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For an integer $t \geq 1$ and a partition λ, we let $\mathcal{H}_t(\lambda)$ be the multiset of hook lengths of λ which are divisible by t. Then, define $a_t^{\text{even}}(n)$ and $a_t^{\text{odd}}(n)$ to be the number of partitions of n such that $|\mathcal{H}_t(\lambda)|$ is even or odd, respectively. In a recent paper, Han generalized the Nekrasov–Okounkov formula to obtain a generating function for $a_t(n) = a_t^{even}(n) - a_t^{odd}(n)$. We use this generating function to prove congruences for the coefficients $a_t(n)$.

1. Introduction and statement of results

Let $p(n)$ denote the number of integer partitions of n. Ramanujan proved the following important congruence relations for the partition function, which hold for all nonnegative n:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1-1}$$

These congruences can be proven through q-series identities or with the theory of modular forms; both methods rely on the following generating function for $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots. \tag{1-2}$$

Recently, Nekrasov and Okounkov [2006] generalized Equation (1-2) by discovering a combinatorial interpretation of $\prod_{n=1}^{\infty}(1-q^n)^b$, for $b \in \mathbb{C}$, in terms of partition hook lengths. Here we briefly recall their results, beginning by introducing the necessary notation.

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A Ferrers diagram, a pictorial representation of a partition, allows us to define the *hook length* of a box in the partition. The hook length of a box in the Ferrers diagram is the sum of the number of boxes in the same column below it, the number of boxes in the same row and to the right, and one for the box. For example, consider the partition $5+3+2$ of 10. Its Ferrers diagram, with hook lengths filled in, is:

7	6	4	2	1
4	3	1		
2	1			

We define $\mathcal{H}(\lambda)$, the *hook length multiset* of a partition λ, to be the multiset of hook lengths in each box in the Ferrers diagram of λ. We can then define $\mathcal{H}_t(\lambda) \subseteq \mathcal{H}(\lambda)$ to be the multiset of hook lengths of boxes in the partition that are multiples of t.

Nekrasov and Okounkov proved the following formula which uses these combinatorial objects, and holds for any complex b:

$$\sum_{\lambda \in P} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{b}{h^2}\right) = \prod_{n \geq 1} (1-q^n)^{b-1}.$$

More recently, Han obtained a generalization of this formula. A specialization of it gives useful infinite-product generating functions for the series

$$\sum_{n=0}^{\infty} a_t(n) q^n = \sum_{\lambda \in P} q^{|\lambda|} (-1)^{\#\mathcal{H}_t(\lambda)} = \prod_{n=1}^{\infty} \frac{(1-q^{4tn})^t (1-q^{tn})^{2t}}{(1-q^{2tn})^{3t}(1-q^n)}. \tag{1-3}$$

Remark 1.1. For $t=1$, the sum over partitions is easy to understand directly: we have $\mathcal{H}_1(\lambda) = \mathcal{H}(\lambda)$, so $a_1(n) = (-1)^n p(n)$.

A number of congruences of the coefficients $a_t(n)$ are a direct consequence of the Ramanujan congruences in (1-1) combined with Han's generating function (1-3). Namely, for all $n \geq 0$, one has

$$\begin{aligned} a_t(5n+4) &\equiv 0 \pmod{5} && \text{if } t=1 \text{ or } 5 \mid t, \\ a_t(7n+5) &\equiv 0 \pmod{7} && \text{if } t=1 \text{ or } 7 \mid t, \\ a_t(11n+6) &\equiv 0 \pmod{11} && \text{if } t=1 \text{ or } 11 \mid t. \end{aligned}$$

Here, we are interested in further congruences of the form $a_t(An+B)$.

Our search for such congruences over small arithmetic progressions and with small prime moduli yielded just one Ramanujan-type congruence that was not of the above form.

Theorem 1.2. *If* $n \geq 0$, *then* $a_2(5n+4) \equiv 0 \pmod{5}$.

This congruence can be proven through q-series identities, which can in turn be proven using the theory of modular forms. More generally, we have:

Theorem 1.3. *If $3 \nmid t$ and $\ell > 3t^3 + t$ is prime, a positive proportion of primes p satisfy*

$$a_t\left(\frac{p^3\ell n+1}{24}\right) \equiv 0 \pmod{\ell},$$

for all n coprime to p.

Remark 1.4. Saying that *a positive proportion of primes* satisfy a condition means that the limit

$$\lim_{n\to\infty} \frac{\#\{p \leq n : p \text{ prime, and } p \text{ satisfies the condition}\}}{\#\{p \leq n : p \text{ prime}\}}$$

exists and is strictly positive.

Corollary 1.5. *For t, ℓ, p satisfying the previous theorem, there are linear congruences*

$$a_t(p^4\ell n + b_p) \equiv 0 \pmod{\ell},$$

for a fixed $b_p < p^4\ell$ and all nonnegative integers n.

To prove this, we also use methods of the theory of modular forms. Such methods were first employed by Ono [2000] and Ahlgren and Ono [2001] to prove the existence of classes of congruences for the partition function. We apply similar arguments to the generating functions for the $a_t(n)$.

Remark 1.6. Treneer [2006] extended the arguments in [Ono 2000] and [Ahlgren and Ono 2001] in a general way, to prove congruences for the coefficients of all weakly holomorphic modular forms. This result can be applied to our generating function to obtain similar conclusions. However, we proceed by other methods to obtain explicit constructions.

In Section 2, we discuss Han's generating function, and relate it to the theory of modular forms. In Section 3, we prove Theorem 1.2, and in Sections 4 and 5, we prove Theorem 1.3.

2. Han's generating function and modular forms

Han's Generating Function. Han [2008] proved the Nekrasov–Okounkov formula using combinatorial methods, and obtained the following generalization:

Theorem 2.1. *For any positive integer t, and complex numbers b, y, we have*

$$\sum_{\lambda\in P} q^{|\lambda|} \prod_{h\in\mathcal{H}_t(\lambda)} \left(y - \frac{tyb}{h^2}\right) = \prod_{n=1}^{\infty} \frac{(1-q^{tn})}{(1-(yq^t)^n)^{t-b}(1-q^n)}.$$

Taking $b=0$ and $y=-1$, the left side reduces to the generating functions $\sum a_t(n)q^n$ that we are interested in:

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})}{(1-(-q^t)^n)^t(1-q^n)}.$$

Manipulating the terms of the infinite product gives the following formula:

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{4tn})^t(1-q^{tn})^{2t}}{(1-q^{2tn})^{3t}(1-q^n)}.$$

Our results depend on the modularity of these series, which we explain in the next section.

Modularity of $\sum a_t(n)q^n$. Recall the definition of Dedekind's η-function, where we let $q=e^{2\pi iz}$:

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Using this definition and the infinite product generating function we can write

$$\sum_{n=0}^{\infty} a_t(n)q^n = q^{1/24}\frac{\eta(4tz)^t\eta(tz)^{2t}}{\eta(2tz)^{3t}\eta(z)}.$$

Replacing z by $24z$, we have

$$\sum_{n=0}^{\infty} a_t(n)q^{24n-1} = \frac{\eta(96tz)^t\eta(24tz)^{2t}}{\eta(48tz)^{3t}\eta(24z)}.$$

Combining Theorem 1.65 in [Ono 2004] about integer-weight η-quotients with the transformation properties for $\eta(24z)$, we have:

Theorem 2.2. *$\sum_{n=0}^{\infty} a_t(n)q^{24n-1}$ is a weakly holomorphic modular form of weight $-1/2$ on the congruence subgroup $\Gamma_0(2304t)$, with character $\chi(d) = \big((2^t3)/d\big)$.*

3. Proof of Theorem 1.2

A q-series identity. The key step to the proof that $a_2(5n+4) \equiv 0 \pmod 5$ is the following q-series identity, which can be proven from the theory of modular forms:

Theorem 3.1. *The following identity is true*:

$$\prod_{n=1}^{\infty} \frac{(1-q^{4n})^2(1-q^n)^2}{1-q^{2n}} = \sum_{n\in\mathbb{Z}}(1-3n)q^{3n^2-2n}.$$

Proof. Let $q = e^{2\pi i z}$, and define

$$f(z) = \prod_{n\geq 1} \frac{(1-q^{4n})^2(1-q^n)^2}{1-q^{2n}}, \qquad g(z) = \sum_{n\in\mathbb{Z}} (1-3n)q^{3n^2-2n}.$$

We will prove that $q\, f(3z)$ and $q\, g(3z)$ are both modular forms in the same finite-dimensional space. Thus, to show equality it suffices to show that a finite number of terms in the q-expansion of $q(f(3z)-g(3z))$ are zero; this implies $f(z) = g(z)$.

We can write $q\, f(3z)$ as a quotient of Dedekind's eta-functions:

$$q\, f(3z) = q\prod_{n=1}^{\infty} \frac{(1-q^{12n})^2(1-q^{3n})^2}{1-q^{6n}} = \frac{\eta(12z)^2\eta(3z)^2}{\eta(6z)}.$$

By the standard theory of eta-quotients (as in [Ono 2004, Section 1.4]), this is a cusp form of weight 3/2, level 144, and character $\chi(d) = (3/d)$.

On the other hand, $q\, g(3z)$ can be expressed as a Jacobi theta function. Define $\psi(n)$ to be the Dirichlet character $(n/3)$. As in [Ono 2004, Section 1.3.1], define

$$\theta(\psi, 1, z) = \sum_{n=1}^{\infty} \psi(n) n q^{n^2},$$

which is a cusp form of weight 3/2, level 36, and character $\chi(d) = (3/d)$. By periodicity of ψ, we have

$$\begin{aligned}
\theta(\psi,1,z) &= \psi(0)\sum_{n=1}^{\infty} 3nq^{(3n)^2} + \psi(1)\sum_{n=0}^{\infty}(3n+1)q^{(3n+1)^2} + \psi(2)\sum_{n=0}^{\infty}(3n+2)q^{(3n+2)^2} \\
&= \sum_{n=0}^{\infty}(3n+1)q^{(3n+1)^2} - \sum_{n=0}^{\infty}(3n+2)q^{(3n+2)^2} \\
&= -\sum_{n=-\infty}^{\infty}(1-3n)q^{(3n-1)^2} = -q\sum_{n\in\mathbb{Z}}(1-3n)q^{3(3n^2-2n)} = -q\, g(3z).
\end{aligned}$$

Therefore, both $q\, f(3z)$ and $q\, g(3z)$ are in $S_{3/2}(\Gamma_0(144), \chi)$, so to check equality it suffices [Sturm 1987, Theorem 1] to check equality of the first $k/24[\Gamma_0(1) : \Gamma_0(144)] = 18$ coefficients. Thus, we have

$$f(z) = g(z) = 1 - 2q + 4q^5 - 5q^8 + 7q^{16} - 8q^{21} \pm \cdots. \qquad \square$$

Proof of the congruence. We can now prove that $a_2(5n+4) \equiv 0 \pmod 5$. First, note that we can formally factor the generating function for $a_2(n)$. By doing so and applying the binomial theorem mod 5, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_2(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{4n})^5} \prod_{n=1}^{\infty} \frac{(1-q^{8n})^2(1-q^{2n})^2}{(1-q^{4n})} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} \\ &\equiv \prod_{n=1}^{\infty} \frac{1}{(1-q^{20n})} \prod_{n=1}^{\infty} \frac{(1-q^{8n})^2(1-q^{2n})^2}{(1-q^{4n})} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} \pmod 5.\end{aligned}$$

Using the partition generating function (1-2), Theorem 3.1, and an identity of Jacobi, we can write

$$\sum_{n=0}^{\infty} a_2(n)q^n = \Bigl(\sum_{i=0}^{\infty} p(i)q^{20i}\Bigr)\Bigl(\sum_{k\in\mathbb{Z}}(1-3k)(q^2)^{3k^2-2k}\Bigr)\Bigl(\sum_{m=0}^{\infty} q^{(m^2+m)/2}\Bigr).$$

A coefficient $a_2(5n+4)$ will thus be a sum of terms $p(i)\cdot(1-3k)\cdot 1$ where

$$20i+6k^2-4k+\frac{m^2+m}{2} \equiv 4 \pmod 5.$$

We can check that this only holds when $m \equiv k \equiv 2 \pmod 5$. For such terms, $1-3k \equiv 0 \pmod 5$, so $a_2(5n+4) \equiv 0 \pmod 5$. $\square$

4. Sieved generating functions and cusp forms mod ℓ

To prove the existence of an infinite class of congruences, we follow similar arguments to those used by Ono [2000] and Ahlgren and Ono [2001] to prove congruences for the partition function. We first construct a cusp form congruent to a sieved version of our original generating function

$$\sum_{n=0}^{\infty} a_t(n)q^n.$$

Theorem 4.1. *If $3 \nmid t$ and $\ell > 3t^3+t$ is prime, there exists a half-integer weight cusp form $g_{t,\ell}(z)$ with a q-series expansion satisfying the congruence*

$$g_{t,\ell}(z) \equiv \sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{24n+\frac{24\beta_\ell-1}{\ell}} \pmod{\ell},$$

where β_ℓ satisfies $24\beta_\ell \equiv 1 \pmod{\ell}$ and $0 < \beta_\ell < \ell$.

We can rewrite the sieved generating function as

$$\sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{24n+\frac{24\beta_\ell-1}{\ell}} = \sum_{\substack{n\ge 0\\ \ell n\equiv -1 \pmod{24}}} a_t\Bigl(\frac{\ell n+1}{24}\Bigr)q^n.$$

If we take $a_t(m) = 0$ for any noninteger m, then the conclusion of Theorem 4.1 can be written as

$$g_{t,\ell}(z) \equiv \sum_{n=0}^{\infty} a_t\Big(\frac{\ell n+1}{24}\Big) q^n \pmod{\ell}. \tag{4-1}$$

Preliminaries for Proof. We define the functions

$$F_t(z) = \frac{\eta(4tz)^t \eta(tz)^{2t}}{\eta(2tz)^{3t}\eta(z)} = q^{-1/24}\sum_{n=0}^{\infty} a_t(n) q^n,$$

$$H_{t,\ell}(z) = \eta(z)^{\ell}\eta(2tz)^{5t\ell}\eta(4tz)^{3t\ell}\eta(tz)^{2t\ell},$$

$$G_{t,\ell}(z) = F_t(z) H_{t,\ell}(z)^{\ell}.$$

By standard facts about eta-quotients (as in [Ono 2004, Section 1.4]), $G_{t,\ell}(z)$ is an integer-weight cusp form on $\Gamma_0(4t)$ and $H_{t,\ell}(24z)$ is a half-integer weight cusp form on $\Gamma_0(2304t)$. We relate the sieved generating function from Theorem 4.1 to these functions by:

Lemma 4.2. *The following congruence between q-series expansions holds*:

$$G_{t,\ell}(z) | T(\ell) \equiv H_{t,\ell}(z) \sum_{n=0}^{\infty} a_t(\ell n + \beta_\ell) q^{n+\frac{24\beta_\ell - 1}{24\ell}} \pmod{\ell}.$$

Here we let $T(\ell)$ denote the ℓ-th Hecke operator

$$\Big(\sum_{n=0}^{\infty} b(n) q^n\Big) | T(\ell) = \sum_{n=0}^{\infty} \Big(b(\ell n) + \chi(\ell)\ell^{k-1} b(n/\ell)\Big) q^n,$$

where k and χ are the weight and character of the form $\sum_{n=0}^{\infty} b(n)q^n$.

Proof. Define $\delta_\ell = (\ell^2 - 1)/24$, which is an integer for all primes $\ell \geq 5$. By definition of $G_{t,\ell}(z)$ and η, we have

$$G_{t,\ell}(z) = \Big(\sum_{n=0}^{\infty} a_t(n) q^{n+\delta_l}\Big) q^{t^2\ell^2} \prod_{n=1}^{\infty} (1-q^n)^{\ell^2} (1-q^{2tn})^{5t\ell^2} (1-q^{4tn})^{3t\ell^2} (1-q^{tn})^{2t\ell^2}.$$

Applying the binomial theorem mod ℓ gives

$$G_{t,\ell}(z) \equiv q^{t^2\ell^2} \prod_{n=1}^{\infty} (1-q^{n\ell^2})(1-q^{2t\ell^2 n})^{5t}(1-q^{4t\ell^2 n})^{3t}(1-q^{t\ell^2 n})^{2t} \cdot \Big(\sum_{n=0}^{\infty} a_t(n) q^{n+\delta_l}\Big) \pmod{\ell}.$$

The $T(\ell)$ operator is equivalent mod ℓ to the $U(\ell)$ operator, which is defined as

$$\Big(\sum_{n=0}^{\infty} b(n)q^n\Big)|U(\ell) = \sum_{n=0}^{\infty} b(\ell n)q^n.$$

We apply $T(\ell)$ to the left side and $U(\ell)$ to the right side to obtain

$$\begin{aligned}
G_{t,\ell}(z)|T(\ell) &\equiv q^{t^2\ell}\prod_{n=1}^{\infty}(1-q^{n\ell})(1-q^{2t\ell n})^{5t}(1-q^{4t\ell n})^{3t}(1-q^{t\ell n})^{2t} \\
&\quad\cdot\Big(\sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{n+\frac{\beta_\ell+\delta_\ell}{\ell}}\Big) \\
&\equiv H_{t,\ell}(z)\Big(\sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{n+\frac{24\beta_\ell-1}{24\ell}}\Big) \pmod{\ell}. \qquad \square
\end{aligned}$$

We define

$$g_{t,\ell}(z) = \frac{G_{t,\ell}(z)|T(\ell)|V(24)}{H_{t,\ell}(24z)}.$$

Lemma 4.2 tells us that $g_{t,\ell}(z)$ is congruent mod ℓ to our sieved generating function. Thus, to prove Theorem 4.1 we need to show that $g_{t,\ell}(z)$ is a cusp form.

Proof of Theorem 4.1. It suffices to prove that $(G_{t,\ell}(z)|T(\ell))/H_{t,\ell}(z)$ vanishes at all of the cusps, since applying $V(24)$ will preserve cuspidality. Since the Hecke operator preserves the level, $G_{t,\ell}(z)|T(\ell)$ is a form on $\Gamma_0(4t)$; standard facts about eta-quotients show that $H_{t,\ell}(z)^{24}$ is a form on $\Gamma_0(4t)$ as well.

Since the order of vanishing of $H_{t,\ell}(z)$ at any cusp is 1/24-th of the order of vanishing of $H_{t,\ell}(z)^{24}$ at that cusp, it suffices to consider orders of vanishing on a set of cusps containing a representative for each equivalence class on $\Gamma_0(4t)$. The cusps of the form c/d, where $d|4t$ and $(c,d)=1$, form such a set. We can divide the allowed values of d into three classes: $d=T$, $d=2T$, and $d=4T$, where $T|t$ and, for the latter two cases, $2T\nmid t$.

Let $\operatorname{ord}_{c/d} f$ denote the invariant order of vanishing of a function f at a cusp c/d. We can compute:

$$\begin{aligned}
d=T:&\quad \operatorname{ord}_{c/d} G_{t,\ell} = \frac{3T^2-4+21T^2\ell^2+4\ell^2}{96}, &\quad \operatorname{ord}_{c/d} H_{t,\ell} &= \frac{21T^2\ell+4\ell}{96},\\
d=2T:&\quad \operatorname{ord}_{c/d} G_{t,\ell} = \frac{-3T^2-1+15T^2\ell^2+\ell^2}{24}, &\quad \operatorname{ord}_{c/d} H_{t,\ell} &= \frac{15T^2\ell+\ell}{24},\\
d=4T:&\quad \operatorname{ord}_{c/d} G_{t,\ell} = \frac{-1+24T^2\ell^2+\ell^2}{24}, &\quad \operatorname{ord}_{c/d} H_{t,\ell} &= \frac{24T^2\ell+\ell}{24}.
\end{aligned}$$

Applying a Hecke operator $T(\ell)$ to a function takes the q-series expansion at a cusp c/d to a linear combination of q-series expansions around cusps of the form

c'/d, with q replaced by $q^{1/\ell}$. Because the order of vanishing depends only on the denominator, we have

$$\operatorname{ord}_{c/d} G_{t,\ell}(z)|T(\ell) \geq \frac{1}{\ell} \operatorname{ord}_{c/d} G_{t,\ell}(z).$$

Since $G_{t,\ell}(z)|T(\ell)$ is a form on $\Gamma_0(4t)$, we know that its order of vanishing must be of the form $A/4t$, where A is an integer. Using this fact, we can analyze the behavior at each cusp, and show that $\operatorname{ord}_{c/d} G_{t,\ell}(z)|T(\ell) > \operatorname{ord}_{c/d} H_{t,\ell}(z)$. For instance, at cusps c/d where $d = 2T$, we have

$$\operatorname{ord}_{c/d} G_{t,\ell}(z)|T(\ell) = \frac{A}{4t} \geq \frac{1}{24}\Big(\frac{-3T^2-1}{\ell} + 15T^2\ell + \ell\Big).$$

This gives

$$6A \geq \frac{-3T^2t - t}{\ell} + 15T^2t\ell + t\ell.$$

By hypothesis, $\ell > 3t^3 + t \geq 3T^2t + t$; hence

$$0 > \frac{-3T^2t - t}{\ell} > -1.$$

Since the other terms in the inequality are integers, we must have

$$6A \geq 15T^2t\ell + t\ell.$$

If equality held, the equation would reduce to $0 \equiv t\ell \pmod 3$; since t, ℓ are coprime to 3, we must have the strict inequality $6A > 15T^2t\ell + t\ell$. We therefore obtain the desired inequality

$$\operatorname{ord}_{c/d} G_{t,\ell}(z)|T(\ell) = \frac{A}{4t} > \frac{15T^2\ell + \ell}{24} = \operatorname{ord}_{c/d} H_{t,\ell}.$$

A similar analysis at cusps c/d where $d = T$ and $d = 4T$ shows that

$$\operatorname{ord}_{c/d} G_{t,\ell}(z)|T(\ell) > \operatorname{ord}_{c/d} H_{t,\ell}(z),$$

as well. Therefore,

$$\operatorname{ord}_{c/d} \frac{G_{t,\ell}(z)|T(\ell)}{H_{t,\ell}(z)} > 0$$

at all cusps. □

5. Proof of Theorem 1.3

We can now consider applying Hecke operators to function $g_{t,\ell}(z)$; modulo ℓ, this is equivalent to applying them to the sieved generating function

$$\sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{24n+\frac{24\beta_\ell-1}{\ell}}.$$

For a half-integral weight modular form

$$f(z)=\sum_{n=0}^{\infty} b(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N),\chi)$$

and a prime p, the Hecke operator $T(p^2)$ is defined by

$$f(z)|T(p^2)=\sum_{n=0}^{\infty}\left(b(p^2n)+\chi(p)\left(\frac{(-1)^\lambda n}{p}\right)p^{\lambda-1}b(n)+\chi(p^2)p^{2\lambda-1}b(n/p^2)\right)q^n.$$

Following the methods of Ono [2000], we will prove the following theorem, from which we can obtain congruences of the desired type.

Theorem 5.1. *If $(t,3)=1$ and $\ell>3t^3+3$ is prime, then for a positive proportion of primes p,*

$$\sum_{n=0}^{\infty} a_t(\ell n+\beta_\ell)q^{24n+\frac{24\beta_\ell-1}{\ell}}\equiv 0 \pmod{\ell}.$$

A Theorem of Serre and the Shimura Correspondence. The proof of Theorem 5.1 relies on two important theorems, one of Serre and one of Shimura. Serre [1976] proves that many Hecke operators annihilate modulo ℓ an integer weight space of cusp forms.

Theorem 5.2 (Serre). *Consider a fixed space of cusp forms $S_k(\Gamma_0(N),\chi)$, where k is an integer. The set of primes $p\equiv -1 \pmod{N}$ such that $f|T(p)\equiv 0 \pmod{\ell}$ for all $f\in S_k(\Gamma_0(N),\chi)$ has positive density.*

To apply this to the half-integer weight case, we use the *Shimura correspondence* [Shimura 1973] to relate integer weight and half-integer weight forms.

Theorem 5.3 (Shimura). *Let $f=\sum_{n=1}^{\infty} b(n)q^n$ be a half-integer weight cusp form in $S_{\lambda+1/2}(\Gamma_0(4N),\psi)$. For a positive integer r, define $S_r(f)$ by*

$$S_r(f)(z)=\sum_{n=1}^{\infty} A_r(n)q^n,\qquad \sum_{n=1}^{\infty}\frac{A_r(n)}{n^s}=L(s-\lambda+1,\psi\chi_{-1}^{\lambda}\chi_t)\sum_{n=1}^{\infty}\frac{b(rn^2)}{n^s},$$

where χ_{-1} and χ_t are the Kronecker characters for $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{t})$. Then

$$S_r(f)\in S_{2\lambda}(\Gamma_0(4N),\psi^2).$$

Moreover, if $p \nmid 4N$ is prime, then $S_r\big(f|T(p^2)\big) = S_r(f)|T(p)$.

Combining these two theorems will give us an analogue to Serre's theorem for half-integer weight modular forms, which proves the existence of primes that annihilate our sieved generating function.

Proof of Theorem 5.1. Let $\mathcal{P}_{t,\ell}$ be the set of primes $p \equiv -1 \pmod{2304t}$ such that $f|T(p^2) \equiv 0 \pmod{\ell}$ for all $f \in S_{2\lambda}(\Gamma_0(2304t), \chi_0)$, where χ_0 is the trivial Dirichlet character, and $\lambda + 1/2$ is the weight of the form $g_{t,\ell}(z)$ constructed in Section 4. By Serre's Theorem, $\mathcal{P}_{t,\ell}$ has positive density in the set of primes.

Furthermore, $S_r(g_{t,\ell})$, the image of g under the t-th Shimura correspondence, is in $S_{2\lambda}(\Gamma_0(2304t), \chi_0)$. So, for any $p \in \mathcal{P}_{t,\ell}$,

$$S_r(g_{t,\ell})|T(p) = S_r(g_{t,\ell}|T(p^2)) \equiv 0 \pmod{\ell}.$$

By construction of the Shimura correspondence, if $S_r(f) \equiv 0 \pmod{\ell}$, then $f \equiv 0 \pmod{\ell}$. So, for all $p \in \mathcal{P}_{t,\ell}$, $g_{t,\ell}|T(p^2) \equiv 0 \pmod{\ell}$. □

Proof of Theorem 1.3. From Theorem 5.1, for a positive proportion of primes p and all m,

$$b_{t,\ell}(p^2m) + \chi(p)\Big(\frac{(-1)^\lambda m}{p}\Big)p^{\lambda-1}b_{t,\ell}(m) + \chi(p^2)p^{2\lambda-1}b_{t,\ell}(m/p^2) \equiv 0 \pmod{\ell},$$

where $b_{t,\ell}(n)$ is the coefficient of q^n in the Fourier expansion of $g_{t,\ell}(z)$.

In particular, consider $m = pn$ for some n coprime to p. Then m/p^2 is not an integer, and $b_{t,\ell}(m/p^2) = 0$; furthermore the Legendre symbol $\big(((-1)^\lambda m)/p\big)$ is zero. Recalling Equation (4-1), we have

$$b_{t,\ell}(p^3n) \equiv a_t\left(\frac{p^3\ell n + 1}{24}\right) \equiv 0 \pmod{\ell},$$

which proves Theorem 1.3. □

Proof of Corollary 1.5. Let $0 \le r \le 24$ satisfy $rp\ell \equiv -1 \pmod{24}$. Replacing n by $24pn + r$, we obtain

$$a_t\Big(\frac{24p^4\ell n - rp^3\ell + 1}{24}\Big) = a_t(p^4\ell n + b_p) \equiv 0 \pmod{\ell},$$

where $b_p = (rp^3\ell + 1)/24$ is an integer. □

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