Some numerical radius inequalities for Hilbert space operators

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We present several numerical radius inequalities for Hilbert space operators. More precisely, we prove that if $A, B, C, D \in B(H)$ and $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ then $\max(w(A), w(D)) \le \frac{1}{2}(\|T\| + \|T^2\|^{1/2})$ and $\max\big((w(BC))^{1/2}, (w(CB))^{1/2}\big) \le \frac{1}{2}(\|T\| + \|T^2\|^{1/2})$. We also show that if $A \in B(H)$ is positive, then

$$w(AX - XA) \le \tfrac{1}{2}\|A\|(\|X\| + \|X^2\|^{1/2}).$$

1. Introduction and preliminaries

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $A \in B(H)$ let

$$\begin{aligned} w(A) &= \sup\{|\langle x, Ax\rangle| : \|x\| = 1\}, \\ \|A\| &= \sup\{\|Ax\| : \|x\| = 1\}, \\ |A| &= (A^*A)^{1/2} \end{aligned}$$

denote the numerical radius, the usual operator norm of A and the absolute value of A. It is well known that $w(\cdot)$ is a norm on $B(H)$, and that for all $A \in B(H)$,

$$\tfrac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1-1}$$

Here are some basic properties of the numerical radius:

$$w(|A|) = \|A\|, \tag{1-2}$$

$$w(A^*A) = w(AA^*), \tag{1-3}$$

$$w(UAU^*) = w(A), \tag{1-4}$$

$$w(A_1 \oplus A_2 \oplus \cdots \oplus A_n) = \max\{w(A_i) : i = 1, 2, \ldots, n\}, \tag{1-5}$$

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for all operators $A, A_1, A_2, \dots, A_n \in B(H)$ and all unitary operators $U \in B(H)$.

Suppose $H = M_1 \oplus M_2$ and $A \in B(H)$. Then we can write A as a block matrix

$$A = \begin{bmatrix} I_1^* A I_1 & I_1^* A I_2 \\ I_2^* A I_1 & I_2^* A I_2 \end{bmatrix}, \tag{1-6}$$

where $I_i \in B(M_i, H)$ such that $I_i(x) = x$ $(i = 1, 2)$. If A and B are operators in $B(H)$ we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$, regarded as an operator on $H \oplus H$. Thus

$$\|A \oplus B\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max(\|A\|, \|B\|). \tag{1-7}$$

Suppose $\mathscr{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where $A_i \in B(H)$ and $x_1, x_2, \dots, x_n \in H$. That is,

$$\mathscr{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix},$$

which we also write $\mathscr{A} = \operatorname{diag}(A_1, \dots, A_n)$. Then

$$\left\langle [x_1, \dots, x_n]^T, \mathscr{A}[x_1, \dots, x_n]^T \right\rangle = \sum_{i=1}^{n} \langle x_i, A_i(x_i) \rangle,$$

$$w(\mathscr{A}) = \sup\left\{ \left| \left\langle [x_1, \dots, x_n]^T, \mathscr{A}[x_1, \dots, x_n]^T \right\rangle \right| : \sum_{i=1}^{n} \|x_i\|^2 = 1 \right\}.$$

For additional properties of the numerical radius, see [Bhatia 1997; Halmos 1982] and references therein.

Consider $A = [A_{ij}]$, where $A_{ij} \in B(H)$ and $i, j = 1, 2, \dots, n$. We define $C(A) = A_{11} \oplus A_{22} \oplus \cdots \oplus A_{nn}$, called the *n-pinching* of A. We set $z = e^{2\pi i/n}$ and $U := \operatorname{diag}(I, zI, \dots, z^{n-1} I)$, where I is the identity operator in $B(H)$. Using the identity $\sum_{k=0}^{n-1} z^k = 0$, one can see that $C(A) = (1/n) \sum_{k=0}^{n-1} U^{*k} A U^k$ (see also [Bhatia 2000; 1997]).

It is shown in [Kittaneh 2005] that if $A, B, C, D, S, T \in B(H)$, then

$$w(ATB + CSD) \\ \le \tfrac{1}{2}\left(\|A|T^*|^{2(1-\alpha)} A^* + B^* |T|^{2\alpha} B + C|S^*|^{2(1-\alpha)} C^* + D^* |S|^{2\alpha} D\| \right),$$

for all α with $0 \le \alpha \le 1$. In particular, if $A, U, P \in B(H)$ such that U is unitary and P is projection, we have

$$w(AU \pm U^*A) \le \tfrac{1}{2}\big\| |A| + |A^*| + U^*(|A| + |A^*|)U \big\| \le \|A\| + \|A^2\|^{1/2}, \quad (1\text{-}8)$$

$$w(AP - PA) \le \tfrac{1}{2}\big\| |A| + |A^*| + P(|A| + |A^*|)P \big\| \le \|A\| + \|A^2\|^{1/2}, \quad (1\text{-}9)$$

$$w(A) \le \tfrac{1}{2}\big(\|A\| + \|A^2\|^{1/2}\big). \quad (1\text{-}10)$$

The last inequality refines the second inequality in (1-1); see also [Kittaneh 2003]. In [Kittaneh 2007; Bhatia and Kittaneh 2008] it is shown that if $A, B, X \in B(H)$ such that A and B are positive, then

$$|||AX - XB||| \le \max(\|A\|, \|B\|)|||X|||,$$

where $|||\cdot|||$ is a unitarily invariant norm.

In particular,

$$\|AX - XA\| \le \|A\|\|X\|. \quad (1\text{-}11)$$

In this paper we establish some inequalities sharper than inequalities (1-9) and (1-11) to the numerical radius and we give a new proof of inequality (1-10). Some applications of these inequalities are considered as well.

2. Main results

In [Bhatia 1997] it is shown that

$$\frac{1}{2}\left|\left|\left|\begin{bmatrix} A+B & 0 \\ 0 & A+B \end{bmatrix}\right|\right|\right| \le \left|\left|\left|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right|\right|\right| \le \left|\left|\left|\begin{bmatrix} |A|+|B| & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right|,$$

where $|||\cdot|||$ is a unitarily invariant norm. In this paper we extend this inequality to the numerical radius. We begin by establishing an interesting property of the numerical radius.

Lemma 2.1. *Let $A \in B(H)$. Then*

$$w(C(A)) \le w(A). \quad (2\text{-}1)$$

Proof. Since $C(A) = \frac{1}{n}\sum_{k=0}^{n-1} U^{*k}AU^k$, we have

$$w(C(A)) \le \frac{1}{n}\sum_{k=0}^{n-1} w(U^{*k}AU^k) = \frac{1}{n}\sum_{k=0}^{n-1} w(A) = w(A),$$

where the inequality follows from property (1-4). □

Theorem 2.2. *Let $A_1, A_2, \dots, A_n \in B(H)$. Then*

$$\frac{1}{n} w\Big(\operatorname{diag}\Big(\sum_{i=1}^n A_i, \dots, \sum_{i=1}^n A_i\Big)\Big) \le w(\mathscr{A}) \le w\Big(\operatorname{diag}\Big(\sum_{i=1}^n |A_i|, 0, \dots, 0\Big)\Big).$$

Proof. For the first inequality, we have, using (1-5),

$$\begin{aligned} w\Big(\operatorname{diag}\Big(\sum_{i=1}^n A_i,\ldots,\sum_{i=1}^n A_i\Big)\Big) &= w\Big(\sum_{i=1}^n A_i\Big) \le \sum_{i=1}^n w(A_i) \\ &\le n\max\{w(A_i) : i=1,2,\ldots,n\} = nw(\mathscr{A}). \end{aligned}$$

For the second inequality first we suppose $A_1, A_2, \ldots, A_n$ to be positive, so

$$\begin{aligned} w\left(\begin{bmatrix} \sum_{i=1}^n A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} A_1^{1/2} & A_2^{1/2} & \cdots & A_n^{1/2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} A_1^{1/2} & 0 & \cdots & 0 \\ A_2^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2} & 0 & \cdots & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} A_1^{1/2} & 0 & \cdots & 0 \\ A_2^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2} & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} A_1^{1/2} & A_2^{1/2} & \cdots & A_n^{1/2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\ A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n \end{bmatrix}\right), \end{aligned}$$

where the second equality follows from (1-3). Using the inequality (2-1), we get

$$\begin{aligned} w\left(\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}\right) &\le w\left(\begin{bmatrix} A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\ A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n \end{bmatrix}\right) \\ &= w\Big(\operatorname{diag}\Big(\sum_{i=1}^n A_i, 0, \ldots, 0\Big)\Big), \end{aligned}$$

Now let $A_1, A_2, \ldots, A_n$ be arbitrary. Then

$$w\left(\begin{bmatrix} |A_1| & 0 & \cdots & 0 \\ 0 & |A_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A_n| \end{bmatrix}\right) \le w\Big(\operatorname{diag}\Big(\sum_{i=1}^n |A_i|, 0, \ldots, 0\Big)\Big).$$

Since

$$w\left(\begin{bmatrix} |A_1| & 0 & \cdots & 0 \\ 0 & |A_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A_n| \end{bmatrix}\right) = w\left(|\mathscr{A}|\right) \geq w\left(\mathscr{A}\right),$$

we have $w(\mathscr{A}) \leq w\big(\operatorname{diag}\big(\sum_{i=1}^{n} |A_i|, 0, \dots, 0\big)\big)$. □

Corollary 2.3. *Let $A \in B(H)$. Then $\frac{1}{2} w((A + A^*) \oplus (A + A^*)) \leq w(A \oplus A^*)$.*

Kittaneh [2006] showed that if $A, B, C, D \in B(H)$ and if $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then

$$\max(r(A), r(D)) \leq \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}), \quad (r(BC))^{1/2} \leq \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}).$$

We show similar inequalities for the numerical radius. To achieve this, we need the following lemma [Kittaneh 2005].

Lemma 2.4. *If $A, B \in B(H)$ and $AB = BA$, then $w(AB) \leq 2w(A)w(B)$.*

Theorem 2.5. *If $A, B, C, D \in B(H)$ and $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then*

$$\max(w(A), w(D)) \leq \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}), \tag{2-2}$$

and

$$\max((w(BC))^{1/2}, (w(CB))^{1/2}) \leq \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}). \tag{2-3}$$

Proof.

By (1-5), we have $\max(w(A), w(D)) = w\big(\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]\big)$. Since D is arbitrary,

$$\max(w(A), w(D)) = w\left(\begin{bmatrix} A & 0 \\ 0 & -D \end{bmatrix}\right).$$

Consider the unitary operator $U = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ on $H \oplus H$. Then $2\left[\begin{smallmatrix} A & 0 \\ 0 & -D \end{smallmatrix}\right] = TU + UT$. Thus

$$\max(w(A), w(D)) \leq \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}),$$

by inequality (1-8). This proves the inequality (2-2).

To prove the inequality (2-3), we note that

$$\begin{aligned} \max(w(BC), w(CB)) &= w\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right) \quad \text{(by (1-5))} \\ &= w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^2\right) \\ &\leq 2w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)^2 \quad \text{(by Lemma 2.4).} \end{aligned}$$

Since B is arbitrary, we have

$$\max(w(BC), w(CB)) \le 2w\left(\begin{bmatrix} 0 & -B \\ C & 0 \end{bmatrix}\right)^2.$$

We observe that $2\left[\begin{smallmatrix} 0 & -B \\ C & 0 \end{smallmatrix}\right] = TU - UT$, so

$$\max\big((w(BC))^{1/2}, (w(CB))^{1/2}\big) \le \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2})$$

by inequality (1-8). □

Corollary 2.6. *If $A \in B(H)$, then*

$$w(A) \le \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}) \le \|A\|.$$

Proof. Let $T = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then

$$\begin{aligned} w(A) &\le \tfrac{1}{2}(\|T\| + \|T^2\|^{1/2}) \quad \text{(by (2-2))} \\ &= \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}) \quad \text{(by (1-7))} \\ &\le \|A\|. \end{aligned}$$ □

Corollary 2.7. *If $A \in B(H)$, then $\|A + A^*\| \le \|A\| + \|A^2\|^{1/2} \le 2\|A\|$.*

Proof. Since $A + A^*$ is self-adjoint, we have

$$\begin{aligned} \tfrac{1}{2}\|A + A^*\| &= \tfrac{1}{2}w((A + A^*) \oplus (A + A^*)) && \text{(by (1-2) and (1-5))} \\ &\le w(A \oplus A^*) && \text{(by Corollary 2.3)} \\ &\le \tfrac{1}{2}\big(\|A \oplus A^*\| + \|(A \oplus A^*)^2\|^{1/2}\big) && \text{(by Corollary 2.6)} \\ &= \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}) && \text{(by (1-7))} \\ &\le \|A\|. \end{aligned}$$ □

We use some similar strategies as in [Kittaneh 2007] to prove the next two results.

Theorem 2.8. *Let $A, P \in B(H)$ such that P is a projection. Then*

$$w(AP - PA) \le \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}). \tag{2-4}$$

Proof. Using the decomposition $H = \operatorname{ran} P \oplus \ker P$ and equality (1-6), we represent P as the form $P = \left[\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where I_1 is the identity operator on $\operatorname{ran} P$. With respect to this decomposition, A can be written as $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$. Then

$$PA - AP = \begin{bmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{bmatrix}.$$

If I_2 is the identity operator on ker P and if $U = \begin{bmatrix} I_1 & 0 \\ 0 & -I_2 \end{bmatrix}$, then U is unitary and $\begin{bmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{bmatrix} = \frac{1}{2}(UA - AU)$. Therefore

$$w(AP - PA) = w\left(\begin{bmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{bmatrix}\right) = \tfrac{1}{2} w(AU - U^* A) \leq \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}),$$

where the inequality follows from (1-8). □

Theorem 2.9. *Suppose that $A \in B(H)$ is positive. Then*

$$w(AX - XA) \leq \tfrac{1}{2}\|A\|(\|X\| + \|X^2\|^{1/2}). \tag{2-5}$$

Proof. First we prove that if A is positive and a contraction, then

$$w(AX - XA) \leq \tfrac{1}{2}(\|X\| + \|X^2\|^{1/2}).$$

If $R = \sqrt{A - A^2}$, the operator

$$P = \begin{bmatrix} A & R \\ R & I - A \end{bmatrix}$$

is a projection on $H \oplus H$, because $A\sqrt{A - A^2} = \sqrt{A - A^2}A$. If $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$, then $PY - YP = \begin{bmatrix} AX - XA & -XR \\ RX & 0 \end{bmatrix}$. By the inequality (2-4), we have

$$w(YP - PY) \leq \tfrac{1}{2}(\|Y\| + \|Y^2\|^{1/2}).$$

Now if $Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} AX - XA & 0 \\ 0 & 0 \end{bmatrix} = Q(PY - YP)Q^*$, so

$$\begin{aligned} w\left(\begin{bmatrix} AX - XA & 0 \\ 0 & 0 \end{bmatrix}\right) &= w(YP - PY) && \text{(by (1-4))} \\ &\leq \tfrac{1}{2}(\|Y\| + \|Y^2\|^{1/2}) && \text{(by (2-4))} \\ &= \tfrac{1}{2}(\|X\| + \|X^2\|^{1/2}) && \text{(by (1-7))}, \end{aligned}$$

whence $w(AX - XA) \leq \frac{1}{2}(\|X\| + \|X^2\|^{1/2})$. Let A be a positive operator. It follows from the inequality

$$w\Big(\frac{A}{\|A\|}X - X\frac{A}{\|A\|}\Big) \leq \frac{1}{2}(\|X\| + \|X^2\|^{1/2})$$

that $w(AX - XA) \leq \frac{1}{2}\|A\|(\|X\| + \|X^2\|^{1/2})$. □

Corollary 2.10. *If $A, B \in B(H)$ such that A is positive and B is self-adjoint, then*

$$\|AB - BA\| \leq \|A\|\|B\|. \tag{2-6}$$

Proof. The inequality (2-6) follows from (2-5) by letting $X = B$. □

Corollary 2.11. *Suppose that $T \in B(H)$ has the cartesian decomposition $T = A + iB$ such that A is positive and B is self-adjoint. Then*

$$\|T^*T - TT^*\| \leq \|A\|^2 + \|B\|^2.$$

Proof. By (2-6) and the arithmetic–geometric mean inequality, we have

$$\|T^*T - TT^*\| = 2\|AB - BA\| \leq 2\|A\|\|B\| \leq \|A\|^2 + \|B\|^2. \qquad \square$$

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