INVOLVE 2:4(2009)

Contributions to Seymour's second neighborhood conjecture

James Brantner, Greg Brockman, Bill Kay and Emma Snively

(Communicated by Vadim Ponomarenko)

Let D be a simple digraph without loops or digons. For any $v \in V(D)$ let $N_1(v)$ be the set of all nodes at out-distance 1 from v and let $N_2(v)$ be the set of all nodes at out-distance 2. We show that if the underlying graph is triangle-free, there must exist some $v \in V(D)$ such that $|N_1(v)| \leq |N_2(v)|$. We provide several properties a "minimal" graph which does not contain such a node must have. Moreover, we show that if one such graph exists, then there exist infinitely many.

1. Introduction

In this article, we consider only simple nonempty finite digraphs (those containing no loops or multiple edges and having a nonempty vertex set), unless stated otherwise. We also require that our digraphs contain no digons, that is, if D is a digraph then $(u, v) \in E(D) \Rightarrow (v, u) \notin E(D)$. If i is a positive integer, we denote the ith neighborhood of a vertex u in D by $N_{i,D}(u) = \{v \in V(D) | \operatorname{dist}_D(u, v) = i\}$, where $\operatorname{dist}_D(u, v)$ is the length of the shortest directed path from u to v in D (if there is no directed path from u to v, we set $\operatorname{dist}_D(u, v) = \infty$). If D is clear from context, we simply write $N_i(u)$ and $\operatorname{dist}(u, v)$. We will also consider the ith in-neighborhood of a node $N_{-i}(u) = \{v \in V(D) | \operatorname{dist}(v, u) = i\}$. In addition, if $V' \subseteq V(D)$, we let $D[V']$ be the subgraph of D induced by V'.

Graph theorists will be familiar with the following conjecture by Seymour.

Conjecture 1.1 (Seymour's second neighborhood conjecture). *Let D be a directed graph. Then there exists a vertex $v_0 \in V(D)$ such that $|N_1(v_0)| \leq |N_2(v_0)|$.*

Dean and Latka [1995] conjectured this to be true when D is a tournament. Dean's conjecture was subsequently proven by Fisher [1996]. Further, Kaneko and Locke [2001] showed Conjecture 1.1 to be true if the minimum out-degree of vertices in D is less than 7, while Cohn, Wright and Godbole [Cohn et al. 2009]

MSC2000: 05C20.

Keywords: graph theory, second neigbhorhood conjecture, graph properties, open problems in graph theory.

showed that it holds for random graphs almost always. Finally, Fidler and Yuster [2007] proved that Conjecture 1.1 holds for graphs with minimum out-degree $|V(D)|-2$, tournaments minus a star, and tournaments minus a subtournament. While over the years there have been several attempts at a proof of Conjecture 1.1, none of these has yet been successful.

For completeness, we introduce the related Caccetta–Häggkvist conjecture.

Conjecture 1.2 (Caccetta–Häggkvist). *If D is a directed graph with minimum out-degree at least $|V(D)|/k$, then D has a directed cycle of length at most k.*

Conjecture 1.1 would imply the $k = 3$ case of Conjecture 1.2. Much work has been done on Conjecture 1.2, including an entire workshop sponsored by the American Institute of Mathematics and the National Science Foundation, but still both Conjectures 1.1 and 1.2 remain wide open.

In this paper, we will show that Conjecture 1.1 holds for digraphs where the underlying graph is triangle-free. We then take a different tack and provide conditions that must be satisfied by any appropriately-defined minimal counterexample to Seymour's second neighborhood conjecture.

2. Definitions

Definition 2.1. Suppose that D is digraph and $u \in V(D)$. We say that u is *satisfactory* if $|N_1(u)| \le |N_2(u)|$. Also, u is a *sink* if $|N_1(u)| = 0$. Note that a sink is trivially satisfactory.

Definition 2.2. Let $\mathscr{A}$ be the set of *Seymour counterexamples*, i.e., simple directed graphs D with no satisfactory vertices (in other words, counterexamples to Seymour's second neighborhood conjecture). Let

$$\mathscr{A}' = \{D \mid |E(D)| = \min_{H \in \mathscr{A}} |E(H)|\}$$

be the set of graphs in $\mathscr{A}$ with the fewest number of edges. Finally, let $\mathscr{A}'' = \{D \mid |V(D)| = \min_{H \in \mathscr{A}'} |V(H)|\}$ be the set of graphs in $\mathscr{A}'$ with the fewest number of vertices. We will refer to any element of $\mathscr{A}''$ as a *minimal counterexample*. Note that $\mathscr{A}''$ is empty if and only if Conjecture 1.1 is true.

Definition 2.3. Define $A_{s,G}(u) = |N_1(u)| - |N_2(u)|$ to be the *antisatisfaction* of u. As usual, if G is clear from context, we simply write $A_s(v)$. Notice that u is satisfactory if and only if $A_s(u) \le 0$.

Definition 2.4. Again let D be a digraph. If $(u, v) \in E(D)$, we say that edge (u, v) is the *base* of a transitive triangle if u and v share a common first neighbor; that is, $|N_1(u) \cap N_1(v)| \ge 1$ (see Figure 1).

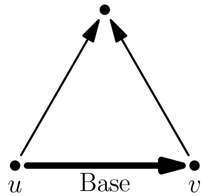

Figure 1. Demonstration of an edge that is the base of a transitive triangle.

If, for distinct $t, u, v, w \in V(D)$, we have (t, u), (u, w), (t, v), $(v, w) \in E$ then we call $\{(t, u), (u, w), (t, v), (v, w)\}$ a *Seymour diamond*. We say the edges (t, u), (t, v) are the *bases* of the Seymour diamond (see Figure 2).

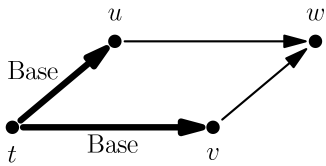

Figure 2. Demonstration of the bases of a Seymour diamond.

3. Directed cycles and underlying girth

In this section we show that certain classes of graphs satisfy Seymour's second neighborhood conjecture. The following theorem shows that directed cycles are necessary for a graph to be a Seymour counterexample.

Observation 3.1. *If a digraph contains no directed cycles, then it must have a satisfactory vertex.*

Proof. Let D be a directed graph. Suppose that D contains no satisfactory vertices. Then D has no sink, as noted in Definition 2.1. Thus if $u \in V(D)$, we have $|N_1(u)| \geq 1$. Now pick an arbitrary vertex $v_0 \in V(D)$, and consider the infinite sequence $\{v_i\}_{i=0}^{\infty}$ defined recursively by $v_{i+1} \in N_1(v_i)$ for $i \geq 0$. Since V is finite, we then have that there exist some $r \neq s$ such that $v_r = v_s$. Then we note that the sequence of edges $(v_r, v_{r+1}), (v_{r+1}, v_{r+2}), \ldots, (v_{s-1}, v_s = v_r)$ defines a dicycle in D, thus completing our proof. $\square$

Recall that the girth of an undirected graph is the length of its shortest cycle. We show that any Seymour counterexample must have underlying girth of exactly 3:

Theorem 3.2. *Let G be a simple graph with girth strictly larger than* 3. *Then any orientation of G will result in a directed graph with a satisfactory vertex.*

Proof. Let D be any orientation of G. Clearly there must exist some vertex v_0 with minimal out-degree. If $|N_1(v_0)| = 0$, then v_0 is a sink and hence a satisfactory vertex. Otherwise, let $v_1 \in N_1(v_0)$. By construction, we have that $|N_1(v_1)| \geq$

$|N_1(v_0)|$. Furthermore, the underlying graph has girth at least 4, so $|N_1(v_0) \cap N_1(v_1)| = 0$. Thus, $|N_2(v_0)| \geq |N_1(v_1)| \geq |N_1(v_0)|$, and by definition v_0 is a satisfactory vertex. □

Remark. A similar argument will show that any digraph D which has no transitive triangle as a subgraph must have a satisfactory vertex. We will prove a stronger version of this result in the following section.

4. Properties of counterexamples to Seymour's second neighborhood conjecture

To this point, we have been showing that classes of graphs satisfy Conjecture 1.1. In this section we reverse course and explore necessary properties of the minimal counterexample graphs of $\mathcal{A}''$ from Definition 2.2.

Theorem 4.1. *Suppose $\mathcal{M} \in \mathcal{A}''$.*

(i) *$\mathcal{M}$ is strongly connected.*

(ii) *For each $u \in V(\mathcal{M})$, $A_s(u) \in \{1, 2\}$.*

(iii) *For every edge $e = (u, v) \in E(\mathcal{M})$, there exists a path of length 1 or 2 avoiding e from u to all but at most 1 element of $\{v\} \cup N_1(v)$.*

(iv) *Every edge of $\mathcal{M}$ is the base of either a transitive triangle or a Seymour diamond.*

(v) *For any node $u \in V(\mathcal{M})$, there exists a node $v \in N_{-1}(u)$ such that $A_s(v) = 1$.*

(vi) *There exists a cycle $C = (v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1)$ in $\mathcal{M}$ such that for $1 \leq i \leq k$, we have that $A_s(v_i) = 1$.*

Proof.
(i) Let D be a digraph with $u \in V(D)$. We define

$$W_D(u) = \{v \,|\, \mathrm{dist}(u, v) \neq \infty\}$$

to be the *reachable vertices from u* with respect to D. If D is clear from context, we simply write $W(u)$. Pick an arbitrary node u from the vertex set of $\mathcal{M}$. Now consider $\mathcal{M}' = \mathcal{M}[W(u)]$. We now pick an arbitrary node $v \in W(u)$. Clearly, $N_{1,\mathcal{M}}(v) \subseteq W(u)$ and $N_{2,\mathcal{M}}(v) \subseteq W(u)$. But this implies that

$$A_{s,\mathcal{M}'} = |N_{1,\mathcal{M}'}(v)| - |N_{2,\mathcal{M}'}(v)| = |N_{1,\mathcal{M}}(v)| - |N_{2,\mathcal{M}}(v)| = A_{s,\mathcal{M}},$$

and hence v is satisfactory in $\mathcal{M}'$ if and only if v is satisfactory in $\mathcal{M}$. Since by construction $\mathcal{M}$ contains no satisfactory vertices, v cannot be satisfactory in $\mathcal{M}'$. Thus $\mathcal{M}'$ contains no satisfactory vertices. But $\mathcal{M}'$ is a subgraph of $\mathcal{M}$, and so by minimality of $\mathcal{M}$ we have that $\mathcal{M} = \mathcal{M}'$. Since u was arbitrary, we are done.

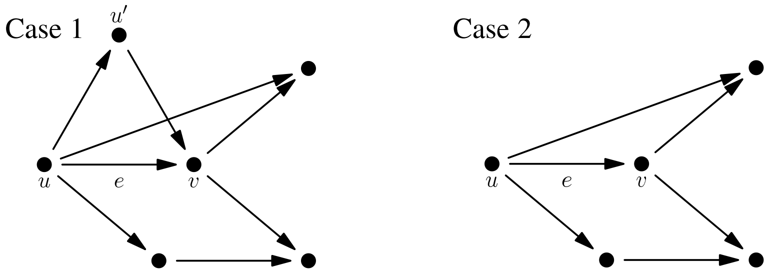

Figure 3. Two possible cases resulting from deleting an edge from $\mathcal{M}$. In case 1, there is a length 2 path from u to v, while in case 2 no such path exists. Note that it is possible that deleting e will increase the size of u's second neighborhood, as shown in case 1.

(ii) Pick an arbitrary edge $e = (u, v) \in E(\mathcal{M})$. Consider the digraph M obtained by deleting e from $\mathcal{M}$. Since M has fewer edges than $\mathcal{M}$, we have that M contains a satisfactory vertex. For each vertex $w \in V(\mathcal{M})$, we note that $|N_{1,M}(w)| = |N_{1,\mathcal{M}}(w)|$ unless $w = u$, in which case $|N_{1,M}(u)| = |N_{1,\mathcal{M}}(u)| - 1$. Furthermore, we have that $|N_{2,M}(w)| \leq |N_{2,\mathcal{M}}(w)|$, except if $w = u$, in which case we have that $|N_{2,M}(u)| \leq |N_{2,\mathcal{M}}(u)| + 1$. (See Figure 3.)

Thus, we obtain that in M for $w \neq u \in V(M)$, $A_{s,M}(w) \geq A_{s,\mathcal{M}}(w)$, and hence all vertices in M besides u are not satisfactory. Thus by process of elimination we have that u is satisfactory in M. Thus

$$0 \geq A_{s,M}(u) = |N_{1,M}(u)| - |N_{2,M}(u)| \geq (|N_{1,\mathcal{M}}(u)| - 1) - (|N_{2,\mathcal{M}}(u)| + 1),$$

and hence we have that $0 < A_{s,\mathcal{M}}(u) = |N_{1,\mathcal{M}}(u)| - |N_{2,\mathcal{M}}(u)| \leq 2$. Result (ii) follows immediately.

(iii) Pick an arbitrary edge $e = (u, v) \in E(\mathcal{M})$. Consider the graph M obtained by deleting e from $\mathcal{M}$. We see that $|N_{2,M}(u)| \geq |N_{2,\mathcal{M}}(u)|$, since otherwise $A_{s,M}(u) \leq 0$ and u is not satisfactory in M, a contradiction. Consider now $X = N_{2,M}(u) \setminus N_{2,\mathcal{M}}(u)$. We note that $X \subseteq \{v\}$, since v is the only vertex that could have been added to u's second neighborhood in M (case 1 in Figure 3). Thus we see that

$$|N_{2,\mathcal{M}}(u) \setminus N_{2,M}(u)| \leq 1,$$

with equality only if $v \in N_{2,M}(u)$.

Note that $N_{1,\mathcal{M}}(v) \subseteq N_{1,\mathcal{M}}(u) \cup N_{2,\mathcal{M}}(u)$. Let $Y = N_{1,\mathcal{M}}(u) \cap N_{1,\mathcal{M}}(v)$ and $Z = N_{2,\mathcal{M}}(u) \cap N_{1,\mathcal{M}}(v)$. For $y \in Y$, we clearly have a path of length 1 from u to y avoiding e (namely the edge (u, y)). If $|N_{2,\mathcal{M}}(u) \setminus N_{2,M}(u)| = 0$, then for $z \in Z$, we have a path of length 2 from u to z in M, and considering this path in $\mathcal{M}$ yields a path from u to z avoiding e. And finally, if $|N_{2,\mathcal{M}}(u) \setminus N_{2,M}(u)| = 1$, then we have a path of length 2 from u to z in M for all but 1 vertex in Z, and as before we have a corresponding path from u to z avoiding e. But in this case, there is a path of length 2 from u to v avoiding e, and hence we have obtained the desired result.

(iv) Paths of length 1 from u to $v' \in N_1(v)$ yield transitive triangles with e as the base, and paths of length 2 from u to $v' \in \{v\} \cup N_1(v)$ yield Seymour diamonds with e as one of the bases. By part 3, at least one of these structures exists, and hence we are done.

(v) In $\mathcal{M}$, pick an arbitrary vertex u. Delete this vertex and label the resulting graph M. Then in a similar manner to before, one of the nodes in $N_{-1,\mathcal{M}}(u)$ must be satisfactory in M by vertex minimality of $\mathcal{M}$. Label this node t. Since $|N_{1,M}(t)| = |N_{1,\mathcal{M}}(t)| - 1$, t is satisfactory, and $|N_{2,M}(t)| \subseteq |N_{2,\mathcal{M}}(t)|$ (note that in contrast to deleting an edge, deleting a vertex does not allow any vertices to add nodes to their second neighborhoods), we see that we must have $|N_{2,M}(t)| = |N_{2,\mathcal{M}}(t)|$. It is then necessary that $A_{s,\mathcal{M}}(t) = 1$. Since u was arbitrary, we have obtained the desired result.

(vi) We apply the same technique as we used Observation 3.1. We present a brief sketch of our proof: by part (v), each node in $\mathcal{M}$ has an in-neighbor having antisatisfaction of exactly 1. If we begin at an arbitrary vertex and choose one of its in-neighbors having antisatisfaction of exactly 1, do the same for the resulting vertex, and iterate this process, at some point we must arrive back at a vertex we have already visited. Thus we have constructed a dicycle of nodes having antisatisfaction exactly 1. $\square$

We now extend some of our results from the previous theorem. In particular, we turn to a count of the number of transitive triangles and Seymour diamonds that certain edges must belong to.

Theorem 4.2. *If $\mathcal{M} \in A''$, suppose that $e = (u, v) \in E(\mathcal{M})$ and $|N_1(u)| \leq |N_1(v)|$. Then e must be the base of at least $|N_1(v)| - |N_1(u)| + 1$ transitive triangles and the base of at least $|N_1(v)| - |N_1(u)| + 1$ Seymour diamonds.*

Proof. Since $N_1(v) \setminus (N_1(u) \cap N_1(v)) \subseteq N_2(u)$, we have

$$|N_2(u)| \geq |N_1(v)| - |N_1(u) \cap N_1(v)|.$$

But since $\mathcal{M}$ contains no satisfactory vertices, we have that $|N_2(u)| < |N_1(u)|$. By transitivity, we obtain $|N_1(v)| - |N_1(v) \cap N_1(u)| < |N_1(u)|$. It then follows that $|N_1(v)| - |N_1(u)| < |N_1(v) \cap N_1(u)|$, but $|N_1(v) \cap N_1(u)|$ is the number of transitive triangles having base e, so we have proved the first half of the theorem.

To prove the second half of the theorem, we consider the following cases.

Case 1. Suppose there exists a vertex u' such that $(u, u'), (u', v) \in E(\mathcal{M})$. By part (iii) of Theorem 4.1, we know that u must be connected to at least $|N_1(v)| - 1$ elements of $N_1(v)$ via a path of length 1 or 2 avoiding e. But we see that u is adjacent to at most $|N_1(u) - 2|$ nodes in $N_1(v)$. Subtracting, we see that u is connected via a path of length 2 avoiding e to at least $|N_1(v)| - 1 - (|N_1(u)| - 2) = (|N_1(v)| - |N_1(u)|) + 1$ nodes in $N_1(v)$; each of which yields a Seymour diamond of which e is the base, which is the desired result.

Case 2. Suppose there is no such u'. Then again applying part (iii) of Theorem 4.1, it must be that there exists a path of length 1 or 2 avoiding e to each node in $N_1(v)$. But u is adjacent to at most $|N_1(u)| - 1$ of these nodes, and as before we count that there is a path of length 2 avoiding e from u to at least $|N_1(v)| - (|N_1(u)| - 1) = |N_1(v)| - |N_1(u)| + 1$ nodes in $|N_1(v)|$. Since each of these paths yields a Seymour diamond with e as the base, we are done. □

Finally, we show that there is not some finite nonzero number of counterexamples to the conjecture. That is, either the conjecture is true, or there are an infinite number of (non-isomorphic) graphs that violate Conjecture 1.1. We provide a constructive proof below.

Theorem 4.3. *If Seymour's second neighborhood conjecture is false, there are infinitely many non-isomorphic strongly-connected counterexamples to Seymour's second neighborhood conjecture.*

Proof. Suppose that Seymour's second neighborhood conjecture is false, and suppose that digraph D is any strongly-connected counterexample to Seymour's second neighborhood conjecture. (By Theorem 4.1(i) , such a D must exist.) Let H be any digraph satisfying the condition $A_s(v) \geq 0$ for all $v \in V(H)$; that is, all of H's vertices have nonnegative antisatisfaction. Note that any dicycle satisfies the relevant condition, and hence there exists a choice of H on any number n of vertices, $n \geq 3$.

We now construct a graph D' on $|V(D)| \cdot |V(H)|$ vertices such that D' is a counterexample to Seymour's second neighborhood conjecture, thus proving our theorem. We define our graph D' as follows:

(i) $V(D') = V(D) \times V(H)$.

(ii) If $u = (d_1, h_1), v = (d_2, h_2) \in V(D')$, then $(u, v) \in E(D')$ if and only if either
   - (a) $d_1 = d_2$ and $(h_1, h_2) \in E(H)$, or
   - (b) $d_1 \neq d_2$ and $(d_1, d_2) \in E(D)$.

For any vertex $v = (d, h) \in V(D')$, we calculate that

$$|N_{1,D'}(v)| = |N_{1,H}(h)| + |V(H)| \cdot |N_{1,D}(d)|,$$

since the neighborhood is equivalent to the set of vertices reachable by stepping in H, holding d constant, or stepping in D and allowing h to be arbitrary.

Similarly, we have

$$|N_{2,D'}(v)| = |N_{2,H}(h)| + |V(H)| \cdot |N_{2,D}(d)|,$$

since we may consider walking two steps in H or two steps in D. Note that taking one step in H and one step in D or one step in D and then one in H will result in reaching a vertex that is in $N_{1,D'}(v)$, and hence this is not an overcount.

We then calculate that

$$\begin{aligned} A_{s,D'}(v) &= |N_{1,D'}(v)| - |N_{2,D'}(v)| \\ &= (|N_{1,H}(h)| - |N_{2,H}(h)|) + |V(H)|(|N_{1,D}(d)| - |N_{2,D}(d)|). \end{aligned}$$

But by our choice of H, we have $|N_{1,H}(h)| - |N_{2,H}(h)| \geq 0$, and by our choice of D we have $|N_{1,D}(d)| - |N_{2,D}(d)| > 0$. Hence we obtain $A_{s,D'}(v) > 0$, thus implying that every vertex in D' has positive antisatisfaction.

Furthermore, D' is strongly connected: fix $(d_1, h_1), (d_2, h_2) \in V(D')$. If $d_1 \neq d_2$, let $d_1, \delta_1, \ldots, \delta_i, d_2$ define a directed path in D from d_1 to d_2. Then

$$(d_1, h_1), (\delta_1, h_2), \ldots, (\delta_i, h_2), (d_2, h_2)$$

defines a directed path in D' from (d_1, h_1) to (d_2, h_2). If $d_1 = d_2$, let $d_3 \in N_{1,D}(d_1)$; we know that (d_1, h_1), (d_3, h_2) are adjacent in D', and since $d_2 \neq d_3$ there is a path from (d_3, h_2) to (d_2, h_2) in D', the existence of a path from (d_1, h_1) to (d_2, h_2) follows.

By definition, we then have that D' is a strongly-connected counterexample to Seymour's second neighborhood conjecture. □

5. Conclusions and future directions

In total, this paper has been an exploration of Seymour's second neighborhood conjecture. We have neither proven nor disproved the conjecture, but instead determined some classes of graphs that do satisfy the conjecture; we have also described some properties of a family of minimal counterexamples. Moreover, we have shown that the existence of one counterexample graph implies the existence of infinitely many such graphs. Our work is intended as a stepping stone for further analysis of Conjecture 1.1, which we hope will ultimately lead to its resolution.

Acknowledgment

This work was done at the East Tennessee State University REU, supported by NSF grant 0552730, under the supervision of Dr. Anant Godbole.

References

[Cohn et al. 2009] Z. Cohn, E. Wright, and A. Godbole, "Probabilistic versions of Seymour's distance two conjecture", preprint, 2009, Available at www.etsu.edu/math/godbole/seymour.pdf.

[Dean and Latka 1995] N. Dean and B. J. Latka, "Squaring the tournament—an open problem", pp. 73–80 in *Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)*, vol. 109, 1995. MR 96h:05085 Zbl 0904.05034

[Fidler and Yuster 2007] D. Fidler and R. Yuster, "Remarks on the second neighborhood problem", *J. Graph Theory* **55**:3 (2007), 208–220. MR 2009b:05122 Zbl 1122.05040

[Fisher 1996] D. C. Fisher, "Squaring a tournament: a proof of Dean's conjecture", *J. Graph Theory* **23**:1 (1996), 43–48. MR 97i:05051 Zbl 0857.05042

[Kaneko and Locke 2001] Y. Kaneko and S. C. Locke, "The minimum degree approach for Paul Seymour's distance 2 conjecture", pp. 201–206 in *Proceedings of the Thirty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, LA, 2001)*, vol. 148, 2001. MR 2002k:05078 Zbl 0996.05042

Received: 2008-08-18 Revised: 2009-08-03 Accepted: 2009-08-13

jbrantne@brandeis.edu *Department of Philosophy, Brandeis University, 415 South Street, Waltham, MA 02453, United States*

brockman@hcs.harvard.edu *Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, United States*

kayw@mailbox.sc.edu *Department of Mathematics, University of South Carolina, Leconte 411, Columbia, SC 29208, United States*

snivelee@rose-hulman.edu *Department of Mathematics, Rose–Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN 47803, United States*