ds + L_2 \sin\int_0^s \kappa\, ds, \tag{17}$$

where L_1 and L_2 are real numbers.

Corollary 3.6. *Let $\psi = \psi(s)$ be a space-like curve with curvatures $\kappa \neq 0$, $\tau \neq 0$, and $\sigma \neq 0$, then ψ is a type-3 slant helix in E_1^4 if and only if there is the following relation among the curvatures of ψ:*

$$\frac{\sigma}{\tau} = L_1 \cos\int_0^s \kappa ds + L_2 \sin\int_0^s \kappa\, ds. \tag{18}$$

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