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On the existence of unbounded solutions for some rational equations

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We resolve several conjectures regarding the boundedness character of the rational difference equation

$$x_n = \frac{\alpha + \delta x_{n-3}}{A + Bx_{n-1} + Cx_{n-2} + Ex_{n-4}}, \quad n \in \mathbb{N}.$$

We show that whenever parameters are nonnegative, $A < \delta$, and $C, E > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. We also partly resolve a conjecture regarding the boundedness character of the rational difference equation

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N}.$$

We show that whenever $B > 2^5$, unbounded solutions exist for some choice of nonnegative initial conditions.

1. Introduction

Palladino [2009a] studies a trichotomy behavior of the k-th order rational difference equation with nonnegative parameters and nonnegative initial conditions,

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

Palladino established that there is a trichotomy behavior which is dependent on the relation between A and $\sum_{i=1}^{k} \beta_i$. In particular, in this paper, it was established that, under certain conditions, when $A < \sum_{i=1}^{k} \beta_i$ unbounded solutions exist. Here we will broaden that proof of unboundedness and show that when $A < \sum_{i=1}^{k} \beta_i$ unbounded solutions exist under different conditions. In Section 2 we present a

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proof based on [Palladino 2009a, Section 5] which serves to generalize this work. An immediate consequence of this, as discussed later, will be to show that whenever parameters are nonnegative, $A < \delta$, and $C, E > 0$, unbounded solutions exist for some choice of nonnegative initial conditions for the rational difference equation,

$$x_n = \frac{\alpha + \delta x_{n-3}}{A + Bx_{n-1} + Cx_{n-2} + Ex_{n-4}}, \quad n \in \mathbb{N}.$$

This resolves the conjectures regarding boundedness character for equations 609, 611, 617, and 619 presented in [Camouzis and Ladas 2008].

In Section 3, we partially resolve Conjecture 2 in [Palladino 2009a]. We show that the rational difference equation

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N},$$

has unbounded solutions whenever $B > 2^5$. In the process, we resolve the conjecture in [Camouzis and Ladas 2008] regarding the boundedness character of equation 584. The proof here will use similar techniques to those presented in [Lugo and Palladino $\geq$ 2009].

2. Preliminary results

During this section we use the ideas of modulo classes. Let us introduce these ideas in the following remark.

Remark 1. We say that a is congruent to b with modulus c and write $a \equiv b \bmod c$ if $c \mid a - b$. It is well known that given $z \in \mathbb{Z}$, there exists $a \in \{0, \dots, c-1\}$ so that $z \equiv a \bmod c$. We call such a the residue of z with respect to the modulus c, and write $a = z \bmod c$.

Here we introduce a condition which allows us to construct unbounded solutions, namely Condition 1. Before doing so let us first introduce some notation. Let us define the following sets of indices:

$$I_\beta = \{i \in \{1, 2, \dots, k\} \mid \beta_i > 0\} \quad \text{and} \quad I_B = \{j \in \{1, 2, \dots, k\} \mid B_j > 0\}.$$

These sets are used extensively in [Palladino 2009b] when referring to the k-th order rational difference equation. Similarly we shall make extensive use of this notation.

Condition 1. We say that Condition 1 is satisfied if, for some $p \in \mathbb{N}$, $p \mid \gcd I_\beta$. We also must have disjoint sets $B, L \subset \{0, \dots, p-1\}$ with $B \neq \varnothing$ and with the following properties.

(1) For all $b \in B$, $\{(b - j) \bmod p : j \in I_B\} \subset L$.

(2) For all $\ell \in L$, there exists $j \in I_B$ so that $(\ell - j) \bmod p \in B$.

We now present Theorem 1 which makes use of Condition 1. In the remainder of this section we will verify Condition 1 for a number of special cases of the fourth-order rational difference equation, thereby confirming several conjectures in [Camouzis and Ladas 2008].

Theorem 1. *Consider the k-th order rational difference equation*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}. \tag{1}$$

Assume nonnegative parameters and nonnegative initial conditions. Further assume that $A < \sum_{i=1}^{k} \beta_i$ and $\sum_{i=1}^{k} \beta_i > 0$, and that Condition 1 is satisfied for Equation (1). Then unbounded solutions of Equation (1) exist for some initial conditions.

Proof. By assumption, we may choose $p \in \mathbb{N}$ and $B, L \subset \{0, \dots, p-1\}$ so that Condition 1 is satisfied. Choose initial conditions x_{-m} where $m \in \{0, \dots, k-1\}$ so that the following holds. If $(-m \bmod p) \in B$, then

$$x_{-m} > \frac{2\alpha \sum_{j=1}^{k} B_j}{(\min_{j \in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B} B_j}.$$

If $(-m \bmod p) \in L$, then

$$x_{-m} < \frac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j}.$$

Also assume $x_{-m} > 0$ for all $m \in \{0, \dots, k-1\}$.

Under this choice of initial conditions our solution $\{x_n\}$ has the following properties.

(a) $x_n > \dfrac{2\alpha \sum_{j=1}^{k} B_j}{(\min_{j \in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \dfrac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B} B_j}$ whenever $(n \bmod p) \in B$.

(b) $x_n < \dfrac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j}$ whenever $(n \bmod p) \in L$.

(c) $x_n > 0$ for all $n \in \mathbb{N}$.

We prove this using induction on n; our initial conditions provide the base case. Assume that the statement is true for all $n \le N-1$. We show the statement for $n = N$.

This induction proof has three cases. Let us begin by assuming $(N \bmod p) \in B$.

Case (a). Condition 1(1) tells us that in this case $\{(N-j) \bmod p : j \in I_B\} \subset L$. Hence

$$x_{N-j} < \frac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j} \quad \text{for all } j \in I_B.$$

Since $p \mid \gcd(I_\beta)$, $N \bmod p = (N-i) \bmod p$ for all $i \in I_\beta$.Thus for all $i \in I_\beta$,

$$x_{N-i} > \frac{2\alpha \sum_{j=1}^{k} B_j}{(\min_{j\in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \frac{\sum_{i=1}^{k} \beta_i}{\min_{j\in I_B} B_j}.$$

Hence

$$\begin{aligned}
x_N &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{N-i}}{A + \sum_{j=1}^{k} B_j x_{N-j}} \\
&\geq \frac{\sum_{i=1}^{k} \beta_i}{A + (\sum_{j=1}^{k} B_j)\dfrac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j}} \left(\frac{2\alpha \sum_{j=1}^{k} B_j}{(\min\limits_{j\in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \frac{\sum_{i=1}^{k} \beta_i}{\min\limits_{j\in I_B} B_j} \right) \\
&\geq \frac{2\alpha \sum_{j=1}^{k} B_j}{(\min_{j\in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \frac{\sum_{i=1}^{k} \beta_i}{\min_{j\in I_B} B_j}.
\end{aligned}$$

This inequality is obtained by simply replacing the terms in the denominator with their upper bound, and replacing the terms in the numerator with their lower bound. This finishes case (a).

Case (b). We now assume $(N \bmod p) \in L$. Since $p \mid \gcd(I_\beta)$, we have $N \bmod p = (N-i) \bmod p$ for all $i \in I_\beta$. Hence

$$x_{N-i} < \frac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j} \quad \text{for all } i \in I_\beta.$$

Condition 1(2) guarantees that there exists $j \in I_B$ so that

$$x_{N-j} > \frac{2\alpha \sum_{j=1}^{k} B_j}{(\min_{j\in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \frac{\sum_{i=1}^{k} \beta_i}{\min_{j\in I_B} B_j}.$$

Hence

$$x_N = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{N-i}}{A + \sum_{j=1}^{k} B_j x_{N-j}} < \frac{\alpha + (\sum_{i=1}^{k} \beta_i)\dfrac{(\sum_{i=1}^{k} \beta_i) - A}{2\sum_{j=1}^{k} B_j}}{(\min_{j\in I_B} B_j)\left(\dfrac{2\alpha \sum_{j=1}^{k} B_j}{(\min\limits_{j\in I_B} B_j)((\sum_{i=1}^{k} \beta_i) - A)} + \dfrac{\sum_{i=1}^{k} \beta_i}{\min\limits_{j\in I_B} B_j} \right)}$$

$$= \frac{\alpha + (\sum_{i=1}^k \beta_i) \dfrac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j}}{\dfrac{2\alpha \sum_{j=1}^k B_j}{(\sum_{i=1}^k \beta_i) - A} + \sum_{i=1}^k \beta_i}$$

$$= \frac{\dfrac{2\alpha \sum_{j=1}^k B_j}{(\sum_{i=1}^k \beta_i) - A} + \sum_{i=1}^k \beta_i}{\dfrac{2\alpha \sum_{j=1}^k B_j}{(\sum_{i=1}^k \beta_i) - A} + \sum_{i=1}^k \beta_i} \left(\frac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j} \right) = \frac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j}. \qquad (2)$$

This finishes case (b).

Case (c). It is clear that if $x_n > 0$ for $n < N$. Then $x_N > 0$ so case (c) is trivial.

We now use the facts we obtained from our induction to prove that a particular subsequence is unbounded. Take $b \in B$. We now show that $\{x_{mp+b}\}_{m=1}^{\infty}$ diverges to ∞. We explained earlier that

$$x_{mp+b-j} < \frac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j},$$

since $\{(mp + b - j) \bmod p : j \in I_B\} \subset L$. Hence,

$$\begin{aligned} x_{mp+b} &= \frac{\alpha + \sum_{i=1}^k \beta_i x_{mp+b-i}}{A + \sum_{j=1}^k B_j x_{mp+b-j}} > \frac{\sum_{i=1}^k \beta_i x_{mp+b-i}}{A + (\sum_{j=1}^k B_j) \dfrac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j}} \\ &\geq \frac{(\sum_{i=1}^k \beta_i)(\min_{i \in \{1,\dots,\lfloor k/p \rfloor\}} (x_{mp+b-ip}))}{A + (\sum_{j=1}^k B_j) \dfrac{(\sum_{i=1}^k \beta_i) - A}{2\sum_{j=1}^k B_j}} \\ &\geq \frac{2\sum_{i=1}^k \beta_i}{A + \sum_{i=1}^k \beta_i} \min_{i \in \{1,\dots,\lfloor k/p \rfloor\}} (x_{mp+b-ip}), \quad m \geq k. \end{aligned}$$

This is a difference inequality which holds for the subsequence $\{x_{mp+b}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [Palladino 2008]. We set $z_m = x_{mp+b}$ for $m \in \mathbb{N}$. Since we have just shown that $\{z_m\}$ satisfies the difference inequality

$$z_m \geq \frac{2\sum_{i=1}^k \beta_i}{A + \sum_{i=1}^k \beta_i} \min_{i \in \{1,\dots,\lfloor k/p \rfloor\}} (z_{m-i}), \quad m \geq k,$$

we can use the results of [Palladino 2008], particularly Theorem 3, to conclude that for $m \geq k$,

$$\min(z_{m-1}, \ldots, z_{m-\lfloor k/p \rfloor}) \geq \min(y_{\lfloor \frac{m-k}{\lfloor k/p \rfloor} \rfloor}, \ldots, y_{m-k}),$$

where $\{y_m\}_{m=0}^{\infty}$ is a solution of the difference equation

$$y_m = \frac{2\sum_{i=1}^{k} \beta_i}{A + \sum_{i=1}^{k} \beta_i} y_{m-1}, \quad m \in \mathbb{N}, \tag{3}$$

with $y_0 = \min(z_{k-1}, \ldots, z_{k-\lfloor k/p \rfloor})$. Clearly every positive solution diverges to ∞ for the simple difference equation (3), since $A < \sum_{i=1}^{k} \beta_i$. Hence using the inequality we have obtained, $\{z_m\}_{m=1}^{\infty}$ diverges to ∞. Hence with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^{\infty}$, namely $\{x_{mp+b}\}_{m=1}^{\infty}$, which diverges to ∞. Hence our solution $\{x_n\}_{n=1}^{\infty}$ is unbounded. So we have exhibited an unbounded solution whenever $A < \sum_{i=1}^{k} \beta_i$. □

Corollary 1. *Consider the fourth-order order rational difference equation*

$$x_n = \frac{\alpha + \delta x_{n-3}}{A + Bx_{n-1} + Cx_{n-2} + Ex_{n-4}}, \quad n \in \mathbb{N}. \tag{4}$$

Assume nonnegative parameters and nonnegative initial conditions so that the denominator is nonvanishing. Further assume that $\delta, C, E > 0$.

(i) *Whenever $A > \delta$, the unique equilibrium is globally asymptotically stable.*

(ii) *Whenever $A = \delta$ and $\alpha > 0$, the unique equilibrium is globally asymptotically stable.*

(iii) *Whenever $A = \delta$ and $\alpha = 0$, every solution of Equation* (4) *converges to a periodic solution of period* 3.

(iv) *Whenever $A < \delta$, then Equation* (4) *has unbounded solutions for some choice of initial conditions.*

Proof. Cases (i), (ii), and (iii) were shown in [Palladino 2009b].

We now prove case (iv). Let us check Condition 1. Choose $B = \{0\}$ and $L = \{1, 2\}$. Condition 1(1) is satisfied since for all $b \in B$, namely $b = 0$, $\{(0 - j) \bmod 3 : j \in \{2, 4\}\} = \{(0 - j) \bmod 3 : j \in \{1, 2, 4\}\} = \{1, 2\}$. Condition 1(2) is satisfied since for $1 \in L$, there exists $4 \in I_B$ so that $(1 - 4) \bmod 3 = -3 \bmod 3 = 0 \in \{0\}$. Also for $2 \in L$, there exists $2 \in I_B$ so that $(2 - 2) \bmod 3 = 0 \bmod 3 = 0 \in \{0\}$. Furthermore

$$A < \delta = \sum_{i=1}^{k} \beta_i \quad \text{and} \quad \sum_{i=1}^{k} \beta_i = \delta > 0.$$

Thus Theorem 1 applies and so in case (iv) Equation (4) has unbounded solutions for some choice of initial conditions. □

Notice that Corollary 1 resolves conjectures 609, 611, 617, and 619 in [Camouzis and Ladas 2008] regarding boundedness character.

3. The equation $x_n = \dfrac{x_{n-3}}{Bx_{n-1} + x_{n-4}}$

In [Palladino 2009a] it is conjectured that the difference equation

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N},$$

has unbounded solutions whenever $B > 0$. We show that whenever $B > 2^5$ unbounded solutions exist for some choice of nonnegative initial conditions. This does not fully establish the conjecture in [Palladino 2009a]. It does however establish the Conjecture 584 in [Camouzis and Ladas 2008]. We make use of the argument structure presented in Lemma 1 of [Lugo and Palladino ≥ 2009]. Let us repeat this lemma for the sake of the reader.

Lemma 1. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[0, \infty)$. Suppose that there exists $D > 1$ and hypotheses $H_1, \dots, H_k$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ so that the following holds. Whenever x_{n-i} satisfies H_i for all $i \in \{1, \dots, k\}$, then x_{n+p_n-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{n+p_n-1} \geq Dx_{n-1}$. Further assume that for some $N \in \mathbb{N}$, x_{N-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{N-1} > 0$. Then $\{x_n\}_{n=1}^{\infty}$ is unbounded. Particularly $\{x_{z_m-1}\}_{m=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ which diverges to ∞, where $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$.*

Proof. Let $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$. Using induction, we prove that given $m \in \mathbb{N}$ the following holds. $x_{z_m-1} \geq D^m x_{N-1}$ and x_{z_m-i} satisfies H_i for all $i \in \{1, \dots, k\}$. By assumption, x_{N-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{N-1} \geq D^0 x_{N-1}$. This provides the base case. Assume $x_{z_{m-1}-i}$ satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{z_{m-1}-1} \geq D^{m-1} x_{N-1}$. Using our earlier assumption this implies that there exists $p_{z_{m-1}}$ so that $x_{z_{m-1}+p_{z_{m-1}}-i}$ satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{z_{m-1}+p_{z_{m-1}}-1} \geq Dx_{z_{m-1}-1} \geq (D)D^{m-1} x_{N-1} = D^m x_{N-1}$.

So we have shown that $x_{z_m-1} \geq D^m x_{N-1}$ for all $m \in \mathbb{N}$. Hence the subsequence $\{x_{z_m-1}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ clearly diverges to ∞ since $D > 1$. □

Theorem 2. *Consider the fourth order rational difference equation,*

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N}. \tag{5}$$

Suppose $B > 2^5$. Then Equation (5) has unbounded solutions for some initial conditions.

Proof. We choose initial conditions so that

$$x_{-2} > B, \qquad x_{-3} < \tfrac{1}{4},$$

and one of the following holds:

(1) $x_0 < \dfrac{1}{4B}$ and $x_{-1} < \dfrac{1}{B}$;

(2) $\dfrac{1}{4B} \le x_0 \le 2x_{-2}$ and $x_{-1} < \dfrac{1}{B^2 x_{-2}}$;

(3) $x_0 > 2x_{-2}$ and $x_{-1} < \dfrac{1}{B^2 x_{-2}}$.

We show that there exists $D = 2$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \{2, 3, 5\}$ so that the following holds.

Whenever

$$x_{n-3} > B, \qquad x_{n-4} < \tfrac{1}{4},$$

and one of the following holds:

(1) $x_{n-1} < \dfrac{1}{4B}$ and $x_{n-2} < \dfrac{1}{B}$;

(2) $\dfrac{1}{4B} \le x_{n-1} \le 2x_{n-3}$ and $x_{n-2} < \dfrac{1}{B^2 x_{n-3}}$;

(3) $x_{n-1} > 2x_{n-3}$ and $x_{n-2} < \dfrac{1}{B^2 x_{n-3}}$;

then we have

$$x_{n+p_n-3} > Dx_{n-3} > B, \qquad x_{n+p_n-4} < \tfrac{1}{4},$$

and one of the following holds:

(1) $x_{n+p_n-1} < \dfrac{1}{4B}$ and $x_{n+p_n-2} < \dfrac{1}{B}$;

(2) $\dfrac{1}{4B} \le x_{n+p_n-1} \le 2x_{n+p_n-3}$ and $x_{n+p_n-2} < \dfrac{1}{B^2 x_{n+p_n-3}}$;

(3) $x_{n+p_n-1} > 2x_{n+p_n-3}$ and $x_{n+p_n-2} < \dfrac{1}{B^2 x_{n+p_n-3}}$.

First assume

$$x_{n-1} < \frac{1}{4B}, \qquad x_{n-2} < \frac{1}{B}, \qquad x_{n-3} > B, \qquad x_{n-4} < \tfrac{1}{4}.$$

In this case $p_n = 3$. Since $B > 2^5$ we have

$$x_{n+p_n-4} = x_{n-1} < \frac{1}{4B} < \tfrac{1}{4}.$$

Since $x_{n-4} < \frac{1}{4}$ and $x_{n-1} < \frac{1}{4B}$ we have

$$x_{n+p_n-3} = x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}} \ge \frac{x_{n-3}}{2\max\left(Bx_{n-1}, x_{n-4}\right)} > 2x_{n-3} > B.$$

Since $x_{n-2} < \dfrac{1}{B}$,

$$x_{n+p_n-2} = x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-3}} \leq \frac{x_{n-2}}{Bx_n} < \frac{1}{B^2 x_n} < \frac{1}{B^3} < \frac{1}{B}.$$

Hence regardless of the value of x_{n+p_n-1} one of our requirements is satisfied. If $x_{n+p_n-1} < \frac{1}{4B}$ then requirement (1) is satisfied. If $\frac{1}{4B} \leq x_{n+p_n-1} \leq 2x_{n+p_n-3}$ then requirement (2) is satisfied. If $x_{n+p_n-1} > 2x_{n+p_n-3}$ then requirement (3) is satisfied.

Next assume

$$\frac{1}{4B} \leq x_{n-1} \leq 2x_{n-3}, \qquad x_{n-2} < \frac{1}{B^2 x_{n-3}}, \qquad x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4}.$$

In this case $p_n = 5$. Since $B > 2^5$ we have

$$x_{n+p_n-4} = x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-3}} < \frac{x_{n-2}}{x_{n-3}} < \frac{1}{B^2 x_{n-3}^2} < \frac{1}{4}.$$

Since $x_{n-2} < \dfrac{1}{B^2 x_{n-3}}$ and $B > 2^5$ we have

$$\begin{aligned} x_{n+p_n-3} = x_{n+2} &= \frac{x_{n-1}}{Bx_{n+1} + x_{n-2}} \geq \frac{x_{n-1}}{2\max\left(Bx_{n+1}, x_{n-2}\right)} \\ &> \frac{x_{n-1}}{2\max\left(\dfrac{1}{Bx_{n-3}^2}, \dfrac{1}{B^2 x_{n-3}}\right)} \geq \frac{B^2 x_{n-3}}{8B} > 2x_{n-3} > B. \end{aligned}$$

Also notice that

$$\begin{aligned} x_{n+p_n-2} = x_{n+3} &= \frac{x_n}{Bx_{n+2} + x_{n-1}} = \frac{x_{n-3}}{(Bx_{n+2} + x_{n-1})(Bx_{n-1} + x_{n-4})} \\ &< \frac{x_{n-3}}{Bx_{n+2}(Bx_{n-1}+x_{n-4})} < \frac{8x_{n-3}}{B^2 x_{n-3}(Bx_{n-1}+x_{n-4})} < \frac{8}{B^3 x_{n-1}} < \frac{2^5}{B^2} < \frac{1}{B}. \end{aligned}$$

Notice that

$$\begin{aligned} x_{n+p_n-1} = x_{n+4} &= \frac{x_{n+1}}{Bx_{n+3} + x_n} < \frac{1}{(B^2 x_{n-3}^2)(Bx_{n+3} + x_n)} < \frac{1}{B^2 x_{n-3}^2 x_n} \\ &= \frac{Bx_{n-1} + x_{n-4}}{B^2 x_{n-3}^3} < \frac{2Bx_{n-3} + .25}{B^2 x_{n-3}^3} < \frac{3}{Bx_{n-3}^2} < \frac{1}{4B}. \end{aligned}$$

Hence requirement (1) is satisfied in this case. Finally assume

$$x_{n-1} > 2x_{n-3}, \qquad x_{n-2} < \frac{1}{B^2 x_{n-3}}, \qquad x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4}.$$

In this case $p_n = 2$. Immediately we have

$$x_{n+p_n-4} = x_{n-2} < \frac{1}{B^2 x_{n-3}} < \frac{1}{4}.$$

Also by assumption,

$$x_{n+p_n-3} = x_{n-1} > 2x_{n-3} > B.$$

Further since $x_{n-1} > 2x_{n-3}$,

$$x_{n+p_n-2} = x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}} < \frac{x_{n-3}}{Bx_{n-1}} < \frac{1}{2B} < \frac{1}{B}.$$

Furthermore

$$x_{n+p_n-1} = x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-3}} < \frac{x_{n-2}}{x_{n-3}} < \frac{1}{B^2 x_{n-3}^2} < \frac{1}{4B}.$$

Hence requirement (1) is satisfied in this case, so after an application of Lemma 1 the proof is complete. □

4. Conclusion

As noted in the introduction, Theorem 2 partly resolves Conjecture 2 in [Palladino 2009a]; the latter, however, is only part of a larger conjecture, namely Conjecture 1 in the same reference. For convenience we restate this conjecture.

Conjecture 1. Consider the k-th order rational difference equation

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}. \tag{6}$$

Assume nonnegative parameters and nonnegative initial conditions so that the denominator is nonvanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$ and that there does not exist $j \in I_B$ such that $\gcd(I_\beta) \mid j$. Then unbounded solutions of Equation (6) exist for some initial conditions.

It would be interesting to study this conjecture further utilizing techniques similar to that used in Theorem 2.

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